The discrete de Rham complex, and its application to the Navier–Stokes equations

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from joint works with Daniele Di Pietro and several others...

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Outline

Stokes in "standard" formulation

- Ω domain, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Find $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
-\nu \boldsymbol{\Delta u} + \mathbf{grad}\, p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, && \text{(boundary condition)}
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L^2(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\nu(\mathbf{grad}\,\boldsymbol{u}, \mathbf{grad}\,\boldsymbol{v})_{L^2} - (p, \operatorname{div} \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in H_0^1(\Omega),\\
(\operatorname{div} \boldsymbol{u}, q)_{L^2} &= 0 && \forall q \in L^2(\Omega)
\end{aligned}
$$

- A priori estimates require: ([1])

  - Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{grad}\cdot\|_{L^2}$ on $H_0^1(\Omega)$,
  - inf-sup $\sup_{\boldsymbol{v}\in H_0^1} \frac{(p, \operatorname{div} \boldsymbol{v})_{L^2}}{\|\boldsymbol{v}\|_{H_0^1}} \geq C\|p\|_{L^2}$, equivalent to $\operatorname{Im}\operatorname{div} = L^2(\Omega)$.

[1] $a \lesssim b$ means $a \leq Cb$ with C independent of a, b.

Stokes in curl-curl formulation: weak form

- Recasting the Stokes equations:

$$\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\textstyle\int_\Omega p &= 0
\end{aligned}$$

Note: These are natural BCs. Essential BCs are: $\boldsymbol{u} \times \boldsymbol{n} = 0$ and $p = 0$ on $\partial\Omega$.

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\mathrm{div}\,\boldsymbol{u} = 0 \quad & \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad & \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p = 0 \quad &
\end{aligned}
$$

Note: These are natural BCs. Essential BCs are: $\boldsymbol{u}\times\boldsymbol{n} = 0$ and $p = 0$ on $\partial\Omega$.

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega)\times H^1_\star(\Omega)$ and

$$
\begin{aligned}
\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} = (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad & \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} = 0 \quad & \forall q \in H^1_\star(\Omega),
\end{aligned}
$$

where $H^1_\star(\Omega) := \{r \in H^1(\Omega) \;:\; \int_\Omega r = 0\}$.

Stokes equations in curl-curl formulation: stability

$$\begin{aligned} \nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H^1_\star(\Omega) \end{aligned}$$

- Make $\boldsymbol{v} = \mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad} = 0$.
- Make $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$:

$$\nu\|\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2} \leq \|\boldsymbol{f}\|_{L^2}\|\boldsymbol{u}\|_{L^2}.$$

Stokes equations in curl-curl formulation: stability

$$\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} = (\boldsymbol{f}, \boldsymbol{v})_{L^2} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),$$
$$-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} = 0 \quad \forall q \in H^1_\star(\Omega)$$

- Make $\boldsymbol{v} = \mathbf{grad}\,p$ to get $\|\mathbf{grad}\,p\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ since $\mathbf{curl}\,\mathbf{grad} = 0$.
- Make $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$:

$$\nu\|\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2} \leq \|\boldsymbol{f}\|_{L^2}\|\boldsymbol{u}\|_{L^2}.$$

- If Ω does not have any tunnel,

$$\mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl}.$$

The incompressibility gives $\boldsymbol{u} \perp \mathrm{Im}\,\mathbf{grad}$, so $\boldsymbol{u} \in (\mathrm{Ker}\,\mathbf{curl})^\perp$ and the

Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{curl}\cdot\|_{L^2}$ on $(\mathrm{Ker}\,\mathbf{curl})^\perp$

yields

$$\|\boldsymbol{u}\|_{L^2} \lesssim \|\mathbf{curl}\,\boldsymbol{u}\|_{L^2}.$$

Take-home message

Stability for these models requires:

- Poincaré inequalities: $\|\cdot\|_{L^2} \lesssim \|\mathcal{D}\cdot\|_{L^2}$ on $(\operatorname{Ker}\mathcal{D})^\perp$.
- Image/kernel relations for some operators in the sequence (possibly with suitable BCs):

$$H^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl};\Omega) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

Outline

The de Rham complex I

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.

The de Rham complex I

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.
- The complex is exact if we have equalities:

$$\mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl}, \quad \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div}, \quad \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

This imposes some topological properties on Ω...

The de Rham complex II

- Precisely:

$$\text{no ``tunnels''} \implies \operatorname{Im}\mathbf{grad} = \operatorname{Ker}\mathbf{curl} \quad \text{(Stokes in curl-curl)}$$

$$\text{no ``voids''} \implies \operatorname{Im}\mathbf{curl} = \operatorname{Ker}\operatorname{div} \quad \text{(magnetostatics)}$$

$$\text{always:} \qquad \operatorname{Im}\operatorname{div} = L^2(\Omega) \quad \text{(Stokes)}$$

- For non-trivial topologies, de Rham's cohomology characterizes

$$\operatorname{Ker}\mathbf{curl}/\operatorname{Im}\mathbf{grad} \quad \text{and} \quad \operatorname{Ker}\operatorname{div}/\operatorname{Im}\mathbf{curl}$$

The de Rham complex II

- Precisely:

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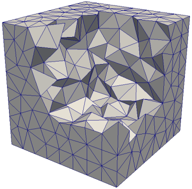
- For non-trivial topologies, de Rham's cohomology characterizes

$$\operatorname{Ker}\mathbf{curl}/\operatorname{Im}\mathbf{grad} \quad \text{and} \quad \operatorname{Ker}\operatorname{div}/\operatorname{Im}\mathbf{curl}$$

