

# A polytopal exterior calculus framework with application to the Maxwell equations on manifolds

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# Main references for this presentation

- *An exterior calculus framework for polytopal methods*. F. Bonaldi, D. A. Di Pietro, J. Droniou, and K. Hu. J. Eur. Math. Soc., 55p, 2025. doi: 10.4171/JEMS/1602. <https://arxiv.org/abs/2303.11093>.
- *A polytopal discrete de Rham complex on manifolds, with application to the Maxwell equations*, J. Droniou, M. Hanot, and T. Oliynyk. J. Comput. Phys. 529, Paper No. 113886, 31p, 2025. doi: 10.1016/j.jcp.2025.113886. <https://arxiv.org/abs/2401.16130>.

These slides:



# Outline

- 1 Maxwell equations and the de Rham complex
- 2 Exterior calculus
- 3 A polytopal discretisation of the de Rham complex in exterior calculus
- 4 DDR complex on a manifold
- 5 Polynomial spaces on manifolds
- 6 Maxwell equations and numerical simulations

Slides:



# Maxwell's equations and preservation of divergence

**Maxwell:**  $E$  electric field,  $B$  magnetic field,  $\epsilon_0$  and  $\mu_0$  vacuum permittivity and permeability:

$$\begin{array}{ll} \partial_t B + \mathbf{curl} E = 0 & \text{div } B = 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl} B = 0 & \text{div } E = 0 \\ \textit{Evolutions} & \textit{Constraints} \end{array}$$

# Maxwell's equations and preservation of divergence

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Evolution preserves constraints:

$$0 = \text{div}(\partial_t B + \mathbf{curl} E) = \partial_t(\text{div } B) + \cancel{\text{div } \mathbf{curl} E} = \partial_t(\text{div } B).$$

Key property:  $\text{Im } \mathbf{curl} \subset \text{ker } \text{div}$ .

# The de Rham complex

$$H(\mathbf{grad}; \Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega) \xrightarrow{0} \{0\}.$$

**Complex:** the image of each operator is included in the kernel of the next one.

$$\text{Im } \mathbf{grad} \subset \ker \mathbf{curl}, \quad \boxed{\text{Im } \mathbf{curl} \subset \ker \mathbf{div}}, \quad \text{Im } \mathbf{div} \subset L^2(\Omega).$$

# The de Rham complex

$$H(\mathbf{grad}; \Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega) \xrightarrow{0} \{0\}.$$

**Complex:** the image of each operator is included in the kernel of the next one.

$$\text{Im } \mathbf{grad} \subset \ker \mathbf{curl}, \quad \boxed{\text{Im } \mathbf{curl} \subset \ker \mathbf{div}}, \quad \text{Im } \mathbf{div} \subset L^2(\Omega).$$

**Exactness of the complex:** the image of each operator is *equal* to the kernel of the next one.

$$\text{Im } \mathbf{grad} = \ker \mathbf{curl}, \quad \text{Im } \mathbf{curl} = \ker \mathbf{div}, \quad \text{Im } \mathbf{div} = L^2(\Omega).$$

Requires **topological property on  $\Omega$** :

- No tunnel for  $\text{Im } \mathbf{grad} = \ker \mathbf{curl}$ .
- No void for  $\text{Im } \mathbf{curl} = \ker \mathbf{div}$ .

# Exactness of the complex gives Maxwell in potential form

$$\begin{aligned}\partial_t B + \mathbf{curl} E &= 0 & \operatorname{div} B &= 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl} B &= 0 & \operatorname{div} E &= 0\end{aligned}$$

$$\Downarrow (B = \mathbf{curl} A, E = -\partial_t A)$$

$$\mu_0 \epsilon_0 \partial_t^2 A + \mathbf{curl} \mathbf{curl} A = 0$$

Requires:  $\operatorname{Im} \mathbf{curl} = \operatorname{Ker} \operatorname{div}$  and  $\operatorname{Im} \mathbf{grad} = \operatorname{Ker} \mathbf{curl}$ .

*(So, in particular,  $\Omega$  must have no voids, no tunnels)*

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Slides:



# Alternating forms I

- **$k$ -alternating form:**  $\omega : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}$  multilinear and fully antisymmetric, i.e.

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_k) = -\omega(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k).$$

Space:  $\text{Alt}^k(\mathbb{R}^n)$ . Note that  $\boxed{\text{Alt}^k(\mathbb{R}^n) = \{0\} \text{ if } k > n.}$

- **Exterior product:**

$$\wedge : \text{Alt}^k(\mathbb{R}^n) \times \text{Alt}^\ell(\mathbb{R}^n) \rightarrow \text{Alt}^{k+\ell}(\mathbb{R}^n).$$

- **Hodge star operator:** isomorphism

$$\star : \text{Alt}^k(\mathbb{R}^n) \rightarrow \text{Alt}^{n-k}(\mathbb{R}^n).$$

## Alternating forms II

- **Trace:**  $\omega \in \text{Alt}^k(\mathbb{R}^n)$  and  $V$  subspace of  $\mathbb{R}^n$ . Then  $\text{tr}_V \omega \in \text{Alt}^k(V)$  such that

$$(\text{tr}_V \omega)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(\mathbf{v}_1, \dots, \mathbf{v}_k) \text{ for all } \mathbf{v}_1, \dots, \mathbf{v}_k \in V.$$

- **Basis:** the dual basis of linear forms

$$\{dx^1, \dots, dx^n\} \text{ s.t. } dx^i(\mathbf{a}) = a_i$$

gives a canonical basis of  $\text{Alt}^k(\mathbb{R}^n)$ :

$$\{dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_k} : \sigma \text{ s.t. } 1 \leq \sigma_1 < \dots < \sigma_k \leq n\}.$$

- $k = 0$ :  $\{1\}$ ,
- $k = 1$ :  $\{dx^1, dx^2, dx^3\}$ ,
- $k = 2$ :  $\{dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3\}$ ,
- $k = 3$ :  $\{dx^1 \wedge dx^2 \wedge dx^3\}$ .

# Alternating forms III

- **Vector proxy** of alternating form: if  $n = 3$  and  $V$  hyperplane,

$k$	$k$ -form $\omega$	proxy for $\omega$	proxy for $\text{tr}_V \omega$
0	$\omega = a$	$\omega_{\sharp} = a$	$\omega_{\sharp}$
1	$\omega = a dx^1 + b dx^2 + c dx^3$	$\omega_{\sharp} = (a, b, c)$	tangential trace of $\omega_{\sharp}$
2	$\omega = a(dx^2 \wedge dx^3)$ $-b(dx^1 \wedge dx^3)$ $+c(dx^1 \wedge dx^2)$	$\omega_{\sharp} = (a, b, c)$	normal trace of $\omega_{\sharp}$
3	$\omega = a(dx^1 \wedge dx^2 \wedge dx^3)$	$\omega_{\sharp} = a$	N/A

# Exterior calculus I

**Differential forms:** a  $k$ -differential form on a domain  $\Omega \subset \mathbb{R}^n$  is

$$\omega : \Omega \rightarrow \text{Alt}^k(\mathbb{R}^n).$$

*Space of  $k$ -differential forms:*  $\Lambda^k(\Omega)$ .

