Conforming lifting, quasi-interpolators and Maxwell compactness for a polytopal de Rham complex

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joint works with: T. Chaumont–Frelet & S. Lemaire
(and D. Di Pietro & S. Pitassi)

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References for this presentation

- *Commuting quasi-interpolators and Maxwell compactness for a polytopal de Rham complex*,
  T. Chaumont-Frelet, J. Droniou and S. Lemaire (2025), 29p.
  https://hal.science/hal-05304175

- *Conforming lifting and adjoint consistency for the Discrete de Rham complex of differential forms*,
  D. A. Di Pietro, J. Droniou and S. Pitassi (2025), 28p.
  https://arxiv.org/abs/2509.21449

Outline

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Minimal-regularity de Rham complex

Ω open bounded set of $\mathbb{R}^3$.

- $H(\mathbf{grad},\Omega) = \{q \in L^2(\Omega) \,:\, \mathbf{grad}\, q \in L^2(\Omega)^3\}$.
- $\boldsymbol{H}(\mathbf{curl};\Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3 \,:\, \mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3\}$.
- $\boldsymbol{H}(\mathrm{div};\Omega) = \{\boldsymbol{w} \in L^2(\Omega)^3 \,:\, \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega)\}$.

$$\boxed{H(\mathbf{grad},\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)}$$

Rellich compactness

Theorem

If $(q_n)_n$ is bounded in $H(\mathbf{grad}, \Omega)$ then $(q_n)_n$ is relatively compact in $L^2(\Omega)$.

- *Relatively* easy because $\mathbf{grad}$ controls the variations in all directions.

Maxwell compactness

Theorem

If $(\boldsymbol{v}_n)_n$ *is bounded in* $\boldsymbol{H}(\mathbf{curl};\Omega)$ *and*

$$\int_\Omega \boldsymbol{v}_n \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega),$$

then $(\boldsymbol{v}_n)_n$ *is relatively compact in* $L^2(\Omega)^3$*.*

- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.

Maxwell compactness

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- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.
- Much more challenging than Rellich: $\mathbf{curl}$ does not control the variations of the function [Weber, 1980], [Jochmann, 1997].

Maxwell compactness

Theorem

If $(\boldsymbol{v}_n)_n$ is bounded in $\boldsymbol{H}(\mathbf{curl};\Omega)$ and

$$\int_\Omega \boldsymbol{v}_n \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad},\Omega),$$

then $(\boldsymbol{v}_n)_n$ is relatively compact in $L^2(\Omega)^3$.

- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.
- Much more challenging than Rellich: $\mathbf{curl}$ does not control the variations of the function [Weber, 1980], [Jochmann, 1997].
- Orthogonality condition equivalent to $\mathrm{div}\, \boldsymbol{v}_n = 0$ and $\boldsymbol{v}_n \cdot \mathbf{n}_\Omega = 0$.

Uses of Maxwell compactness

- Bound of $\mathbf{curl}$ and zero div classical in curl-div problems, such as models in electromagnetism (possibly using vector potential fixed by gauge).
- Compactness required for eigenvalue analysis and nonlinear models.
- Convergence analysis of schemes requires discrete versions of this compactness; see, e.g., [Kikuchi, 1987] for eigenvalue problems.
- Discrete compactness also allows for fine convergence analysis of schemes, possibly with models with rough coefficients [Chaumont-Frelet and Ern, 2023], [Chaumont-Frelet and Ern, 2024].

Outline

1. Maxwell compactness: why?
2. Polytopal meshes: why?
3. Overview of the Discrete De Rham method
   - Generic principles
   - Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space and curl/potential reconstructions
   - The DDR complex and its properties
4. Quasi-interpolator for DDR
5. Conforming lifting for DDR (with D. Pietro and S. Pitassi)
6. Maxwell compactness for DDR

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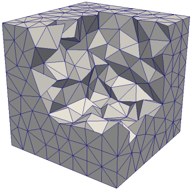
The Finite Element way
Global complex



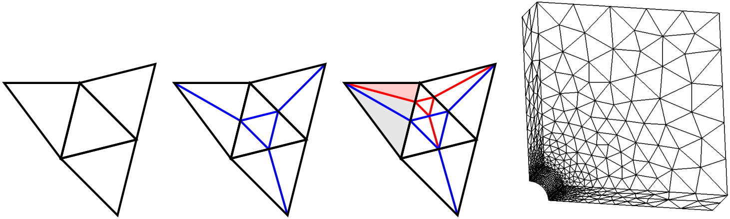
$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{tikzcd}
V_h^0 \arrow[r, "\mathbf{grad}"] \arrow[d, hook] & V_h^1 \arrow[r, "\mathbf{curl}"] \arrow[d, hook] & V_h^2 \arrow[r, "\mathrm{div}"] \arrow[d, hook] & V_h^3 \arrow[d, hook] \\
H(\mathbf{grad}, \Omega) \arrow[r, "\mathbf{grad}"] & \boldsymbol{H}(\mathbf{curl}; \Omega) \arrow[r, "\mathbf{curl}"] & \boldsymbol{H}(\mathrm{div}; \Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega)
\end{tikzcd}$$

Example: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

- Gluing only works on special meshes...



