Polytopal schemes for flows in fractured porous media

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Slides:

References for this presentation

- Design of the method, application to poromechanics with Coulomb friction:

  Droniou, J., Enchéry, G., Faille, I., Haidar, A., and Masson, R. *A bubble VEM–fully discrete polytopal scheme for mixed-dimensional poromechanics with frictional contact at matrix-fracture interfaces.* Computer Methods in Applied Mechanics and Engineering, 422:116838, 2024.
  `https://arxiv.org/abs/2312.09319`

- Analysis for purely mechanical model with Tresca friction:

  Droniou, J., Haidar, A., and Masson, R. *Analysis of a VEM–fully discrete polytopal scheme with bubble stabilisation for contact mechanics with tresca friction.* ESAIM: M2AN Math. Model. Numer. Anal. 59 (2), pp. 1043–1074, 2025. `http://arxiv.org/abs/2404.03045`.

See also references inside.

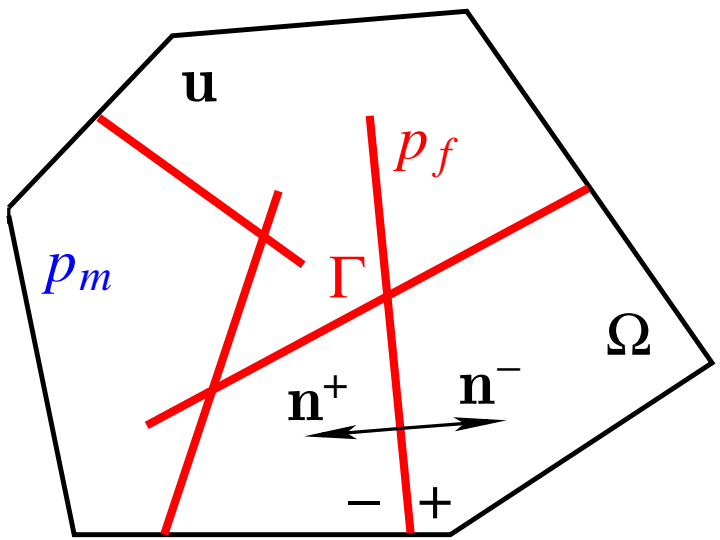
Outline

Slides:

Matrix and fracture network

Notations: Ω domain (matrix), Γ fracture, two sides $\pm$ with outward normals $\mathbf{n}^{\pm}$.

Unknowns: displacement $\mathbf{u}$ in matrix (discontinuous across fractures), pressure p_m in matrix, pressure p_f in fracture.



Poromechanics I

Flow equations: Darcy law for p_m, Poiseuille law for p_f.

$$\begin{cases} \partial_t \phi_m + \operatorname{div} \mathbf{V}_m = h_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \mathbf{V}_m = -\frac{\mathbb{K}_m}{\eta}\nabla p_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \partial_t \mathrm{d}_f + \operatorname{div}_{\boldsymbol{\tau}} \mathbf{V}_f - [\![\mathbf{V}_m]\!]_{\mathbf{n}} = h_f & \text{on } (0,T)\times\Gamma, \\ \mathbf{V}_f = \frac{(d_f)^3}{12\eta}\nabla_{\boldsymbol{\tau}} p_f & \text{on } (0,T)\times\Gamma, \\ \gamma^{\pm}_{\mathbf{n}}\mathbf{V}_m = \frac{2\mathbf{K}_{f,\mathbf{n}}}{d_f}[\![p]\!]^{\pm} & \text{on } (0,T)\times\Gamma \end{cases}$$

with

$$\partial_t \phi_m = b\operatorname{div}(\partial_t \mathbf{u}) + \frac{1}{M}\partial_t p_m, \qquad \mathrm{d}_f = \mathrm{d}_f^c - [\![\mathbf{u}]\!]_{\mathbf{n}}.$$

Notations: $[\![\mathbf{V}_m]\!]_{\mathbf{n}} = \gamma^+_{\mathbf{n}}\mathbf{V}_m - \gamma^-_{\mathbf{n}}\mathbf{V}_m, \quad [\![p]\!]^{\pm} = \gamma^{\pm} p_m - p_f.$

Poromechanics II

$$
\begin{aligned}
&\varpi(\mathbf{u}) = 2\mu\mathfrak{e}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\ \mathbb{I} && \text{(Effective stress)},\\
&\varpi^{\top}(\mathbf{u}, p_m) = \varpi(\mathbf{u}) - b p_m\ \mathbb{I} && \text{(Total stress)},\\
&\mathbf{T}^{\pm} = \gamma_{\mathbf{n}}^{\pm}\varpi^{\top}(\mathbf{u}, p_m) + p_f \mathbf{n}^{\pm} && \text{(Traction)}.
\end{aligned}
$$

Mechanical equations: quasi-static contact-mechanics for $\mathbf{u}$ with Coulomb friction.

$$
\begin{cases}
-\operatorname{div}\varpi^{\top}(\mathbf{u}, p_m) = \mathbf{f} & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma},\\
\mathbf{T}^{+} + \mathbf{T}^{-} = 0 & \text{on } (0,T)\times\Gamma,\\
T_{\mathbf{n}} \leqslant 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} \leqslant 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} T_{\mathbf{n}} = 0, & \text{on } (0,T)\times\Gamma,\\
|\mathbf{T}_{\boldsymbol{\tau}}| \leqslant -F\ T_{\mathbf{n}} & \text{on } (0,T)\times\Gamma,\\
\mathbf{T}_{\boldsymbol{\tau}} \cdot \partial_t [\![\mathbf{u}]\!]_{\boldsymbol{\tau}} - F\ T_{\mathbf{n}}\, |\partial_t [\![\mathbf{u}]\!]_{\boldsymbol{\tau}}| = 0 & \text{on } (0,T)\times\Gamma.
\end{cases}
$$

Notations: $\boldsymbol{X}_{\boldsymbol{\tau}}$ and $X_{\mathbf{n}}$: tangential and normal components of $\boldsymbol{X}$ along Γ.

Other contact models: no friction ($F = 0$); Tresca friction ($-F\ T_{\mathbf{n}} \rightsquigarrow \mathrm{g}$ and $\partial_t [\![\mathbf{u}]\!] \rightsquigarrow [\![\mathbf{u}]\!]$).

Poromechanics III

Weak formulation for mechanical equations: using Lagrange multiplier $\boldsymbol{\lambda} = -\mathbf{T}^+$ to impose the contact conditions.

