Topics around a polytopal complex: from design to Maxwell compactness

Jérôme Droniou

Institute of Mathematics Alexander Grothendieck, CNRS, France,
School of Mathematics, Monash University, Australia

https://imag.umontpellier.fr/~droniou/

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Outline

Slides

Stokes in "standard" formulation

- Ω domain, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Find $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} -\nu \boldsymbol{\Delta} \boldsymbol{u} + \mathbf{grad}\, p &= \boldsymbol{f} \quad \text{in } \Omega, && \text{(momentum conservation)} \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, && \text{(mass conservation)} \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega, && \text{(boundary condition)} \end{aligned}$$

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\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in H_0(\mathbf{grad}, \Omega)^d \times L^2(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\nu(\mathbf{grad}\, \boldsymbol{u}, \mathbf{grad}\, \boldsymbol{v})_{L^2} - (p, \operatorname{div} \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in H_0(\mathbf{grad}, \Omega)^d, \\
(\operatorname{div} \boldsymbol{u}, q)_{L^2} &= 0 && \forall q \in L^2(\Omega)
\end{aligned}
$$

Note: for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \operatorname{div}\}$,

$$
H(\bullet, \Omega) := \{z \in L^2(\Omega) \,:\, \bullet z \in L^2(\Omega)\}
$$

and 0 indicates zero boundary conditions.

Stokes in "standard" formulation

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- A priori estimates require: ($a \lesssim b$ means $a \leq Cb$ with C independent of a, b.)

  - Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{grad} \cdot\|_{L^2}$ on $H_0(\mathbf{grad}, \Omega)$,
  - inf-sup $\sup_{\boldsymbol{v} \in H_0(\mathbf{grad}, \Omega)} \frac{(p, \operatorname{div} \boldsymbol{v})_{L^2}}{\|\boldsymbol{v}\|_{H_0(\mathbf{grad}, \Omega)}} \geq C\|p\|_{L^2}$, equivalent to:

$$\operatorname{Im} \operatorname{div} = L^2(\Omega)$$

Stokes in curl-curl formulation: weak form

- Recasting the Stokes equations:

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,u)}^{-\nu\boldsymbol{\Delta}\boldsymbol{u}} + \mathbf{grad}\,p = \boldsymbol{f} &\quad \text{in } \Omega, && \text{(momentum cons.)}\\
\mathrm{div}\,\boldsymbol{u} = 0 &\quad \text{in } \Omega, && \text{(mass cons.)}\\
\mathbf{curl}\,\boldsymbol{u} \times \mathbf{n} = \mathbf{0} \text{ and } \boldsymbol{u}\cdot\mathbf{n} = 0 &\quad \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\textstyle\int_\Omega p = 0 &
\end{aligned}
$$

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\textstyle\int_\Omega p &= 0
\end{aligned}$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H(\mathbf{grad},\Omega)$ such that $\int_\Omega p = 0$ and

$$\begin{aligned}
\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H(\mathbf{grad},\Omega).
\end{aligned}$$

Stokes equations in curl-curl formulation: stability

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H(\mathbf{grad}, \Omega) \end{aligned}$$

- Make $\boldsymbol{v} = \mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad} = 0$.
- Make $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$:

$$\nu\|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}^2 \leq \|\boldsymbol{f}\|_{L^2}\|\boldsymbol{u}\|_{L^2}.$$

Stokes equations in curl-curl formulation: stability

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H(\mathbf{grad}, \Omega) \end{aligned}$$

- Make $\boldsymbol{v} = \mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2} \le \|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad} = 0$.
- Make $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$:

$$\nu\|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}^2 \le \|\boldsymbol{f}\|_{L^2}\|\boldsymbol{u}\|_{L^2}.$$

- If Ω does not have any tunnel,

$$\boxed{\mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl}.}$$

The incompressibility gives $\boldsymbol{u} \perp \mathrm{Im}\,\mathbf{grad}$, so $\boldsymbol{u} \in (\mathrm{Ker}\,\mathbf{curl})^\perp$ and the

Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{curl}\,\cdot\|_{L^2}$ on $(\mathrm{Ker}\,\mathbf{curl})^\perp$

yields

$$\|\boldsymbol{u}\|_{L^2} \lesssim \|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}.$$

Take-home message

Analysis of these models requires:

- Poincaré inequalities: $\|z\|_{L^2} \lesssim \|\bullet z\|_{L^2}$ for $z \in (\mathrm{Ker}\,\bullet)^\perp$.
- Structural properties of the de Rham complex

$$\boxed{H(\mathbf{grad},\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)}$$

Take-home message

Analysis of these models requires:

- Poincaré inequalities: $\|z\|_{L^2} \lesssim \|\bullet z\|_{L^2}$ for $z \in (\mathrm{Ker}\,\bullet)^\perp$.
- Structural properties of the de Rham complex

$$H(\mathbf{grad}, \Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}; \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

Preserving those at the discrete level is essential to design robust schemes.

