A discrete trace theory for polytopal methods, with application to BDDC preconditionners

Jérôme Droniou

joint work with S. Badia, J. Manyer and J. Tushar (Monash University).

IMAG, CNRS, France,
School of Mathematics, Monash University, Australia
https://imag.umontpellier.fr/~droniou/

International conference on
Advanced Numerical Schemes for Challenging PDEs
CityU Hong Kong, 9-11 June 2025

NEMESIS
New generation methods for numerical simulations

Funded by the European Union

European Research Council

References for this presentation

- *A discrete trace theory for non-conforming hybrid discretisation methods*.
  S. Badia, J. Droniou, and J. Tushar. 34p, 2024.
  `http://arxiv.org/abs/2409.15863`

- *Analysis of BDDC preconditioners for non-conforming polytopal hybrid discretisation methods*.
  S. Badia, J. Droniou, J. Manyer and J. Tushar. 17p, 2025.

- Details on BDDC implementation:
  `https://jordimanyer.deno.dev/slides/2025_matrix_bddchho.pdf`

Outline

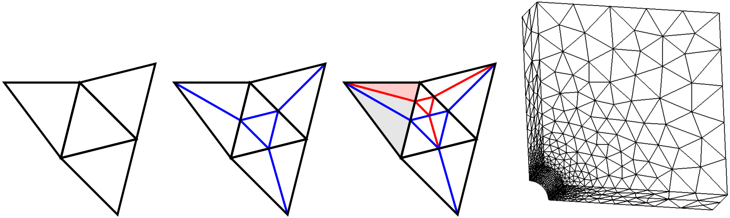
Slides

Finite Elements

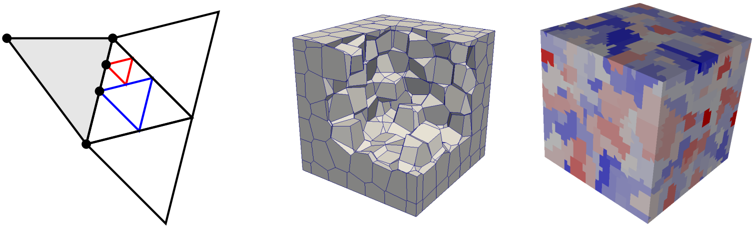
- Local polynomial spaces on each element, glued to create global conforming basis functions (e.g., in H^1).

  Gluing requires special meshes: tetrahedra, hexahedra.

- Shortcomings:
  - local refinement trades mesh size for mesh quality.
  - complex geometries require large number of elements.
  - global conforming basis functions require large number of DOFs.



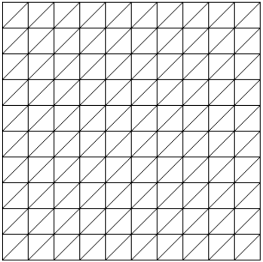
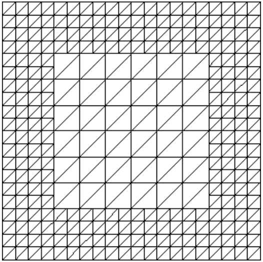
Polytopal methods I



- No need for global conforming piecewise polynomial basis functions. Starts from DOFs and build local polynomial operators (differentials, potentials).

- Immediate benefits:
  - ☐ Seamless local refinement, preserving mesh regularity.
  - ☐ Agglomerated elements are easy to handle (e.g., for multi-grid methods).
  - ☐ High-level approach can lead to leaner methods (fewer DOFs).

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathbb{P}^2$-($\mathbb{P}^1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

Outline

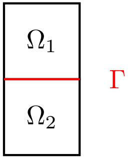
Slides

Outline

Slides

Substructuring non-overlapping DD methods I

- *Transfer of information* between domains.



(i) Take v_1 in Ω_1.

(ii) Consider the trace $(v_1)_{|\Gamma}$ of v_1 on Γ.

(iii) Define v_2 in Ω_2 as the harmonic extension of $(v_1)_{|\Gamma}$.

- $v_1 \to v_2$ is continuous for the H^1 norms, as composition of continuous maps:
  - Trace: $H^1(\Omega_1) \to H^{1/2}(\Gamma)$,
  - Lifting: $H^{1/2}(\Gamma) \to H^1(\Omega_2)$.

Substructuring non-overlapping DD methods II

Conforming methods: discrete spaces $V_h(\Omega_i) \subset H^1(\Omega_i)$ and $V_h(\Gamma) = (V_h(\Omega_i))|_\Gamma \subset H^{1/2}(\Gamma)$.

$\rightsquigarrow$ continuous trace and lifting prove the continuity of $v_1 \to v_2$ *(modulo truncation estimate)*.

Non-conforming methods: discrete spaces $U_h(\Omega_i)$ and $U_h(\Gamma) = (U_h(\Omega_i))|_\Gamma$ not contained in $H^1(\Omega_i)$ and $H^{1/2}(\Gamma)$.

$\rightsquigarrow$ need to create a suitable norm on $U_h(\Gamma)$ that is:

- ☐ *not too strong*, to get a continuous discrete trace $U_h(\Omega_1) \to U_h(\Gamma)$,
- ☐ *strong enough*, to get a continuous discrete lifting $U_h(\Gamma) \to U_h(\Omega_2)$.

Outline

1 Why polytopal methods?

2 Discrete trace theory
- What is it, and why is that useful?
- Continuous trace and lifting
- Discrete trace and lifting

3 BDDC preconditionner

4 Numerical illustration

Slides

Continuous setting I

Ω bounded Lipschitz domain of $\mathbb{R}^d$.