- **Emulating these properties is key for stable discretizations.**

The Finite Element way

Global complex



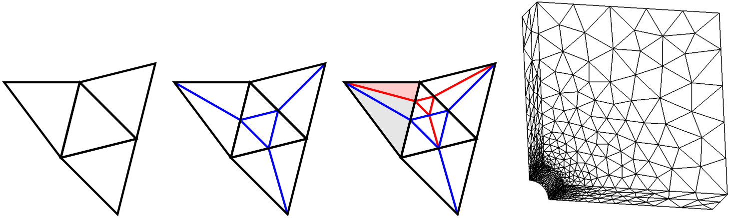
$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\quad \mathbf{grad} \quad} & V_h^1 & \xrightarrow{\quad \mathbf{curl} \quad} & V_h^2 & \xrightarrow{\quad \mathrm{div} \quad} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \end{array}$$

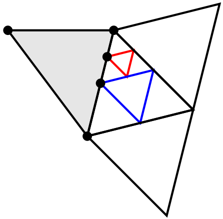
Example: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].

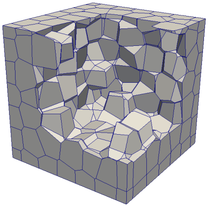
- **Gluing only works on special meshes!**

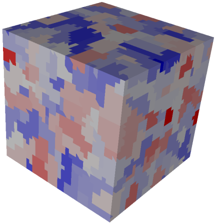


- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes I



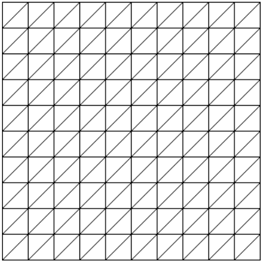
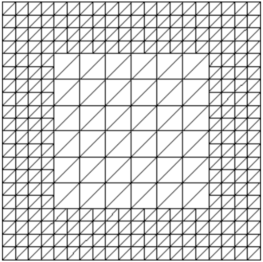




- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

Benefits of polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathcal{P}_2$-($\mathcal{P}_1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

Outline

Outline

Two "PAMIR" approaches for polytopal complexes I

Virtual Element Method

- Construct a sub-complex of finite-dimensional but not piecewise polynomial spaces:

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & V^{\mathrm{n}}_{k+1} & \xrightarrow{\ \mathbf{grad}\ } & V^{\mathrm{e}}_{k} & \xrightarrow{\ \mathbf{curl}\ } & V^{\mathrm{f}}_{k} & \xrightarrow{\ \mathrm{div}\ } & V^{\mathrm{v}}_{k} & \xrightarrow{\ 0\ } & \{0\} \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\ \mathbf{grad}\ } & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\ \mathbf{curl}\ } & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\ \mathrm{div}\ } & L^2(\Omega) & \xrightarrow{\ 0\ } & \{0\}
\end{array}
$$

- Functions in each $V_h^\bullet$ are not explicitly described, but:

  - $V_h^\bullet$ have DOFs that are polynomials moments on the vertices, edges, faces or elements.
  - Some polynomial projections of the functions/their derivatives can be computed from these DOFs.

Ref: [Beirão da Veiga et al., 2017], [Beirão da Veiga et al., 2018].

Two “PAMIR” approaches for polytopal complexes II

Discrete De Rham (fully discrete approach)

- Construct a fully discrete complex of bespoke finite-dimensional spaces and operators:

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{G}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \uparrow{\scriptstyle \underline{I}^k_{\mathbf{grad},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},h}} & & \uparrow{\scriptstyle I^k_{L^2,h}} & & \\
\mathbb{R} & \hookrightarrow & C^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{\Omega}) & \xrightarrow{0} & \{0\}
\end{array}
$$

- Discrete spaces are not made of functions but:

  - $\underline{X}^k_{\bullet,h}$ made of vectors of polynomials on vertices, edges, faces, elements.
  - Interpolators $\underline{I}^k_{\bullet,h}$ give meaning to these polynomials/DOFs as moments.
  - Discrete operators (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023b].

Two "PAMIR" approaches for polytopal complexes III

VEM–DDR bridges

- The VEM and DDR complexes from the (original) literature are different.
- The DDR complex can be "virtualized":

$$\begin{array}{ccccccc}
X^{\mathrm{n}}_{k+1,h} & \xrightarrow{\mathbf{grad}} & X^{\mathrm{e}}_{k,h} & \xrightarrow{\mathbf{curl}} & X^{\mathrm{f}}_{k,h} & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) \\
\downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \mathrm{Id} \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h)
\end{array}$$

and the VEM complex can be "fully-discretized":

$$\begin{array}{ccccccc}
V^{\mathrm{n}}_{k+1,h} & \xrightarrow{\mathbf{grad}} & V^{\mathrm{e}}_{k,h} & \xrightarrow{\mathbf{curl}} & V^{\mathrm{f}}_{k,h} & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) \\
\downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \mathrm{Id} \\
\underline{V}^{\mathrm{n}}_{k+1,h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{V}^{\mathrm{e}}_{k,h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{V}^{\mathrm{f}}_{k,h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h)
\end{array}$$

Ref: [Beirão da Veiga et al., 2022].

Outline

Guiding principles

Collaborator: Francesca Rapetti

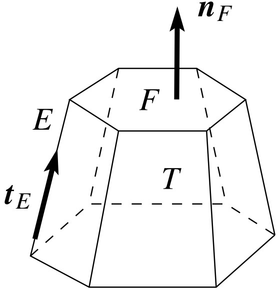
- Hierarchical constructions: from lowest-dimensional mesh entity to higher-dimensional entities.
- Enhancement:
  - discrete differential operator first,
  - potential reconstruction using the discrete differential operator.

  (both polynomially consistent, both based on IBP formulas.)
- The definition of the spaces (DOFs) also guided by these IBP formulas.

