# Introduction to the Discrete De Rham method, and application to a (weird) formulation of the Navier-Stokes equations

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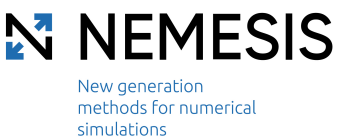

# Outline


1 Why Hibert complexes for PDEs?

2 The de Rham complexe and the finite element approach

3 The discrete de Rham complex on polytopal meshes
- Generic principles
- Construction and properties of the DDR complex
- Properties
- Exterior calculus formulation

4 Application to Navier–Stokes

5 Numerical results

# Maxwell equations: preservation of constraint

- Maxwell equations in the void:

| Evolution | Constraints |
| --- | --- |
| $\partial_t E = \mathbf{curl}\, B$ | $\operatorname{div} E = 0$ |
| $\partial_t B = -\mathbf{curl}\, E$ | $\operatorname{div} B = 0$ |

# Maxwell equations: preservation of constraint

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|---|---|
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| $\partial_t B = -\mathbf{curl}\, E$ | $\operatorname{div} B = 0$ |

- Constraint preservation:

$$\partial_t (\operatorname{div} E) = \operatorname{div}(\partial_t E) = \operatorname{div} \mathbf{curl}\, B = 0$$

since

$$\operatorname{div} \mathbf{curl} = 0$$

# Stokes in "standard" formulation

- $\Omega$ domain, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Find $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
-\nu \boldsymbol{\Delta} \boldsymbol{u} + \mathbf{grad}\, p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)} \\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)} \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, && \text{(boundary condition)}
\end{aligned}
$$

[1] $a \lesssim b$ means $a \leq Cb$ with $C$ independent of $a, b$.

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$$\begin{aligned} -\nu \boldsymbol{\Delta u} + \mathbf{grad}\, p &= \boldsymbol{f} \quad \text{in } \Omega, && \text{(momentum conservation)} \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, && \text{(mass conservation)} \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega, && \text{(boundary condition)} \end{aligned}$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L^2(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} \nu(\mathbf{grad}\, \boldsymbol{u}, \mathbf{grad}\, \boldsymbol{v})_{L^2} - (p, \operatorname{div} \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in H_0^1(\Omega), \\ (\operatorname{div} \boldsymbol{u}, q)_{L^2} &= 0 && \forall q \in L^2(\Omega) \end{aligned}$$

- A priori estimates require: [1]

  - Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{grad}\, \cdot\|_{L^2}$ on $H_0^1(\Omega)$,
  - inf-sup $\sup_{\boldsymbol{v} \in H_0^1} \frac{(p, \operatorname{div} \boldsymbol{v})_{L^2}}{\|\boldsymbol{v}\|_{H_0^1}} \geq C \|p\|_{L^2}$, equivalent to $\operatorname{Im} \operatorname{div} = L^2(\Omega)$.

[1] $a \lesssim b$ means $a \leq Cb$ with $C$ independent of $a, b$.

# Stokes in curl-curl formulation: weak form

- Recasting the Stokes equations:

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)} \\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)} \\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)} \\
\int_\Omega p &= 0
\end{aligned}
$$

*Note*: These are natural BCs. Essential BCs are: $\boldsymbol{u} \times \boldsymbol{n} = 0$ and $p = 0$ on $\partial\Omega$.

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*Note*: These are natural BCs. Essential BCs are: $\boldsymbol{u}\times\boldsymbol{n} = 0$ and $p = 0$ on $\partial\Omega$.

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1_\star(\Omega)$ and

$$
\begin{aligned}
\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H^1_\star(\Omega),
\end{aligned}
$$

where $H^1_\star(\Omega) := \{r \in H^1(\Omega) \,:\, \int_\Omega r = 0\}$.

# Stokes equations in curl-curl formulation: stability

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H^1_\star(\Omega) \end{aligned}$$

- Make $\boldsymbol{v} = \mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad} = 0$.
- Make $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$:

$$\nu \|\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2} \leq \|\boldsymbol{f}\|_{L^2} \|\boldsymbol{u}\|_{L^2}.$$

# Stokes equations in curl-curl formulation: stability

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_{L^2}+(\mathbf{grad}\,p,\boldsymbol{v})_{L^2}&=(\boldsymbol{f},\boldsymbol{v})_{L^2} && \forall \boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl};\Omega),\\ -(\boldsymbol{u},\mathbf{grad}\,q)_{L^2}&=0 && \forall q\in H^1_\star(\Omega)\end{aligned}$$

- Make $\boldsymbol{v}=\mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2}\le\|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad}=0$.
- Make $(\boldsymbol{v},q)=(\boldsymbol{u},p)$:

$$\nu\|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}^2\le\|\boldsymbol{f}\|_{L^2}\|\boldsymbol{u}\|_{L^2}.$$

- If $\Omega$ does not have any tunnel,

$$\mathrm{Im}\,\mathbf{grad}=\mathrm{Ker}\,\mathbf{curl}.$$

The incompressibility gives $\boldsymbol{u}\perp\mathrm{Im}\,\mathbf{grad}$, so $\boldsymbol{u}\in(\mathrm{Ker}\,\mathbf{curl})^\perp$ and the

Poincaré inequality: $\|\cdot\|_{L^2}\lesssim\|\mathbf{curl}\,\cdot\|_{L^2}$ on $(\mathrm{Ker}\,\mathbf{curl})^\perp$

yields

$$\|\boldsymbol{u}\|_{L^2}\lesssim\|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}.$$

# Take-home message

Analysis of these models requires:

- Poincaré inequalities: $\|\cdot\|_{L^2} \lesssim \|\mathcal{D}\cdot\|_{L^2}$ on $(\operatorname{Ker}\mathcal{D})^\perp$.
- Image/kernel relations for some operators in the sequence (possibly with suitable BCs):

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

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# The de Rham complex I

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.

# The de Rham complex I

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.
- The complex is exact if we have equalities:

$$\mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl}, \quad \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div}, \quad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

This imposes some topological properties on $\Omega$...

# The de Rham complex II

- Precisely:

$$\text{no “tunnels”} \implies \operatorname{Im}\mathbf{grad} = \operatorname{Ker}\mathbf{curl} \quad \text{(Stokes in curl-curl)}$$
$$\text{no “voids”} \implies \operatorname{Im}\mathbf{curl} = \operatorname{Ker}\operatorname{div} \quad \text{(magnetostatics)}$$
$$\text{always:} \quad \operatorname{Im}\operatorname{div} = L^2(\Omega) \quad \text{(Stokes)}$$

- For non-trivial topologies, de Rham's cohomology characterizes

$$\operatorname{Ker}\mathbf{curl}/\operatorname{Im}\mathbf{grad} \quad \text{and} \quad \operatorname{Ker}\operatorname{div}/\operatorname{Im}\mathbf{curl}$$

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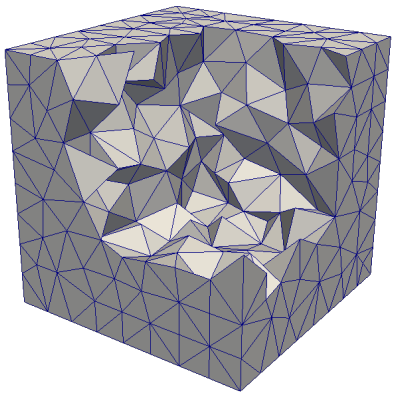
- For non-trivial topologies, de Rham's cohomology characterizes

$$\operatorname{Ker}\mathbf{curl}/\operatorname{Im}\mathbf{grad} \quad \text{and} \quad \operatorname{Ker}\operatorname{div}/\operatorname{Im}\mathbf{curl}$$

