A bubble-enriched polytopal method for contact mechanics

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ACSIOM seminar, 2 April 2024

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Funded by the European Union

erc European Research Council Established by the European Commission

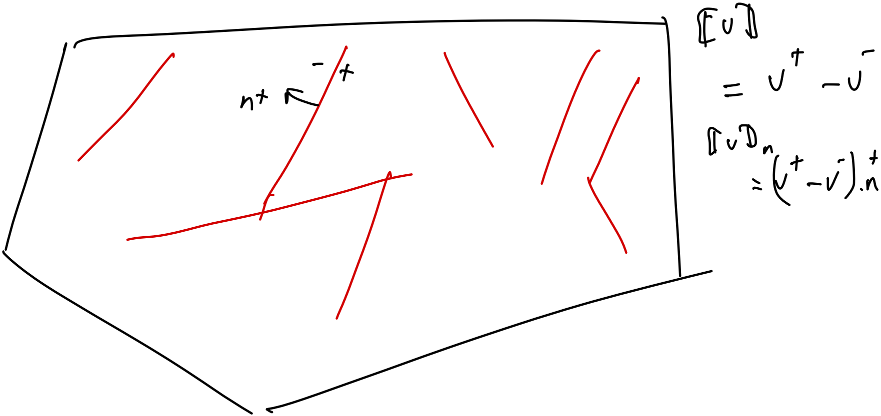
References for this presentation

- Design of the method, application to poromechanics with Coulomb friction: [Droniou et al., 2024a].
- Analysis for Tresca friction: [Droniou et al., 2024b].

Outline

Matrix and fracture network

Notations: Γ fracture, two sides $\pm$ with outward normals $\mathbf{n}^{\pm}$, jump $[\![\cdot]\!]$, tangential $\boldsymbol{\tau}$ and normal $\mathbf{n}$ component of vectors.

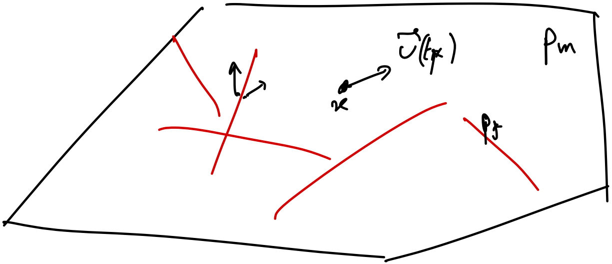


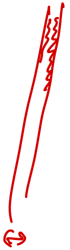
Poromechanics

Uknowns: displacement $\mathbf{u}$ in matrix (discontinuous at fractures), pressure p_m in matrix, pressure p_f in fracture.

Physics: quasi-static contact-mechanics for $\mathbf{u}$, Darcy law for p_m, Poiseuille law for p_f.

Contact: without friction, with Tresca friction, with Coulomb friction...





Mechanics with Tresca friction: strong form

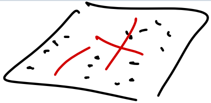
σ(u) n+

$$
\begin{cases}
-\mathrm{div}\,\sigma(\mathbf{u}) = \mathbf{f} & \text{on } \Omega\backslash\overline{\Gamma}, \\
\gamma_{\mathbf{n}}^{+}\sigma(\mathbf{u}) + \gamma_{\mathbf{n}}^{-}\sigma(\mathbf{u}) = \mathbf{0} & \text{on } \Gamma, \\
T_{\mathbf{n}}(\mathbf{u}) \leqslant 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} \leqslant 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} T_{\mathbf{n}}(\mathbf{u}) = 0 & \text{on } \Gamma, \\
|\mathbf{T}_{\boldsymbol{\tau}}(\mathbf{u})| \leqslant \mathrm{g} & \text{on } \Gamma, \\
\mathbf{T}_{\boldsymbol{\tau}}(\mathbf{u}) \cdot [\![\mathbf{u}]\!]_{\boldsymbol{\tau}} + \mathrm{g}\,|[\![\mathbf{u}]\!]_{\boldsymbol{\tau}}| = 0 & \text{on } \Gamma,
\end{cases}
$$

where:

