The Discrete De Rham method for electromagnetic models: from Maxwell to Yang–Mills to manifolds

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NEMESIS New generation methods for numerical simulations

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Outline

Slides

Maxwell's equations and preservation of divergence

Maxwell: E electric field, B magnetic field, ϵ_0 and μ_0 vacuum permittivity and permeability:

$$
\begin{array}{rr}
\partial_t B + \mathbf{curl}\, E = 0 & \operatorname{div} B = 0 \\
\mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B = 0 & \operatorname{div} E = 0 \\
\textit{Evolutions} & \textit{Constraints}
\end{array}
$$

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$$\begin{aligned} \partial_t B + \mathbf{curl}\, E &= 0 & \operatorname{div} B &= 0 \\ \mu_0\epsilon_0\partial_t E - \mathbf{curl}\, B &= 0 & \operatorname{div} E &= 0 \\ \textit{Evolutions} & & \textit{Constraints} & \end{aligned}$$

Evolution preserves constraints:

$$0 = \operatorname{div}(\partial_t B + \mathbf{curl}\, E) = \partial_t(\operatorname{div} B) + \cancel{\operatorname{div}\mathbf{curl}\, E} = \partial_t(\operatorname{div} B).$$

Key property: $\operatorname{Im}\mathbf{curl} \subset \ker\operatorname{div}$.

The de Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}.$$

Complex: the image of each operator is included in the kernel of the next one.

$$\mathrm{Im}\,\mathbf{grad} \subset \ker\mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} \subset \ker\mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

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$$\mathrm{Im}\,\mathbf{grad} \subset \ker \mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} \subset \ker \mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

Exactness of the complex: the image of each operator is *equal* to the kernel of the next one.

$$\mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

Requires topological property on Ω:

- ◦ No tunnel for $\mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}$.
- ◦ No void for $\mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}$.

Exactness of the complex $\Rightarrow$ Maxwell in potential form

Assumption: Ω is topologically trivial (no void, no tunnel).

Magnetic vector potential: We have $\operatorname{Im} \mathbf{curl} = \ker \operatorname{div}$ so

$$\operatorname{div} B = 0 \Longrightarrow \exists A \text{ s.t. } B = \mathbf{curl}\, A.$$

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Temporal gauge: A is not unique, but the *Hodge* decomposition gives

$$A = A_0 + \mathbf{grad}\, \varphi \quad \text{ with } A_0 \perp \operatorname{Im} \mathbf{grad}.$$

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Write $E = E_0 + \mathbf{grad}\,\psi$ and choose φ such that $\partial_t \varphi = -\psi$. Then

$$0 = \partial_t B + \mathbf{curl}\, E = \mathbf{curl}(\partial_t A + E) = \mathbf{curl}(\partial_t A_0 + E_0).$$

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$$0 = \partial_t B + \mathbf{curl}\, E = \mathbf{curl}(\partial_t A + E) = \mathbf{curl}(\partial_t A_0 + E_0).$$

Since $(\partial_t A_0 + E_0) \perp \mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}$, we infer $\partial_t A_0 + E_0 = 0$ and thus

$$\partial_t A = -E.$$

Maxwell equations in magnetic potential: $\mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B = 0$ is recast as

$$\mu_0 \epsilon_0 \partial_t^2 A + \mathbf{curl}\,\mathbf{curl}\, A = 0$$

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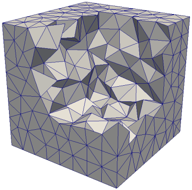


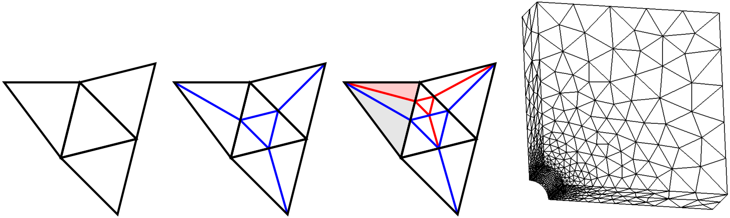
$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\textbf{grad}} & V_h^1 & \xrightarrow{\textbf{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl};\Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \end{array}$$

Example: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

- **Gluing only works on special meshes!**



- Approach limited to conforming meshes with standard elements

$\Longrightarrow$ local refinement requires to trade mesh size for mesh quality

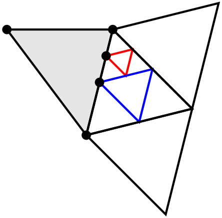
$\Longrightarrow$ complex geometries may require a large number of elements

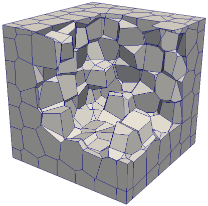
$\Longrightarrow$ the element shape cannot be adapted to the solution

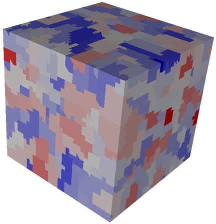
- Need for (global) basis functions

$\Longrightarrow$ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes





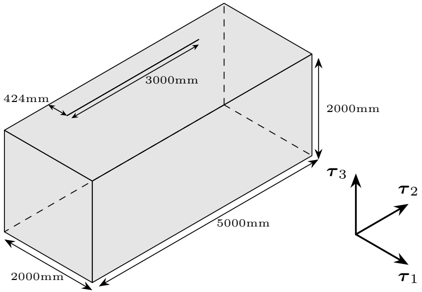


- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

A practical example in an industrial environment I

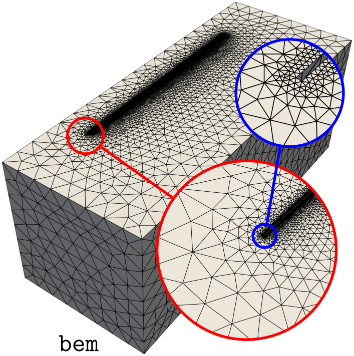
[Touzalin, 2025]

Problem: use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.

