A polytopal complex on manifolds, and applications to the Maxwell equations

Jérôme Droniou

joint work with M. Hanot (U. Lille) and T. Oliynyk (Monash University).

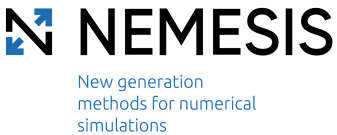
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References for this presentation

- *A polytopal discrete de Rham complex on manifolds, with application to the Maxwell equations*, J. Droniou, M. Hanot, and T. Oliynyk. J. Comput. Phys. 529, Paper No. 113886, 31p, 2025. doi: 10.1016/j.jcp.2025.113886. `https://arxiv.org/abs/2401.16130`.

- *An exterior calculus framework for polytopal methods*. F. Bonaldi, D. A. Di Pietro, J. Droniou, and K. Hu. Journal of the European Mathematical Society, 55p, 2025. doi:10.4171/JEMS/1602. `https://arxiv.org/abs/2303.11093`.

These slides:



Outline

1. Electromagnetism and the de Rham complex
2. Why polytopal methods?
3. Exterior calculus
4. Building blocks: mesh and setting for polynomial complexes on a manifold
5. Discrete De Rham on a manifold
6. Constructing local polynomial spaces
7. Maxwell equations and numerical simulations

Slides



Maxwell's equations and preservation of divergence

Maxwell: E electric field, B magnetic field, ϵ_0 and μ_0 vacuum permittivity and permeability:

$$\begin{aligned} \partial_t B + \mathbf{curl}\, E &= 0 & \operatorname{div} B &= 0 \\ \mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B &= 0 & \operatorname{div} E &= 0 \\ \textit{Evolutions} & & \textit{Constraints} & \end{aligned}$$

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Evolution preserves constraints:

$$0 = \operatorname{div}(\partial_t B + \mathbf{curl}\, E) = \partial_t(\operatorname{div} B) + \cancel{\operatorname{div} \mathbf{curl}\, E} = \partial_t(\operatorname{div} B).$$

Key property: $\operatorname{Im} \mathbf{curl} \subset \ker \operatorname{div}$.

The de Rham complex

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}.$$

Complex: the image of each operator is included in the kernel of the next one.

$$\mathrm{Im}\,\mathbf{grad} \subset \ker \mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} \subset \ker \mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

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$$\mathrm{Im}\,\mathbf{grad} \subset \ker \mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} \subset \ker \mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

Exactness of the complex: the image of each operator is *equal* to the kernel of the next one.

$$\mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}\,, \qquad \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,, \qquad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

Requires topological property on Ω:

- No tunnel for $\mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}$.
- No void for $\mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}$.

Exactness of the complex $\Rightarrow$ Maxwell in potential form

Assumption: Ω is topologically trivial (no void, no tunnel).

Magnetic vector potential: We have $\operatorname{Im} \mathbf{curl} = \ker \operatorname{div}$ so

$$\operatorname{div} B = 0 \Longrightarrow \exists A \text{ s.t. } B = \mathbf{curl}\, A.$$

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Temporal gauge: A is not unique, but the *Hodge* decomposition gives

$$A = A_0 + \mathbf{grad}\, \varphi \quad \text{with } A_0 \perp \operatorname{Im}\mathbf{grad}.$$

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$$A = A_0 + \mathbf{grad}\, \varphi \qquad \text{with } A_0 \perp \operatorname{Im}\mathbf{grad}.$$

Write $E = E_0 + \mathbf{grad}\, \psi$ and choose φ such that $\partial_t \varphi = -\psi$. Then

$$0 = \partial_t B + \mathbf{curl}\, E = \mathbf{curl}(\partial_t A + E) = \mathbf{curl}(\partial_t A_0 + E_0).$$

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$$0 = \partial_t B + \mathbf{curl}\, E = \mathbf{curl}(\partial_t A + E) = \mathbf{curl}(\partial_t A_0 + E_0).$$

Since $(\partial_t A_0 + E_0) \perp \operatorname{Im}\mathbf{grad} = \ker \mathbf{curl}$, we infer $\partial_t A_0 + E_0 = 0$ and thus

$$\partial_t A = -E.$$

Maxwell equations in magnetic potential: $\mu_0 \epsilon_0 \partial_t E - \mathbf{curl}\, B = 0$ is recast as

$$\mu_0 \epsilon_0 \partial_t^2 A + \mathbf{curl}\,\mathbf{curl}\, A = 0$$

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4 Building blocks: mesh and setting for polynomial complexes on a manifold

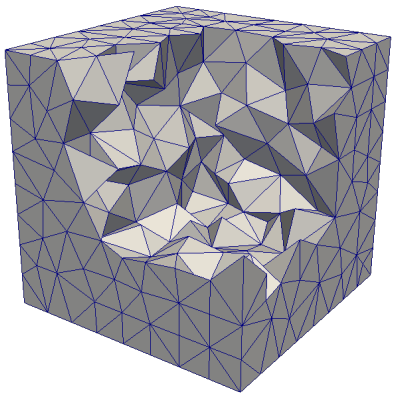
5 Discrete De Rham on a manifold

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Slides





$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

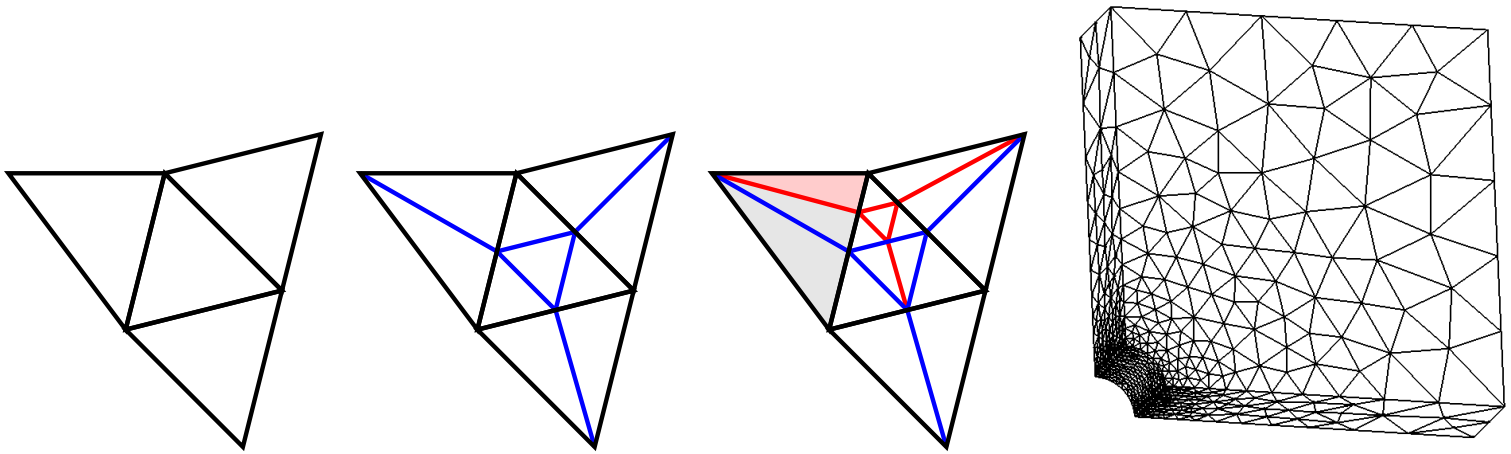
$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\mathbf{grad}} & V_h^1 & \xrightarrow{\mathbf{curl}} & V_h^2 & \xrightarrow{\mathrm{div}} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \end{array}$$

Example: conforming $\mathcal{P}_k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

- **Gluing only works on special meshes!**

The Finite Element way

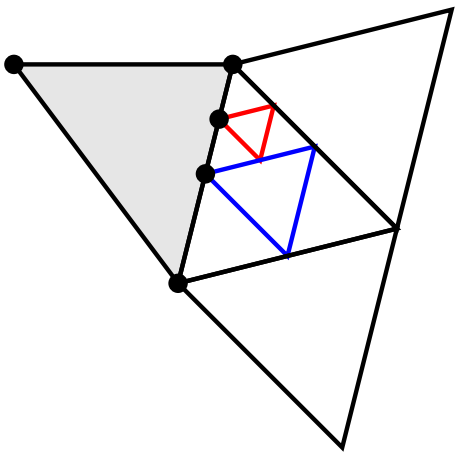
Shortcomings

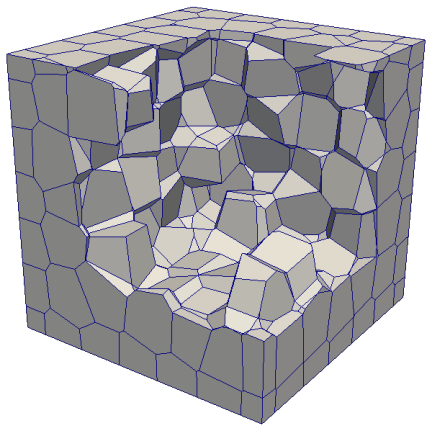


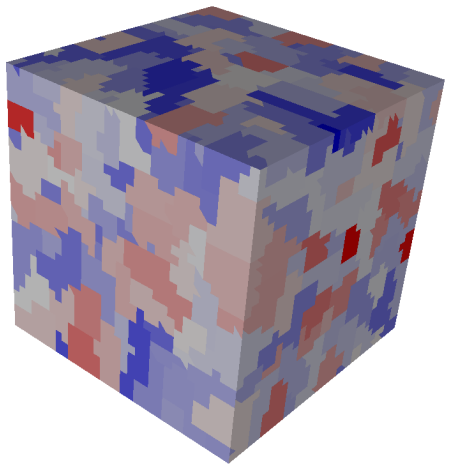
- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes





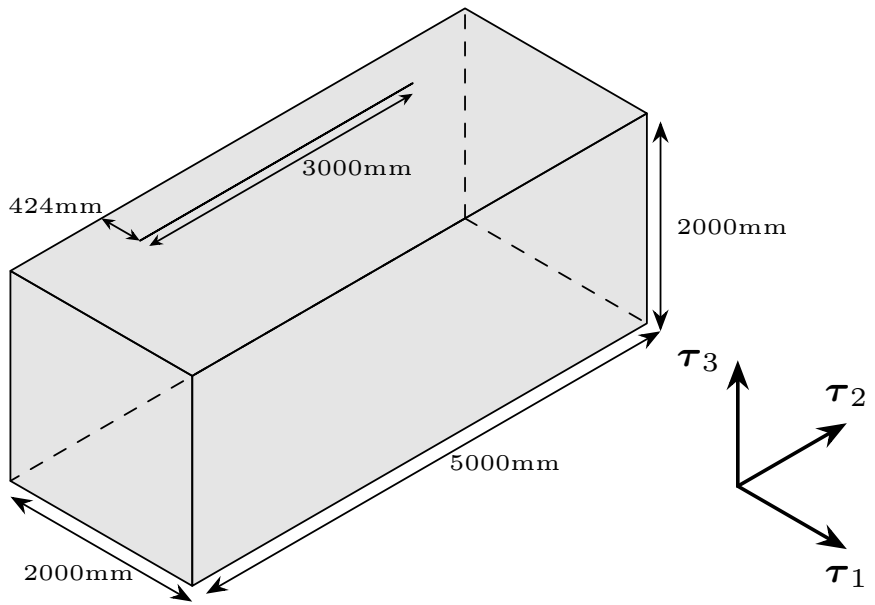


- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

A practical example in an industrial environment I

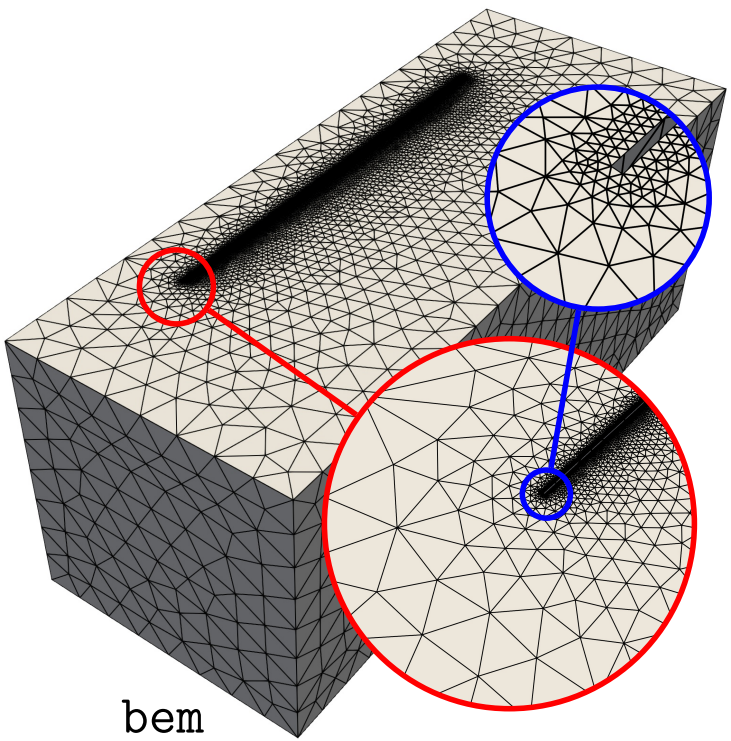
[Touzalin, 2025]

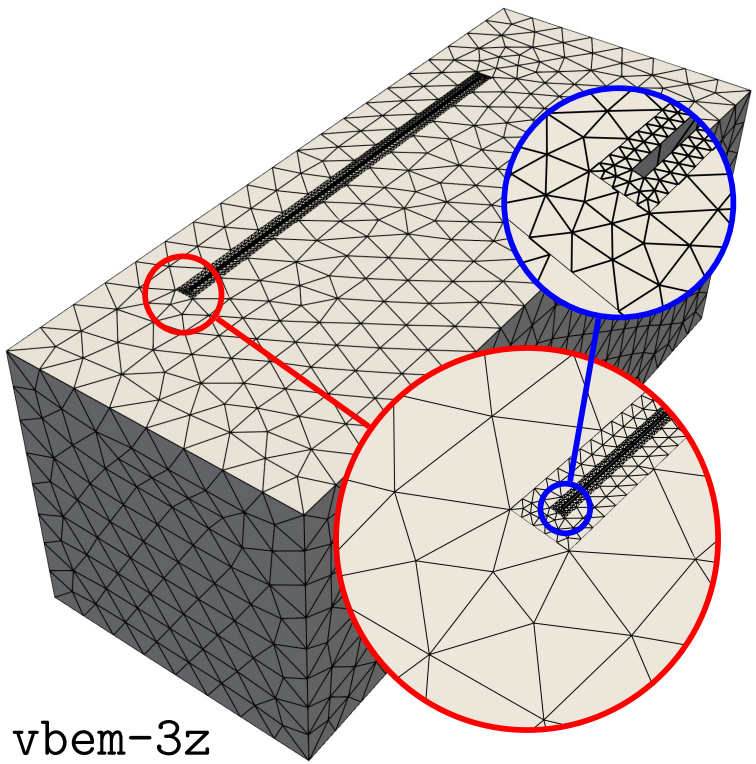
Problem: use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



A practical example in an industrial environment II

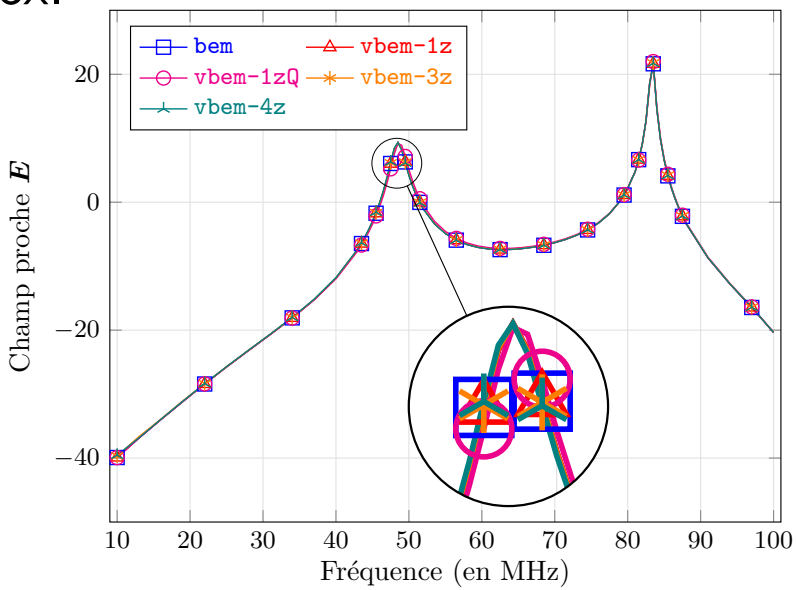
Meshes: conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).

