Polytopal schemes for contact problems in fractured media

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Slides:

References for this presentation

- Low order scheme: design, application to poromechanics with Coulomb friction, analysis for contact problem with Tresca friction:

  • Droniou, J., Enchéry, G., Faille, I., Haidar, A., and Masson, R. *A bubble VEM–fully discrete polytopal scheme for mixed-dimensional poromechanics with frictional contact at matrix-fracture interfaces.* Computer Methods in Applied Mechanics and Engineering, 422:116838, 2024. `https://arxiv.org/abs/2312.09319`

  • Droniou, J., Haidar, A., and Masson, R. *Analysis of a VEM–fully discrete polytopal scheme with bubble stabilisation for contact mechanics with tresca friction.* ESAIM: M2AN Math. Model. Numer. Anal. 59 (2), pp. 1043–1074, 2025. `http://arxiv.org/abs/2404.03045`.

- High-order scheme: design and analysis for Tresca friction:

  • J. Droniou, R. Kumar, R. Masson and R. Singla *A higher order polytopal methods for contact mechanics with Tresca friction*. J. Comput. Phys. 560, 114923, 26p, 2026. `http://arxiv.org/abs/2601.07586`.

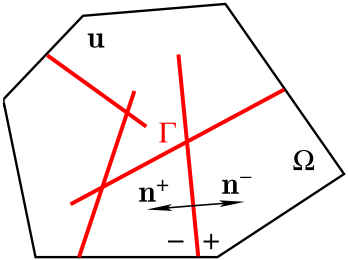
Outline

Slides:

Matrix and fracture network

Notations: Ω domain (matrix), Γ fracture, two sides $\pm$ with outward normals $\mathbf{n}^{\pm}$.

Unknowns: displacement $\boldsymbol{u}$ in matrix (discontinuous across fractures).



Contact mechanics with Tresca friction

$$
\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{on } \Omega \setminus \overline{\Gamma},\\
\boldsymbol{\sigma}(\mathbf{u}) &= \frac{E}{1+\nu}\left(\boldsymbol{\epsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu}(\mathrm{tr}\boldsymbol{\epsilon}(\mathbf{u}))\mathrm{I}_3\right) && \text{on } \Omega \setminus \overline{\Gamma},\\
\gamma_{\mathbf{n}}^{+}\boldsymbol{\sigma}(\mathbf{u}) + \gamma_{\mathbf{n}}^{-}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{0} && \text{on } \Gamma,\\
T_{\mathbf{n}}(\mathbf{u}) \le 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} \le 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} T_{\mathbf{n}}(\mathbf{u}) &= 0 && \text{on } \Gamma,\\
|\mathbf{T}_\tau(\mathbf{u})| &\le g && \text{on } \Gamma,\\
\mathbf{T}_\tau(\mathbf{u}) \cdot [\![\mathbf{u}]\!]_\tau + g|[\![\mathbf{u}]\!]_\tau| &= 0 && \text{on } \Gamma,
\end{aligned}
$$

where:

- $g \ge 0$ Tresca threshold.
- ν Poisson ratio, E Young modulus.
- $[\![\mathbf{u}]\!] = \gamma^{+}\mathbf{u} - \gamma^{-}\mathbf{u}$ jump across fracture.
- $\mathbf{T}(\mathbf{u}) = \gamma_{\mathbf{n}}^{+}\boldsymbol{\sigma}(\mathbf{u})$ surface traction on positive side ($\mathbf{T}_\tau$ and $T_{\mathbf{n}}$ tangential and normal^{+} components).

Contact mechanics with Tresca friction

$$
\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{on } \Omega \setminus \overline{\Gamma}, \\
\boldsymbol{\sigma}(\mathbf{u}) = \frac{E}{1+\nu}\Big(\boldsymbol{\epsilon}(\mathbf{u}) &+ \frac{\nu}{1-2\nu}(\operatorname{tr}\boldsymbol{\epsilon}(\mathbf{u}))\mathrm{I}_3\Big) && \text{on } \Omega \setminus \overline{\Gamma}, \\
\gamma_{\mathbf{n}}^{+}\boldsymbol{\sigma}(\mathbf{u}) + \gamma_{\mathbf{n}}^{-}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{0} && \text{on } \Gamma, \\
T_{\mathbf{n}}(\mathbf{u}) \leq 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} \leq 0,\ [\![\mathbf{u}]\!]_{\mathbf{n}} T_{\mathbf{n}}(\mathbf{u}) &= 0 && \text{on } \Gamma, \\
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\end{aligned}
$$

