A complete theory of discrete trace and lifting for hybrid polytopal methods

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joint work with S. Badia and J. Tushar (Monash University).

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New generation methods for numerical simulations

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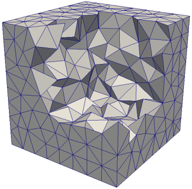
Reference for this presentation

A discrete trace theory for non-conforming hybrid discretisation methods.
S. Badia, J. Droniou, and J. Tushar. 34p, 2024.
http://arxiv.org/abs/2409.1863

Outline

The Finite Element way

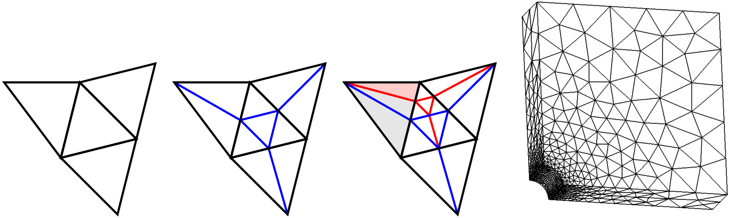


$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form discrete subspaces of the energy space (e.g., $H^1(\Omega)$ for 2nd-order elliptic problems).
  Example: conforming $\mathbb{P}^k$ spaces.

- **Gluing only works on special meshes!**

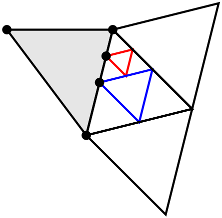
The Finite Element way
Shortcomings

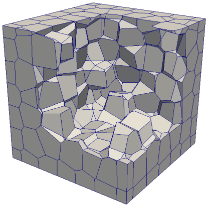


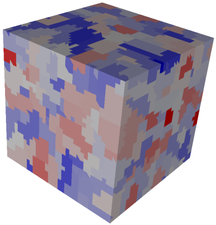
- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes I



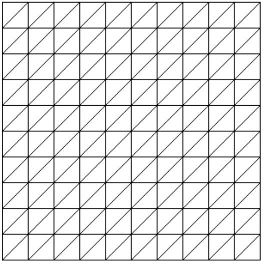
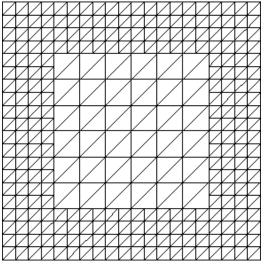




- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

Benefits of polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathbb{P}^2$-($\mathbb{P}^1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

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Continuous setting I

Ω bounded Lipschitz domain of $\mathbb{R}^d$.

$H^1(\Omega)$-seminorm: for $v \in H^1(\Omega)$,

$$|v|_{1,\Omega} := \|\nabla v\|_{L^2(\Omega)}.$$

$H^{1/2}(\partial\Omega)$-seminorm: for $w \in H^{1/2}(\partial\Omega)$:

$$|w|_{1/2,\partial\Omega} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^d} \, dxdy \right)^{1/2}.$$

Continuous setting II

Trace operator: $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\gamma(v) = v_{|\partial\Omega}$ when v is smooth.

Theorem (Trace inequality)

$$|\gamma(v)|_{1/2,\partial\Omega} \lesssim |v|_{1,\Omega} \qquad \forall v \in H^1(\Omega).$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that:

$$\gamma(\mathcal{L}(w)) = w \quad \textit{and} \quad |\mathcal{L}(w)|_{1,\Omega} \lesssim |w|_{1/2,\partial\Omega} \qquad \forall w \in H^{1/2}(\partial\Omega).$$

Polytopal mesh

Ω polytopal. Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ with $\mathcal{T}_h$ set of elements, $\mathcal{F}_h$ set of faces.

- standard mesh regularity assumption (elements/faces do not become too elongated), and
- quasi-uniformity: with $h_X = \mathrm{diam}(X)$,

$$\exists \rho > 0 \,:\, \rho h_{t'} \leq h_t \quad \forall t, t' \in \mathcal{T}_h.$$

Set $h := \max_{t \in \mathcal{T}_h} h_t$ and write $a \lesssim b$ for "$a \leq Cb$ with C depending only on the mesh regularity parameters".