## Exterior derivative

- o Decomposition on canonical basis:

$$\omega_x = \sum_{\sigma} a_{\sigma}(x) dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}.$$

- o If  $\omega \in C^1 \Lambda^k(\Omega)$ ,  $d^k \omega \in \Lambda^{k+1}(\Omega)$  such that

$$d^k \omega_x = \sum_{\sigma} \sum_{i=1}^n \frac{\partial a_{\sigma}}{\partial x_i}(x) dx^i \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}.$$

*Note:*  $\omega \in C^1 \Lambda^k(\Omega)$  means  $a_{\sigma} \in C^1(\Omega)$ .

# Exterior calculus II

Vector proxy:

$k$	$k$ -form $\omega$	proxy for $\omega$	$\text{tr}_V \omega$	$d^k \omega$
0	$a$	$\omega_{\#} = a$	$\omega_{\#}$	$\text{grad } \omega_{\#}$
1	$a dx^1 + b dx^2 + c dx^3$	$\omega_{\#} = (a, b, c)$	tang. trace of $\omega_{\#}$	$\text{curl } \omega_{\#}$
2	$a(dx^2 \wedge dx^3)$ $-b(dx^1 \wedge dx^3)$ $+c(dx^1 \wedge dx^2)$	$\omega_{\#} = (a, b, c)$	normal trace of $\omega_{\#}$	$\text{div } \omega_{\#}$
3	$a(dx^1 \wedge dx^2 \wedge dx^3)$	$\omega_{\#} = a$	N/A	0

# Exterior calculus de Rham complex

Set  $H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) : d^k\omega \in L^2\Lambda^{k+1}(\Omega)\}$ .

In dimension  $n = 3$ :

$$\begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{d^0} & H\Lambda^1(\Omega) & \xrightarrow{d^1} & H\Lambda^2(\Omega) & \xrightarrow{d^2} & H\Lambda^3(\Omega) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ H(\mathbf{grad}; \Omega) & \xrightarrow{\mathbf{grad}} & H(\mathbf{curl}; \Omega) & \xrightarrow{\mathbf{curl}} & H(\mathbf{div}; \Omega) & \xrightarrow{\mathbf{div}} & L^2(\Omega). \end{array}$$

In any dimension  $n$ :

$$H\Lambda^0(\Omega) \rightarrow \dots \xrightarrow{d^{k-1}} H\Lambda^k(\Omega) \xrightarrow{d^k} H\Lambda^{k+1}(\Omega) \xrightarrow{d^{k+1}} \dots \rightarrow H\Lambda^n(\Omega)$$

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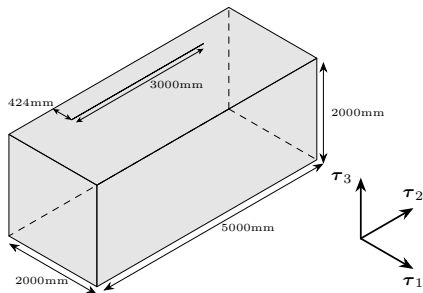
Slides:



# Why polytopal methods? I

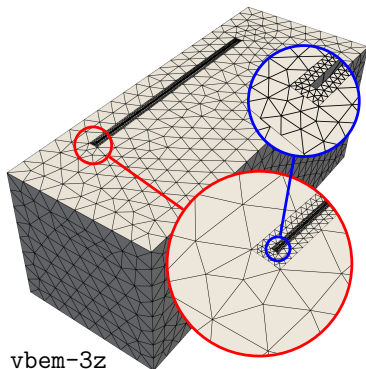
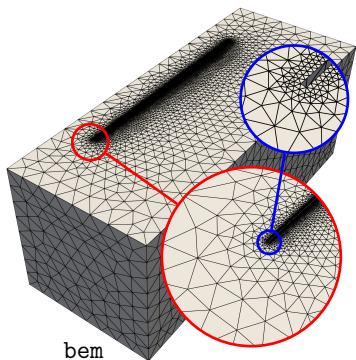
Reference: [24]

**Problem:** use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



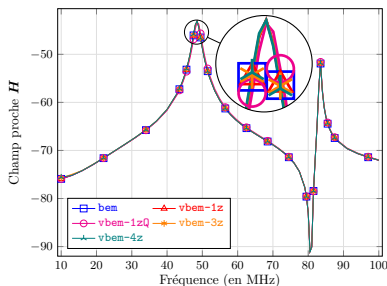
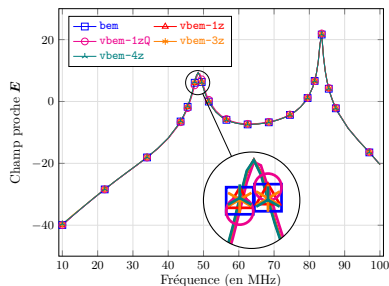
# Why polytopal methods? II

**Meshes:** conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).



# Why polytopal methods? III

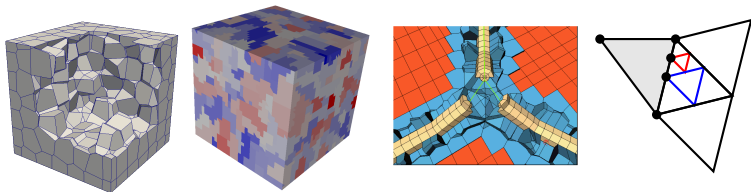
**Accuracy:** comparison of modulus of reflected near fields at the top.



**Computational cost**

<i>Method</i>	<i>Assembly</i>	<i>Resolution</i>
bem	813s	125s
vbem-3z	321s	19s

# Domain and polytopal mesh



- Assume  $\Omega \subset \mathbb{R}^n$  polytopal (polygon if  $n = 2$ , polyhedron if  $n = 3, \dots$ )
- We consider a **polytopal mesh**  $\mathcal{M}_h$  with  $d$ -cells,  $0 \leq d \leq n$
- $d$ -cells in  $\mathcal{M}_h$  are collected in  $\Delta_d(\mathcal{M}_h)$ .

When  $n = 3$ :

- $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$ : set of **vertices**
- $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$ : set of **edges**
- $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$ : set of **faces**
- $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$ : set of **elements**

# Towards the Discrete De Rham (DDR)

## Stokes formulae

We remove  $k$  from  $d^k$

- Stokes formula (IBP):

$$\int_f d\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge d\mu + \int_{\partial f} \text{tr}_{\partial f} \omega \wedge \text{tr}_{\partial f} \mu$$

- Inner product on  $L^2\Lambda^k(f)$ :

$$(\omega, \beta)_f = \int_f \omega \wedge \star\beta.$$

- Stokes formula with inner products: for  $\omega \in H\Lambda^k(f)$  and  $\beta \in H\Lambda^{k+1}(f)$ ,

$$(d\omega, \beta)_f = (-1)^{k+1} (\omega, \delta\beta)_f + (\text{tr}_{\partial f} \omega, \star^{-1} \text{tr}_{\partial f} (\star\beta))_{\partial f}$$

with co-derivative  $\delta = \star^{-1}d\star : H\Lambda^{k+1}(f) \rightarrow L^2\Lambda^k(f)$ .