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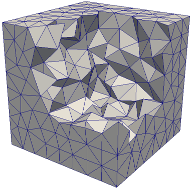


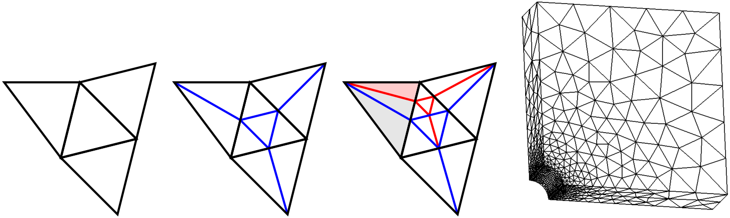
$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\quad \mathbf{grad} \quad} & V_h^1 & \xrightarrow{\quad \mathbf{curl} \quad} & V_h^2 & \xrightarrow{\quad \mathrm{div} \quad} & V_h^3 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ H(\mathbf{grad},\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \end{array}$$

Example: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

- Gluing only works on special meshes...



- ∘ Approach limited to conforming meshes with standard elements

$\Longrightarrow$ local refinement requires to trade mesh size for mesh quality

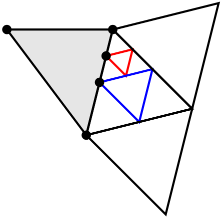
$\Longrightarrow$ complex geometries may require a large number of elements

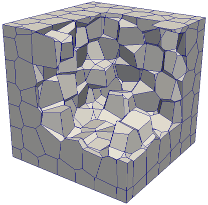
$\Longrightarrow$ the element shape cannot be seamlessly adapted to the solution (e.g. hexahedra in boundary layers + tetrahedra in the bulk for CFD simulations)

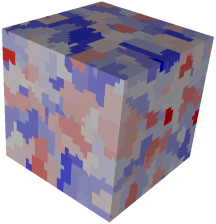
- ∘ Need for (global) basis functions

$\Longrightarrow$ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes





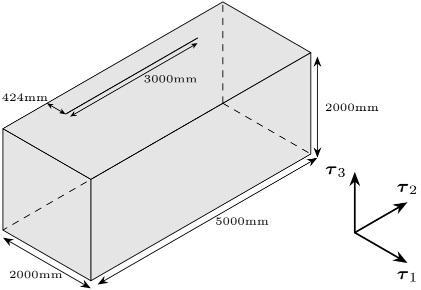


- Local refinement is easy, and preserves mesh regularity.
- Agglomeration of elements (e.g., for multigrid methods) is seamless.
- High-level approach can lead to leaner methods (fewer DOFs).
- Can be combined with standard Finite Elements on hybrid meshes (made of tetrahedra/hexahedra + polyhedral elements).

A practical example from CEA-CESTA

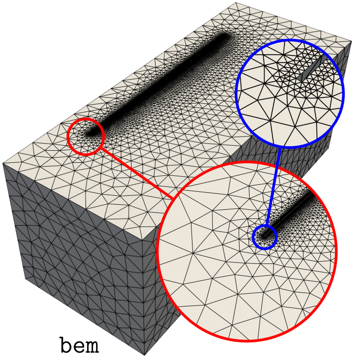
[Touzalin, 2025]

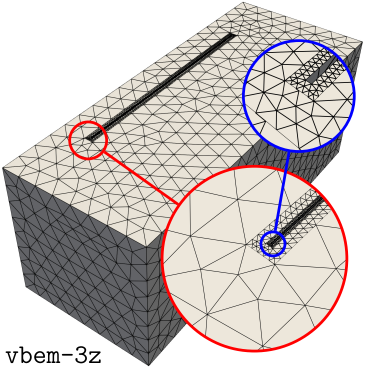
Problem: use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



A practical example from CEA-CESTA

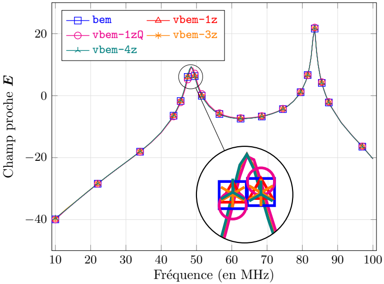
Meshes: conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).

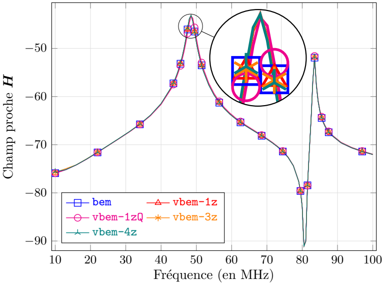




A practical example from CEA-CESTA

Accuracy: comparison of modulus of reflected near fields at the top.





Computational cost

Method	*Assembly*	*Resolution*
bem	813s	125s
vbem-3z	321s	19s

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Overview of the Discrete De Rham (DDR) complex

$$\mathbb{R} \longrightarrow \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0$$

- Fully discrete complex (not sub-complex) of bespoke finite-dimensional spaces and operators.
- Discrete spaces not made of functions but:
  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Overview of the Discrete De Rham (DDR) complex

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & C^\infty(\overline{\Omega}) \arrow[r, "\mathbf{grad}"] \arrow[d, "\underline{I}^k_{\mathbf{grad},h}"] & C^\infty(\overline{\Omega})^3 \arrow[r, "\mathbf{curl}"] \arrow[d, "\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}"] & C^\infty(\overline{\Omega})^3 \arrow[r, "\mathrm{div}"] \arrow[d, "\underline{\boldsymbol{I}}^k_{\mathrm{div},h}"] & C^\infty(\overline{\Omega}) \arrow[r, "0"] \arrow[d, "I^k_{L^2,h}"] & 0 \\
\mathbb{R} \arrow[r] & \underline{X}^k_{\mathbf{grad},h} \arrow[r, "\underline{\boldsymbol{G}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \arrow[r, "\underline{\boldsymbol{C}}^k_h"] & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \arrow[r, "D_h"] & \mathcal{P}^k(\mathcal{T}_h) \arrow[r, "0"] & 0
\end{tikzcd}
$$

- Fully discrete complex (not sub-complex) of bespoke finite-dimensional spaces and operators.
- Discrete spaces not made of functions but:
  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.
  - Interpolators $\underline{\boldsymbol{I}}^k_{\bullet,h}$ give meaning to these polynomials/DOFs as moments.