Discrete $H^1(\Omega)$ space and seminorm

Hybrid space: unknowns are polynomials in the elements and on the faces. Fix $k \geq 0$ and set

$$\underline{U}_h := \{\underline{v}_h = ((v_t)_{t\in\mathcal{T}_h}, (v_f)_{f\in\mathcal{F}_h}) \,:\, v_t \in \mathbb{P}^k(t)\,, \quad v_f \in \mathbb{P}^k(f)\}.$$

Discrete $H^1(\Omega)$-seminorm: with $\underline{v}_t = (v_t, (v_f)_{f\in\mathcal{F}_t})$ restriction of $\underline{v}_h$ to t,

$$|\underline{v}_h|_{1,h}^2 := \sum_{t\in\mathcal{T}_h} |\underline{v}_t|_{1,t}^2$$

$$\text{with} \quad |\underline{v}_t|_{1,t}^2 := \|\nabla v_t\|_{L^2(t)}^2 + \sum_{f\in\mathcal{F}_t} h_t^{-1}\|v_f - v_t\|_{L^2(f)}^2.$$

Discrete $H^{1/2}(\partial\Omega)$ space and seminorm

Boundary space: restriction to boundary of hybrid space (piecewise polynomial functions).

$$U_h^{\rm bd} := \{w_h = ((w_f)_{f\in\mathcal{F}_h^{\rm bd}}) \,:\, w_f \in \mathbb{P}^k(f)\} \subset L^2(\partial\Omega).$$

Trace (restriction): $\gamma_h : \underline{U}_h \to U_h^{\rm bd}$ such that

$$\gamma_h(\underline{v}_h) = (v_f)_{f\in\mathcal{F}_h^{\rm bd}} \qquad \forall \underline{v}_h \in \underline{U}_h.$$

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Discrete $H^{1/2}(\partial\Omega)$-seminorm:

$$|w_h|_{1/2,h}^2 := \underbrace{\sum_{f\in\mathcal{F}_h^{\rm bd}} h_f^{-1}\|w_f - \overline{w}_f\|_{L^2(f)}^2}_{\text{local variation in each } f} + \underbrace{\sum_{(f,f')\in\mathcal{FF}_h^{\rm bd}} |f|_{d-1}|f'|_{d-1}\frac{|\overline{w}_f - \overline{w}_{f'}|^2}{\delta_{ff'}^d}}_{\text{medium–long range interactions}}$$

($\mathcal{FF}_h^{\rm bd}$ = pairs of all faces on $\partial\Omega$).

Outline

Main results

Theorem (Trace inequality)

$$|\gamma_h(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h. \tag{1}$$

Theorem (Lifting)

There exists a linear operator $\mathcal{L}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ *such that:*

$$\gamma(\mathcal{L}_h(w_h)) = w_h \quad \textit{and} \quad |\mathcal{L}_h(w_h)|_{1,h} \lesssim |w_h|_{1/2,h} \qquad \forall w_h \in U_h^{\mathrm{bd}}. \tag{2}$$

Main results

Theorem (Trace inequality)

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- Hidden constant independent of $\mathrm{diam}(\Omega)$.
- Directly gives trace/lifting for Hybridizable Discontinuous Galerkin [Cockburn et al., 2009], Hybrid High-Order [Di Pietro and Droniou, 2020], non-conforming Virtiual Elements [de Dios et al., 2016], etc.

Why is that new?

Previous results: using only L^2-norms on $\partial\Omega$
[Eymard et al., 2000, Droniou et al., 2018].

- Allows for a trace inequality

$$\|\gamma(\underline{v}_h)\|_{L^2(\partial\Omega)} \lesssim |\underline{v}_h|_{1,h} + \|v_h\|_{L^2(\Omega)} \qquad \forall \underline{v}_h \in \underline{U}_h$$

  (where $(v_h)_{|t} = v_t$ for all $t \in \mathcal{T}_h$).

- Does not allow for a (uniformly bounded) lifting.

Why is that useful?

Domain decomposition methods: exchange information by trace and lifting.

Consider two domains Ω_1, Ω_2 with interface Γ. A typical construction in substructuring non-overlapping DD is:

(i) Take v_1 in Ω_1.

(ii) Consider the trace $(v_1)_{|\Gamma}$ of v_1 on Γ.

(iii) Define v_2 in Ω_2 as the harmonic extension of $(v_1)_{|\Gamma}$.