Other models:

- *No friction is $g = 0$.*
- *Coulomb is $g \rightsquigarrow -F\mathbf{T}_{\mathbf{n}}$ and $[\![\mathbf{u}]\!]_{\tau} \rightsquigarrow \partial_t [\![\mathbf{u}]\!]_{\tau}$.*

Variational formulation

Weak formulation for mechanical equations: using Lagrange multiplier $\boldsymbol{\lambda} = -\mathbf{T}$ to impose the contact conditions.

Space and cone:

$$\mathbf{U}_0 = \{\boldsymbol{v} \in H^1(\Omega \backslash \overline{\Gamma})^d \ : \ \boldsymbol{v}_{|\partial\Omega} = 0\},$$
$$\boldsymbol{C}_f = \left\{\boldsymbol{\mu} \in [\![\mathbf{U}_0]\!]' \ : \ \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle_\Gamma \leq \int_\Gamma g|\boldsymbol{w}_\tau| \quad \forall \boldsymbol{w} \in [\![\mathbf{U}_0]\!] \text{ s.t. } w_\mathbf{n} \leq 0\right\}.$$

Mixed variational formulation: find $\boldsymbol{u} \in \mathbf{U}_0$ and $\boldsymbol{\lambda} \in \boldsymbol{C}_f$ such that

$$\int_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) + \langle \boldsymbol{\lambda}, [\![\boldsymbol{v}]\!] \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \boldsymbol{v} \qquad \forall \mathbf{v} \in \mathbf{U}_0,$$

$$\langle \boldsymbol{\mu} - \boldsymbol{\lambda}, [\![\boldsymbol{u}]\!] \rangle_\Gamma \leq 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{C}_f.$$

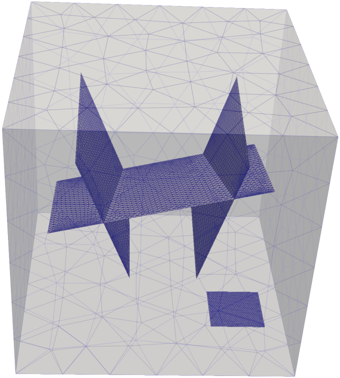
Outline

Slides:

General principles of the two schemes

- Displacement DOFs:
  - Low order: one per vertex (+bubble on fracture faces).
  - High order: one per vertex, edge, face and cell.
- Traction DOF: one per fracture face (both low and high order).
- Reconstruction of (broken) piecewise polynomial displacement and strain in each cell $\rightsquigarrow$ consistent contributions to the discrete bilinear forms.
  - Low order: $\mathbb{P}^1$ displacement.
  - High order: $\mathbb{P}^2$ displacement.
- Weak conformity imposed by a stabilisation term penalising differences between displacement reconstruction in cells and on faces.

Example of practical gain I



- $\Omega = (0,1)^3$.
- Elasticity parameters: $E = 4 \times 10^9$, $\nu = 0.2$.
- Tresca with $g = 4 \times 10^6$.
- BC: $\boldsymbol{u} = 0$ at the bottom, $\boldsymbol{u} = [1.5, -1.5, -2] \times 10^{-3}$ at the top. Neumann elsewhere.

Example of practical gain II

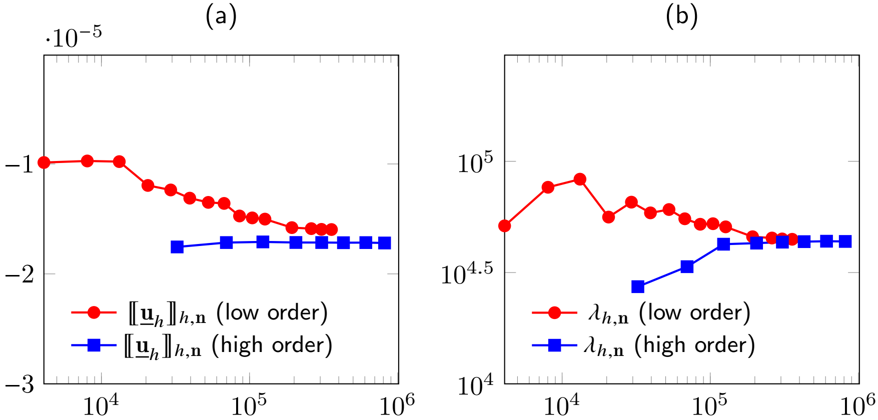


Figure: Normal displacement jump and traction vs. nb DOFs on leftmost vertical fracture.

