A complete theory of discrete trace and lifting for hybrid polytopal methods

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joint work with S. Badia and J. Tushar (Monash University).

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Laboratoire Jacques-Louis Lions, 25 Octobre 2024

NEMESIS
New generation methods for numerical simulations

Funded by the European Union

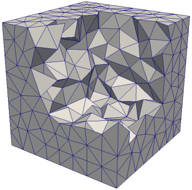
European Research Council
Established by the European Commission

Reference for this presentation

A discrete trace theory for non-conforming hybrid discretisation methods.
S. Badia, J. Droniou, and J. Tushar. 34p, 2024.
`http://arxiv.org/abs/2409.15863`

Outline

The Finite Element way

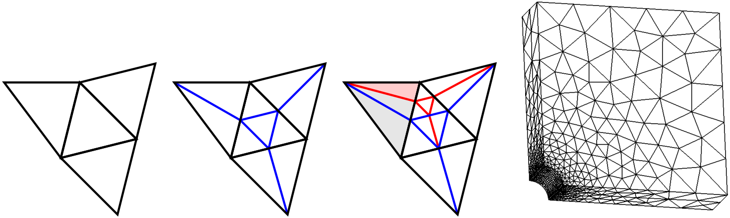


$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form discrete subspaces of the energy space (e.g., $H^1(\Omega)$ for 2nd-order elliptic problems).
  Example: conforming $\mathbb{P}^k$ spaces.
- **Gluing only works on special meshes!**

The Finite Element way

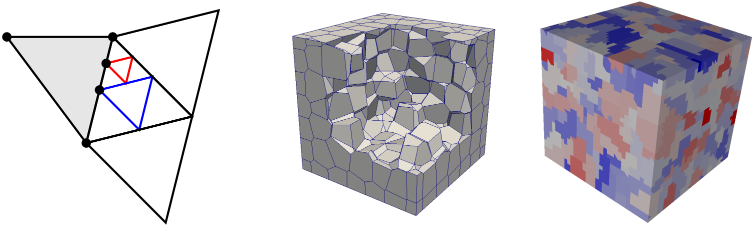
Shortcomings



- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

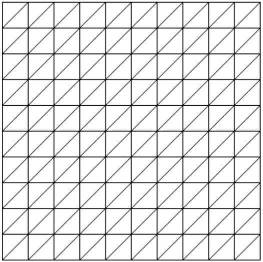
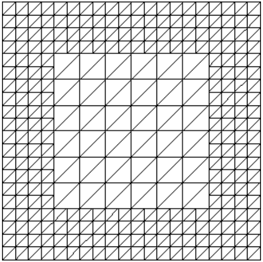
Benefits of polytopal meshes I



- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

Benefits of polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathbb{P}^2$-$(\mathbb{P}^1 + \mathcal{B}^3)$ scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

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1 Why polytopal methods?

2 Discrete trace theory
- Spaces and seminorms
- Trace and lifting

3 Elements of proofs
- Trace inequality
  - Continuous
  - Discrete
- Lifting
  - Continuous
  - Discrete

4 Numerical illustration

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Continuous setting I

Ω bounded Lipschitz domain of $\mathbb{R}^d$.

$H^1(\Omega)$-seminorm: for $v \in H^1(\Omega)$,

$$|v|_{1,\Omega} := \|\nabla v\|_{L^2(\Omega)}.$$

$H^{1/2}(\partial\Omega)$-seminorm: for $w \in H^{1/2}(\partial\Omega)$:

$$|w|_{1/2,\partial\Omega} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^d} \, dx dy \right)^{1/2}.$$

Continuous setting II

Trace operator: $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\gamma(v) = v_{|\partial\Omega}$ when v is smooth.

Theorem (Trace inequality)

$$|\gamma(v)|_{1/2,\partial\Omega} \lesssim |v|_{1,\Omega} \qquad \forall v \in H^1(\Omega).$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that:

$$\gamma(\mathcal{L}(w)) = w \quad \textit{and} \quad |\mathcal{L}(w)|_{1,\Omega} \lesssim |w|_{1/2,\partial\Omega} \qquad \forall w \in H^{1/2}(\partial\Omega).$$

Polytopal mesh

Ω polytopal. Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ with $\mathcal{T}_h$ set of elements, $\mathcal{F}_h$ set of faces.