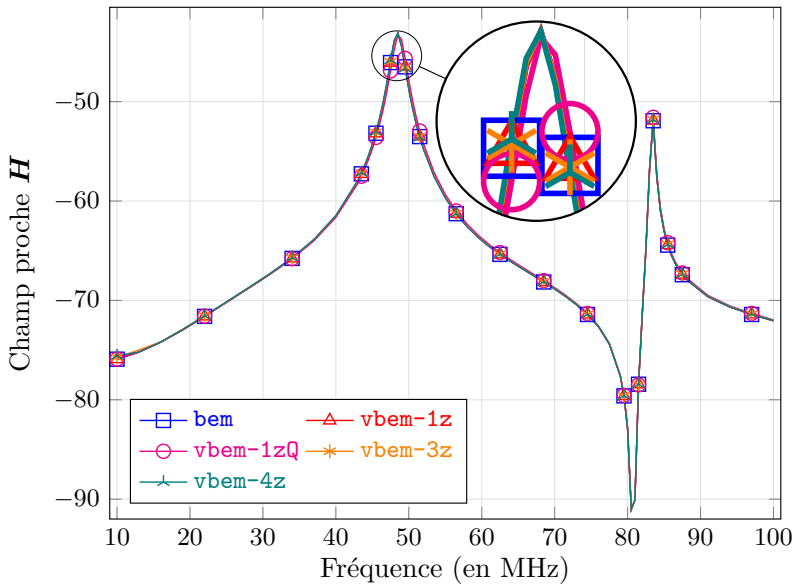




A practical example in an industrial environment III

Accuracy: comparison of modulus of reflected near fields at the center of the box.





Computational cost

Method	*Assembly*	*Resolution*
bem	813s	125s
vbem-3z	321s	19s

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Slides



Alternating forms I

- k-alternating form: $\omega : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ multilinear and fully antisymmetric, i.e.

$$\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k).$$

  Space: $\mathrm{Alt}^k(\mathbb{R}^n)$. Note that $\boxed{\mathrm{Alt}^k(\mathbb{R}^n) = \{0\} \text{ if } k > n.}$

- Exterior product:

$$\wedge : \mathrm{Alt}^k(\mathbb{R}^n) \times \mathrm{Alt}^\ell(\mathbb{R}^n) \to \mathrm{Alt}^{k+\ell}(\mathbb{R}^n).$$

- Hodge star operator: isomorphism

$$\star : \mathrm{Alt}^k(\mathbb{R}^n) \to \mathrm{Alt}^{n-k}(\mathbb{R}^n).$$

Alternating forms II

- Trace: $\omega \in \mathrm{Alt}^k(\mathbb{R}^n)$ and V subspace of $\mathbb{R}^n$. Then $\mathrm{tr}_V\,\omega \in \mathrm{Alt}^k(V)$ such that

$$(\mathrm{tr}_V\,\omega)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_k) = \omega(\boldsymbol{v}_1,\dots,\boldsymbol{v}_k) \text{ for all } \boldsymbol{v}_1,\dots,\boldsymbol{v}_k \in V.$$

- Basis: the dual basis of linear forms

$$\{\mathrm{d}x^1,\dots,\mathrm{d}x^n\} \text{ s.t. } \mathrm{d}x^i(\boldsymbol{a}) = a_i$$

gives a canonical basis of $\mathrm{Alt}^k(\mathbb{R}^n)$:

$$\{\mathrm{d}x^{\sigma_1}\wedge\cdots\wedge\mathrm{d}x^{\sigma_k} \;:\; \sigma \text{ s.t. } 1\le\sigma_1<\cdots<\sigma_k\le n\}.$$

- $k=0$: $\{1\}$,
- $k=1$: $\{\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3\}$,
- $k=2$: $\{\mathrm{d}x^1\wedge\mathrm{d}x^2,\mathrm{d}x^1\wedge\mathrm{d}x^3,\mathrm{d}x^2\wedge\mathrm{d}x^3\}$,
- $k=3$: $\{\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\}$.

Alternating forms III

- Vector proxy of alternating form: if $n = 3$ and V hyperplane,

k	k-form ω	proxy for ω	proxy for $\mathrm{tr}_V\,\omega$
0	$\omega = a$	$\omega_\sharp = a$	$\omega_\sharp$
1	$\omega = a\mathrm{d}x^1 + b\mathrm{d}x^2 + c\mathrm{d}x^3$	$\omega_\sharp = (a, b, c)$	tangential trace of $\omega_\sharp$
2	$\omega = a(\mathrm{d}x^2 \wedge \mathrm{d}x^3)$ $-b(\mathrm{d}x^1 \wedge \mathrm{d}x^3)$ $+c(\mathrm{d}x^1 \wedge \mathrm{d}x^2)$	$\omega_\sharp = (a, b, c)$	normal trace of $\omega_\sharp$
3	$\omega = a(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3)$	$\omega_\sharp = a$	N/A

Exterior calculus I

Differential forms: a k-differential form on a domain $\Omega \subset \mathbb{R}^n$ is

$$\omega : \Omega \to \mathrm{Alt}^k(\mathbb{R}^n).$$

Space of k-differential forms: $\Lambda^k(\Omega)$.

Exterior derivative

- Decomposition on canonical basis:

$$\omega_{\boldsymbol{x}} = \sum_{\sigma} a_\sigma(\boldsymbol{x}) \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}.$$

- If $\omega \in C^1\Lambda^k(\Omega)$, $\mathrm{d}^k\omega \in \Lambda^{k+1}(\Omega)$ such that

$$\mathrm{d}^k\omega_{\boldsymbol{x}} = \sum_{\sigma} \sum_{i=1}^{n} \frac{\partial a_\sigma}{\partial x_i}(\boldsymbol{x}) \mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}.$$

Koszul operator: if $\omega \in \Lambda^k(\Omega)$, define $\kappa\omega \in \Lambda^{k-1}(\Omega)$ by

$$(\kappa\omega)_{\boldsymbol{x}}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}) = \omega_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}).$$

Exterior calculus II

Vector proxy:

k	k-form ω	proxy for ω	$\mathrm{tr}_V\,\omega$	$\mathrm{d}^k\omega$
0	a	$\omega_\sharp = a$	$\omega_\sharp$	$\mathbf{grad}\,\omega_\sharp$
1	$a\mathrm{d}x^1 + b\mathrm{d}x^2 + c\mathrm{d}x^3$	$\omega_\sharp = (a,b,c)$	tang. trace of $\omega_\sharp$	$\mathbf{curl}\,\omega_\sharp$
2	$a(\mathrm{d}x^2\wedge\mathrm{d}x^3)$ $-b(\mathrm{d}x^1\wedge\mathrm{d}x^3)$ $+c(\mathrm{d}x^1\wedge\mathrm{d}x^2)$	$\omega_\sharp = (a,b,c)$	normal trace of $\omega_\sharp$	$\mathrm{div}\,\omega_\sharp$
3	$a(\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3)$	$\omega_\sharp = a$	N/A	0

Exterior calculus de Rham complex

Set $H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) \,:\, \mathrm{d}^k\omega \in L^2\Lambda^{k+1}(\Omega)\}$.

In dimension $n = 3$:

$$\begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ H(\mathbf{grad};\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega). \end{array}$$

In any dimension n:

$$H\Lambda^0(\Omega) \to \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} H\Lambda^{k+1}(\Omega) \xrightarrow{\mathrm{d}^{k+1}} \cdots \to H\Lambda^n(\Omega)$$

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Mesh I

Ω manifold of dimension n. Seen from ambiant coordinates (embedded manifold), the mesh will be curved... Handling curved meshes, even in flat spaces, is often difficult.

Flat space

- Not very difficult for dG, but other drawbacks (no static condensation, etc.) [Cangiani et al., 2021].
- VEM on curved meshes (flat space): polynomials on edges by parametrisation [Beirão da Veiga et al., 2019, Dassi et al., 2021].
- HHO on curved meshes (flat space):
  - ☐ parametrisation of faces [Botti and Di Pietro, 2018].
  - ☐ polynomials on edges by restriction (extrinsic approach) [Yemm, 2024].

Mesh II

Manifolds

- FEM: extrinsic approach
  [Demlow, 2009, Demlow and Dziuk, 2007, Arnold, 2018].
- VEM: extrinsic approach [Frittelli et al., 2023].
- FEM: intrinsic approach, with global homeomorphism with flat space
  [Bachini et al., 2021a, Bachini et al., 2021b].

  Construction of polynomial functions can be local, but continuity requirements make it difficult to glue over the entire manifold
  [Licht, 2023].
- Also approaches from the General Relativity community (lattice based), but limited to specific shapes
  [Gentle, 2002, Brewin, 1998, Frauendiener, 2006]...

No previous polytopal complex on manifolds...

Mesh III

Notations:

- $\mathcal{M}_h$ finite collection of pairwise disjoint submanifolds of dimension $d = 0, \ldots, n$ ($d = 0$: vertices, $d = 1$: edges; etc.)
- $\Delta_d(\mathcal{M}_h)$ collection of all d-dimensional manifolds.
- Boundary of $f \in \Delta_d(\mathcal{M}_h)$ made of $(d-1)$-manifolds $f' \in \Delta_{d-1}(f)$.

