

The exterior calculus Discrete De Rham method on manifolds and application to the Maxwell equations

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joint work with M. Hanot (U. Lille) and T. Oliynyk (Monash University).

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Reference for this presentation

A polytopal discrete de Rham complex on manifolds, with application to the Maxwell equations, J. Droniou, M. Hanot, and T. Oliynyk, 33p, 2024.
<https://arxiv.org/abs/2401.16130>.

These slides:



Outline

- 1 Building blocks: mesh and setting for polynomial complexes on a manifold
- 2 Discrete De Rham on a manifold
- 3 Constructing local polynomial spaces
- 4 Maxwell equations and numerical simulations

Article

<https://arxiv.org/abs/2401.16130>

Slides



Ω manifold of dimension n . Seen from ambient coordinates (embedded manifold), the mesh will be curved... Handling curved meshes, even in flat spaces, is often difficult.

Flat space

- Not very difficult for **dG**, but other drawbacks (no static condensation, etc.) [Cangiani et al., 2021].
- **VEM on curved meshes** (flat space): polynomials on edges by parametrisation [Beirão da Veiga et al., 2019, Dassi et al., 2021].
- **HHO on curved meshes** (flat space):
 - parametrisation of faces [Botti and Di Pietro, 2018].
 - polynomials on edges by restriction (intrinsic approach) [Yemm, 2024].

Manifolds

- **FEM**: **extrinsic** approach
[Demlow, 2009, Demlow and Dziuk, 2007, Arnold, 2018].
- **VEM**: **extrinsic** approach [Frittelli et al., 2023].
- **FEM**: **intrinsic** approach, with global homeomorphism with flat space
[Bachini et al., 2021a, Bachini et al., 2021b].

Construction of polynomial functions can be local, but continuity requirements make it difficult to glue over the entire manifold [Licht, 2023].

- Also approaches from the **General Relativity** community (lattice based), but limited to specific shapes [Gentle, 2002, Brewin, 1998, Frauendiener, 2006]...

No previous polytopal complex on manifolds...

Notations:

- \mathcal{M}_h finite collection of pairwise disjoint **submanifolds** of dimension $d = 0, \dots, n$ ($d = 0$: vertices, $d = 1$: edges; etc.)
- $\Delta_d(\mathcal{M}_h)$ collection of all d -dimensional manifolds.
- Boundary of $f \in \Delta_d(\mathcal{M}_h)$ made of $(d - 1)$ -manifolds $f' \in \Delta_{d-1}(f)$.

Regular mesh sequence:

- Let M_h be a flat polytopal mesh.
- For all d , there is a bijection $\Phi_d : \Delta_d(\mathcal{M}_h) \rightarrow \Delta_d(M_h)$ that respect boundaries.
- For all $f \in \Delta_d(\mathcal{M}_h)$, there is a diffeomorphism $\phi_f : f \rightarrow \Phi_d(f)$ such that

$$\|\nabla\phi_f\|_\infty \approx \|\det(\nabla\phi_f)\|_\infty^{\frac{1}{d}} \approx \|\det(\nabla\phi_f^{-1})\|_\infty^{-\frac{1}{d}} \approx \|\nabla\phi_f^{-1}\|_\infty^{-1}.$$

- $(\mathcal{M}_h)_h$ is regular if there are such bijections and $(M_h)_h$ is regular.

Abstract setting for local polynomial forms

Complexes for the exterior derivative and (some) Koszul operator:

$$\dots \xrightarrow{d} \mathcal{P}_r \Lambda^l(f) \xrightarrow{d} \mathcal{P}_{r-1} \Lambda^{l+1}(f) \xrightarrow{d} \dots$$

$$\dots \xleftarrow{\kappa} \mathcal{P}_r \Lambda^l(f) \xleftarrow{\kappa} \mathcal{P}_{r-1} \Lambda^{l+1}(f) \xleftarrow{\kappa} \dots$$

Assumption (Homogeneous polynomials decomposition)

- *There is a decomposition $\mathcal{P}_r \Lambda^l(f) = \bigoplus_{s \leq r} \mathcal{H}_s \Lambda^l(f)$.*
- *We have $d\kappa + \kappa d = \lambda_{s,l} \text{Id}$ on $\mathcal{H}_s \Lambda^l(f)$.*
- *Eigenvalues identify polynomial degree: $\lambda_{s,l} = 0$ if and only if $s = l = 0$, and if $\lambda_{s,l} = \lambda_{s',l}$ then $s = s'$.*
- *$d\mathcal{H}_s \Lambda^l(f) \neq \{0\}$ whenever $s > 0$ and $l < d$.*

In the flat case, $d\kappa + \kappa d = (s + l) \text{Id}$ on $\mathcal{H}_s \Lambda^l(f)$.