- standard mesh regularity assumption (elements/faces do not become too elongated), and
- quasi-uniformity: with $h_X = \mathrm{diam}(X)$,

$$\exists \rho > 0 \,:\, \rho h_{t'} \le h_t \quad \forall t, t' \in \mathcal{T}_h.$$

Set $h := \max_{t \in \mathcal{T}_h} h_t$ and write $a \lesssim b$ for "$a \le Cb$ with C depending only on the mesh regularity parameters".

Discrete $H^1(\Omega)$ space and seminorm

Hybrid space: unknowns are polynomials in the elements and on the faces. Fix $k \geq 0$ and set

$$\underline{U}_h := \{\underline{v}_h = ((v_t)_{t\in\mathcal{T}_h}, (v_f)_{f\in\mathcal{F}_h}) \,:\, v_t \in \mathbb{P}^k(t)\,, \quad v_f \in \mathbb{P}^k(f)\}.$$

Discrete $H^1(\Omega)$-seminorm: with $\underline{v}_t = (v_t, (v_f)_{f\in\mathcal{F}_t})$ restriction of $\underline{v}_h$ to t,

$$|\underline{v}_h|_{1,h}^2 := \sum_{t\in\mathcal{T}_h} |\underline{v}_t|_{1,t}^2$$

$$\text{with} \quad |\underline{v}_t|_{1,t}^2 := \|\nabla v_t\|_{L^2(t)}^2 + \sum_{f\in\mathcal{F}_t} h_t^{-1}\|v_f - v_t\|_{L^2(f)}^2.$$

Discrete $H^{1/2}(\partial\Omega)$ space and seminorm

Boundary space: restriction to boundary of hybrid space (piecewise polynomial functions).

$$U_h^{\mathrm{bd}} := \{w_h = ((w_f)_{f\in\mathcal{F}_h^{\mathrm{bd}}})\,:\, w_f \in \mathbb{P}^k(f)\} \subset L^2(\partial\Omega).$$

Trace (restriction): $\gamma_h : \underline{U}_h \to U_h^{\mathrm{bd}}$ such that

$$\gamma_h(\underline{v}_h) = (v_f)_{f\in\mathcal{F}_h^{\mathrm{bd}}} \qquad \forall \underline{v}_h \in \underline{U}_h.$$

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Discrete $H^{1/2}(\partial\Omega)$-seminorm:

$$|w_h|^2_{1/2,h} := \underbrace{\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} h_f^{-1}\|w_f - \overline{w}_f\|^2_{L^2(f)}}_{\text{local variation in each } f} + \underbrace{\sum_{(f,f')\in\mathcal{FF}_h^{\mathrm{bd}}} |f|_{d-1}|f'|_{d-1}\frac{|\overline{w}_f - \overline{w}_{f'}|^2}{\delta^d_{ff'}}}_{\text{medium–long range interactions}}$$

($\mathcal{FF}_h^{\mathrm{bd}} =$ pairs of all faces on $\partial\Omega$).

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Main results

Theorem (Trace inequality)

$$|\gamma_h(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h. \tag{1}$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ such that:

$$\gamma(\mathcal{L}_h(w_h)) = w_h \quad \textit{and} \quad |\mathcal{L}_h(w_h)|_{1,h} \lesssim |w_h|_{1/2,h} \qquad \forall w_h \in U_h^{\mathrm{bd}}. \tag{2}$$

Main results

Theorem (Trace inequality)

$$|\gamma_h(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h. \tag{1}$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ *such that:*

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- Hidden constant independent of $\mathrm{diam}(\Omega)$.
- Directly gives trace/lifting for Hybridizable Discontinuous Galerkin [Cockburn et al., 2009], Hybrid High-Order [Di Pietro and Droniou, 2020], non-conforming Virtiual Elements [de Dios et al., 2016], etc.

Why is that new?

Previous results: using only L^2-norms on $\partial\Omega$
[Eymard et al., 2000, Droniou et al., 2018].

- Allows for a trace inequality

$$\|\gamma(\underline{v}_h)\|_{L^2(\partial\Omega)} \lesssim |\underline{v}_h|_{1,h} + \|v_h\|_{L^2(\Omega)} \qquad \forall \underline{v}_h \in \underline{U}_h$$

  (where $(v_h)_{|t} = v_t$ for all $t \in \mathcal{T}_h$).

- Does not allow for a (uniformly bounded) lifting.

Why is that useful?

Domain decomposition methods: exchange information by trace and lifting.

Consider two domains Ω_1, Ω_2 with interface Γ. A typical construction in substructuring non-overlapping DD is:

(i) Take v_1 in Ω_1.