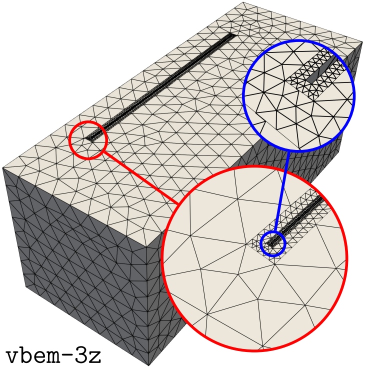


A practical example in an industrial environment II

Meshes: conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).



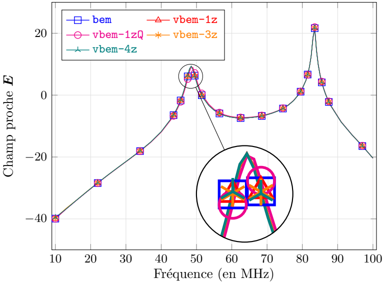
bem

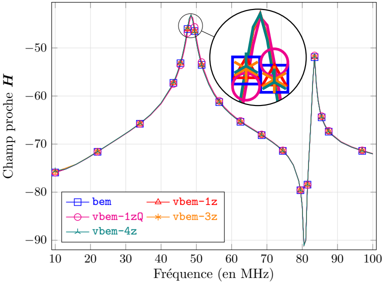


vbem-3z

A practical example in an industrial environment III

Accuracy: comparison of modulus of reflected near fields at the center of the box.





Computational cost

Method	*Assembly*	*Resolution*
bem	813s	125s
vbem-3z	321s	19s

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Overview of the Discrete De Rham (DDR) complex I

- Construct a fully discrete complex of bespoke finite-dimensional spaces and operators:

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \uparrow{\scriptstyle \underline{I}^k_{\mathbf{grad},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},h}} & & \uparrow{\scriptstyle I^k_{L^2,h}} & & \\
\mathbb{R} & \hookrightarrow & C^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{\Omega}) & \xrightarrow{0} & \{0\}
\end{array}
$$

- Discrete spaces are not made of functions but:
  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Interpolators $\underline{\boldsymbol{I}}^k_{\bullet,h}$ give meaning to these polynomials/DOFs as moments.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Guiding principles for the construction

Joint work with D. Di Pietro and F. Rapetti.
(Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023a].)

- Hierarchical construction: from vertices, to edges, to faces, to elements.

- Enhancement: on each (relevant) mesh entity,
  - discrete differential operator first,
  - potential reconstruction using the discrete differential operator.

  (both polynomially consistent, both based on IBP formulas.)

- The definition of the spaces (DOFs) also guided by these IBP formulas.

Same guiding principles as the Hybrid High-Order (HHO) method [Di Pietro and Droniou, 2020].

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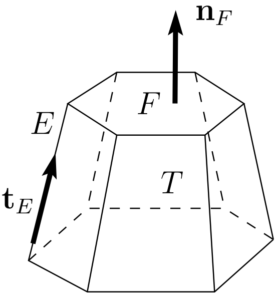
Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements (T), faces (F), edges (E), vertices (V), with intrinsic orientations (tangent, normal).
  - $|X|$: measure of X (length of edge, area of face, volume of element).
  - $\omega_{TF} \in \{+1, -1\}$ such that $\omega_{TF}\mathbf{n}_F$ outer normal to T.
  - $\omega_{FE} \in \{+1, -1\}$ such that $\omega_{FE}\mathbf{t}_E$ clockwise on F.



$\mathcal{P}^k$-consistent rot on a face F ($k \geq 0$)

- Image of vector rotor on F:

$$\boldsymbol{\mathcal{R}}^\ell(F) = \mathbf{rot}_F\, \mathcal{P}^{\ell+1}(F).$$

- IBP is the starting point: if $\boldsymbol{v} \in \boldsymbol{C}^1(\overline{T})$, with $\boldsymbol{v}_{\mathrm{t},F}$ the tangential component on F, we have

$$\int_F (\mathrm{rot}_F\, \boldsymbol{v}_{\mathrm{t},F})\, r = \int_F \boldsymbol{v}_{\mathrm{t},F} \cdot \underbrace{\mathbf{rot}_F\, r}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \mathbf{t}_E) \underbrace{r}_{\in \mathcal{P}^k(E)} \qquad \forall r \in \mathcal{P}^k(F)$$

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$$\int_F (\mathrm{rot}_F\, \boldsymbol{v}_{\mathrm{t},F})\, r = \int_F \pi_{\boldsymbol{\mathcal{R}},F}^{k-1} \boldsymbol{v}_{\mathrm{t},F} \cdot \underbrace{\mathbf{rot}_F\, r}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^k (\boldsymbol{v} \cdot \mathbf{t}_E) \underbrace{r}_{\in \mathcal{P}^k(E)} \qquad \forall r \in \mathcal{P}^k(F)$$

with $\pi_{\boldsymbol{\mathcal{Z}},\mathsf{P}}^k$ the L^2-projection on $\boldsymbol{\mathcal{Z}}^k(\mathsf{P})$.

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$$\int_F (\mathrm{rot}_F \, \boldsymbol{v}_{\mathrm{t},F}) \, r = \int_F \pi^{k-1}_{\boldsymbol{\mathcal{R}},F} \boldsymbol{v}_{\mathrm{t},F} \cdot \, \mathbf{rot}_F \, r - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi^k_{\mathcal{P},E} (\boldsymbol{v} \cdot \mathbf{t}_E) r \qquad \forall r \in \mathcal{P}^k(F)$$

- Space and interpolator:

$$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F} = \left\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, (v_E)_{E \in \mathcal{E}_F}) \, : \, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F) \, , \; v_E \in \mathcal{P}^k(E) \right\},$$

$$\underline{\boldsymbol{I}}^k_{\mathrm{rot},E} \boldsymbol{v} = (\pi^{k-1}_{\boldsymbol{\mathcal{R}},F} \boldsymbol{v}_{\mathrm{t},F}, (\pi^k_{\mathcal{P},E}(\boldsymbol{v} \cdot \mathbf{t}_E))_{E \in \mathcal{E}_F}) \qquad \forall \boldsymbol{v} \in \boldsymbol{C}(\overline{F}).$$

$\mathcal{P}^k$-consistent rot on a face F ($k \geq 0$)