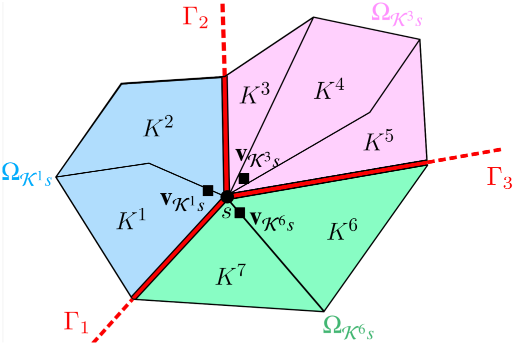
Outline

Slides:

Mesh

Mesh $\mathcal{M}$: polytopal and compatible with fractures.

- $\mathcal{F}^{+}_{\Gamma,K}$ faces of K on positive side of fracture.
- For $s \in \mathcal{V}$, $\mathcal{K}s$: set of cells on the same side of K.
- If $\sigma \in \mathcal{F}_{\Gamma}$: K on positive side, L on negative side.
- Main displacement unknowns: one per vertex per side of fracture.



Discrete spaces I

Displacement: nodal unknowns (discontinuous across fracture) and one bubble on each fracture face (positive side).

$$\mathbf{U}_{0,h} = \Big\{ \boldsymbol{v}_h = ((\boldsymbol{v}_{\mathcal{K}s})_{K\in\mathcal{M},\, s\in\mathcal{V}_K}, (\boldsymbol{v}_{K\sigma})_{K\in\mathcal{M},\, \sigma\in\mathcal{F}^+_{\Gamma,K}}) \ : \\ \boldsymbol{v}_{\mathcal{K}s} \in \mathbb{R}^d,\ \boldsymbol{v}_{K\sigma} \in \mathbb{R}^d,\ \boldsymbol{v}_{\mathcal{K}s} = 0 \text{ if } s \in \mathcal{V}^{\text{ext}} \\ \boldsymbol{v}_{\mathcal{K}s} = \boldsymbol{v}_{\mathcal{L}s} \text{ if } K, L \text{ are on the same side of } \Gamma \Big\}.$$

Discrete spaces II

Lagrange multipliers: piecewise constant on fracture faces.

$$\mathbf{M}_h = \left\{ \boldsymbol{\lambda}_h \in L^2\left(\Gamma\right)^d \,:\, \boldsymbol{\lambda}_\sigma := \left(\boldsymbol{\lambda}_h\right)_{|\sigma} \text{ is constant for all } \sigma \in \mathcal{F}_\Gamma \right\}.$$

Interpolator: averaging on each face: if $\boldsymbol{\lambda} \in \boldsymbol{L}^2(\Gamma)$ then

$$\mathcal{I}_{\mathbf{M}_h} \boldsymbol{\lambda} = \left(\frac{1}{|\sigma|} \int_\sigma \boldsymbol{\lambda} \right)_{\sigma \in \mathcal{F}_K} \in \mathbf{M}_h.$$

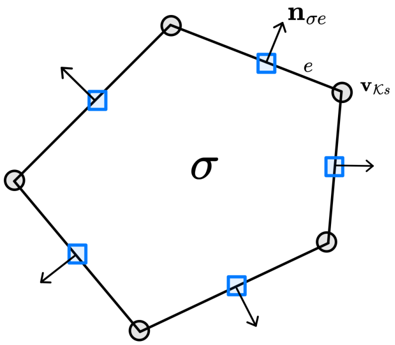
Discrete dual cone:

$$\mathbf{C}_h = \left\{ \boldsymbol{\mu}_h = \left(\mu_{\mathcal{D},\mathbf{n}}, \boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}\right) \in \mathbf{M}_h \;:\; \mu_{\mathcal{D},\mathbf{n}} \geq 0,\ |\boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}| \leq g \right\}.$$