Same guiding principles as the Hybrid High-Order (HHO) method [Di Pietro and Droniou, 2020].

Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements/faces/edges/vertices, with intrinsic orientations (tangent, normal).
  - $\omega_{TF} \in \{+1, -1\}$ such that $\omega_{TF}\boldsymbol{n}_F$ outer normal to T.
  - $\omega_{FE} \in \{+1, -1\}$ such that $\omega_{FE}\boldsymbol{t}_E$ clockwise on F.



$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

with derivatives in the direction $\boldsymbol{t}_E$.

$\mathcal{P}^k$-consistent gradient
Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_E^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

  with π_E^{k-1} the L^2-projection on $\mathcal{P}^{k-1}(E)$.

$\mathcal{P}^k$-consistent gradient

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- Space and interpolator:

$$\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \, : \, q_E \in \mathcal{P}^{k-1}(E) \, , \; q_V \in \mathbb{R} \right\},$$

$$\underline{I}_{\mathbf{grad},E}^k q = (\pi_E^{k-1} q, (q(\boldsymbol{x}_V))_{V \in \mathcal{V}_E}) \qquad \forall q \in C(\overline{E}).$$

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_E^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \ : \ q_E \in \mathcal{P}^{k-1}(E), \ q_V \in \mathbb{R} \right\}$.
- Edge gradient $G_E^k : \underline{X}_{\mathbf{grad},E}^k \to \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

$\mathcal{P}^k$-consistent gradient

Edge E

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- Potential reconstruction $\gamma_E^{k+1} : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma_E^{k+1} \underline{q}_E) z' = -\int_E (G_E^k \underline{q}_E) z + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}) \qquad \forall z \in \mathcal{P}^{k+2}(E).$$

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = -\int_E \pi_E^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}^k_{\mathbf{grad},E} = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \;:\; q_E \in \mathcal{P}^{k-1}(E),\; q_V \in \mathbb{R} \right\}.$
- Edge gradient $G_E^k : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}) \qquad \forall r \in \mathcal{P}^k(E).$$

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Could be used to transport averages in Active Flux methods on polygons...

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F q \underbrace{\mathrm{div}_F\,\boldsymbol{v}}_{\in\mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall\boldsymbol{v}\in\mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_F^{k-1} q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$-consistent gradient

Face F

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$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_F^{k-1} q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Space and interpolator:

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \;:\; q_F \in \mathcal{P}^{k-1}(F),\; q_E \in \mathcal{P}^{k-1}(E),\; q_V \in \mathbb{R}\Big\},$$

$$\underline{I}^k_{\mathbf{grad},F} q = (\pi_F^{k-1} q, (\pi_E^{k-1} q_{|E})_{E\in\mathcal{E}_F}, (q(\boldsymbol{x}_V))_{V\in\mathcal{V}_F}) \qquad \forall q \in C(\overline{F}).$$

$\mathcal{P}^k$-consistent gradient

Face F

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$$\int_F (\mathbf{grad}_F\, q)\cdot\boldsymbol{v} = -\int_F \pi_F^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in\mathcal{P}^{k-1}(F)} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \qquad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

- Space :

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ : \\ q_F \in \mathcal{P}^{k-1}(F)\,,\ q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R}\Big\},$$

- Face gradient $\mathbf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}^k_F \underline{q}_F)\cdot\boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\gamma_E^{k+1} \underline{q}_E)\boldsymbol{v}\cdot\boldsymbol{n}_{FE} \quad \forall \boldsymbol{v} \in \mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$-consistent gradient

Face F

- Space :

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ : \\ q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R}\Big\},$$

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- Potential reconstruction $\gamma^{k+1}_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma^{k+1}_F \underline{q}_F) \operatorname{div}_F \boldsymbol{z} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{z} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{z}\cdot\boldsymbol{n}_{FE} \\ \forall \boldsymbol{z} \in \mathcal{R}^{\mathrm{c},k+2}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

($\operatorname{div}_F : \mathcal{R}^{\mathrm{c},k+2}(F) \to \mathcal{P}^{k+1}(F)$ *is an isomorphism.*)

$\mathcal{P}^k$-consistent gradient

Face F

- Space :

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \; : \; q_F \in \mathcal{P}^{k-1}(F),\; q_E \in \mathcal{P}^{k-1}(E),\; q_V \in \mathbb{R} \Big\},$$

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$$\int_F (\gamma^{k+1}_F \underline{q}_F)\,\mathrm{div}_F\, \boldsymbol{z} = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{z} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F (\gamma^{k+1}_E \underline{q}_E)\boldsymbol{z}\cdot\boldsymbol{n}_{FE}$$
$$\forall \boldsymbol{z} \in \mathcal{R}^{\mathrm{c},k+2}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k+1}(F).$$

Could be used to transport averages in Active Flux methods on polygotopes...

$\mathcal{P}^k$-consistent gradient

Element T

Same principle! Based on IBP we determine:

- An additional unknown ($q_T \in \mathcal{P}^{k-1}(T)$) to get the space $\underline{X}^k_{\mathbf{grad},T}$, and its meaning (polynomial moment on T) to get the interpolator $\underline{I}^k_{\mathbf{grad},T}$.
- A formula for the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3$.
- A potential reconstruction $P^{k+1}_{\mathbf{grad},T} : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T)$.

$\mathcal{P}^k$-consistent gradient

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- A formula for the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3$.
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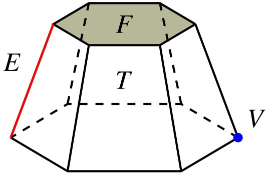
Element gradient could be used to transport point values in Active Flux methods on polygotopes...