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(iii) Define v_2 in Ω_2 as the harmonic extension of $(v_1)_{|\Gamma}$.

The map $v_1 \to v_2$ must be continuous for the H^1 norms. We must therefore set up a norm on Γ which is

- not too strong, for the continuity of the trace $v_1 \to (v_1)_{|\Gamma}$,
- strong enough, for the continuity of the lifting $(v_1)_{|\Gamma} \to v_2$.

Approach

- Previous approaches attempted to interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014].

  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

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- Previous approaches attempted to interpolate discrete functions on H^1 functions, to use the continuous trace/lifting [Cowsar et al., 1995, Diosady and Darmofal, 2012, Cockburn et al., 2014].

  $\rightsquigarrow$ restriction to FE meshes (triangular/tetrahedral or rectangular/hexahedral).

- Here, following principles of Discrete Functional Analysis [Eymard et al., 2010, Droniou et al., 2018], we do not use continuous trace/lifting results but mimic their proofs in the fully discrete setting.

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Estimate I

Starting point: set $y = x + \rho$ and write

$$
\begin{aligned}
& u(0, x+\rho) - u(0, x) \\
& \quad = u(0, x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0, x) \\
& \quad = \int_\rho^0 \partial_1 u(s, x+\rho) ds + \int_0^\rho \partial_2 u(\rho, x+s) ds + \int_0^\rho \partial_1 u(s, x) ds.
\end{aligned}
$$

Estimate II

Starting point: set $y = x + \rho$ and write

$$\begin{aligned} &u(0, x+\rho) - u(0,x) \\ &\quad = u(0,x+\rho) - u(\rho, x+\rho) + u(\rho, x+\rho) - u(\rho, x) + u(\rho, x) - u(0,x) \\ &\quad = \int_\rho^0 \partial_1 u(s, x+\rho) ds + \int_0^\rho \partial_2 u(\rho, x+s) ds + \int_0^\rho \partial_1 u(s,x) ds. \end{aligned}$$

Take L^2-norms w.r.t. x (swap integrals):

$$\begin{aligned} &\|u(0, \cdot + \rho) - u(0, \cdot)\|_{L^2(\mathbb{R})} \\ &\qquad \le \int_0^\rho \|\partial_1 u(s, \cdot+\rho)\|_{L^2(\mathbb{R})} \, ds + \int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})} \, ds \\ &\qquad\qquad + \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds \\ &\qquad \le \rho \Big(\underbrace{\frac{2}{\rho} \int_0^\rho \|\partial_1 u(s, \cdot)\|_{L^2(\mathbb{R})} \, ds}_{=: F_1(\rho)} + \underbrace{\|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}}_{=: F_2(\rho)} \Big). \end{aligned}$$

Estimate III

Change of variable ($y = x + \rho$) in the $H^{1/2}$ semi-norm:

$$\begin{aligned} |u(0,\cdot)|^2_{1/2,\mathbb{R}} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(0,x) - u(0,y)|^2}{|x-y|^2}\, dxdy \\ &= \int_{\mathbb{R}} \frac{\|u(0,\cdot) - u(0,\cdot+\rho)\|^2_{L^2(\mathbb{R})}}{\rho^2} d\rho \\ &\leq C(\|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})}) \end{aligned}$$

where

$$F_1(\rho) = \frac{2}{\rho} \int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})}\, ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Estimate IV

$$|u(0,\cdot)|^2_{1/2,\mathbb{R}} \leq C(\|F_1\|^2_{L^2(\mathbb{R})} + \|F_2\|^2_{L^2(\mathbb{R})})$$

$$F_1(\rho) = \frac{2}{\rho}\int_0^\rho \|\partial_1 u(s,\cdot)\|_{L^2(\mathbb{R})}\, ds, \quad F_2(\rho) = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Conclusion:

$$\|F_2\|^2_{L^2(\mathbb{R})} = \|\partial_2 u\|^2_{L^2(\mathbb{R}^2)} \leq |u|^2_{H^1(\mathbb{R}^2)}.$$

By Hardy inequality:

$$\|F_1\|^2_{L^2(\mathbb{R})} \leq C\|\partial_1 u\|^2_{L^2(\mathbb{R}^2)} \leq C|u|^2_{H^1(\mathbb{R}^2)}.$$

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Driving principles

- Points become faces.
  ■ *Continuous* $H^{1/2}$ *seminorm integrates over* x, y, *discrete* $H^{1/2}$*-seminorm sums over pairs of faces*.