Reconstructions in $\mathbf{U}_{0,h}$: faces

- Nodal values $\rightsquigarrow$ reconstructed edge values $\rightsquigarrow$ reconstructed face gradient:

$$\nabla_{K\sigma}\boldsymbol{v}_h = \frac{1}{|\sigma|}\sum_{e=s_1s_2\in\mathcal{E}_\sigma}|e|\frac{\mathbf{v}_{\mathcal{K}s_1}+\mathbf{v}_{\mathcal{K}s_2}}{2}\otimes\mathbf{n}_{\sigma e}\in\mathbb{R}^3.$$



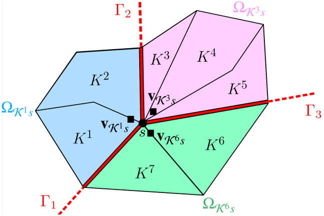
- Nodal values $\rightsquigarrow$ reconstructed face averaged displacement:

$$\Pi^{K\sigma} \boldsymbol{v}_h = \sum_{s \in \mathcal{V}_\sigma} \omega_s^\sigma \boldsymbol{v}_{\mathcal{K}s}.$$

- Interpolator: for $\boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega} \backslash \Gamma)$, $\mathcal{I}_{\mathbf{U}_{0,h}} \boldsymbol{v} \in \mathbf{U}_{0,h}$ given by

$$(\mathcal{I}_{\mathbf{U}_{0,h}} \boldsymbol{v})_{\mathcal{K}s} = \boldsymbol{v}_{|K}(\mathbf{x}_s) \qquad \forall K \in \mathcal{M},\ \forall s \in \mathcal{V}_K,$$

$$(\mathcal{I}_{\mathbf{U}_{0,h}} \boldsymbol{v})_{K\sigma} = \frac{1}{|\sigma|} \int_\sigma (\gamma^{K\sigma} \boldsymbol{v} - \Pi^{K\sigma}(\mathcal{I}_{\mathbf{U}_{0,h}} \boldsymbol{v})) \qquad \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{F}_{\Gamma,K}^+.$$



Reconstructions in $\mathbf{U}_{0,h}$: faces

- Reconstructed jump reconstructions, with bubble:

$$[\![\boldsymbol{v}_h]\!]_\sigma = \underbrace{\Pi_{K\sigma}\boldsymbol{v} - \Pi_{L\sigma}\boldsymbol{v}}_{\text{fixed by vertex DOFs}} + \underbrace{\boldsymbol{v}_{K\sigma}}_{\text{free}}.$$

- Key property for inf-sup stability: for all $\sigma \in \mathcal{F}_\Gamma$ (with K cell on positive side, L cell on negative side), if $\boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega}\backslash\Gamma)$ then

$$\int_\sigma [\![\mathcal{I}_{\mathbf{U}_{0,h}}\boldsymbol{v}]\!]_\sigma = \int_\sigma (\gamma^{K\sigma}\boldsymbol{v} - \Pi^{L\sigma}\mathcal{I}_{\mathbf{U}_{0,h}}\boldsymbol{v}).$$

Reconstructions in $\mathbf{U}_{0,h}$: cells

Same principles...

- Reconstructed face values and bubbles give cell gradient and displacement:

$$\nabla_K \boldsymbol{v}_h = \frac{1}{|K|}\sum_{\sigma\in\mathcal{F}_K} |\sigma| \Pi_{K\sigma}\boldsymbol{v} \otimes \mathbf{n}_{K\sigma} + \frac{1}{|K|}\sum_{\sigma\in\mathcal{F}^+_{\Gamma,K}} |\sigma| \boldsymbol{v}_{K\sigma} \otimes \mathbf{n}_{K\sigma}, \tag{1}$$

$$\Pi_K \boldsymbol{v}_h = \sum_{s\in\mathcal{V}_K} \omega_s^K \boldsymbol{v}_{\mathcal{K}s}. \tag{2}$$

($\mathbb{P}^1$ *displacement easily obtained from these*)

Global reconstructions: $[\![\cdot]\!]_h$, ∇_h, Π_h, $\mathbb{e}_h$, div_h, $\mathbb{\sigma}_h$.