Abstract setting for local polynomial forms

Complexes for the exterior derivative and (some) Koszul operator:

$$\cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r\Lambda^l(f) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^{l+1}(f) \xrightarrow{\mathrm{d}} \cdots .$$
$$\cdots \xleftarrow{\kappa} \mathcal{P}_r\Lambda^l(f) \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^{l+1}(f) \xleftarrow{\kappa} \cdots .$$

Assumption (Homogeneous polynomials decomposition)

- *There is a decomposition* $\mathcal{P}_r\Lambda^l(f) = \oplus_{s\leq r}\mathcal{H}_s\Lambda^l(f)$.
- *We have* $\mathrm{d}\kappa + \kappa\mathrm{d} = \lambda_{s,l}\mathrm{Id}$ *on* $\mathcal{H}_s\Lambda^l(f)$.
- *Eigenvalues identify polynomial degree:* $\lambda_{s,l} = 0$ *if and only if* $s = l = 0$, *and if* $\lambda_{s,l} = \lambda_{s',l}$ *then* $s = s'$.
- $\mathrm{d}\mathcal{H}_s\Lambda^l(f) \neq \{0\}$ *whenever* $s > 0$ *and* $l < d$.

In the flat case:

- ☐ (0-forms) For $g(x) = x^\alpha$, we have $x \cdot \mathbf{grad}\, g(x) = |\alpha| g(x)$.
- ☐ (l-forms) $\mathrm{d}\kappa + \kappa\mathrm{d} = (s+l)\mathrm{Id}$ on $\mathcal{H}_s\Lambda^l(f)$.

Consequences of the design assumption

Bijectivity of d and κ: the following mappings are bijective:

$$\mathrm{d} : \kappa\mathcal{H}_s\Lambda^l(f) \to \mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f) \qquad (\subset \mathcal{H}_s\Lambda^l(f))$$
$$\kappa : \mathrm{d}\mathcal{H}_s\Lambda^l(f) \to \kappa\mathcal{H}_{s-1}\Lambda^{l+1} \qquad (\subset \mathcal{H}_s\Lambda^l(f))\,.$$

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$$\kappa : \mathrm{d}\mathcal{H}_s\Lambda^l(f) \to \kappa\mathcal{H}_{s-1}\Lambda^{l+1} \qquad (\subset \mathcal{H}_s\Lambda^l(f))\,.$$

Direct decomposition:

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^1(f)$$
$$\mathcal{P}_r\Lambda^l(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1.$$

♢ *Vector proxy, flat setting on an element $f = T$ in dimension $n = 3$:*

$$\mathcal{P}_r(T) = \mathcal{P}_0(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}_{r-1}(T)$$
$$\mathcal{P}_r(T)^3 = \mathbf{grad}\,\mathcal{P}_{r+1}(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}_{r-1}(T)^3,$$
$$\mathcal{P}_r(T)^3 = \mathbf{curl}\,\mathcal{P}_{r+1}(T)^3 \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}_{r-1}(T),$$
$$\mathcal{P}_r(T) = \mathrm{div}\,\mathcal{P}_{r+1}(T)^3.$$

Consequences of the design assumption

Direct decomposition:

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Definition (Trimmed polynomial spaces)

$$\mathcal{P}_r^-\Lambda^0(f) := \mathcal{P}_r\Lambda^0(f)$$
$$\mathcal{P}_r^-\Lambda^l(f) := \mathrm{d}\mathcal{P}_r\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1.$$

♢ *Vector proxy, flat setting on an element $f = T$ in dimension $n = 3$:*

$$\mathcal{P}_r^-\Lambda^0(T) \sim \mathcal{P}_r(T)$$
$$\mathcal{P}_r^-\Lambda^1(T) \sim \mathbf{grad}\,\mathcal{P}_r(T) \oplus (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}_{r-1}(T)^3 = \boldsymbol{\mathcal{N}}_r(T),$$
$$\mathcal{P}_r^-\Lambda^2(T) \sim \mathbf{curl}\,\mathcal{P}_r(T)^3 \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}_{r-1}(T) = \boldsymbol{\mathcal{RT}}_r(T),$$
$$\mathcal{P}_r^-\Lambda^3(T) \sim \mathrm{div}\,\mathcal{P}_r(T)^3 = \mathcal{P}_{r-1}(T).$$

Example: $\boldsymbol{\mathcal{N}}_1(T) = \mathcal{P}_0(T) + (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}_0(T)^3$.

Outline

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2 Why polytopal methods?

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Slides



Space and discrete operators ($r \geq 0$)

Space: vectors of trimmed polynomials on vertices, edges, faces, etc. (almost)

$$\underline{\boldsymbol{X}}_{r,h}^k := \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f).$$

Notation: for $f \in \Delta_d(\mathcal{M}_h)$,

$$\underline{\omega}_f = (\omega_f, (\omega_{f'})_{f' \in \Delta_{d-1}(f)}, (\omega_{f''})_{f'' \in \Delta_{d-2}(f)}, \text{etc.})$$

♢ Vector proxies:

$$\underline{\boldsymbol{X}}_{r,h}^0 \sim \bigtimes_{V \in \mathcal{V}_h} \mathbb{R} \times \bigtimes_{E \in \mathcal{E}_h} \mathcal{P}_{r-1}(E) \times \bigtimes_{F \in \mathcal{F}_h} \mathcal{P}_{r-1}(F) \times \bigtimes_{T \in \mathcal{T}_h} \mathcal{P}_{r-1}(T),$$

$$\underline{\boldsymbol{X}}_{r,h}^1 \sim \bigtimes_{E \in \mathcal{E}_h} \underbrace{\mathcal{P}_r(E)}_{\text{tangential trace}} \times \bigtimes_{F \in \mathcal{F}_h} \boldsymbol{\mathcal{RT}}_r(F) \times \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{\mathcal{RT}}_r(T),$$

$$\underline{\boldsymbol{X}}_{r,h}^2 \sim \bigtimes_{F \in \mathcal{F}_h} \underbrace{\mathcal{P}_r(F)}_{\text{normal trace}} \times \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{\mathcal{N}}_r(T).$$

Space and discrete operators ($r \geq 0$)

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Local exterior derivative and potential reconstruction in full spaces: for $f_\ell \in \Delta_\ell(\mathcal{M}_h)$,

$$\mathrm{d}_{r,f_\ell}^k : \underline{\boldsymbol{X}}_{r,f_\ell}^k \to \star^{-1}\mathcal{P}_r\Lambda^{\ell-k-1}(f_\ell) \text{ and } P_{r,f_\ell}^k : \underline{\boldsymbol{X}}_{r,f_\ell}^k \to \star^{-1}\mathcal{P}_r\Lambda^{\ell-k}(f_\ell).$$

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DDR complex: discrete exterior derivative by projecting $\mathrm{d}_{r,f}^k$:

$$\cdots \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k-1}} \underline{\boldsymbol{X}}_{r,h}^k \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k}} \underline{\boldsymbol{X}}_{r,h}^{k+1} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k+1}} \cdots.$$

$$\underline{\mathrm{d}}_{r,h}^k\underline{\omega}_h = \left(\star^{-1}\,\pi_{r,f}^{-,d-(k+1)}\star\,\mathrm{d}_{r,f}^k\underline{\omega}_h\right)_{d=k+1,\ldots,n;f\in\Delta_d(\mathcal{M}_h)}.$$

Construction of discrete operators

- Hierarchical and recursive process:

$$P^k_{r,f_k} \rightsquigarrow \mathrm{d}^k_{r,f_{k+1}} \rightsquigarrow P^k_{r,f_{k+1}} \rightsquigarrow \mathrm{d}^k_{r,f_{k+2}} \rightsquigarrow \cdots \rightsquigarrow P^k_{r,f_n}.$$

- Whole construction based on integration-by-parts, i.e., Stokes formula:

$$(\mathrm{d}^k\omega, \mu)_\Omega = (-1)^{k+1}(\omega, \delta^{n-k-1}\mu)_\Omega + (\mathrm{tr}_{\partial\Omega}\,\omega, \star^{-1}\,\mathrm{tr}_{\partial\Omega}\,\mu)_{\partial\Omega}.$$

Potential, $\boxed{d = k}$

If $f \in \Delta_d(\mathcal{M}_h)$ *and* $\underline{\omega}_f \in \underline{X}^k_{r,f}$ *then*

$$\underline{\omega}_f = (\omega_f) \text{ with } \omega_f \in \mathcal{P}_r\Lambda^k(f).$$

Potential:

$$P^k_{r,f}\underline{\omega}_f = \omega_f \in \mathcal{P}_r\Lambda^k(f).$$

Vector proxies:

$$\begin{aligned}
P^0_{r,V}\underline{\omega}_V &= \omega_V \in \mathbb{R} && \forall V \in \mathcal{V}_h, && \text{(pointwise value)}\\
P^1_{r,E}\underline{\omega}_E &= \omega_E \in \mathcal{P}_r(E) && \forall E \in \mathcal{E}_h, && \text{(tangential trace)}\\
P^2_{r,F}\underline{\omega}_F &= \omega_F \in \mathcal{P}_r(F) && \forall F \in \mathcal{F}_h. && \text{(normal trace)}
\end{aligned}$$