# Towards DDR

## Computing projections on polynomial spaces

A) Compute the projection of  $d\omega$  on  $\mathcal{P}_r\Lambda^{k+1}(f)$ :

$$(d\omega, \beta)_f = (-1)^{k+1}(\omega, \delta\beta)_f + \underbrace{(\text{tr}_{\partial f} \omega, \star^{-1} \text{tr}_{\partial f}(\star\beta))}_{\in \mathcal{P}_r\Lambda^k(\partial f)}_{\partial f} \quad \forall \beta \in \mathcal{P}_r\Lambda^{k+1}(f).$$

*Requires:*

- projection of  $\omega$  on  $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$ ,
- projection of  $\text{tr}_{\partial f} \omega$  on  $\mathcal{P}_r\Lambda^k(\partial f)$  (*hierarchical construction*).

# Towards DDR

## Computing projections on polynomial spaces

A) Compute the projection of  $d\omega$  on  $\mathcal{P}_r\Lambda^{k+1}(f)$ :

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Requires:

- projection of  $\omega$  on  $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$ ,
- projection of  $\text{tr}_{\partial f} \omega$  on  $\mathcal{P}_r\Lambda^k(\partial f)$  (hierarchical construction).

B) Compute the projection of  $\omega$  on  $\mathcal{P}_r\Lambda^k(f)$ :

$$(-1)^{k+1}(\omega, \delta\mu)_f = (d\omega, \mu)_f - (\text{tr}_{\partial f} \omega, \star^{-1} \text{tr}_{\partial f}(\star\mu))_{\partial f} \quad \forall \mu \in \mathcal{P}_{r+1}\Lambda^{k+1}(f).$$

$\Rightarrow d\omega$  and  $\text{tr}_{\partial f} \omega$  give the projection of  $\omega$  on  $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f) \subset \mathcal{P}_r\Lambda^k(f)$ .

# Towards DDR

## Computing projections on polynomial spaces

A) Compute the projection of  $d\omega$  on  $\mathcal{P}_r\Lambda^{k+1}(f)$ :

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Requires:

- projection of  $\omega$  on  $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$ ,
- projection of  $\text{tr}_{\partial f} \omega$  on  $\mathcal{P}_r\Lambda^k(\partial f)$  (hierarchical construction).

B) Compute the projection of  $\omega$  on  $\mathcal{P}_r\Lambda^k(f)$ :

$$(-1)^{k+1}(\omega, \delta\mu)_f = (d\omega, \mu)_f - (\text{tr}_{\partial f} \omega, \star^{-1} \text{tr}_{\partial f}(\star\mu))_{\partial f} \quad \forall \mu \in \mathcal{P}_{r+1}\Lambda^{k+1}(f).$$

$\Rightarrow d\omega$  and  $\text{tr}_{\partial f} \omega$  give the projection of  $\omega$  on  $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f) \subset \mathcal{P}_r\Lambda^k(f)$ .

Whole projection of  $\omega$  on  $\mathcal{P}_r\Lambda^k(f)$  additionally requires:

- projection of  $\omega$  on a complement of  $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$  in  $\mathcal{P}_r\Lambda^k(f)$ .

\* *Conclusion*: to get the projection of  $d\omega$  and  $\omega$  on  $\mathcal{P}_r$ , we need the projections of  $\omega$  on:

- $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$ ,
- a complement of  $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$  in  $\mathcal{P}_r\Lambda^k(f)$ .

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- a complement of  $\delta\mathcal{P}_{r+1}\Lambda^{k+1}(f)$  in  $\mathcal{P}_r\Lambda^k(f)$ .

**Koszul complement**: with  $\kappa$  Koszul operator and  $\mathcal{K}_r^\ell(f) = \kappa\mathcal{P}_{r-1}\Lambda^{\ell+1}(f)$ ,

$$\begin{aligned}\mathcal{P}_r\Lambda^0(f) &= \mathcal{P}_0\Lambda^0(f) \oplus \mathcal{K}_r^0(f), \\ \mathcal{P}_r\Lambda^{d-k}(f) &= d\mathcal{P}_{r+1}\Lambda^{d-k-1}(f) \oplus \mathcal{K}_r^{d-k}(f) \quad (\text{for } k < d).\end{aligned}$$

# Towards the DDR

## Trimmed spaces

\* *Conclusion*: to get the projection of  $d\omega$  and  $\omega$  on  $\mathcal{P}_r$ , we need the projections of  $\omega$  on:

- $\delta\mathcal{P}_r\Lambda^{k+1}(f) \subset \mathcal{P}_{r-1}\Lambda^k(f)$ ,
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Since  $\delta = \star^{-1}d\star$  and  $\star : \Lambda^{d-k}(f) \rightarrow \Lambda^k(f)$  is an isomorphism:

$$\begin{aligned}\mathcal{P}_r\Lambda^d(f) &= \mathcal{P}_0\Lambda^d(f) \oplus \star^{-1}\mathcal{K}_r^0(f), \\ \mathcal{P}_r\Lambda^k(f) &= \delta\mathcal{P}_{r+1}\Lambda^{k+1}(f) \oplus \star^{-1}\mathcal{K}_r^{d-k}(f) \quad (\text{for } k < d).\end{aligned}$$

\* *Conclusion, revisited*: we need the projection of  $\omega$  on

$$\begin{aligned} \mathcal{P}_r \Lambda^d(f) & \quad (\text{if } k = d), \\ \delta \mathcal{P}_r \Lambda^{k+1}(f) \oplus \star^{-1} \mathcal{K}_r^{d-k}(f) & \quad (\text{if } k < d). \end{aligned}$$

# Towards the DDR

## Trimmed spaces

\* *Conclusion, revisited:* we need the projection of  $\omega$  on

$$\begin{aligned} \mathcal{P}_r \Lambda^d(f) & \quad (\text{if } k = d), \\ \delta \mathcal{P}_r \Lambda^{k+1}(f) \oplus \star^{-1} \mathcal{K}_r^{d-k}(f) & \quad (\text{if } k < d). \end{aligned}$$

Trimmed space: re-apply  $\star$  to define

$$\begin{aligned} \mathcal{P}_r^- \Lambda^0(f) & := \mathcal{P}_r \Lambda^0(f), \\ \mathcal{P}_r^- \Lambda^{d-k}(f) & := d\mathcal{P}_{\square r} \Lambda^{d-k-1}(f) \oplus \mathcal{K}_r^{d-k}(f) \subsetneq \mathcal{P}_r \Lambda^{d-k}(f) \quad (\text{if } k < d). \end{aligned}$$

\* *Conclusion, final:* we need the projection of  $\omega$  on  $\star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f)$ .