Consequences of the design assumption

Bijection of d and κ : the following mappings are bijective:

$$d : \kappa\mathcal{H}_s\Lambda^l(f) \rightarrow d\mathcal{H}_{s+1}\Lambda^{l-1}(f) \quad (\subset \mathcal{H}_s\Lambda^l(f))$$

$$\kappa : d\mathcal{H}_s\Lambda^l(f) \rightarrow \kappa\mathcal{H}_{s-1}\Lambda^{l+1} \quad (\subset \mathcal{H}_s\Lambda^l(f)).$$

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Direct decomposition:

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^1(f)$$

$$\mathcal{P}_r\Lambda^l(f) = d\mathcal{P}_{r+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1.$$

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Definition (Trimmed polynomial spaces)

$$\mathcal{P}_r^- \Lambda^0(f) := \mathcal{P}_r\Lambda^0(f)$$

$$\mathcal{P}_r^- \Lambda^l(f) := d\mathcal{P}_r\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1.$$

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Space and discrete operators

Space: vectors of trimmed polynomials on vertices, edges, faces, etc. (almost)

$$\underline{\mathbf{X}}_{r,h}^k := \prod_{d=k}^n \prod_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f).$$

Notation: for $f \in \Delta_d(\mathcal{M}_h)$,

$$\underline{\omega}_f = (\omega_f, (\omega_{f'})_{f' \in \Delta_{d-1}(f)}, (\omega_{f''})_{f'' \in \Delta_{d-2}(f)}, \text{etc.})$$

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Local exterior derivative and potential reconstruction: for $f_\ell \in \Delta_\ell(\mathcal{M}_h)$ with $\ell = k, \dots, n$,

$$d_{r,f_\ell}^k : \underline{\mathbf{X}}_{r,f_\ell}^k \rightarrow \star^{-1} \mathcal{P}_r \Lambda^{\ell-k-1}(f_\ell) \text{ and } P_{r,f_\ell}^k : \underline{\mathbf{X}}_{r,f_\ell}^k \rightarrow \star^{-1} \mathcal{P}_r \Lambda^{\ell-k}(f_\ell)$$

in a hierarchically and recursive process:

$$P_{r,f_k}^k \rightsquigarrow d_{r,f_{k+1}}^k \rightsquigarrow P_{r,f_{k+1}}^k \rightsquigarrow d_{r,f_{k+2}}^k \rightsquigarrow \dots \rightsquigarrow P_{r,f_n}^k.$$

Assumption (Trace of trimmed polynomials)

$$\mathrm{tr}_{f'} \mathcal{P}_r^- \Lambda^l(f) \subset \mathcal{P}_r^- \Lambda^l(f') \quad \forall f' \in \Delta_{d'}(f),$$

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DDR complex: in exterior calculus presentation on the manifold.

$$\cdots \xrightarrow{\underline{d}_{r,h}^{k-1}} \underline{\mathbf{X}}_{r,h}^k \xrightarrow{\underline{d}_{r,h}^k} \underline{\mathbf{X}}_{r,h}^{k+1} \xrightarrow{\underline{d}_{r,h}^{k+1}} \cdots .$$

Components of $\underline{d}_{r,h}^k \underline{\omega}_h$ obtained projecting the local exterior derivatives $\underline{d}_{r,f}^k \underline{\omega}_f \in \star^{-1} \mathcal{P}_r \Lambda^{d-k-1}(f)$ on the trimmed spaces $\star^{-1} \mathcal{P}_r^- \Lambda^{d-k-1}(f)$.

Theorem (Commutation property)

$$\underline{d}_{r,f}^k(\underline{I}_{r,f}^k \omega) = \underline{I}_{r,f}^{k+1}(d\omega) \quad \forall \omega \in C^1 \Lambda^k(\bar{f}).$$

Theorem (Cohomology)

The DDR complex has the same cohomology as the continuous de Rham complex.

