

# The Hybrid High-Order method as an extension of Crouzeix–Raviart

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# References for this presentation

- Primer (basis for this talk, first chapter of upcoming book):

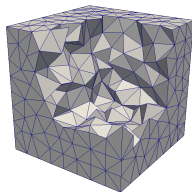
D.A. Di Pietro, J. Droniou and R. Dubois. From Finite Elements to Hybrid High-Order methods.  
<https://arxiv.org/abs/2503.00425>

- Advanced textbook with several variants and applications:

D. A. Di Pietro and J. Droniou. The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications. Modeling, Simulation and Applications, vol. 19. Springer International Publishing, 2020, xxi + 525p. doi: 10.1007/978-3-030-37203-3.  
<https://hal.archives-ouvertes.fr/hal-02151813>.

- 1 Why polytopal methods?
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# The Finite Element way

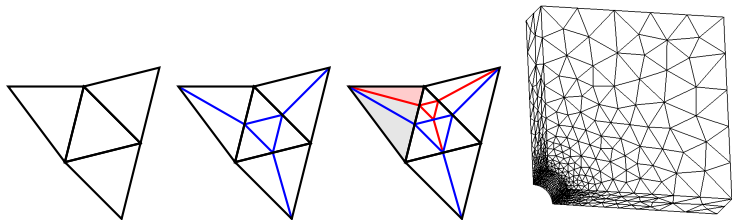


$\mathcal{T}_h = \{T\}$  conforming tetrahedral/hexahedral mesh.

- Define **local polynomial spaces** on each element, and **glue them together** to form a subspace of the continuous space associated with the model.  
*Example:* conforming  $\mathcal{P}^1$  spaces in  $H^1(\Omega)$ .
- **Gluing only works on special meshes...**

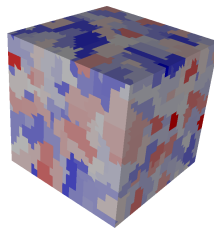
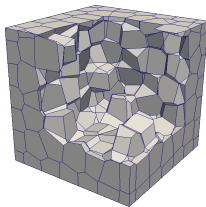
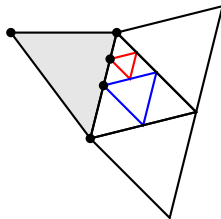
# The Finite Element way

## Shortcomings



- Approach limited to **conforming meshes** with **standard elements**
  - ⇒ local refinement requires to **trade mesh size for mesh quality**
  - ⇒ complex geometries may require a **large number of elements**
  - ⇒ the element shape cannot be seamlessly **adapted to the solution** (e.g. hexahedra in boundary layers + tetrahedra in the bulk for CFD simulations)
- Need for (global) basis functions
  - ⇒ significant increase of DOFs on hexahedral elements

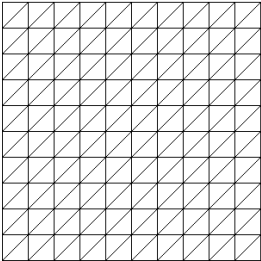
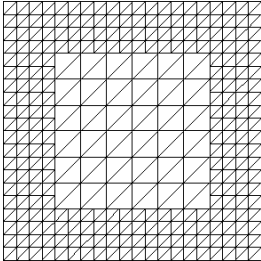
# Benefits of polytopal meshes



- **Local refinement** is easy, and preserves mesh regularity.
- **Agglomeration of elements** (e.g., for multigrid methods) is seamless.
- High-level approach can lead to **leaner methods** (fewer DOFs).
- Can be **combined with standard Finite Elements** on hybrid meshes (made of tetrahedra/hexahedra + polyhedral elements).

