An HHO–DDR polytopal method for the Brinkman problem that is robust in pure Stokes and Darcy regimes

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Reference for this presentation

A polytopal method for the Brinkman problem robust in all regimes,
D. A. Di Pietro and J. Droniou.
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Outline

1. What regime-dependent error estimates should be
2. The Brinkman model and its limiting regimes
3. Scheme and error estimate
   - An HHO scheme for Brinkman
   - Error analysis
   - HHO–DDR scheme
4. Numerical tests

Slides

Regime identified by physical parameters

Model: with α, β physical parameters,

$$\mathcal{F}(u; \alpha, \beta) = 0.$$

Regime 1	Regime 2
$\alpha = 0,\ \beta > 0$	$\alpha > 0,\ \beta = 0$

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Model: with α, β physical parameters,

$$\mathcal{F}(u; \alpha, \beta) = 0.$$

Regime 1	Regime 2
$\alpha = 0,\ \beta > 0$	$\alpha > 0,\ \beta = 0$

Dominant regime?

- Depends on relative magnitude of α and β, i.e., on $\mathcal{C}^{\alpha,\beta} := \alpha/\beta$.
- What does "Regime 1 dominates" mean? $\mathcal{C}^{\alpha,\beta} \leq 1$? $\mathcal{C}^{\alpha,\beta} \leq 10$?
- Requires a dimensionless number $\mathcal{C}^{\alpha,\beta}$, that can then be compared to 1.

Numerical analysis

Scheme:

$$\mathcal{F}_h(u_h; \alpha, \beta) = 0.$$

Error estimates:

- Should capture all regimes (not just $\mathcal{C}^{\alpha,\beta} \to 0$, $\mathcal{C}^{\alpha,\beta} \to \infty$).

Numerical analysis

Scheme:

$$\mathcal{F}_h(u_h; \alpha, \beta) = 0.$$

Error estimates:

- Should capture all regimes (not just $\mathcal{C}^{\alpha,\beta} \to 0$, $\mathcal{C}^{\alpha,\beta} \to \infty$).
- Should take into account local dominant regime: $\mathcal{C}_T^{\alpha,\beta}$ attached to element T, treated according to its regime ($\mathcal{C}_T^{\alpha,\beta} \geq 1$ or $\mathcal{C}_T^{\alpha,\beta} < 1$).

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- Should take into account local dominant regime: $\mathcal{C}_T^{\alpha,\beta}$ attached to element T, treated according to its regime ($\mathcal{C}_T^{\alpha,\beta} \geq 1$ or $\mathcal{C}_T^{\alpha,\beta} < 1$).
- $\mathcal{C}_T^{\alpha,\beta}$ often involves the local length h_T and can actually improve the error estimates.

Capture transitional regimes

No transitional regimes: with $g_i(0) = 0$,

$$\|u - u_h\| \lesssim \left[\sum_T g_1(\alpha_T) h_T^{r_2} |u|^2_{H^{\ell_2}(T)} + \sum_T g_2(\beta_T) h_T^{r_1} |u|^2_{H^{\ell_1}(T)} \right]^{\frac{1}{2}}.$$

[7, 8] (DG for advection–diffusion)
[1] (FE for Brinkman, robustness in Darcy limit)
[9] (VEM for Brinkman, robustness in Darcy limit)

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[7, 8] (DG for advection–diffusion)
[1] (FE for Brinkman, robustness in Darcy limit)
[9] (VEM for Brinkman, robustness in Darcy limit)

Transitional regimes:

$$\|u - u_h\| \lesssim \left[\sum_T \min(1, \mathcal{C}_T^{\alpha,\beta}) h_T^{r_2} |u|^2_{H^{\ell_2}(T)} + \sum_T \min(1, (\mathcal{C}_T^{\alpha,\beta})^{-1}) h_T^{r_1} |u|^2_{H^{\ell_1}(T)}\right]^{\frac{1}{2}}$$

Transitional regimes when $\mathcal{C}_T^{\alpha,\beta}$ such that both terms have same magnitude.

[5] (Advection–diffusion with HHO)
[2] (Brinkman with HHO on triangles).

Outline

Slides

Brinkman model

Flow of viscous fluid in porous matrix with fractures, bubbles, or channels.