(ii) Consider the trace $(v_1)_{|\Gamma}$ of v_1 on Γ.

(iii) Define v_2 in Ω_2 as the harmonic extension of $(v_1)_{|\Gamma}$.

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(iii) Define v_2 in Ω_2 as the harmonic extension of $(v_1)_{|\Gamma}$.

The map $v_1 \to v_2$ must be continuous for the H^1 norms. We must therefore set up a norm on Γ which is

- not too strong, for the continuity of the trace $v_1 \to (v_1)_{|\Gamma}$,
- strong enough, for the continuity of the lifting $(v_1)_{|\Gamma} \to v_2$.

Approach

- Previous approaches attempted to interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014].

  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

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- Previous approaches attempted to interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014].

  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

- Here, following principles of Discrete Functional Analysis [Eymard et al., 2010, Droniou et al., 2018], we do not use continuous trace/lifting results but mimic their proofs in the fully discrete setting.

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Estimate I

Starting point: set $y = x + \rho$ and write

$$\begin{aligned}
&u(0, x+\rho) - u(0,x) \\
&\quad = u(0,x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0,x) \\
&\quad = \int_{\rho}^{0} \partial_1 u(s, x+\rho) ds + \int_0^{\rho} \partial_2 u(\rho, x+s) ds + \int_0^{\rho} \partial_1 u(s,x) ds.
\end{aligned}$$

Estimate II

Starting point: set $y = x + \rho$ and write

$$\begin{aligned} & u(0, x+\rho) - u(0,x) \\ &\quad = u(0,x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0,x) \\ &\quad = \int_\rho^0 \partial_1 u(s, x+\rho) ds + \int_0^\rho \partial_2 u(\rho, x+s) ds + \int_0^\rho \partial_1 u(s,x) ds. \end{aligned}$$

Take L^2-norms w.r.t. x (swap integrals):

$$\begin{aligned} & \|u(0, \cdot + \rho) - u(0, \cdot)\|_{L^2(\mathbb{R})} \\ &\qquad \le \int_0^\rho \|\partial_1 u(s, \cdot + \rho)\|_{L^2(\mathbb{R})} \, ds + \int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})} \, ds \\ &\qquad\qquad + \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds \\ &\qquad \le \rho \Big(\underbrace{\frac{2}{\rho} \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds}_{=: F_1(\rho)} + \underbrace{\|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}}_{=: F_2(\rho)} \Big). \end{aligned}$$

Estimate III

Change of variable ($y = x + \rho$) in the $H^{1/2}$ semi-norm:

$$
\begin{aligned}
|u(0,\cdot)|^2_{1/2,\mathbb{R}} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(0,x) - u(0,y)|^2}{|x-y|^2}\, dx dy \\
&= \int_{\mathbb{R}} \frac{\|u(0,\cdot) - u(0,\cdot+\rho)\|^2_{L^2(\mathbb{R})}}{\rho^2} d\rho \\
&\leq C(\|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})})
\end{aligned}
$$

where

$$
F_1(\rho) = \frac{2}{\rho} \int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})}\, ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.
$$

Estimate IV

$$|u(0,\cdot)|^2_{1/2,\mathbb{R}} \leq C(\|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})})$$

$$F_1(\rho) = \frac{2}{\rho}\int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})}\, ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Conclusion:

$$\|F_2\|^2_{L^2(\mathbb{R})} = \|\partial_2 u\|^2_{L^2(\mathbb{R}^2)} \leq |u|^2_{H^1(\mathbb{R}^2)}.$$

By Hardy inequality:

$$\|F_1\|^2_{L^2(\mathbb{R})} \leq C\|\partial_1 u\|^2_{L^2(\mathbb{R}^2)} \leq C|u|^2_{H^1(\mathbb{R}^2)}.$$

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Driving principles

- Points become faces.
  - *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

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  ■ *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

- Integrate along lines ⇝ sum over cells/faces that intersect the line.

$$\begin{aligned}\int_0^\rho \partial_1 u(s,x)dx &\rightsquigarrow \overline{v}_{t_N} - \overline{v}_{f_{N-1}} + \overline{v}_{f_{N-1}} - \overline{v}_{t_{N-1}} + \cdots + \overline{v}_{t_1} - \overline{v}_f \\ &\lesssim \sum_{t \in \mathrm{Li}(f,t_N)} h^{\frac{2-d}{2}} |\underline{v}_t|_{1,t}.\end{aligned}$$