# Example of efficiency (I): Reissner–Mindlin plate problem

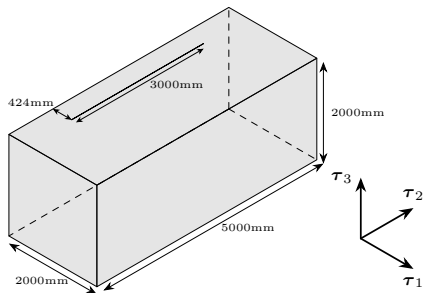
[Di Pietro and Droniou, 2022]

| Stabilised $\mathcal{P}^2$ -( $\mathcal{P}^1 + \mathcal{B}^3$ ) scheme            |         | DDR (polytopal) scheme                                                             |         |
|-----------------------------------------------------------------------------------|---------|------------------------------------------------------------------------------------|---------|
|  |         |  |         |
| nb. DOFs                                                                          | Error   | nb. DOFs                                                                           | Error   |
| 2403                                                                              | 0.138   | 550                                                                                | 0.161   |
| 9603                                                                              | 6.82e-2 | 2121                                                                               | 6.77e-2 |
| 38402                                                                             | 3.40e-2 | 8329                                                                               | 3.1e-2  |

# Example of efficiency (II): from CEA-CESTA

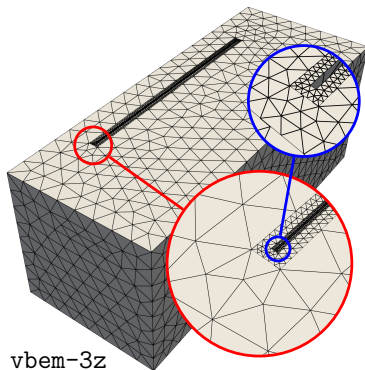
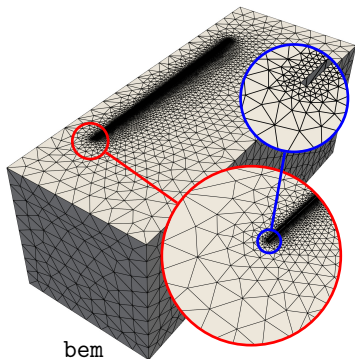
[Touzalin, 2025]

**Problem:** use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



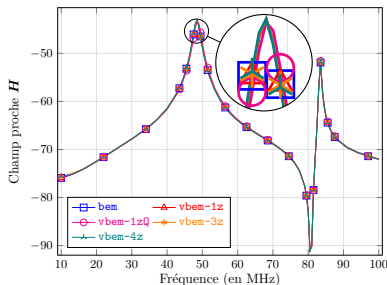
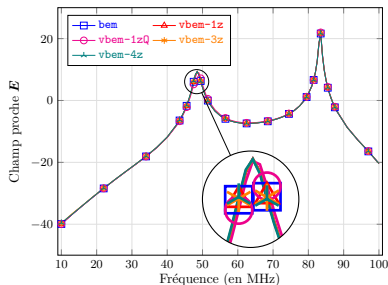
## Example of efficiency (II): from CEA-CESTA

**Meshes:** conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).



# Example of efficiency (II): from CEA-CESTA

**Accuracy:** comparison of modulus of reflected near fields at the top.



**Computational cost**

| <i>Method</i> | <i>Assembly</i> | <i>Resolution</i> |
|---------------|-----------------|-------------------|
| bem           | 813s            | 125s              |
| vbem-3z       | 321s            | 19s               |

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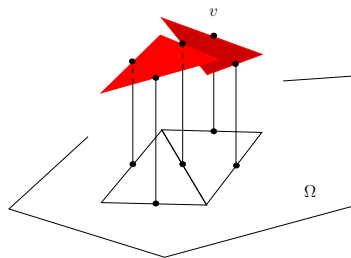
# Crouzeix–Raviart functions

$\mathcal{T}_h$ : triangulation of a domain  $\Omega$ .

**Space:** piecewise linear functions on  $\mathcal{T}_h$ , with continuity at edge midpoints.

$$V_h = \{v : \Omega \rightarrow \mathbb{R} : v_T \in \mathcal{P}^1(T) \quad \forall T \in \mathcal{T}_h, \\ v_T(x_F) = v_{T'}(x_F) \quad \forall F \text{ edge between } T, T'\},$$

where  $v_T := v|_T$  and  $x_F$  = center of mass of  $F$ .