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- Integrate along lines $\rightsquigarrow$ sum over cells/faces that intersect the line.

$$\int_0^\rho \partial_1 u(s,x)dx \rightsquigarrow \overline{v}_{t_N} - \overline{v}_{f_{N-1}} + \overline{v}_{f_{N-1}} - \overline{v}_{t_{N-1}} + \cdots + \overline{v}_{t_1} - \overline{v}_f$$
$$\lesssim \sum_{t \in \mathrm{Li}(f, t_N)} h^{\frac{2-d}{2}} |\underline{v}_t|_{1,t}.$$

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- Need a distance between faces/cells: has to be up to h.
  ■ *Cannot consider all $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ such that $\delta_{ff'} = \rho$ for a given ρ... Instead, $(f, f') \in \mathcal{F}_h^{\mathrm{bd}}$ are "at distance ℓh of each other" if $\ell h \leq \delta_{ff'} < (\ell + 1)h$.*
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- Need to be able to swap integrals.
  ■ *Integrate vertically to* $\partial\Omega$ *then parallel to* $\partial\Omega$ $\rightsquigarrow$ *layers along* $\partial\Omega$*.*

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- Need a discrete Hardy inequality: $r_m \geq 0$ and $R_l := \frac{1}{l}\sum_{m=0}^{l} r_m$, then

$$\sum_{l=1}^{L} R_l^2 \leq 32 \sum_{l=0}^{L} r_l^2.$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot{+}s)\, ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot{+}s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Take (f, f') "within distance ℓh" and consider

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{t \in \mathrm{Li}(ff';\delta_{ff'})} |\underline{v}_t|_{1,t}.$$

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$$|v_{t_{ff',f}} - v_{t_{ff',f'}}| \lesssim h^{\frac{2-d}{2}} \sum_{r=1}^{\ell} \sum_{\substack{t\in \mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}.$$

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- Estimate cardinality $\#\{t \in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t)-x_f| \simeq rh\} \lesssim 1$, so

$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t\in \mathrm{Li}(ff';\delta_{ff'}) \\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}^2.$$

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$$|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2 \lesssim h^{2-d}\ell \sum_{r=1}^{\ell} \sum_{\substack{t\in\mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

- Multiply by $|f|\,|f'|/\delta_{ff'}^2 \lesssim h^{d-2}/\ell^d$, sum over (f,f'):

$$\sum_{(f,f'),\,\delta_{ff'}\simeq\ell h} |f|\,|f'|\frac{|v_{t_{ff',f}} - v_{t_{ff',f'}}|^2}{\delta_{ff'}^2}$$

$$\lesssim \frac{1}{\ell^{d-1}} \sum_{r=1}^{\ell} \sum_{(f,f'),\,\delta_{ff'}\simeq\ell h} \sum_{\substack{t\in\mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|^2_{1,t}.$$

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$$\frac{1}{\rho}\|\int_0^\rho \partial_2 u(\rho,\cdot+s)\,ds\|_{L^2(\mathbb{R})} \le \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho,\cdot+s)\|_{L^2(\mathbb{R})}\,ds = \|\partial_2 u(\rho,\cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Multiply by $|f|\,|f'|/\delta_{ff'}^2 \lesssim h^{d-2}/\ell^d$, sum over (f,f'):

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{(f,f'),\,\delta_{ff'}\simeq \ell h}\ \sum_{\substack{t\in \mathrm{Li}(ff';\delta_{ff'})\\ |p(x_t)-x_f|\simeq rh}} |\underline{v}_t|_{1,t}^2.$$

- Swap sums over faces and cells:

$$\frac{1}{\ell^{d-1}}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\le \mathrm{dist}(p(x_t),\partial\Omega)\le \ell h\}} |\underline{v}_t|_{1,t}^2\ \#\mathfrak{F}(t,r)$$

where $\mathfrak{F}(t,r) := \{(f,f')\ :\ \delta_{ff'}\simeq \ell h,\ t\in \mathrm{Li}(ff';\delta_{ff'}),\ |p(x_t)-x_f|\simeq rh\}$.