Exterior derivative, $\boxed{d = k+1}$

If $f \in \Delta_d(\mathcal{M}_h)$ *and* $\underline{\omega}_f \in \underline{X}^k_{r,f}$ *then, with* $\delta = \star^{-1}\mathrm{d}\star$ *(adjoint of* d*),*

$$\underline{\omega}_f = ((\omega_{f'})_{f'\subset\partial f}, \omega_f) \text{ with } \omega_f \in \star^{-1}\mathcal{P}_r^-\Lambda^1(f) = \delta\mathcal{P}_r\Lambda^d(f) \oplus \star^{-1}\kappa\mathcal{P}_{r-1}\Lambda^2(f).$$

Exterior derivative: $\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^{k+1}(f)$ such that, for all $\mu \in \mathcal{P}_r\Lambda^{k+1}(f)$,

$$(\mathrm{d}^k_{r,f}\underline{\omega}_f, \mu)_f = (-1)^{k+1}(\omega_f, \delta\mu)_f + (\underbrace{P^k_{r,\partial f}\underline{\omega}_{\partial f}}_{\text{already built}}, \star^{-1}\,\mathrm{tr}_{\partial f}\star\mu)_{\partial f}$$

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Vector proxies, 0-forms:

$$\underline{\omega}_E = ((\omega_V)_{V\in\mathcal{V}_E}, \omega_E) \in \Big(\bigtimes_{V\in\mathcal{V}_E}\mathbb{R}\Big) \times \mathcal{P}_{r-1}(E)$$

and, for all $E \in \mathcal{E}_h$, define $G_{r,E}\underline{\omega}_E \in \mathcal{P}_r(E)$ such that

$$\int_E G_{r,E}\underline{\omega}_E\,\mu = -\int_E \omega_E\partial_E\mu + \sum_{V\in\mathcal{V}_E}\underbrace{\Xi_{VE}}_{\text{orientation } V}\omega_V\mu(\boldsymbol{x}_V) \qquad \forall\mu\in\mathcal{P}_r(E).$$

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Vector proxies, 1-forms:

$$\underline{\omega}_F = ((\omega_E)_{E\in\mathcal{E}_F}, \boldsymbol{\omega}_F) \in \Big(\bigtimes_{E\in\mathcal{E}_F}\mathcal{P}_r(E)\Big) \times \boldsymbol{\mathcal{RT}}_r(F)$$

and, for all $F \in \mathcal{F}_h$, define $C_{r,F}\underline{\omega}_F \in \mathcal{P}_r(F)$ such that

$$\int_F C_{r,F}\underline{\omega}_F\,\mu = \int_F \boldsymbol{\omega}_F \cdot \mathbf{rot}_F\,\mu - \sum_{E\in\mathcal{E}_F}\Xi_{EF}\int_E \omega_E\mu \qquad \forall\mu \in \mathcal{P}_r(F).$$

Potential, $\boxed{d = k+1}$

If $f \in \Delta_d(\mathcal{M}_h)$ *and* $\underline{\omega}_f \in \underline{X}^k_{r,f}$ *then*

$\underline{\omega}_f = ((\omega_{f'})_{f' \subset \partial f}, \omega_f)$ *with* $\omega_f \in \star^{-1}\mathcal{P}_r^- \Lambda^1(f) = \delta\mathcal{P}_r\Lambda^d(f) \oplus \star^{-1}\kappa\mathcal{P}_{r-1}\Lambda^2(f)$.

Potential: $P^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^k(f) = \boxed{\delta(\star^{-1}\kappa\mathcal{P}_r\Lambda^1(f)) \oplus \star^{-1}\kappa\mathcal{P}_{r-1}\Lambda^2(f)}$ s.t.:

- For all $\mu \in \star^{-1}\kappa\mathcal{P}_r\Lambda^1(f)$,

$$(-1)^{k+1}(P^k_{r,f}\underline{\omega}_f, \delta\mu)_f = (\mathrm{d}^k_{r,f}\underline{\omega}_f, \mu)_f - (P^k_{r,\partial f}\underline{\omega}_{\partial f}, \star^{-1}\operatorname{tr}_{\partial f} \star\mu)_{\partial f}.$$

- For all $\nu \in \star^{-1}\kappa\mathcal{P}_{r-1}\Lambda^2(f)$:

$$(P^k_{r,f}\underline{\omega}_f, \nu)_f = (\omega_f, \nu)_f$$

Potential, $d = k+1$

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Vector proxies, 0-forms: for all $E \in \mathcal{E}_h$, define $P_{r,E}\underline{\omega}_E \in \mathcal{P}_r(E)$ such that

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Vector proxies, 1-forms: for all $F \in \mathcal{F}_h$, define $\boldsymbol{P}_{r,F}\underline{\omega}_F \in \mathcal{P}_r(F)^2$ such that

$$\int_F \boldsymbol{P}_{r,F}\underline{\omega}_F \cdot \mathbf{rot}_F\, \mu = \int_F C_{r,F}\underline{\omega}_F\, \mu + \sum_{E\in\mathcal{E}_F} \Xi_{EF} \int_E \omega_E \mu \qquad \forall \mu \in \mathcal{P}^0_{r+1}(F),$$

$$\int_F \boldsymbol{P}_{r,F}\underline{\omega}_F \cdot \boldsymbol{\nu} = \int_F \boldsymbol{\omega}_F \cdot \boldsymbol{\nu} \qquad \forall \boldsymbol{\nu} \in (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}_{r-1}(F).$$

DDR on manifolds

In the previous slides, only the vector proxies are written in the flat space. The entire construction is already done on curved spaces...

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Assumption (Trace of trimmed polynomials)

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Vector proxies on flat spaces:

- $\boldsymbol{\mathcal{N}}_r(F)\cdot\mathbf{t}_E \subset \mathcal{P}_{r-1}(E)$,
- $\boldsymbol{\mathcal{RT}}_r(T)\cdot\mathbf{n}_F \subset \mathcal{P}_{r-1}(F)$,
- $\boldsymbol{\mathcal{N}}_r(T)\times\mathbf{n}_F \subset \boldsymbol{\mathcal{RT}}_r(F)$.

Properties

Theorem (Commutation property)

$$\underline{\mathrm{d}}^k_{r,f}(\underline{\boldsymbol{I}}^k_{r,f}\omega) = \underline{\boldsymbol{I}}^{k+1}_{r,f}(\mathrm{d}\omega) \qquad \forall \omega \in C^1\Lambda^k(\overline{f}).$$

Theorem (Cohomology)

The DDR complex has the same cohomology as the continuous de Rham complex.

Theorem (Consistency for "regular" mesh sequences)

If the local polynomial spaces satisfy a Poincaré inequality, a topological decomposition property, a discrete trace inequality and polynomial approximation properties, then, for smooth ω,

$$\|P^k_{r,f}\underline{\boldsymbol{I}}^k_{r,f}\omega - \omega\|_f \lesssim C_\omega h_f^{r+1},$$
$$\|\mathrm{d}^k_{r,f}\underline{\boldsymbol{I}}^k_{r,f}\omega - \mathrm{d}\omega\|_f \lesssim C_\omega h_f^{r+1}.$$

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Condition on local charts

- On principle, defining polynomial spaces on cells can be done via local coordinates (local charts).
- However, we need compatibility conditions on the charts, to ensure:
  - a suitable construction of homogeneous polynomial spaces and Koszul operators (first assumption)
  - trace properties of trimmed polynomial spaces (second assumption).