- $\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}$ with λ, μ Lamé coefficients.
- $T_{\mathbf{n}}(\mathbf{u}) = \gamma_{\mathbf{n}}^{+}\sigma(\mathbf{u}) \cdot \mathbf{n}^{+}$ and $\mathbf{T}_{\boldsymbol{\tau}}(\mathbf{u}) = \gamma_{\mathbf{n}}^{+}\sigma(\mathbf{u}) - T_{\mathbf{n}}(\mathbf{u})\mathbf{n}^{+}$ tangential and normal surface tractions.
- g Tresca threshold.

Mechanics with Tresca friction: spaces



$v = 0$ sur $\partial\Omega$

Spaces:

- For the displacement:

$$\mathbf{U}_0 = H^1_0(\Omega\backslash\overline{\Gamma})^d$$

Mechanics with Tresca friction: spaces

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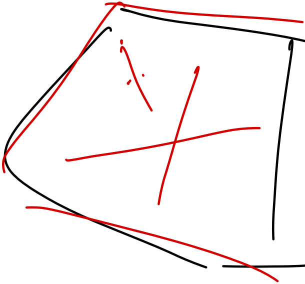
$$\mathbf{U}_0 = H^1_0(\Omega\backslash\overline{\Gamma})^d$$

- Space of jumps:

$$H^{1/2}_{0,\mathrm{j}}(\Gamma) = \{[\![\mathbf{v}]\!] \;:\; \mathbf{v} \in \mathbf{U}_0\}$$

with $H^{-1/2}_{0,\mathrm{j}}(\Gamma)$ dual space, and $\langle\cdot,\cdot\rangle_\Gamma$ duality pairing.



Mechanics with Tresca friction: spaces

Spaces:

- For the displacement:

$$\mathbf{U}_0 = H_0^1(\Omega \backslash \overline{\Gamma})^d$$

- Space of jumps:

$$H_{0,\mathrm{j}}^{1/2}(\Gamma) = \{[\![\mathbf{v}]\!] \,:\, \mathbf{v} \in \mathbf{U}_0\}$$

with $H_{0,\mathrm{j}}^{-1/2}(\Gamma)$ dual space, and $\langle \cdot, \cdot \rangle_\Gamma$ duality pairing.

- Lagrange multiplier (for contact inequalities):

$$\boldsymbol{C}_f = \left\{ \boldsymbol{\mu} \in H_{0,\mathrm{j}}^{-1/2}(\Gamma) \,:\, \langle \boldsymbol{\mu}, \mathbf{v} \rangle_\Gamma \leq \langle \mathrm{g}, |\mathbf{v}_{\boldsymbol{\tau}}| \rangle_\Gamma \; \forall \mathbf{v} \in H_{0,\mathrm{j}}^{1/2}(\Gamma) \text{ with } \mathbf{v} \cdot \mathbf{n}^+ \leq 0 \right\}.$$

Formally equivalent to $\boldsymbol{\mu}_{\mathbf{n}} \leq 0$ *and* $|\boldsymbol{\mu}_{\boldsymbol{\tau}}| \leq g$.

Mechanics with Tresca friction: mixed-variational formulation

Weal formulation: find $\mathbf{u} \in \mathbf{U}_0$ and $\boldsymbol{\lambda} \in \boldsymbol{C}_f$ s.t.

$$\int_\Omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) + \langle \boldsymbol{\lambda}, [\![\mathbf{v}]\!] \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \mathbf{U}_0,$$

$$\langle \boldsymbol{\mu} - \boldsymbol{\lambda}, [\![\mathbf{u}]\!] \rangle_\Gamma \leq 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{C}_f$$

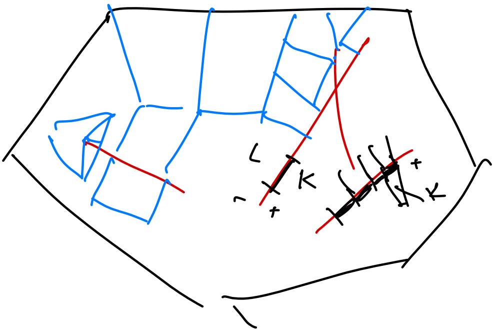
Note: $\boldsymbol{\lambda} = -\gamma_{\mathbf{n}}^{+} \sigma(\mathbf{u}) = \gamma_{\mathbf{n}}^{-} \sigma(\mathbf{u})$.