$H^1(\Omega)$-seminorm: for $v \in H^1(\Omega)$,

$$|v|_{1,\Omega} := \|\nabla v\|_{L^2(\Omega)}.$$

$H^{1/2}(\partial\Omega)$-seminorm: for $w \in H^{1/2}(\partial\Omega)$,

$$|w|_{1/2,\partial\Omega} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^d} \, dxdy \right)^{1/2}.$$

Continuous setting II

Trace operator: $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\gamma(v) = v_{|\partial\Omega}$ when v is smooth.

Theorem (Trace inequality)

$$|\gamma(v)|_{1/2,\partial\Omega} \lesssim |v|_{1,\Omega} \qquad \forall v \in H^1(\Omega).$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that:

$$\gamma(\mathcal{L}(w)) = w \quad \textit{and} \quad |\mathcal{L}(w)|_{1,\Omega} \lesssim |w|_{1/2,\partial\Omega} \qquad \forall w \in H^{1/2}(\partial\Omega).$$

Outline

Slides

Polytopal mesh

Ω polytopal. Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ with $\mathcal{T}_h$ set of elements, $\mathcal{F}_h$ set of faces.

- standard mesh regularity assumption (elements/faces do not become too elongated), and
- quasi-uniformity: with $h_X = \mathrm{diam}(X)$,

$$\exists \rho > 0 \,:\, \rho h_{t'} \leq h_t \quad \forall t, t' \in \mathcal{T}_h.$$

Set $h := \max_{t \in \mathcal{T}_h} h_t$ and write $a \lesssim b$ for "$a \leq Cb$ with C depending only on the mesh regularity parameters".

Discrete $H^1(\Omega)$ space and seminorm

Hybrid space: unknowns are polynomials in the elements and on the faces. Fix $k \geq 0$ and set

$$\underline{U}_h := \{\underline{v}_h = ((v_t)_{t\in\mathcal{T}_h}, (v_f)_{f\in\mathcal{F}_h}) \,:\, v_t \in \mathbb{P}^k(t)\,, \quad v_f \in \mathbb{P}^k(f)\}.$$

Discrete $H^1(\Omega)$-seminorm: with $\underline{v}_t = (v_t, (v_f)_{f\in\mathcal{F}_t})$ restriction of $\underline{v}_h$ to t,

$$|\underline{v}_h|_{1,h}^2 := \sum_{t\in\mathcal{T}_h} |\underline{v}_t|_{1,t}^2$$

$$\text{with} \quad |\underline{v}_t|_{1,t}^2 := \|\nabla v_t\|_{L^2(t)}^2 + \sum_{f\in\mathcal{F}_t} h_t^{-1}\|v_f - v_t\|_{L^2(f)}^2.$$

Discrete $H^{1/2}(\partial\Omega)$ space and seminorm

Boundary space: restriction to boundary of hybrid space (piecewise polynomial functions).

$$U_h^{\mathrm{bd}} := \{w_h = ((w_f)_{f\in\mathcal{F}_h^{\mathrm{bd}}}) \,:\, w_f \in \mathbb{P}^k(f)\} \subset L^2(\partial\Omega).$$

Trace (restriction): $\gamma_h : \underline{U}_h \to U_h^{\mathrm{bd}}$ such that

$$\gamma_h(\underline{v}_h) = (v_f)_{f\in\mathcal{F}_h^{\mathrm{bd}}} \qquad \forall \underline{v}_h \in \underline{U}_h.$$

Discrete $H^{1/2}(\partial\Omega)$ space and seminorm

Boundary space: restriction to boundary of hybrid space (piecewise polynomial functions).

$$U_h^{\mathrm{bd}} := \{w_h = ((w_f)_{f\in\mathcal{F}_h^{\mathrm{bd}}})\,:\,w_f\in\mathbb{P}^k(f)\}\subset L^2(\partial\Omega).$$

Discrete $H^{1/2}(\partial\Omega)$-seminorm: with $\overline{w}_f = \frac{1}{|f|_{d-1}}\int_f w$,

$$|w_h|_{1/2,h}^2 := \underbrace{\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} h_f^{-1}\|w_f-\overline{w}_f\|_{L^2(f)}^2}_{\text{local variation in each } f} + \underbrace{\sum_{(f,f')\in\mathcal{FF}_h^{\mathrm{bd}}} |f|_{d-1}|f'|_{d-1}\frac{|\overline{w}_f-\overline{w}_{f'}|^2}{|x_f-x_{f'}|^d}}_{\text{medium–long range interactions}}$$

($\mathcal{FF}_h^{\mathrm{bd}}$ = pairs of all faces on $\partial\Omega$).

Main results

Theorem (Trace inequality)

$$|\gamma_h(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h. \tag{1}$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ *such that:*

$$\gamma(\mathcal{L}_h(w_h)) = w_h \quad \textit{and} \quad |\mathcal{L}_h(w_h)|_{1,h} \lesssim |w_h|_{1/2,h} \qquad \forall w_h \in U_h^{\mathrm{bd}}. \tag{2}$$

Main results

Theorem (Trace inequality)

$$|\gamma_h(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h. \tag{1}$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ such that:

$$\gamma(\mathcal{L}_h(w_h)) = w_h \quad \textit{and} \quad |\mathcal{L}_h(w_h)|_{1,h} \lesssim |w_h|_{1/2,h} \qquad \forall w_h \in U_h^{\mathrm{bd}}. \tag{2}$$

- Hidden constant independent of $\mathrm{diam}(\Omega)$.
- Directly gives trace/lifting for (Hybridizable) Discontinuous Galerkin [Cockburn et al., 2009], Hybrid High-Order [Di Pietro and Ern, 2015, Di Pietro and Droniou, 2020], non-conforming Virtual Elements [de Dios et al., 2016], etc.

Why is that new?

Previous discrete trace results: using only $L^2(\partial\Omega)$-norms on trace space [Eymard et al., 2000, Droniou et al., 2018].