Spaces and cone:

$$\mathbf{U}_0 = \{\mathbf{v} \in H^1(\Omega \backslash \overline{\Gamma})^d \ : \ \mathbf{v}_{|\partial\Omega} = 0\},$$

$$\boldsymbol{C}_f(\lambda_\mathbf{n}) = \Big\{\boldsymbol{\mu} \in H^{-1/2}(\Gamma)^d \ : \ \langle \boldsymbol{\mu}, \mathbf{v}\rangle_\Gamma \leq \langle F\lambda_\mathbf{n}, |\mathbf{v}_{\boldsymbol{\tau}}|\rangle_\Gamma \quad \forall \mathbf{v} \in (H^{1/2}(\Gamma))^d \text{ s.t. } v_\mathbf{n} \leq 0\Big\}.$$

Equations: find $\mathbf{u} : [0,T] \to \mathbf{U}_0$ and $\boldsymbol{\lambda} : [0,T] \to \boldsymbol{C}_f(\lambda_n)$ s.t., for all $\mathbf{v} : [0,T] \to \mathbf{U}_0$ and $\boldsymbol{\mu} : [0,T] \to \boldsymbol{C}_f(\lambda_n)$,

$$\int_\Omega (\boldsymbol{\sigma}(\mathbf{u}) : \mathbb{e}(\mathbf{v}) - b\ p_m \mathrm{div}(\mathbf{v})) + \langle \boldsymbol{\lambda}, [\![\mathbf{v}]\!]\rangle_\Gamma + \int_\Gamma p_f\ [\![\mathbf{v}]\!]_\mathbf{n} = \int_\Omega \mathbf{f}\cdot\mathbf{v},$$

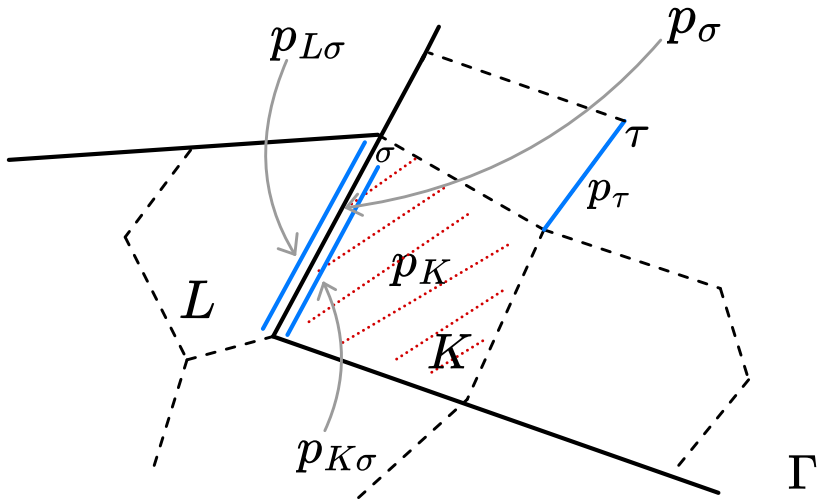
$$\langle \mu_\mathbf{n} - \lambda_\mathbf{n}, [\![\mathbf{u}]\!]_\mathbf{n}\rangle_\Gamma + \langle \boldsymbol{\mu}_{\boldsymbol{\tau}} - \boldsymbol{\lambda}_{\boldsymbol{\tau}}, [\![\partial_t \mathbf{u}]\!]_{\boldsymbol{\tau}}\rangle_\Gamma \leq 0.$$

Outline

Slides:

Mesh and spaces

$\mathcal{M}, \mathcal{F}, \mathcal{V}$ cells (K), faces (σ) and vertices (s). $\mathcal{X}_{\mathrm{z}}$ entities $\mathcal{X}$ on z.



Spaces for matricial and interface pressures: $X_{\mathcal{D}} = X_{\mathcal{D}_m} \times X_{\mathcal{D}_f}$ with

$$X_{\mathcal{D}_m} = \Big\{ p_{\mathcal{D}_m} = \Big((p_K)_{K\in\mathcal{M}}, (p_\sigma)_{\sigma\in\mathcal{F}\setminus\mathcal{F}_\Gamma}, (p_{K\sigma})_{\sigma\in\mathcal{F}_\Gamma, K\in\mathcal{M}_\sigma} \Big) : p_K \in \mathbb{R},\ p_\sigma \in \mathbb{R},\ p_{K\sigma} \in \mathbb{R} \Big\},$$

$$X_{\mathcal{D}_f} = \Big\{ p_{\mathcal{D}_f} = \Big((p_\sigma)_{\sigma\in\mathcal{F}_\Gamma}, (p_e)_{e\in\mathcal{E}_\Gamma} \Big) : p_\sigma \in \mathbb{R},\ p_e \in \mathbb{R} \Big\}.$$

$X^0_{\mathcal{D}}$: with zero boundary conditions.

Reconstruction operators

Gradients: $\nabla_{\mathcal{D}_m} : X_{\mathcal{D}_m} \longrightarrow L^2(\Omega)^d$ such that (with $p_{K\sigma} = p_\sigma$ if $\sigma \notin \Gamma$)

$$(\nabla_{\mathcal{D}_m} p_{\mathcal{D}_m})|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| p_{K\sigma} \mathbf{n}_{K\sigma} + \text{stabilisation}.$$

Similar for $\nabla_{\mathcal{D}_f} : X_{\mathcal{D}_f} \longrightarrow L^2(\Gamma)^d$ along the fracture.

Function reconstructions: $\Pi_{\mathcal{D}_m} : X_{\mathcal{D}_m} \to L^2(\Omega)$ such that

$$(\Pi_{\mathcal{D}_m} p_{\mathcal{D}_m})|_K = p_K.$$

Similar for $\Pi_{\mathcal{D}_f} : X_{\mathcal{D}_f} \to L^2(\Gamma)$.

Jump reconstruction: $[\![\cdot]\!]^\pm : X_{\mathcal{D}} \to L^2(\Gamma)$ such that

$$([\![p_{\mathcal{D}}]\!]^\pm)|_\sigma = p_{K\sigma} - p_\sigma \quad \text{with } K \text{ on the } \pm \text{ side.}$$

Flow scheme

Model:

$$\begin{cases} \partial_t \phi_m - \operatorname{div}\left(\frac{\mathbb{K}_m}{\eta}\nabla p_m\right) = h_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \partial_t \mathrm{d}_f - \operatorname{div}_{\boldsymbol{\tau}}\left(\frac{(d_f)^3}{12\eta}\nabla_{\boldsymbol{\tau}} p_f\right) - [\![\mathbf{V}_m]\!]_{\mathbf{n}} = h_f & \text{on } (0,T)\times\Gamma, \\ \gamma_{\mathbf{n}}^{\pm}\mathbf{V}_m = \frac{2\mathbf{K}_{f,\mathbf{n}}}{d_f}[\![p]\!]^{\pm} & \text{on } (0,T)\times\Gamma \end{cases}$$

Flow scheme

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HFV scheme: Find $(p_{\mathcal{D}}^n)_{n=1,\dots,N} \in (X_{\mathcal{D}}^0)^N$ s.t., $\forall q_{\mathcal{D}} \in X_{\mathcal{D}}^0$, $\forall n = 1,\cdots,N$,

$$\begin{aligned} &\int_\Omega \left(\delta_t^n \phi_{\mathcal{D}}\, \Pi_{\mathcal{D}_m} q_{\mathcal{D}_m} + \frac{\mathbb{K}_m}{\eta}\nabla_{\mathcal{D}_m} p_{\mathcal{D}_m}^n \cdot \nabla_{\mathcal{D}_m} q_{\mathcal{D}_m}\right) \\ &+ \int_\Gamma \left(\delta_t^n \mathrm{d}_{f,\mathcal{D}}\, \Pi_{\mathcal{D}_f} q_{\mathcal{D}_f} + \frac{(d_{f,\mathcal{D}}^{n-1})^3}{12\eta}\nabla_{\mathcal{D}_f} p_{\mathcal{D}_f}^n \cdot \nabla_{\mathcal{D}_f} q_{\mathcal{D}_f}\right) \\ &+ \sum_{\mathfrak{a}\in\{+,-\}} \int_\Gamma \frac{\mathbf{K}_{f,\mathbf{n}}}{d_{f,\mathcal{D}}^{n-1}} [\![p_{\mathcal{D}}^n]\!]_{\mathcal{D}}^{\mathfrak{a}}\, [\![q_{\mathcal{D}}]\!]_{\mathcal{D}}^{\mathfrak{a}} = \int_\Omega h_m \Pi_{\mathcal{D}_m} q_{\mathcal{D}_m} + \int_\Gamma h_f \Pi_{\mathcal{D}_f} q_{\mathcal{D}_f}, \end{aligned}$$

where $\delta_t^n A = \frac{A^n - A^{n-1}}{\delta t}$.