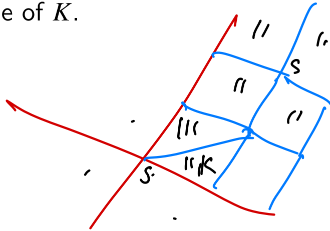
Outline

Mesh

Polytopal mesh, compatible with fractures

- $\mathcal{M}, \mathcal{F}, \mathcal{V}$ cells, faces and vertices. $\mathcal{X}_{\mathrm{z}}$ entities $\mathcal{X}$ on z.
- $\mathcal{F}^{+}_{\Gamma,K}$ faces of K on positive side of fracture.
- For $s \in \mathcal{V}$, $\mathcal{K}_s$: set of cells in $\mathcal{M}_s$ on the same side of K.
- If $\sigma \in \mathcal{F}_{\Gamma}$: K on positive side, L on negative side.

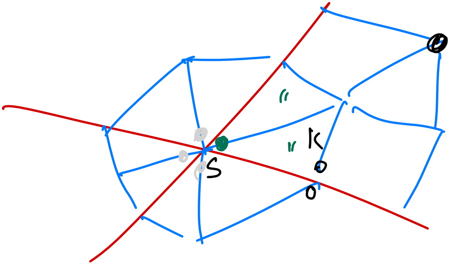


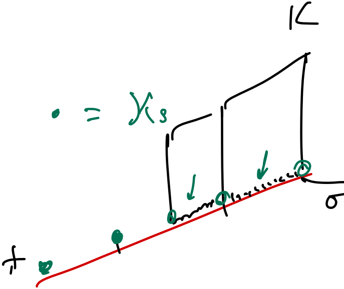


Discrete spaces

Displacement: nodal unknowns (discontinuous across fracture) and one bubble on each fracture face (positive side).

$$\mathbf{U}_{0,\mathcal{D}} = \Big\{ \mathbf{v}_{\mathcal{D}} = \big((\mathbf{v}_{\mathcal{K}s})_{K\in\mathcal{M},\, s\in\mathcal{V}_K}, (\mathbf{v}_{K\sigma})_{K\in\mathcal{M},\, \sigma\in\mathcal{F}^+_{\Gamma,K}}\big) \ : \\ \mathbf{v}_{\mathcal{K}s} \in \mathbb{R}^d,\ \mathbf{v}_{K\sigma} \in \mathbb{R}^d,\ \mathbf{v}_{\mathcal{K}s} = 0 \text{ if } s \in \mathcal{V}^{\text{ext}} \\ \mathbf{v}_{\mathcal{K}s} = \mathbf{v}_{\mathcal{L}s} \text{ if } K, L \in \mathcal{M}_s \text{ are on the same side of } \Gamma \Big\}.$$





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Lagrange multipliers: piecewise constant on fracture faces.

$$\mathbf{M}_{\mathcal{D}} = \left\{ \boldsymbol{\lambda}_{\mathcal{D}} \in L^2(\Gamma)^d \ : \ \boldsymbol{\lambda}_\sigma := (\boldsymbol{\lambda}_{\mathcal{D}})_{|\sigma} \text{ is constant for all } \sigma \in \mathcal{F}_\Gamma \right\}.$$