Degrees of freedom: values at edge midpoints.

Remark:

$$v_T(x_F) = \frac{1}{|F|} \int_F v_T$$

**Space with zero BC:**  $V_{h,0} := \{v \in V_h : v_T(x_F) = 0 \quad \forall F \in \mathcal{F}_h^b\}$ .

# CR scheme for the Laplace problem

Laplace problem: find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$-\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

Weak formulation:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

with

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{and} \quad \ell(v) = \int_{\Omega} f v.$$

# CR scheme for the Laplace problem

**Laplace problem:** find  $u : \Omega \rightarrow \mathbb{R}$  such that

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with

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{and} \quad \ell(v) = \int_{\Omega} f v.$$

**Crouzeix–Raviart scheme:** standard gradient is not an  $L^2$ -function, so we use broken gradient.

$$\text{Find } u_h \in V_{h,0} \text{ such that } a_h(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{h,0}$$

where

$$a_h(u_h, v_h) = \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h = \sum_{T \in \mathcal{T}_h} \int_T \nabla u_T \cdot \nabla v_T.$$

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# Stability analysis

## Theorem (Poincaré inequality for CR functions)

*It holds*

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)} \quad \forall v_h \in V_{h,0}.$$

## Corollary (Well-posedness of the scheme)

*If  $u_h$  is a solution to the Crouzeix–Raviart scheme, then*

$$\|\nabla_h u_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

**Proof of the corollary:** make  $v_h = u_h$ ,

$$\begin{aligned} \|\nabla_h u_h\|_{L^2(\Omega)}^2 &= \int_{\Omega} \nabla_h u_h \cdot \nabla_h u_h = \int_{\Omega} f u_h \leq \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} \\ &\lesssim \|f\|_{L^2(\Omega)} \|\nabla_h u_h\|_{L^2(\Omega)}. \end{aligned}$$

# Proof of the Poincaré inequality

Pre-image by divergence: there exists  $\tau \in H^1(\Omega)$  s.t.

$$\operatorname{div} \tau = v_h \quad \text{and} \quad \|\tau\|_{H^1(\Omega)} \lesssim \|v_h\|_{L^2(\Omega)}.$$

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Local integration-by-parts: we have

$$\begin{aligned} \|v_h\|_{L^2(\Omega)}^2 &= \sum_{T \in \mathcal{T}_h} \int_T v_T \operatorname{div} \tau \\ &\stackrel{IBP}{=} \sum_{T \in \mathcal{T}_h} \left( - \int_T \nabla v_T \cdot \tau + \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau \cdot n_{TF}) \right) \end{aligned}$$

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where  $v_F = v(x_F)$  (does not depend on  $T$ ) and we have used

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h} \int_F v_F \tau \cdot n_{TF} = \sum_{F \in \mathcal{F}_h^i} v_F (\tau \cdot n_{TF} + \tau \cdot n_{T'F}) = 0.$$

# Proof of the Poincaré inequality

$$\|v_h\|_{L^2(\Omega)}^2 = - \sum_{T \in \mathcal{T}_h} \int_T \nabla v_T \cdot \tau + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F (v_T - v_F)(\tau \cdot n_{TF}),$$

Cauchy–Schwarz inequalities:

$$\begin{aligned} \|v_h\|_{L^2(\Omega)}^2 &\leq \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}} \\ &\quad \times \left[ \sum_{T \in \mathcal{T}_h} \left( \|\tau\|_{L^2(T)}^2 + h_T \|\tau\|_{L^2(\partial T)}^2 \right) \right]^{\frac{1}{2}} \end{aligned}$$

# Proof of the Poincaré inequality

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Trace inequalities:

$$\left[ \sum_{T \in \mathcal{T}_h} \left( \|\tau\|_{L^2(T)}^2 + h_T \|\tau\|_{L^2(\partial T)}^2 \right) \right]^{\frac{1}{2}} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \|\tau\|_{H^1(T)}^2 \right]^{\frac{1}{2}} \lesssim \|v_h\|_{L^2(\Omega)}$$