Data:

- Ω polytopal domain in $\mathbb{R}^d$, $d = 2, 3$.
- $\mu : \Omega \to (0, \infty)$ viscosity, $\nu : \Omega \to [0, \infty)$ inverse permeability.
- $\boldsymbol{f} : \Omega \to \mathbb{R}^3$, $g : \Omega \to \mathbb{R}$ volumetric source terms.

Model: find velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\mathbf{div}(\mu\,\mathbf{grad}\,\boldsymbol{u}) + \nu\boldsymbol{u} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} &= g && \text{in } \Omega,\\
\boldsymbol{u} &= \mathbf{0} && \text{on } \partial\Omega,\\
\int_\Omega p &= 0.
\end{aligned}
$$

Note: $\mathbf{grad}\,\boldsymbol{u}$ could be replaced by $\mathbf{grad}_s\,\boldsymbol{u}$, and μ, ν could be tensors.

Limiting models I

Stokes:

$$
\begin{aligned}
-\mathbf{div}(\mu\,\mathbf{grad}\,\boldsymbol{u}) \cancelto{0}{+\nu}\,\boldsymbol{u} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} &= \cancelto{0}{g} && \text{in } \Omega,\\
\boldsymbol{u} &= \mathbf{0} && \text{on } \partial\Omega,\\
\int_\Omega p &= 0.
\end{aligned}
$$

Characteristics influencing the discretisation:

- Primal formulation.
- L^2-estimate on $\mathbf{grad}\,\boldsymbol{u}$.
- Requires inf–sup condition for p.

Limiting models II

Darcy:

$$
\begin{aligned}
-\,\mathbf{div}(\cancelto{0}{\mu}\,\mathbf{grad}\,\boldsymbol{u}) + \nu\boldsymbol{u} + \mathbf{grad}\,p &= \cancelto{\mathbf{0}}{\boldsymbol{f}} && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} &= g && \text{in } \Omega,\\
\boldsymbol{u}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega,\\
\int_\Omega p &= 0.
\end{aligned}
$$

Characteristics influencing the discretisation:

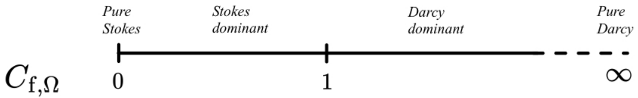
- Mixed formulation.
- L^2-estimate on $\boldsymbol{u}$.
- Requires inf–sup condition for p.

Friction coefficient: measure balance between regimes

Dimensionless number: with L a characteristic length,

$$C_{\mathrm{f},\Omega} = \frac{\nu L^2}{\mu}.$$

Regimes:



Outline

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Mesh and data

Mesh: polytopal mesh of Ω.

- $\mathcal{T}_h$: set of elements T.
- $\mathcal{F}_h$: set of faces F ($\mathcal{F}_T$: faces of T).
- $\mathcal{P}^\ell(X)$: polynomials of degree $\leq \ell$ on $X = T, F$.

Data:

- μ, ν piecewise constant on $\mathcal{T}_h$ (values μ_T, ν_T on $T \in \mathcal{T}_h$).

Discrete spaces I

Fix $k \geq 0$ polynomial degree.

Velocity space: Hybrid High-Order (HHO) space.

$$\underline{\boldsymbol{U}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h} \big) : \\ \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^k(T)^d \text{ for all } T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \boldsymbol{\mathcal{P}}^k(F)^d \text{ for all } F \in \mathcal{F}_h \Big\}.$$

With boundary conditions:

$$\underline{\boldsymbol{U}}_{h,0}^k := \{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \,:\, \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \subset \partial\Omega \}.$$

Velocity interpolator: for $\boldsymbol{v} \in H^1(\Omega)^d$, with $\boldsymbol{\pi}_X^\ell$ the $L^2(X)$-projector on $\mathcal{P}^\ell(X)^d$,

$$\underline{\boldsymbol{I}}_h^k \boldsymbol{v} := \big((\boldsymbol{\pi}_T^k \boldsymbol{v})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_F^k \boldsymbol{v})_{F\in\mathcal{F}_h} \big) \in \underline{\boldsymbol{U}}_h^k.$$

(Replace subscript h with T $\rightsquigarrow$ space and interpolator on element T.)