The Discrete de Rham method

- Contrary to FE, do not seek explicit (or any!) basis functions.
- Replace continuous spaces by fully discrete ones made of vectors of polynomials, representing polynomial moments when interpreted through the interpolator.
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Create discrete operators (differential, potential reconstruction) between the spaces.



DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ : \\ \boldsymbol{v}_T \in \mathcal{RT}^k(T),\ \boldsymbol{v}_F \in \mathcal{RT}^k(F),\ v_E \in \mathcal{P}^k(E) \Big\},$$

- Interpolator: $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = \underline{\boldsymbol{v}}_h$ with

$$\begin{aligned} v_E &= L^2\text{-projection on } \mathcal{P}^k(E) \text{ of } \boldsymbol{v}\cdot\boldsymbol{t}_E, \\ \boldsymbol{v}_F &= L^2\text{-projection on } \mathcal{RT}^k(F) \text{ of } \boldsymbol{v}_{\mathrm{t},F}, \\ \boldsymbol{v}_T &= L^2\text{-projection on } \mathcal{RT}^k(T) \text{ of } \boldsymbol{v}. \end{aligned}$$

- Potential reconstructions for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$:
  - tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$,
  - element potential $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3$.

Discrete gradient

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{G}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete gradient: project the face/element/edge gradients

$$\mathbf{G}^k_T : \underline{X}^k_{\mathrm{grad},T} \to \mathcal{P}^k(T)^3, \qquad \mathbf{G}^k_F : \underline{X}^k_{\mathrm{grad},F} \to \mathcal{P}^k(F)^2,$$
$$G^k_E : \underline{X}^k_{\mathrm{grad},E} \to \mathcal{P}^k(E)$$

onto the proper spaces:

$$\underline{\boldsymbol{G}}^k_h \underline{q}_h = \Big((\boldsymbol{\pi}^k_{\mathcal{RT},T} \mathbf{G}^k_T \underline{q}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}^k_{\mathcal{RT},F} \mathbf{G}^k_F \underline{q}_F)_{F\in\mathcal{F}_h}, (G^k_E \underline{q}_E)_{E\in\mathcal{E}_h} q \Big).$$

DOF by mesh entities

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{N}}^k(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

DDR vs. Serendipity-DDR vs. Raviart-Thomas-Nédélec

Discrete space	$k=0$	$k=1$	$k=2$
$H^1(T)$	4 ♢ 4 ♢ 4	15 ♢ 10 ♢ 10	32 ♢ 20 ♢ 20
$\boldsymbol{H}(\mathbf{curl};T)$	6 ♢ 6 ♢ 6	28 ♢ 23 ♢ 20	65 ♢ 53 ♢ 45
$\boldsymbol{H}(\mathrm{div};T)$	4 ♢ 4 ♢ 4	18 ♢ 18 ♢ 15	44 ♢ 44 ♢ 36
$L^2(T)$	1 ♢ 1 ♢ 1	4 ♢ 4 ♢ 4	10 ♢ 10 ♢ 10

Table: Tetrahedron: local number of DOFs for DDR ♢ SDDR ♢ RTN.

Discrete space	$k=0$	$k=1$	$k=2$
$H^1(T)$	8 ♢ 8 ♢ 8	27 ♢ 20 ♢ 27	54 ♢ 32 ♢ 64
$\boldsymbol{H}(\mathbf{curl};T)$	12 ♢ 12 ♢ 12	46 ♢ 39 ♢ 54	99 ♢ 77 ♢ 144
$\boldsymbol{H}(\mathrm{div};T)$	6 ♢ 6 ♢ 6	24 ♢ 24 ♢ 36	56 ♢ 56 ♢ 108
$L^2(T)$	1 ♢ 1 ♢ 1	4 ♢ 4 ♢ 8	10 ♢ 10 ♢ 27

Table: Hexahedron: local number of DOFs for DDR ♢ SDDR ♢ RTN.

Ref. for serendipity reduction: [Di Pietro and Droniou, 2023a].

Outline

Algebraic properties

DDR sequence:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Theorem (Complex property and exactness [Di Pietro et al., 2020], [Di Pietro and Droniou, 2021])

The DDR sequence is a complex, which is exact if the topology of Ω is trivial:

$$\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h, \quad \mathrm{Im}\,\underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\,D^k_h, \quad \mathrm{Im}\,D^k_h = \mathcal{P}^k(\mathcal{T}_h).$$

Algebraic properties

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Theorem (Cohomology [Di Pietro et al., 2023])

For a generic domain Ω, the DDR complex has the same cohomology as the continuous de Rham complex.

Collaborator: Silvano Pitassi.

L^2-like inner products

- Local L^2-like inner product on the DDR spaces:

  for $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ or (div, k),
  $$(x_T, y_T)_{\bullet,T} = \int_T \boldsymbol{P}^{\ell}_{\bullet,T} x_T \cdot \boldsymbol{P}^{\ell}_{\bullet,T} y_T + \mathrm{s}_{\bullet,T}(x_T, y_T) \qquad \forall x_T, y_T \in \underline{X}^k_{\bullet,T},$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face/edge potentials*).

- Global L^2-like product by standard assembly of local ones.