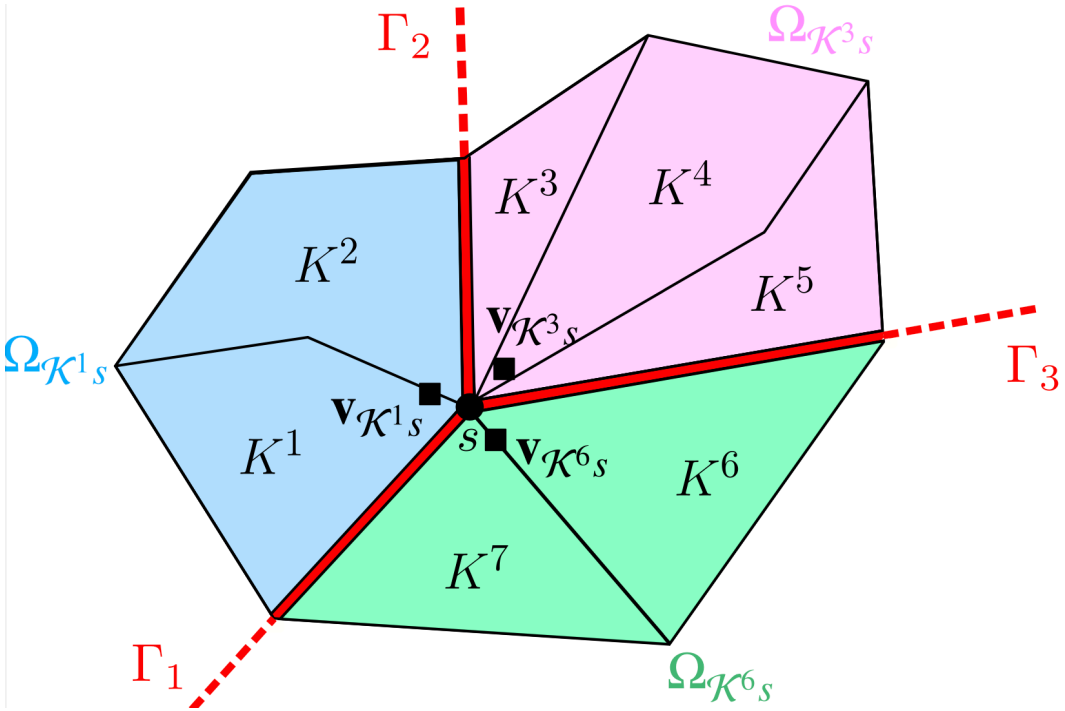
Outline

Slides:

Mesh

Polytopal mesh, compatible with fractures

- $\mathcal{F}_{\Gamma,K}^{+}$ faces of K on positive side of fracture.
- For $s \in \mathcal{V}$, $\mathcal{K}s$: set of cells on the same side of K.
- If $\sigma \in \mathcal{F}_\Gamma$: K on positive side, L on negative side.



Discrete spaces

Displacement: nodal unknowns (discontinuous across fracture) and one bubble on each fracture face (positive side).

$$\mathbf{U}_{0,\mathcal{D}} = \Big\{ \mathbf{v}_{\mathcal{D}} = \left((\mathbf{v}_{\mathcal{K}s})_{K\in\mathcal{M},\, s\in\mathcal{V}_K}, (\mathbf{v}_{K\sigma})_{K\in\mathcal{M},\, \sigma\in\mathcal{F}^+_{\Gamma,K}} \right) : \\ \mathbf{v}_{\mathcal{K}s} \in \mathbb{R}^d,\ \mathbf{v}_{K\sigma} \in \mathbb{R}^d,\ \mathbf{v}_{\mathcal{K}s} = 0 \text{ if } s \in \mathcal{V}^{\text{ext}} \\ \mathbf{v}_{\mathcal{K}s} = \mathbf{v}_{\mathcal{L}s} \text{ if } K, L \text{ are on the same side of } \Gamma \Big\}.$$

Lagrange multipliers: piecewise constant on fracture faces.

$$\mathbf{M}_{\mathcal{D}} = \left\{ \boldsymbol{\lambda}_{\mathcal{D}} \in L^2(\Gamma)^d \ :\ \boldsymbol{\lambda}_\sigma := (\boldsymbol{\lambda}_{\mathcal{D}})_{|\sigma} \text{ is constant for all } \sigma \in \mathcal{F}_\Gamma \right\}.$$

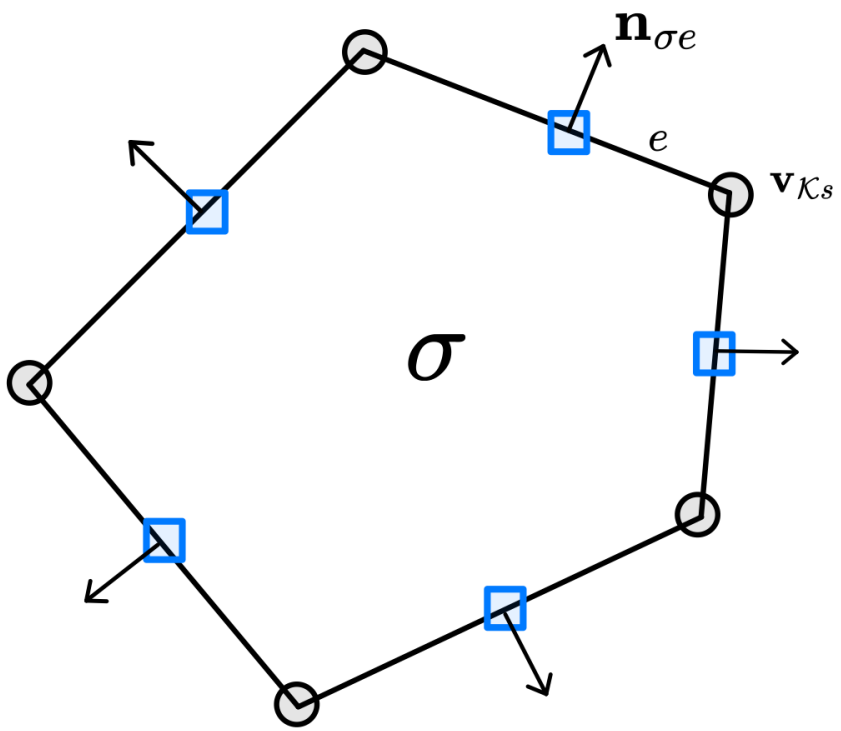
Discrete dual cone:

$$\mathbf{C}_{\mathcal{D}}\left(\lambda_{\mathcal{D},\mathbf{n}}\right) = \left\{ \boldsymbol{\mu}_{\mathcal{D}} = \left(\mu_{\mathcal{D},\mathbf{n}}, \boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}\right) \in \mathbf{M}_{\mathcal{D}} \ :\ \mu_{\mathcal{D},\mathbf{n}} \geq 0,\ |\boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}| \leq F\lambda_{\mathcal{D},\mathbf{n}} \right\}.$$