Discrete spaces II

Pressure:

$$P_h^k := \left\{ q \in L^2(\Omega) \, : \, q_{|T} \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h \, , \quad \int_\Omega q = 0 \right\} .$$

Discretisation of diffusive term $-\operatorname{div}(\mu \, \mathbf{grad}\, \boldsymbol{u})$ [4]

Discrete gradient reconstruction: $\boldsymbol{G}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathcal{P}^k(T)^{d\times d}$ s.t.

$$\int_T \boldsymbol{G}_T^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = \int_T \mathbf{grad}\, \boldsymbol{v}_T : \boldsymbol{\tau} + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F - \boldsymbol{v}_T)\cdot \boldsymbol{\tau} \mathbf{n}_{TF} \qquad \forall \boldsymbol{\tau} \in \mathcal{P}^k(T)^{d\times d}.$$

Bilinear form:

$$a_{\mu,h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \sum_{T\in\mathcal{T}_h} \mu_T a_{\mathrm{S},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T),$$

$$a_{\mathrm{S},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \int_T \boldsymbol{G}_T^k \underline{\boldsymbol{w}}_T : \boldsymbol{G}_T^k \underline{\boldsymbol{v}}_T + \mathrm{Stabs}_{\mathrm{S},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T).$$

Local weighted H^1-like norm:

$$\|\underline{\boldsymbol{v}}_T\|_{\mu,T} \coloneqq \mu_T^{1/2} a_{\mathrm{S},T}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{1/2}.$$

$$\left(\sum_{F\in\mathcal{F}_T} \mu_T h_T^{-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F)}^2\right)^{1/2} \lesssim \|\underline{\boldsymbol{v}}_T\|_{\mu,T}.$$

Discretisation of Darcy term $\nu\boldsymbol{u}$

Bilinear form:

$$a_{\nu,h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h} \nu_T a_{\mathrm{D},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)$$

$$a_{\mathrm{D},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) := \int_T \boldsymbol{w}_T \cdot \boldsymbol{v}_T + \mathrm{Stab}_{\mathrm{D},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T).$$

Local weighted L^2-like norm:

$$\|\underline{\boldsymbol{v}}_T\|_{\nu,T} := \nu_T^{1/2} a_{\mathrm{D},T}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{1/2}$$

$$\left(\sum_{F\in\mathcal{F}_T} \nu_T h_T \|\boldsymbol{v}_F - \boldsymbol{v}_T\|^2_{L^2(F)}\right)^{1/2} \lesssim \|\underline{\boldsymbol{v}}_T\|_{\nu,T}.$$

Discretisation of pressure term ∇p and source term $\boldsymbol{f}$

Discrete divergence: $D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathcal{P}^k(T)$ s.t.

$$D_T^k \underline{\boldsymbol{v}}_T = \mathrm{tr}(\boldsymbol{G}_T^k \underline{\boldsymbol{v}}_T).$$

Coupling bilinear form: For $(\underline{\boldsymbol{v}}_h, q_h) \in \underline{\boldsymbol{U}}_h^k \times \mathcal{P}^k(\mathcal{T}_h)$,

$$b_h(\underline{\boldsymbol{v}}_h, q_h) \coloneqq -\sum_{T\in\mathcal{T}_h} \int_T D_T^k \underline{\boldsymbol{v}}_T \; q_T,$$

Source term:

$$\sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{f} \cdot \boldsymbol{v}_T.$$

Outline

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Error estimate, using a local friction coefficient

$$C_{\mathrm{f},T} := \frac{\nu_T h_T^2}{\mu_T}$$

- $C_{\mathrm{f},T} \ll 1$: T Stokes-dominated.
- $C_{\mathrm{f},T} \gg 1$: T Darcy-dominated.
- Intermediate regimes also measured by $C_{\mathrm{f},T}$ (e.g. $C_{\mathrm{f},T} \approx 1$).