Guiding principles for the construction

Joint work with D. Di Pietro and F. Rapetti.
(Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023].)

- Hierarchical construction: from vertices, to edges, to faces, to elements.

- Enhancement: on each (relevant) mesh entity,
  - discrete differential operator first,
  - potential reconstruction using the discrete differential operator.

  (both polynomially consistent, both based on IBP formulas.)

- The definition of the spaces (DOFs) also guided by these IBP formulas.

Same guiding principles as the Hybrid High-Order (HHO) method [Di Pietro et al., 2014], [Di Pietro and Droniou, 2020].

Outline

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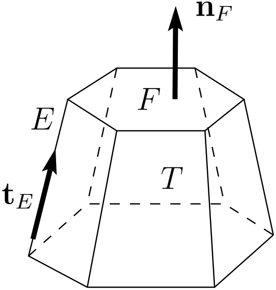
Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements (T), faces (F), edges (E), vertices (V), with intrinsic orientations (tangent, normal).
- $\mathcal{P}^\ell(X)$ polynomial of degree $\leq \ell$ on $X = T, F, E$.



$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

  with derivatives in the direction $\mathbf{t}_E$.

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi^{k-1}_{\mathcal{P},E} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

  with $\pi^{k-1}_{\mathcal{P},E}$ the L^2-projection on $\mathcal{P}^{k-1}(E)$.

$\mathcal{P}^k$-consistent gradient
Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q \, r' + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \,:\, q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R} \right\},$$

$$\underline{I}_{\mathbf{grad},E}^k q = (\pi_{\mathcal{P},E}^{k-1} q, (q(\boldsymbol{x}_V))_{V \in \mathcal{V}_E}) \qquad \forall q \in C(\overline{E}).$$

$\mathcal{P}^k$-consistent gradient
Edge E

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$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q\, r' + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \,:\, q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R} \right\}$.
- Edge gradient $G_E^k \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

$\mathcal{P}^k$-consistent gradient
Edge E

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- Potential reconstruction $\gamma^{k+1}_E \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma^{k+1}_E \underline{q}_E) z' = -\int_E (G^k_E \underline{q}_E) z + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}) \qquad \forall z \in \mathcal{P}^{k+2}(E).$$

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q\, r' + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \, : \, q_E \in \mathcal{P}^{k-1}(E)\,,\; q_V \in \mathbb{R} \right\}$.
- Edge gradient $G_E^k \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

- Potential reconstruction $\gamma_E^{k+1} \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma_E^{k+1} \underline{q}_E) z' = -\int_E (G_E^k \underline{q}_E) z + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}) \qquad \forall z \in \mathcal{P}^{k+2}(E).$$

> Polynomially consistent: $G_E^k \underline{I}_{\mathbf{grad},E}^k q = q'$ and $\gamma_E^{k+1} \underline{I}_{\mathbf{grad},E}^k q = q$ for all $q \in \mathcal{P}^{k+1}(E)$.

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F q \underbrace{\mathrm{div}_F\,\boldsymbol{v}}_{\in\mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall\boldsymbol{v}\in\mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$-consistent gradient
Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in\mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$$

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$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \;\operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},F}^k = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) : \\ q_F \in \mathcal{P}^{k-1}(F)\,,\; q_E \in \mathcal{P}^{k-1}(E)\,,\; q_V \in \mathbb{R} \Big\},$$

$$\underline{I}_{\mathbf{grad},F}^k q = (\pi_{\mathcal{P},F}^{k-1} q, (\pi_{\mathcal{P},E}^{k-1} q_{|E})_{E\in\mathcal{E}_F}, (q(\boldsymbol{x}_V))_{V\in\mathcal{V}_F}) \qquad \forall q \in C(\overline{F}).$$

$\mathcal{P}^k$-consistent gradient
Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi^{k-1}_{\mathcal{P},F} q\ \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v}\in\mathcal{P}^k(F)^2.$$

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ :$
$$q_F \in \mathcal{P}^{k-1}(F)\,,\ q_E\in\mathcal{P}^{k-1}(E)\,,\ q_V\in\mathbb{R}\Big\},$$
- Face gradient $\mathsf{G}^k_F : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathsf{G}^k_F \underline{q}_F)\cdot\boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v}\in\mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$-consistent gradient

Face F

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ :$
$$q_F \in \mathcal{P}^{k-1}(F)\,,\ q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R} \Big\},$$
- Face gradient $\mathbf{G}^k_F : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.
$$\int_F (\mathbf{G}^k_F \underline{q}_F)\cdot\boldsymbol{v} = -\int_F q_F \,\mathrm{div}_F\, \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$
- Potential reconstruction $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.
$$\int_F (\gamma^{k+1}_F \underline{q}_F)\,\mathrm{div}_F\, \boldsymbol{z} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{z} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{z}\cdot\mathbf{n}_{FE}$$
$$\forall \boldsymbol{z} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k+2}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