- **Emulating these properties is key for stable discretizations.**



$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

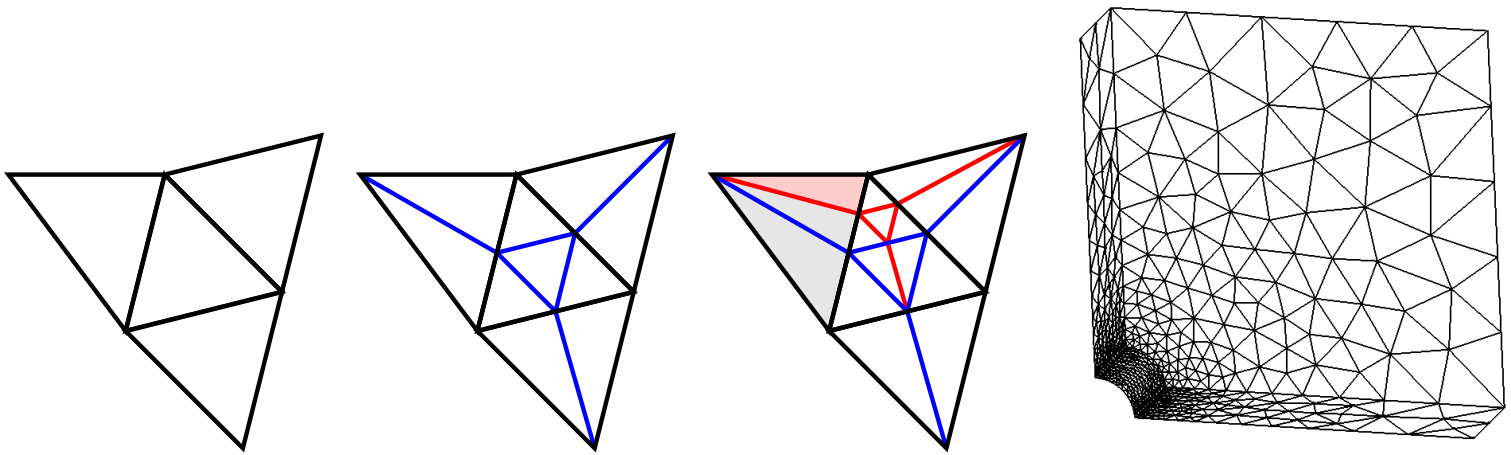
$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\mathbf{grad}} & V_h^1 & \xrightarrow{\mathbf{curl}} & V_h^2 & \xrightarrow{\mathrm{div}} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \end{array}$$

*Example*: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

- **Gluing only works on special meshes!**

# The Finite Element way

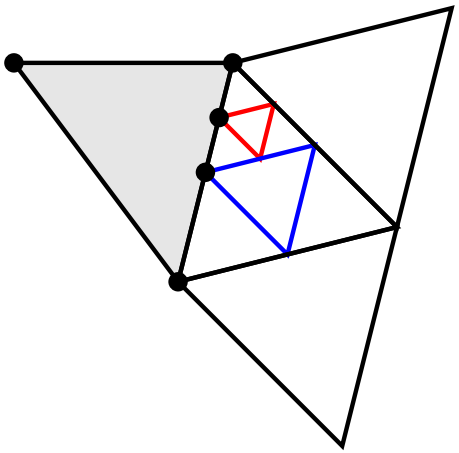
## Shortcomings

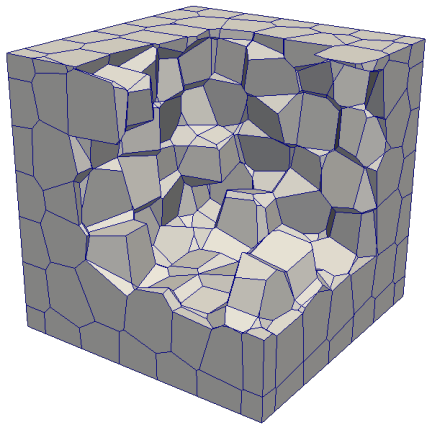


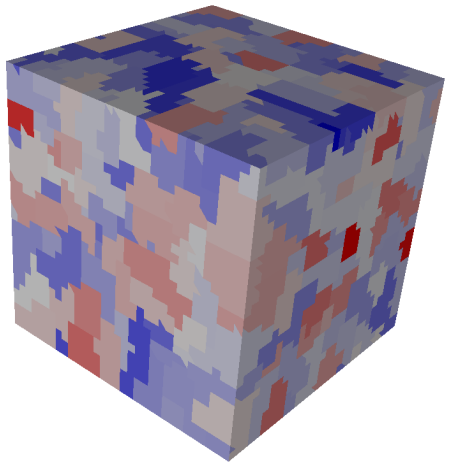
- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

# Benefits of polytopal meshes I



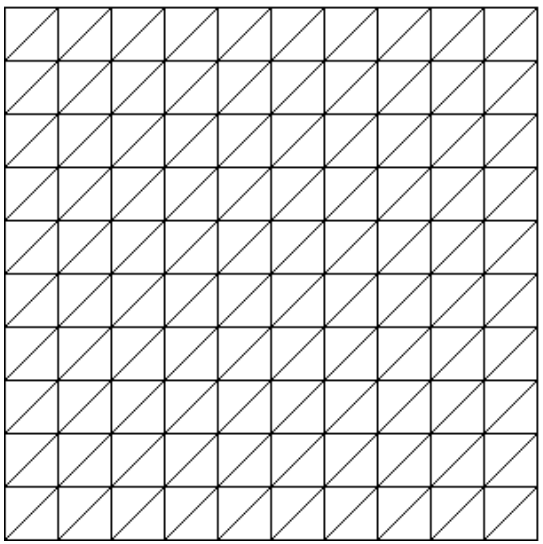
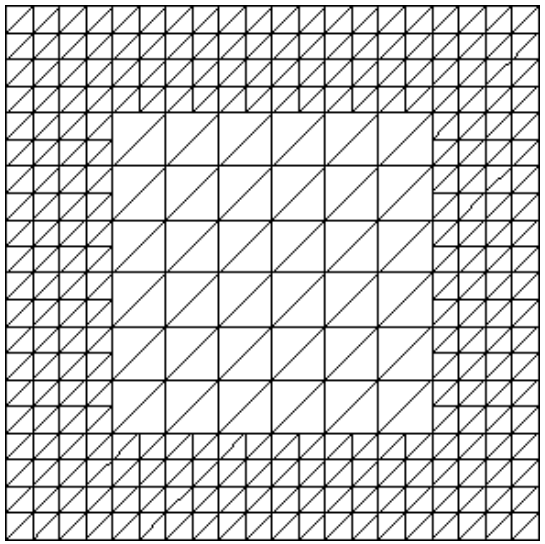




- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

# Benefits of polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

| Stabilised $\mathcal{P}_2$-$(\mathcal{P}_1 + \mathcal{B}^3)$ scheme | | DDR scheme | |
|---|---|---|---|
|  | |  | |
| nb. DOFs | Error | nb. DOFs | Error |
| 2403 | 0.138 | 550 | 0.161 |
| 9603 | 6.82e-2 | 2121 | 6.77e-2 |
| 38402 | 3.40e-2 | 8329 | 3.1e-2 |

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# Two approaches for polytopal complexes I

## Virtual Element Method

- Construct a sub-complex of finite-dimensional but not piecewise polynomial spaces:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & V_{k+1}^{\mathrm{n}} & \xrightarrow{\mathbf{grad}} & V_k^{\mathrm{e}} & \xrightarrow{\mathbf{curl}} & V_k^{\mathrm{f}} & \xrightarrow{\mathrm{div}} & V_k^{\mathrm{v}} & \xrightarrow{0} & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}$$

- Functions in each $V_h^\bullet$ are not explicitly described, but:
  - $V_h^\bullet$ have DOFs that are polynomials moments on the vertices, edges, faces or elements.
  - Some polynomial projections of the functions/their derivatives can be computed from these DOFs.

Ref: [Beirão da Veiga et al., 2017], [Beirão da Veiga et al., 2018].