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where $\mathfrak{F}(t,r) := \{(f,f')\ :\ \delta_{ff'}\simeq \ell h,\, t\in \mathrm{Li}(ff';\delta_{ff'}),\, |p(x_t)-x_f|\simeq rh\}$.

- Estimate cardinality: $\#\mathfrak{F}(t,r)\lesssim \ell^{d-2}$, so

$$\frac{1}{\ell}\sum_{r=1}^{\ell}\sum_{\{t\,:(\ell-2)h\leq \mathrm{dist}(p(x_t),\partial\Omega)\leq \ell h\}} |\underline{v}_t|_{1,t}^2 \lesssim \sum_{\{t\,:(\ell-2)h\leq \mathrm{dist}(p(x_t),\partial\Omega)\leq \ell h\}} |\underline{v}_t|_{1,t}^2$$

Swap integrals and translate

Continuous manipulations: (for F_2)

$$\frac{1}{\rho}\| \int_0^\rho \partial_2 u(\rho, \cdot + s)\, ds\|_{L^2(\mathbb{R})} \leq \frac{1}{\rho}\int_0^\rho \|\partial_2 u(\rho, \cdot + s)\|_{L^2(\mathbb{R})}\, ds = \|\partial_2 u(\rho, \cdot)\|_{L^2(\mathbb{R})}.$$

Discrete manipulations:

- Estimate cardinality: $\#\mathfrak{F}(t,r) \lesssim \ell^{d-2}$, so

$$\sum_{\{t : (\ell-2)h \leq \mathrm{dist}(p(x_t), \partial\Omega) \leq \ell h\}} |\underline{v}_t|^2_{1,t}$$

- Conclude by summing over ℓ (each layer appears 3 times):

$$3 \sum_t |\underline{v}_t|^2_{1,t} = 3|\underline{v}_h|^2_{1,h}$$

Outline

Outline

Construction of the lifting

Take $w \in H^{1/2}(\mathbb{R}^{d-1})$ and define $v \in H^1([0,1) \times \mathbb{R}^{d-1})$ by by averaging w over the base of a cone, which becomes more and more narrow as we get close to the boundary.

With $\rho_x(\boldsymbol{y}) = x^{-(d-1)}\rho(x^{-1}\boldsymbol{y})$ usual smoothing kernel,

$$v(x, \boldsymbol{y}) = (\rho_x \star w)(\boldsymbol{y}).$$

Some arguments in the estimates on partial derivatives

- $\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y}) d\boldsymbol{y} = 0$ to write (for $i \geq 2$)

$$\partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d} \int_{\mathbb{R}} (w(\boldsymbol{z}) - w(\boldsymbol{y})) \partial_i \rho(x^{-1}(\boldsymbol{y} - \boldsymbol{z})) d\boldsymbol{z}.$$

- $\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{z}))| d\boldsymbol{z} \leq C/x$ to write, using Cauchy–Schwarz:

$$\left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})| |\partial_i(\rho_x(\boldsymbol{y} - \boldsymbol{z})) d\boldsymbol{z}| \right)^2 \leq \frac{C}{x} \int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2 |\partial_i(\rho_x(\boldsymbol{y} - \boldsymbol{z}))| d\boldsymbol{z}.$$

Outline

Construction of the lifting

Also average on the base of a cone...

- For $t \in \mathcal{T}_h$, set $\delta_t = \operatorname{dist}(x_t, \partial\Omega)$ and

$$\mathcal{A}_t = \{f \in \mathcal{F}_h^{\mathrm{bd}} \,:\, \operatorname{dist}(p(x_t), f) \leq \delta_t)\}.$$

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- Give to each $f \in \mathcal{A}_t$ an identical weight:

$$\rho_t(f) = \begin{cases} \frac{1}{\#\mathcal{A}_t} & \text{if } f \in \mathcal{A}_t, \\ 0 & \text{otherwise.} \end{cases}$$

Construction of the lifting

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$$\rho_t(f) = \begin{cases} \frac{1}{\#\mathcal{A}_t} & \text{if } f \in \mathcal{A}_t, \\ 0 & \text{otherwise.} \end{cases}$$