- Allows for a trace inequality

$$\|\gamma(\underline{v}_h)\|_{L^2(\partial\Omega)/\mathbb{R}} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h$$

  (where $(v_h)_{|t} = v_t$ for all $t \in \mathcal{T}_h$).

- Does not allow for a (uniformly continuous) lifting.

Approach

Previous approaches (for BDDC analysis)

- Interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014, Brenner et al., 2017].

  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

Approach

Previous approaches (for BDDC analysis)

- Interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014, Brenner et al., 2017].

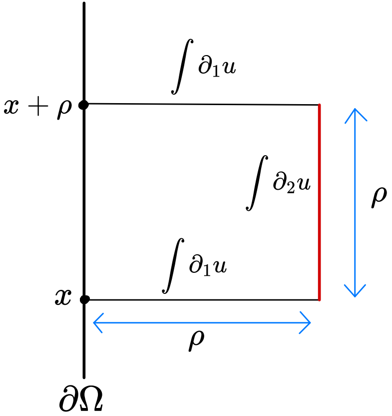
  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

Our approach:

Follow principles of Discrete Functional Analysis [Droniou et al., 2018]: do not use continuous trace/lifting results but mimic their proofs in the discrete setting.

An example of (challenging) “continuous $\to$ discrete” I

Continuous estimate for the trace

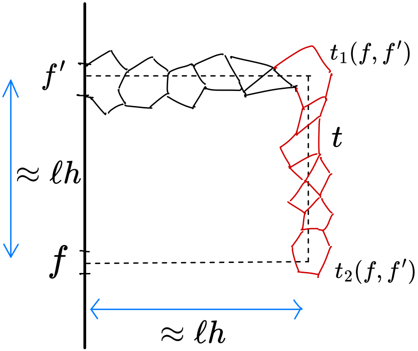


$$u(0,x+\rho) - u(0,x) = \int \partial_1 u(s,x) + \int \partial_2 u(\rho,s) + \int \partial_1 u(s,x+\rho).$$

Take L^2-norm over x: vertical section contributes with $\rho\|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}$.

An example of (challenging) “continuous $\to$ discrete” II

Discrete estimate for the trace

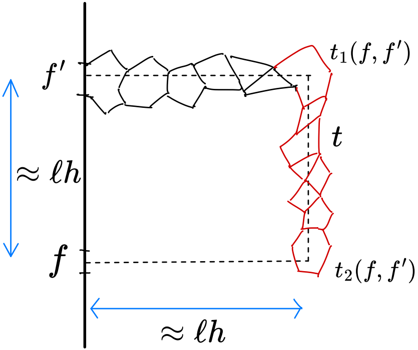


- “$f' = f + \rho$” does not make sense

  $\rightsquigarrow$ distances between faces up to $O(h)$.

An example of (challenging) “continuous $\to$ discrete” III

Discrete estimate for the trace

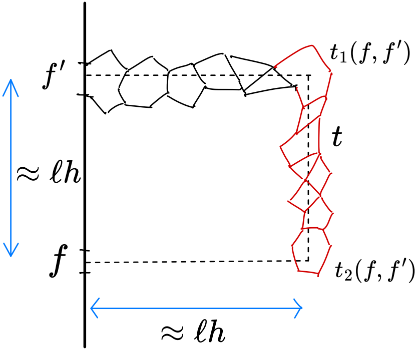


- $|v_{t_1(f,f')} - v_{t_2(f,f')}| \lesssim h^{\frac{2-d}{d}} \sum_{t} |\underline{v}_t|_{1,t}$

  $\rightsquigarrow$ Cauchy–Schwarz requires to count the t (split into ℓ horizontal layers...)

An example of (challenging) "continuous $\to$ discrete" IV

Discrete estimate for the trace



- $|v_{t_1(f,f')} - v_{t_2(f,f')}|^2 \lesssim h^{2-d}\ell \sum_{t} |\underline{v}_t|^2_{1,t}$.

$\rightsquigarrow$ sum over (f,f') within distance $\approx \ell h$: how many times each cell t appears? *(response: $\approx \ell^{d-2}$...)*

Outline

Slides

Discrete problem

- Given $\mathcal{A}_h : \underline{\widehat{U}}_h \to \underline{\widehat{U}}_h'$ coercive operator and $\ell_h \in \underline{\widehat{U}}_h'$, the discrete problem is:

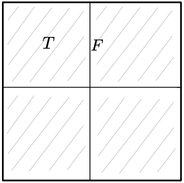
$$\text{Find } \underline{u}_h \in \underline{\widehat{U}}_h \text{ such that}$$
$$\langle \mathcal{A}_h \underline{u}_h, \underline{v}_h \rangle = \ell_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{\widehat{U}}_h.$$

- Algebraically, this is written:

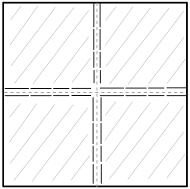
$$\mathsf{A}_h U_h = b_h.$$

Preconditionner I

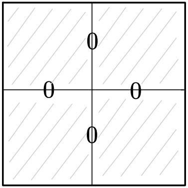
Coarse mesh (size H, elements T, faces F) and fine mesh (size h, elements t, faces f).



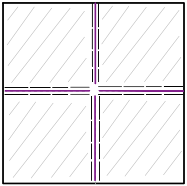
$\widehat{\underline{U}}_h$: one value on $f \subset F$



$\underline{U}_h$: two values on $f \subset F$



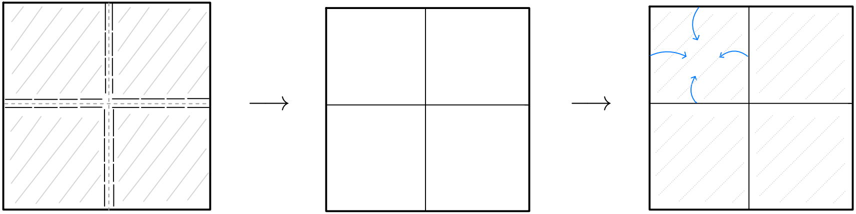
$\underline{U}_{h,0}$: zero on each $f \subset F$



$\widetilde{\underline{U}}_h$: two values on $f \subset F$, continuous average over F.