Analytical properties

$$\begin{array}{ccccccc}
C^\infty(\overline{\Omega}) & & C^\infty(\overline{\Omega})^3 & & C^\infty(\overline{\Omega})^3 & & \\
\big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{grad},h} & & \big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \big\downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h) \\
(\cdot,\cdot)_{\mathbf{grad},h} & & (\cdot,\cdot)_{\mathbf{curl},h} & & (\cdot,\cdot)_{\mathrm{div},h} & & (\cdot,\cdot)_{L^2}
\end{array}$$

- For stability:

Poincaré inequalities: for $\underline{\mathrm{d}}^k_h = \underline{\boldsymbol{G}}^k_h, \underline{\boldsymbol{C}}^k_h, D^k_h$,

$$\|\underline{x}_h\|_h \lesssim \|\underline{\mathrm{d}}^k_h \underline{x}_h\|_h \qquad \forall \underline{x}_h \in (\mathrm{Ker}\,\underline{\mathrm{d}}^k_h)^\perp.$$

Analytical properties

$$
\begin{array}{ccccccc}
C^\infty(\overline{\Omega}) & & C^\infty(\overline{\Omega})^3 & & C^\infty(\overline{\Omega})^3 & & \\
\downarrow \underline{\boldsymbol{I}}_{\mathbf{grad},h}^k & & \downarrow \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k & & \downarrow \underline{\boldsymbol{I}}_{\mathrm{div},h}^k & & \\
\underline{X}_{\mathbf{grad},h}^k & \xrightarrow{\underline{\boldsymbol{G}}_h^k} & \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k & \xrightarrow{\underline{\boldsymbol{C}}_h^k} & \underline{\boldsymbol{X}}_{\mathrm{div},h}^k & \xrightarrow{D_h^k} & \mathcal{P}^k(\mathcal{T}_h) \\
(\cdot,\cdot)_{\mathbf{grad},h} & & (\cdot,\cdot)_{\mathbf{curl},h} & & (\cdot,\cdot)_{\mathrm{div},h} & & (\cdot,\cdot)_{L^2}
\end{array}
$$

- For stability:

> Poincaré inequalities: for $\underline{\mathrm{d}}_h^k = \underline{\boldsymbol{G}}_h^k, \underline{\boldsymbol{C}}_h^k, D_h^k$,
>
> $$\|\underline{x}_h\|_h \lesssim \|\underline{\mathrm{d}}_h^k \underline{x}_h\|_h \qquad \forall \underline{x}_h \in (\mathrm{Ker}\, \underline{\mathrm{d}}_h^k)^\perp.$$

- For consistency:
  - Primal consistency: approximation properties of $\boldsymbol{P}_{\bullet,h}^\ell \circ \underline{\boldsymbol{I}}_{\bullet,h}^k$ and $\mathrm{d}_h^k \circ \underline{\boldsymbol{I}}_{\bullet,h}^k$.
  - Adjoint consistency: control error in global discrete integration-by-parts.

Poincaré inequalities

Theorem (Poincaré inequalities [Di Pietro and Droniou, 2021], [Di Pietro and Droniou, 2023b], [Di Pietro and Hanot, 2023])

It holds:

$$
\begin{aligned}
\|\underline{q}_h\|_{\mathbf{grad},h} &\lesssim \|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h} && \forall \underline{q}_h \in (\operatorname{Ker} \underline{\boldsymbol{G}}_h^k)^\perp,\\
\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} &\lesssim \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\zeta}}_h\|_{\mathrm{div},h} && \forall \underline{\boldsymbol{\zeta}}_h \in (\operatorname{Ker} \underline{\boldsymbol{C}}_h^k)^\perp,\\
\|\underline{\boldsymbol{w}}_h\|_{\mathrm{div},h} &\lesssim \|D_h^k \underline{\boldsymbol{w}}_h\|_{L^2(\Omega)} && \forall \underline{\boldsymbol{w}}_h \in (\operatorname{Ker} D_h^k)^\perp.
\end{aligned}
$$

- Essential to use the complex exactness to get stability of numerical discretisations.

Primal consistency

Theorem (Consistency of potential reconstruction and stabilisation [Di Pietro and Droniou, 2023b])

It holds, for $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ *or* (div, k),

$$\|\boldsymbol{P}^{\ell}_{\bullet,T}\underline{\boldsymbol{I}}^{k}_{\bullet,T} f - f\|_{L^2(T)} + \mathrm{s}_{\bullet,T}(\underline{\boldsymbol{I}}^{k}_{\bullet,T} f, \underline{\boldsymbol{I}}^{k}_{\bullet,T} f) \lesssim h_T^{\ell+1} |f|_{H^{\ell+1}(T)} \qquad \forall f \in H^{\ell+1}(T)$$

(caveat for $\bullet = \mathbf{curl}$*).*

- Comes from local polynomial consistency: $\boldsymbol{P}^{\ell}_{\bullet,T}\underline{\boldsymbol{I}}^{k}_{\bullet,T} x_T = x_T$ and $\mathrm{s}_{\bullet,T}(\underline{\boldsymbol{I}}^{k}_{\bullet,T} x_T, \cdot) = 0$ if $x_T \in \mathcal{P}^{\ell}(T)$.
- Gives consistency of discrete L^2 inner products.

Commutation properties

Theorem (Commutation properties [Di Pietro and Droniou, 2023b])

$$
\begin{array}{ccc}
C^1(\overline{\Omega}) & \xrightarrow{\ \mathbf{grad}\ } & C^0(\overline{\Omega}) \\
\Big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{grad},h} & & \Big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \\
\underline{\boldsymbol{X}}^k_{\mathbf{grad},h} & \xrightarrow{\ \underline{\boldsymbol{G}}^k_h\ } & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}
\end{array}
\qquad
\begin{array}{ccc}
H^2(\Omega) & \xrightarrow{\ \mathbf{curl}\ } & H^1(\Omega) \\
\Big\downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \Big\downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} \\
\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\ \underline{\boldsymbol{C}}^k_h\ } & \underline{\boldsymbol{X}}^k_{\mathrm{div},h}
\end{array}
\qquad
\begin{array}{ccc}
H^1(\Omega) & \xrightarrow{\ \mathrm{div}\ } & L^2(\Omega) \\
\Big\downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \Big\downarrow \pi^k_h \\
\underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{\ D^k_h\ } & \mathcal{P}^k(\mathcal{T}_h)
\end{array}
$$

- Together with the consistency of potential reconstruction, provides optimal approximation properties of the differential operators.
- Essential for robust approximations (e.g. pressure-robust for Stokes, locking-free for Reissner-Mindlin...).