Consistency

Standard approach: starts with proving an exact consistency result for polynomial functions.

$$P_{r,f}^k \mathbf{I}_{r,f}^k \omega = \omega \quad \forall \omega \in \mathcal{P}_r \Lambda^k(f).$$

Based on the following: in the flat case, for $d = k$, $\star \omega \in \mathcal{P}_r \Lambda^0(f)$ so

$$\star^{-1} \pi_{r,f}^0 \star \omega = \star^{-1} \star \omega = \omega \quad \forall \omega \in \mathcal{P}_r \Lambda^k(f).$$

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Approach: show an approximation property by induction on the dimension of the face, based on: For all $\mu_f \in \mathcal{P}_r \Lambda^{d-k-1}(f)$,

$$\int_f d_{r,f}^k \underline{\mathbf{I}}_{r,f}^k \omega \wedge \mu_f = \int_f d\omega \wedge \mu_f + \int_{\partial f} (P_{r,\partial f}^k \underline{\mathbf{I}}_{r,\partial f}^k \omega - \text{tr}_{\partial f} \omega) \wedge \text{tr}_{\partial f} \mu_f$$

(obtained from definition of $d_{r,f}^k$, $\underline{\mathbf{I}}_{r,f}^k$ and IBP).

Assumption

We have the following local Poincaré inequality, topological decomposition, discrete trace and polynomial approximation properties:

$$\|\mu\|_f \lesssim h_f \|\mathrm{d}\mu\|_f, \quad \forall \mu \in \kappa \mathcal{P}_{r-1} \Lambda^k(f),$$

$$\|\alpha\|_f + \|\beta\|_f \lesssim \|\alpha + \beta\|_f, \quad \forall (\alpha, \beta) \in \kappa \mathcal{P}_{r-1} \Lambda^{k+1}(f) \oplus \mathrm{d} \mathcal{P}_{r+1} \Lambda^{k-1}(f),$$

$$\|\mathrm{tr}_{\partial f} \mu\|_{\partial f} \lesssim h_f^{-\frac{1}{2}} \|\mu\|_f, \quad \forall \mu \in \mathcal{P}_r \Lambda^k(f),$$

$$\|\star^{-1} \pi_{r,f}^{d-k} \star \omega - \omega\|_f \lesssim h_f^{t+1} |\omega|_{H^{t+1}}, \quad \forall \omega \in H^{t+1} \Lambda^k(f), \forall t \in [0, r],$$

Consistency

Hierarchical regularity and norm:

$$H^{r+1}\Lambda^k(f; \Delta) := \left\{ \omega \in H^{r+1}\Lambda^k(f) : \right. \\ \left. \operatorname{tr}_{f'} \omega \in H^{r+1}\Lambda^k(f') \quad \forall f' \in \Delta_{d'}(f), \forall d' \in [k, d-1] \right\},$$
$$|\omega|_{r,f,\Delta} := \sum_{d'=k}^d h_f^{\frac{d-d'}{2}} |\operatorname{tr}_{f'} \omega|_{H^{r+1}\Lambda^k(f')} \quad \forall \omega \in H^{r+1}\Lambda^k(f; \Delta).$$

Theorem (Primal consistency)

$$\|P_{r,f}^k \mathbf{I}_{r,f}^k \omega - \omega\|_f \lesssim h_f^{r+1} |\omega|_{r,f,\Delta} \quad \forall \omega \in H^{r+1}\Lambda^k(f; \Delta),$$
$$\|d_{r,f}^k \mathbf{I}_{r,f}^k \omega - d\omega\|_f \lesssim h_f^{r+1} |d\omega|_{r,f,\Delta} \quad \forall \omega \in C^1\Lambda^k(\bar{f})$$

s.t. $d\omega \in H^{r+1}\Lambda^{k+1}(f; \Delta).$

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Condition on local charts

- On principle, defining polynomial spaces on cells can be done via **local coordinates** (local charts).
- However, for hybrid methods we need $\text{tr}_{f'} \mathcal{P}_r \Lambda^l(f) \subset \mathcal{P}_r \Lambda^l(f')$: this requires **compatibility conditions** for the charts.