Preconditionner II

Weighting and harmonic extension: $\mathcal{Q}_h : \underline{U}_h \to \widehat{\underline{U}}_h$ defined by



fully discontinuous

averaging values on each $f \subset F$

discrete harmonic lifting

Preconditionner III

BDDC preconditionner: with $\widetilde{\mathcal{A}}_h$ and $\mathcal{A}_{h,0}$ the restrictions of $\mathcal{A}_h$ to $\underline{\widetilde{U}}_h$ and $\underline{U}_{h,0}$,

$$\mathcal{B}_h = \mathcal{Q}_h \widetilde{\mathcal{A}}_h^{-1} \mathcal{Q}_h^\top + \mathcal{A}_{h,0}^{-1}.$$

Theorem (Preconditionner estimates)

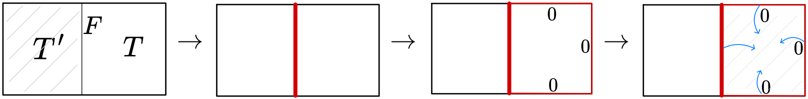
The preconditionned operator $\mathcal{B}_h \mathcal{A}_h$ satisfies the condition number estimate

$$\kappa(\mathcal{B}_h \mathcal{A}_h) \lesssim \left(1 + \ln \frac{H}{h}\right)^2.$$

Note: $\kappa(\mathcal{A}_h) \simeq h^{-2}$.

Truncation estimate

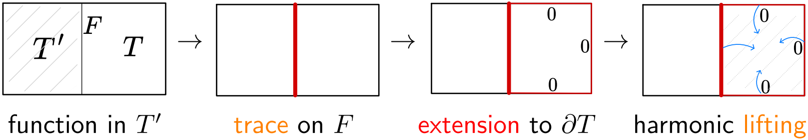
- The proof requires exchange of information $T' \to F \to T$ (where T, T' neighbouring cells of F).



function in T' trace on F extension to ∂T harmonic lifting

Truncation estimate

- The proof requires exchange of information $T' \to F \to T$ (where T, T' neighbouring cells of F).



- Norms estimates consequence of trace inequality, lifting property and:

Theorem (Boundary extension estimate)

Let $w_h \in U_h^{\mathrm{bd}}(F)$ be such that $\int_F w_h = 0$. Define $z_h \in U_h^{\mathrm{bd}}(\partial T)$ as the extension of w_h by 0 on $\partial T \backslash F$. Then,

$$|z_h|_{1/2,h,\partial T} \lesssim \left(1 + \ln \frac{H}{h}\right) |w_h|_{1/2,h,F}.$$

Proof: by mimicking at the discrete level the proof in the continuous setting, strongly inspired by [Brenner et al., 2017, Brenner and Scott, 2008].

Outline

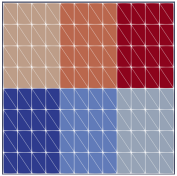
Slides

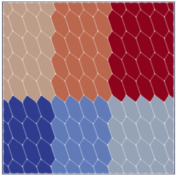
Setting

- $\Omega = (0,1)^2$, Laplace problem with exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)x(1-x)y(1-y).$$

- Mesh construction:
  - ☐ Select number of processors $N_p = N_p^x \times N_p^y$;
  - ☐ Split the domain in $\frac{H}{h}N_p^x \times \frac{H}{h}N_p^y$ subdomains;
  - ☐ Loosely mesh each one using simplicial and polygonal meshes.





- Two scales $\frac{H}{h} = 8$ and $\frac{H}{h} = 16$.

- Three methods: Hybridized Discontinuous Galerkin (HDG), Hybrid High-Order (HHO) and mixed-order HHO.

Number of BDDC-GMRES iterations I

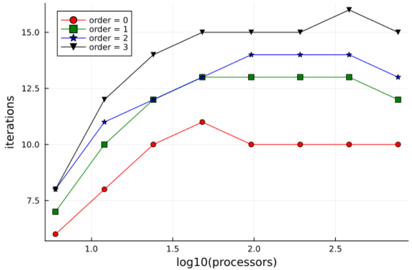


Figure: HDG on simplicial meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Number of BDDC-GMRES iterations II

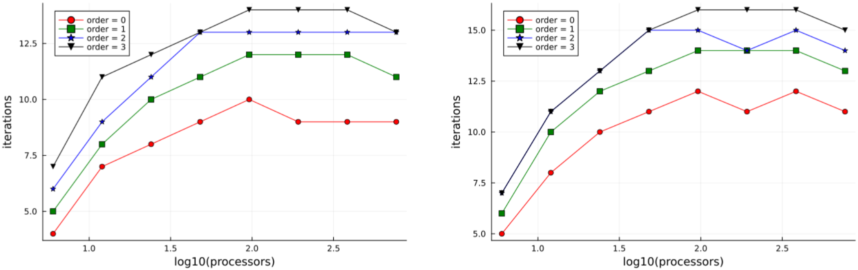


Figure: HHO on simplicial meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Number of BDDC-GMRES iterations III