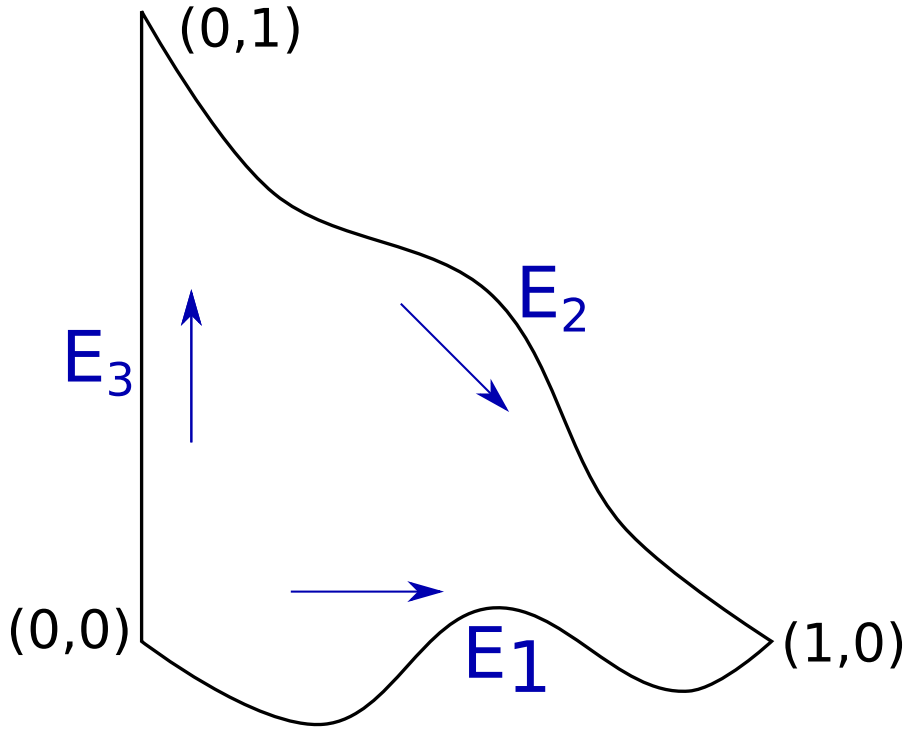
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Assumption

There are C^2-diffeomorphisms $I_f : U_f \subset \mathbb{R}^d \to \overline{f}$ for all $f \in \Delta_d(\mathcal{M}_h)$ and $d \in [0, n]$.
Moreover, if $f' \in \Delta_{d-1}(f)$, with $\mathfrak{I}_{f,f'} : f' \to \overline{f}$ the inclusion of f' into $\overline{f}$, the mapping $(I_f)^{-1} \circ \mathfrak{I}_{f,f'} \circ I_{f'} : U_{f'} \to \mathbb{R}^d$ is affine.

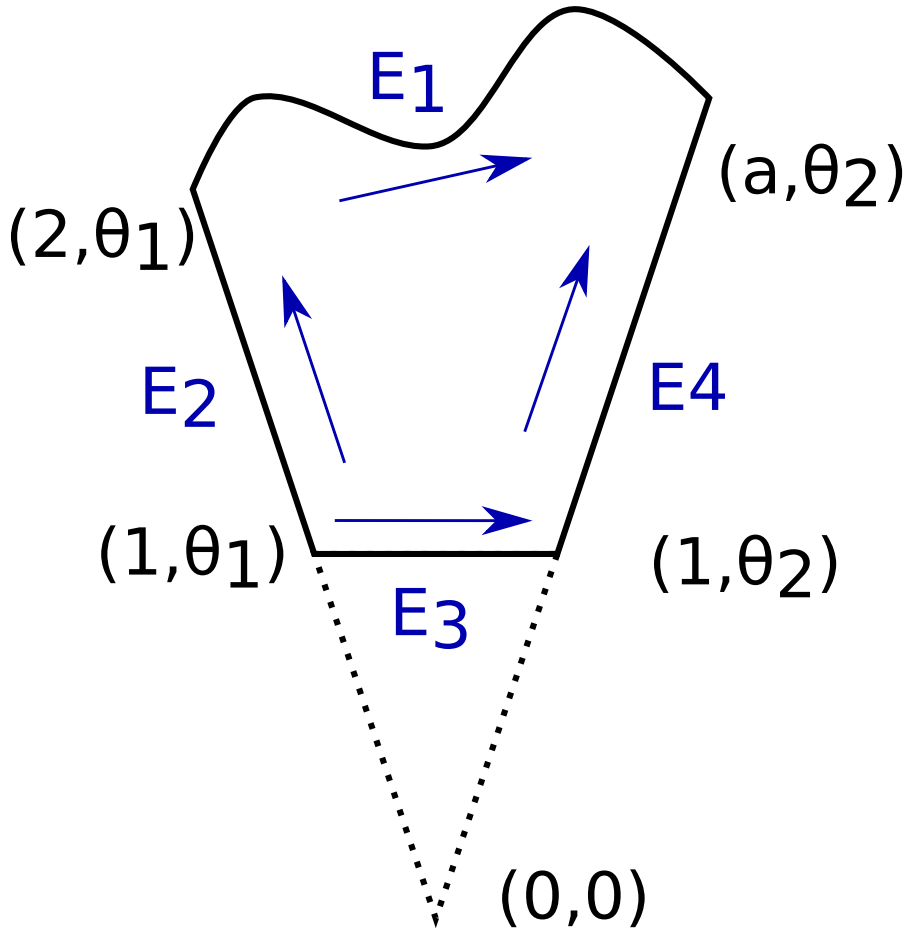
From there: case by case basis



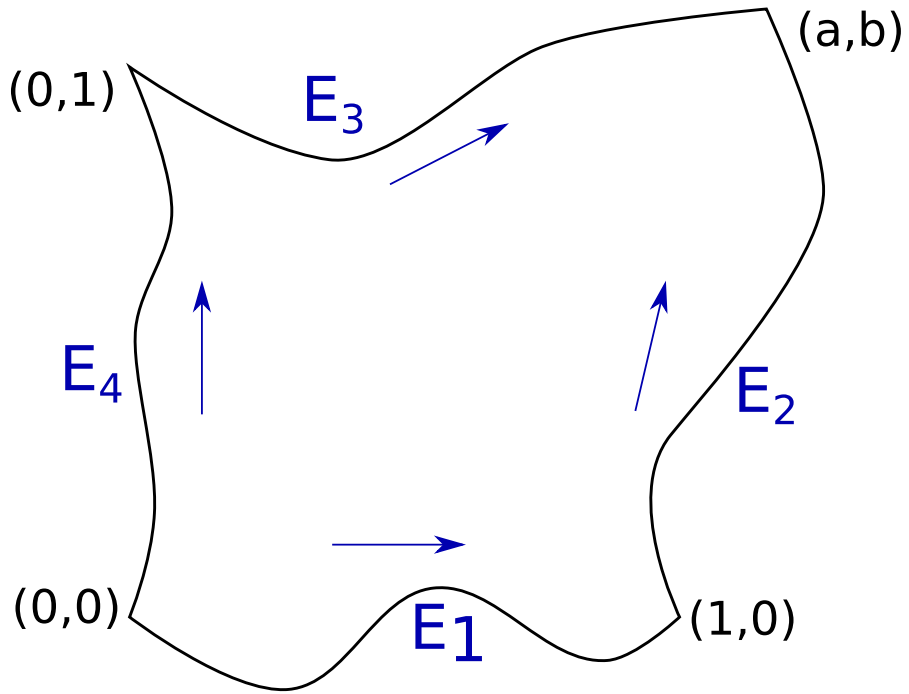
$$I_1(t) = (h_1(t), g_1(t))$$
$$I_2(t) = (h_2(t), g_2(t)) \rightsquigarrow J_f(x,y) = \frac{1}{G_2(x) - G_1(x)} \begin{bmatrix} (y - G_1(x))h_2^{-1}(x) + (G_2(x) - y)h_1^{-1}(x) \\ (y - G_1(x))(1 - h_2^{-1}(x)) \end{bmatrix}$$
$$I_3(t) = (0, t),$$

with $G_i = g_i \circ h_i^{-1}$.

From there: case by case basis



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Maxwell equations in differential forms

- Ω a 3-dimensional Lorentzian manifold with metric decomposed in 2+1 with zero-shift and constant lapse c:

$$(g_{\mu\nu}) := \begin{pmatrix} -c^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix},$$

- d: spatial exterior derivative (2-dim); δ: co-differential; vol: spatial volume form.

- B: $(1/c)\times$ magnetic field (2-form), E: electric field (1-form).

$$\begin{aligned} \partial_t B &= -\mathrm{d}E, \\ -\delta E &= \rho\epsilon_0^{-1}, \\ \frac{1}{c^2}\partial_t E &= \delta B - \mu_0 J, \end{aligned}$$

with μ_0 vacuum permeability, ρ electric charge density and J electric current density.
Note the compatibility condition on the data: $\delta J = \partial_t \rho$.