Reconstructions in $\mathbf{U}_{0,\mathcal{D}}$: faces

- From nodes, reconstruct edge values and use them to define the face gradient:

$$\nabla_{K\sigma}\mathbf{v}_{\mathcal{D}} = \frac{1}{|\sigma|}\sum_{e=s_1s_2\in\mathcal{E}_\sigma}|e|\frac{\mathbf{v}_{\mathcal{K}s_1}+\mathbf{v}_{\mathcal{K}s_2}}{2}\otimes\mathbf{n}_{\sigma e}.$$



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- Reconstruct face averaged displacement:

$$\Pi_{K\sigma}\mathbf{v}_{\mathcal{D}} = \sum_{s\in\mathcal{V}_\sigma}\omega_s^\sigma\mathbf{v}_{\mathcal{K}s}.$$

- Jump reconstructions, with the bubble:

$$[\![\mathbf{v}_{\mathcal{D}}]\!]_\sigma = \underbrace{\Pi_{K\sigma}\mathbf{v}-\Pi_{L\sigma}\mathbf{v}}_{\text{fixed by vertex DOFs}} + \underbrace{\mathbf{v}_{K\sigma}}_{\text{free}}.$$

Reconstructions in $\mathbf{U}_{0,\mathcal{D}}$: cells

Same principles...

- Using reconstructed face values and bubble, define the cell gradient:

$$\nabla_K \mathbf{v}_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| \Pi_{K\sigma} \mathbf{v} \otimes \mathbf{n}_{K\sigma} + \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_{\Gamma,K}^+} |\sigma| \mathbf{v}_{K\sigma} \otimes \mathbf{n}_{K\sigma}.$$

- Reconstruct cell averaged displacement:

$$\Pi_K \mathbf{v}_{\mathcal{D}} = \sum_{s \in \mathcal{V}_K} \omega_s^K \mathbf{v}_{\mathcal{K}s}.$$

Global reconstructions: $[\![\cdot]\!]_{\mathcal{D}}$, $\nabla_{\mathcal{D}}$, $\Pi_{\mathcal{D}}$, $\mathfrak{e}_{\mathcal{D}}$, $\mathrm{div}_{\mathcal{D}}$, $\mathfrak{o}_{\mathcal{D}}$.

Mechanics scheme

Model:

$$\int_\Omega (\varpi(\mathbf{u}) : \mathbb{e}(\mathbf{v}) - b\ p_m \mathrm{div}(\mathbf{v})) + \langle \boldsymbol{\lambda}, [\![\mathbf{v}]\!] \rangle_\Gamma + \int_\Gamma p_f\ [\![\mathbf{v}]\!]_\mathbf{n} = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

$$\langle \mu_\mathbf{n} - \lambda_\mathbf{n}, [\![\mathbf{u}]\!]_\mathbf{n} \rangle_\Gamma + \langle \boldsymbol{\mu}_\boldsymbol{\tau} - \boldsymbol{\lambda}_\boldsymbol{\tau}, [\![\partial_t \mathbf{u}]\!]_\boldsymbol{\tau} \rangle_\Gamma \leq 0.$$

Scheme:

$$\int_\Omega \varpi_\mathcal{D}(\mathbf{u}_\mathcal{D}^n) : \mathbb{e}_\mathcal{D}(\mathbf{v}_\mathcal{D}) + S_{\mu,\lambda,\mathcal{D}}(\mathbf{u}_\mathcal{D}^n, \mathbf{v}_\mathcal{D}) - \int_\Omega b \Pi_{\mathcal{D}_m} p_{\mathcal{D}_m}^n \mathrm{div}_\mathcal{D} \mathbf{v}_\mathcal{D}$$
$$+ \int_\Gamma \Pi_{\mathcal{D}_f} p_{\mathcal{D}_f}^n [\![\mathbf{v}_\mathcal{D}]\!]_{\mathcal{D},\mathbf{n}} + \int_\Gamma \boldsymbol{\lambda}_\mathcal{D}^n \cdot [\![\mathbf{v}_\mathcal{D}]\!]_\mathcal{D} = \sum_{K \in \mathcal{M}} \int_K \mathbf{f}_K^n \cdot \Pi_\mathcal{D} \mathbf{v}_\mathcal{D},$$

$$\int_\Gamma \left((\mu_{\mathcal{D},\mathbf{n}} - \lambda_{\mathcal{D},\mathbf{n}}^n) [\![\mathbf{u}_\mathcal{D}^n]\!]_{\mathcal{D},\mathbf{n}} + (\boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}} - \boldsymbol{\lambda}_{\mathcal{D},\boldsymbol{\tau}}^n) \cdot [\![\delta_t^n \mathbf{u}_\mathcal{D}]\!]_{\mathcal{D},\boldsymbol{\tau}} \right) \leq 0.$$

Note: can also be written in virtual elements framework.

Coupling equations

Model:

$$\partial_t \phi_m = b \mathrm{div}(\partial_t \mathbf{u}) + \frac{1}{M} \partial_t p_m,$$
$$\mathrm{d}_f = \mathrm{d}_f^c - [\![\mathbf{u}]\!]_{\mathbf{n}}.$$

Discretisation:

$$\delta_t^n \phi_{\mathcal{D}} = b \ \mathrm{div}_{\mathcal{D}} \left(\delta_t^n \mathbf{u}_{\mathcal{D}} \right) + \frac{1}{M} \delta_t^n \Pi_{\mathcal{D}_m} p_{\mathcal{D}_m},$$
$$\mathrm{d}_{f,\mathcal{D}}^n = \mathrm{d}_{f,\mathcal{D}}^c - [\![\mathbf{u}_{\mathcal{D}}^n]\!]_{\mathcal{D},\mathbf{n}},$$

Outline

Slides:

On the fully coupled system

Theorem (Discrete energy estimate)

For all $n = 1, \cdots, N$,

$$
\begin{aligned}
&\delta_t^n \int_\Omega \frac{1}{2}\Big(\varpi_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) : \mathfrak{C}_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) + S_{\mu,\lambda,\mathcal{D}}(\mathbf{u}_{\mathcal{D}}, \mathbf{u}_{\mathcal{D}}) + \frac{1}{M}|\Pi_{\mathcal{D}_m} p_{\mathcal{D}_m}|^2\Big) \\
&+ \int_\Gamma F \lambda^n_{\mathcal{D},\mathbf{n}} |[\![\delta_t^n \mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D},\boldsymbol{\tau}}| \\
&+ \int_\Omega \frac{\mathbb{K}_m}{\eta} \nabla_{\mathcal{D}_m} p^n_{\mathcal{D}_m} \cdot \nabla_{\mathcal{D}_m} p^n_{\mathcal{D}_m} + \int_\Gamma \frac{(\mathrm{d}^{n-1}_{f,\mathcal{D}})^3}{12\eta} |\nabla_{\mathcal{D}_f} p^n_{\mathcal{D}_f}|^2 \\
&+ \sum_{\mathfrak{a}\in\{+,-\}} \int_\Gamma \frac{\mathbf{K}_{f,\mathbf{n}}}{\mathrm{d}^{n-1}_{f,\mathcal{D}}} ([\![p^n_{\mathcal{D}}]\!]^{\mathfrak{a}}_{\mathcal{D}})^2 \\
&\leq \int_\Omega h_m \Pi_{\mathcal{D}_m} p^n_{\mathcal{D}_m} + \int_\Gamma h_f \nabla_{\mathcal{D}_f} p^n_{\mathcal{D}_f} + \sum_{K\in\mathcal{M}} \int_K \mathbf{f}^n_K \cdot \Pi_{\mathcal{D}} \delta_t^n \mathbf{u}_{\mathcal{D}}.
\end{aligned}
$$

Moreover, $\mathrm{d}^n_{f,\mathcal{D}} \geq \mathrm{d}^c_{f,\mathcal{D}}$.