Scheme

Continuous formulation: find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbf{U}_0 \times \boldsymbol{\lambda} \in \boldsymbol{C}_f$ such that

$$\int_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) + \langle \boldsymbol{\lambda}, [\![\boldsymbol{v}]\!] \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \boldsymbol{v} \qquad \forall \mathbf{v} \in \mathbf{U}_0,$$

$$\langle \boldsymbol{\mu} - \boldsymbol{\lambda}, [\![\boldsymbol{u}]\!] \rangle_\Gamma \leq 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{C}_f.$$

Discrete formulation: find $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h) \in \mathbf{U}_{0,h} \times \mathbf{M}_h$ such that

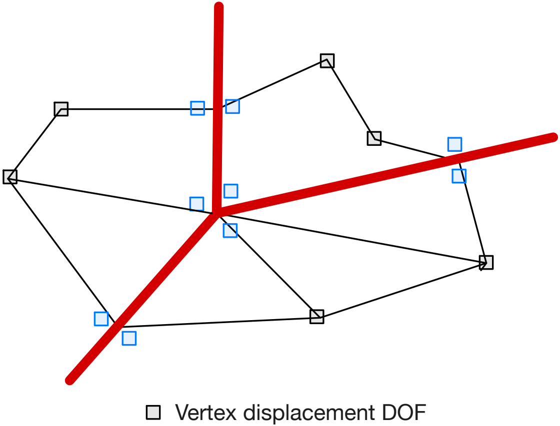
$$\int_\Omega \sigma_h(\boldsymbol{u}_h) : \varepsilon_h(\boldsymbol{v}_h) + S_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_\Gamma \boldsymbol{\lambda}_h \cdot [\![\boldsymbol{v}_h]\!]_h = \sum_{K \in \mathcal{M}} \int_K \mathbf{f} \cdot \Pi_h \boldsymbol{v}_h, \qquad \forall \boldsymbol{v}_h \in \mathbf{U}_{0,h},$$

$$\int_\Gamma (\boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) \cdot [\![\boldsymbol{u}_h]\!]_h \leq 0 \qquad \forall \boldsymbol{\mu}_h \in \mathbf{M}_h.$$

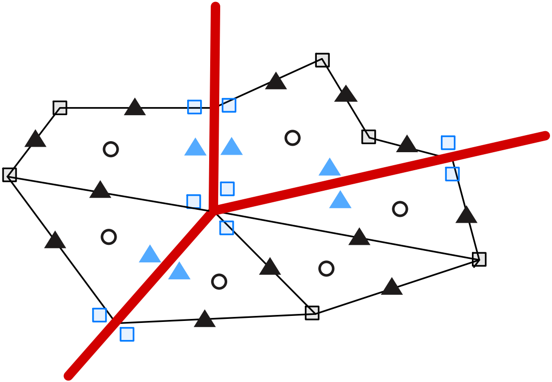
Outline

Slides:

Low-order DOFs



High-order DOFs



- Vertex displacement DOF
- Edge displacement DOF
- Cell displacement DOF

Reconstructions

Principles

- Hierarchical: from edges to faces to cells.
- Enhancement: gradient first, then displacement.
- All based on mimicking integration-by-parts formulas.

Reconstructions

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Example on a face. If $\underline{\mathbf{v}}_F = (v_F, (v_E)_{E\in\mathcal{E}_F}, (v_V)_{V\in\mathcal{V}_F})$ is a vector of DOFs on a face F (with edges, vertices), then $G_F(\underline{\mathbf{v}}_F) \in \mathbb{P}^1(F)^2$ and $\gamma_F(\underline{\mathbf{v}}_F) \in \mathbb{P}^2(F)$ are defined by

$$\int_F G_F(\underline{\mathbf{v}}_F)\cdot \boldsymbol{w} = -\int_F v_F \operatorname{div}\boldsymbol{w} + \sum_{E\in\mathcal{E}_F}\int_E \gamma_E(\underline{\mathbf{v}}_E)\boldsymbol{w}\cdot\mathbf{n}_{FE} \qquad \forall \boldsymbol{w}\in\mathbb{P}^1(F)^2,$$

$$\int_F \gamma_F(\underline{\mathbf{v}}_F)\operatorname{div}\boldsymbol{z} = -\int_F G_F(\underline{\mathbf{v}}_F)\cdot\boldsymbol{z} + \sum_{E\in\mathcal{E}_F}\int_E \gamma_E(\underline{\mathbf{v}}_E)\boldsymbol{z}\cdot\mathbf{n}_{FE} \qquad \forall \boldsymbol{z}\in(\boldsymbol{x}-\boldsymbol{x}_F)\mathbb{P}^2(F)$$