- Starting point:

$$\int_F (\mathrm{rot}_F\, \boldsymbol{v}_{\mathrm{t},F})\, r = \int_F \pi^{k-1}_{\boldsymbol{\mathcal{R}},F} \boldsymbol{v}_{\mathrm{t},F} \cdot\, \mathbf{rot}_F\, r - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \pi^k_{\mathcal{P},E}(\boldsymbol{v}\cdot \mathbf{t}_E) r \qquad \forall r \in \mathcal{P}^k(F)$$

- Space:
$$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F} = \left\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, (v_E)_{E\in\mathcal{E}_F}) \,:\, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F)\,,\; v_E \in \mathcal{P}^k(E) \right\}.$$
- Face curl (rot) reconstruction on $\underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$: define $C^k_F \underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F)$ s.t.

$$\int_F (C^k_F \underline{\boldsymbol{v}}_F) r = \int_F \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \cdot \mathbf{rot}_F\, r - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E\, r \qquad \forall r \in \mathcal{P}^k(F).$$

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Consistency: $C^k_F \underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v} = \pi^k_{\mathcal{P},F}(\mathrm{rot}_F\, \boldsymbol{v}_{\mathrm{t},F})$.

$\mathcal{P}^k$-consistent tangential potential on a face F $(k \geq 0)$

- IBP is the starting point: for $\boldsymbol{v} \in \boldsymbol{C}^1(\overline{T})$,

$$\int_F \boldsymbol{v}_{\mathrm{t},F} \cdot \mathbf{rot}_F\, r = \int_F (\mathrm{rot}_F\, \boldsymbol{v}_{\mathrm{t},F})\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \mathbf{t}_E) r \quad \forall r \in \mathcal{P}^{k+1}(F).$$

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- $\underline{\boldsymbol{X}}^k_{\mathrm{rot},F} = \Big\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, (v_E)_{E \in \mathcal{E}_F}) \,:\, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F)\,,\ v_E \in \mathcal{P}^k(E) \Big\}$.
- For $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ define $\boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{P}}^k(F)$ such that

$$\int_F \boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \cdot \mathbf{rot}_F\, r = \int_F C^k_F \underline{\boldsymbol{v}}_F\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r \qquad \forall r \in \mathcal{P}^{k+1}(F).$$

$\mathcal{P}^k$-consistent tangential potential on a face F $(k \geq 0)$

- $\displaystyle\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot \mathbf{rot}_F\, r = \int_F C_F^k \underline{\boldsymbol{v}}_F\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r \qquad \forall r \in \mathcal{P}^{k+1}(F).$

  $\rightsquigarrow$ only defines $\pi_{\boldsymbol{\mathcal{R}},F}^k(\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F)$ (projection on $\boldsymbol{\mathcal{R}}^k(F)$).

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- Enrich the discrete space by a complement space: considering

$$\boldsymbol{\mathcal{P}}^k(F) = \boldsymbol{\mathcal{R}}^k(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F) \quad \text{with} \quad \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F) \coloneqq (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F),$$

  we re-define

$$\underline{\boldsymbol{X}}_{\mathrm{rot},F}^k = \Big\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c}}, (v_E)_{E\in\mathcal{E}_F}) : \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c}} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E) \Big\}$$

$$\underline{\boldsymbol{I}}_{\mathrm{rot},E}^k \boldsymbol{v} = (\pi_{\boldsymbol{\mathcal{R}},F}^{k-1} \boldsymbol{v}_{\mathrm{t},F}, \pi_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c},k} \boldsymbol{v}_{\mathrm{t},F}, (\pi_{\mathcal{P},E}^k(\boldsymbol{v}\cdot\mathbf{t}_E))_{E\in\mathcal{E}_F}).$$

$\mathcal{P}^k$-consistent tangential potential on a face F $(k \geq 0)$

- $\displaystyle\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot \mathbf{rot}_F\, r = \int_F C_F^k \underline{\boldsymbol{v}}_F\, r + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r \qquad \forall r \in \mathcal{P}^{k+1}(F).$

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$$\underline{\boldsymbol{X}}_{\mathrm{rot},F}^k = \Big\{ \underline{\boldsymbol{v}}_F = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c}}, (v_E)_{E\in\mathcal{E}_F}) \,: \\ \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c}} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E) \Big\}$$

$$\underline{\boldsymbol{I}}_{\mathrm{rot},E}^k \boldsymbol{v} = (\pi_{\boldsymbol{\mathcal{R}},F}^{k-1} \boldsymbol{v}_{\mathrm{t},F}, \pi_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c},k} \boldsymbol{v}_{\mathrm{t},F}, (\pi_{\mathcal{P},E}^k(\boldsymbol{v}\cdot\mathbf{t}_E))_{E\in\mathcal{E}_F}).$$

- Complete definition of $\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F$ by setting $\pi_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c},k}(\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F) = \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^{\mathrm{c}}$.

> Polynomial consistency: $\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{I}}_{\mathrm{rot},F}^k \boldsymbol{v} = \boldsymbol{v}_{\mathrm{t},F}$ for all $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(T)$.

$\mathcal{P}^k$-consistent curl on an element T $(k \geq 0)$

Same principle, based on IBP...

- Space:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T} = \Big\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}, \boldsymbol{v}^{\rm c}_{\boldsymbol{\mathcal{R}},T}, (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}^{\rm c}_{\boldsymbol{\mathcal{R}},F})_{F\in\mathcal{F}_T}, (v_E)_{E\in\mathcal{E}_T}) \;:\; \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \in \boldsymbol{\mathcal{R}}^{k-1}(T)\,,\ \boldsymbol{v}^{\rm c}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{{\rm c},k}(T),\ \text{etc.} \Big\}$$

- Curl reconstruction $\mathsf{C}^k_T \underline{\boldsymbol{v}}_T \in \boldsymbol{\mathcal{P}}^k(T)$ such that, for all $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}^k(T)$,

$$\int_T \mathsf{C}^k_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} = \int_T \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \cdot \mathbf{curl}\,\boldsymbol{w} + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}^k_{{\rm t},F} \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w} \times \mathbf{n}_F).$$

- Potential reconstruction $\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T \in \boldsymbol{\mathcal{P}}^k(T)$ such that, for all $\boldsymbol{w} \in (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}^k(T) \subset \boldsymbol{\mathcal{P}}^{k+1}(T)$,

$$\int_T \boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T \cdot \mathbf{curl}\,\boldsymbol{w} = \int_T \mathsf{C}^k_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} - \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}^k_{{\rm t},F} \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w} \times \mathbf{n}_F)$$

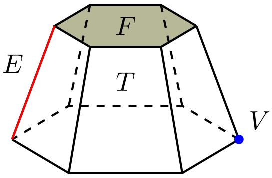
and complete by $\pi^{{\rm c},k}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T) = \boldsymbol{v}^{\rm c}_{\boldsymbol{\mathcal{R}},T}$.