- Approach limited to conforming meshes with standard elements

$\implies$ local refinement requires to trade mesh size for mesh quality

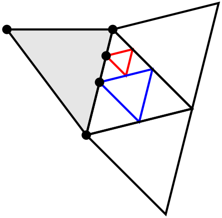
$\implies$ complex geometries may require a large number of elements

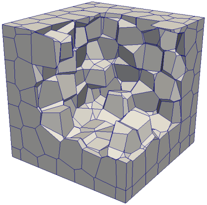
$\implies$ the element shape cannot be seamlessly adapted to the solution (e.g. hexahedra in boundary layers + tetrahedra in the bulk for CFD simulations)

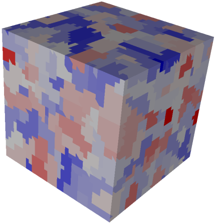
- Need for (global) basis functions

$\implies$ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes





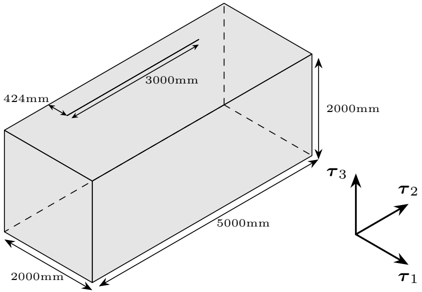


- Local refinement is easy, and preserves mesh regularity.
- Agglomeration of elements (e.g., for multigrid methods) is seamless.
- High-level approach can lead to leaner methods (fewer DOFs).
- Can be combined with standard Finite Elements on hybrid meshes (made of tetrahedra/hexahedra + polyhedral elements).

A practical example from CEA-CESTA

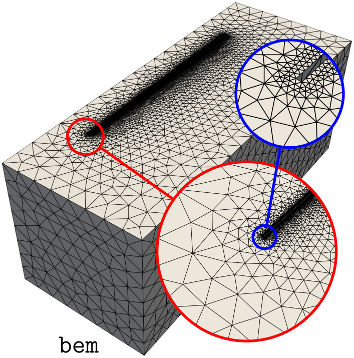
[Touzalin, 2025]

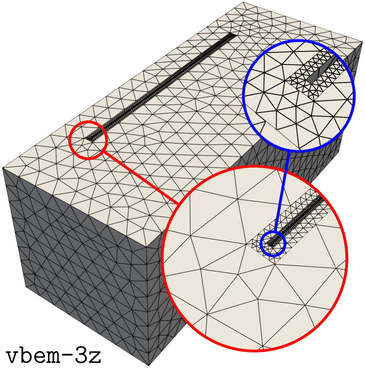
Problem: use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



A practical example from CEA-CESTA

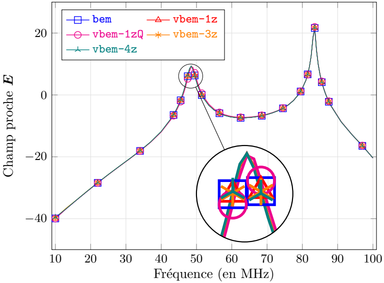
Meshes: conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).

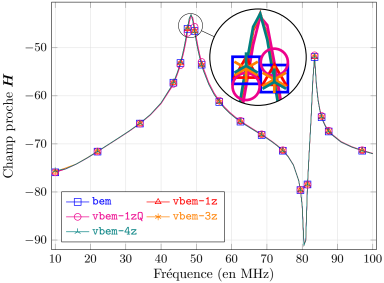




A practical example from CEA-CESTA

Accuracy: comparison of modulus of reflected near fields at the top.





Computational cost

Method	*Assembly*	*Resolution*
bem	813s	125s
vbem-3z	321s	19s

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Overview of the Discrete De Rham (DDR) complex

$$\mathbb{R} \longrightarrow \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Fully discrete complex (not sub-complex) of bespoke finite-dimensional spaces and operators.
- Discrete spaces not made of functions but:
  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Overview of the Discrete De Rham (DDR) complex

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & C^\infty(\overline{\Omega}) \arrow[r, "\mathbf{grad}"] \arrow[d, "\underline{I}^k_{\mathbf{grad},h}"] & C^\infty(\overline{\Omega})^3 \arrow[r, "\mathbf{curl}"] \arrow[d, "\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}"] & C^\infty(\overline{\Omega})^3 \arrow[r, "\mathrm{div}"] \arrow[d, "\underline{\boldsymbol{I}}^k_{\mathrm{div},h}"] & C^\infty(\overline{\Omega}) \arrow[r, "0"] \arrow[d, "I^k_{L^2,h}"] & \{0\} \\
\mathbb{R} \arrow[r] & \underline{X}^k_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \arrow[r, "D_h"] & \mathcal{P}^k(\mathcal{T}_h) \arrow[r, "0"] & \{0\}
\end{tikzcd}
$$

- Fully discrete complex (not sub-complex) of bespoke finite-dimensional spaces and operators.
- Discrete spaces not made of functions but:
  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.
  - Interpolators $\underline{\boldsymbol{I}}^k_{\bullet,h}$ give meaning to these polynomials/DOFs as moments.