Outline

Slides:

Low-order scheme

Theorem

If $\boldsymbol{u} \in H^2(\mathcal{M})$ *and* $\boldsymbol{\lambda} \in H^1(\mathcal{F}_\Gamma)$, *then*

$$\|\nabla_h \boldsymbol{u}_h - \nabla \boldsymbol{u}\|_{L^2(\Omega\setminus\overline{\Gamma})} + \|\boldsymbol{\lambda}_h - \boldsymbol{\lambda}\|_{-1/2,\Gamma} \leq C_{\boldsymbol{u},\lambda} h.$$

- $\|\cdot\|_{-1/2,\Gamma}$ discrete $H^{-1/2}$-like seminorm.
- Error analysis based on consistency and stability.

  Tools: Körn inequality and inf-sup conditions, ensured by the bubble degrees of freedom.

Tools: Korn and inf-sup

Discrete H^1-norm:

$$\|\boldsymbol{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 := \|\nabla_{\mathcal{D}}\boldsymbol{v}_{\mathcal{D}}\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{M}} S_K(\boldsymbol{v}_{\mathcal{D}},\boldsymbol{v}_{\mathcal{D}}).$$

Theorem (Discrete Korn inequality)

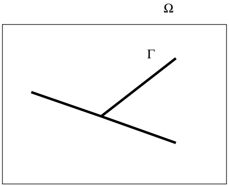
For all $\boldsymbol{v} \in \mathbf{U}_{0,h}$,

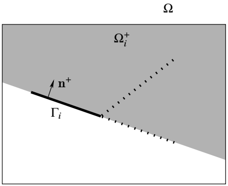
$$\|\boldsymbol{v}_{\mathcal{D}}\|_{1,\mathcal{D}}^2 \lesssim \|\mathbb{\epsilon}_{\mathcal{D}}(\boldsymbol{v}_{\mathcal{D}})\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{M}} S_K(\boldsymbol{v}_{\mathcal{D}},\boldsymbol{v}_{\mathcal{D}}).$$

Theorem (Discrete inf-sup condition)

$$\sup_{\boldsymbol{v}_{\mathcal{D}}\in\mathbf{U}_{0,h}\setminus\{0\}} \frac{\int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}}\cdot [\![\boldsymbol{v}_{\mathcal{D}}]\!]_{\mathcal{D}}}{\|\boldsymbol{v}_{\mathcal{D}}\|_{1,\mathcal{D}}} \gtrsim \|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma} \qquad \forall \boldsymbol{\lambda}_{\mathcal{D}} \in \mathbf{M}_h.$$

Ideas for discrete inf–sup property I





$$\|\boldsymbol{\lambda}_{\mathcal{D}}\|_{-1/2,\Gamma} := \sum_{i\in I} \sup_{\boldsymbol{v}_i \in H^1(\Omega_i^+;\Gamma_i)^d\setminus\{0\}} \frac{\int_{\Gamma_i} \boldsymbol{\lambda}_{\mathcal{D}}\cdot\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|_{H^1(\Omega_i^+)}}.$$

Ingredients for inf-sup:

- Clément-like H^1-stable interpolator adapted to fractures.
- Fortin property for jump: for $\boldsymbol{v}_i \in H^1(\Omega_i^+;\Gamma_i)$ and $\boldsymbol{v}_{\mathcal{D}}$ = interpolant of extension by 0 of $\boldsymbol{v}_i$ by 0,

$$\int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}}\cdot [\![\boldsymbol{v}_{\mathcal{D}}]\!]_{\mathcal{D}} = \int_{\Gamma_i} \boldsymbol{\lambda}_{\mathcal{D}}\cdot\boldsymbol{v}_i.$$

High-order scheme

Discrete energy norm:

$$\|\boldsymbol{v}_h\|_{\mathrm{en},h}^2 := \int_\Omega \sigma_h(\boldsymbol{v}_h) : \varepsilon_h(\boldsymbol{v}_h) + S_h(\boldsymbol{v}_h, \boldsymbol{v}_h).$$