Weak formulation and scheme

- Weak formulation: find $(\widetilde{E}, \widetilde{B}) \in C^1([0,T];\Lambda^1(\Omega)) \times C^1([0,T];\Lambda^2(\Omega))$ s.t., for all $(v^1, v^2) \in \Lambda^1(\Omega) \times \Lambda^2(\Omega)$,

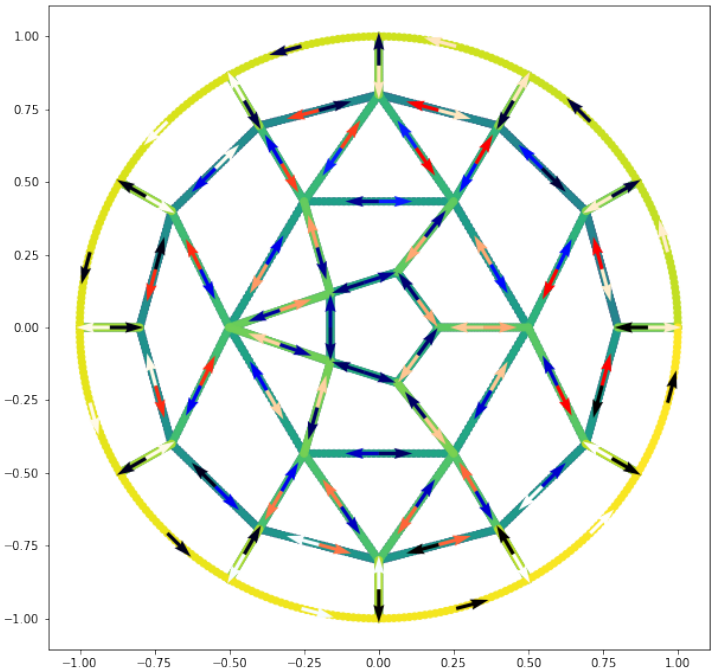
$$\begin{aligned} \langle \partial_t B, v^2 \rangle &= -\langle \mathrm{d}E, v^2 \rangle, \\ \frac{1}{c^2}\langle \partial_t E, v^1 \rangle &= \langle B, \mathrm{d}v^1 \rangle - \mu_0 \langle J, v^1 \rangle. \end{aligned}$$

- DDR scheme, with $X_h := \underline{\boldsymbol{X}}^1_{r,h} \times \underline{\boldsymbol{X}}^2_{r,h}$: find $(\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{B}}_h) \in C^1([0,T];X_h)$ such that for all $(\underline{\boldsymbol{v}}^1_h, \underline{v}^2_h) \in X_h$ and all $t \in (0,T)$,

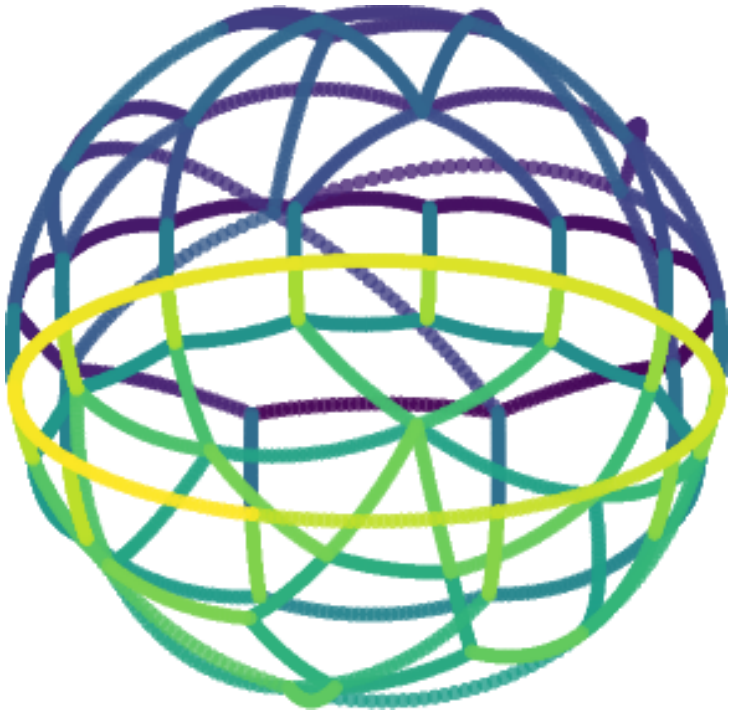
$$\begin{aligned} \langle \partial_t \underline{\boldsymbol{B}}_h(t), \underline{v}^2_h \rangle_h &= -\langle \underline{\mathrm{d}}^1_{r,h} \underline{\boldsymbol{E}}_h(t), \underline{v}^2_h \rangle_h, \\ \frac{1}{c^2}\langle \partial_t \underline{\boldsymbol{E}}_h(t), \underline{\boldsymbol{v}}^1_h \rangle_h &= \langle \underline{\boldsymbol{B}}_h(t), \underline{\mathrm{d}}^1_{r,h} \underline{\boldsymbol{v}}^1_h \rangle_h - \mu_0 \langle \underline{\boldsymbol{I}}^1_{r,h} \widetilde{J}(t), \underline{\boldsymbol{v}}^1_h \rangle_h. \end{aligned}$$

- Both preserve the constraint (dropped equation).

Mesh on the sphere

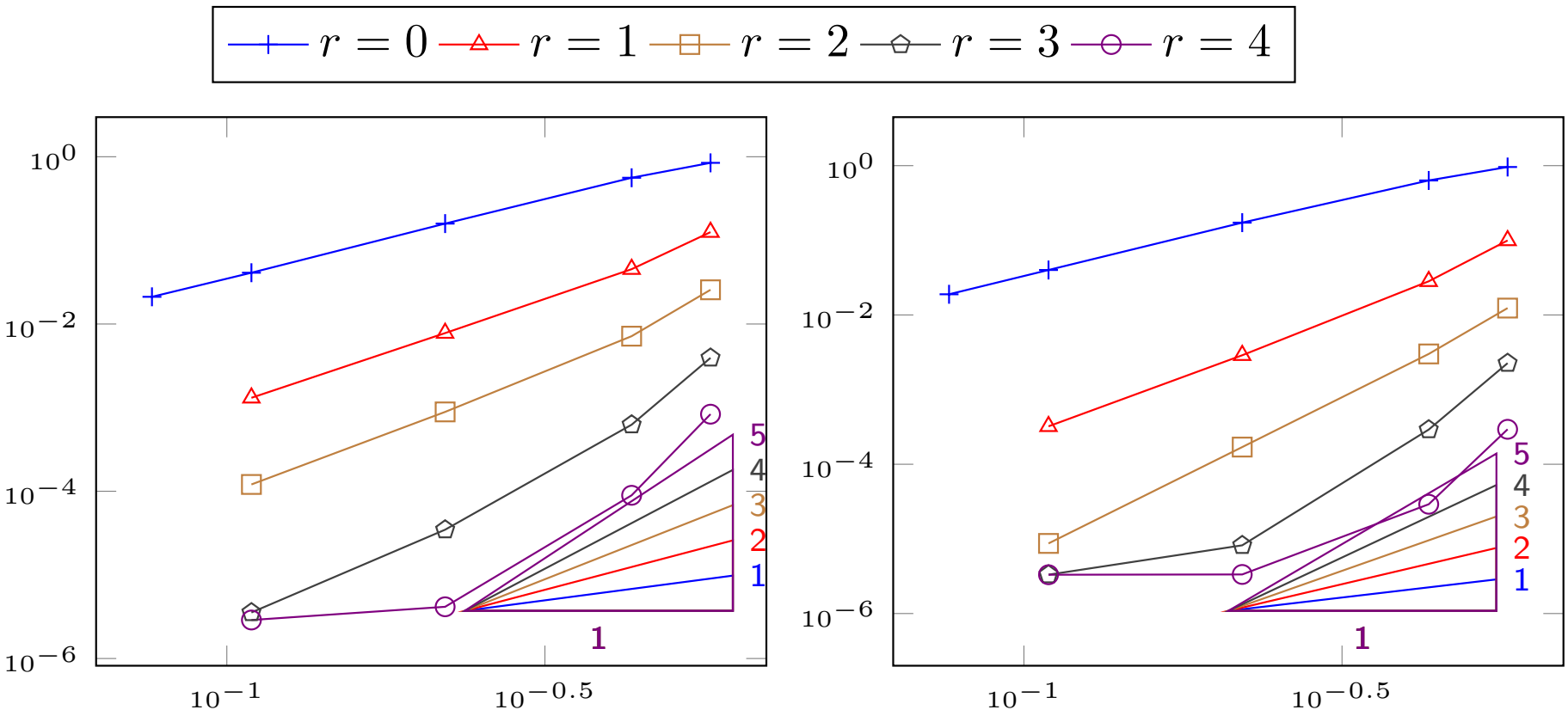


(a) Mesh in one chart and orientations.



(b) Embedding of the 1-skeleton of the mesh into $\mathbb{R}^3$.