# DDR: Discrete $H\Lambda^k$ space

Discrete space of vectors of polynomial forms attached to mesh entities:

$$\underline{X}_{r,h}^k = \prod_{d=k}^n \prod_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f).$$

Space	$f_0 \equiv V$	$f_1 \equiv E$	$f_2 \equiv F$	$f_3 \equiv T$
$\underline{X}_{r,h}^0$	$\mathbb{R} = \mathcal{P}_r \Lambda^0(f_0)$	$\star^{-1} \mathcal{P}_r^- \Lambda^1(f_1)$	$\star^{-1} \mathcal{P}_r^- \Lambda^2(f_2)$	$\star^{-1} \mathcal{P}_r^- \Lambda^3(f_3)$
$\underline{X}_{r,h}^1$		$\mathcal{P}_r \Lambda^1(f_1)$	$\star^{-1} \mathcal{P}_r^- \Lambda^1(f_2)$	$\star^{-1} \mathcal{P}_r^- \Lambda^2(f_3)$
$\underline{X}_{r,h}^2$			$\mathcal{P}_r \Lambda^2(f_2)$	$\star^{-1} \mathcal{P}_r^- \Lambda^1(f_3)$
$\underline{X}_{r,h}^3$				$\mathcal{P}_r \Lambda^3(f_3)$
$(\underline{X}_{r,h}^0)^\sharp$	$\mathbb{R} = \mathcal{P}_r(V)$	$\mathcal{P}_{r-1}(E)$	$\mathcal{P}_{r-1}(F)$	$\mathcal{P}_{r-1}(T)$
$(\underline{X}_{r,h}^1)^\sharp$		$\mathcal{P}_r(E)$	$\mathcal{RT}_r(F)^\perp$	$\mathcal{RT}_r(T)$
$(\underline{X}_{r,h}^2)^\sharp$			$\mathcal{P}_r(F)$	$\mathcal{N}_r(T)$
$(\underline{X}_{r,h}^3)^\sharp$				$\mathcal{P}_r(T)$

**Goal:** reconstruct on each cell  $f$  a polynomial **exterior derivative** and a polynomial **potential**:

$$d_{r,f}^k : \underline{X}_{r,f}^k \rightarrow \mathcal{P}_r \Lambda^{k+1}(f) \quad \text{and} \quad P_{r,f}^k : \underline{X}_{r,f}^k \rightarrow \mathcal{P}_r \Lambda^k(f).$$

**Hierarchical and recursive** process (from lowest-dimensional  $f$  to highest-dimensional  $f$ ).

Notice: if  $f \in \Delta_d(\mathcal{M}_h)$  and  $\underline{\omega}_f \in \underline{X}_{r,f}^k$  then

$$\underline{\omega}_f = (\omega_f) \text{ with } \omega_f \in \mathcal{P}_r \Lambda^k(f).$$

Potential:

$$P_{r,f}^k \underline{\omega}_f = \omega_f \in \mathcal{P}_r \Lambda^k(f).$$

[purple = “already constructed”]

Notice: if  $f \in \Delta_d(\mathcal{M}_h)$  and  $\underline{\omega}_f \in \underline{X}_{r,f}^k$  then

$$\underline{\omega}_f = ((\omega_{f'})_{f' \subset \partial f}, \omega_f) \text{ with } \omega_f \in \star^{-1} \mathcal{P}_r^- \Lambda^1(f) = \delta \mathcal{P}_r \Lambda^d(f) \oplus \star^{-1} \mathcal{K}_r^1(f).$$

Exterior derivative:  $d_{r,f}^k \underline{\omega}_f \in \mathcal{P}_r \Lambda^{k+1}(f)$  such that, for all  $\mu \in \mathcal{P}_r \Lambda^{k+1}(f)$ ,

$$(d_{r,f}^k \underline{\omega}_f, \mu)_f = (-1)^{k+1} (\omega_f, \delta \mu)_f + (P_{r,\partial f}^k \underline{\omega}_{\partial f}, \star^{-1} \text{tr}_{\partial f} \star \mu)_{\partial f}$$

$\rightsquigarrow$  Use  $L^2$ -projection of  $\omega_f$  on  $\delta \mathcal{P}_r \Lambda^d(f)$ .

# DDR: $d = k + 1$ , potential reconstruction

[purple = "already constructed"]

Notice: if  $f \in \Delta_d(\mathcal{M}_h)$  and  $\underline{\omega}_f \in \underline{X}_{r,f}^k$  then

$$\underline{\omega}_f = ((\omega_{f'})_{f' \subset \partial f}, \omega_f) \text{ with } \omega_f \in \star^{-1} \mathcal{P}_r^- \Lambda^1(f) = \delta \mathcal{P}_r \Lambda^d(f) \oplus \star^{-1} \mathcal{K}_r^1(f).$$

Potential:  $P_{r,f}^k \underline{\omega}_f \in \mathcal{P}_r \Lambda^k(f) = \delta(\star^{-1} \mathcal{K}_{r+1}^0(f)) \oplus \star^{-1} \mathcal{K}_r^1(f)$  such that:

- For all  $\mu \in \star^{-1} \mathcal{K}_{r+1}^0(f)$ ,

$$(-1)^{k+1} (P_{r,f}^k \underline{\omega}_f, \delta \mu)_f = (d_{r,f}^k \underline{\omega}_f, \mu)_f - (P_{r,\partial f}^k \underline{\omega}_{\partial f}, \star^{-1} \text{tr}_{\partial f} \star \mu)_{\partial f}.$$

- For all  $\nu \in \star^{-1} \mathcal{K}_r^1(f)$ :

$$(P_{r,f}^k \underline{\omega}_f, \nu)_f = (\omega_f, \nu)_f$$

$\rightsquigarrow$  use  $L^2$ -projection of  $\omega_f$  on  $\star^{-1} \mathcal{K}_r^1(f)$ .

# Discrete De Rham complex

- Global discrete exterior derivative  $\underline{d}_{r,h}^k : \underline{X}_{r,h}^k \rightarrow \underline{X}_{r,h}^{k+1}$  by projecting:

$$\underline{d}_{r,h}^k \underline{\omega}_h := \left( \star^{-1} \pi_{r,f}^{-,d-k-1} (\star \underline{d}_{r,f}^k \underline{\omega}_f) \right)_{f \in \Delta_{[k+1 \dots n]}(\mathcal{M}_h)}$$

- The DDR complex then reads

$$\underline{X}_{r,h}^0 \xrightarrow{\underline{d}_{r,h}^0} \underline{X}_{r,h}^1 \xrightarrow{\underline{d}_{r,h}^1} \dots \xrightarrow{\underline{d}_{r,h}^{n-2}} \underline{X}_{r,h}^{n-1} \xrightarrow{\underline{d}_{r,h}^{n-1}} \underline{X}_{r,h}^n \longrightarrow \{0\}$$

## Theorem (Cohomology of the DDR complex [14])

*The DDR sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all  $k$ ,*

$$\text{Ker } \underline{d}_{r,h}^k / \text{Im } \underline{d}_{r,h}^{k-1} \cong \text{Ker } d^k / \text{Im } d^{k-1}.$$

Interpolator:  $\underline{I}_{r,h}^k : C^0 \Lambda^k(\overline{\Omega}) \rightarrow \underline{X}_{r,h}^k$  by  $L^2$ -projections:

$$\underline{I}_{r,h}^k \omega = (\star^{-1} \pi_{r,f}^{-,d-k} (\star \operatorname{tr}_f \omega))_{f \in \Delta_d(\mathcal{M}_h), d \in [k,n]}.$$

Theorem (Polynomial consistency)

$$\begin{aligned} P_{r,f}^k \underline{I}_{r,f}^k \omega &= \omega & \forall \omega \in \mathcal{P}_r \Lambda^k(f), \\ d_{r,f}^k \underline{I}_{r,f}^k \omega &= d\omega & \forall \omega \in \mathcal{P}_{r+1}^- \Lambda^k(f). \end{aligned}$$

- Using the potential reconstructions, we can define **consistent discrete  $L^2$ -products**  $(\cdot, \cdot)_{k,h}$  on  $\underline{X}_{r,h}^k$ .
  - **Numerical schemes**: replace spaces/derivatives/ $L^2$ -products by discrete counterparts.
  - Analysis requires, for all form degrees:
    - **discrete Poincaré inequalities** [12, 15, 17],
    - **adjoint consistency properties** [12, 18].
- $\rightsquigarrow$  stable schemes, with optimal  $\mathcal{O}(h^{k+1})$  rates of convergence in energy norm [4, 12, 13, 16, 19].