Guiding principles for the construction

Joint work with D. Di Pietro and F. Rapetti.
(Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023].)

- Hierarchical construction: from vertices, to edges, to faces, to elements.

- Enhancement: on each (relevant) mesh entity,
  - discrete differential operator first,
  - potential reconstruction using the discrete differential operator.

  (both polynomially consistent, both based on IBP formulas.)

- The definition of the spaces (DOFs) also guided by these IBP formulas.

Same guiding principles as the Hybrid High-Order (HHO) method [Di Pietro et al., 2014], [Di Pietro and Droniou, 2020].

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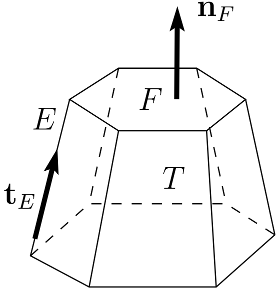
Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements (T), faces (F), edges (E), vertices (V), with intrinsic orientations (tangent, normal).
- $\mathcal{P}^\ell(X)$ polynomial of degree $\leq \ell$ on $X = T, F, E$.
- Raviart–Thomas space on $X = T, F$: for $\ell \geq 0$,

$$\boldsymbol{\mathcal{RT}}^\ell(X) = \boldsymbol{\mathcal{P}}^{\ell-1}(X) \oplus (\boldsymbol{x} - \boldsymbol{x}_X)\mathcal{P}^{\ell-1}(X).$$



Space and interpolator

Space: fix $k \geq 0$ and set

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big),$$
$$\boldsymbol{v}_T \in \boldsymbol{\mathcal{RT}}^k(T) \quad \forall T \in \mathcal{T}_h\,, \quad \boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^k(F) \quad \forall F \in \mathcal{F}_h\,,$$
$$v_E \in \mathcal{P}^k(E) \quad \forall E \in \mathcal{E}_h\Big\}.$$

Interpolators: for $\boldsymbol{v} : \overline{\Omega} \to \mathbb{R}^3$ such that the tangential traces $\boldsymbol{v}_{\mathrm{t},E}$, $\boldsymbol{v}_{\mathrm{t},F}$ of $\boldsymbol{v}$ on each E, F are single-valued and integrable,

$$\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} := \big((\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},T}\boldsymbol{v})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},F}\boldsymbol{v}_{\mathrm{t},F})_{F\in\mathcal{F}_h}, (\pi^k_{\mathcal{P},E}(\boldsymbol{v}_{\mathrm{t},E}))_{E\in\mathcal{E}_h}\big),$$

where $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},X}$ and $\pi^k_{\mathcal{P},X}$ are the $L^2(X)$-orthogonal projection on $\boldsymbol{\mathcal{RT}}^k(X)$ and $\mathcal{P}^k(X)$.

Operators and potential reconstructions

$$\underline{\boldsymbol{X}}_{\mathbf{curl},h}^{k} \coloneqq \Big\{\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big),$$
$$\boldsymbol{v}_T \in \boldsymbol{\mathcal{RT}}^k(T) \quad \forall T\in\mathcal{T}_h\,, \quad \boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^k(F) \quad \forall F\in\mathcal{F}_h\,,$$
$$v_E \in \mathcal{P}^k(E) \quad \forall E\in\mathcal{E}_h\Big\}.$$

Face curl: For $F\in\mathcal{F}_h$ and $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^{k}$, define $C_F^k\underline{\boldsymbol{v}}_h \in \mathcal{P}^k(F)$ by mimicking IBP:

$$\int_F (C_F^k\underline{\boldsymbol{v}}_h) r = \int_F \boldsymbol{v}_F\cdot\mathbf{rot}_F\, r - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E v_E\, r \qquad \forall r\in\mathcal{P}^k(F).$$

Operators and potential reconstructions

Face curl:

$$\int_F (C_F^k \underline{\boldsymbol{v}}_h) r = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F\, r - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E\, r \qquad \forall r \in \mathcal{P}^k(F).$$

Reconstructed face tangential trace: Define $\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{P}}^k(F)$ such that, for all $r \in \mathcal{P}^{k+1}(F)$ and $\boldsymbol{w} \in (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)$,

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_h \cdot (\mathbf{rot}_F\, r + \boldsymbol{w}) = \int_F C_F^k \underline{\boldsymbol{v}}_h\, r + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w}.$$

Operators and potential reconstructions

Face curl:

$$\int_F (C_F^k \underline{\boldsymbol{v}}_h) r = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F\, r - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, r \qquad \forall r \in \mathcal{P}^k(F).$$

Reconstructed face tangential trace:

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_h \cdot (\mathbf{rot}_F\, r + \boldsymbol{w}) = \int_F C_F^k \underline{\boldsymbol{v}}_h\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r + \int_F \boldsymbol{v}_F \cdot \boldsymbol{w}.$$

Element curl and potential: $\mathbf{C}_T^k \underline{\boldsymbol{v}}_h \in \boldsymbol{\mathcal{P}}^k(T)$ and $\boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{v}}_T \in \boldsymbol{\mathcal{P}}^k(T)$ also by mimicking IBP.