Theorem (Error estimates for the HHO scheme)

$$\begin{aligned}
\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{\mu,h}^2 + \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{\nu,h}^2 + \|p_h - \pi_h^k p\|_{L^2(\Omega)}^2 \\
\lesssim \Bigg[\sum_{T\in\mathcal{T}_h} \mu_T \min(1, C_{\mathrm{f},T}^{-1}) h_T^{2(k+1)} |\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(T)}^2 \\
+ \sum_{T\in\mathcal{T}_h} \nu_T \min(1, C_{\mathrm{f},T}) h_T^{2(k+1)} |\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(T)}^2 \\
+ \textit{Errors from coupling term} \Bigg].
\end{aligned}$$

Consistency of diffusive term: $C_{\mathrm{f},T}$ saves the day (and h)

Consistency error:

$$\sum_{T\in\mathcal{T}_h}\int_T(-\mathbf{div}(\mu_T\,\mathbf{grad}\,\boldsymbol{u}))\cdot\boldsymbol{v}_T-\underbrace{\mu_T\int_T\boldsymbol{G}_T^k\underline{\boldsymbol{I}}_T^k\boldsymbol{u}:\boldsymbol{G}_T^k\underline{\boldsymbol{v}}_T+\mu_T\mathrm{Stab}}_{a_{\mu,\nu}(\underline{\boldsymbol{I}}_h^k\boldsymbol{u},\underline{\boldsymbol{v}}_h)}$$

After IBP, element contribution (without stabilisation):

$$\sum_{F\in\mathcal{F}_T}\int_F\mu_T(\mathbf{grad}\,\boldsymbol{u}-\boldsymbol{G}_T^k\underline{\boldsymbol{I}}_T^k\boldsymbol{u})\mathbf{n}_F\cdot(\boldsymbol{v}_F-\boldsymbol{v}_T)$$

$$\leq\mu_T^{1/2}\underbrace{h_T^{1/2}\|\,\mathbf{grad}\,\boldsymbol{u}-\boldsymbol{G}_T^k\underline{\boldsymbol{I}}_T^k\boldsymbol{u}\|_{L^2(\partial T)}}_{\lesssim h_T^{k+1}|\boldsymbol{u}|_{H^{k+2}(T)}}\underbrace{\left(\mu_T h_T^{-1}\sum_{F\in\mathcal{F}_T}\|\boldsymbol{v}_F-\boldsymbol{v}_T\|^2_{\boldsymbol{L}^2(F)}\right)^{1/2}}_{\mathfrak{T}_T}.$$

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Stokes regime: $\mathfrak{T}_T\lesssim\|\underline{\boldsymbol{v}}_T\|_{\mu,T}$.

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Stokes regime: $\mathfrak{T}_T \lesssim \|\underline{\boldsymbol{v}}_T\|_{\mu,T}$.

Darcy regime:

$C_{\mathrm{f},T}\geq 1$ so $\mu_T\leq \nu_T h_T^2$.

Thus:

$$\mathfrak{T}_T\leq\left(\nu_T h_T\sum_{F\in\mathcal{F}_T}\|\boldsymbol{v}_F-\boldsymbol{v}_T\|^2_{\boldsymbol{L}^2(F)}\right)^{1/2}\lesssim\|\underline{\boldsymbol{v}}_T\|_{\nu,T}.$$

Consistency of coupling term

Consistency error:

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\, p\cdot \boldsymbol{v}_T + \underbrace{\sum_{T\in\mathcal{T}_h}\int_T D_T^k \underline{\boldsymbol{v}}_T\ \pi_T^k p}_{-b_h(\underline{\boldsymbol{v}}_h,\pi_h^k p)}$$

After IBP, element contribution:

$$\sum_{F\in\mathcal{F}_T}\int_F (\pi_T^k p - p)\ (\boldsymbol{v}_F - \boldsymbol{v}_T)\cdot \mathbf{n}_{TF}$$

$$\leq \underbrace{h_T^{1/2}\|\pi_T^k p - p\|_{L^2(\partial T)}}_{\lesssim h_T^{k+1}|p|_{H^{k+1}(T)}} \underbrace{\left(\sum_{F\in\mathcal{F}_T} h_T^{-1}\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F)}^2\right)^{1/2}}_{\mathfrak{T}_T}$$

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Stokes regime: $\mathfrak{T}_T \lesssim \mu_T^{-1/2}\|\underline{\boldsymbol{v}}_T\|_{\mu,T}$.

Darcy regime: $\mathfrak{T}_T \lesssim \nu_T^{-1/2} h_T^{-1}\|\underline{\boldsymbol{v}}_T\|_{\mu,T}$.

No $C_{\mathrm{f},T}$ can help us gain an h_T here...