# Two approaches for polytopal complexes II

## Discrete De Rham (fully discrete approach)

- Construct a fully discrete complex of bespoke finite-dimensional spaces and operators:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \uparrow{\scriptstyle \underline{I}^k_{\mathbf{grad},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},h}} & & \uparrow{\scriptstyle I^k_{L^2,h}} & & \\
\mathbb{R} & \hookrightarrow & C^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{\Omega}) & \xrightarrow{0} & \{0\}
\end{array}$$

- Discrete spaces are not made of functions but:

  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Interpolators $\underline{\boldsymbol{I}}^k_{\bullet,h}$ give meaning to these polynomials/DOFs as moments.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023b].

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# Guiding principles

- Hierarchical constructions: from lowest-dimensional mesh entity to higher-dimensional entities.

- Enhancement:
  - discrete differential operator first,
  - potential reconstruction using the discrete differential operator.

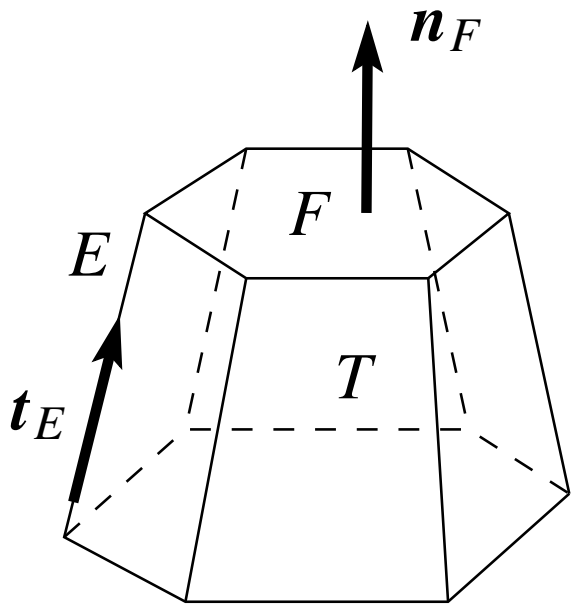
  *(both polynomially consistent, both based on IBP formulas.)*

- The definition of the spaces (DOFs) also guided by these IBP formulas.

*Same guiding principles as the Hybrid High-Order (HHO) method [Di Pietro and Droniou, 2020].*

# Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements/faces/edges/vertices, with intrinsic orientations (tangent, normal).
  - $\omega_{TF} \in \{+1, -1\}$ such that $\omega_{TF}\boldsymbol{n}_F$ outer normal to $T$.
  - $\omega_{FE} \in \{+1, -1\}$ such that $\omega_{FE}\boldsymbol{t}_E$ clockwise on $F$.



# $\mathcal{P}^k$-consistent gradient

## Edge $E$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

  with derivatives in the direction $\boldsymbol{t}_E$.

# $\mathcal{P}^k$-consistent gradient

## Edge $E$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_E^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

with $\pi_E^{k-1}$ the $L^2$-projection on $\mathcal{P}^{k-1}(E)$.

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- Space and interpolator:

$$\underline{X}^k_{\mathbf{grad},E} = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \; : \; q_E \in \mathcal{P}^{k-1}(E), \; q_V \in \mathbb{R} \right\},$$

$$\underline{I}^k_{\mathbf{grad},E} q = (\pi_E^{k-1} q, (q(\boldsymbol{x}_V))_{V \in \mathcal{V}_E}) \qquad \forall q \in C(\overline{E}).$$

# $\mathcal{P}^k$-consistent gradient

## Edge $E$

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$$\int_E q' r = -\int_E \pi_E^{k-1} q \; r' + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}^k_{\mathbf{grad},E} = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \; : \; q_E \in \mathcal{P}^{k-1}(E) \, , \; q_V \in \mathbb{R} \right\}$.
- Edge gradient $G^k_E \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G^k_E \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

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- Edge gradient $G^k_E \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

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- Potential reconstruction $\gamma^{k+1}_E \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma^{k+1}_E \underline{q}_E) z' = -\int_E (G^k_E \underline{q}_E) z + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}) \qquad \forall z \in \mathcal{P}^{k+2}(E).$$

# $\mathcal{P}^k$-consistent gradient

## Edge $E$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_E^{k-1} q\, r' + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}^k_{\mathbf{grad},E} = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \,:\, q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \right\}$.
- Edge gradient $G^k_E \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G^k_E \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

- Potential reconstruction $\gamma^{k+1}_E \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma^{k+1}_E \underline{q}_E) z' = -\int_E (G^k_E \underline{q}_E) z + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}) \qquad \forall z \in \mathcal{P}^{k+2}(E).$$

Polynomially consistent: $G^k_E \underline{I}^k_{\mathbf{grad},E} q = q'$ and $\gamma^{k+1}_E \underline{I}^k_{\mathbf{grad},E} q = q$ for all $q \in \mathcal{P}^{k+1}(E)$.

# $\mathcal{P}^k$-consistent gradient

## Face $F$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot \boldsymbol{v} = -\int_F q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

# $\mathcal{P}^k$-consistent gradient

## Face $F$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_F^{k-1} q\, \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

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- Space and interpolator:

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ : \ q_F \in \mathcal{P}^{k-1}(F)\,,\ q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R} \Big\},$$

$$\underline{I}^k_{\mathbf{grad},F} q = (\pi_F^{k-1} q, (\pi_E^{k-1} q_{|E})_{E\in\mathcal{E}_F}, (q(\boldsymbol{x}_V))_{V\in\mathcal{V}_F}) \qquad \forall q \in C(\overline{F}).$$

# $\mathcal{P}^k$-consistent gradient

## Face $F$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot \boldsymbol{v} = -\int_F \pi_F^{k-1} q\ \mathrm{div}_F\, \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) :$

$$q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R}\Big\},$$

- Face gradient $\mathbf{G}_F^k$ : $\underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}_F^k \underline{q}_F)\cdot \boldsymbol{v} = -\int_F q_F\, \mathrm{div}_F\, \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma_E^{k+1} \underline{q}_E)\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

# $\mathcal{P}^k$-consistent gradient

## Face $F$

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) :$

$$q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R}\Big\},$$

- Face gradient $\mathbf{G}^k_F : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}^k_F \underline{q}_F)\cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Potential reconstruction $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma^{k+1}_F \underline{q}_F) \operatorname{div}_F z = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot z + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E) z\cdot \boldsymbol{n}_{FE}$$

$$\forall z \in \boldsymbol{\mathcal{R}}^{c,k+2}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

# $\mathcal{P}^k$-consistent gradient

## Face $F$

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) : q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R}\Big\}$,
- Face gradient $\mathbf{G}^k_F : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}^k_F \underline{q}_F)\cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Potential reconstruction $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma^{k+1}_F \underline{q}_F) \operatorname{div}_F z = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot z + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E) z\cdot \boldsymbol{n}_{FE}$$

$$\forall z \in \mathcal{R}^{c,k+2}(F) := (\boldsymbol{x}-\boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

Polynomially consistent: $\mathbf{G}^k_F \underline{I}^k_{\mathbf{grad},E} q = \mathbf{grad}\, q$ and $\gamma^{k+1}_F \underline{I}^k_{\mathbf{grad},E} q = q$ for all $q \in \mathcal{P}^{k+1}(F)$.

# $\mathcal{P}^k$-consistent gradient

## Element $T$

Same principle! Based on IBP we determine:

- An additional unknown ($q_T \in \mathcal{P}^{k-1}(T)$) to get the space $\underline{X}^k_{\mathbf{grad},T}$, and its meaning (polynomial moment on $T$) to get the interpolator $\underline{I}^k_{\mathbf{grad},T}$.
- A formula for the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3$.
- A potential reconstruction $P^{k+1}_{\mathbf{grad},T} : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T)$.