- Lift $w_h = (w_f)_{f \in \mathcal{F}_h^{\mathrm{bd}}} \in U_h^{\mathrm{bd}}$ into $v_h = ((v_t)_{t \in \mathcal{T}_h}, (v_f)_{f \in \mathcal{F}_h})$ such that

$$v_t = \frac{1}{\#\mathcal{A}_t} \sum_{f \in \mathcal{A}_t} \overline{w}_f = \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} \overline{w}_f \rho_t(f)$$

$$v_f = \begin{cases} \frac{v_t + v_{t'}}{2} & \text{if } f \text{ internal face between } t, t' \in \mathcal{T}_h, \\ w_f & \text{if } f \in \mathcal{F}_h^{\mathrm{bd}}. \end{cases}$$

Estimate of $|\underline{v}_h|_{1,h}$ I

Low order reconstruction: all v_t are constant, so

$$|\underline{v}_h|^2_{1,h} \simeq \sum_{(t,t')\text{ neighbours}} h^{d-2}|v_t - v_{t'}|^2.$$

Estimate of $|\underline{v}_h|_{1,h}$ II

Adaptation of arguments

☐ Continuous:

$$\int_{\mathbb{R}^{d-1}} \partial_i \rho(x^{-1}\boldsymbol{y})d\boldsymbol{y} = 0$$
$$\rightsquigarrow \quad \partial_i(\rho_x \star w)(\boldsymbol{y}) = \frac{1}{x^d}\int_{\mathbb{R}}(w(\boldsymbol{z}) - w(\boldsymbol{y}))\partial_i\rho(x^{-1}(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z} \quad \forall \boldsymbol{y}$$

☐ Discrete:

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}(\rho_t(f) - \rho_{t'}(f))\Big(= \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}\rho_t(f) - \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}\rho_{t'}(f) = 1 - 1\Big) = 0$$
$$\rightsquigarrow \quad v_t - v_{t'} = \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}}(\overline{w}_f - \overline{w}_{f'})D_g\rho(f) \quad \forall f' \in \mathcal{F}_h^{\mathrm{bd}},$$

where $D_g\rho(f) = \rho_t(f) - \rho_{t'}(f)$ with (t,t') cells on each side of $g \in \mathcal{F}_h^{\mathrm{in}}$.

Estimate of $|\underline{v}_h|_{1,h}$ III

- Continuous:

$$\int_{\mathbb{R}^{d-1}} |\partial_i(\rho_x(\boldsymbol{y})|d\boldsymbol{y} \leq \frac{C}{x}$$

$$\rightsquigarrow \quad \left(\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})||\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))d\boldsymbol{z}|\right)^2 \leq \frac{C}{x}\int_{\mathbb{R}} |w(\boldsymbol{y}) - w(\boldsymbol{z})|^2|\partial_i(\rho_x(\boldsymbol{y}-\boldsymbol{z}))|d\boldsymbol{z}.$$

- Discrete:

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |D_g\rho(f)| \lesssim \frac{h}{\delta_t}$$

$$\rightsquigarrow |\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}}\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2|D_g\rho(f)|\frac{h^{d-1}}{\delta_t}.$$

with $f' \in \mathcal{F}_h^{\mathrm{bd}}$ such that g "projects close to f'".

Estimate of $|\underline{v}_h|_{1,h}$ IV

$$\sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |D_g\rho_t(f)| = \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} |\rho_t(f) - \rho_{t'}(f)| \lesssim \frac{h}{\delta_t}.$$

Requires:

- ☐ $\#\mathcal{A}_t \simeq \left(\frac{\delta_t}{h}\right)^{d-1}$
- ☐ $\#(\mathcal{A}_t \Delta \mathcal{A}_{t'}) \lesssim \left(\frac{\delta_t}{h}\right)^{d-2}$
- ☐ $\forall f \in \mathcal{A}_t \cap \mathcal{A}_{t'}\ ,\ |\rho_t(f) - \rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d}$
- ☐ $|\rho_t(f) - \rho_{t'}(f)| \lesssim \left(\frac{h}{\delta_t}\right)^{d-1}$.