Outline

Slides

The Discrete de Rham method

- Contrary to FE, do not seek explicit (or any!) basis functions.
- Replace continuous spaces by fully discrete ones made of vectors of polynomials, representing polynomial moments when interpreted through the interpolator.
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Build polynomial reconstructions (differential operator, potential) between the spaces.



DDR complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

DDR complex

$$\mathbb{R} \xrightarrow{I^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete divergence space:

$$\underline{\boldsymbol{X}}^k_{\mathrm{div},h} = \Big\{ \underline{\boldsymbol{z}}_T = ((\boldsymbol{z}_{\boldsymbol{\mathcal{G}},T}, \boldsymbol{z}^{\mathrm{c}}_{\boldsymbol{\mathcal{G}},T})_{T\in\mathcal{T}_h}, (z_F)_{F\in\mathcal{F}_h}) \ : \\ \boldsymbol{z}_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{k-1}(T)\,,\ \boldsymbol{z}^{\mathrm{c}}_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)\,,\ z_F \in \mathcal{P}^k(F) \Big\}$$

with $\boldsymbol{\mathcal{G}}^{k-1}(T) = \mathbf{grad}\,\mathcal{P}^k(T)\,, \quad \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) = (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}^{k-1}(T).$

Discrete curl: project the face/element curl reconstructions

$$\mathbf{C}^k_T \underline{\boldsymbol{v}}_T \in \boldsymbol{\mathcal{P}}^k(T), \qquad C^k_F \underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F)$$

on the proper spaces:

$$\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h = \Big((\pi^{k-1}_{\boldsymbol{\mathcal{G}},T} \mathbf{C}^k_T \underline{\boldsymbol{v}}_T, \pi^{\mathrm{c},k}_{\boldsymbol{\mathcal{G}},T} \mathbf{C}^k_T \underline{\boldsymbol{v}}_T)_{T\in\mathcal{T}_h}, (C^k_F \underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_h} \Big).$$

L^2-like inner products

- Potential reconstructions in each space: $P^{k+1}_{\mathbf{grad},T}$, $\boldsymbol{P}^k_{\mathbf{curl},T}$, $\boldsymbol{P}^k_{\mathrm{div},T}$.
- Local L^2-like inner product on the DDR spaces:

  for $(\bullet,\ell)=(\mathbf{grad},k+1)$, $(\mathbf{curl},k)$ or (div,k),

  $$(x_T,y_T)_{\bullet,T}=\int_T \boldsymbol{P}^\ell_{\bullet,T}x_T\cdot\boldsymbol{P}^\ell_{\bullet,T}y_T+\mathrm{s}_{\bullet,T}(x_T,y_T)\qquad\forall x_T,y_T\in\underline{X}^k_{\bullet,T},$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face/edge potentials*).

- Global L^2-like product $(\cdot,\cdot)_{\bullet,h}$ by standard assembly of local ones.

DOF by mesh entities

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\boldsymbol{\mathcal{RT}}^k(F)$	$\boldsymbol{\mathcal{RT}}^k(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

$\boldsymbol{\mathcal{RT}}^k$: Raviart–Thomas space, $\boldsymbol{\mathcal{N}}^k$: Nédélec space.

The DDR complex and its properties

- It is a complex, with the same cohomology as the continuous de Rham complex. [Di Pietro et al., 2020], [Di Pietro et al., 2023]
- Poincaré inequalities. [Di Pietro and Droniou, 2021a], [Di Pietro and Droniou, 2023a], [Di Pietro and Hanot, 2024b]
- Consistency (both primal and adjoint). [Di Pietro and Droniou, 2023a]
- Commutation properties between the interpolators and the continuous/discrete operators. [Di Pietro and Droniou, 2021b]

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- Commutation properties between the interpolators and the continuous/discrete operators. [Di Pietro and Droniou, 2021b]

$\rightsquigarrow$ optimally-convergent scheme (error in $\mathcal{O}(h^{k+1})$) for a range of models: magnetostatics, Stokes & Navier–Stokes, etc.

$\rightsquigarrow$ robust error estimates with respect to some physical parameters. [Di Pietro and Droniou, 2021b, Beirão da Veiga et al., 2022]

Outline

Slides

Weak form of Maxwell's equations

Strong form: initial conditions E_0, B_0, essential BC on $E \times n$ and $B \cdot n$, and

$$\begin{aligned} \partial_t B + \mathbf{curl}\, E &= 0 & \operatorname{div} B &= 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B &= 0 & \operatorname{div} E &= 0 \\ &\textit{Evolutions} & &\textit{Constraints} \end{aligned}$$

Weak form of Maxwell's equations

Strong form: initial conditions E_0, B_0, essential BC on $E \times n$ and $B \cdot n$, and

$$\begin{array}{cc} \partial_t B + \mathbf{curl}\, E = 0 & \operatorname{div} B = 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B = 0 & \operatorname{div} E = 0 \\ \textit{Evolutions} & \textit{Constraints} \end{array}$$

Weak form: find $E : [0, T] \to \boldsymbol{H}(\mathbf{curl}; \Omega)$ and $B : [0, T] \to \boldsymbol{H}(\operatorname{div}; \Omega)$ that satisfy the ICs and BCs and, at all time,

$$\begin{aligned} (\partial_t B, \boldsymbol{w})_{L^2(\Omega)} + (\mathbf{curl}\, E, \boldsymbol{w})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\operatorname{div}; \Omega), \\ \mu_0 \epsilon_0 (\partial_t E, \boldsymbol{v})_{L^2(\Omega)} - (B, \mathbf{curl}\, \boldsymbol{v})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega). \end{aligned}$$