# $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big) \ : \\ \boldsymbol{v}_T \in \boldsymbol{\mathcal{RT}}^k(T),\ \boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^k(F),\ v_E \in \mathcal{P}^k(E) \Big\},$$

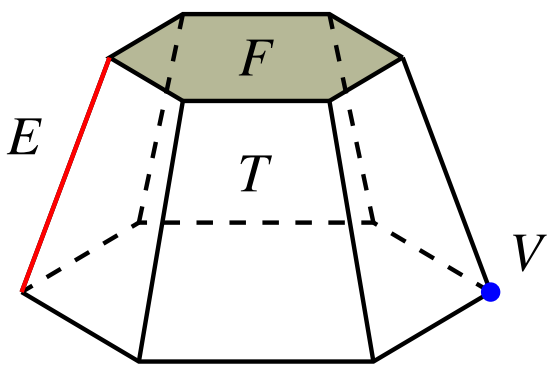
- Interpolator: $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = \underline{\boldsymbol{v}}_h$ with

$$\begin{aligned} v_E &= L^2\text{-projection on } \mathcal{P}^k(E) \text{ of } \boldsymbol{v}\cdot\boldsymbol{t}_E, \\ \boldsymbol{v}_F &= L^2\text{-projection on } \boldsymbol{\mathcal{RT}}^k(F) \text{ of } \boldsymbol{v}_{\mathrm{t},F} \text{ (tangential projection)}, \\ \boldsymbol{v}_T &= L^2\text{-projection on } \boldsymbol{\mathcal{RT}}^k(T) \text{ of } \boldsymbol{v}. \end{aligned}$$

- Operator reconstructions for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$:
  - face (scalar) curl $C^k_F\underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F) \rightsquigarrow$ tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F)^2$,
  - element curl $\boldsymbol{C}^k_T\underline{\boldsymbol{v}}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow$ element potential $\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \in \mathcal{P}^k(T)^3$.

# The Discrete de Rham method

- Contrary to FE, do not seek explicit (or any!) basis functions.
- Replace continuous spaces by fully discrete ones made of vectors of polynomials, representing polynomial moments when interpreted through the interpolator.
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Create discrete operators (differential, potential reconstruction) between the spaces.



# Discrete gradient

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathrm{curl},h} \xrightarrow{\underline{C}^k_h} \underline{X}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete gradient: project the face/element/edge gradients

$$\mathbf{G}^k_T \underline{q}_T \in \mathcal{P}^k(T)^3, \qquad \mathbf{G}^k_F \underline{q}_F \in \mathcal{P}^k(F)^2, \qquad G^k_E \underline{q}_E \in \mathcal{P}^k(E)$$

in the proper spaces:

$$\underline{\boldsymbol{G}}^k_h \underline{q}_h = \Big( (\boldsymbol{\pi}^k_{\mathcal{RT},T} \mathbf{G}^k_T \underline{q}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}^k_{\mathcal{RT},F} \mathbf{G}^k_F \underline{q}_F)_{F\in\mathcal{F}_h}, (G^k_E \underline{q}_E)_{E\in\mathcal{E}_h} q \Big).$$

# DDR vs. Serendipity-DDR vs. Raviart-Thomas-Nédélec

| Discrete space | $k=0$ | $k=1$ | $k=2$ |
|---|---|---|---|
| $H^1(T)$ | 4 ◇ 4 ◇ 4 | 15 ◇ 10 ◇ 10 | 32 ◇ 20 ◇ 20 |
| $\boldsymbol{H}(\mathbf{curl};T)$ | 6 ◇ 6 ◇ 6 | 28 ◇ 23 ◇ 20 | 65 ◇ 53 ◇ 45 |
| $\boldsymbol{H}(\mathrm{div};T)$ | 4 ◇ 4 ◇ 4 | 18 ◇ 18 ◇ 15 | 44 ◇ 44 ◇ 36 |
| $L^2(T)$ | 1 ◇ 1 ◇ 1 | 4 ◇ 4 ◇ 4 | 10 ◇ 10 ◇ 10 |

Table: Tetrahedron: local number of DOFs for DDR ◇ SDDR ◇ RTN.

| Discrete space | $k=0$ | $k=1$ | $k=2$ |
|---|---|---|---|
| $H^1(T)$ | 8 ◇ 8 ◇ 8 | 27 ◇ 20 ◇ 27 | 54 ◇ 32 ◇ 64 |
| $\boldsymbol{H}(\mathbf{curl};T)$ | 12 ◇ 12 ◇ 12 | 46 ◇ 39 ◇ 54 | 99 ◇ 77 ◇ 144 |
| $\boldsymbol{H}(\mathrm{div};T)$ | 6 ◇ 6 ◇ 6 | 24 ◇ 24 ◇ 36 | 56 ◇ 56 ◇ 108 |
| $L^2(T)$ | 1 ◇ 1 ◇ 1 | 4 ◇ 4 ◇ 8 | 10 ◇ 10 ◇ 27 |

Table: Hexahedron: local number of DOFs for DDR ◇ SDDR ◇ RTN.

Ref. for serendipity reduction: [Di Pietro and Droniou, 2023a].

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# Algebraic properties

DDR sequence:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

**Theorem (Complex property and exactness [Di Pietro et al., 2020], [Di Pietro and Droniou, 2021])**

*The DDR sequence is a complex, which is exact if the topology of $\Omega$ is trivial:*

$$\operatorname{Im} \underline{\boldsymbol{G}}^k_h = \operatorname{Ker} \underline{\boldsymbol{C}}^k_h, \quad \operatorname{Im} \underline{\boldsymbol{C}}^k_h = \operatorname{Ker} D^k_h, \quad \operatorname{Im} D^k_h = \mathcal{P}^k(\mathcal{T}_h).$$

# Algebraic properties

DDR sequence:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{\boldsymbol{D}^k_h} \boldsymbol{\mathcal{P}}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

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**Theorem (Cohomology [Di Pietro et al., 2023])**

*For a generic domain $\Omega$, the DDR complex has the same cohomology as the continuous de Rham complex.*

# $L^2$-like inner products

- Local $L^2$-like inner product on the DDR spaces:

  for $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ or $(\mathrm{div}, k)$,

  $$(x_T, y_T)_{\bullet,T} = \int_T \boldsymbol{P}^{\ell}_{\bullet,T} x_T \cdot \boldsymbol{P}^{\ell}_{\bullet,T} y_T + \mathrm{s}_{\bullet,T}(x_T, y_T) \qquad \forall x_T, y_T \in \underline{X}^k_{\bullet,T},$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face/edge potentials*).

- Global $L^2$-like product by standard assembly of local ones.

# Analytical properties

$$\begin{array}{ccccccc}
C^\infty(\overline{\Omega}) & & C^\infty(\overline{\Omega})^3 & & C^\infty(\overline{\Omega})^3 & & \\
\downarrow{\scriptstyle \underline{I}^k_{\mathbf{grad},h}} & & \downarrow{\scriptstyle \underline{I}^k_{\mathbf{curl},h}} & & \downarrow{\scriptstyle \underline{I}^k_{\mathrm{div},h}} & & \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h) \\
(\cdot,\cdot)_{\mathbf{grad},h} & & (\cdot,\cdot)_{\mathbf{curl},h} & & (\cdot,\cdot)_{\mathrm{div},h} & & (\cdot,\cdot)_{L^2}
\end{array}$$

- For stability:

  Poincaré inequalities: for $\underline{\mathrm{d}}^k_h = \underline{\boldsymbol{G}}^k_h, \underline{\boldsymbol{C}}^k_h, D^k_h$,

  $$\|\underline{x}_h\|_h \lesssim \|\underline{\mathrm{d}}^k_h \underline{x}_h\|_h \qquad \forall \underline{x}_h \in (\operatorname{Ker} \underline{\mathrm{d}}^k_h)^\perp.$$

- For consistency:
  - Primal consistency: approximation properties of $\boldsymbol{P}^\ell_{\bullet,h} \circ \underline{\boldsymbol{I}}^k_{\bullet,h}$ and $\mathrm{d}^k_h \circ \underline{\boldsymbol{I}}^k_{\bullet,h}$.
  - Adjoint consistency: control error in global discrete integration-by-parts.