Outline

Slides

Stronger IBP by choosing appropriate potential

New source term:

$$\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{f}\cdot\boldsymbol{P}_{\mathrm{D},T}^k\underline{\boldsymbol{v}}_T.$$

Consistency error:

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\,p\cdot\boldsymbol{P}_{\mathrm{D},T}^k\boldsymbol{v}_T+\sum_{T\in\mathcal{T}_h}\int_T D_T^k\underline{\boldsymbol{v}}_T\ \pi_T^k p.$$

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$$\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{f}\cdot \boldsymbol{P}_{\mathrm{D},T}^k\underline{\boldsymbol{v}}_T.$$

Consistency error:

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\,p\cdot \boldsymbol{P}_{\mathrm{D},T}^k\boldsymbol{v}_T+\sum_{T\in\mathcal{T}_h}\int_T D_T^k\underline{\boldsymbol{v}}_T\ \pi_T^k p.$$

Discrete divergence-based potential: $\boldsymbol{P}_{\mathrm{D},T}^k:\underline{\boldsymbol{U}}_T^k\to\mathcal{P}^k(T)^d$ such that

$$\int_T \mathbf{grad}\,q\cdot \boldsymbol{P}_{\mathrm{D},T}^k\underline{\boldsymbol{v}}_T=-\int_T qD_T^k\underline{\boldsymbol{v}}_T+\sum_{F\in\mathcal{F}_T}\int_F q\,(\boldsymbol{v}_F\cdot\mathbf{n}_{TF})\quad\forall q\in\mathcal{P}^{k+1}(T).$$

(Completed by fixing $\boldsymbol{P}_{\mathrm{D},T}^k\underline{\boldsymbol{v}}_T$ on a complement of $\mathbf{grad}\,\mathcal{P}^{k+1}(T)$ in $\mathcal{P}^k(T)^d$.)

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(Completed by fixing $\boldsymbol{P}_{\mathrm{D},T}^k\underline{\boldsymbol{v}}_T$ on a complement of $\mathbf{grad}\,\mathcal{P}^{k+1}(T)$ in $\mathcal{P}^k(T)^d$.)

D_T^k and $\boldsymbol{P}_{\mathrm{D},T}^k$ are the Discrete De Rham (DDR) divergence and potential. $\underline{\boldsymbol{U}}_T^k$ is an enrichment of the corresponding DDR space (that only has normal fluxes as face unknowns) [3, 6].

Consistency in Darcy regime

Consistency error and definition of $\boldsymbol{P}^k_{\mathrm{D},T}$:

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\, p \cdot \boldsymbol{P}^k_{\mathrm{D},T}\boldsymbol{v}_T + \sum_{T\in\mathcal{T}_h}\int_T D^k_T \underline{\boldsymbol{v}}_T\ \pi^k_T p - \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F p\ (\boldsymbol{v}_F\cdot\mathbf{n}_{TF}).$$

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\, \pi^{k+1}_T p \cdot \boldsymbol{P}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T + \sum_{T\in\mathcal{T}_h}\int_T \pi^{k+1}_T p D^k_T \underline{\boldsymbol{v}}_T - \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \pi^{k+1}_T p\ (\boldsymbol{v}_F\cdot\mathbf{n}_{TF}) = 0$$

Consistency in Darcy regime

Consistency error and definition of $\boldsymbol{P}^k_{\mathrm{D},T}$:

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$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{grad}\,\pi^{k+1}_T p\cdot \boldsymbol{P}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T + \sum_{T\in\mathcal{T}_h}\int_T \pi^{k+1}_T p D^k_T\underline{\boldsymbol{v}}_T - \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \pi^{k+1}_T p\ (\boldsymbol{v}_F\cdot\mathbf{n}_{TF}) = 0$$

Subtract:

$$\int_T \underbrace{\mathbf{grad}(p-\pi^{k+1}_T p)}_{\mathcal{O}(h^{k+1})}\cdot \underbrace{\boldsymbol{P}^k_{\mathrm{D},T}\boldsymbol{v}_T}_{\mathcal{O}(\nu_T^{-1/2}\|\underline{\boldsymbol{v}}_T\|_{\nu,T})} + \int_T \cancel{(\pi^k_T p - \pi^{k+1}_T p)} D^k_T\underline{\boldsymbol{v}}_T - \sum_{F\in\mathcal{F}_T}\int_F \underbrace{(p-\pi^{k+1}_T p)}_{\mathcal{O}(h^{k+3/2})} \underbrace{(\boldsymbol{v}_F\cdot\mathbf{n}_{TF})}_{\mathcal{O}(\nu_T^{-1/2}h_T^{-1/2}\|\underline{\boldsymbol{v}}_T\|_{\nu,T})}$$