Weak form of Maxwell's equations

Strong form: initial conditions E_0, B_0, essential BC on $E \times n$ and $B \cdot n$, and

$$\begin{array}{rr} \partial_t B + \mathbf{curl}\, E = 0 & \operatorname{div} B = 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B = 0 & \operatorname{div} E = 0 \\ \textit{Evolutions} & \textit{Constraints} \end{array}$$

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Preservation of constraints:

- Strong on B since $\partial_t B + \mathbf{curl}\, E = 0$ in strong sense.
- Weak on E: $\partial_t (E, \mathbf{grad}\, q)_{L^2(\Omega)} = 0$ for all $q \in H^1_0(\Omega)$

DDR scheme for Maxwell's equations

Weak form: find $E : [0,T] \to \boldsymbol{H}(\mathbf{curl};\Omega)$ and $B : [0,T] \to \boldsymbol{H}(\mathrm{div};\Omega)$ that satisfy the ICs and BCs and, at all time,

$$\begin{aligned}(\partial_t B, \boldsymbol{w})_{L^2(\Omega)} + (\mathbf{curl}\, E, \boldsymbol{w})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div};\Omega),\\ \mu_0\epsilon_0(\partial_t E, \boldsymbol{v})_{L^2(\Omega)} - (B, \mathbf{curl}\, \boldsymbol{v})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).\end{aligned}$$

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$$
\begin{aligned}
(\partial_t B, \boldsymbol{w})_{L^2(\Omega)} + (\mathbf{curl}\, E, \boldsymbol{w})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div};\Omega),\\
\mu_0\epsilon_0(\partial_t E, \boldsymbol{v})_{L^2(\Omega)} - (B, \mathbf{curl}\, \boldsymbol{v})_{L^2(\Omega)} &= 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).
\end{aligned}
$$

Scheme: find $\underline{\boldsymbol{E}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{B}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ that satisfy the *interpolated* ICs and BCs and, at all time,

$$
\begin{aligned}
(\partial_t \underline{\boldsymbol{B}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} &= 0 \quad \forall \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h,0},\\
\mu_0\epsilon_0(\partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{B}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} &= 0 \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h,0}.
\end{aligned}
$$

DDR scheme for Maxwell's equations

Scheme: find $\underline{\boldsymbol{E}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{B}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ that satisfy the *interpolated* ICs and BCs and, at all time,

$$
\begin{aligned}
(\partial_t \underline{\boldsymbol{B}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} &= 0 \quad \forall \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h,0},\\
\mu_0\epsilon_0(\partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{B}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} &= 0 \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h,0}.
\end{aligned}
$$

Preservation of constraints:

- Strong on $\underline{\boldsymbol{B}}_h$ since $\partial_t \underline{\boldsymbol{B}}_h + \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h = 0$ in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ and, by complex property,

$$
0 = D^k_h(\partial_t \underline{\boldsymbol{B}}_h + \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h) = \partial_t(D^k_h \underline{\boldsymbol{B}}_h) + \cancel{D^k_h \circ \underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{B}}_h = \partial_t(D^k_h \underline{\boldsymbol{B}}_h).
$$

DDR scheme for Maxwell's equations

Scheme: find $\underline{\boldsymbol{E}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{B}}_h : [0,T] \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ that satisfy the *interpolated* ICs and BCs and, at all time,

$$\begin{aligned}(\partial_t \underline{\boldsymbol{B}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{w}}_h)_{\mathrm{div},h} = 0 \quad \forall \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h,0},\\ \mu_0\epsilon_0(\partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{B}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} = 0 \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h,0}.\end{aligned}$$

Preservation of constraints:

- Strong on $\underline{\boldsymbol{B}}_h$ since $\partial_t \underline{\boldsymbol{B}}_h + \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h = 0$ in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ and, by complex property,

$$0 = D^k_h(\partial_t \underline{\boldsymbol{B}}_h + \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{E}}_h) = \partial_t(D^k_h \underline{\boldsymbol{B}}_h) + \cancel{D^k_h \circ \underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{B}}_h = \partial_t(D^k_h \underline{\boldsymbol{B}}_h).$$

- Weak on $\underline{\boldsymbol{E}}_h$. $D^k_h \underline{\boldsymbol{E}}_h$ does not exist, but, for all $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}$,

$$0 = \mu_0\epsilon_0(\partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{B}}_h, \cancel{\underline{\boldsymbol{C}}^k_h \circ \underline{\boldsymbol{G}}^k_h} \underline{q}_h)_{\mathrm{div},h} = \mu_0\epsilon_0 \partial_t(\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h}$$

Outline

Slides

Overview

Joint works with J.J. Qian and T. Oliynyk
Ref: [Droniou and Qian, 2024], [Droniou and Qian, 2024].

- Model of elementary particles in the universe.
- Non-linear extension of electromagnetism.
- Based on Lie algebras.

Outline

Slides

Notations

- Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$
  - Vector space $\mathfrak{g}$
  - Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, antisymmetric, bilinear etc.
- $(e_I)_I$ a basis of the Lie algebra

Example

- $\mathfrak{g} = \mathfrak{su}(2)$ matrix Lie algebra, $[A, B] := AB - BA$

$$e_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, e_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Decomposition of $\mathfrak{g}$-valued function $f = \sum_I f^I \otimes e_I$
- Inner product $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ s.t.

$$\langle [a,b], c\rangle = \langle a, [b,c]\rangle \qquad \forall a,b,c \in \mathfrak{g}.$$

Yang–Mills equations

- A: Gauge potential
- Electric field E and magnetic field B defined by (temporal gauge)

$$E = -\partial_t A$$

$$B = \mathbf{curl}\, A$$

Maxwell equations:

$$\partial_t E = \mathbf{curl}\, B \qquad \operatorname{div} E = 0$$

$$\partial_t B = -\mathbf{curl}\, E \qquad \operatorname{div} B = 0$$

Yang–Mills equations

- A: Gauge potential
- Electric field E and magnetic field B defined by (temporal gauge)

$$E = -\partial_t A$$
$$B = \mathbf{curl}\, A + \frac{1}{2}\star[A, A]$$

Yang–Mills equations:

$$\partial_t E = \mathbf{curl}\, B + \star[A, B] \qquad \mathrm{div}\, E + \star[A, \star E] = 0$$
$$\partial_t B = -\mathbf{curl}\, E - \star[A, E] \qquad \mathrm{div}\, B + \star[A, \star B] = 0$$