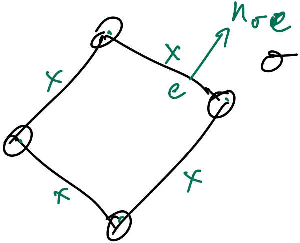
Discrete dual cone:

$$\mathbf{C}_{\mathcal{D}} = \left\{ \boldsymbol{\lambda}_{\mathcal{D}} \in \mathbf{M}_{\mathcal{D}} \ : \ \lambda_{\mathcal{D},\mathbf{n}} \geq 0,\ |\boldsymbol{\lambda}_{\mathcal{D},\boldsymbol{\tau}}| \leq \mathrm{g} \right\} \subset \boldsymbol{C}_f.$$

Reconstructions in $\mathbf{U}_{0,\mathcal{D}}$: faces

- From nodes, reconstruct edge values and use them to define the face gradient:

$$\nabla^{K\sigma}\mathbf{v}_{\mathcal{D}} = \frac{1}{|\sigma|}\sum_{e=s_1s_2\in\mathcal{E}_\sigma}|e|\frac{\mathbf{v}_{\mathcal{K}s_1}+\mathbf{v}_{\mathcal{K}s_2}}{2}\otimes\mathbf{n}_{\sigma e}.$$



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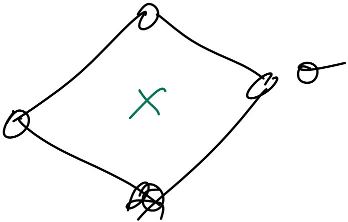
- Reconstruct face averaged value from nodes, and face displacement:

$$\overline{\mathbf{v}}_{K\sigma} = \sum_{s\in\mathcal{V}_\sigma}\omega_s^\sigma\mathbf{v}_{\mathcal{K}s} \quad \text{and} \quad \Pi^{K\sigma}\mathbf{v}_{\mathcal{D}}(\mathbf{x}) = \nabla^{K\sigma}\mathbf{v}_{\mathcal{D}}(\mathbf{x}-\overline{\mathbf{x}}_\sigma)+\overline{\mathbf{v}}_{K\sigma} \quad \forall\mathbf{x}\in\sigma.$$



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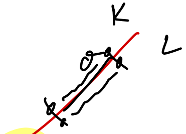
- Jump reconstructions:

$$[\![\mathbf{v}_{\mathcal{D}}]\!]_\sigma = \frac{1}{|\sigma|}\int_\sigma(\Pi^{K\sigma}\mathbf{v}_{\mathcal{D}}-\Pi^{L\sigma}\mathbf{v}_{\mathcal{D}})+\mathbf{v}_{K\sigma} = \overline{\mathbf{v}}_{K\sigma}-\overline{\mathbf{v}}_{L\sigma}+\mathbf{v}_{K\sigma}.$$



Reconstructions in $\mathbf{U}_{0,\mathcal{D}}$: cells

Same principles...

- Using reconstructed face values and bubble, define the cell gradient:

$$\nabla^K \mathbf{v}_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \overline{\mathbf{v}}_{K\sigma} \otimes \mathbf{n}_{K\sigma} + \sum_{\sigma \in \mathcal{F}^+_{\Gamma,K}} \frac{|\sigma|}{|K|} \mathbf{v}_{K\sigma} \otimes \mathbf{n}_{K\sigma}.$$

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Global reconstructions: $[\![\cdot]\!]_{\mathcal{D}}$, $\nabla^{\mathcal{D}}$, $\Pi_{\mathcal{D}}$, $\varepsilon_{\mathcal{D}}$, $\mathrm{div}_{\mathcal{D}}$, $\sigma_{\mathcal{D}}$.

Interpolator

Displacement: $\mathcal{I}_{\mathbf{U}_{0,\mathcal{D}}} : C_0^0(\Omega\backslash\Gamma) \longrightarrow \mathbf{U}_{0,\mathcal{D}}$ defined by:

$$(\mathcal{I}_{\mathbf{U}_{0,\mathcal{D}}}\mathbf{v})_{\mathcal{K}s} = \mathbf{v}_{|K}(\mathbf{x}_s) \qquad \forall K \in \mathcal{M},\ \forall s \in \mathcal{V}_K,$$

$$(\mathcal{I}_{\mathbf{U}_{0,\mathcal{D}}}\mathbf{v})_{K\sigma} = \frac{1}{|\sigma|}\int_\sigma (\gamma^{K\sigma}\mathbf{v} - \Pi^{K\sigma}(\mathcal{I}_{\mathbf{U}_{0,\mathcal{D}}}\mathbf{v})) \qquad \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{F}^+_{\Gamma,K}.$$