Final HHO–DDR scheme

Regime-dependent potential: with $\langle P \rangle$ the truth value of P (1 if P is true, 0 otherwise),

$$\widetilde{\boldsymbol{P}}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T \coloneq \langle C_{\mathrm{f},T} < 1 \rangle \boldsymbol{v}_T + \langle C_{\mathrm{f},T} \geq 1 \rangle \boldsymbol{P}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T.$$

Scheme: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}^k_{h,0} \times P^k_h$ such that

$$\begin{aligned}
a_{\mu,h}(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + a_{\nu,h}(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h)& \\
= \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{f} \cdot \widetilde{\boldsymbol{P}}^k_{\mathrm{D},T}\underline{\boldsymbol{v}}_T & \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}^k_{h,0}, \\
-b_h(\underline{\boldsymbol{u}}_h, q_h) = \int_\Omega g q_h & \qquad \forall q_h \in P^k_h.
\end{aligned}$$

Regime-dependent error estimate

Theorem (Error estimates for the HHO–DDR scheme)

With C depending only on the mesh regularity and $\max(\mu,\nu)$, it holds

$$
\begin{aligned}
\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{\mu,h}^2 + \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{\nu,h}^2 + \|p_h - \pi_h^k p\|_{L^2(\Omega)}^2 \\
\le C \Bigg[\sum_{T\in\mathcal{T}_h} \mu_T \min(1, C_{\mathrm{f},T}^{-1}) h_T^{2(k+1)} |\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(T)}^2 \\
+ \sum_{T\in\mathcal{T}_h} \nu_T \min(1, C_{\mathrm{f},T}) h_T^{2(k+1)} |\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(T)}^2 \Bigg] \\
+ C \Bigg[\sum_{T\in\mathcal{T}_h} \mu_T^{-1} \langle C_{\mathrm{f},T} < 1 \rangle h_T^{2(k+1)} |p|_{H^{k+1}(T)}^2 \\
+ \sum_{T\in\mathcal{T}_h} \nu_T^{-1} \langle C_{\mathrm{f},T} \ge 1 \rangle h_T^{2(k+1)} |p|_{H^{k+2}(T)}^2 \Bigg].
\end{aligned}
$$

- Works in pure Stokes ($\mu = 0$) and pure Darcy ($\nu = 0$) regimes, respectively removing the terms involving μ_T^{-1} or ν_T^{-1}.

Outline

Slides

Global transition from Stokes to Darcy I

Domain and meshes: $\Omega = (0,1)^3$, tetrahedral and Voronoi meshes.

Exact solution: set $C_{\mathrm{f},\Omega} = \nu/\mu$ and

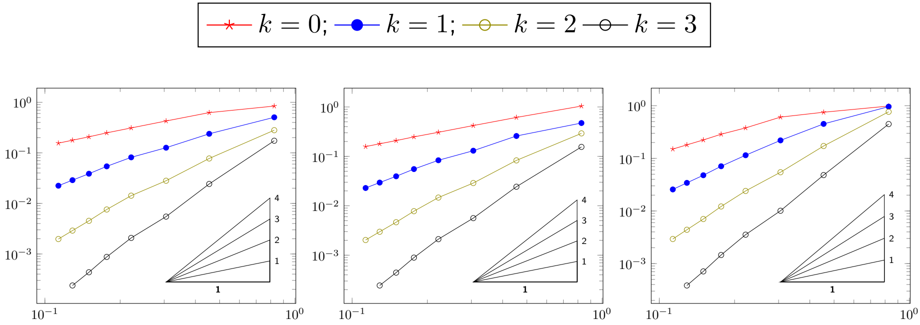
$$\begin{aligned} p(x,y,z) &= \sin(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad \forall (x,y,z) \in \Omega, \\ \boldsymbol{u} &= \exp(-C_{\mathrm{f},\Omega})\boldsymbol{u}_{\mathrm{S}} + (1 - \exp(-C_{\mathrm{f},\Omega}))\boldsymbol{u}_{\mathrm{D}}, \end{aligned}$$