$\mathcal{P}^k$-consistent gradient
Face F

- Space: $\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \; : \; q_F \in \mathcal{P}^{k-1}(F)\,,\; q_E \in \mathcal{P}^{k-1}(E)\,,\; q_V \in \mathbb{R} \Big\},$
- Face gradient $\mathbf{G}^k_F : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}^k_F \underline{q}_F)\cdot\boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{v}\cdot\mathbf{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Potential reconstruction $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma^{k+1}_F \underline{q}_F) \operatorname{div}_F \boldsymbol{z} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{z} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{z}\cdot\mathbf{n}_{FE}$$
$$\forall \boldsymbol{z} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k+2}(F) := (\boldsymbol{x}-\boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

Polynomially consistent: $\mathbf{G}^k_F \underline{I}^k_{\mathbf{grad},E} q = \mathbf{grad}\, q$ and $\gamma^{k+1}_F \underline{I}^k_{\mathbf{grad},E} q = q$ for all $q \in \mathcal{P}^{k+1}(F)$.

Same principle! Based on IBP we determine:

- An additional unknown ($q_T \in \mathcal{P}^{k-1}(T)$) to get the space $\underline{X}^k_{\mathbf{grad},T}$, and its meaning (polynomial moment on T) to get the interpolator $\underline{I}^k_{\mathbf{grad},T}$.
- A formula for the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3$.
- A potential reconstruction $P^{k+1}_{\mathbf{grad},T} : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T)$.

$\underline{X}^k_{\mathbf{grad},h}$, the discrete $H(\mathbf{grad},\Omega)$ space

- Discrete $H(\mathbf{grad},\Omega)$ space:

$$\underline{X}^k_{\mathbf{grad},h} := \Big\{ \underline{q}_h = \big((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}, (q_E)_{E\in\mathcal{E}_h}, (q_V)_{V\in\mathcal{V}_h} \big) \;: \\ q_T \in \mathcal{P}^{k-1}(T),\ q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \Big\}.$$

- Interpolator: $\underline{I}^k_{\mathbf{grad},h} q = \underline{q}_h$ with

$$\begin{aligned} q_V &= q(\boldsymbol{x}_V), \\ q_X &= L^2\text{-projection on } \mathcal{P}^{k-1}(X) \text{ of } q, \text{ for } X \in \{E,F,T\}. \end{aligned}$$

- Operator reconstructions:
  - ⋆ edge gradient $G^k_E \underline{q}_E \in \mathcal{P}^k(E) \rightsquigarrow$ edge trace $\gamma^{k+1}_E \underline{q}_E \in \mathcal{P}^{k+1}(E)$,
  - ⋆ face gradient $\mathbf{G}^k_F \underline{q}_F \in \mathcal{P}^k(F)^2 \rightsquigarrow$ face trace $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$,
  - ⋆ element gradient $\mathbf{G}^k_T \underline{q}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow$ el. potential $P^{k+1}_{\mathbf{grad},T} \underline{q}_T \in \mathcal{P}^{k+1}(T)$,

$\underline{X}^k_{\mathbf{grad},h}$, the discrete $H(\mathbf{grad},\Omega)$ space

- Discrete $H(\mathbf{grad},\Omega)$ space:

$$\underline{X}^k_{\mathbf{grad},h} \coloneqq \Big\{ \underline{q}_h = \big((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}, (q_E)_{E\in\mathcal{E}_h}, (q_V)_{V\in\mathcal{V}_h}\big) \;: \\ q_T \in \mathcal{P}^{k-1}(T),\ q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \Big\}.$$

- Interpolator: $\underline{I}^k_{\mathbf{grad},h} q = \underline{q}_h$ with

$$\begin{aligned} q_V &= q(\boldsymbol{x}_V),\\ q_X &= L^2\text{-projection on } \mathcal{P}^{k-1}(X) \text{ of } q, \text{ for } X \in \{E,F,T\}. \end{aligned}$$

- Operator reconstructions:
  - ⋆ edge gradient $G^k_E \underline{q}_E \in \mathcal{P}^k(E) \rightsquigarrow$ edge trace $\gamma^{k+1}_E \underline{q}_E \in \mathcal{P}^{k+1}(E)$,
  - ⋆ face gradient $\mathbf{G}^k_F \underline{q}_F \in \mathcal{P}^k(F)^2 \rightsquigarrow$ face trace $\gamma^{k+1}_F \underline{q}_F \in \mathcal{P}^{k+1}(F)$,
  - ⋆ element gradient $\mathbf{G}^k_T \underline{q}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow$ el. potential $P^{k+1}_{\mathbf{grad},T} \underline{q}_T \in \mathcal{P}^{k+1}(T)$,

We want:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \coloneqq \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \,:\, \boldsymbol{v}_T \in \boldsymbol{\mathcal{RT}}^k(T),\ \boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^k(F),\ v_E \in \mathcal{P}^k(E) \Big\},$$

  with $\boldsymbol{\mathcal{RT}}^k(X) = \boldsymbol{\mathcal{P}}^{k-1}(X) \oplus (\boldsymbol{x} - \boldsymbol{x}_X)\mathcal{P}^{k-1}(X)$ the Raviart–Thomas space.
- Interpolator: $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = \underline{\boldsymbol{v}}_h$ with

$$\begin{aligned} v_E &= L^2\text{-projection on } \mathcal{P}^k(E) \text{ of } \boldsymbol{v}\cdot\mathbf{t}_E, \\ \boldsymbol{v}_F &= L^2\text{-projection on } \boldsymbol{\mathcal{RT}}^k(F) \text{ of } \boldsymbol{v}_{\mathrm{t},F} \text{ (tangential projection)}, \\ \boldsymbol{v}_T &= L^2\text{-projection on } \boldsymbol{\mathcal{RT}}^k(T) \text{ of } \boldsymbol{v}. \end{aligned}$$

- Operator reconstructions:
  - ⋆ face (scalar) curl $C^k_F\underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F) \rightsquigarrow$ tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F)^2$,
  - ⋆ element curl $\boldsymbol{C}^k_T\underline{\boldsymbol{v}}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow$ element potential $\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \in \mathcal{P}^k(T)^3$.