On the pure mechanical model with Tresca friction

Theorem (Error estimate for Tresca)

If $\mathbf{u} \in H^2(\mathcal{M})$ *and* $\boldsymbol{\lambda} \in H^1(\mathcal{F}_\Gamma)$, *then*

$$\|\nabla_{\mathcal{D}} \mathbf{u}_{\mathcal{D}} - \nabla \mathbf{u}\|_{L^2(\Omega \setminus \overline{\Gamma})} + \|\boldsymbol{\lambda}_{\mathcal{D}} - \boldsymbol{\lambda}\|_{-1/2,\Gamma} \lesssim C_{\mathbf{u},\boldsymbol{\lambda}} h_{\mathcal{D}}.$$

- $\|\cdot\|_{-1/2,\Gamma}$ discrete $H^{-1/2}$-like seminorm.
- Error analysis based on consistency and stability.

Tools: Körn inequality and inf-sup conditions, ensured by the bubble degrees of freedom.

Tools: Korn and inf-sup

Discrete H^1-norm:

$$\|\mathbf{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 := \|\nabla_{\mathcal{D}}\mathbf{v}_{\mathcal{D}}\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{M}} S_K(\mathbf{v}_{\mathcal{D}},\mathbf{v}_{\mathcal{D}}).$$

Theorem (Discrete Korn inequality)

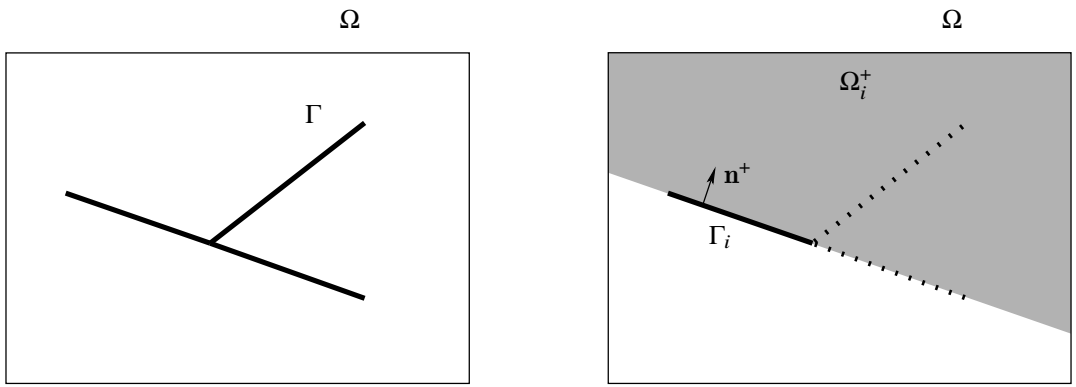
For all $\mathbf{v}\in\mathbf{U}_{0,\mathcal{D}}$,

$$\|\mathbf{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 \lesssim \|\mathfrak{E}_{\mathcal{D}}(\mathbf{v}_{\mathcal{D}})\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{M}} S_K(\mathbf{v}_{\mathcal{D}},\mathbf{v}_{\mathcal{D}}).$$

Theorem (Discrete inf-sup condition)

$$\sup_{\mathbf{v}_{\mathcal{D}}\in\mathbf{U}_{0,\mathcal{D}}\setminus\{0\}} \frac{\int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}}\cdot[\![\mathbf{v}_{\mathcal{D}}]\!]_{\mathcal{D}}}{\|\mathbf{v}_{\mathcal{D}}\|_{1,\mathcal{D}}} \gtrsim \|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma} \qquad \forall\boldsymbol{\lambda}_{\mathcal{D}}\in\mathbf{M}_{\mathcal{D}}.$$

Ideas for discrete inf–sup property I



$$\|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma} := \sum_{i\in I} \sup_{\mathbf{v}_i \in H^1(\Omega_i^+;\Gamma_i)^d\setminus\{0\}} \frac{\int_{\Gamma_i} \boldsymbol{\lambda}_{\mathcal{D}}\cdot\mathbf{v}_i}{\|\mathbf{v}_i\|_{H^1(\Omega_i^+)}}.$$

Ingredients for inf-sup:

- Clément-like H^1-stable interpolator adapted to fractures.
- Fortin property for jump: for $\mathbf{v}_i \in H^1(\Omega_i^+;\Gamma_i)$ and $\mathbf{v}_{\mathcal{D}}$ = interpolant of extension by 0 of $\mathbf{v}_i$ by 0,

$$\int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}}\cdot[\![\mathbf{v}_{\mathcal{D}}]\!]_{\mathcal{D}} = \int_{\Gamma_i} \boldsymbol{\lambda}_{\mathcal{D}}\cdot\mathbf{v}_i.$$

Outline

Slides:

Outline

Static contact (Tresca), 3D manufactured solution I

Setting:

- $\Omega = (-1, 1)^3$, $\Gamma = \{0\} \times (-1, 1)^2$.
- $\mathrm{g} = 1$, $\mu = \lambda = 1$.
- Explicit analytical solution such that:
  - sticky-contact for $z < 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $[\![u]\!]_{\boldsymbol{\tau}} = 0$)
  - slippy-contact for $z > 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $|[\![u]\!]_{\boldsymbol{\tau}}| > 0$)
- Cartesian, tetrahedral and generalised hexahedral meshes.