with

$$\boldsymbol{u}_{\mathrm{S}}(x,y,z) = \frac{1}{2}\begin{bmatrix} \sin(2\pi x)\cos(2\pi y)\cos(2\pi z) \\ \cos(2\pi x)\sin(2\pi y)\cos(2\pi z) \\ -2\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) \end{bmatrix} \quad \forall (x,y,z) \in \Omega,$$

$$\boldsymbol{u}_{\mathrm{D}} = \begin{cases} -\nu^{-1}\,\mathbf{grad}\,p & \text{if } \nu > 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

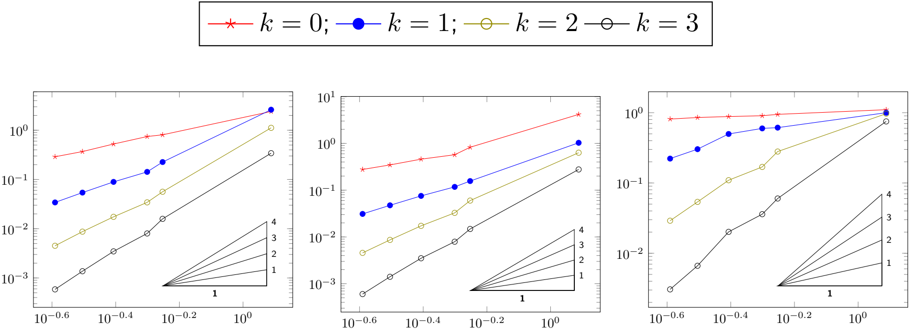
Global transition from Stokes to Darcy II



(a) $\mu = \nu = 1$ (Brinkman) (b) $\mu = 1,\ \nu = 0$ (Stokes) (c) $\mu = 0,\ \nu = 1$ (Darcy)

Figure: Voronoi meshes, relative errors in energy norm on (u, p) w.r.t. h

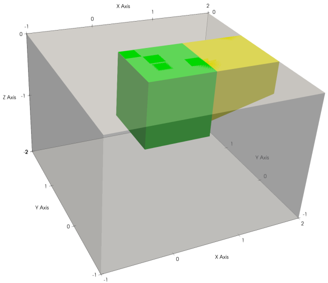
Global transition from Stokes to Darcy III

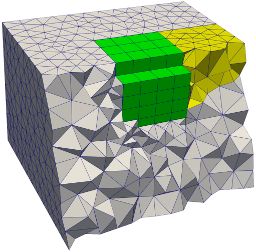


(a) $\mu = \nu = 1$ (Brinkman) (b) $\mu = 1,\ \nu = 0$ (Stokes) (c) $\mu = 0,\ \nu = 1$ (Darcy)

Figure: Tetrahedral meshes, relative errors in (u, p) vs. h

Lid-driven cavity with porous surroundings I





- Green cavity: pure Stokes flow, $\mu = 10^{-2}$.
- Surroundings: pure Darcy flow with $\nu^{-1} = 10^{-7}$ in grey box, $\nu^{-1} = 10^{-2}$ in yellow wedge.
- Forcing term: $\boldsymbol{f} = (0, 0, -0.98)$ (gravity).
- Boundary conditions: $\boldsymbol{u} = (x(1-x), 0, 0)$ on top of cavity, $\mathbf{0}$ elsewhere.