Driving principles

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  ■ *Continuous $H^{1/2}$ seminorm integrates over x, y, discrete $H^{1/2}$-seminorm sums over pairs of faces.*

- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

- Need a distance between faces/cells: has to be up to h.
  ■ *Cannot consider all $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ such that $\delta_{ff'} = \rho$ for a given ρ... Instead, $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ are "at distance ℓh of each other" if $\ell h \leq \delta_{ff'} < (\ell + 1)h$.*
  ■ *Makes "change of variable" $x + \rho \to x$ less straightforward.*

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- Need to be able to swap integrals.
  - *Integrate vertically to $\partial\Omega$ then parallel to $\partial\Omega$ ⇝ layers along $\partial\Omega$.*

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  - *Makes "change of variable" $x + \rho \to x$ less straightforward.*
- Need to be able to swap integrals.
  - *Integrate vertically to $\partial\Omega$ then parallel to $\partial\Omega$ ⇝ layers along $\partial\Omega$.*
- Need a discrete Hardy inequality: $r_m \ge 0$ and $R_l := \frac{1}{l}\sum_{m=0}^{l} r_m$, then

$$\sum_{l=1}^{L} R_l^2 \le 32 \sum_{l=0}^{L} r_l^2.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot{+}s)\, ds\|_{L^2(\mathbb{R})} \leq \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot{+}s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Take (f, f') "within distance ℓh" and consider

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{t \in \mathrm{Li}(ff';\delta_{ff'})} |\underline{v}_t|_{1,t}.$$

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- ○ Split into layers:

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{r=1}^{\ell} \sum_{\substack{t \in \mathrm{Li}(ff'; \delta_{ff'}) \\ |p(x_t) - x_f| \simeq rh}} |\underline{v}_t|_{1,t}.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot + s)\, ds \|_{L^2(\mathbb{R})} \leq \frac{1}{\rho} \int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

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- Split into layers:

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- Estimate cardinality $\#\{t \in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t) - x_f| \simeq rh\} \lesssim 1$, so

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t \in \mathrm{Li}(ff';\delta_{ff'}) \\ |p(x_t) - x_f| \simeq rh}} |\underline{v}_t|_{1,t}^2.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot+s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot+s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Estimate cardinality $\#\{t\in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t)-x_f|\simeq rh\}\lesssim 1$, so

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t\in \mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

- Multiply by $|f|\,|f'|/\delta_{ff'}^2 \lesssim h^{d-2}/\ell^d$, sum over (f,f'):

$$\sum_{(f,f'),\,\delta_{ff'}\simeq \ell h} |f|\,|f'|\frac{|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2}{\delta_{ff'}^2}$$

$$\lesssim \frac{1}{\ell^{d-1}} \sum_{r=1}^{\ell} \sum_{(f,f'),\,\delta_{ff'}\simeq \ell h} \sum_{\substack{t\in \mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

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Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot + s)\, ds \|_{L^2(\mathbb{R})} \leq \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Multiply by $|f|\,|f'|/\delta_{ff'}^2 \lesssim h^{d-2}/\ell^d$, sum over (f, f'):

$$\frac{1}{\ell^{d-1}} \sum_{r=1}^{\ell} \sum_{(f,f'),\, \delta_{ff'} \simeq \ell h} \sum_{\substack{t \in \mathrm{Li}(ff';\delta_{ff'}) \\ |p(x_t) - x_f| \simeq rh}} |\underline{v}_t|_{1,t}^2.$$

- Swap sums over faces and cells:

$$\frac{1}{\ell^{d-1}} \sum_{r=1}^{\ell} \sum_{\{t\,:(\ell-2)h \leq \mathrm{dist}(p(x_t), \partial\Omega) \leq \ell h\}} |\underline{v}_t|_{1,t}^2 \, \#\mathfrak{F}(t, r)$$

where $\mathfrak{F}(t,r) := \{(f, f') \,:\, \delta_{ff'} \simeq \ell h,\, t \in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t) - x_f| \simeq rh\}$.