# Poincaré inequalities

**Theorem (Poincaré inequalities [Di Pietro and Droniou, 2021], [Di Pietro and Hanot, 2023], [Di Pietro et al., 2025])**

*It holds:*

$$\|\underline{q}_h\|_{\mathbf{grad},h} \lesssim \|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h} \qquad \forall \underline{q}_h \in (\mathrm{Ker}\, \underline{\boldsymbol{G}}_h^k)^\perp,$$
$$\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\zeta}}_h\|_{\mathrm{div},h} \qquad \forall \underline{\boldsymbol{\zeta}}_h \in (\mathrm{Ker}\, \underline{\boldsymbol{C}}_h^k)^\perp,$$
$$\|\underline{\boldsymbol{w}}_h\|_{\mathrm{div},h} \lesssim \|D_h^k \underline{\boldsymbol{w}}_h\|_{L^2(\Omega)} \qquad \forall \underline{\boldsymbol{w}}_h \in (\mathrm{Ker}\, D_h^k)^\perp.$$

- Essential to use the complex exactness to get stability of numerical discretisations.

# Primal consistency

**Theorem (Consistency of potential reconstruction and stabilisation [Di Pietro and Droniou, 2023b])**

*It holds, for* $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ *or* $(\mathrm{div}, k)$,

$$\|\boldsymbol{P}^{\ell}_{\bullet,T}\underline{\boldsymbol{I}}^{k}_{\bullet,T} f - f\|_{L^2(T)} + \mathrm{s}_{\bullet,T}(\underline{\boldsymbol{I}}^{k}_{\bullet,T} f, \underline{\boldsymbol{I}}^{k}_{\bullet,T} f) \lesssim h_T^{\ell+1} |f|_{H^{\ell+1}(T)} \qquad \forall f \in H^{\ell+1}(T)$$

*(caveat for* $\bullet = \mathbf{curl}$*).*

- Comes from local polynomial consistency: $\boldsymbol{P}^{\ell}_{\bullet,T}\underline{\boldsymbol{I}}^{k}_{\bullet,T} x_T = x_T$ and $\mathrm{s}_{\bullet,T}(\underline{\boldsymbol{I}}^{k}_{\bullet,T} x_T, \cdot) = 0$ if $x_T \in \mathcal{P}^{\ell}(T)$.
- Gives consistency of discrete $L^2$ inner products.

# Commutation properties

**Theorem (Commutation properties [Di Pietro and Droniou, 2023b])**

$$\begin{array}{ccc} C^1(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^0(\overline{\Omega}) \\ \downarrow \underline{\boldsymbol{I}}^k_{\mathbf{grad},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \\ \underline{\boldsymbol{X}}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \end{array} \qquad \begin{array}{ccc} H^2(\Omega) & \xrightarrow{\mathbf{curl}} & H^1(\Omega) \\ \downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} \\ \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \end{array} \qquad \begin{array}{ccc} H^1(\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \downarrow \pi^k_h \\ \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h) \end{array}$$

- Together with the consistency of potential reconstruction, provides optimal approximation properties of the differential operators.
- Essential for robust approximations (e.g. pressure-robust for Stokes, locking-free for Reissner-Mindlin...).

# Adjoint consistency

**Theorem (Adjoint consistency for the discrete gradient [Di Pietro and Droniou, 2021])**

*For all* $\boldsymbol{v} \in C^0(\overline{\Omega}) \cap \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \boldsymbol{H}^{\max(k+1,2)}(\mathcal{T}_h)$ *and* $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$\left| (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h\underline{q}_h)_{\mathbf{curl},h} + \int_\Omega \mathrm{div}\,\boldsymbol{v}\ P^{k+1}_{\mathbf{grad},h}\underline{q}_h \right| \lesssim h^{k+1}|\boldsymbol{v}|_{H^{(k+1,2)}(\mathcal{T}_h)} \|\underline{\boldsymbol{G}}^k_h\underline{q}_h\|_{\mathbf{curl},h}.$$

- Similar adjoint consistencies for the curl, divergence.
- Essential for error estimates when IBP are involved in the weak formulations.

# Schemes

- Numerical schemes are obtained replacing spaces, differential operators, and $L^2$-products with their discrete DDR counterparts.

- Poincaré inequalities, primal and adjoint consistencies yield stable schemes, with $O(h^{k+1})$ rates of convergence in energy norm.
  [Di Pietro and Droniou, 2021], [Beirão da Veiga et al., 2022],
  [Droniou and Qian, 2023], [Di Pietro and Droniou, 2023b],
  [Di Pietro and Droniou, 2022], etc.

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# The de Rham complex in exterior calculus formulation

- Differential forms:
  - $k$-forms: mappings $\omega$ on $\Omega$ s.t. $\omega_x \in \Lambda^k(\mathbb{R}^n)$, $k$-alternate linear forms on $\mathbb{R}^n$.
  - $\mathrm{d}^k$: exterior derivative of $k$-forms.
  - $H\Lambda^k(\Omega)$: $k$-forms $\omega \in L^2$ s.t. $\mathrm{d}^k\omega \in L^2$.
- The continuous de Rham complex with differential forms:

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the following links are established through vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

# Construction of a DDR exterior calculus complex

- Discrete spaces $\underline{X}_{r,f}^k$ with polynomial components attached to mesh entities, representing projections of traces of $k$-forms.
- Recursive and hierarchical construction on $d$-cells $f$ (for $d = k+1, \ldots, n$):
  - Discrete exterior derivative

$$\mathrm{d}_{r,f}^k : \underline{X}_{r,f}^k \to \mathcal{P}_r\Lambda^{k+1}(f)$$

  - Discrete potential (playing the role of a $k$-form inside $f$), using the discrete exterior derivative

$$P_{r,f}^k : \underline{X}_{r,f}^k \to \mathcal{P}_r\Lambda^{k}(f)$$

  - Reconstructions mimic the Stokes formula: $\forall(\omega,\mu) \in \Lambda^\ell(f)\times\Lambda^{n-\ell-1}(f)$,

$$\int_f \mathrm{d}^\ell\omega\wedge\mu = (-1)^{\ell+1}\int_f \omega\wedge\mathrm{d}^{n-\ell-1}\mu + \int_{\partial f}\mathrm{tr}_{\partial f}\,\omega\wedge\mathrm{tr}_{\partial f}\,\mu$$

- Benefit: unified construction and algebraic proofs for any space dimension, and all along the sequence (no specific argument for 2D/3D or $\mathbf{grad}$, $\mathbf{curl}$, $\mathrm{div}$). [Bonaldi et al., 2023].

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# DDR scheme for Stokes in curl-curl formulation

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1_\star(\Omega)$ s.t.

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H^1_\star(\Omega). \end{aligned}$$

- DDR scheme: Find $(\underline{\boldsymbol{u}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{X}^k_{\mathbf{grad},h,\star}$ such that

$$\begin{aligned} \nu(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\ -(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} &= 0 && \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,\star}, \end{aligned}$$

where $\underline{X}^k_{\mathbf{grad},h,\star} := \{\underline{r}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{I}^k_{\mathbf{grad},h} 1, \underline{r}_h) = 0\}$.

# Navier–Stokes equations in curl-curl formulation

- Additional convective term:

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\operatorname{div} \boldsymbol{u})\boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2} \mathbf{grad}\, |\boldsymbol{u}|^2.$$

so

$$-\nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \mathbf{grad}\, p = \nu\, \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{u} + \underbrace{(\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u}}_{\text{additional term}} + \mathbf{grad} \underbrace{\left( p + \frac{1}{2} |\boldsymbol{u}|^2 \right)}_{\text{new pressure } p'}$$

- Additional term in weak formulation (vanishes for $\boldsymbol{v} = \boldsymbol{u}$)

$$\int_\Omega [(\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u}] \cdot \boldsymbol{v}.$$

# DDR scheme for Navier–Stokes in curl-curl formulation

- Same as Stokes, but we need to discretize

$$\int_\Omega [(\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u}] \cdot \boldsymbol{v}.$$

- Natural choice: replace continuous curl and functions by discrete ones.

$$\sum_{T \in \mathcal{T}_h} \int_T \left[ (\boldsymbol{C}_T^k \underline{\boldsymbol{u}}_T) \times \boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{u}}_T \right] \cdot \boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{v}}_T$$

*Non-dissipativity: this term vanishes if* $\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{u}}_h$.