Note: $\star[\boldsymbol{v}, \boldsymbol{w}] \coloneqq \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \otimes [e_I, e_J]$,

$\star[\boldsymbol{v}, \star\boldsymbol{w}] \coloneqq \sum_{I,J} (\boldsymbol{v}^I \cdot \boldsymbol{w}^J) \otimes [e_I, e_J]$

Weak formulation

- Find $(A, E) : [0, T] \to (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g}) \times (\boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g})$ s.t.

$$\partial_t A = -E,$$
$$\int_\Omega \langle \partial_t E, \boldsymbol{v} \rangle = \int_\Omega \langle B, \mathbf{curl}\, \boldsymbol{v} + \star[A, \boldsymbol{v}] \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; U) \otimes \mathfrak{g},$$

  where $B = \mathbf{curl}\, A + \frac{1}{2}\star[A, A]$

- Initial conditions: $A(0) = A_0$, $E(0) = E_0$.
- Constraint (was $\operatorname{div} E + \star[A, \star E] = 0$):

$$\int_\Omega \langle E, \mathbf{grad}\, q + [A, q] \rangle = 0, \qquad \forall q \in H^1(U) \otimes \mathfrak{g}.$$

Preservation of constraint?

- Much more challenging than in the linear case, requires:

$$\mathrm{div} \circ \mathbf{curl} = 0, \ \mathbf{curl} \circ \mathbf{grad} = 0 \text{ and}$$
$$\mathbf{curl}[A, q] + \star[A, \mathbf{grad}\, q] = [\mathbf{curl}\, A, q]$$

- Finite Elements schemes: algebra ok, but test functions not of correct degree (*requires to set* $[A_h, q_h]$ *as test function*).

- Polytopal (DDR) scheme: discrete operators do not satisfy the additional algebraic relation.

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- Finite Elements schemes: algebra ok, but test functions not of correct degree (*requires to set* $[A_h, q_h]$ *as test function*).

- Polytopal (DDR) scheme: discrete operators do not satisfy the additional algebraic relation.

$\rightsquigarrow$ Workaround: use a constrained formulation, enforcing via Lagrange multiplier the evolution relation

$$\int_\Omega \langle \partial_t E, \mathbf{grad}\, q + [A, q] \rangle = 0 \qquad \forall q \in H^1(\Omega) \otimes \mathfrak{g}.$$

DDR scheme

- Lie-algebra valued discrete spaces by tensorisation.
  Example: $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}$.
- Same for operators. *Example:* $\gamma^{\mathfrak{g},k}_{\mathrm{t},F} = \gamma^k_{\mathrm{t},F} \otimes \mathrm{Id}$, $\underline{\boldsymbol{C}}^{k,\mathfrak{g}}_h = \underline{\boldsymbol{C}}^k_h \otimes \mathrm{Id}$.
- Discretisation of linear terms as for Maxwell.
- Discretisation of nonlinear brackets? (e.g., $\star[A, \boldsymbol{v}]$).
  - Option 1: design a fully discrete bracket between the DDR spaces.
    Complex, but naturally provides a discrete B.
  - Option 2: use the polynomial reconstructions in all nonlinearities.
    Easy, but no associated "physical" discrete B.

DDR scheme: discretisation of cross-product bracket

$$\partial_t A = -E$$

$$\int_U \langle \partial_t E, \boldsymbol{v} \rangle + \int_U \langle \mathbf{grad}\,\lambda + [A, \lambda], \boldsymbol{v} \rangle = \int_U \langle B, \mathbf{curl}\,\boldsymbol{v} + \star[A, \boldsymbol{v}] \rangle$$

$$\int_U \langle \partial_t E, \mathbf{grad}\,q + [A, q] \rangle = 0$$

To be considered:

- $B = \mathbf{curl}\,A + \frac{1}{2}\star[A, A] \in \boldsymbol{H}(\mathrm{div}; \Omega) \otimes \mathfrak{g}$.
- $\star[\boldsymbol{v}, \boldsymbol{w}] = \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \otimes [e_I, e_J]$.

Option 1 (O1): design discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$.

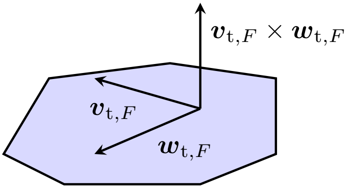
$$\underline{\boldsymbol{X}}^k_{\mathrm{div},h} = \Big\{ \underline{\boldsymbol{z}}_T = ((\boldsymbol{z}_{\boldsymbol{\mathcal{G}},T}, \boldsymbol{z}^{\mathrm{c}}_{\boldsymbol{\mathcal{G}},T})_{T\in\mathcal{T}_h}, (z_F)_{F\in\mathcal{F}_h}) \ : \\ \boldsymbol{z}_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{k-1}(T)\,,\ \boldsymbol{z}^{\mathrm{c}}_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)\,,\ z_F \in \mathcal{P}^k(F) \Big\}$$

Option 1 (O1): design discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

- Face value in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$ represents a flux:

$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \cdot \mathbf{n}_F \otimes [e_I, e_J]$$



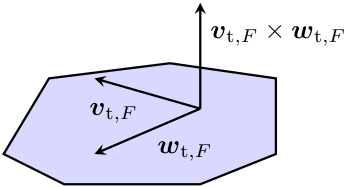
$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \star[\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{w}_{\mathrm{t},F}] \cdot \mathbf{n}_F$$

Option 1 (O1): design discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

- Face value in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$ represents a flux:

$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \cdot \mathbf{n}_F \otimes [e_I, e_J]$$



$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \star[\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{w}_{\mathrm{t},F}] \cdot \mathbf{n}_F$$