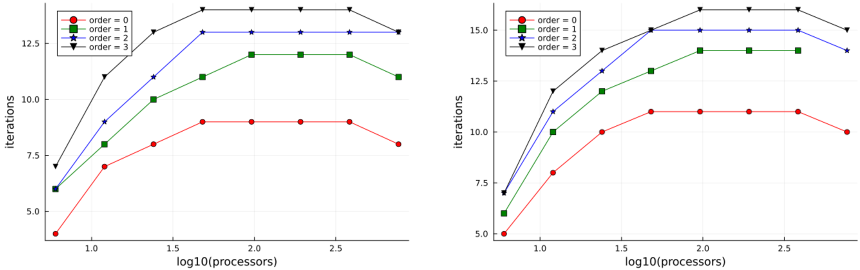


Figure: Mixed-order HHO on simplicial meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Number of BDDC-GMRES iterations IV

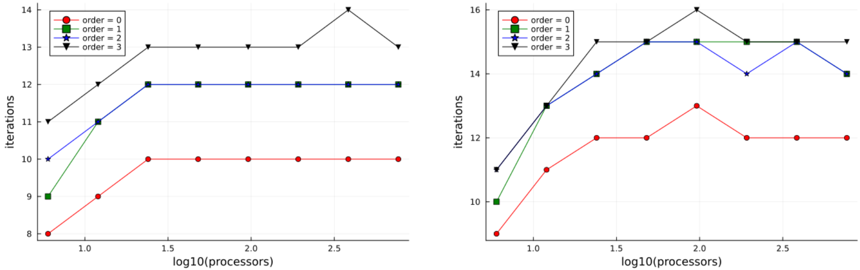


Figure: HDG on polytopal meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Number of BDDC-GMRES iterations V

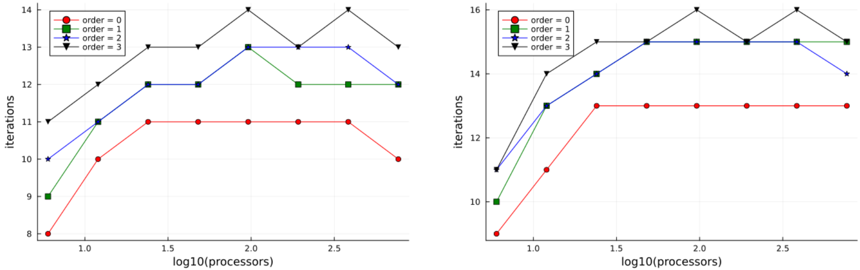


Figure: HHO on polytopal meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Number of BDDC-GMRES iterations VI

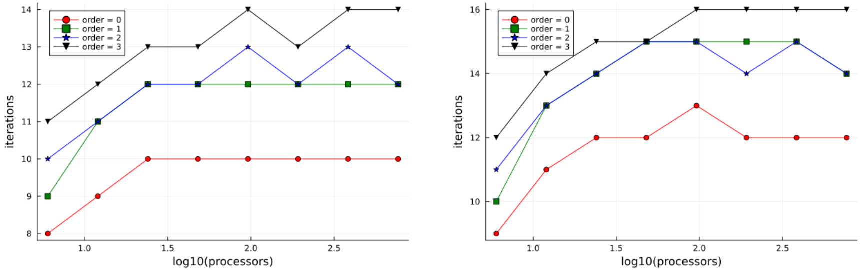


Figure: Mixed-order HHO on polytopal meshes, for $H/h = 8$ (left) and $H/h = 16$ (right).

Conclusions

- Complete discrete trace theory, with definition of boundary norm, trace inequality and lifting in discrete spaces of polytopal hybrid methods.
- Applicable to a range of schemes: HHO, VEM, HDG, etc. (and even FEM).
- Constructive proofs, obtained by mimicking proofs in the continuous setting (more flexible than looking for lifting in conforming spaces).
- For the moment, requires quasi-uniform meshes, but with elements of generic shapes.
- Allows for a complete analysis of BDDC (and possibly other DDM) for a range of polytopal methods.

Funded by
the European Union

European Research Council
Established by the European Commission

NEMESIS
New generation
methods for numerical
simulations

Funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Thank you for your attention!

References I

Brenner, S. C., Park, E.-H., and Sung, L.-Y. (2017).
A BDDC preconditioner for a symmetric interior penalty Galerkin method.
Electron. Trans. Numer. Anal., 46:190–214.

Brenner, S. C. and Scott, L. R. (2008).
The mathematical theory of finite element methods, volume 15 of *Texts in Applied Mathematics*.
Springer, New York, third edition.

Cockburn, B., Dubois, O., Gopalakrishnan, J., and Tan, S. (2014).
Multigrid for an HDG method.
IMA J. Numer. Anal., 34(4):1386–1425.

Cockburn, B., Gopalakrishnan, J., and Lazarov, R. (2009).
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365.

Cowsar, L. C., Mandel, J., and Wheeler, M. F. (1995).
Balancing domain decomposition for mixed finite elements.
Math. Comp., 64(211):989–1015.

de Dios, B. A., Lipnikov, K., and Manzini, G. (2016).
The nonconforming virtual element method.
ESAIM: Mathematical Modelling and Numerical Analysis, 50(3):879–904.

References II

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications, volume 19 of *Modeling, Simulation and Applications*.
Springer International Publishing, https://hal.archives-ouvertes.fr/hal-02151813.

Di Pietro, D. A. and Ern, A. (2015).
A hybrid high-order locking-free method for linear elasticity on general meshes.
Computer Methods in Applied Mechanics and Engineering, 283:1–21.

Diosady, L. T. and Darmofal, D. L. (2012).
A unified analysis of balancing domain decomposition by constraints for discontinuous Galerkin discretizations.
SIAM J. Numer. Anal., 50(3):1695–1712.

Droniou, J., Eymard, R., Gallouët, T., Guichard, C., and Herbin, R. (2018).
The gradient discretisation method, volume 82 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*.
Springer, Cham.