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Slides:



# Considerations around meshes I

## Curved faces/edges in flat space

- OK for **dG**, but other drawbacks (no static condensation, etc.) [8].
- **VEM**: polynomials on edges by parametrisation [5, 9].
- **HHO**:
  - parametrisation of faces [6].
  - polynomials on faces by restriction (intrinsic approach) [25].

## Meshes for manifolds

- **Extrinsic** approach for **FEM** [1, 10, 11] and **VEM** [21].
- **Intrinsic approach** for **FEM**, global homeomorphism with flat space [2, 3].
- Polynomial construction can be local, but continuity requirements challenges gluing over the entire manifold [23].
- Lattice-based approaches from the **General Relativity** community, limited to specific shapes [7, 20, 22]...

No previous polytopal complex on manifolds...

## Notations:

- $\mathcal{M}_h$  finite collection of pairwise disjoint **submanifolds** of dimension  $d = 0, \dots, n$  ( $d = 0$ : vertices,  $d = 1$ : edges; etc.)
- $\Delta_d(\mathcal{M}_h)$  collection of all  $d$ -dimensional manifolds.
- Boundary of  $f \in \Delta_d(\mathcal{M}_h)$  made of  $(d - 1)$ -manifolds  $f' \in \Delta_{d-1}(f)$ .

# Reminder: space and reconstructions

**Space:** vectors of trimmed polynomials on vertices, edges, faces, etc.

$$\underline{X}_{r,h}^k := \bigtimes_{d=k}^n \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f).$$

**Local reconstructions:** same as in the flat case!

○ If  $d = k$ ,  $P_{r,f}^k \underline{\omega}_f = \omega_f$ .

○ If  $d \geq k + 1$ :

– Exterior derivative: for all  $\mu \in \mathcal{P}_r \Lambda^{k+1}(f)$ ,

$$(\underline{d}_{r,f}^k \underline{\omega}_f, \mu)_f = (-1)^{k+1} (\omega_f, \delta \mu)_f + (P_{r,\partial f}^k \underline{\omega}_{\partial f}, \star^{-1} \text{tr}_{\partial f} \star \mu)_{\partial f}.$$

– Potential: for all  $(\mu, \nu) \in \mathcal{K}_{r+1}^{k-1}(f) \times \mathcal{K}_r^k(f)$ ,

$$\begin{aligned} (-1)^{k+1} (P_{r,f}^k \underline{\omega}_f, \delta \mu)_f &= (\underline{d}_{r,f}^k \underline{\omega}_f, \mu)_f - (P_{r,\partial f}^k \underline{\omega}_{\partial f}, \star^{-1} \text{tr}_{\partial f} \star \mu)_{\partial f}, \\ (P_{r,f}^k \underline{\omega}_f, \nu)_f &= (\omega_f, \nu)_f. \end{aligned}$$

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## Back to the flat case

Koszul operator:  $\kappa : \Lambda^{k+1}(f) \rightarrow \Lambda^k(f)$  such that

$$(\kappa\omega)_x(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega_x(x - x_f, \mathbf{v}_1, \dots, \mathbf{v}_k).$$

Homotopy formula: with  $\mathcal{H}_r\Lambda^k(f)$  = homogeneous polynomials of degree  $r$ ,

$$(d\kappa + \kappa d)\omega = (r + k)\omega \text{ for all } \omega \in \mathcal{H}_r\Lambda^k(f).$$

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$$(d\kappa + \kappa d)\omega = (r + k)\omega \text{ for all } \omega \in \mathcal{H}_r \Lambda^k(f).$$

Consequences:

$$\begin{aligned} d : \kappa \mathcal{H}_r \Lambda^k(f) &\rightarrow d \mathcal{H}_{r+1} \Lambda^{k-1}(f) \text{ is an isomorphism,} \\ \kappa : d \mathcal{H}_r \Lambda^k(f) &\rightarrow \kappa \mathcal{H}_{r-1} \Lambda^{k+1}(f) \text{ is an isomorphism,} \\ \mathcal{P}_r \Lambda^k(f) &= d \mathcal{P}_{r+1} \Lambda^{k-1}(f) \oplus \mathcal{K}_r^k(f) \quad (\text{if } k \geq 1). \end{aligned}$$

$\rightsquigarrow$  Justifies the definition of trimmed spaces

$$\mathcal{P}_r^- \Lambda^k(f) = d \mathcal{P}_r \Lambda^{k-1}(f) \oplus \kappa \mathcal{P}_{r-1} \Lambda^{k+1}(f).$$

# Framework for suitable polynomials on manifold

## Assumption (Homogeneous polynomials decomposition)

There is a decomposition  $\mathcal{P}_r \Lambda^\ell(f) = \bigoplus_{s \leq r} \mathcal{H}_s \Lambda^\ell(f)$  and an operator  $\kappa$  s.t.

- We have  $d\kappa + \kappa d = \lambda_{s,\ell} \text{Id}$  on  $\mathcal{H}_s \Lambda^\ell(f)$ .
- Eigenvalues identify polynomial degree:  $\lambda_{s,\ell} = 0$  if and only if  $s = \ell = 0$ , and if  $\lambda_{s,\ell} = \lambda_{s',\ell}$  then  $s = s'$ .
- $d\mathcal{H}_s \Lambda^\ell(f) \neq \{0\}$  whenever  $s > 0$  and  $\ell < d$ .

$\rightsquigarrow$  Same consequences as in the flat case, and same definition of trimmed polynomials.

## Assumption (Trace of trimmed polynomials)

$$\mathrm{tr}_{f'} \mathcal{P}_r^- \Lambda^\ell(f) \subset \mathcal{P}_r^- \Lambda^\ell(f') \quad \forall f' \in \Delta_{d'}(f),$$

**DDR complex:** define  $\underline{d}_{r,h}^k$  by projecting reconstructions  $d_{r,f}^k$  on components of the discrete space:

$$\cdots \xrightarrow{\underline{d}_{r,h}^{k-1}} \underline{X}_{r,h}^k \xrightarrow{\underline{d}_{r,h}^k} \underline{X}_{r,h}^{k+1} \xrightarrow{\underline{d}_{r,h}^{k+1}} \cdots$$

# Consistency

**Standard technique:** starts with proving an exact consistency result for polynomial functions.

$$P_{r,f}^k \underline{I}_{r,f}^k \omega = \omega \quad \forall \omega \in \mathcal{P}_r \Lambda^k(f).$$

Based on the following: in the flat case, for  $d = k$ ,  $\star \omega \in \mathcal{P}_r \Lambda^0(f)$  so

$$\star^{-1} \pi_{r,f}^0 \star \omega = \star^{-1} \star \omega = \omega \quad \forall \omega \in \mathcal{P}_r \Lambda^k(f).$$

**Does not work here**, because  $\star$  not constant so  $\star \omega \notin \mathcal{P}_r \Lambda^0(f)$ .

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**Does not work here**, because  $\star$  not constant so  $\star \omega \notin \mathcal{P}_r \Lambda^0(f)$ .