Lagrange multiplier: $\mathcal{I}_{\mathbf{M}_\mathcal{D}} : L^2(\Gamma) \longrightarrow \mathbf{M}_\mathcal{D}$ defined by

$$(\mathcal{I}_{\mathbf{M}_\mathcal{D}}\boldsymbol{\lambda})_\sigma = \frac{1}{|\sigma|}\int_\sigma \boldsymbol{\lambda} \qquad \forall \sigma \in \mathcal{F}_\Gamma.$$

Scheme

Find $\mathbf{u}_{\mathcal{D}} \in \mathbf{U}_{0,\mathcal{D}}$ and $\boldsymbol{\lambda}_{\mathcal{D}} \in \mathbf{M}_{\mathcal{D}}$ s.t.

$$\int_\Omega \mathbb{\sigma}_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) : \mathbb{\varepsilon}_{\mathcal{D}}(\mathbf{v}_{\mathcal{D}}) + \sum_{K\in\mathcal{M}} (2\mu_K + \lambda_K) S_K(\mathbf{u}_{\mathcal{D}}, \mathbf{v}_D)$$
$$+ \int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}} \cdot [\![\mathbf{v}_{\mathcal{D}}]\!]_{\mathcal{D}} = \int_\Omega \mathbf{f} \cdot \widetilde{\Pi}^{\mathcal{D}} \mathbf{v}_{\mathcal{D}} \qquad \forall \mathbf{v}_{\mathcal{D}} \in \mathbf{U}_{0,\mathcal{D}}$$

$$\int_\Gamma (\boldsymbol{\mu}_{\mathcal{D}} - \boldsymbol{\lambda}_{\mathcal{D}}) \cdot [\![\mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D}} \le 0 \qquad \forall \boldsymbol{\mu}_{\mathcal{D}} \in \mathbf{M}_{\mathcal{D}},$$

where

$$S_K(\mathbf{u}_{\mathcal{D}}, \mathbf{v}_{\mathcal{D}}) = h_K^{d-2} \sum_{s\in\mathcal{V}_K} \Big(\mathbf{u}_{\mathcal{K}s} - \Pi^K \mathbf{u}_{\mathcal{D}}(\mathbf{x}_s)\Big) \cdot \Big(\mathbf{v}_{\mathcal{K}s} - \Pi^K \mathbf{v}_{\mathcal{D}}(\mathbf{x}_s)\Big)$$
$$+ h_K^{d-2} \sum_{\sigma\in\mathcal{F}^+_{\Gamma,K}} \mathbf{u}_{K\sigma} \cdot \mathbf{v}_{K\sigma}.$$

Outline

Error estimate

Theorem (Error estimate)

If $\mathbf{u} \in H^2(\mathcal{M})$ *and* $\boldsymbol{\lambda} \in H^1(\mathcal{F}_\Gamma)$*, then*

$$\|\nabla^{\mathcal{D}} \mathbf{u}_{\mathcal{D}} - \nabla \mathbf{u}\|_{L^2(\Omega\setminus\overline{\Gamma})} + \|\boldsymbol{\lambda}_{\mathcal{D}} - \boldsymbol{\lambda}\|_{-1/2,\Gamma} \lesssim C_{\mathbf{u},\lambda} h_{\mathcal{D}}.$$

- $\|\cdot\|_{-1/2,\Gamma}$ discrete $H^{-1/2}$-like seminorm, defined later.

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- $\|\cdot\|_{-1/2,\Gamma}$ discrete $H^{-1/2}$-like seminorm, defined later.
- Error estimate comes from a more abstract version that only requires $\boldsymbol{\lambda} \in L^2(\Gamma)$.
- Error analysis based on consistency and stability.