Condition on local charts

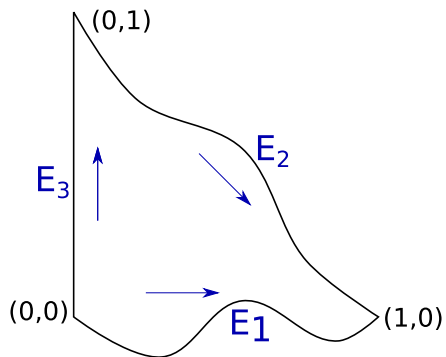
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- However, for hybrid methods we need $\text{tr}_{f'} \mathcal{P}_r \Lambda^l(f) \subset \mathcal{P}_r \Lambda^l(f')$: this requires **compatibility conditions** for the charts.

Assumption

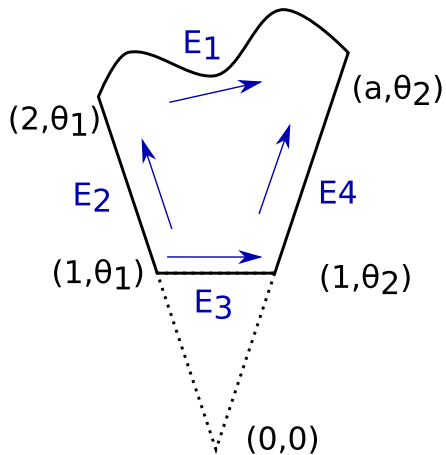
There are C^2 -diffeomorphisms $I_f : U_f \subset \mathbb{R}^d \rightarrow \bar{f}$ for all $f \in \Delta_d(\mathcal{M}_h)$ and $d \in [0, n]$.

Moreover, if $f' \in \Delta_{d-1}(f)$, with $\mathfrak{J}_{f,f'} : f' \rightarrow \bar{f}$ the inclusion of f' into \bar{f} , the mapping $(I_f)^{-1} \circ \mathfrak{J}_{f,f'} \circ I_{f'} : U_{f'} \rightarrow \mathbb{R}^d$ is affine.

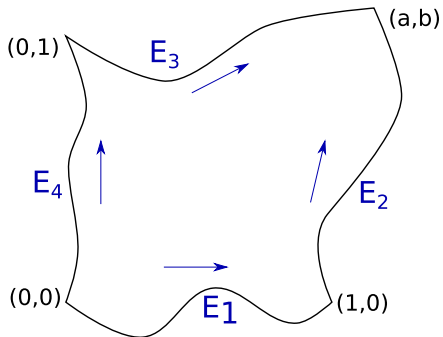
From there: case by case basis



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Maxwell equations in differential forms

- Ω a 3-dimensional Lorentzian manifold with metric decomposed in 2+1 with zero-shift and constant lapse c :

$$(g_{\mu\nu}) := \begin{pmatrix} -c^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix},$$

- d : spatial exterior derivative (2-dim); δ : co-differential; vol : spatial volume form.

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- d : spatial exterior derivative (2-dim); δ : co-differential; vol : spatial volume form.
- B : $(1/c) \times$ magnetic field (2-form), E : electric field (1-form).

$$\begin{aligned} \partial_t B &= -dE, \\ -\delta E &= \rho \epsilon_0^{-1}, \\ \frac{1}{c^2} \partial_t E &= \delta B - \mu_0 J, \end{aligned}$$

with μ_0 vacuum permeability, ρ electric charge density and J electric current density.

Note the compatibility condition on the data: $\delta J = \partial_t \rho$.

Weak formulation and scheme

- Weak formulation: find $(\tilde{E}, \tilde{B}) \in C^1([0, T]; \Lambda^1(\Omega)) \times C^1([0, T]; \Lambda^2(\Omega))$ s.t., for all $(v^1, v^2) \in \Lambda^1(\Omega) \times \Lambda^2(\Omega)$,

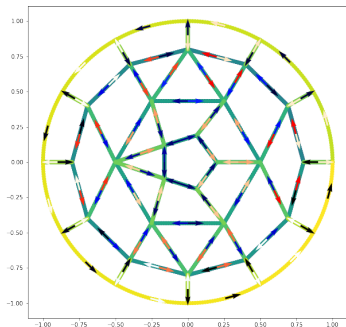
$$\begin{aligned}\langle \partial_t B, v^2 \rangle &= - \langle dE, v^2 \rangle, \\ \frac{1}{c^2} \langle \partial_t E, v^1 \rangle &= \langle B, dv^1 \rangle - \mu_0 \langle J, v^1 \rangle.\end{aligned}$$