Lid-driven cavity with porous surroundings II

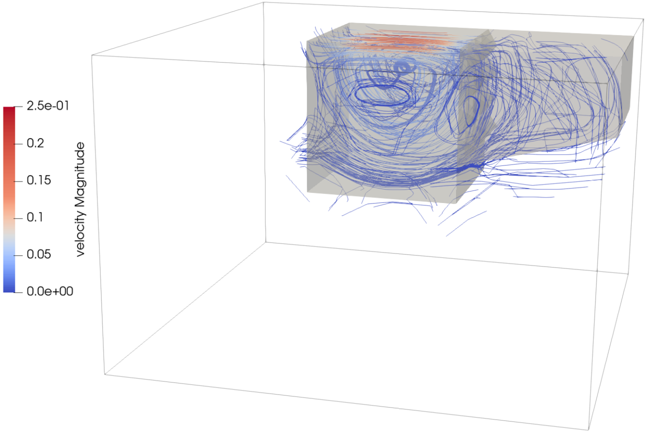
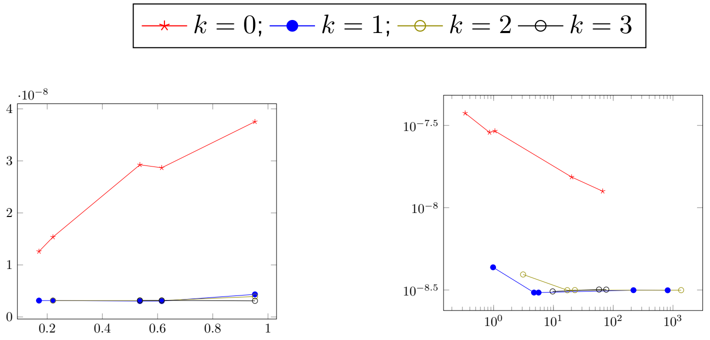


Figure: Streamlines (cavity and wedge displayed in shadow).

Lid-driven cavity with porous surroundings III



(a) w.r.t. mesh size

(b) w.r.t. wall time (seconds)

Figure: Convergence of flux values from the cavity to the wedge.

Increasing order is better than refining mesh.

Conclusions

- Polytopal scheme of arbitrary order from the Brinkman model.
- Regime (Stokes / Brinkman / Darcy) identified by local dimensionless friction coefficients.
- Robust error estimate across the whole range of regimes, including intermediate ones; clearly identifies contributions of each regime.
- Clear computational gain in going above lowest order scheme.

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NEMESIS
New generation
methods for numerical
simulations

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Thank you for your attention!

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More tests: various regimes in various parts of the domain I

Domain: $\Omega = (0,1)^3$ split in
- $\Omega_{\mathrm{S}} = (0, 1/2) \times (0,1)^2$ with $(\mu, \nu) = (1, 10^7)$,
- $\Omega_{\mathrm{D}} = (1/2, 1) \times (0,1)^2$ with $(\mu, \nu) = (0, 10^2)$.

Mesh: Cartesian from 2^3 to 32^3 cubes.

Exact solution: $\boldsymbol{u} = \boldsymbol{u}_0 + \chi_{\mathrm{S}} \boldsymbol{u}_{\mathrm{S}} + \chi_{\mathrm{D}} \boldsymbol{u}_{\mathrm{D}}$ with χ_i characteristic functions of the subdomains and

$$\boldsymbol{u}_0(x,y,z) = \begin{bmatrix} \exp(-y-z) \\ \sin(\pi y)\sin(\pi z) \\ yz \end{bmatrix},$$

$$\boldsymbol{u}_{\mathrm{S}}(x,y,z) = \cos(\pi x)(x-0.5) \begin{bmatrix} y+z \\ y+\cos(\pi z) \\ \sin(\pi y) \end{bmatrix},$$

$$\boldsymbol{u}_{\mathrm{D}}(x,y,z) = \cos(\pi x)(x-0.5) \begin{bmatrix} \sin(\pi y)\sin(\pi z) \\ z^3 \\ y^2 z^2 \end{bmatrix}.$$

More tests: various regimes in various parts of the domain II

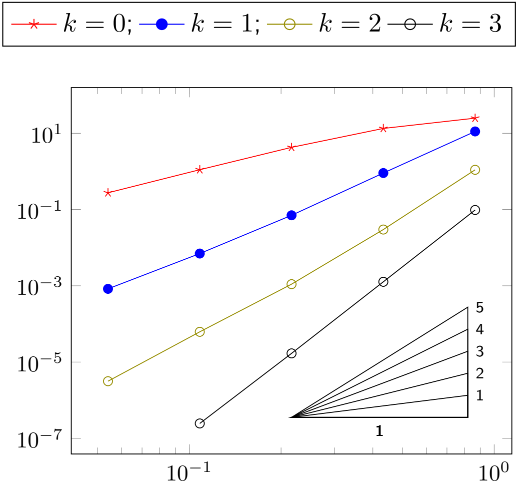

Figure: Relative errors vs. h