# Convergence result

**Theorem (Error estimates [Di Pietro et al., 2024])**

*Define the discrete $L^4$-Sobolev constant by*

$$C_{\mathrm{S},h} \coloneqq \max\left\{ \frac{\|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}} \;:\; \underline{\boldsymbol{v}}_h \in (\operatorname{Im}\underline{\boldsymbol{G}}^k_h)^\perp\backslash\{\boldsymbol{0}\} \right\}.$$

*Then, if*

$$C^2_{\mathrm{S},h}\|\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{R}_{\boldsymbol{u}})\|_{\mathbf{curl},h} \text{ is small enough,}$$

*we have*

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- $\mathbf{R}_{\boldsymbol{u}}$: solenoidal part of forcing term $\boldsymbol{f}$, depends only on $\boldsymbol{u}$.
- Robust estimate with respect to the pressure (RHS does not depend on $p'$).

# Convergence result

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- Based on discrete Poincaré inequalities, primal/adjoint consistencies, and estimates obtained through the unified polytopal exterior calculus framework.

# Convergence result

**Theorem (Error estimates [Di Pietro et al., 2024])**

*Define the discrete $L^4$-Sobolev constant by*

$$C_{\mathrm{S},h} := \max\left\{ \frac{\|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}} \ : \ \underline{\boldsymbol{v}}_h \in (\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h)^\perp\backslash\{\boldsymbol{0}\} \right\}.$$

*Then, if*

$$C^2_{\mathrm{S},h}\|\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathrm{R}_{\boldsymbol{u}})\|_{\mathbf{curl},h} \text{ is small enough,}$$

*we have*

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- Valid for Stokes without smallness assumption (no $C_{\mathrm{S},h}$, and robust estimate on the pressure).
- Boundedness of $C_{S,h}$ w.r.t. $h$ still an open question (but expected for convex domains).

# Outline


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# Pressure-robustness I

- Analytical solution on $\Omega = (0,1)^3$:

$$p(x,y,z) = \lambda \sin(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad \text{with } \lambda \in \{1, 100\},$$

$$\boldsymbol{u}(x,y,z) = \begin{bmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) \\ \frac{1}{2}\cos(2\pi x)\sin(2\pi y)\cos(2\pi z) \\ -\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) \end{bmatrix}.$$

- Measured errors (discrete and potential-based):

$$E_{\boldsymbol{u}}^{\mathrm{d}} := \left( \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|^2_{\mathrm{div},h} \right)^{1/2},$$

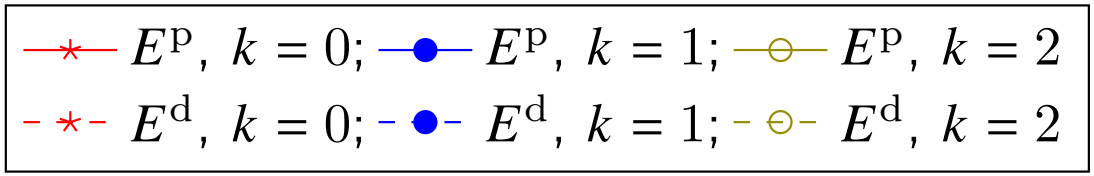
$$E_p^{\mathrm{d}} := \|\underline{\boldsymbol{G}}^k_h(\underline{p}_h - \underline{I}^k_{\mathbf{grad},h}p)\|_{\mathbf{curl},h},$$

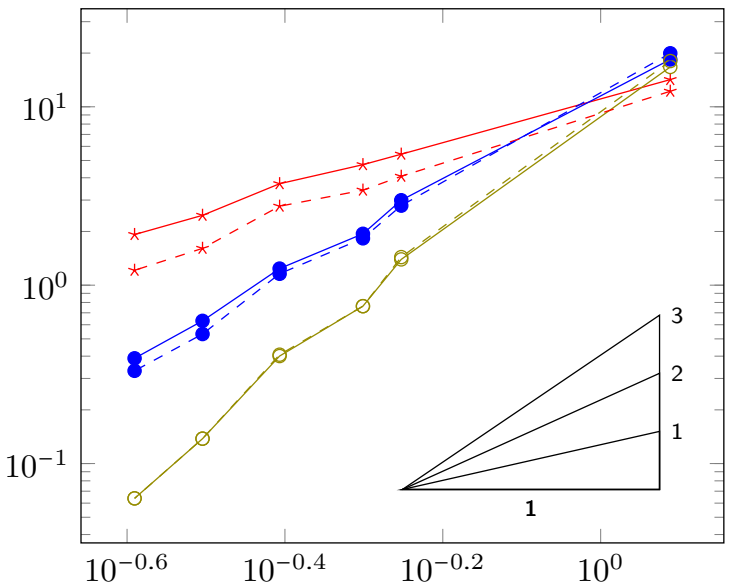
$$E_{\boldsymbol{u}}^{\mathrm{p}} := \left( \|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{u}}_h - \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{u}}_h - \mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \right)^{1/2},$$

$$E_p^{\mathrm{p}} := \|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{G}}^k_h\underline{p}_h - \mathbf{grad}\,p\|_{\boldsymbol{L}^2(\Omega)}.$$

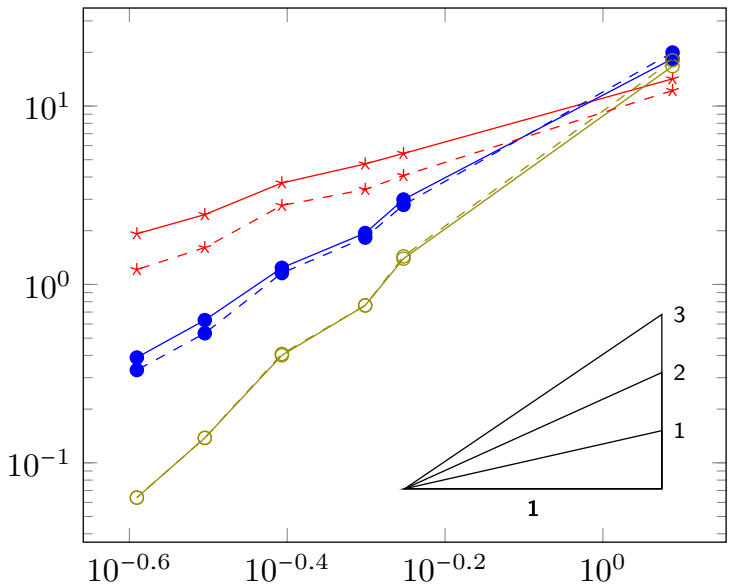
- Meshes: tetrahedral and Voronoi meshes.