Adjoint consistency

Theorem (Adjoint consistency for the discrete gradient [Di Pietro and Droniou, 2021])

For all $\boldsymbol{v} \in C^0(\overline{\Omega}) \cap \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \boldsymbol{H}^{\max(k+1,2)}(\mathcal{T}_h)$ *and* $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$\left| (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} + \int_\Omega \mathrm{div}\,\boldsymbol{v}\ \ P^{k+1}_{\mathbf{grad},h}\underline{q}_h \right| \lesssim h^{k+1} |\boldsymbol{v}|_{H^{(k+1,2)}(\mathcal{T}_h)} \|\underline{\boldsymbol{G}}^k_h \underline{q}_h\|_{\mathbf{curl},h}.$$

- Similar adjoint consistencies for the curl, divergence.
- Essential for error estimates when IBP are involved in the weak formulations.

Schemes

- Numerical schemes are obtained replacing spaces, differential operators, and L^2-products with their discrete DDR counterparts.

Schemes

- Numerical schemes are obtained replacing spaces, differential operators, and L^2-products with their discrete DDR counterparts.

- Poincaré inequalities, primal and adjoint consistencies yield stable schemes, with $\mathcal{O}(h^{k+1})$ rates of convergence in energy norm.
  [Di Pietro and Droniou, 2021], [Beirão da Veiga et al., 2022],
  [Droniou and Qian, 2023], [Di Pietro and Droniou, 2023b],
  [Di Pietro and Droniou, 2022], etc.

Outline

The de Rham complex in exterior calculus formulation

Collaborators: Francesco Bonaldi, Kaibo Hu

- Differential forms:
  - k-forms: mappings ω on Ω s.t. $\omega_x \in \Lambda^k(\mathbb{R}^n)$, k-alternate linear forms on $\mathbb{R}^n$.
  - d^k: exterior derivative of k-forms.
  - $H\Lambda^k(\Omega)$: k-forms $\omega \in L^2$ s.t. $\mathrm{d}^k\omega \in L^2$.
- The continuous de Rham complex with differential forms:

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the following links are established through vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

Construction of a DDR exterior calculus complex

- Discrete spaces $\underline{X}^k_{r,f}$ with polynomial components attached to mesh entities, representing projections of traces of k-forms.
- Recursive and hierarchical construction on d-cells f (for $d = k+1, \ldots, n$):
  - Discrete exterior derivative

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f)$$

  - Discrete potential (playing the role of a k-form inside f), using the discrete exterior derivative

$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^k(f)$$

  - Reconstructions mimic the Stokes formula: $\forall(\omega,\mu) \in \Lambda^\ell(f) \times \Lambda^{n-\ell-1}(f)$,

$$\int_f \mathrm{d}^\ell\omega \wedge \mu = (-1)^{\ell+1}\int_f \omega \wedge \mathrm{d}^{n-\ell-1}\mu + \int_{\partial f} \mathrm{tr}_{\partial f}\,\omega \wedge \mathrm{tr}_{\partial f}\,\mu$$

- Benefit: unified construction and algebraic proofs for any space dimension, and all along the sequence (no specific argument for 2D/3D or $\mathbf{grad}$, $\mathbf{curl}$, div). [Bonaldi et al., 2023].

Outline

DDR scheme for Stokes in curl-curl formulation

Collaborators: Louenço Beirão da Veiga, Franco Dassi

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1_\star(\Omega)$ s.t.

$$
\begin{aligned}
\nu(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2} + (\mathbf{grad}\,p, \boldsymbol{v})_{L^2} &= (\boldsymbol{f}, \boldsymbol{v})_{L^2} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-(\boldsymbol{u}, \mathbf{grad}\,q)_{L^2} &= 0 && \forall q \in H^1_\star(\Omega).
\end{aligned}
$$

- DDR scheme: Find $(\underline{\boldsymbol{u}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{X}^k_{\mathbf{grad},h,\star}$ such that

$$
\begin{aligned}
\nu(\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h\underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
-(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{G}}^k_h\underline{q}_h)_{\mathbf{curl},h} &= 0 && \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,\star},
\end{aligned}
$$

where $\underline{X}^k_{\mathbf{grad},h,\star} := \{\underline{r}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{I}^k_{\mathbf{grad},h}1, \underline{r}_h) = 0\}$.

Navier–Stokes equations in curl-curl formulation

Collaborator: Jia Jia Qian

- Additional convective term:

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\operatorname{div} \boldsymbol{u})\boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2}\, \mathbf{grad}\, |\boldsymbol{u}|^2.$$

so

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathbf{grad}\, p = \nu\, \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{u} + \underbrace{(\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u}}_{\text{additional term}} + \mathbf{grad} \underbrace{\left(p + \frac{1}{2}|\boldsymbol{u}|^2\right)}_{\text{new pressure } p'}$$

- Additional term in weak formulation (vanishes for $\boldsymbol{v} = \boldsymbol{u}$)

$$\int_\Omega [(\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u}] \cdot \boldsymbol{v}.$$

DDR scheme for Navier–Stokes in curl-curl formulation

- Same as Stokes, but we need to discretize

$$\int_\Omega [(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}] \cdot \boldsymbol{v}.$$

- Natural choice: replace continuous curl and functions by discrete ones.

$$\int_\Omega \left[(\boldsymbol{C}_h^k \underline{\boldsymbol{u}}_h) \times \boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{u}}_h\right] \cdot \boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{v}}_h$$

(where $\boldsymbol{C}_h^k$ and $\boldsymbol{P}_{\mathbf{curl},h}^k$ are the global piecewise-polynomial discrete curl and potential reconstructions obtained by patching the local ones together).