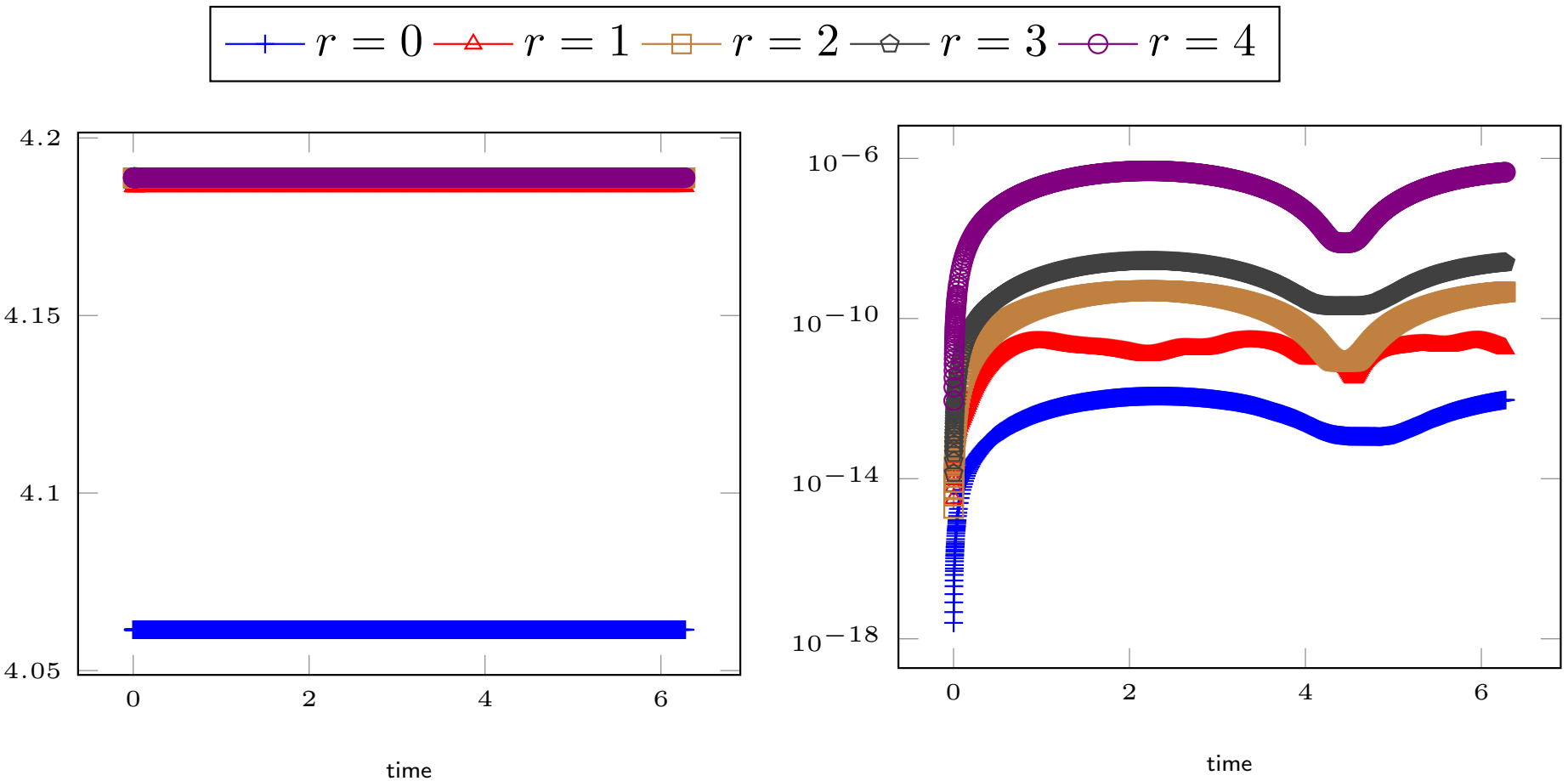
Numerical results: smooth solution



(a) Error on E.

(b) Error on B.

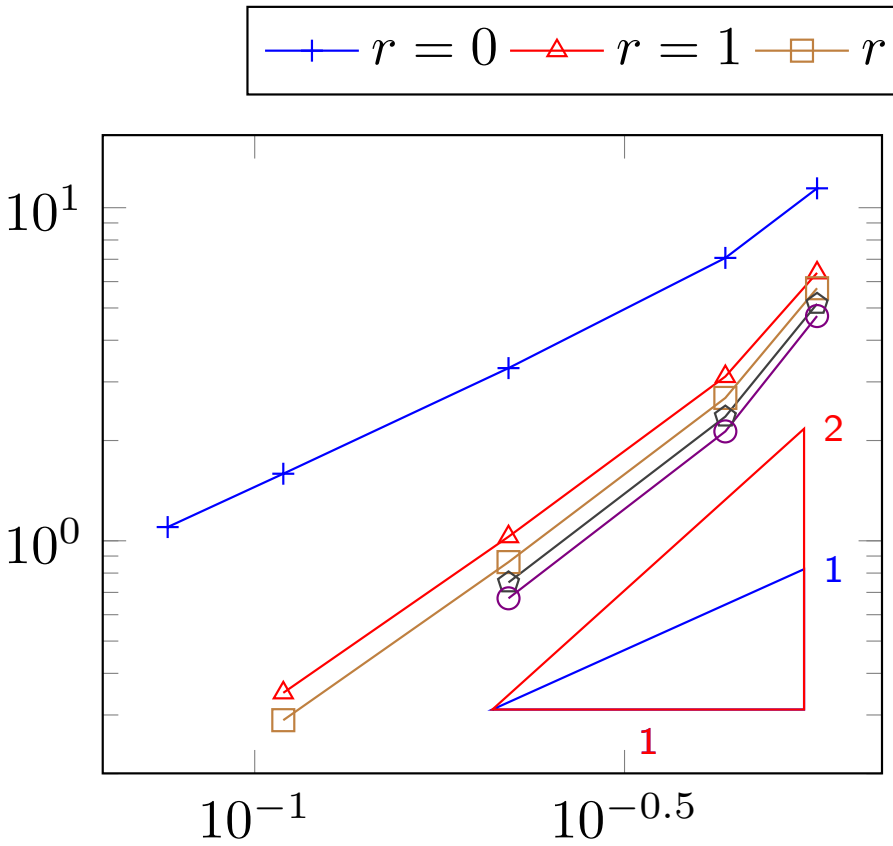
Numerical results: preservation of energy and constraint



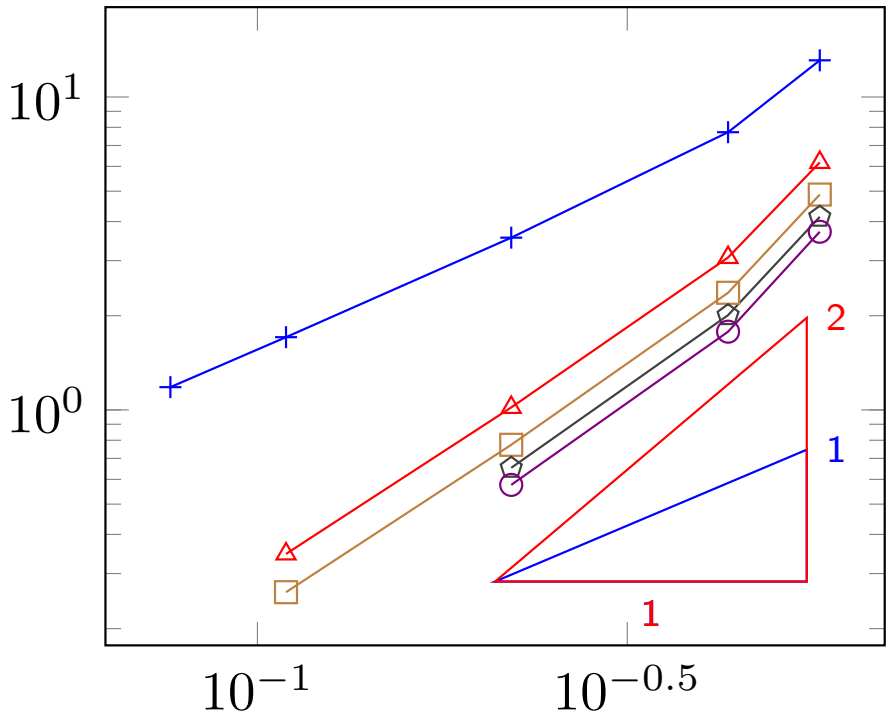
(a) Total energy $E^2 + B^2$.

(b) Error on the discrete constraint.

Numerical results: piecewise smooth C^0 solution



(a) Error on E.



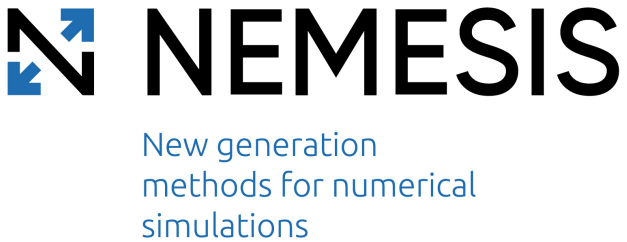
(b) Error on B.

Conclusion

- Discrete de Rham complex, of arbitrary order of accuracy and applicable on generic (and curved) polytopal meshes.
- Abstract setting for polynomial spaces on polytopal meshes on manifolds, identifies the key properties for developing discrete de Rham complexes.
- Practical construction from charts in special cases.
- Set of algebraic results, and consistency.
- To do: better/more general construction of local polynomial spaces; meshes with hanging nodes; adjoint consistency.







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Thank you for your attention!

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