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Cauchy–Schwarz inequalities:

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Intermediate conclusion:

$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}}$$

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$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}}$$

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$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}}$$

Variation of  $v$ : since  $v_T$  is linear,

$$\begin{aligned} \|v_F - v_T\|_{L^2(F)} &= \|v_T(x_F) - v_T\|_{L^2(F)} = \|(x_F - x) \cdot \nabla v_T\|_{L^2(F)} \\ &\leq h_T |F|^{\frac{1}{2}} |\nabla v_T| \lesssim h_T^{\frac{1}{2}} \|\nabla v_T\|_{L^2(T)}. \end{aligned}$$

# Proof of the Poincaré inequality

$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}}$$

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Final conclusion

$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \|\nabla v_T\|_{L^2(T)}^2 \right]^{\frac{1}{2}} = \|\nabla_h v_h\|_{L^2(\Omega)}.$$

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# Everything can be expressed in terms of DOFs

DOFs: for  $v \in \mathcal{P}^1(T)$  set

$$v_F = v(x_F) = \frac{1}{|F|} \int_F v.$$

Recovering the gradient: for all  $w \in \mathcal{P}^1(T)$  we have

$$\int_T \nabla v \cdot \nabla w \stackrel{IBP}{=} - \int_T v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v (\nabla w \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}).$$

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Recovering the average: with  $x_T$  center of mass of  $T$  (i.e.  $\int_T (x - x_T) = 0$ ),

$$\int_T v = \frac{1}{d} \int_T v \operatorname{div}(x - x_T)$$

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$$\begin{aligned} \int_T v &= \frac{1}{d} \int_T v \operatorname{div}(x - x_T) \\ &\stackrel{IBP}{=} - \frac{1}{d} \int_T \cancel{\nabla v \cdot (x - x_T)} + \frac{1}{d} \sum_{F \in \mathcal{F}_T} \int_F v (x - x_T) \cdot n_{TF} \\ &= \sum_{F \in \mathcal{F}_T} \frac{d_{TF} |F|}{d} v_F. \end{aligned}$$

with  $d_{TF} = \operatorname{dist}(x_T, F)$ .

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# Summary

Formulas for recovery: If  $v \in \mathcal{P}^1(T)$  and  $v_F = \frac{1}{|F|} \int_F v$  then

$$\int_T \nabla v \cdot \nabla w = \sum_{F \in \mathcal{F}_T} \int_F \mathbf{v}_F (\nabla w \cdot \mathbf{n}_{TF}) \quad \forall w \in \mathcal{P}^1(T),$$
$$\int_T v = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}|F|}{d} \mathbf{v}_F.$$

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$$\int_T v = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}|F|}{d} \mathbf{v}_F.$$

These formulas are valid on any polygon/polyhedron  $T$ !

*They identify key DOFs (face values) and ways to reconstruct linear functions from these DOFs.*

# Discrete space and reconstructions

$\mathcal{T}_h$ : polygonal mesh on  $\Omega$ .

Space:

$$\underline{V}_h := \{ \underline{v}_h = (\underline{v}_F)_{F \in \mathcal{F}_h} : v_F \in \mathbb{R} \quad \forall F \in \mathcal{F}_h \}$$

with restriction to  $T \in \mathcal{T}_h$  denoted by  $\underline{V}_T$ .