Discrete Korn inequality

Discrete H^1-norm:

$$\|\mathbf{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 := \sum_{K\in\mathcal{M}} \left(\|\nabla^K \mathbf{v}_{\mathcal{D}}\|_{L^2(K)}^2 + S_K(\mathbf{v}_{\mathcal{D}}, \mathbf{v}_{\mathcal{D}}) \right).$$

Theorem (Discrete Korn inequality)

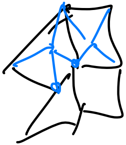
For all $\mathbf{v} \in \mathbf{U}_{0,\mathcal{D}}$,

$$\|\mathbf{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 \lesssim \|\mathbb{e}_{\mathcal{D}}(\mathbf{v}_{\mathcal{D}})\|_{L^2(\Omega\backslash\overline{\Gamma})}^2 + \sum_{K\in\mathcal{M}} S_K(\mathbf{v}_{\mathcal{D}}, \mathbf{v}_{\mathcal{D}}).$$

Discrete Korn inequality: idea of proof

- Start from the $\mathbb{P}^1$-conforming reconstruction $\mathcal{R}_{av,h}\mathbf{v}_h$ on a triangular submesh (averaging cell values at each node) and

$$\|\nabla_h(\mathbf{v}_h - \mathcal{R}_{av,h}\mathbf{v}_h)\|^2_{L^2} \lesssim \sum_{\sigma\in\mathcal{F}} h_\sigma^{-1} \|[\![\mathbf{v}_h]\!]_\sigma\|^2_{L^2(\sigma)}.$$



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- Using Korn for $\mathcal{R}_{av,h}\mathbf{v}_h$ gives

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- Do the same procedure but building $\mathcal{R}_{av,h}\mathbf{v}_h$ from averages over cell values the same side of Γ:

$$\|\nabla_h\mathbf{v}_h\|_{L^2}^2 \lesssim \|\mathbb{e}_h(\mathbf{v}_h)\|_{L^2}^2 + \sum_{\sigma\in\mathcal{F}\setminus\mathcal{F}_\Gamma} h_\sigma^{-1}\|[\![\mathbf{v}_h]\!]_\sigma\|_{L^2(\sigma)}^2.$$

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- Conclude by proving that, for $\sigma = K|L$ not in $\mathcal{F}_\Gamma$,

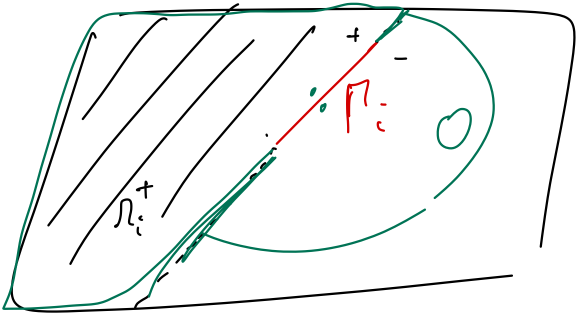
$$h_\sigma^{-1}\|[\![\mathbf{v}_h]\!]_\sigma\|_{L^2(\sigma)}^2 \lesssim S_K(\mathbf{v}_\mathcal{D},\mathbf{v}_\mathcal{D}) + S_L(\mathbf{v}_\mathcal{D},\mathbf{v}_\mathcal{D}).$$

Discrete inf–sup property I

Definition (Discrete $H^{-1/2}(\Gamma)$-norm)

For $\boldsymbol{\lambda}_{\mathcal{D}} \in \mathbf{M}_{\mathcal{D}}$:

$$\|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma} = \sum_{i \in I} \|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma_i} \quad \text{with} \quad \|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma_i} = \sup_{\mathbf{v}_i \in H^1(\Omega_i^+;\Gamma_i)^d \setminus \{0\}} \frac{\int_{\Gamma_i} \boldsymbol{\lambda}_{\mathcal{D}} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|_{H^1(\Omega_i^+)}}.$$