Eymard, R., Gallouët, T., and Herbin, R. (2000).
Finite volume methods.
In Ciarlet, P. G. and Lions, J.-L., editors, *Techniques of Scientific Computing, Part III*, Handbook of Numerical Analysis, VII, pages 713–1020. North-Holland, Amsterdam.

Outline

Slides

Outline

5 Elements of proof

- Trace inequality
  - Continuous
  - Discrete
- Lifting
  - Continuous
  - Discrete

Slides

Outline

5 Elements of proof

- Trace inequality
  - Continuous
  - Discrete
- Lifting
  - Continuous
  - Discrete

Estimate I

Starting point: set $y = x + \rho$ and write

$$
\begin{aligned}
&u(0, x+\rho) - u(0, x) \\
&\quad = u(0, x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0, x) \\
&\quad = \int_\rho^0 \partial_1 u(s, x+\rho) ds + \int_0^\rho \partial_2 u(\rho, x+s) ds + \int_0^\rho \partial_1 u(s, x) ds.
\end{aligned}
$$

Estimate II

Starting point: set $y = x + \rho$ and write

$$
\begin{aligned}
&u(0, x+\rho) - u(0,x) \\
&\quad = u(0, x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0, x) \\
&\quad = \int_\rho^0 \partial_1 u(s, x+\rho) ds + \int_0^\rho \partial_2 u(\rho, x+s) ds + \int_0^\rho \partial_1 u(s, x) ds.
\end{aligned}
$$

Take L^2-norms w.r.t. x (swap integrals):

$$
\begin{aligned}
&\|u(0, \cdot + \rho) - u(0, \cdot)\|_{L^2(\mathbb{R})} \\
&\qquad \le \int_0^\rho \|\partial_1 u(s, \cdot + \rho)\|_{L^2(\mathbb{R})} \, ds + \int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})} \, ds \\
&\qquad\qquad + \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds \\
&\qquad \le \rho \Big(\underbrace{\frac{2}{\rho} \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds}_{=: F_1(\rho)} + \underbrace{\|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}}_{=: F_2(\rho)} \Big).
\end{aligned}
$$

Estimate III

Change of variable $(y = x + \rho)$ in the $H^{1/2}$ semi-norm:

$$\begin{aligned} |u(0,\cdot)|^2_{1/2,\mathbb{R}} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(0,x) - u(0,y)|^2}{|x-y|^2} \, dx dy \\ &= \int_{\mathbb{R}} \frac{\|u(0,\cdot) - u(0,\cdot + \rho)\|^2_{L^2(\mathbb{R})}}{\rho^2} d\rho \\ &\leq C(\|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})}) \end{aligned}$$

where

$$F_1(\rho) = \frac{2}{\rho} \int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})} \, ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Estimate IV

$$|u(0,\cdot)|_{1/2,\mathbb{R}}^2 \leq C(\|F_1\|_{L^2(\mathbb{R})}^2 + \|F_2\|_{L^2(\mathbb{R})}^2)$$

$$F_1(\rho) = \frac{2}{\rho}\int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})}\,ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Conclusion:

$$\|F_2\|_{L^2(\mathbb{R})}^2 = \|\partial_2 u\|_{L^2(\mathbb{R}^2)}^2 \leq |u|_{H^1(\mathbb{R}^2)}^2.$$

By Hardy inequality:

$$\|F_1\|_{L^2(\mathbb{R})}^2 \leq C\|\partial_1 u\|_{L^2(\mathbb{R}^2)}^2 \leq C|u|_{H^1(\mathbb{R}^2)}^2.$$

Outline

Driving principles

- Points become faces.
  - *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

Driving principles

- Points become faces.
  ■ *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

$$\int_0^\rho \partial_1 u(s,x)dx \rightsquigarrow \overline{v}_{t_N} - \overline{v}_{f_{N-1}} + \overline{v}_{f_{N-1}} - \overline{v}_{t_{N-1}} + \cdots + \overline{v}_{t_1} - \overline{v}_f$$
$$\lesssim \sum_{t \in \mathrm{Li}(f,t_N)} h^{\frac{2-d}{2}} |\underline{v}_t|_{1,t}.$$

Driving principles

- Points become faces.
  ■ *Continuous* $H^{1/2}$ *seminorm integrates over* x, y*, discrete* $H^{1/2}$*-seminorm sums over pairs of faces.*

- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

- Need a distance between faces/cells: has to be up to h.
  ■ *Cannot consider all* $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ *such that* $|x_f - x_{f'}| = \rho$ *for a given* ρ*... Instead,* $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ *are "at distance* ℓh *of each other" if* $\ell h \leq |x_f - x_{f'}| < (\ell + 1)h$*.*
  ■ *Makes "change of variable"* $x + \rho \to x$ *less straightforward.*

Driving principles

- Points become faces.
  ■ *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

- Need a distance between faces/cells: has to be up to h.
  ■ *Cannot consider all $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ such that $|x_f - x_{f'}| = \rho$ for a given ρ... Instead, $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ are "at distance ℓh of each other" if $\ell h \leq |x_f - x_{f'}| < (\ell + 1)h$.*
  ■ *Makes "change of variable" $x + \rho \to x$ less straightforward.*

- Need to be able to swap integrals.
  ■ *Integrate vertically to $\partial\Omega$ then parallel to $\partial\Omega \rightsquigarrow$ layers along $\partial\Omega$.*

Driving principles

- Points become faces.
  ■ *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

- Need a distance between faces/cells: has to be up to h.
  ■ *Cannot consider all $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ such that $|x_f - x_{f'}| = \rho$ for a given ρ... Instead, $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ are "at distance ℓh of each other" if $\ell h \leq |x_f - x_{f'}| < (\ell + 1)h$.*
  ■ *Makes "change of variable" $x + \rho \to x$ less straightforward.*