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot{+}s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot{+}s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Swap sums over faces and cells:

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|^2_{1,t}\,\#\mathfrak{F}(t,r)$$

  where $\mathfrak{F}(t,r) := \{(f,f')\ :\ \delta_{ff'}\simeq \ell h,\, t\in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t)-x_f|\simeq rh\}$.
- Estimate cardinality: $\#\mathfrak{F}(t,r)\lesssim \ell^{d-2}$, so

$$\frac{1}{\ell}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|^2_{1,t} \lesssim \sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|^2_{1,t}$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot{+}s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot{+}s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Estimate cardinality: $\#\mathfrak{F}(t,r) \lesssim \ell^{d-2}$, so

$$\sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|_{1,t}^2$$

- Conclude by summing over ℓ (each layer appears 3 times):

$$3\sum_t |\underline{v}_t|_{1,t}^2 = 3|\underline{v}_h|_{1,h}^2$$

Outline

1. Why polytopal methods?
2. Discrete trace theory
   - Spaces and seminorms
   - Trace and lifting
3. Elements of proofs
   - Trace inequality
     - Continuous
     - Discrete
   - Lifting
     - Continuous
     - Discrete
4. Numerical illustration

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Construction of the lifting

Take $w \in H^{1/2}(\mathbb{R}^{d-1})$ and define $v \in H^1([0,1) \times \mathbb{R}^{d-1})$ by by averaging w over the base of a cone, which becomes more and more narrow as we get close to the boundary.

With $\rho_x(\boldsymbol{y}) = x^{-(d-1)}\rho(x^{-1}\boldsymbol{y})$ usual smoothing kernel,

$$v(x, \boldsymbol{y}) = (\rho_x \star w)(\boldsymbol{y}).$$

Some arguments in the estimates on partial derivatives

- $\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y})d\boldsymbol{y} = 0$ to write (for $i \geq 2$)

$$\partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d}\int_{\mathbb{R}} (w(\boldsymbol{z}) - w(\boldsymbol{y}))\partial_i \rho(x^{-1}(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z}.$$

- $\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{z}))|d\boldsymbol{z} \leq C/x$ to write, using Cauchy–Schwarz:

$$\left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})||\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z}|\right)^2 \leq \frac{C}{x}\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2|\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))|d\boldsymbol{z}.$$

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Construction of the lifting

Also average on the base of a cone...

- For $t \in \mathcal{T}_h$, set $\delta_t = \mathrm{dist}(x_t, \partial\Omega)$ and

$$\mathcal{A}_t = \{f \in \mathcal{F}_h^{\mathrm{bd}} \,:\, \mathrm{dist}(p(x_t), f) \leq \delta_t)\}.$$

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- Give to each $f \in \mathcal{A}_t$ an identical weight:

$$\rho_t(f) = \begin{cases} \frac{1}{\#\mathcal{A}_t} & \text{if } f \in \mathcal{A}_t, \\ 0 & \text{otherwise.} \end{cases}$$

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- Lift $w_h = (w_f)_{f \in \mathcal{F}_h^{\mathrm{bd}}} \in U_h^{\mathrm{bd}}$ into $v_h = ((v_t)_{t \in \mathcal{T}_h}, (v_f)_{f \in \mathcal{F}_h})$ such that

$$v_t = \frac{1}{\#\mathcal{A}_t} \sum_{f \in \mathcal{A}_t} \overline{w}_f = \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} \overline{w}_f \rho_t(f)$$

$$v_f = \begin{cases} \frac{v_t + v_{t'}}{2} & \text{if } f \text{ internal face between } t, t' \in \mathcal{T}_h, \\ w_f & \text{if } f \in \mathcal{F}_h^{\mathrm{bd}}. \end{cases}$$

Estimate of $|\underline{v}_h|_{1,h}$ I

Low order reconstruction: all v_t are constant, so

$$|\underline{v}_h|_{1,h}^2 \simeq \sum_{(t,t')\text{ neighbours}} h^{d-2}|v_t - v_{t'}|^2.$$

Estimate of $|\underline{v}_h|_{1,h}$ II

Adaptation of arguments

☐ Continuous:

$$\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y}) d\boldsymbol{y} = 0$$

$$\rightsquigarrow \quad \partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d} \int_{\mathbb{R}} (w(\boldsymbol{z}) \color{red}{- w(\boldsymbol{y})}\color{black}{)} \partial_i \rho(x^{-1}(\boldsymbol{y} - \boldsymbol{z})) d\boldsymbol{z} \quad \forall \boldsymbol{y}$$

☐ Discrete:

$$\sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} (\rho_t(f) - \rho_{t'}(f)) \Big(= \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} \rho_t(f) - \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} \rho_{t'}(f) = 1 - 1 \Big) = 0$$

$$\rightsquigarrow \quad v_t - v_{t'} = \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'}) D_g \rho(f) \quad \forall f' \in \mathcal{F}_h^{\mathrm{bd}},$$

where $D_g\rho(f) = \rho_t(f) - \rho_{t'}(f)$ with (t, t') cells on each side of $g \in \mathcal{F}_h^{\mathrm{in}}$.