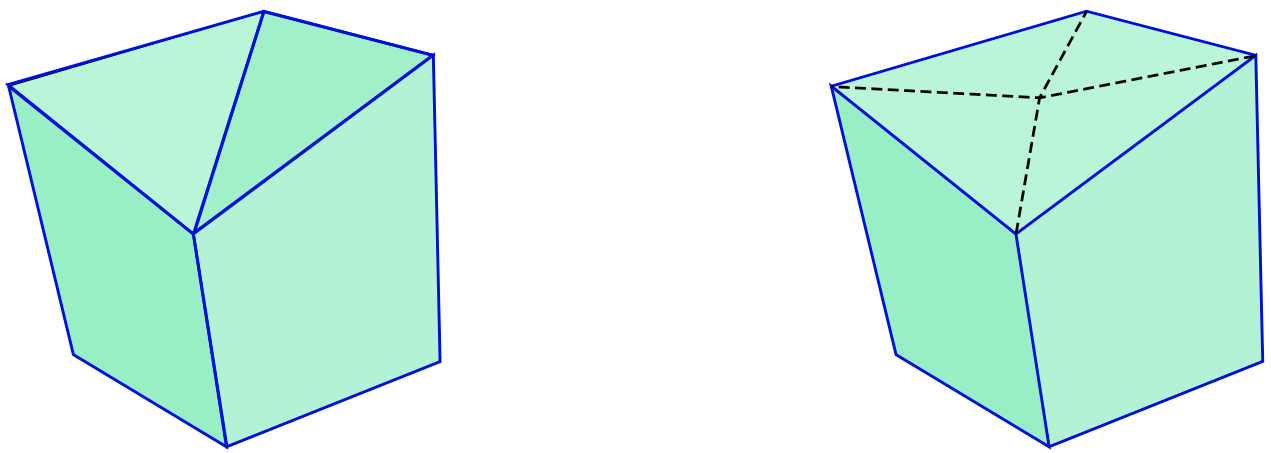
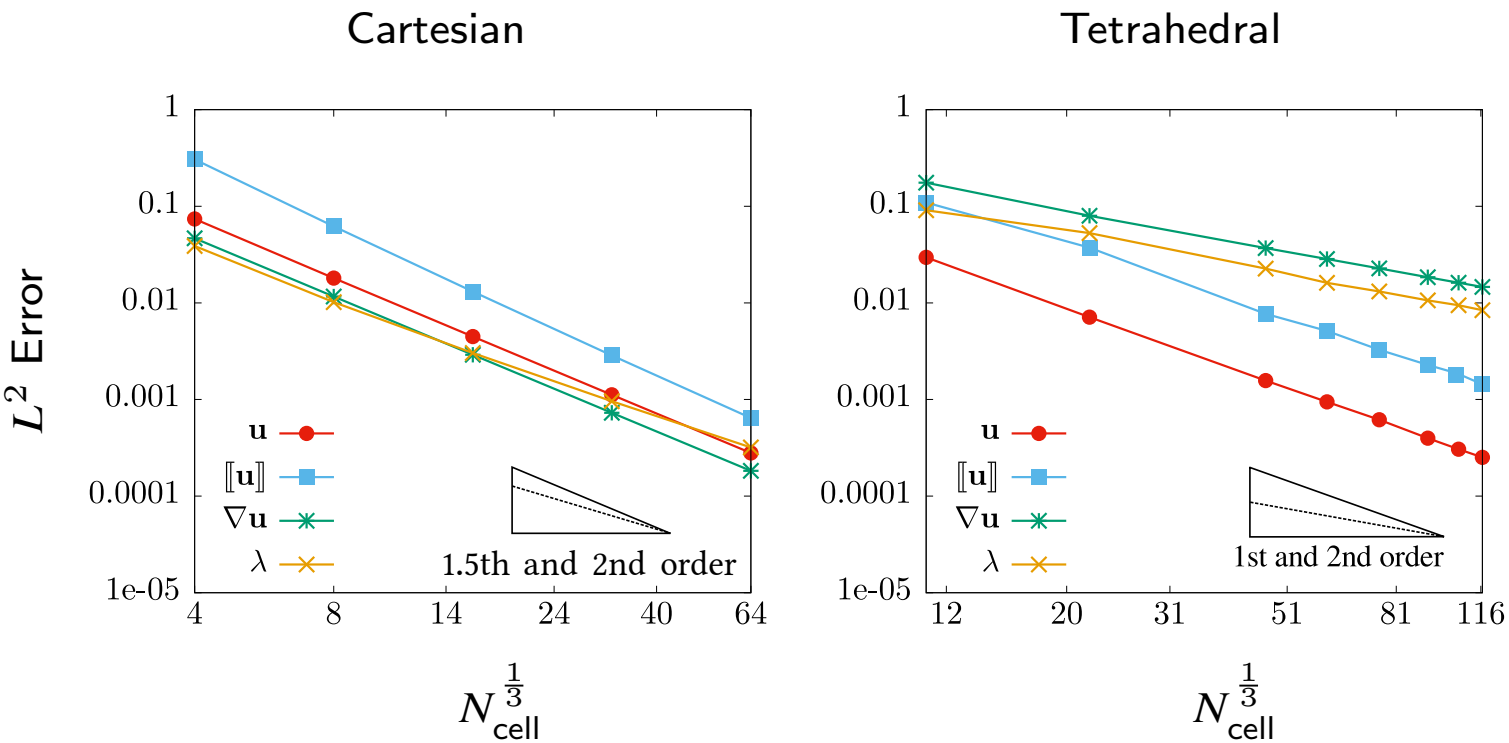


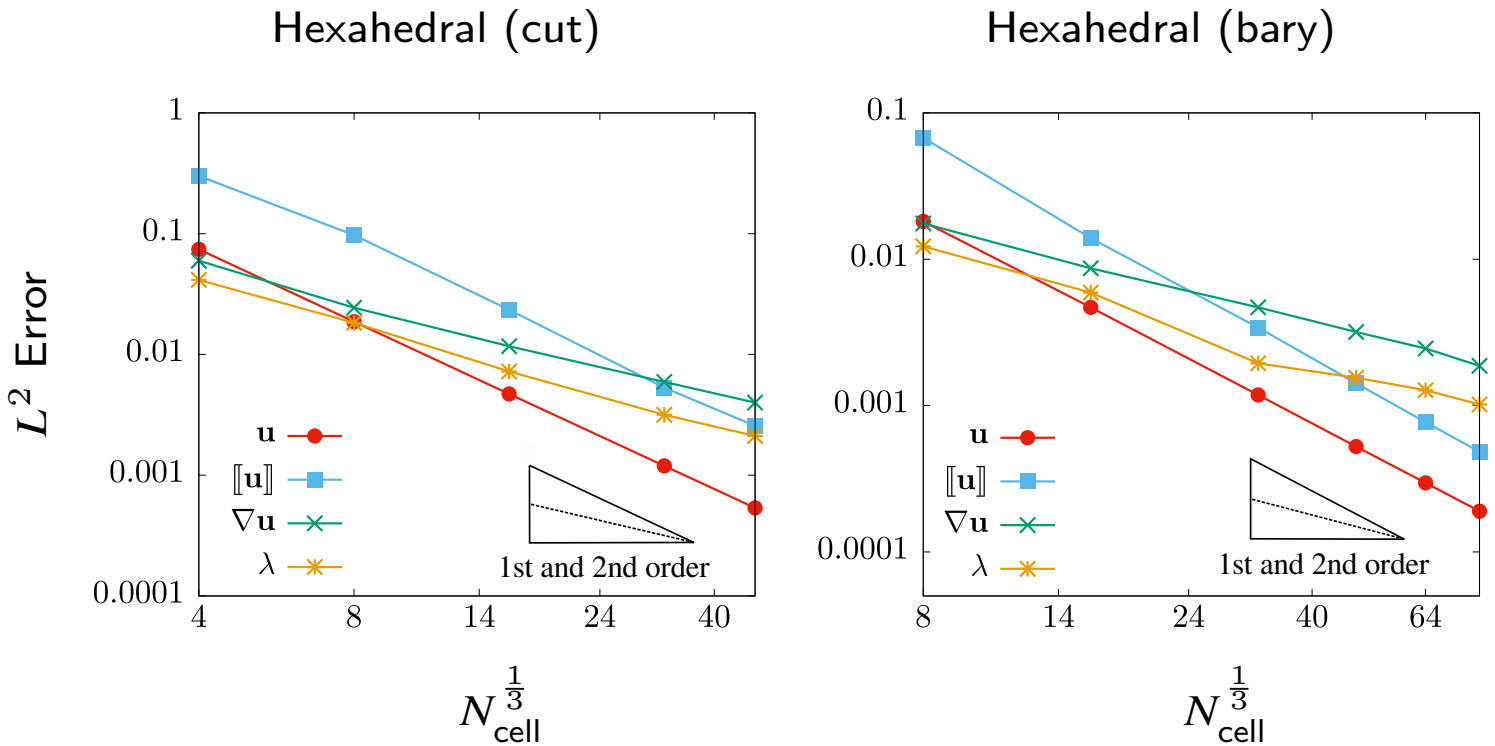
Figure: Generalised hexahedral meshes: cut (left) and barycentric subdivisions (right).