# Pressure-robustness II



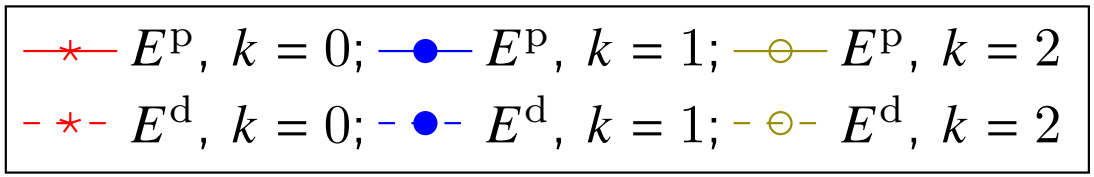

(a) Errors on $\boldsymbol{u}$, $\lambda = 1$

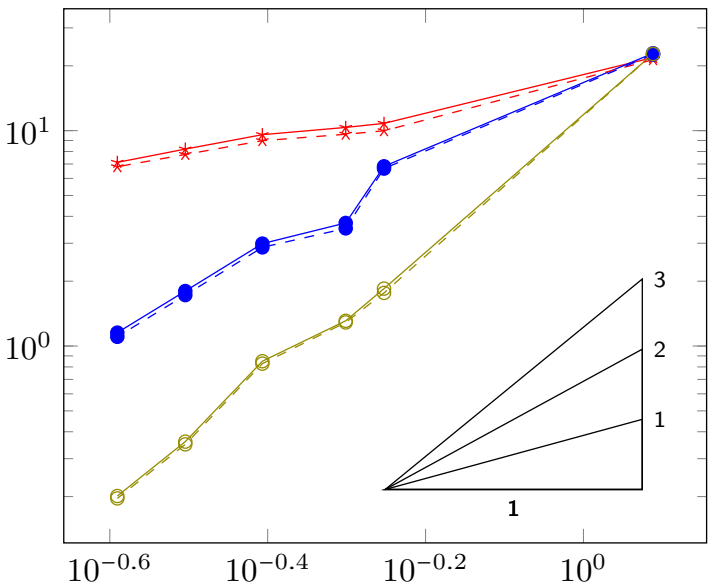



(b) Errors on $\boldsymbol{u}$, $\lambda = 10^{2}$

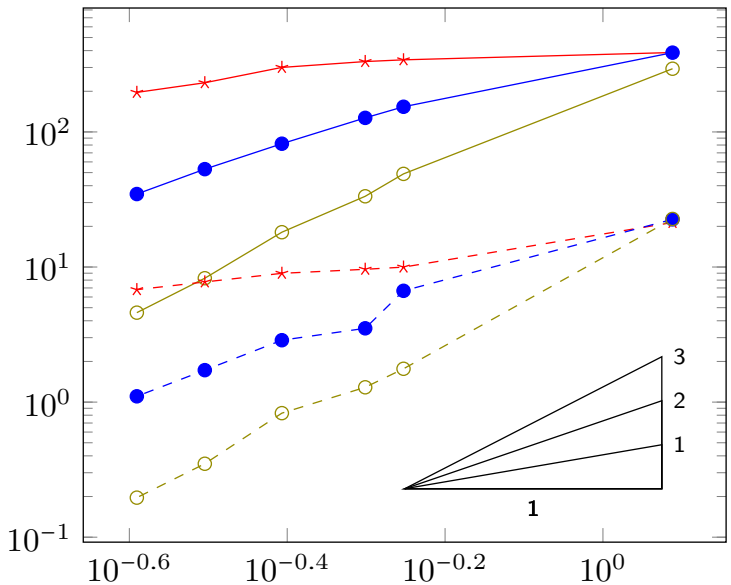
Figure: Tetrahedral meshes: errors with respect to $h$