Potential reconstruction: define  $p_T^1 = \underline{V}_T \rightarrow \mathcal{P}^1(T)$  by: for all  $\underline{v}_T \in \underline{V}_T$ ,

$$\begin{aligned} \int_T \nabla p_T^1 \underline{v}_T \cdot \nabla w &= \sum_{F \in \mathcal{F}_T} \int_F \underline{v}_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^1(T), \\ \int_T p_T^1 \underline{v}_T &= \sum_{F \in \mathcal{F}_T} \frac{d_{TF} |F|}{d} \underline{v}_F. \end{aligned}$$

# Interpolator and consistency

**Interpolator:** gives meaning to the components in the discrete space.

$$\begin{aligned}\underline{I}_h^0 : H^1(\Omega) &\rightarrow \underline{V}_h \text{ such that, for all } v \in H^1(\Omega), \\ \underline{I}_h^0 v &= (\pi_{\mathcal{P},F}^0 v|_F)_{F \in \mathcal{F}_h},\end{aligned}$$

where  $\pi_{\mathcal{P},F}^0 : L^2(F) \rightarrow \mathcal{P}^0(F)$  is the  $L^2$ -orthogonal projector (average on  $F$ ).

*Restriction to  $T$  denoted by  $\underline{I}_T^0 : H^1(T) \rightarrow \underline{V}_T$ .*

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*Restriction to  $T$  denoted by  $\underline{I}_T^0 : H^1(T) \rightarrow \underline{V}_T$ .*

**Consistency:**

$$\text{If } v \in \mathcal{P}^1(T) \text{ then } p_T^1 \circ \underline{I}_T^0 v = v.$$

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**IBP to recover higher-order polynomials:** let  $k \geq 1$  and  $v \in \mathcal{P}^{k+1}(T)$ . Then  $\nabla v$  is fully identified by: for all  $w \in \mathcal{P}^{k+1}(T)$ ,

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**Average:** since  $k \geq 1$ ,

$$\int_T v = \int_T \pi_{\mathcal{P},T}^{k-1} v.$$

# Discrete space and reconstructions

Space:

$$\underline{V}_h := \{ \underline{v}_h = ((\textcolor{red}{v}_T)_{T \in \mathcal{T}_h}, (\textcolor{red}{v}_F)_{F \in \mathcal{F}_h}) : v_T \in \mathcal{P}^{k-1}(T), v_F \in \mathcal{P}^k(F) \}$$

Potential reconstruction: define  $p_T^{k+1} : \underline{V}_T \rightarrow \mathcal{P}^{k+1}(T)$  by: for all  $\underline{v}_T \in \underline{V}_T$ ,

$$\int_T \nabla \textcolor{red}{p}_T^{k+1} \underline{v}_T \cdot \nabla w = - \int_T \textcolor{red}{v}_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \textcolor{red}{v}_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\int_T \textcolor{red}{p}_T^{k+1} \underline{v}_T = \int_T \textcolor{red}{v}_T.$$

# Interpolator and consistency

**Interpolator:** gives meaning to the components in the discrete space.

$$\begin{aligned} \underline{I}_h^k : H^1(\Omega) &\rightarrow \underline{V}_h \text{ such that, for all } v \in H^1(\Omega), \\ \underline{I}_h^k v &= ((\pi_{\mathcal{P},T}^{k-1} v)_{T \in \mathcal{T}_h}, \pi_{\mathcal{P},F}^k v|_F)_{F \in \mathcal{F}_h}), \end{aligned}$$

**Consistency:**

$$\text{If } v \in \mathcal{P}^{k+1}(T) \text{ then } p_T^{k+1} \circ \underline{I}_T^k v = v.$$

# Naive scheme

Weak form of Laplace problem:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

with

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{and} \quad \ell(v) = \int_{\Omega} f v.$$

First idea for HHO scheme: (for  $k \geq 1$ )

$$\text{Find } \underline{u}_h \in \underline{V}_{h,0}^k \text{ such that } a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

where

$$a_h(\underline{u}_h, \underline{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla \underline{p}_T^{k+1} \underline{u}_T \cdot \nabla \underline{p}_T^{k+1} \underline{v}_T \quad \text{and} \quad \ell_h(\underline{v}_h) = \int_{\Omega} f v_h,$$

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Bad choice: the kernel of this  $a_h$  is non-trivial...