Discrete inf–sup property II

Theorem (Discrete inf-sup condition)

$$\sup_{\mathbf{v}_\mathcal{D}\in\mathbf{U}_{0,\mathcal{D}}\setminus\{0\}} \frac{\int_\Gamma \boldsymbol{\lambda}_\mathcal{D}\cdot[\![\mathbf{v}_\mathcal{D}]\!]_\mathcal{D}}{\|\mathbf{v}_\mathcal{D}\|_{1,\mathcal{D}}} \gtrsim \|\boldsymbol{\lambda}_\mathcal{D}\|_{-1/2,\Gamma} \qquad \forall \boldsymbol{\lambda}_\mathcal{D}\in\mathbf{M}_\mathcal{D}.$$

Discrete inf–sup property III

Tools:

- Clément-like H^1-stable interpolator $\mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}$, adapted to fractures.

Discrete inf–sup property IV

Tools:

- Clément-like H^1-stable interpolator $\mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}$, adapted to fractures.
- Fortin property for jump:

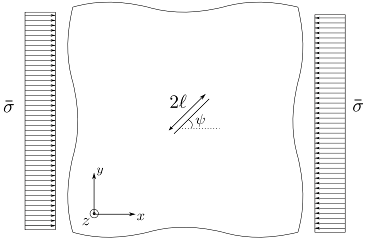
$$\int_\sigma [\![\mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}\mathbf{v}]\!]_\sigma = \begin{cases} \int_\sigma (\gamma^{K\sigma}\mathbf{v} - \Pi^{L\sigma}\mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}\mathbf{v}) & \text{if } \sigma \in \mathcal{F}^+_{\Gamma_i,K}, \\ 0 & \text{if } \sigma \in \mathcal{F}^+_{\Gamma,K} \setminus \mathcal{F}^+_{\Gamma_i,K}. \end{cases}$$

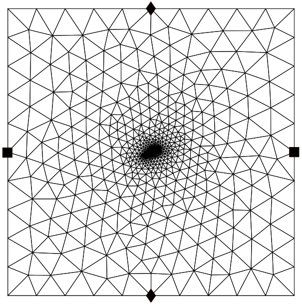
Consequence: if $\mathbf{v}_i \in H^1(\Omega_i^+;\Gamma_i)$, $\widetilde{\mathbf{v}}_i$ extension of $\mathbf{v}_i$ by 0 and $\mathbf{v}_\mathcal{D} = \mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}\widetilde{\mathbf{v}}_i$,

$$\int_\Gamma \boldsymbol{\lambda}_\mathcal{D} \cdot [\![\mathbf{v}_\mathcal{D}]\!]_\mathcal{D} = \sum_{\sigma\in\mathcal{F}_\Gamma} \boldsymbol{\lambda}_\sigma \cdot \int_\sigma [\![\mathcal{I}^{i,\mathrm{a}}_{\mathbf{U}_{0,\mathcal{D}}}\widetilde{\mathbf{v}}_i]\!]_\sigma = \int_{\Gamma_i} \boldsymbol{\lambda}_\mathcal{D} \cdot \mathbf{v}_i.$$

Outline

2D domain with fracture under compression I





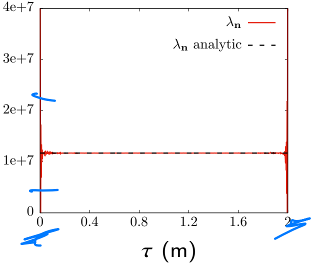
Analytical solution (τ coordinate along fracture):

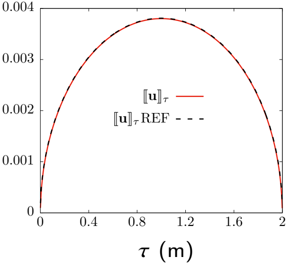
$$\lambda_{\mathbf{n}} = \sigma \sin^2(\psi),$$

$$|[\![\mathbf{u}]\!]_{\boldsymbol{\tau}}| = \frac{4(1-\nu)}{E}\sigma \sin(\psi)\left(\cos(\psi) - \frac{\mathrm{g}}{\lambda_{\mathbf{n}}}\sin(\psi)\right)\sqrt{\ell^2 - (\ell^2 - \tau^2)}.$$