- Need to be able to swap integrals.
  ■ *Integrate vertically to $\partial\Omega$ then parallel to $\partial\Omega$ $\rightsquigarrow$ layers along $\partial\Omega$.*

- Need a discrete Hardy inequality: $r_m \geq 0$ and $R_l := \frac{1}{l}\sum_{m=0}^{l} r_m$, then

$$\sum_{l=1}^{L} R_l^2 \leq 32 \sum_{l=0}^{L} r_l^2.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho, \cdot{+}s)\, ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot{+}s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Take (f, f') "within distance ℓh" and consider

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{t\in \mathrm{Li}(ff';|x_f - x_{f'}|)} |\underline{v}_t|_{1,t}.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot{+}s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot{+}s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Take (f,f') "within distance ℓh" and consider

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{t\in\mathrm{Li}(ff';|x_f-x_{f'}|)} |\underline{v}_t|_{1,t}.$$

- Split into layers:

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{r=1}^{\ell} \sum_{\substack{t\in\mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot+s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot+s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Split into layers:

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{r=1}^{\ell} \sum_{\substack{t\in \mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}.$$

- Estimate cardinality $\#\{t\in \mathrm{Li}(ff';|x_f-x_{f'}|),\, |p(x_t)-x_f|\simeq rh\} \lesssim 1$, so

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t\in \mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}^2.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot+s)\,ds\|_{L^2(\mathbb{R})} \leq \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot+s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Estimate cardinality $\#\{t \in \mathrm{Li}(ff';|x_f - x_{f'}|),\, |p(x_t)-x_f| \simeq rh\} \lesssim 1$, so

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t\in\mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

- Multiply by $|f|\,|f'|/|x_f - x_{f'}|^2 \lesssim h^{d-2}/\ell^d$, sum over (f,f'):

$$\sum_{(f,f'),\,|x_f-x_{f'}|\simeq\ell h} |f|\,|f'|\frac{|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2}{|x_f-x_{f'}|^2}$$

$$\lesssim \frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell} \sum_{(f,f'),\,|x_f-x_{f'}|\simeq\ell h} \sum_{\substack{t\in\mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot{+}s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot{+}s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Multiply by $|f|\,|f'|/|x_f - x_{f'}|^2 \lesssim h^{d-2}/\ell^d$, sum over (f,f'):

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{(f,f'),\,|x_f-x_{f'}|\simeq \ell h}\ \sum_{\substack{t\in \mathrm{Li}(ff';|x_f-x_{f'}|)\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

- Swap sums over faces and cells:

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|^2_{1,t}\ \#\mathfrak{F}(t,r)$$

where $\mathfrak{F}(t,r) := \{(f,f')\ :\ |x_f - x_{f'}| \simeq \ell h,\ t \in \mathrm{Li}(ff';|x_f-x_{f'}|),\ |p(x_t)-x_f| \simeq rh\}$.

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot{+}s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot{+}s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Swap sums over faces and cells:

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \operatorname{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|_{1,t}^2\,\#\mathfrak{F}(t,r)$$

  where $\mathfrak{F}(t,r) := \{(f,f') \,:\, |x_f - x_{f'}| \simeq \ell h,\, t \in \mathrm{Li}(ff'; |x_f - x_{f'}|),\, |p(x_t) - x_f| \simeq rh\}$.
- Estimate cardinality: $\#\mathfrak{F}(t,r) \lesssim \ell^{d-2}$, so

$$\frac{1}{\ell}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \operatorname{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|_{1,t}^2 \lesssim \sum_{\{t\,:(\ell-2)h\le \operatorname{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|_{1,t}^2$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot + s)\, ds\|_{L^2(\mathbb{R})} \leq \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Estimate cardinality: $\#\mathfrak{F}(t,r) \lesssim \ell^{d-2}$, so

$$\sum_{\{t\,:(\ell-2)h \leq \operatorname{dist}(p(x_t),\partial\Omega) \leq \ell h\}} |\underline{v}_t|_{1,t}^2$$

- Conclude by summing over ℓ (each layer appears 3 times):

$$3\sum_t |\underline{v}_t|_{1,t}^2 = 3|\underline{v}_h|_{1,h}^2$$

Outline

5 Elements of proof

- Trace inequality
  - Continuous
  - Discrete
- Lifting
  - Continuous
  - Discrete

Slides

Outline

Construction of the lifting

Take $w \in H^{1/2}(\mathbb{R}^{d-1})$ and define $v \in H^1([0,1) \times \mathbb{R}^{d-1})$ by by averaging w over the base of a cone, which becomes more and more narrow as we get close to the boundary.

With $\rho_x(\boldsymbol{y}) = x^{-(d-1)}\rho(x^{-1}\boldsymbol{y})$ usual smoothing kernel,

$$v(x, \boldsymbol{y}) = (\rho_x \star w)(\boldsymbol{y}).$$

Some arguments in the estimates on partial derivatives

- $\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y}) d\boldsymbol{y} = 0$ to write (for $i \geq 2$)

$$\partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d} \int_{\mathbb{R}} (w(\boldsymbol{z}) - w(\boldsymbol{y})) \partial_i \rho(x^{-1}(\boldsymbol{y} - \boldsymbol{z})) d\boldsymbol{z}.$$

- $\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{z}))| d\boldsymbol{z} \leq C/x$ to write, using Cauchy–Schwarz:

$$\left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})| |\partial_i(\rho_x(\boldsymbol{y} - \boldsymbol{z})) d\boldsymbol{z}| \right)^2 \leq \frac{C}{x} \int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2 |\partial_i(\rho_x(\boldsymbol{y} - \boldsymbol{z}))| d\boldsymbol{z}.$$

Outline

Construction of the lifting

Also average on the base of a cone...