Theorem (Error estimate)

$$\|\underline{\mathbf{u}}_h - \mathbb{I}_h \mathbf{u}\|_{\mathrm{en},h} \leq C_{1,\boldsymbol{u},\lambda} h^{\frac{1}{2}+\mathfrak{r}}$$
$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-1/2,\Gamma} \lesssim C_{2,\boldsymbol{u},\lambda} h^{\frac{1}{2}+\mathfrak{r}},$$

where

- $\mathfrak{r} < 1$ *if* $g = 0$ *(no friction),*
- $\mathfrak{r} = 1/2$ *if* $g \neq 0$ *(friction).*

and $C_{1,\boldsymbol{u},\lambda}$ *does not depend on the Young modulus* E *or the Poisson ratio* ν*.*

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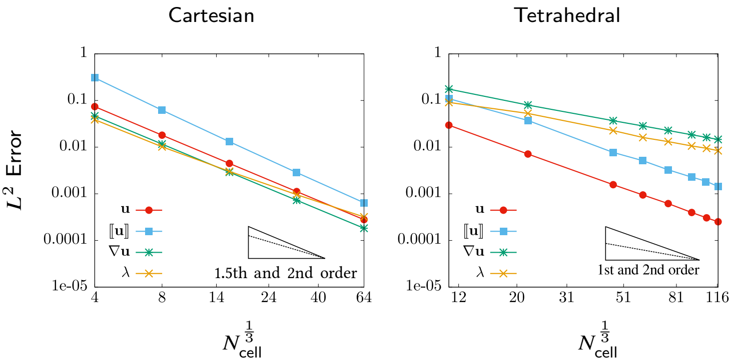
Tresca with 3D manufactured solution I

Setting:

- $\Omega = (-1,1)^3$, $\Gamma = \{0\} \times (-1,1)^2$.
- $g = 1$, $\mu = \lambda = 1$.
- Explicit analytical solution such that:
  - sticky-contact for $z < 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $[\![u]\!]_{\boldsymbol{\tau}} = 0$)
  - slippy-contact for $z > 0$ ($[\![u]\!]_{\mathbf{n}} = 0$, $|[\![u]\!]_{\boldsymbol{\tau}}| > 0$)
- Cartesian, tetrahedral and generalised hexahedral meshes.

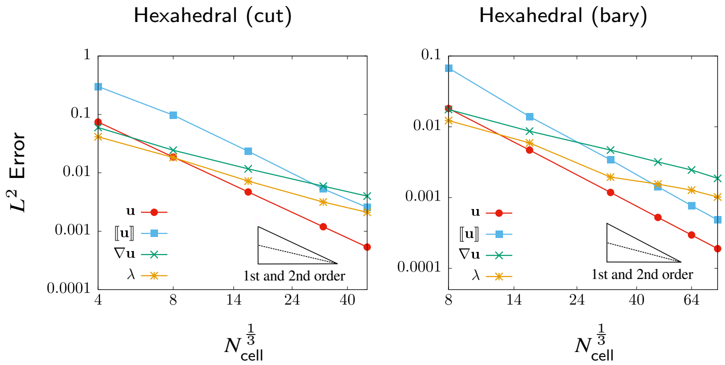
Figure: Generalised hexahedral meshes: cut (left) and barycentric subdivisions (right).

Tresca with 3D manufactured solution II



Note: 10^{-2} accuracy for $\boldsymbol{u}$ achieved with $\sim 10^3$ Cartesian cells vs. $\sim 20^3$ triangular cells.

Tresca with 3D manufactured solution III



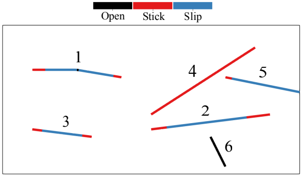
Note: 10^{-2} accuracy for $\boldsymbol{u}$ achieved with $\sim 10^3$ Hexahedral cells.