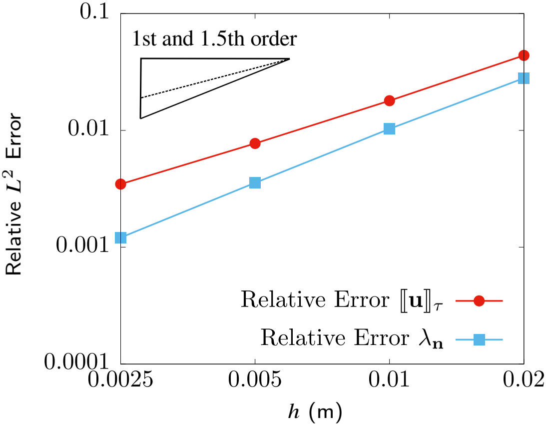
$\psi = \pi/9$, $2\ell = 2$ m, $F = 1/\sqrt{3}$ (so $\mathrm{g} = \lambda_{\mathbf{n}}/F$), $E = 25$ GPa and $\nu = 0.25$.

2D domain with fracture under compression II

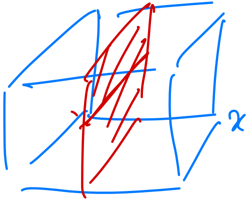




2D domain with fracture under compression III



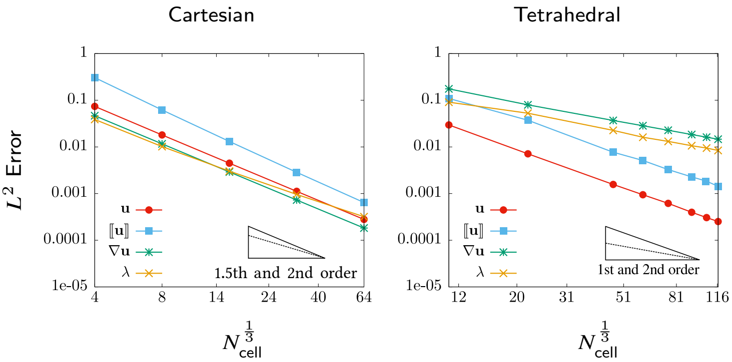
3D manufactured solution I



Setting:

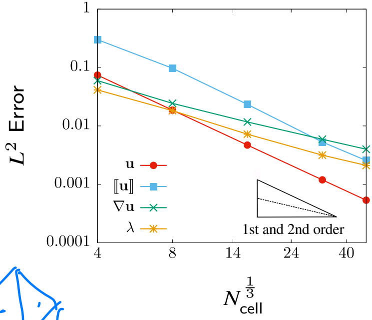
- $\Omega = (-1, 1)^3$, $\Gamma = \{0\} \times (-1, 1)^2$.
- $g = 1$, $\mu = \lambda = 1$.
- Explicit analytical solution such that:
  - sticky-contact for $z < 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $[\![u]\!]_{\boldsymbol{\tau}} = 0$)
  - slippy-contact for $z > 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $|[\![u]\!]_{\boldsymbol{\tau}}| > 0$)
- Cartesian, tetrahedral and hexahedral (randomly perturbed Cartesian) meshes.

3D manufactured solution II

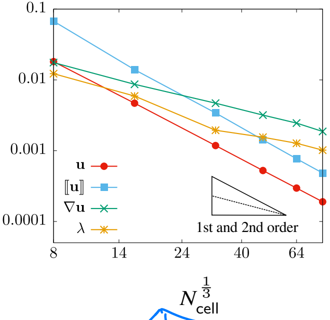


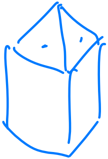
3D manufactured solution III

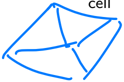
Hexahedral (cut)



Hexahedral (bary)







Thank you!

References I

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