# Pressure-robustness III



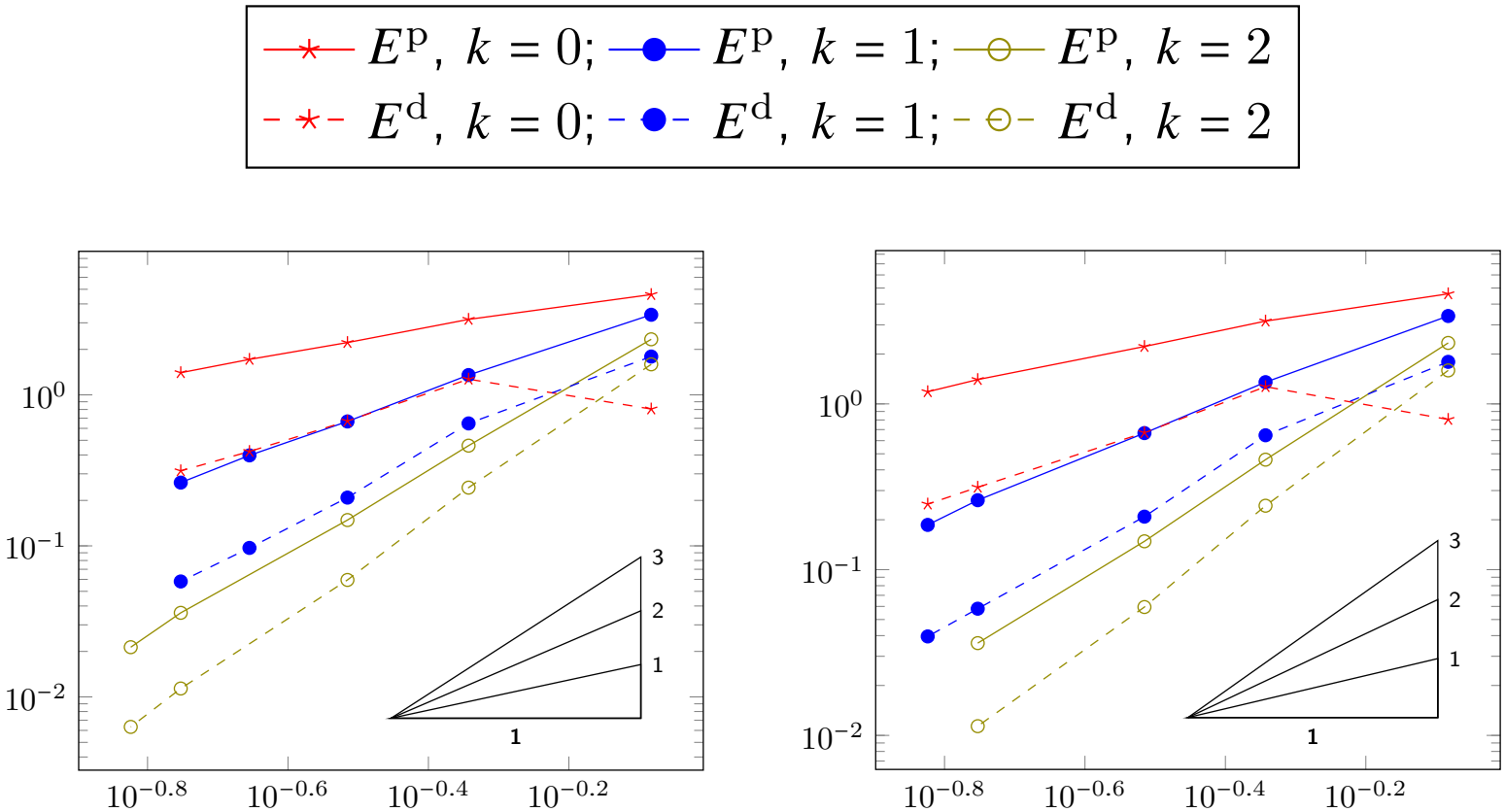

(a) Errors on $\nabla p$, $\lambda = 1$




(b) Errors on $\nabla p$, $\lambda = 10^2$

Figure: Tetrahedral meshes: errors with respect to $h$

# Pressure-robustness IV

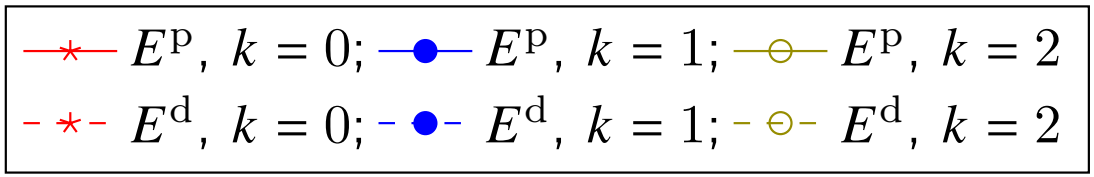


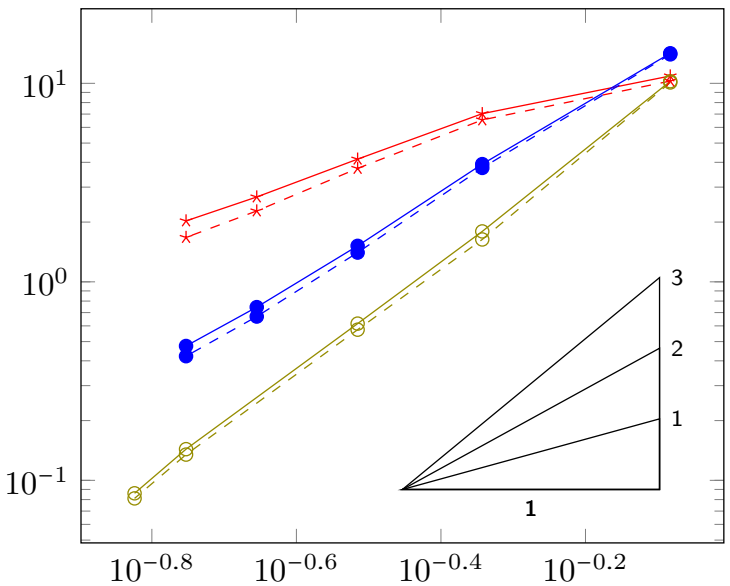

(a) Errors on $\boldsymbol{u}$, $\lambda = 1$

(b) Errors on $\boldsymbol{u}$, $\lambda = 10^2$

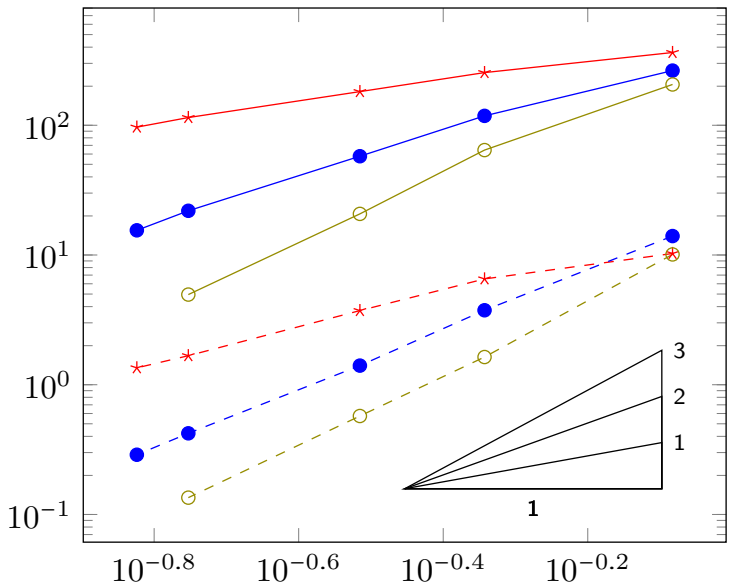
Figure: Voronoi meshes: errors with respect to $h$

# Pressure-robustness V




(a) Errors on $\nabla p$, $\lambda = 1$




(b) Errors on $\nabla p$, $\lambda = 10^2$

Figure: Voronoi meshes: errors with respect to $h$

# Flow in cavity – mixed BCs I

In the unit cube $\Omega = (0,1)^3$:

- Essential BCs (pressure and tangential velocity):

$$p(x,y,z) = -z \quad \text{and} \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$$

  on the bottom corner $\{0\} \times (0, 0.25) \times (0, 0.25)$ of the face $x = 0$.

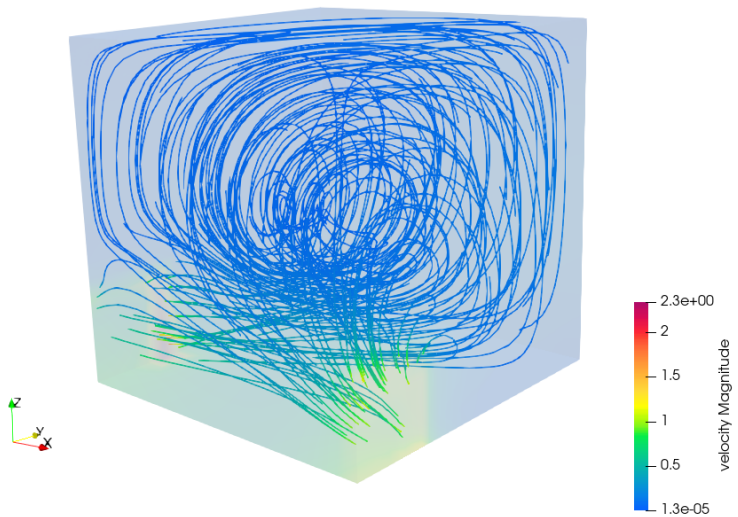
- Natural BCs (tangential vorticity and flux):

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 1$$

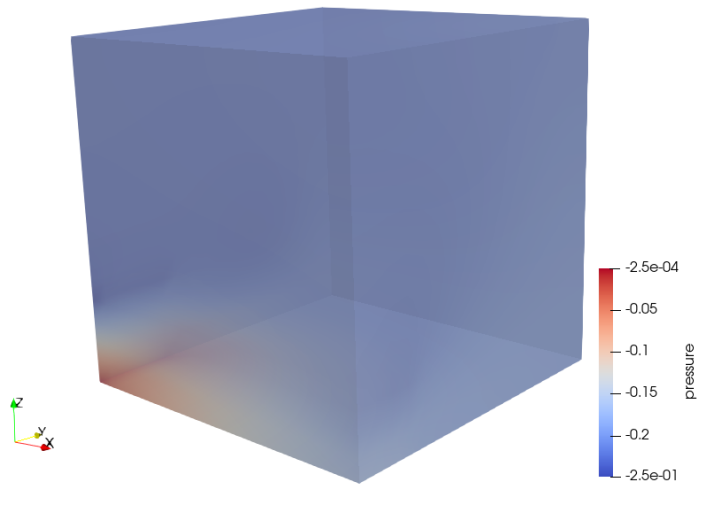
  on the bottom corner $\{1\} \times (0, 0.25) \times (0, 0.25)$ of the face $x = 1$,

- Homogeneous natural BCs elsewhere.

# Flow in cavity – mixed BCs II




(a) Velocity




(b) Pressure

Figure: Velocity streamlines and pressure

# Conclusion I

- DDR:
  - fully discrete complex, with stability properties as the continuous complex;
  - arbitrary degree of accuracy;
  - applicable on polytopal meshes.

- DOFs:
  - polynomial moments attached to mesh entities (vertices, edges, faces, elements);
  - come fom the hierarchical construction based on integration-by-parts.

- Systematic techniques to reduce the number of DOFs without loosing accuracy:
  - enhancement (reconstruction of potential from discrete differential operator),
  - serendipity (on any polytopal mesh).

  $\rightsquigarrow$ leaner complexes than FE approches on certain meshes (*and fully compatible with FE complexes on hybrid meshes*).

# Conclusion II

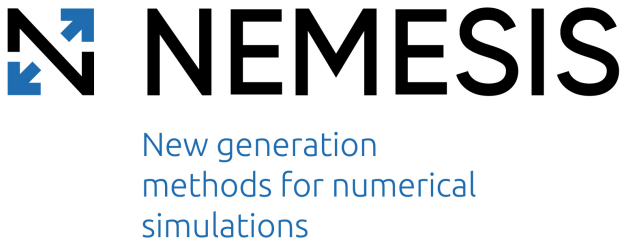
- Full set of homological and analytical results: cohomology, Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.
- Polytopal exterior calculus approach to unify the construction.
- Some other applications/complexes:
  - div-div plates complex and serendipity version [Di Pietro and Droniou, 2023a], [Botti et al., 2023].
  - Magnetostatics equations [Di Pietro and Droniou, 2021].
  - Yang–Mills equations [Droniou et al., 2023], [Droniou and Qian, 2023].
  - Stokes complex [Hanot, 2023].
  - Rot-rot complex [Di Pietro, 2023].
  - etc.

- Notes and series of introductory lectures to DDR:

  `https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html`

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

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**Thank you for your attention!**

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