Static contact (Tresca), 3D manufactured solution II



Note: 10^{-2} accuracy for $\mathbf{u}$ achieved with $\sim 30^3$ Cartesian cells, $\sim 60^3$ triangular cells.

Static contact (Tresca), 3D manufactured solution III



Note: 10^{-2} accuracy for $\mathbf{u}$ achieved with $\sim 30^3$ Hexahedral cells.

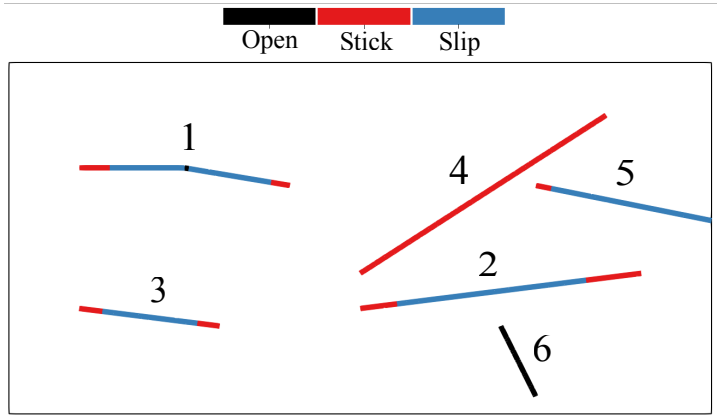
Outline

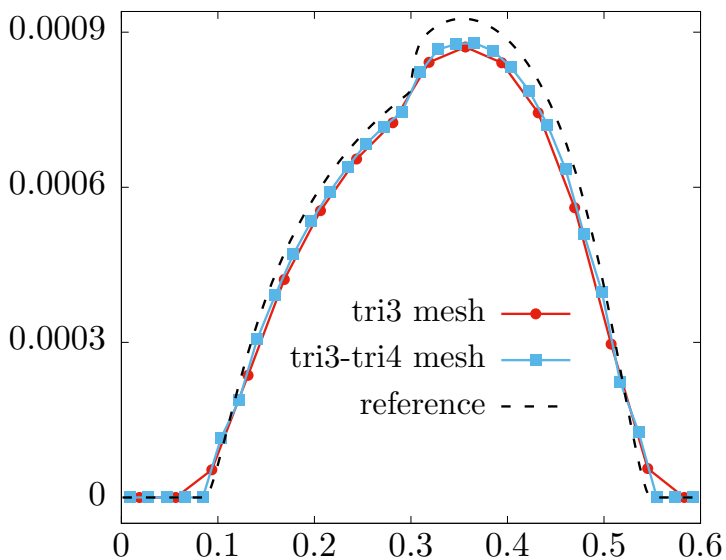
Setting

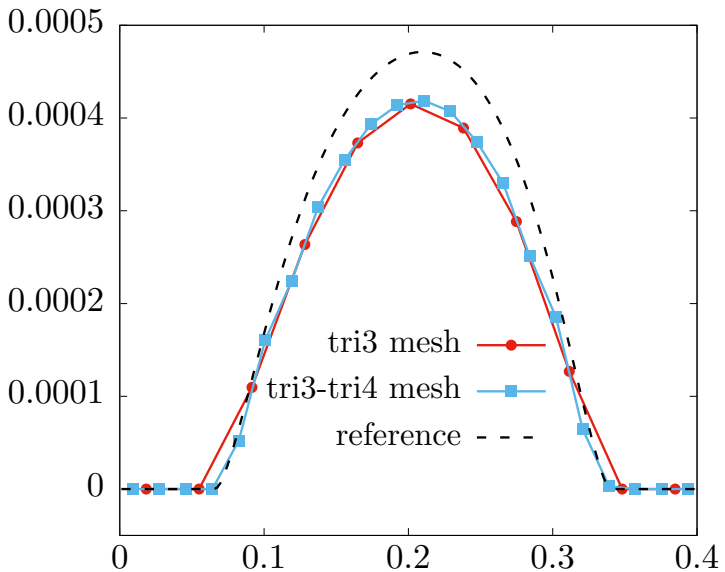
- 6 fractures in a 2m×1m quadrangle.
- Displacement: $\mathbf{u} = 0$ at the bottom, $\mathbf{u} = [0.005m, -0.002m]$ at the top.
- $E = 4$GPa, $\nu = 0.2$, friction variable in the fractures.
- Triangular meshes, reference solution computed with 730 880 triangles.

Focus on stick-slip states of the fractures

Results







tri3: triangles.
tri3-tri4: triangles with one additional node at fracture edges (polygons).

Outline

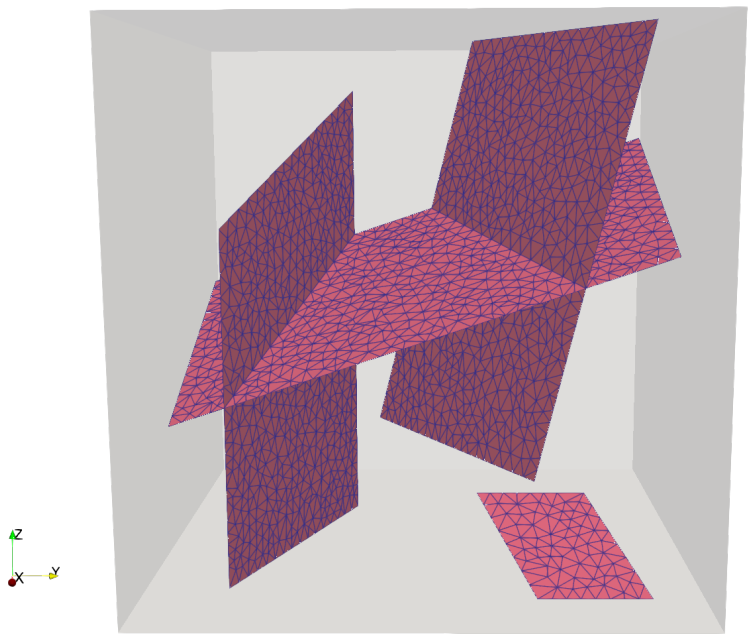
Setting

Data: $E = 4$Gpa, $\nu = 0.2$, $F = 0.5$, $b = 0.5$, $M = 10$GPa.

Dirichlet BC: $\mathbf{u} = 0$ at bottom,

$$\mathbf{u}(t, x, y, 1) = \begin{cases} 0.002 \times [1, 1, -1]^\top \frac{2t}{T} & \text{if } t \leq T/2, \\ 0.002 \times [1, 1, -1]^\top & \text{if } t \geq T/2. \end{cases}$$

Fracture network:



Results I

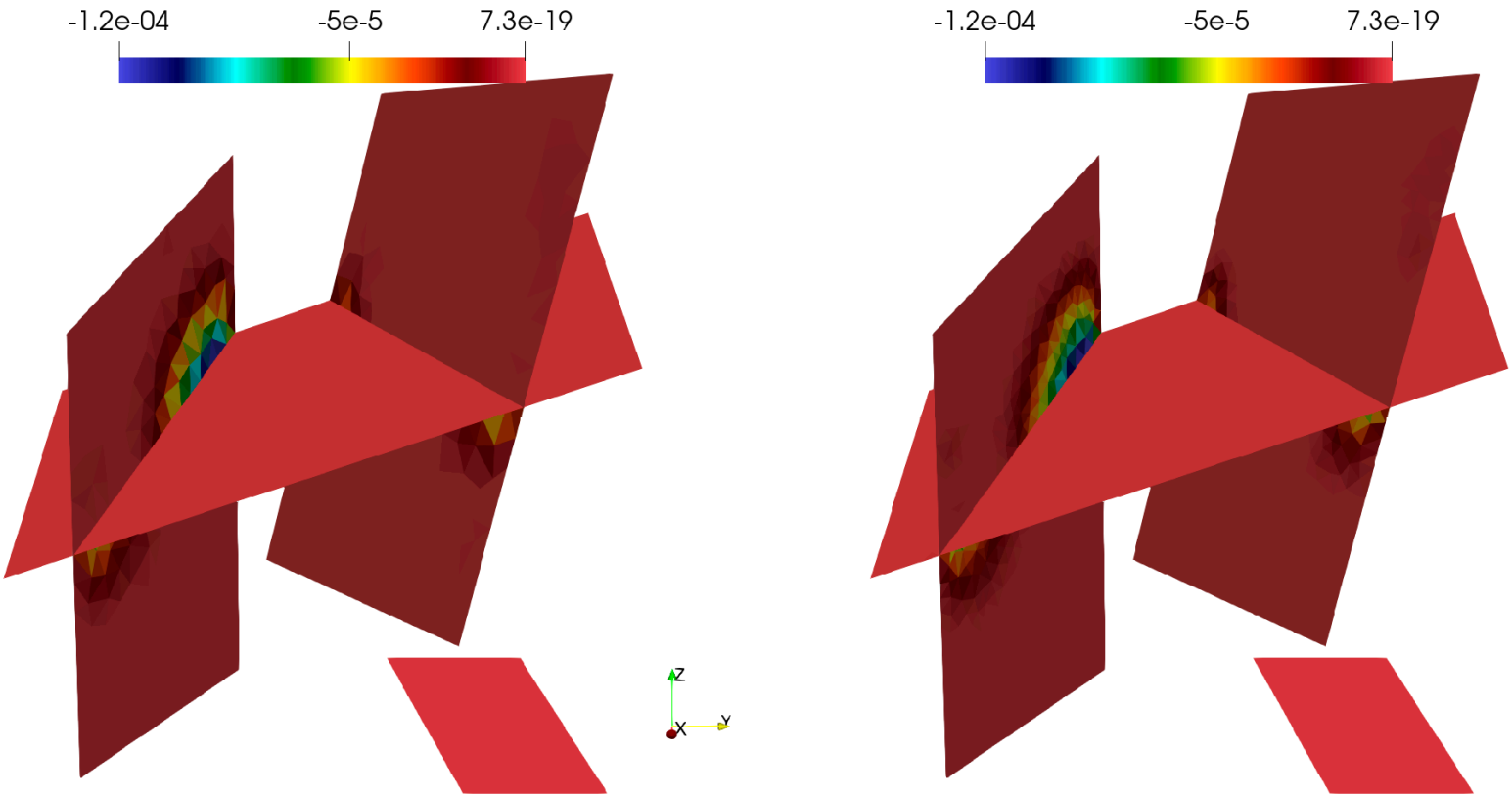


Figure: Normal displacement jumps using 47k cells (left) and 127k cells (right).

Results II

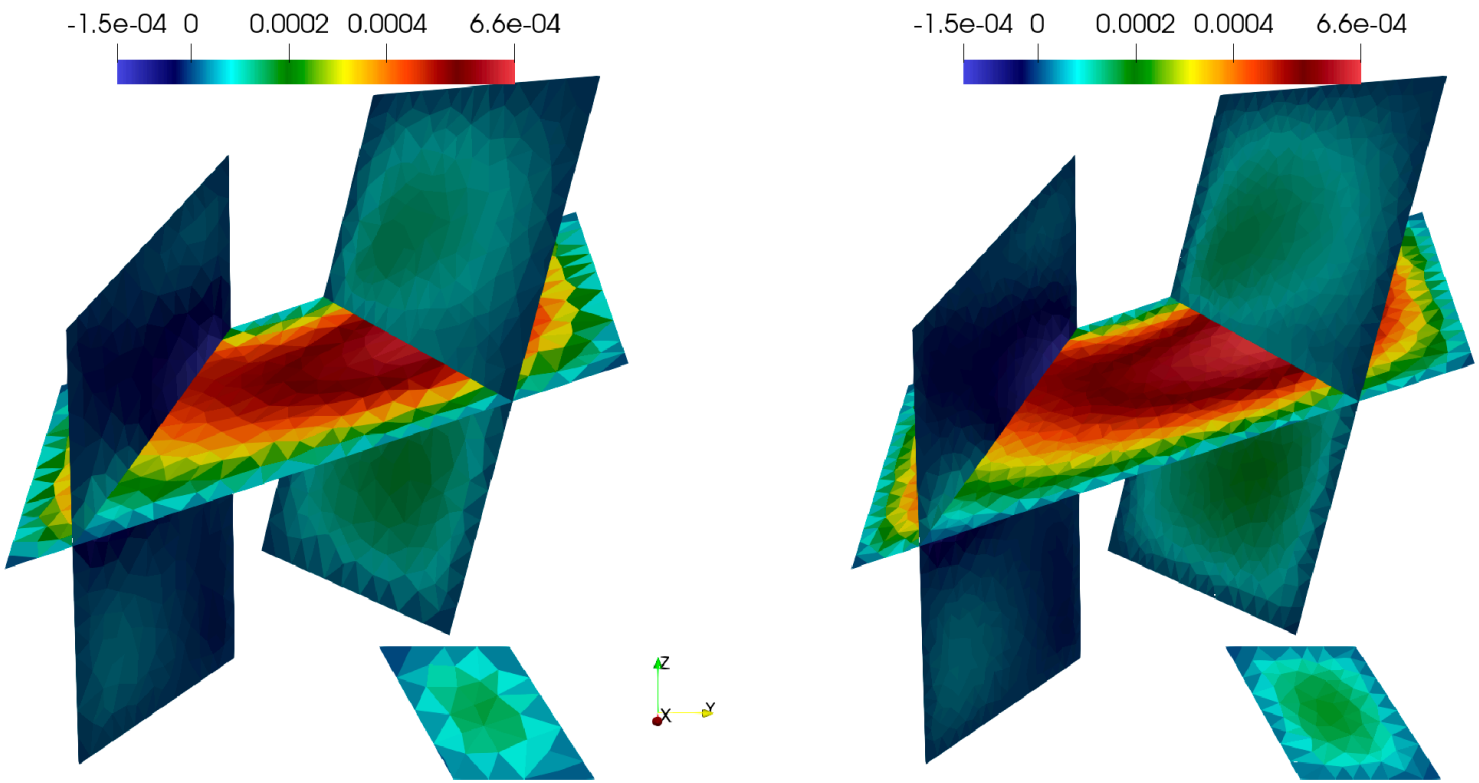


Figure: Tangential displacement jumps (one direction) using 47k cells (left) and 127k cells (right).

Conclusions

- Polytopal scheme, applicable on generic meshes (including hanging nodes, cut cells, local refinements). Seamlessly handles crossing fractures, etc.
- Bubble enrichment (first one for polytopal methods) to ensure inf-sup conditions to bound Lagrange multipliers.
- Complete analysis for mechanical models.
- Robust simulations (including solver behaviour) for 3D poromechanical model with network of fractures.
- Ongoing work: extension to arbitrary order of approximation.

  $\rightsquigarrow$ **Ritesh's presentation, MS113, Friday 10am-12noon (Leacock 232)**

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Thank you for your attention!