Non-dissipative: this term vanishes if $\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{u}}_h$.

Convergence result

Theorem (Error estimates [Di Pietro et al., 2024])

Define the discrete L^4-Sobolev constant by

$$C_{\mathrm{S},h} \coloneqq \max\left\{ \frac{\|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}} \;:\; \underline{\boldsymbol{v}}_h \in (\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h)^\perp \backslash \{\mathbf{0}\} \right\}.$$

Then, if

$$C^2_{\mathrm{S},h}\|\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{R}_{\boldsymbol{u}})\|_{\mathbf{curl},h} \text{ is small enough,}$$

we have

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- $\mathbf{R}_{\boldsymbol{u}}$: solenoidal part of forcing term $\boldsymbol{f}$, depends only on $\boldsymbol{u}$.
- Robust estimate with respect to the pressure (RHS does not depend on p').

Convergence result

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we have

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- Based on discrete Poincaré inequalities, primal/adjoint consistencies, and estimates obtained through the unified polytopal exterior calculus framework.

Convergence result

Theorem (Error estimates [Di Pietro et al., 2024])

Define the discrete L^4-Sobolev constant by

$$C_{\mathrm{S},h} \coloneqq \max\left\{\frac{\|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}} \,:\, \underline{\boldsymbol{v}}_h \in (\operatorname{Im}\underline{\boldsymbol{G}}^k_h)^\perp\backslash\{\mathbf{0}\}\right\}.$$

Then, if

$$C^2_{\mathrm{S},h}\|\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{R}_{\boldsymbol{u}})\|_{\mathbf{curl},h} \text{ is small enough,}$$

we have

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- Valid for Stokes without smallness assumption (and no $C_{\mathrm{S},h}$).

Convergence result

Theorem (Error estimates [Di Pietro et al., 2024])

Define the discrete L^4-Sobolev constant by

$$C_{\mathrm{S},h} \coloneqq \max\left\{ \frac{\|\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}} \;:\; \underline{\boldsymbol{v}}_h \in (\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h)^\perp \backslash \{\mathbf{0}\} \right\}.$$

Then, if

$$C^2_{\mathrm{S},h}\|\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{R}_{\boldsymbol{u}})\|_{\mathbf{curl},h} \text{ is small enough,}$$

we have

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u})\|_{\mathrm{div},h} \lesssim C(\boldsymbol{u})h^{k+1}.$$

- Boundedness of $C_{S,h}$ w.r.t. h still an open question (but expected for convex domains).

Outline

Pressure-robustness I

- Analytical solution on $\Omega = (0,1)^3$:

$$p(x,y,z) = \lambda \sin(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad \text{with } \lambda \in \{1, 100\},$$

$$\boldsymbol{u}(x,y,z) = \begin{bmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) \\ \frac{1}{2}\cos(2\pi x)\sin(2\pi y)\cos(2\pi z) \\ -\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) \end{bmatrix}.$$

- Measured errors (discrete and potential-based):

$$E_{\boldsymbol{u}}^{\mathrm{d}} := \left(\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u})\|_{\mathrm{div},h}^2 \right)^{1/2},$$

$$E_p^{\mathrm{d}} := \|\underline{\boldsymbol{G}}_h^k(\underline{p}_h - \underline{I}_{\mathbf{grad},h}^k p)\|_{\mathbf{curl},h},$$

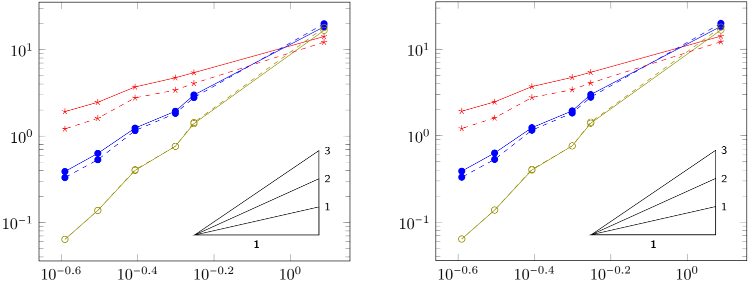
$$E_{\boldsymbol{u}}^{\mathrm{p}} := \left(\|\boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{P}_{\mathrm{div},h}^k \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{u}}_h - \mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2 \right)^{1/2},$$

$$E_p^{\mathrm{p}} := \|\boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{G}}_h^k \underline{p}_h - \mathbf{grad}\, p\|_{\boldsymbol{L}^2(\Omega)}.$$

- Meshes: tetrahedral and Voronoi meshes.