All have polynomial consistency, e.g.:

$$\mathbf{C}_T^k \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(T) \quad (\textit{Nédélec space}).$$

Discrete curl

$\underline{\boldsymbol{C}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \to \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$ by projecting face and element reconstructed curls onto the components in

$$\underline{\boldsymbol{X}}_{\mathrm{div},h}^k := \Big\{ \underline{\boldsymbol{z}}_T = ((\boldsymbol{z}_T)_{T\in\mathcal{T}_h}, (z_F)_{F\in\mathcal{F}_h}) \ : \\ \boldsymbol{z}_T \in \boldsymbol{\mathcal{N}}^k(T)\,,\ z_F \in \mathcal{P}^k(F) \Big\},$$

that is,

$$\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h = ((\boldsymbol{\pi}_{\boldsymbol{\mathcal{N}},T}^k \mathbf{C}_T^k \underline{\boldsymbol{v}}_h)_{T\in\mathcal{T}_h}, (C_F^k \underline{\boldsymbol{v}}_h)_{F\in\mathcal{F}_h}).$$

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4 Quasi-interpolator for DDR

5 Conforming lifting for DDR (with D. Pietro and S. Pitassi)

6 Maxwell compactness for DDR

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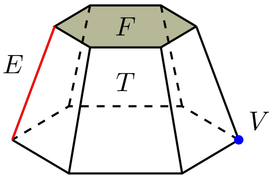
DDR complex: summary

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

- Do not seek any basis functions.
- Fully discrete spaces not made of functions, but of vectors of polynomials (DOFs).
- Polynomials attached to geometric entities (emulates continuity properties of each space).
- Polynomial reconstructions of differential operator and potential by mimicking IBPs.



L^2-like inner products

- Local L^2-like inner product on the DDR spaces:

  for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$ and $k_{\mathbf{grad}} = k+1$, $k_{\mathbf{curl}} = k_{\mathrm{div}} = k$,

$$(x_T, y_T)_{\bullet,T} = \int_T P^{k_\bullet}_{\bullet,T} x_T \cdot P^{k_\bullet}_{\bullet,T} y_T + \mathrm{s}_{\bullet,T}(x_T, y_T) \qquad \forall x_T, y_T \in \underline{X}^k_\bullet(T),$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face/edge potentials*).

- Global L^2-like product $(\cdot,\cdot)_{\bullet,h}$ by standard assembly of local ones.

DOF by mesh entities

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\boldsymbol{\mathcal{RT}}^k(F)$	$\boldsymbol{\mathcal{RT}}^k(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

The DDR complex and its properties

- Complex with the same cohomology as the continuous de Rham complex, applicable on generic polytopal meshes.
- Poincaré inequalities.
- Consistency (both primal and adjoint).
- Commutation properties between the interpolators and the continuous/discrete operators.

[Di Pietro et al., 2020], [Di Pietro et al., 2023], [Di Pietro and Hanot, 2024], [Di Pietro and Droniou, 2021a]

The DDR complex and its properties

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$\rightsquigarrow$ optimally-convergent schemes (error in $\mathcal{O}(h^{k+1})$) for a range of models: magnetostatics, Stokes & Navier–Stokes, etc.

$\rightsquigarrow$ robust error estimates with respect to some physical parameters.
[Di Pietro and Droniou, 2021b], [Beirão da Veiga et al., 2022]

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Where this happened

Where this happened

Construction of the quasi-interpolator

Quasi-interpolator on FE space: with $\mathcal{S}_h$ simplicial mesh and $\ell \geq 1$,

$$
\begin{array}{ccccccc}
H(\mathbf{grad}, \Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}; \Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}; \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathbf{grad},h} & & \Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathbf{curl},h} & & \Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathrm{div},h} & & \Big\downarrow \pi^{\ell}_{\mathcal{P},\mathcal{S}_h} \\
V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{\ell}(\mathcal{S}_h)
\end{array}
$$

where $V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h)$, $\boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h)$ and $\boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h)$ are the conforming Lagrange, Nédélec and Raviart–Thomas finite element spaces.

[Ern et al., 2022], [Chaumont-Frelet and Vohralík, 2024]

Construction of the quasi-interpolator

Quasi-interpolator on FE space: with $\mathcal{S}_h$ simplicial mesh and $\ell \geq 1$,

$$
\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\big\downarrow \widehat{\mathcal{J}}^\ell_{\mathbf{grad},h} & & \big\downarrow \widehat{\mathcal{J}}^\ell_{\mathbf{curl},h} & & \big\downarrow \widehat{\mathcal{J}}^\ell_{\mathrm{div},h} & & \big\downarrow \pi^\ell_{\mathcal{P},\mathcal{S}_h} \\
V^\ell_{\mathbf{grad}}(\mathcal{S}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{V}^\ell_{\mathbf{curl}}(\mathcal{S}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}^\ell_{\mathrm{div}}(\mathcal{S}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^\ell(\mathcal{S}_h)
\end{array}
$$

Quasi-interpolator on DDR: take $\mathcal{S}_h$ matching simplicial submesh of $\mathcal{T}_h$, $\ell \geq k+1$, and interpolate from the FE spaces...