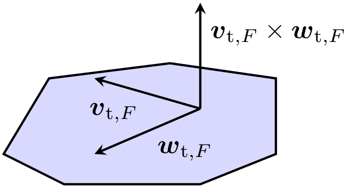
- Leads to setting $\left(\star[\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h]^{\mathrm{div},k,h}\right)_F = \pi^k_{\mathcal{P},F}(\star[\boldsymbol{\gamma}^{\mathfrak{g},k}_{\mathrm{t},F} \underline{\boldsymbol{v}}_F, \boldsymbol{\gamma}^{\mathfrak{g},k}_{\mathrm{t},F} \underline{\boldsymbol{w}}_F] \cdot \mathbf{n}_F)$

Option 1 (O1): design discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

- Face value in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$ represents a flux:

$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \sum_{I,J} (\boldsymbol{v}^I \times \boldsymbol{w}^J) \cdot \mathbf{n}_F \otimes [e_I, e_J]$$



$$\star[\boldsymbol{v},\boldsymbol{w}] \cdot \mathbf{n}_F = \star[\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{w}_{\mathrm{t},F}] \cdot \mathbf{n}_F$$

- Leads to setting $\left(\star[\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h]^{\mathrm{div},k,h}\right)_F = \pi^k_{\mathcal{P},F}(\star[\boldsymbol{\gamma}^{\mathfrak{g},k}_{\mathrm{t},F}\underline{\boldsymbol{v}}_F, \boldsymbol{\gamma}^{\mathfrak{g},k}_{\mathrm{t},F}\underline{\boldsymbol{w}}_F] \cdot \mathbf{n}_F)$
- Element value built using $\boldsymbol{P}^k_{\mathbf{curl},T}$.

Option 1 (O1): design discrete bracket in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}$

$$\star[\cdot,\cdot]^{\mathrm{div},k,h} : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \times (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \otimes \mathfrak{g}) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \otimes \mathfrak{g}.$$

Then

$$\int_\Omega \langle B, \mathbf{curl}\,\boldsymbol{v} + \star[A,\boldsymbol{v}]\rangle \rightsquigarrow (\underline{\boldsymbol{B}}_h, \underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_h + \star[\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{v}}_h]^{\mathrm{div},k,h})_{\mathrm{div},\mathfrak{g},h}$$

with $\underline{\boldsymbol{B}}_h = \underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h + \frac{1}{2}\star[\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{A}}_h]^{\mathrm{div},k,h} \in \underline{\boldsymbol{X}}^k_{\mathrm{div},k} \otimes \mathfrak{g}$.

Option 2 (O2): potentials in continuous bracket

With $B = \mathbf{curl}\, A + \frac{1}{2}\star[A, A]$:

$$\int_\Omega \langle B, \mathbf{curl}\, \boldsymbol{v} + \star[A, \boldsymbol{v}]\rangle = \int_\Omega \langle \mathbf{curl}\, A, \mathbf{curl}\, \boldsymbol{v}\rangle + \int_\Omega \langle \mathbf{curl}\, A, \star[A, \boldsymbol{v}]\rangle + \int_\Omega \langle \frac{1}{2}\star[A, A], \mathbf{curl}\, \boldsymbol{v} + \star[A, \boldsymbol{v}]\rangle$$

discretised as

$$(\underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_h)_{\mathrm{div},\mathfrak{g},h} + \int_\Omega \langle \boldsymbol{C}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h, \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k} \underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k} \underline{\boldsymbol{v}}_h]\rangle + \int_\Omega \langle \frac{1}{2}\star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k} \underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},h} \underline{\boldsymbol{A}}_h], \boldsymbol{C}_h^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_h + \star[\boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k} \underline{\boldsymbol{A}}_h, \boldsymbol{P}_{\mathbf{curl},h}^{\mathfrak{g},k} \underline{\boldsymbol{v}}_h]\rangle.$$

Option 2 (O2): potentials in continuous bracket

With $B = \mathbf{curl}\, A + \frac{1}{2}\star[A, A]$:

$$\int_\Omega \langle B, \mathbf{curl}\, \boldsymbol{v} + \star[A, \boldsymbol{v}]\rangle = \int_\Omega \langle \mathbf{curl}\, A, \mathbf{curl}\, \boldsymbol{v}\rangle + \int_\Omega \langle \mathbf{curl}\, A, \star[A, \boldsymbol{v}]\rangle + \int_\Omega \langle \frac{1}{2}\star[A, A], \mathbf{curl}\, \boldsymbol{v} + \star[A, \boldsymbol{v}]\rangle$$

discretised as

$$(\underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h, \underline{\boldsymbol{C}}_h^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_h)_{\mathrm{div},\mathfrak{g},h} + \int_\Omega \langle \boldsymbol{C}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h, \star[\boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h, \boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{v}}_h]\rangle$$
$$+ \int_\Omega \langle \frac{1}{2}\star[\boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h, \boldsymbol{P}^{\mathfrak{g},h}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h], \boldsymbol{C}_h^{k,\mathfrak{g}} \underline{\boldsymbol{v}}_h + \star[\boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h, \boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{v}}_h]\rangle.$$

- Magnetic field not in the discrete $\boldsymbol{H}(\mathrm{div}; U) \otimes \mathfrak{g}$, but piecewise polynomial:

$$\boldsymbol{B}_h = \boldsymbol{C}_h^{k,\mathfrak{g}} \underline{\boldsymbol{A}}_h + \frac{1}{2}\star[\boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h, \boldsymbol{P}^{\mathfrak{g},k}_{\mathbf{curl},h} \underline{\boldsymbol{A}}_h].$$

Outline

Slides

Numerical tests

- Domain $\Omega = (0,1)^3$, $t \in [0,1]$
- Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$

$$e_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; e_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

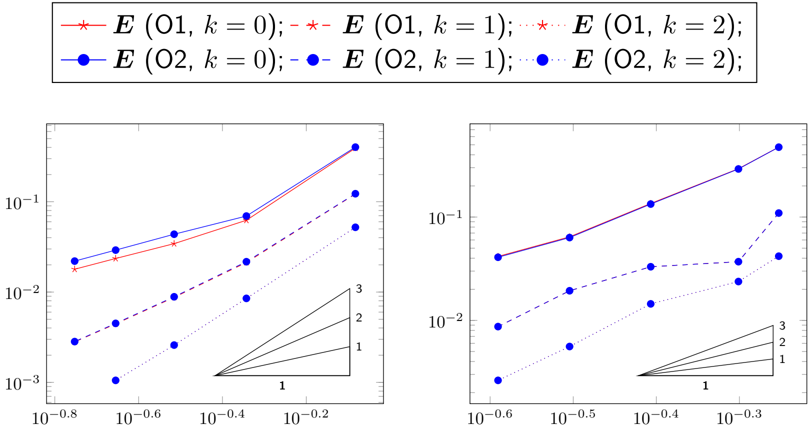
- Mesh sequences
  - Voronoi polytopal meshes
  - Tetrahedral meshes
- Newton iterations
  - Stopping criterion $\epsilon = 10^{-6}$
  - Timestep $\delta t = \min\{0.1, 0.2h^{k+1}\}$
  - Direct solver