Estimate of $|\underline{v}_h|_{1,h}$ III

☐ Continuous:

$$\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{y})|d\boldsymbol{y} \leq \frac{C}{x}$$

$$\rightsquigarrow \quad \left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})||\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z}|\right)^2$$

$$\leq \frac{C}{x}\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2|\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))|d\boldsymbol{z}.$$

☐ Discrete:

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |D_g\rho(f)| \lesssim \frac{h}{\delta_t}$$

$$\rightsquigarrow |\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}}\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2|D_g\rho(f)|\frac{h^{d-1}}{\delta_t}.$$

with $f' \in \mathcal{F}_h^{\mathrm{bd}}$ such that g "projects close to f'".

Estimate of $|\underline{v}_h|_{1,h}$ IV

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |D_g\rho_t(f)| = \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |\rho_t(f)-\rho_{t'}(f)| \lesssim \frac{h}{\delta_t}.$$

Requires:

☐ $\#\mathcal{A}_t \simeq \left(\frac{\delta_t}{h}\right)^{d-1}$

☐ $\#(\mathcal{A}_t\Delta\mathcal{A}_{t'}) \lesssim \left(\frac{\delta_t}{h}\right)^{d-2}$

☐ $\forall f\in\mathcal{A}_t\cap\mathcal{A}_{t'}\,,\ |\rho_t(f)-\rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d}$

☐ $|\rho_t(f)-\rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d-1}$.

Estimate of $|\underline{v}_h|_{1,h}$ V

$$|\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}} \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2 |D_g\rho(f)| \frac{h^{d-1}}{\delta_t}.$$

Write $\sum_{g\in\mathcal{F}_h^{\mathrm{in}}}$ as $\sum_{f'\in\mathcal{F}_h^{\mathrm{bd}}} \sum_{g \text{ above } f'}$ and conclude by proving

$$\sum_{g \text{ above } f'} |D_g\rho(f)| \frac{h^{d-1}}{\delta_t} \lesssim \frac{|f|\,|f'|}{\delta_{ff'}^d}.$$

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Equivalence of norms

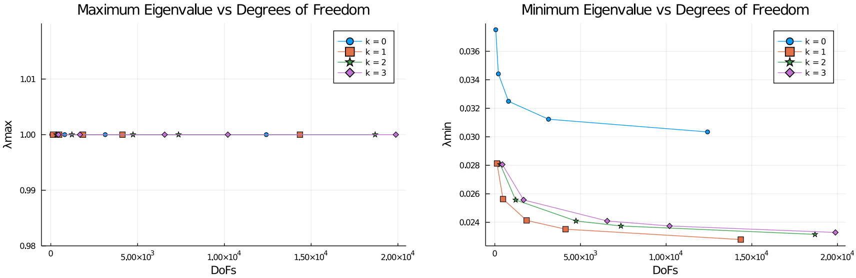
- Let $\mathcal{E}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ be the discrete harmonic extension for the discrete H^1-seminorm (minimises this norm with given boundary conditions).
- The discrete trace and lifting give

$$|\mathcal{E}_h(w_h)|_{1,h} \simeq |w_h|_{1/2,h} \quad \forall w_h \in U_h^{\partial}.$$

- We assess this equivalence by solving a generalised eigenvalue problem to evaluate the constants in the upper and lower bounds.

Results

Ω: square. Cartesian mesh.



(a) Maximum eigenvalues

(b) Minimum eigenvalues

Conclusions

- Complete discrete trace theory, with definition of boundary norm, trace inequality and lifting in discrete spaces of polytopal hybrid methods.
- Applicable to a range of schemes: HHO, VEM, HDG, etc. (and even FEM).
- Constructive proofs, obtained by mimicking proofs in the continuous setting (more flexible than looking for lifting in conforming spaces).
- For the moment, requires quasi-uniform meshes, but with elements of generic shapes.
- Allows for the analysis of BDDC and similar for polytopal methods.

Funded by
the European Union

European Research Council
Established by the European Commission

NEMESIS
New generation
methods for numerical
simulations

Funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Thank you for your attention!

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