$$
\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\big\downarrow \widehat{\mathcal{J}}^\ell_{\mathbf{grad},h} & & \big\downarrow \widehat{\mathcal{J}}^\ell_{\mathbf{curl},h} & & \big\downarrow \widehat{\mathcal{J}}^\ell_{\mathrm{div},h} & & \big\downarrow \pi^\ell_{\mathcal{P},\mathcal{S}_h} \\
V^\ell_{\mathbf{grad}}(\mathcal{S}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{V}^\ell_{\mathbf{curl}}(\mathcal{S}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}^\ell_{\mathrm{div}}(\mathcal{S}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^\ell(\mathcal{S}_h) \\
\big\downarrow \underline{I}^k_{\mathbf{grad},h} & & \big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \big\downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \big\downarrow \pi^k_{\mathcal{P},h} \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{S}_h)
\end{array}
$$

A kind of magic...

$$
\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Big\downarrow \widehat{\mathcal{I}}^{\ell}_{\mathbf{grad},h} & & \Big\downarrow \widehat{\mathcal{I}}^{\ell}_{\mathbf{curl},h} & & \Big\downarrow \widehat{\mathcal{I}}^{\ell}_{\mathrm{div},h} & & \Big\downarrow \pi^{\ell}_{\mathcal{P},\mathcal{S}_h} \\
V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{\ell}(\mathcal{S}_h) \\
\Big\downarrow \underline{I}^{k}_{\mathbf{grad},h} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathbf{curl},h} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathrm{div},h} & & \Big\downarrow \pi^{k}_{\mathcal{P},h} \\
\underline{X}^{k}_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^{k}_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^{k}_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^{k}(\mathcal{S}_h)
\end{array}
$$

A kind of magic...

$$\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathbf{grad},h} & & \Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathbf{curl},h} & & \Big\downarrow \widehat{\mathcal{J}}^{\ell}_{\mathrm{div},h} & & \Big\downarrow \pi^{\ell}_{\mathcal{P},\mathcal{S}_h} \\
V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{\ell}(\mathcal{S}_h) \\
\Big\downarrow \underline{I}^{k}_{\mathbf{grad},h} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathbf{curl},h} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathrm{div},h} & & \Big\downarrow \pi^{k}_{\mathcal{P},h} \\
\underline{X}^{k}_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^{k}_{h}} & \underline{\boldsymbol{X}}^{k}_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^{k}_{h}} & \underline{\boldsymbol{X}}^{k}_{\mathrm{div},h} & \xrightarrow{D^{k}_{h}} & \mathcal{P}^{k}(\mathcal{S}_h)
\end{array}$$

Why does this work?

- $\underline{I}^k_{\mathbf{grad},h}$, $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{I}}^k_{\mathrm{div},h}$ can be applied to functions that have suitable single-valued traces on mesh entities.
  (e.g., single-valued tangential traces on edges and faces for $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}$)

A kind of magic...

$$
\begin{tikzcd}
H(\mathbf{grad},\Omega) \arrow[r, "\mathbf{grad}"] \arrow[d, "\widehat{\mathcal{J}}^{\ell}_{\mathbf{grad},h}"] & \boldsymbol{H}(\mathbf{curl};\Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\widehat{\mathcal{J}}^{\ell}_{\mathbf{curl},h}"] & \boldsymbol{H}(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\widehat{\mathcal{J}}^{\ell}_{\mathrm{div},h}"] & L^2(\Omega) \arrow[d, "\pi^{\ell}_{\mathcal{P},\mathcal{S}_h}"] \\
V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h) \arrow[r, "\mathbf{grad}"] \arrow[d, "\underline{I}^{k}_{\mathbf{grad},h}"] & \boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h) \arrow[r, "\mathbf{curl}"] \arrow[d, "\underline{\boldsymbol{I}}^{k}_{\mathbf{curl},h}"] & \boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h) \arrow[r, "\mathrm{div}"] \arrow[d, "\underline{\boldsymbol{I}}^{k}_{\mathrm{div},h}"] & \mathcal{P}^{\ell}(\mathcal{S}_h) \arrow[d, "\pi^{k}_{\mathcal{P},h}"] \\
\underline{X}^{k}_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^{k}_{h}"] & \underline{\boldsymbol{X}}^{k}_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^{k}_{h}"] & \underline{\boldsymbol{X}}^{k}_{\mathrm{div},h} \arrow[r, "D^{k}_{h}"] & \mathcal{P}^{k}(\mathcal{S}_h)
\end{tikzcd}
$$

Why does this work?

- $\underline{I}^{k}_{\mathbf{grad},h}$, $\underline{\boldsymbol{I}}^{k}_{\mathbf{curl},h}$ and $\underline{\boldsymbol{I}}^{k}_{\mathrm{div},h}$ can be applied to functions that have suitable single-valued traces on mesh entities.
  (e.g., single-valued tangential traces on edges and faces for $\underline{\boldsymbol{I}}^{k}_{\mathbf{curl},h}$)
- The FE spaces $V^{\ell}_{\mathbf{grad}}(\mathcal{S}_h)$, $\boldsymbol{V}^{\ell}_{\mathbf{curl}}(\mathcal{S}_h)$ and $\boldsymbol{V}^{\ell}_{\mathrm{div}}(\mathcal{S}_h)$ have such suitable traces!