**Consequence:** **no polynomial consistency**, but still optimal consistency for smooth forms:

$$\|d_{r,f}^k I_{r,f}^k \omega - d\omega\|_f \leq C_\omega h_f^{r+1} \quad \forall \omega \in C^\infty \Lambda^k(\bar{f}).$$

# Construction of polynomials: condition on local charts

- On principle, defining polynomial spaces on cells can be done via **local coordinates** (local charts).
- However, for hybrid methods we need  $\text{tr}_{f'} \mathcal{P}_r \Lambda^l(f) \subset \mathcal{P}_r \Lambda^l(f')$ : this requires **compatibility conditions** for the charts.

# Construction of polynomials: condition on local charts

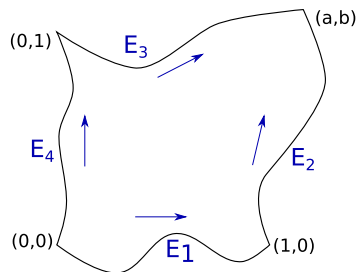
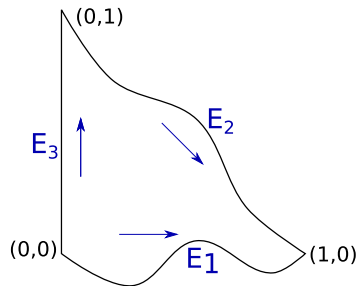
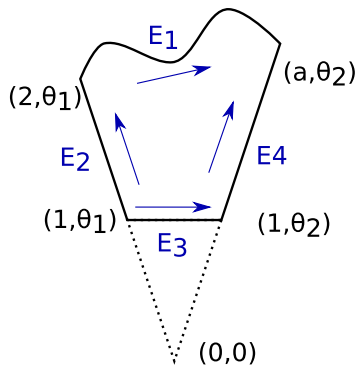
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- However, for hybrid methods we need  $\text{tr}_{f'} \mathcal{P}_r \Lambda^l(f) \subset \mathcal{P}_r \Lambda^l(f')$ : this requires **compatibility conditions** for the charts.

## Assumption

*There are  $C^2$ -diffeomorphisms  $I_f : U_f \subset \mathbb{R}^d \rightarrow \bar{f}$  for all  $f \in \Delta_d(\mathcal{M}_h)$  and  $d \in [0, n]$ .*

*Moreover, if  $f' \in \Delta_{d-1}(f)$ , with  $\mathfrak{I}_{f,f'} : f' \rightarrow \bar{f}$  the inclusion of  $f'$  into  $\bar{f}$ , the mapping  $(I_f)^{-1} \circ \mathfrak{I}_{f,f'} \circ I_{f'} : U_{f'} \rightarrow \mathbb{R}^d$  is affine.*

# From there: case by case basis



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# Maxwell equations in differential forms

- $\Omega$  a 3-dimensional Lorentzian manifold with metric decomposed in 2+1 with zero-shift and constant lapse  $c$ :

$$(g_{\mu\nu}) := \begin{pmatrix} -c^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix},$$

- $d$ : spatial exterior derivative (2-dim);  $\delta$ : co-differential;  $\text{vol}$ : spatial volume form.
- $B$ :  $(1/c) \times$  magnetic field (2-form),  $E$ : electric field (1-form).

$$\begin{aligned} \partial_t B &= -dE, \\ -\delta E &= \rho \epsilon_0^{-1}, \quad (\text{constraint}) \\ \frac{1}{c^2} \partial_t E &= \delta B - \mu_0 J, \end{aligned}$$

with  $\mu_0$  vacuum permeability,  $\rho$  electric charge density and  $J$  electric current density (with  $\delta J = \partial_t \rho$ ).

# Weak formulation and scheme

- **Weak formulation:** find  $(\tilde{E}, \tilde{B}) \in C^1([0, T]; \Lambda^1(\Omega)) \times C^1([0, T]; \Lambda^2(\Omega))$   
s.t., for all  $(v^1, v^2) \in \Lambda^1(\Omega) \times \Lambda^2(\Omega)$ ,

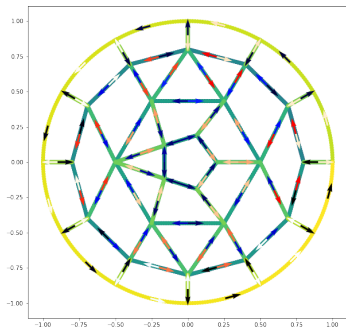
$$\begin{aligned}(\partial_t B, v^2) &= -(\mathrm{d}E, v^2), \\ \frac{1}{c^2}(\partial_t E, v^1) &= (B, \mathrm{d}v^1) - \mu_0(J, v^1).\end{aligned}$$

- **DDR scheme:** with  $X_h := \underline{X}_{r,h}^1 \times \underline{X}_{r,h}^2$ : find  $(\underline{E}_h, \underline{B}_h) \in C^1([0, T]; X_h)$   
such that for all  $(\underline{v}_h^1, \underline{v}_h^2) \in X_h$  and all  $t \in (0, T)$ ,

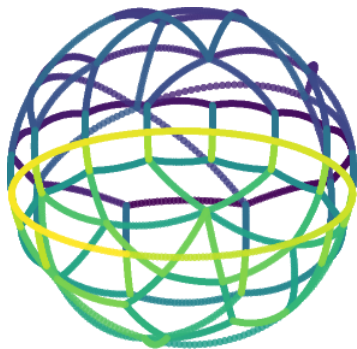
$$\begin{aligned}(\partial_t \underline{B}_h(t), \underline{v}_h^2)_{2,h} &= -(\underline{\mathrm{d}}_{r,h}^1 \underline{E}_h(t), \underline{v}_h^2)_{2,h}, \\ \frac{1}{c^2}(\partial_t \underline{E}_h(t), \underline{v}_h^1)_{1,h} &= (\underline{B}_h(t), \underline{\mathrm{d}}_{r,h}^1 \underline{v}_h^1)_{2,h} - \mu_0(\underline{I}_{r,h}^1 J(t), \underline{v}_h^1)_{1,h}.\end{aligned}$$

- **Both preserve the constraint** (dropped equation).

# Mesh on the sphere



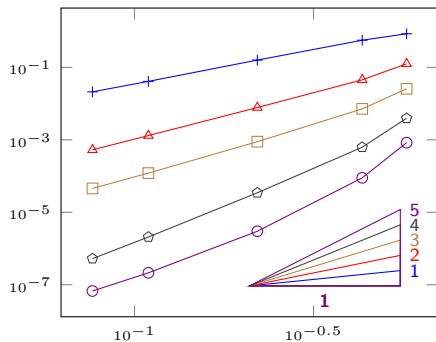
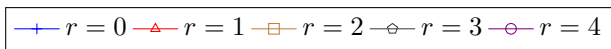
(a) Mesh in one chart and orientations.



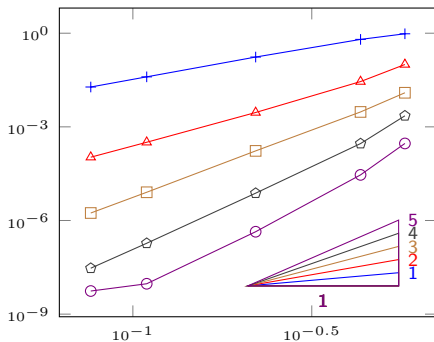
(b) Embedding of the 1-skeleton of the mesh into  $\mathbb{R}^3$ .