Discrete gradient

- We want $\underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$.
- We have:
  - $G^k_E \underline{q}_E \in \mathcal{P}^k(E)$, $\mathbf{G}^k_F \underline{q}_F \in \mathcal{P}^k(F)^2$, $\mathbf{G}^k_T \underline{q}_T \in \mathcal{P}^k(T)^3$,
  - $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ : \ \boldsymbol{v}_T \in \boldsymbol{\mathcal{RT}}^k(T),\ \boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^k(F),\ v_E \in \mathcal{P}^k(E) \Big\}$.

$\rightsquigarrow$ Define $\underline{\boldsymbol{G}}^k_h : \underline{X}^k_{\mathbf{grad},h} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ by projecting edge, face and element reconstructed gradients onto the corresponding components:

$$\underline{\boldsymbol{G}}^k_h \underline{q}_h = ((\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},T} \mathbf{G}^k_T \underline{q}_h)_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},F} \mathbf{G}^k_F \underline{q}_h)_{F\in\mathcal{F}_h}, (G^k_E \underline{q}_h)_{E\in\mathcal{E}_h}).$$

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DDR complex: summary

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

- No basis functions, Fully discrete spaces not made of functions, but of vectors of polynomials (DOFs) attached to geometric entities (emulates continuity properties of each space).

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^{k}(E)$	$\boldsymbol{\mathcal{RT}}^k(F)$	$\boldsymbol{\mathcal{RT}}^k(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^{k}(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

- Polynomial reconstructions of differential operator and potential by mimicking IBPs.

L^2-like inner products

- Local L^2-like inner product on the DDR spaces:

  for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$ and $k_{\mathbf{grad}} = k+1$, $k_{\mathbf{curl}} = k_{\mathrm{div}} = k$,

  $$(x_T, y_T)_{\bullet,T} = \int_T P^{k_\bullet}_{\bullet,T} x_T \cdot P^{k_\bullet}_{\bullet,T} y_T + \mathrm{s}_{\bullet,T}(x_T, y_T) \qquad \forall x_T, y_T \in \underline{X}^k_\bullet(T),$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face/edge potentials*).

- Global L^2-like product $(\cdot,\cdot)_{\bullet,h}$ by standard assembly of local ones.

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The DDR complex: algebraic properties

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

- Complex:

$$\underline{\boldsymbol{C}}^k_h \circ \underline{\boldsymbol{G}}^k_h = 0\,, \quad D^k_h \circ \underline{\boldsymbol{C}}^k_h = 0.$$

- Same cohomology as the continuous de Rham complex; in particular, if Ω has a trivial topology,

$$\operatorname{Im} \underline{\boldsymbol{G}}^k_h = \ker \underline{\boldsymbol{C}}^k_h\,, \quad \operatorname{Im} \underline{\boldsymbol{C}}^k_h = \ker D^k_h.$$

- Commutation properties between the interpolators and the continuous/discrete operators. For example,

$$\underline{\boldsymbol{G}}^k_h \underline{I}^k_{\mathbf{grad},h} q = \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{grad}\, q).$$

[Di Pietro et al., 2020], [Di Pietro et al., 2023], [Di Pietro and Hanot, 2024], [Di Pietro et al., 2025a], [Di Pietro et al., 2025b].

The DDR complex: analytical properties

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

- Poincaré inequalities: for example,

$$\|\underline{\boldsymbol{u}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \quad \forall \underline{\boldsymbol{v}}_h \in (\ker \underline{\boldsymbol{C}}^k_h)^\perp.$$

- Primal consistency (optimal approximation properties): for example, if q is smooth,

$$\|P^{k+1}_{\mathbf{grad},T} \underline{I}^k_{\mathbf{grad},T} q - q\|_{L^2(T)} \lesssim h_T^{k+2} |q|_{H^{k+2}(T)},$$
$$\|(\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{G}}^k_h) \underline{I}^k_{\mathbf{grad},T} q - \mathbf{grad}\, q\|_{L^2(T)} \lesssim h_T^{k+1} |\mathbf{grad}\, q|_{H^{(k+1,2)}(T)}.$$

[Di Pietro et al., 2020], [Di Pietro et al., 2023], [Di Pietro and Hanot, 2024], [Di Pietro et al., 2025a], [Di Pietro et al., 2025b].