Newton iterations (O1)

	Voronoi mesh				
	1	2	3	4	5
h	0.83	0.45	0.31	0.22	0.18
δt	0.1	0.083	0.059	0.043	0.034
N_{avg} $(\epsilon = 10^{-6})$	2	2	2	2.6	1.4
N_{avg} $(\epsilon = 10^{-10})$	2.3	2.3	2.1	3.3	2

	Tetrahedral mesh				
	1	2	3	4	5
h	0.56	0.50	0.39	0.31	0.26
δt	0.1	0.091	0.077	0.063	0.05
N_{avg} $(\epsilon = 10^{-6})$	2	2	2	1.9	1.6
N_{avg} $(\epsilon = 10^{-10})$	2	2	2	2	2

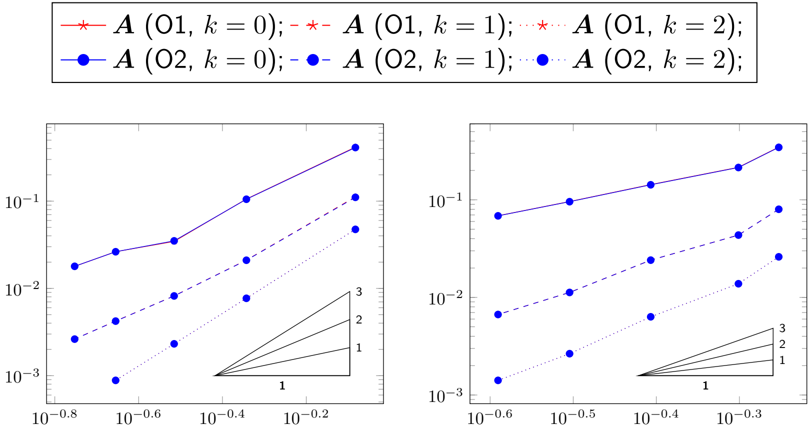
Errors on $\boldsymbol{E}$



(a) "Voro-small-0" mesh

(b) "Tetgen-Cube-0" mesh

Errors on $\boldsymbol{A}$



(a) "Voro-small-0" mesh

(b) "Tetgen-Cube-0" mesh

Constraint preservation

	Voronoi mesh		Tetrahedral mesh	
O1	1	3	2	4
$k = 0$	8.47329e-15	3.05676e-14	2.06362e-14	4.75412e-14
$k = 1$	1.4144e-13	8.93075e-13	3.67781e-13	1.81426e-12
$k = 2$	3.69918e-12	1.18207e-10	3.82037e-12	2.77407e-11
O2	1	3	2	4
$k = 0$	8.16124e-15	3.13617e-14	2.01667e-14	4.77633e-14
$k = 1$	9.8531e-14	8.83056e-13	3.69107e-13	1.81787e-12
$k = 2$	4.16428e-12	7.99416e-11	3.81537e-12	2.77391e-11

Table: Maximum over n of the discrete constraint, measured in the dual norm

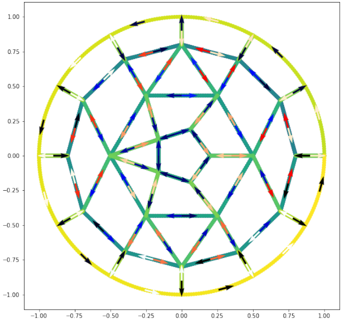
Outline

Slides

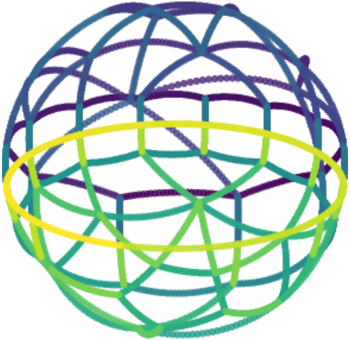
Overview

Joint work with M. Hanot and T. Oliynyk (Ref: [Droniou et al., 2025].)

- Based on DDR in the framework of exterior calculus [Bonaldi et al., 2024, Di Pietro et al., 2025a].
- Introduces intrinsic notion of (trace-compatible) polynomials on polygonal meshes on manifolds.



Mesh in a chart



Mesh on the sphere

Conclusions and extensions I

- An arbitrary order numerical method, applicable on polygonal/polyhedral grids, to discretise the de Rham complex.
- Same cohomology as the de Rham complex (weak formulation $\rightsquigarrow$ scheme, without additional trick).
- Algebraic and analytical properties: commutation properties, Poincaré inequalities, optimal consistency estimates (primal and adjoint), etc.
- Serendipity to systematically reduce the number of DOFs (without loosing in accuracy).
- Can be set up in the exterior calculus framework (unified analysis, easier extension to manifolds).

Conclusions and extensions II

- Range of applications: electro-magnetism models (Maxwell, magnetostatics, etc.), Stokes and Navier–Stokes, Reissner–Mindlin plate problems, etc. Often leads to schemes with specific robustness properties.
  [Di Pietro and Droniou, 2021a, Di Pietro and Droniou, 2021b, Beirão da Veiga et al., 2022, Di Pietro et al., 2024]

- Extended complexes (Stokes complex, Hessian complex, elasticity complex, plates complex, div-div complex, etc.), including by Bernstein–Gelfand–Gelfand construction.
  [Hanot, 2023, Di Pietro and Droniou, 2023b, Di Pietro and Hanot, 2024a, Botti et al., 2023, Di Pietro et al., 2025b]

◦ Notes and series of introductory lectures to DDR:

`https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html`

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

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Thank you for your attention!

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