Properties

$$
\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Big\downarrow \widehat{\underline{I}}^k_{\mathbf{grad},h} & & \Big\downarrow \widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h} & & \Big\downarrow \widehat{\underline{\boldsymbol{I}}}^k_{\mathrm{div},h} & & \Big\downarrow \pi^k_{\mathcal{P},\mathcal{S}_h} \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{S}_h)
\end{array}
$$

- Bounded cochain maps: diagram commutes and the interpolators are continuous.

Properties

$$
\begin{tikzcd}
H(\mathbf{grad},\Omega) \arrow[r, "\mathbf{grad}"] \arrow[d, "\widehat{\underline{I}}^k_{\mathbf{grad},h}"] & \boldsymbol{H}(\mathbf{curl};\Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}"] & \boldsymbol{H}(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\widehat{\underline{\boldsymbol{I}}}^k_{\mathrm{div},h}"] & L^2(\Omega) \arrow[d, "\pi^k_{\mathcal{P},\mathcal{S}_h}"] \\
\underline{X}^k_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \arrow[r, "D^k_h"] & \mathcal{P}^k(\mathcal{S}_h)
\end{tikzcd}
$$

- Primal consistency: for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$ and $\widetilde{T}$ set of neighbours (by vertices) of T,

$$
\|z - P^{k_\bullet}_{\bullet,T}\widehat{\underline{I}}^k_{\bullet,T} z\|_{L^2(T)} \lesssim \left(\sum_{T'\in\widetilde{T}}\left[\|z - \pi^{k_\bullet}_{\mathcal{P},T'} z\|^2_{L^2(T')} + h_T^2\|(\bullet z) - \pi^{k_\bullet}_{\mathcal{P},T'}(\bullet z)\|^2_{L^2(T')}\right]\right)^{1/2}.
$$

Properties

$$\begin{tikzcd}
H(\mathbf{grad},\Omega) \arrow[r, "\mathbf{grad}"] \arrow[d, "\widehat{\underline{I}}^k_{\mathbf{grad},h}"] & \boldsymbol{H}(\mathbf{curl};\Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}"] & \boldsymbol{H}(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\widehat{\underline{\boldsymbol{I}}}^k_{\mathrm{div},h}"] & L^2(\Omega) \arrow[d, "\pi^k_{\mathcal{P},\mathcal{S}_h}"] \\
\underline{X}^k_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \arrow[r, "D^k_h"] & \mathcal{P}^k(\mathcal{S}_h)
\end{tikzcd}$$

- Adjoint consistency: measures defect of discrete IBP, using minimal regularity.
  Example: for $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\Omega)$ such that $\boldsymbol{v}\cdot\mathbf{n}_\Omega \in L^2(\partial\Omega)$, and $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$\left|(\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h\underline{q}_h)_{\mathbf{curl},h} + \int_\Omega \mathrm{div}(\boldsymbol{v})\, P^{k+1}_{\mathbf{grad},h}\underline{q}_h - \int_{\partial\Omega} \boldsymbol{v}\cdot\mathbf{n}_\Omega\, \gamma^{k+1}_{\partial\Omega}\underline{q}_h\right| \lesssim A_h(\boldsymbol{v})\left(\|\underline{q}_h\|_{\mathbf{grad},h} + \|\underline{\boldsymbol{G}}^k_h\underline{q}_h\|_{\mathbf{curl},h}\right),$$

  with $A_h(\boldsymbol{v}) \to 0$ as $h \to 0$.

Outline

Slides

The curl case

Theorem (Conforming lifting)

There exists $\mathcal{L}^k_{\mathbf{curl},h} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \boldsymbol{H}(\mathbf{curl};\Omega)$ *such that, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$
\begin{aligned}
\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h) &= \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_h \quad \forall T \in \mathcal{T}_h,\\
\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h &= \underline{\boldsymbol{v}}_h,\\
\|\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},T},\\
\|\,\mathbf{curl}\,\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h)\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},T}.
\end{aligned}
$$

- $\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h \in \boldsymbol{\mathcal{N}}^{k+3}(\mathcal{S}_h) \cap \boldsymbol{H}(\mathbf{curl};\Omega)$, with $\mathcal{S}_h$ matching simplicial submesh of $\mathcal{T}_h$. Hence, tangential traces of $\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h$ are well-defined, and

$$
\widehat{\mathcal{J}}^{k+3}_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h, \quad \text{so} \quad \widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{v}}_h.
$$

The curl case

Theorem (Conforming lifting)

There exists $\mathcal{L}^k_{\mathbf{curl},h} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \boldsymbol{H}(\mathbf{curl};\Omega)$ *such that, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$*,*

$$\begin{aligned}
\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h) &= \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_h \quad \forall T \in \mathcal{T}_h,\\
\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h &= \underline{\boldsymbol{v}}_h,\\
\|\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},T},\\
\|\,\mathbf{curl}\,\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h)\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},T}.
\end{aligned}$$

- Not based on local solutions to PDEs, but rather by solving local algebraic problems in cochains and trimmed polynomial spaces.