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A DDR scheme for the Laplacian

Weak formulation Find $p \in H_0(\mathbf{grad}, \Omega)$ such that

$$(\mathbf{grad}\, p, \mathbf{grad}\, q)_{L^2(\Omega)} = \int_\Omega f\, q \quad \forall q \in H_0(\mathbf{grad}, \Omega)$$

DDR scheme: set

$$\underline{X}^k_{\mathbf{grad},h,0} \coloneq \left\{ \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, q_V = q_E = q_F = 0 \text{ for all boundary } V, E, F \right\}.$$

and write: Find $\underline{p}_h \in \underline{X}^k_{\mathbf{grad},h,0}$ such that

$$(\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} = \int_\Omega f\, P^{k+1}_{\mathbf{grad},h} \underline{q}_h \quad \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}.$$

Analysis of the scheme: error estimate

$$(\underline{\boldsymbol{G}}_h^k \underline{p}_h, \underline{\boldsymbol{G}}_h^k \underline{q}_h)_{\mathbf{curl},h} = \int_\Omega f\, P_{\mathbf{grad},h}^{k+1} \underline{q}_h \quad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h,0}^k.$$

- Introduce the consistency error

$$\mathcal{E}_h(p, \underline{q}_h) \coloneqq \int_\Omega f\, P_{\mathbf{grad},h}^{k+1} \underline{q}_h - (\underline{\boldsymbol{G}}_h^k \underline{I}_{\mathbf{grad},h}^k p, \underline{\boldsymbol{G}}_h^k \underline{q}_h)_{\mathbf{curl},h}.$$

- Subtract from the scheme to get the error equation

$$(\underline{\boldsymbol{G}}_h^k (\underline{p}_h - \underline{I}_{\mathbf{grad},h}^k p), \underline{\boldsymbol{G}}_h^k \underline{q}_h)_{\mathbf{curl},h} = \mathcal{E}_h(p, \underline{q}_h)$$

- Divide by $\|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h}$ and take the supremum over $\underline{q}_h$:

$$\|\underline{\boldsymbol{G}}_h^k (\underline{p}_h - \underline{I}_{\mathbf{grad},h}^k p)\|_{\mathbf{curl},h} \le \max_{\underline{q}_h} \frac{\mathcal{E}_h(p, \underline{q}_h)}{\|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h}}.$$

Analysis of the scheme: bounding the consistency error

- Consistency error:

$$\mathcal{E}_h(p, \underline{q}_h) := \int_\Omega f \, P^{k+1}_{\mathbf{grad},h} \underline{q}_h - (\underline{\boldsymbol{G}}^k_h \underline{I}^k_{\mathbf{grad},h} p, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h}.$$

- Set $\boldsymbol{u} = -\,\mathbf{grad}\, p$ and use $f = \operatorname{div} \boldsymbol{u}$ and $\underline{\boldsymbol{G}}^k_h \underline{I}^k_{\mathbf{grad},h} p = -\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{u}$:

$$\mathcal{E}_h(p, \underline{q}_h) = (\operatorname{div} \boldsymbol{u}, P^{k+1}_{\mathbf{grad},h} \underline{q}_h)_{L^2(\Omega)} + (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{u}, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h}.$$

$\rightsquigarrow$ *Need to evaluate the non-conformity of the method: defect of integration-by-parts coming from the discrete inner product and potential reconstruction.*

Adjoint consistency

- Adjoint consistency: measures defect in discrete integration-by-parts. For example, if $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}$,

$$\left|(\operatorname{div}\boldsymbol{u}\, P^{k+1}_{\mathbf{grad},h}\underline{q}_h)_{L^2(\Omega)} + (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h}\right| \lesssim h^{k+1}|\boldsymbol{u}|_{H^{(k+1,2)}(\mathcal{T}_h)}\|\underline{\boldsymbol{G}}^k_h \underline{q}_h\|_{\mathbf{curl},h}.$$

Note: proof uses a conforming lifting. [Di Pietro and Droniou, 2023], [Di Pietro et al., 2025b].

Discrete weak curl–curl formulations of Stokes

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H(\mathbf{grad},\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned}
\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H(\mathbf{grad},\Omega).
\end{aligned}$$

- Set

$$\underline{X}^k_{\mathbf{grad},h,\star} \coloneqq \left\{ \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{q}_h, \underline{I}^k_{\mathbf{grad},h} 1)_{\mathbf{grad},h} = 0 \right\}.$$

- DDR scheme: Find $(\underline{\boldsymbol{u}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{X}^k_{\mathbf{grad},h,\star}$ such that, for all $(\underline{\boldsymbol{v}}_h, \underline{q}_h)$,

$$\begin{aligned}
\nu(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h},\\
-(\underline{\boldsymbol{G}}^k_h \underline{q}_h, \underline{\boldsymbol{u}}_h)_{\mathbf{curl},h} &= 0.
\end{aligned}$$

Error estimates

Theorem

Setting

$$\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},1,h}^2 = \|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}^2\,,$$
$$\|\underline{q}_h\|_{\mathbf{grad},1,h}^2 = \|\underline{q}_h\|_{\mathbf{grad},h}^2 + \|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h},$$

we have:

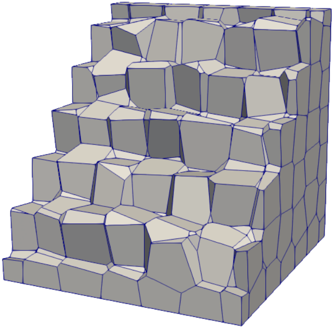
$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},1,h} + \|\underline{p}_h - \underline{I}_{\mathbf{grad},p}^h\|_{\mathbf{grad},1,h} \lesssim C_1(\boldsymbol{u}) h^{k+1}.$$

with $C_1(\boldsymbol{u})$ depending $\boldsymbol{u}$ and some of its derivatives, but not p.