Pressure-robustness II

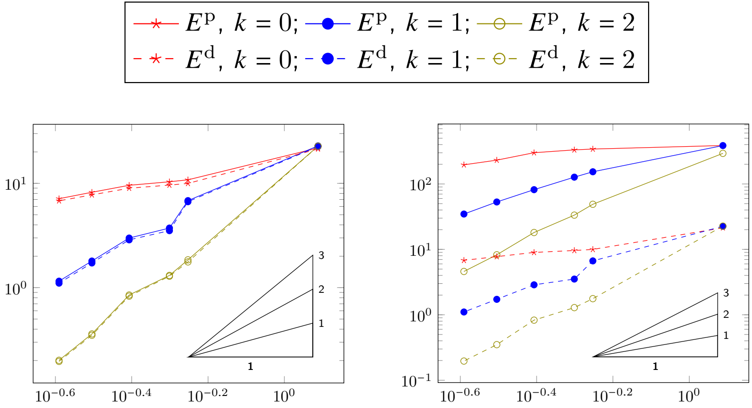
E^{p}, $k = 0$; E^{p}, $k = 1$; E^{p}, $k = 2$
E^{d}, $k = 0$; E^{d}, $k = 1$; E^{d}, $k = 2$



(a) Errors on $\boldsymbol{u}$, $\lambda = 1$ (b) Errors on $\boldsymbol{u}$, $\lambda = 10^2$

Figure: Tetrahedral meshes: errors with respect to h

Pressure-robustness III

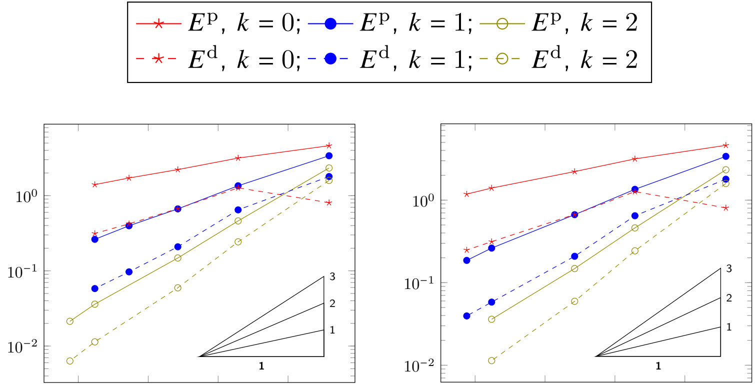


(a) Errors on ∇p, $\lambda = 1$

(b) Errors on ∇p, $\lambda = 10^2$

Figure: Tetrahedral meshes: errors with respect to h

Pressure-robustness IV



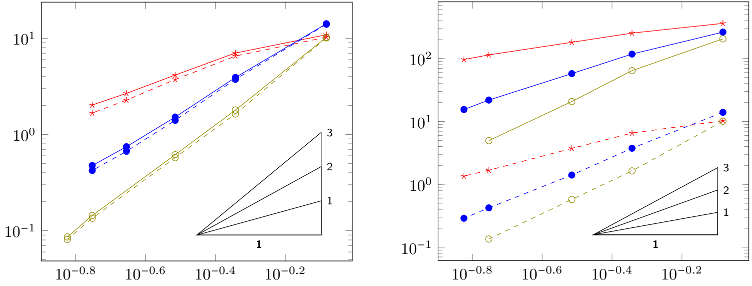
(a) Errors on $\boldsymbol{u}$, $\lambda = 1$

(b) Errors on $\boldsymbol{u}$, $\lambda = 10^2$

Figure: Voronoi meshes: errors with respect to h

Pressure-robustness V

E^{p}, $k=0$;	E^{p}, $k=1$;	E^{p}, $k=2$
E^{d}, $k=0$;	E^{d}, $k=1$;	E^{d}, $k=2$



(a) Errors on ∇p, $\lambda = 1$

(b) Errors on ∇p, $\lambda = 10^2$

Figure: Voronoi meshes: errors with respect to h

Flow in cavity – mixed BCs I

In the unit cube $\boldsymbol{\Omega} = (0,1)^3$:

- Essential BCs (pressure and tangential velocity):

$$p(x,y,z) = -z \quad \text{and} \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$$

  on the bottom corner $\{0\} \times (0,0.25) \times (0,0.25)$ of the face $x = 0$.

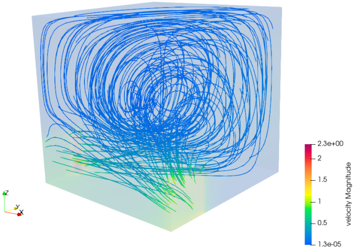
- Natural BCs (tangential vorticity and flux):

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 1$$

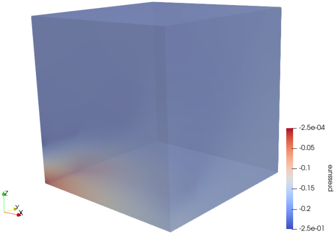
  on the bottom corner $\{1\} \times (0,0.25) \times (0,0.25)$ of the face $x = 1$,

- Homogeneous natural BCs elsewhere.

Flow in cavity – mixed BCs II



(a) Velocity



(b) Pressure

Figure: Velocity streamlines and pressure

Conclusion I

- DDR:
  - fully discrete complex, with stability properties as the continuous complex;
  - arbitrary degree of accuracy;
  - applicable on polytopal meshes.

- DOFs:
  - polynomial moments attached to mesh entities (vertices, edges, faces, elements);
  - come fom the hierarchical construction based on integration-by-parts.

- Systematic techniques to reduce the number of DOFs without loosing accuracy:
  - enhancement (reconstruction of potential from discrete differential operator),
  - serendipity (on any polytopal mesh).

  $\rightsquigarrow$ leaner complexes than FE approches on certain meshes (*and fully compatible with FE complexes on hybrid meshes*).

Conclusion II

- Full set of homological and analytical results: cohomology, Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.
- Polytopal exterior calculus approach to unify the construction.
- Some other applications/complexes:
  - div-div plates complex and serendipity version [Di Pietro and Droniou, 2023a], [Botti et al., 2023].
  - Magnetostatics equations [Di Pietro and Droniou, 2021].
  - Yang–Mills equations [Droniou et al., 2023], [Droniou and Qian, 2023].
  - Stokes complex [Hanot, 2023].
  - Rot-rot complex [Di Pietro, 2023].
  - etc.

- Notes and series of introductory lectures to DDR:

`https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html`

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

Thank you!

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