# Numerical results (smooth exact solution) I

Errors in  $L^2$ -norms vs. mesh size:



(a) Error on  $E$ .



(b) Error on  $B$ .

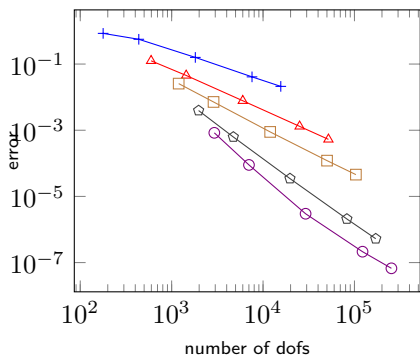
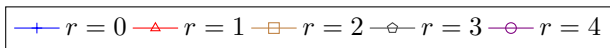
## Numerical results (smooth exact solution) II

Degree $r$	0	1	2	3	4
Error on $E$	1.85	2.66	3.08	4.32	4.56
Error on $dE$	1.73	2.20	3.27	4.37	4.75
Error on $B$	1.96	3.34	4.36	5.49	5.51

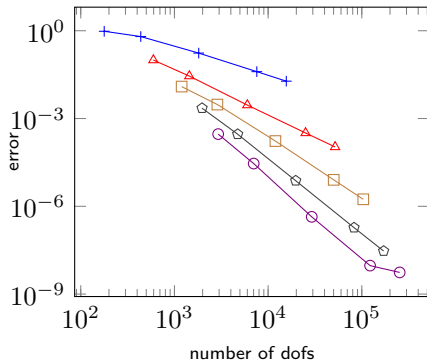
Table: Convergence rates.

# Numerical results (smooth exact solution) III

Errors in  $L^2$ -norms vs. number of DOFs:



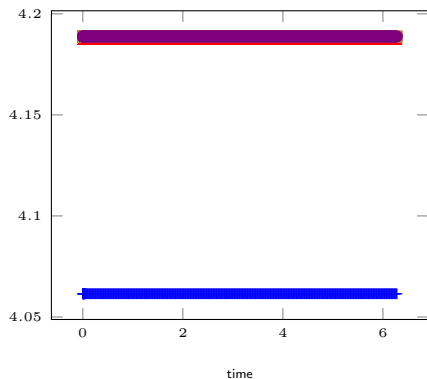
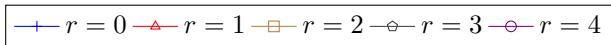
(a) Error on  $E$ .



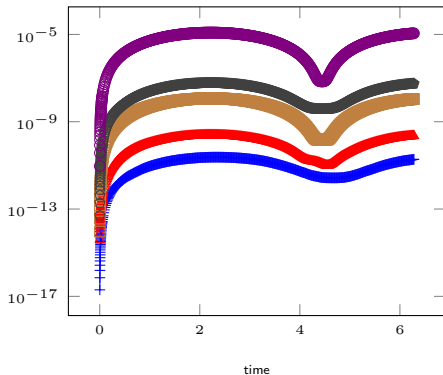
(b) Error on  $B$ .

# Numerical results (smooth exact solution) IV

## Preservations of energy and constraint



(a) Total energy  $E^2 + B^2$ .



(b) Error on the discrete constraint.

# Conclusion

- DDR: Discrete version of the de Rham complex, applicable on **generic meshes** and with an **arbitrary order of accuracy**.
- Construction in **exterior calculus framework**: unifies all spaces and differential operators.
- Unified presentation makes it easy to transfer to **manifolds** (with an **intrinsic mesh**), modulo definition of polynomial spaces.
- Identified **abstract properties** to guide the definition of polynomial spaces, put in practice for several mesh elements shapes.
- Set of **algebraic** results, and **consistency** (without exact polynomial consistency).
- **To do**: better/more general construction of local polynomial spaces; Poincaré inequalities (ongoing); adjoint consistency (ongoing).



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 **NEMESIS**

New generation  
methods for numerical  
simulations

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**Thank you for your attention!**

- [1] D. Arnold. *Finite Element Exterior Calculus*. SIAM, 2018. DOI: 10.1137/1.9781611975543.
- [2] E. Bachini, M. W. Farthing, and M. Putti. "Intrinsic finite element method for advection-diffusion-reaction equations on surfaces". In: *J. Comput. Phys.* 424 (2021), Paper No. 109827, 18. ISSN: 0021-9991,1090-2716. DOI: 10.1016/j.jcp.2020.109827. URL: <https://doi.org/10.1016/j.jcp.2020.109827>.
- [3] E. Bachini, G. Manzini, and M. Putti. "Arbitrary-order intrinsic virtual element method for elliptic equations on surfaces". In: *Calcolo* 58.3 (2021), Paper No. 30, 28. ISSN: 0008-0624,1126-5434. DOI: 10.1007/s10092-021-00418-5. URL: <https://doi.org/10.1007/s10092-021-00418-5>.
- [4] L. Beirão da Veiga, F. Dassi, D. A. Di Pietro, and J. Droniou. "Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes". In: *Comput. Meth. Appl. Mech. Engrg.* 397.115061 (2022). DOI: 10.1016/j.cma.2022.115061. URL: <https://authors.elsevier.com/a/1fChmAQEIzVqH>.
- [5] L. Beirão da Veiga, A. Russo, and G. Vacca. "The virtual element method with curved edges". In: *ESAIM Math. Model. Numer. Anal.* 53.2 (2019), pp. 375–404. ISSN: 2822-7840,2804-7214. DOI: 10.1051/m2an/2018052. URL: <https://doi.org/10.1051/m2an/2018052>.
- [6] L. Botti and D. A. Di Pietro. "Numerical assessment of Hybrid High-Order methods on curved meshes and comparison with discontinuous Galerkin methods". In: *J. Comput. Phys.* 370 (2018), pp. 58–84. DOI: 10.1016/j.jcp.2018.05.017.
- [7] L. Brewin. "Riemann normal coordinates, smooth lattices and numerical relativity". In: *Classical and Quantum Gravity* 15.10 (1998), p. 3085. DOI: 10.1088/0264-9381/15/10/014.

- [8] A. Cangiani, Z. Dong, and E. H. Georgoulis. “ $hp$ -version discontinuous Galerkin methods on essentially arbitrarily-shaped elements”. In: *Math. Comp.* 91.333 (2021), pp. 1–35. ISSN: 0025-5718,1088-6842. DOI: 10.1090/mcom/3667. URL: <https://doi.org/10.1090/mcom/3667>.
- [9] F. Dassi, A. Fumagalli, D. Losapio, S. Scialò, A. Scotti, and G. Vacca. “The mixed virtual element method on curved edges in two dimensions”. In: *Comput. Methods Appl. Mech. Engrg.* 386 (2021), Paper No. 114098, 25. ISSN: 0045-7825,1879-2138. DOI: 10.1016/j.cma.2021.114098. URL: <https://doi.org/10.1016/j.cma.2021.114098>.
- [10] A. Demlow. “Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces”. In: *SIAM J. Numer. Anal.* 47.2 (2009), pp. 805–827. ISSN: 0036-1429,1095-7170. DOI: 10.1137/070708135. URL: <https://doi.org/10.1137/070708135>.
- [11] A. Demlow and G. Dziuk. “An adaptive finite element method for the Laplace-Beltrami operator on implicitly defined surfaces”. In: *SIAM J. Numer. Anal.* 45.1 (2007), pp. 421–442. ISSN: 0036-1429,1095-7170. DOI: 10.1137/050642873. URL: <https://doi.org/10.1137/050642873>.
- [12] D. A. Di Pietro and J. Droniou. “An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency”. In: *Found. Comput. Math.* 23 (2023), pp. 85–164. DOI: 10.1007/s10208-021-09542-8.
- [13] D. A. Di Pietro and J. Droniou. “An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence”. In: *J. Comput. Phys.* 429.109991 (2021). DOI: 10.1016/j.jcp.2020.109991.
- [14] D. A. Di Pietro, J. Droniou, and S. Pitassi. “Cohomology of the discrete de Rham complex on domains of general topology”. In: *Calcolo* (2023). Published online. DOI: 10.1007/s10092-023-00523-7.