- Stability: using exactness of the complex and Poincaré inequalities, as for the continuous model.
- Robustness with respect to the pressure: from the commutation $\underline{\boldsymbol{G}}_h^k(\underline{I}_{\mathbf{grad},h}^k p) = \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k(\mathbf{grad}\, p)$.

Numerical tests: setting

- $\Omega = (0, 1)^3$.
- Voronoi mesh families (similar results on tetrahedral meshes):



(a) Voronoi mesh

Numerical tests: setting

- Exact solution: for some $\lambda \geq 0$,

$$p(x,y,z) = \lambda \sin(2\pi x)\sin(2\pi y)\sin(2\pi z)\,,$$

$$\boldsymbol{u}(x,y,z) = \begin{bmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) \\ \frac{1}{2}\cos(2\pi x)\sin(2\pi y)\cos(2\pi z) \\ -\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) \end{bmatrix}.$$

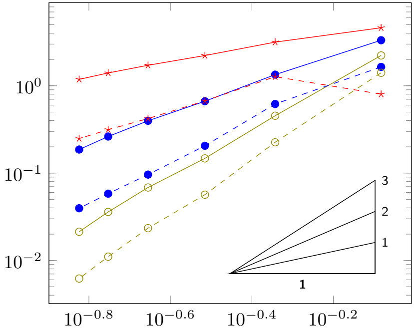
- Measured errors:
  - ⋆ $E_{\boldsymbol{u}}^{\mathrm{d}}$ and E_p^{d} in discrete norms between the approximate solutions and the *interpolates* of the exact solution (as in the theorems).

    E.g: $E_{\boldsymbol{u}}^{\mathrm{d}} = \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},1,h}$.
  - ⋆ $E_{\boldsymbol{u}}^{\mathrm{c}}$ and E_p^{c} in continuous norms between reconstructed potentials/projections of the approximate solutions and the exact solution.

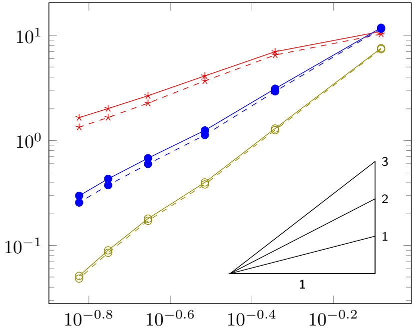
    E.g: $E_p^{\mathrm{c}} = \|p_h - p\|_{H^1(\Omega)}$.

Numerical tests: results for $\lambda = 1$

E^{c}, $k=0$; E^{c}, $k=1$; E^{c}, $k=2$
E^{d}, $k=0$; E^{d}, $k=1$; E^{d}, $k=2$

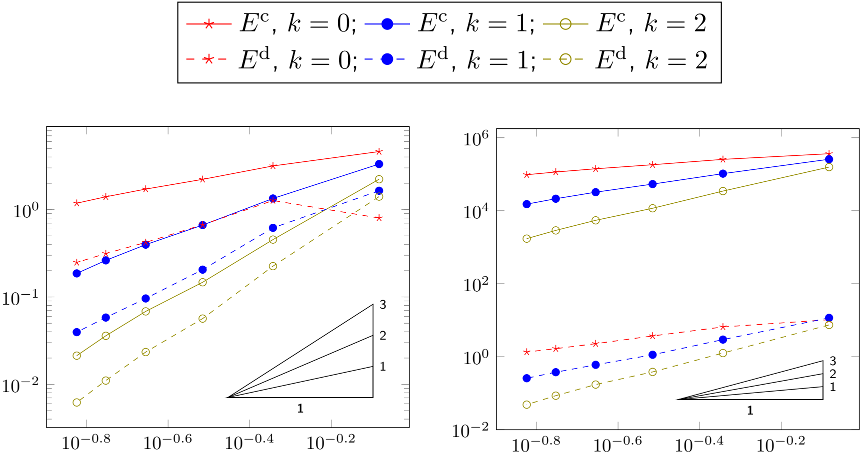


(a) Errors on $\boldsymbol{u}$



(b) Errors on p

Numerical tests: results for $\lambda = 10^5$



(a) Errors on $\boldsymbol{u}$

(b) Errors on p

Outline

Slides

Rellich compactness

Theorem

If $(q_n)_n$ is bounded in $H(\mathbf{grad}, \Omega)$ then $(q_n)_n$ is relatively compact in $L^2(\Omega)$.

- *Relatively* easy because $\mathbf{grad}$ controls the variations in all directions.

Maxwell compactness

Theorem

If $(\boldsymbol{v}_n)_n$ *is bounded in* $\boldsymbol{H}(\mathbf{curl};\Omega)$ *and*

$$\int_\Omega \boldsymbol{v}_n \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad},\Omega),$$

then $(\boldsymbol{v}_n)_n$ *is relatively compact in* $L^2(\Omega)^3$*.*

- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.

Maxwell compactness

Theorem

If $(\boldsymbol{v}_n)_n$ is bounded in $\boldsymbol{H}(\mathbf{curl};\Omega)$ and

$$\int_\Omega \boldsymbol{v}_n \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega),$$

then $(\boldsymbol{v}_n)_n$ is relatively compact in $L^2(\Omega)^3$.

- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.
- Much more challenging than Rellich: $\mathbf{curl}$ does not control the variations of the function [Weber, 1980], [Jochmann, 1997].

Maxwell compactness

Theorem

If $(\boldsymbol{v}_n)_n$ *is bounded in* $\boldsymbol{H}(\mathbf{curl};\Omega)$ *and*

$$\int_\Omega \boldsymbol{v}_n \cdot \mathbf{grad}\, z = 0 \qquad \forall z \in H(\mathbf{grad}, \Omega),$$

then $(\boldsymbol{v}_n)_n$ *is relatively compact in* $L^2(\Omega)^3$*.*

- Also a version for sequences in $\boldsymbol{H}(\mathrm{div};\Omega)$ that are orthogonal to curls.
- Much more challenging than Rellich: $\mathbf{curl}$ does not control the variations of the function [Weber, 1980], [Jochmann, 1997].
- Orthogonality condition equivalent to $\mathrm{div}\, \boldsymbol{v}_n = 0$ and $\boldsymbol{v}_n \cdot \mathbf{n}_\Omega = 0$.

Uses of Maxwell compactness

- Bound of $\mathbf{curl}$ and zero div classical in curl-div problems, such as models in electromagnetism (possibly using vector potential fixed by gauge).
- Compactness required for eigenvalue analysis and nonlinear models.
- Convergence analysis of schemes requires discrete versions of this compactness; see, e.g., [Kikuchi, 1987] for eigenvalue problems.
- Discrete compactness also allows for fine convergence analysis of schemes, possibly with models with rough coefficients [Chaumont-Frelet and Ern, 2023], [Chaumont-Frelet and Ern, 2024].

Maxwell compactness for DDR

Theorem

Let $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ *be such that*

$$\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right)_{h\in\mathcal{H}} \text{ is bounded,}$$
$$(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{G}}_h^k \underline{q}_h)_{\mathbf{curl},h} = 0 \qquad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k.$$

Then, there exists $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}_0(\mathrm{div};\Omega)$ *such that* $\mathrm{div}\,\boldsymbol{v} = 0$ *and, up to a subsequence as* $h \to 0$, $\boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{v}}_h \to \boldsymbol{v}$ *in* $L^2(\Omega)^3$.

Proof relies on:

- *commuting quasi-interpolators (interpolators defined on minimal-regularity spaces),*
- *conforming lifting (the same as for adjoint consistency).*

[Chaumont-Frelet et al., 2025]

Conclusions

- Polytopal complexes essential to design stable schemes for certain PDE models on generic meshes (flexibility for local refinement etc.).
- Construction is hierarchical on mesh entities (vertices $\rightarrow$ edges $\rightarrow$ faces $\rightarrow$ cells).
- DDR: based on suitably chosen degrees of freedom and reconstruction operators inspired by integration-by-parts formula.
- Range of results for analysis of schemes: Poincaré inequalities, primal/adjoint consistency, commutation properties, compactness, etc.
- Fully discrete approach already extended to other complexes (e.g., div-div complex).
- Maxwell compactness for DDR, allows for analysis of eigenvalue problems and nonlinear PDEs.

https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

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Thank you for your attention!

References

Arnold, D. (2018).
Finite Element Exterior Calculus.
SIAM.

Chaumont-Frelet, T., Droniou, J., and Lemaire, S. (2025).
Commuting quasi-interpolators and maxwell compactness for a polytopal de rham complex.
page 30p.

Chaumont-Frelet, T. and Ern, A. (2023).
Asymptotic optimality of the edge finite element approximation of the time-harmonic maxwell's equations.

Chaumont-Frelet, T. and Ern, A. (2024).
A priori and a posteriori analysis of the discontinuous galerkin approximation of the time-harmonic maxwell's equations under minimal regularity assumptions.

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order method for polytopal meshes.
Number 19 in Modeling, Simulation and Application. Springer International Publishing.

Di Pietro, D. A. and Droniou, J. (2023).
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency.
Found. Comput. Math., 23:85–164.

References

Di Pietro, D. A., Droniou, J., Hanot, M.-L., and Pitassi, S. (2025a).
Uniform poincaré inequalities for the discrete de rham complex of differential forms.
page 32p.
Submitted.

Di Pietro, D. A., Droniou, J., and Pitassi, S. (2023).
Cohomology of the discrete de Rham complex on domains of general topology.
Calcolo.
Published online.

Di Pietro, D. A., Droniou, J., and Pitassi, S. (2025b).
Conforming lifting and adjoint consistency for the discrete de rham complex of differential forms.
page 28p.

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Meth. Appl. Math., 14(4):461–472.

Di Pietro, D. A. and Hanot, M.-L. (2024).
Uniform poincaré inequalities for the discrete de rham complex on general domains.
Results Appl. Math., 23(100496).

References

Jochmann, F. (1997).
A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions.
Appl. Anal., 66(1-2):189–203.

Kikuchi, F. (1987).
Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism.
In *Proceedings of the first world congress on computational mechanics (Austin, Tex., 1986)*, volume 64, pages 509–521.

Touzalin, A. (2025).
Méthode des éléments virtuels pour la discrétisation des équations intégrales de frontière en électromagnétisme dans le domaine fréquentiel.
PhD Thesis, CEA-CESTA.

Weber, C. (1980).
A local compactness theorem for Maxwell's equations.
Math. Methods Appl. Sci., 2(1):12–25.