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## A reminder

$$\|v_h\|_{L^2(\Omega)} \lesssim \left[ \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_T\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}}$$

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**Only relies on:**  $v_T \in H^1(T)$ ,  $v_F \in L^2(F)$ , and  $v_F = 0$  if  $F \in \mathcal{F}_h^b$ .

$\rightsquigarrow$  Can be applied to  $((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) = \underline{v}_h \in \underline{V}_{h,0}^k$ .

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Discrete  $H^1$ -like seminorm on  $\underline{V}_{h,0}^k$ :

$$\|\underline{v}_h\|_{1,h} := \left[ \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2 \right]^{\frac{1}{2}}, \quad \|\underline{v}_T\|_{1,T}^2 = \|\nabla v_T\|_{L^2(T)}^2 + \frac{1}{h_T} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2.$$

Theorem (Poincaré for HHO)

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h} \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k.$$

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# Abstract design of the scheme

Scheme structure:

Find  $\underline{u}_h \in \underline{V}_{h,0}^k$  such that  $a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$

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**Stabilisation:** assuming  $a_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,T}^2$  and using Poincaré gives

**Theorem (Stability of HHO)**

*If  $\underline{u}_h \in \underline{V}_{h,0}^k$  solves the scheme then  $\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}.$*

# Design of stabilisation

$$a_T(\underline{u}_T, \underline{v}_T) = \int_T \nabla \underline{p}_T^{k+1} \underline{u}_T \cdot \nabla \underline{p}_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

**Symmetry:**  $s_T$  symmetric to preserve the symmetry of the scheme.

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**Consistency:**  $s_T$  must not “break” the consistency of  $p_T^{k+1}$ :

$$s_T(\underline{I}_T^k w, \cdot) = 0 \quad \forall w \in \mathcal{P}^{k+1}(T).$$

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$\rightsquigarrow$  forces a particular structure, in which  $s_T$  only penalises the “non-polynomial” part of vectors in  $\underline{V}_T^k$ :

$$s_T(\underline{u}_T, \underline{v}_T) = \mathcal{S}_T(\delta_T^k \underline{u}_T, \delta_T^k \underline{v}_T) \quad \text{with} \quad \delta_T^k \underline{v}_T = \underline{v}_T - \underline{I}_T^k p_T^{k+1} \underline{v}_T \in \underline{V}_T^k$$

*Note:*  $\delta_T^k(\underline{I}_T^k w) = \underline{0}$  if  $w \in \mathcal{P}^{k+1}(T)$ .

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**Coercivity:** condition  $a_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,T}^2$  imposes:

For all  $\underline{z}_T \in \underline{V}_T^k$ ,

$$h_T^{-1} \sum_{F \in \mathcal{F}_T} \|z_F - z_T\|_{L^2(F)}^2 \lesssim \mathcal{S}_T(\underline{z}_T, \underline{z}_T) \lesssim \|\underline{z}_T\|_{1,T}^2.$$

# HHO: practical session with DSEL

Potential reconstruction:

$$\int_T \nabla p_T^{k+1} \underline{v}_T \cdot \nabla w = - \int_T \textcolor{red}{v}_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$
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Bilinear and linear forms:

$$a_h(\underline{u}_h, \underline{v}_h) = \sum_{T \in \mathcal{T}_h} \left( \int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T) \right)$$
$$\ell_h(\underline{v}_h) = \int_{\Omega} f v_h.$$

Theorem (Rates of convergence)

*If the solution  $u$  to the continuous model is smooth, then*

$$\|\underline{u}_h - \underline{I}_h^k u\|_{a,h} \lesssim h^{k+1}, \quad \|p_h^{k+1} \underline{u}_h - u\|_{L^2(\Omega)} \lesssim h^{k+2}.$$

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# HHO: practical session with DSEL

Full gradient reconstruction:  $G_T^k \underline{v}_T \in \mathcal{P}^k(T)^d$  s.t.

$$\int_T G_T^k \underline{v}_T \cdot \tau = - \int_T v_T \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^k(T)^d.$$

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**Thank you for your attention!**

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