  $\rightsquigarrow$ *fine estimates, not a virtual function, no limit on dimension....*

The curl case

Theorem (Conforming lifting)

There exists $\mathcal{L}^k_{\mathbf{curl},h} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \boldsymbol{H}(\mathbf{curl};\Omega)$ *such that, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$
\begin{aligned}
\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h) &= \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_h \quad \forall T \in \mathcal{T}_h,\\
\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h &= \underline{\boldsymbol{v}}_h,\\
\|\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},T},\\
\|\,\mathbf{curl}\,\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h)\|_{L^2(T)} &\lesssim \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},T}.
\end{aligned}
$$

- Used to prove adjoint consistency, as well as Maxwell compactness.

Exterior calculus de Rham

$$\begin{array}{ccccccc}
H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & L^2\Lambda^3(\Omega)
\end{array}$$

- $\Lambda^\ell(\Omega)$: (alternate multilinear) ℓ-forms on Ω.
- $\mathrm{d}^\ell : \Lambda^\ell(\Omega) \to \Lambda^{\ell+1}(\Omega)$: exterior derivative.
- Allows for a unified analysis of all spaces/operators along the complex.

Exterior calculus DDR

$$
\begin{tikzcd}
\underline{X}^k_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^k_h"] \arrow[d, leftrightarrow] & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^k_h"] \arrow[d, leftrightarrow] & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \arrow[r, "D^k_h"] \arrow[d, leftrightarrow] & \mathcal{P}^k(\mathcal{T}_h) \arrow[d, leftrightarrow] \\
\underline{X}^k_{0,h} \arrow[r, "\underline{\mathrm{d}}_{0,h}"] & \underline{X}^k_{1,h} \arrow[r, "\underline{\mathrm{d}}^{1,h}"] & \underline{X}^k_{2,h} \arrow[r, "\underline{\mathrm{d}}^{2,h}"] & \underline{X}^k_{3,h}
\end{tikzcd}
$$

- $\underline{X}^\ell_h$: discrete DDR space of ℓ-forms on Ω.
- $\underline{\mathrm{d}}^\ell_h$: discrete exterior derivative.
- Allows for a unified design and analysis of DDR and related tools.

Outline

Slides

Statement

Theorem (Maxwell compactness for DDR)

Let $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ *be such that*

$$\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right)_{h\in\mathcal{H}} \text{ is bounded,}$$
$$(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} = 0 \qquad \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}.$$

Then, there exists $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}_0(\mathrm{div};\Omega)$ *such that* $\mathrm{div}(\boldsymbol{v}) \equiv 0$ *and, up to a subsequence as* $h \to 0$, $\boldsymbol{P}^k_{\mathbf{curl},h} \underline{\boldsymbol{v}}_h \to \boldsymbol{v}$ *in* $L^2(\Omega)^3$.

Also valid in $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$*, for sequence of vectors with bounded discrete divergence and that are orthogonal to discrete curls.*

Sketch of proof: initial stages

- *Weak convergence*: of $\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h$ and $\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h$ towards the same limit $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$.

  (Same limit for both because $\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h) = \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_h$.)

- $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$: for all $q \in C^\infty(\overline{\Omega})$, use primal consistency on

$$(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\,\mathbf{grad}\,q)_{\mathbf{curl},h} = (\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}^k_h \underline{I}^k_{\mathbf{grad},h} q)_{\mathbf{curl},h} = 0$$

  to get

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\,q = 0.$$

Sketch of proof: strong convergence

- *Hodge decomposition*:

$$\boldsymbol{v} - \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \boldsymbol{w}(h) + \mathbf{grad}\, q(h)$$

with $q(h) \in H(\mathbf{grad}, \Omega)$ and $\boldsymbol{w}(h) \in \boldsymbol{H}(\mathbf{curl}; \Omega)$ bounded in their spaces, and

$$\int_\Omega \boldsymbol{w}(h) \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega).$$

Sketch of proof: strong convergence

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$$\boldsymbol{v} - \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \boldsymbol{w}(h) + \mathbf{grad}\, q(h)$$

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$$\int_\Omega \boldsymbol{w}(h) \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega).$$

- $(\boldsymbol{w}(h))_h$ relatively compact in L^2 by continuous Maxwell compactness.

Sketch of proof: strong convergence

- *Hodge decomposition*:

$$\boldsymbol{v} - \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \boldsymbol{w}(h) + \mathbf{grad}\, q(h)$$

  with $q(h) \in H(\mathbf{grad}, \Omega)$ and $\boldsymbol{w}(h) \in \boldsymbol{H}(\mathbf{curl}; \Omega)$ bounded in their spaces, and

$$\int_\Omega \boldsymbol{w}(h) \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega).$$

- $(\boldsymbol{w}(h))_h$ relatively compact in L^2 by continuous Maxwell compactness.

- We have $\boldsymbol{v} \perp \mathbf{grad}\, q(h)$ and, morally,

$$\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h \underset{\sim}{\perp} \mathbf{grad}\, q(h)$$

  since $(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}^k_h \cdot)_{\mathbf{curl},h} = 0$. This should give $\mathbf{grad}\, q(h) \simeq 0$...