- For $t \in \mathcal{T}_h$, set $\delta_t = \mathrm{dist}(x_t, \partial\Omega)$ and

$$\mathcal{A}_t = \{f \in \mathcal{F}_h^{\mathrm{bd}} \, : \, \mathrm{dist}(p(x_t), f) \leq \delta_t)\}.$$

Construction of the lifting

Also average on the base of a cone...

- For $t \in \mathcal{T}_h$, set $\delta_t = \mathrm{dist}(x_t, \partial\Omega)$ and

$$\mathcal{A}_t = \{f \in \mathcal{F}_h^{\mathrm{bd}} \,:\, \mathrm{dist}(p(x_t), f) \leq \delta_t)\}.$$

- Give to each $f \in \mathcal{A}_t$ an identical weight:

$$\rho_t(f) = \begin{cases} \frac{1}{\#\mathcal{A}_t} & \text{if } f \in \mathcal{A}_t, \\ 0 & \text{otherwise.} \end{cases}$$

Construction of the lifting

Also average on the base of a cone...

- For $t \in \mathcal{T}_h$, set $\delta_t = \mathrm{dist}(x_t, \partial\Omega)$ and

$$\mathcal{A}_t = \{f \in \mathcal{F}_h^{\mathrm{bd}} \,:\, \mathrm{dist}(p(x_t), f) \leq \delta_t)\}.$$

- Give to each $f \in \mathcal{A}_t$ an identical weight:

$$\rho_t(f) = \begin{cases} \frac{1}{\#\mathcal{A}_t} & \text{if } f \in \mathcal{A}_t, \\ 0 & \text{otherwise.} \end{cases}$$

- Lift $w_h = (w_f)_{f \in \mathcal{F}_h^{\mathrm{bd}}} \in U_h^{\mathrm{bd}}$ into $v_h = ((v_t)_{t \in \mathcal{T}_h}, (v_f)_{f \in \mathcal{F}_h})$ such that

$$v_t = \frac{1}{\#\mathcal{A}_t} \sum_{f \in \mathcal{A}_t} \overline{w}_f = \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} \overline{w}_f \rho_t(f)$$

$$v_f = \begin{cases} \frac{v_t + v_{t'}}{2} & \text{if } f \text{ internal face between } t, t' \in \mathcal{T}_h, \\ w_f & \text{if } f \in \mathcal{F}_h^{\mathrm{bd}}. \end{cases}$$

Estimate of $|\underline{v}_h|_{1,h}$ I

Low order reconstruction: all v_t are constant, so

$$|\underline{v}_h|_{1,h}^2 \simeq \sum_{(t,t')\ \text{neighbours}} h^{d-2}|v_t - v_{t'}|^2.$$

Estimate of $|\underline{v}_h|_{1,h}$ II

Adaptation of arguments

☐ Continuous:

$$\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y})d\boldsymbol{y} = 0$$

$$\rightsquigarrow \quad \partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d}\int_{\mathbb{R}}(w(\boldsymbol{z}) - w(\boldsymbol{y}))\partial_i\rho(x^{-1}(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z} \quad \forall \boldsymbol{y}$$

☐ Discrete:

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}(\rho_t(f) - \rho_{t'}(f))\Big(= \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}\rho_t(f) - \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}\rho_{t'}(f) = 1 - 1\Big) = 0$$

$$\rightsquigarrow \quad v_t - v_{t'} = \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}(\overline{w}_f - \overline{w}_{f'})D_g\rho(f) \quad \forall f' \in \mathcal{F}_h^{\mathrm{bd}},$$

where $D_g\rho(f) = \rho_t(f) - \rho_{t'}(f)$ with (t,t') cells on each side of $g \in \mathcal{F}_h^{\mathrm{in}}$.

Estimate of $|\underline{v}_h|_{1,h}$ III

- Continuous:

$$\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{y})|d\boldsymbol{y} \leq \frac{C}{x}$$

$$\rightsquigarrow \quad \left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})||\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z}|\right)^2 \leq \frac{C}{x}\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2|\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))|d\boldsymbol{z}.$$

- Discrete:

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |D_g\rho(f)| \lesssim \frac{h}{\delta_t}$$

$$\rightsquigarrow |\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}} \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2 |D_g\rho(f)| \frac{h^{d-1}}{\delta_t}.$$

with $f' \in \mathcal{F}_h^{\mathrm{bd}}$ such that g "projects close to f'".

Estimate of $|\underline{v}_h|_{1,h}$ IV

$$\sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} |D_g \rho_t(f)| = \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} |\rho_t(f) - \rho_{t'}(f)| \lesssim \frac{h}{\delta_t}.$$

Requires:

- ☐ $\#\mathcal{A}_t \simeq \left(\frac{\delta_t}{h}\right)^{d-1}$
- ☐ $\#(\mathcal{A}_t \Delta \mathcal{A}_{t'}) \lesssim \left(\frac{\delta_t}{h}\right)^{d-2}$
- ☐ $\forall f \in \mathcal{A}_t \cap \mathcal{A}_{t'}\,,\ |\rho_t(f) - \rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d}$
- ☐ $|\rho_t(f) - \rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d-1}$.

Estimate of $|\underline{v}_h|_{1,h}$ V

$$|\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}} \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2 |D_g\rho(f)| \frac{h^{d-1}}{\delta_t}.$$

Write $\sum_{g\in\mathcal{F}_h^{\mathrm{in}}}$ as $\sum_{f'\in\mathcal{F}_h^{\mathrm{bd}}} \sum_{g \text{ above } f'}$ and conclude by proving

$$\sum_{g \text{ above } f'} |D_g\rho(f)| \frac{h^{d-1}}{\delta_t} \lesssim \frac{|f|\,|f'|}{|x_f - x_{f'}|^d}.$$