Estimate of $|\underline{v}_h|_{1,h}$ V

$$|\underline{v}_h|_{1,h}^2 \lesssim \sum_{g\in\mathcal{F}_h^{\mathrm{in}}} \sum_{f\in\mathcal{F}_h^{\mathrm{bd}}} (\overline{w}_f - \overline{w}_{f'})^2 |D_g\rho(f)| \frac{h^{d-1}}{\delta_t}.$$

Write $\sum_{g\in\mathcal{F}_h^{\mathrm{in}}}$ as $\sum_{f'\in\mathcal{F}_h^{\mathrm{bd}}} \sum_{g \text{ above } f'}$ and conclude by proving

$$\sum_{g \text{ above } f'} |D_g\rho(f)| \frac{h^{d-1}}{\delta_t} \lesssim \frac{|f|\,|f'|}{\delta_{ff'}^d}.$$

Outline

Equivalence of norms

- Let $\mathcal{E}_h : U_h^{\mathrm{bd}} \to \underline{U}_h$ be the discrete harmonic extension for the discrete H^1-seminorm (minimises this norm with given boundary conditions).

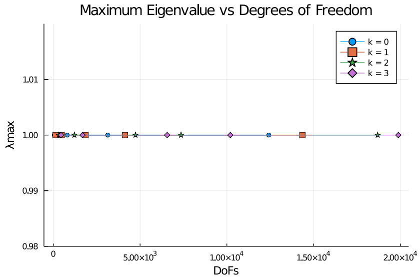
- The discrete trace and lifting give

$$|\mathcal{E}_h(w_h)|_{1,h} \simeq |w_h|_{1/2,h} \quad \forall w_h \in U_h^{\partial}.$$

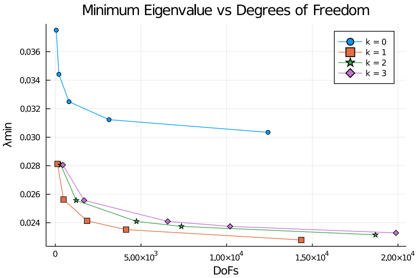
- We assess this equivalence by solving a generalised eigenvalue problem to evaluate the constants in the upper and lower bounds.

Results

Ω: square. Cartesian mesh.



(a) Maximum eigenvalues



(b) Minimum eigenvalues

Conclusions

- Complete discrete trace theory, with definition of boundary norm, trace inequality and lifting in discrete spaces of polytopal hybrid methods.
- Applicable to a range of schemes: HHO, VEM, HDG, etc. (and even FEM).
- Constructive proofs, obtained by mimicking proofs in the continuous setting (more flexible than looking for lifting in conforming spaces).
- For the moment, requires quasi-uniform meshes, but with elements of generic shapes.
- Allows for the analysis of BDDC and similar for polytopal methods.

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Thank you for your attention!

References I

Cockburn, B., Dubois, O., Gopalakrishnan, J., and Tan, S. (2014).
Multigrid for an HDG method.
IMA J. Numer. Anal., 34(4):1386–1425.

Cockburn, B., Gopalakrishnan, J., and Lazarov, R. (2009).
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365.

Cowsar, L. C., Mandel, J., and Wheeler, M. F. (1995).
Balancing domain decomposition for mixed finite elements.
Math. Comp., 64(211):989–1015.

de Dios, B. A., Lipnikov, K., and Manzini, G. (2016).
The nonconforming virtual element method.
ESAIM: Mathematical Modelling and Numerical Analysis, 50(3):879–904.

Di Pietro, D. A. and Droniou, J. (2020).
The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications, volume 19 of *Modeling, Simulation and Applications*.
Springer International Publishing, https://hal.archives-ouvertes.fr/hal-02151813.

Diosady, L. T. and Darmofal, D. L. (2012).
A unified analysis of balancing domain decomposition by constraints for discontinuous Galerkin discretizations.
SIAM J. Numer. Anal., 50(3):1695–1712.

References II

Droniou, J., Eymard, R., Gallouët, T., Guichard, C., and Herbin, R. (2018).
The gradient discretisation method, volume 82 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*.
Springer, Cham.

Eymard, R., Gallouët, T., and Herbin, R. (2000).
Finite volume methods.
In Ciarlet, P. G. and Lions, J.-L., editors, *Techniques of Scientific Computing, Part III*, Handbook of Numerical Analysis, VII, pages 713–1020. North-Holland, Amsterdam.

Eymard, R., Gallouët, T., and Herbin, R. (2010).
Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces.
IMA J. Numer. Anal., 30(4):1009–1043.