Sketch of proof: strong convergence

- *Hodge decomposition*:

$$\boldsymbol{v} - \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \boldsymbol{w}(h) + \mathbf{grad}\, q(h)$$

with $q(h) \in H(\mathbf{grad},\Omega)$ and $\boldsymbol{w}(h) \in \boldsymbol{H}(\mathbf{curl};\Omega)$ bounded in their spaces, and

$$\int_\Omega \boldsymbol{w}(h) \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad},\Omega).$$

- $(\boldsymbol{w}(h))_h$ relatively compact in L^2 by continuous Maxwell compactness.

- We have $\boldsymbol{v} \perp \mathbf{grad}\, q(h)$ and, morally,

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since $(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}^k_h \cdot)_{\mathbf{curl},h} = 0$. This should give $\mathbf{grad}\, q(h) \simeq 0$...

- However, we only have

$$\left| \int_\Omega \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h \cdot \mathbf{grad}\, q(h) \right| \le C(q(h))o(1)$$

where $C(q(h))$ bounded if $(q(h))_h$ is relatively compact in $H(\mathbf{grad},\Omega)$.

Sketch of proof: strong convergence

So, what do we do?

Sketch of proof: strong convergence

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Sketch of proof: strong convergence

So, what do we do?

We go fully discrete, to avoid trading the exact orthogonality $(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}_h^k \cdot)_{\mathbf{curl},h} = 0$ for an approximate orthogonality.

Sketch of proof: strong convergence

$$\boldsymbol{v} - \mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \boldsymbol{w}(h) + \mathbf{grad}\, q(h)$$

- Quasi-interpolate and use $\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\mathcal{L}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{v}}_h$ as well as the commuting properties:

$$\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h = \widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{w}(h) + \underline{\boldsymbol{G}}^k_h(\widehat{\underline{I}}^k_{\mathbf{grad},h}q(h)).$$

Sketch of proof: strong convergence

- Quasi-interpolate and use $\underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \mathcal{L}_{\mathbf{curl},h}^k \underline{\boldsymbol{v}}_h = \underline{\boldsymbol{v}}_h$ as well as the commuting properties:

$$\underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{v} - \underline{\boldsymbol{v}}_h = \underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{w}(h) + \underline{\boldsymbol{G}}_h^k(\underline{\widehat{I}}_{\mathbf{grad},h}^k q(h)).$$

- Take the $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$-inner product and use the discrete orthogonality:

$$\begin{aligned}
\|\underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{v} - \underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h}^2 &= (\underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{v} - \underline{\boldsymbol{v}}_h, \underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{w}(h))_{\mathbf{curl},h} \\
&\quad + (\underline{\widehat{\boldsymbol{I}}}_{\mathbf{curl},h}^k \boldsymbol{v} - \cancel{\underline{\boldsymbol{v}}_h}, \underline{\boldsymbol{G}}_h^k(\underline{\widehat{I}}_{\mathbf{grad},h}^k q(h)))_{\mathbf{curl},h} \\
&= \mathfrak{T}_1 + \mathfrak{T}_2.
\end{aligned}$$

Sketch of proof: strong convergence

- Take the $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$-inner product and use the discrete orthogonality:

$$\begin{aligned}\|\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h\|^2_{\mathbf{curl},h} &= (\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h, \widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{w}(h))_{\mathbf{curl},h} \\ &\quad + (\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h(\widehat{\underline{I}}^k_{\mathbf{grad},h}q(h)))_{\mathbf{curl},h} \\ &= \mathfrak{T}_1 + \mathfrak{T}_2.\end{aligned}$$

- $\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h$ converges weakly to 0, $(\boldsymbol{w}_h)_h$ relatively compact in L^2: by primal consistency,

$$\mathfrak{T}_1 \to 0.$$

Sketch of proof: strong convergence

- Take the $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$-inner product and use the discrete orthogonality:

$$
\begin{aligned}
\|\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h\|^2_{\mathbf{curl},h} &= (\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h, \widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{w}(h))_{\mathbf{curl},h} \\
&\quad + (\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h(\widehat{\underline{I}}^k_{\mathbf{grad},h}q(h)))_{\mathbf{curl},h} \\
&= \mathfrak{T}_1 + \mathfrak{T}_2.
\end{aligned}
$$

- $\widehat{\underline{\boldsymbol{I}}}^k_{\mathbf{curl},h}\boldsymbol{v} - \underline{\boldsymbol{v}}_h$ converges weakly to 0, $(\boldsymbol{w}_h)_h$ relatively compact in L^2: by primal consistency,

$$\mathfrak{T}_1 \to 0.$$

- Use adjoint consistency on the second term together with $\operatorname{div}\boldsymbol{v} = 0$ to get

$$\mathfrak{T}_2 \to 0.$$

Conclusions

- DDR: Arbitrary-order de Rham complex, applicable on generic polyhedra, but compatible with FE methods. Full set of algebraic and analytic results (cohomology, consistencies, Poincaré inequalities, etc.)
- Quasi-interpolator to interpolate minimal-regularity functions onto DDR.
- Conforming lifting into finite element spaces (on submesh): tool to import results of conforming space into DDR.
- Maxwell compactness for DDR, allows for analysis of eigenvalue problems and nonlinear PDEs.

https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

New generation methods for numerical simulations

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Thank you for your attention!

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