- [15] D. A. Di Pietro and M.-L. Hanot. “Uniform Poincaré inequalities for the Discrete de Rham complex on general domains”. In: *Results Appl. Math.* 23.100496 (2024). DOI: 10.1016/j.rinam.2024.100496.
- [16] D. A. Di Pietro and J. Droniou. “A discrete de Rham method for the Reissner-Mindlin plate bending problem on polygonal meshes”. In: *Comput. Math. Appl.* 125 (2022), pp. 136–149. DOI: 10.1016/j.camwa.2022.08.041. URL: <https://arxiv.org/abs/2105.11773>.
- [17] D. A. Di Pietro, J. Droniou, M.-L. Hanot, and S. Pitassi. “Uniform Poincaré inequalities for the discrete de Rham complex of differential forms”. In: (2025). Submitted, 32p. URL: <https://arxiv.org/abs/2501.16116>.
- [18] D. A. Di Pietro, J. Droniou, and S. Pitassi. “Conforming lifting and adjoint consistency for the Discrete de Rham complex of differential forms”. In: (2025), 28p. URL: <https://arxiv.org/abs/2509.21449>.
- [19] J. Droniou and J. J. Qian. “Two arbitrary-order constraint-preserving schemes for the Yang–Mills equations on polyhedral meshes”. In: *Mathematics in Engineering* 6.3 (2024), pp. 468–493. DOI: 10.3934/mine.2024019. URL: <https://arxiv.org/abs/2306.09751>.
- [20] J. Frauendiener. “Discrete differential forms in general relativity”. In: *Classical and Quantum Gravity* 23.16 (2006), S369–S385. DOI: 10.1088/0264-9381/23/16/S05. (Visited on 06/08/2022).
- [21] M. Frittelli, A. Madzvamuse, and I. Sgura. “The bulk-surface virtual element method for reaction-diffusion PDEs: analysis and applications”. In: *Commun. Comput. Phys.* 33.3 (2023), pp. 733–763. ISSN: 1815-2406,1991-7120. DOI: 10.4208/cicp.oa-2022-0204. URL: <https://doi.org/10.4208/cicp.oa-2022-0204>.

- [22] A. Gentle. “Regge Calculus: A Unique Tool for Numerical Relativity”. en. In: *General Relativity and Gravitation* 34.10 (2002), pp. 1701–1718. DOI: 10.1023/A:1020128425143. (Visited on 01/13/2024).
- [23] M. W. Licht. *Smoothed projections over manifolds in finite element exterior calculus*. Submitted. 2023. URL: <https://arxiv.org/abs/2310.14276v1>.
- [24] A. Touzalin. *Méthode des éléments virtuels pour la discrétisation des équations intégrales de frontière en électromagnétisme dans le domaine fréquentiel*. PhD Thesis, CEA-CESTA. 2025. URL: <https://cea.hal.science/tel-05149215>.
- [25] L. Yemm. “A new approach to handle curved meshes in the hybrid high-order method”. In: *Found. Comput. Math.* 24.3 (2024), pp. 1049–1076. DOI: 10.1007/s10208-023-09615-w. URL: <https://doi.org/10.1007/s10208-023-09615-w>.

## Regular mesh sequence:

- Let  $M_h$  be a flat polytopal mesh.
- For all  $d$ , there is a bijection  $\Phi_d : \Delta_d(\mathcal{M}_h) \rightarrow \Delta_d(M_h)$  that respect boundaries.
- For all  $f \in \Delta_d(\mathcal{M}_h)$ , there is a diffeomorphism  $\phi_f : f \rightarrow \Phi_d(f)$  such that

$$\|\nabla\phi_f\|_\infty \approx \|\det(\nabla\phi_f)\|_\infty^{\frac{1}{d}} \approx \|\det(\nabla\phi_f^{-1})\|_\infty^{-\frac{1}{d}} \approx \|\nabla\phi_f^{-1}\|_\infty^{-1}.$$

- $(\mathcal{M}_h)_h$  is regular if there are such bijections and  $(M_h)_h$  is regular.

## Assumption

We have the following local Poincaré inequality, topological decomposition, discrete trace and polynomial approximation properties:

$$\|\mu\|_f \lesssim h_f \|\mathrm{d}\mu\|_f, \quad \forall \mu \in \kappa \mathcal{P}_{r-1} \Lambda^k(f),$$

$$\|\alpha\|_f + \|\beta\|_f \lesssim \|\alpha + \beta\|_f, \quad \forall (\alpha, \beta) \in \kappa \mathcal{P}_{r-1} \Lambda^{k+1}(f) \oplus \mathrm{d} \mathcal{P}_{r+1} \Lambda^{k-1}(f),$$

$$\|\mathrm{tr}_{\partial f} \mu\|_{\partial f} \lesssim h_f^{-\frac{1}{2}} \|\mu\|_f, \quad \forall \mu \in \mathcal{P}_r \Lambda^k(f),$$

$$\|\star^{-1} \pi_{r,f}^{d-k} \star \omega - \omega\|_f \lesssim h_f^{t+1} |\omega|_{H^{t+1}}, \quad \forall \omega \in H^{t+1} \Lambda^k(f), \forall t \in [0, r],$$

Hierarchical regularity and norm:

$$H^{r+1}\Lambda^k(f; \Delta) := \left\{ \omega \in H^{r+1}\Lambda^k(f) : \right. \\ \left. \operatorname{tr}_{f'} \omega \in H^{r+1}\Lambda^k(f') \quad \forall f' \in \Delta_{d'}(f), \forall d' \in [k, d-1] \right\},$$

$$|\omega|_{r,f,\Delta} := \sum_{d'=k}^d h_f^{\frac{d-d'}{2}} |\operatorname{tr}_{f'} \omega|_{H^{r+1}\Lambda^k(f')} \quad \forall \omega \in H^{r+1}\Lambda^k(f; \Delta).$$

Theorem (Primal consistency)

$$\|P_{r,f}^k \underline{I}_{r,f}^k \omega - \omega\|_f \lesssim h_f^{r+1} |\omega|_{r,f,\Delta} \quad \forall \omega \in H^{r+1}\Lambda^k(f; \Delta),$$

$$\|d_{r,f}^k \underline{I}_{r,f}^k \omega - d\omega\|_f \lesssim h_f^{r+1} |d\omega|_{r,f,\Delta} \quad \forall \omega \in C^1\Lambda^k(\bar{f})$$

*s.t.*  $d\omega \in H^{r+1}\Lambda^{k+1}(f; \Delta).$