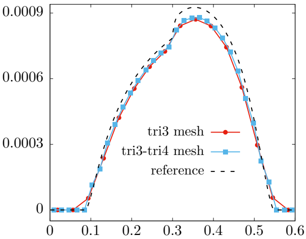
Adding DOFs only on the fractures I

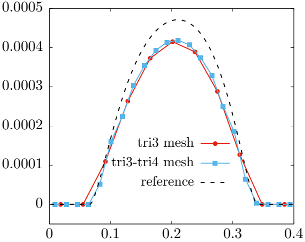
- 6 fractures in a 2m×1m quadrangle.
- Displacement: $\boldsymbol{u} = 0$ at the bottom, $\boldsymbol{u} = [0.005m, -0.002m]$ at the top.
- $E = 4$GPa, $\nu = 0.2$, friction variable in the fractures.
- Triangular meshes, reference solution computed with 730 880 triangles.

Focus on stick-slip states of the fractures

Adding DOFs only on the fractures II







Tangential jumps $[\![\boldsymbol{u}]\!]_\tau$ on fractures 1 and 3.

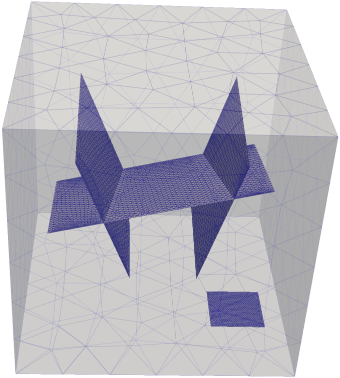
tri3: triangles.

tri3-tri4: triangles with one additional node at fracture edges (polygons).

Outline

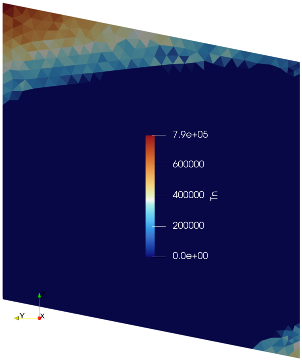
Slides:

Setting

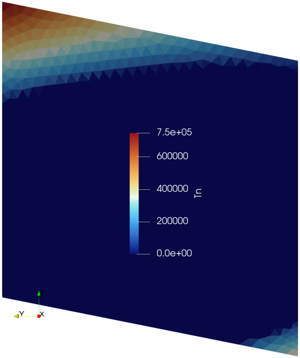


- $\Omega = (0,1)^3$.
- Elasticity parameters: $E = 4 \times 10^9$, $\nu = 0.2$.
- Tresca with $g = 4 \times 10^6$.
- BC: $\boldsymbol{u} = 0$ at the bottom, $\boldsymbol{u} = [1.5, -1.5, -2] \times 10^{-3}$ at the top. Neumann elsewhere.

Normal traction



(a) Low order



(b) High order

Figure: $\lambda_{\mathbf{n}}$ on leftmost vertical fracture.

In both cases, λ approximated by piecewise constant values.

Normal traction

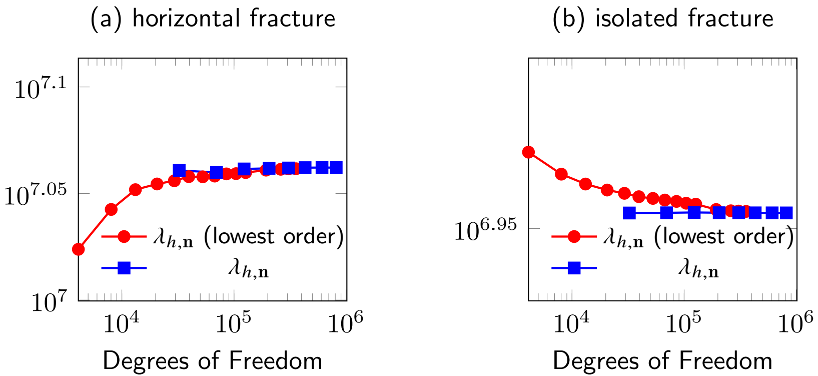


Figure: Average of $\lambda_{\mathbf{n}}$ w.r.t. number of DOFs

Conclusions

- Polytopal schemes, applicable on generic meshes (including hanging nodes, cut cells, local refinements). Seamlessly handles crossing fractures, etc.
- Low-order version, with bubble enrichment (first one for polytopal methods) for inf-sup condition.
  High-order version, only increases order on displacement.
- Convergence analysis with error estimates matching the orders.
- Robust simulations (including solver behaviour) for 3D **poro**mechanical model with network of fractures.
- Simulations show that only one increase of order on displacement already has a strong positive impact.

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Thank you for your attention!