- DDR scheme, with $X_h := \underline{\mathbf{X}}_{r,h}^1 \times \underline{\mathbf{X}}_{r,h}^2$: find $(\underline{\mathbf{E}}_h, \underline{\mathbf{B}}_h) \in C^1([0, T]; X_h)$ such that for all $(\underline{\mathbf{v}}_h^1, \underline{\mathbf{v}}_h^2) \in X_h$ and all $t \in (0, T)$,

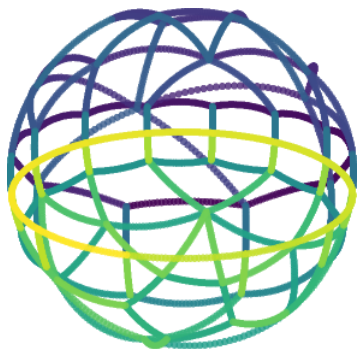
$$\begin{aligned}\langle \partial_t \underline{\mathbf{B}}_h(t), \underline{\mathbf{v}}_h^2 \rangle &= - \langle \underline{\mathbf{d}}_{r,h}^1 \underline{\mathbf{E}}_h(t), \underline{\mathbf{v}}_h^2 \rangle, \\ \frac{1}{c^2} \langle \partial_t \underline{\mathbf{E}}_h(t), \underline{\mathbf{v}}_h^1 \rangle &= \langle \underline{\mathbf{B}}_h(t), \underline{\mathbf{d}}_{r,h}^1 \underline{\mathbf{v}}_h^1 \rangle - \mu_0 \langle \underline{\mathbf{I}}_{r,h}^1 \tilde{\mathbf{J}}(t), \underline{\mathbf{v}}_h^1 \rangle.\end{aligned}$$

- Both preserve the constraint (dropped equation).

Mesh on the sphere

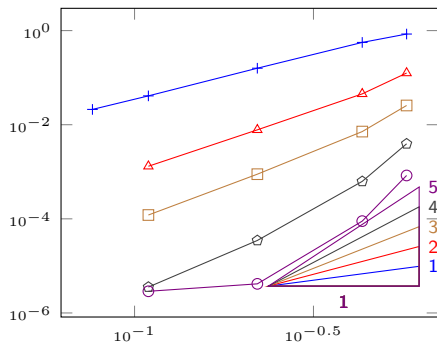
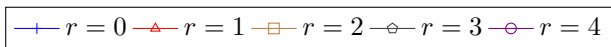


(a) Mesh in one chart and orientations.

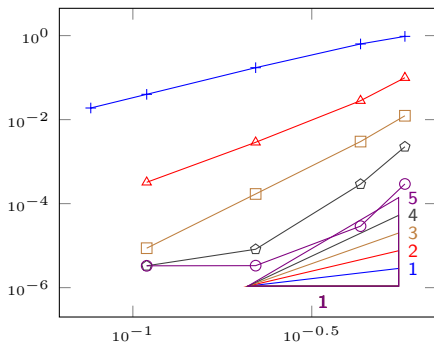


(b) Embedding of the 1-skeleton of the mesh into \mathbb{R}^3 .

Numerical results: smooth solution

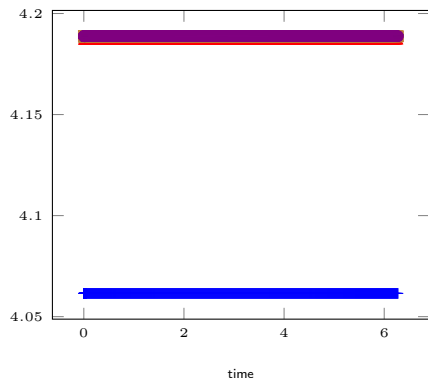
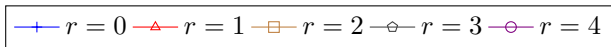


(a) Error on E .

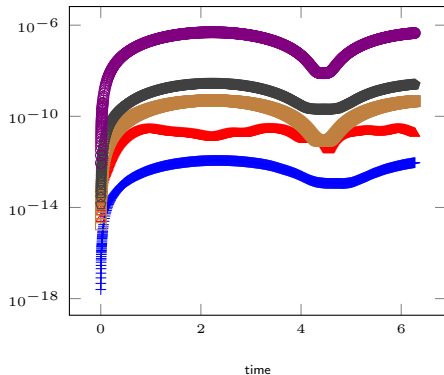


(b) Error on B .

Numerical results: preservation of energy and constraint

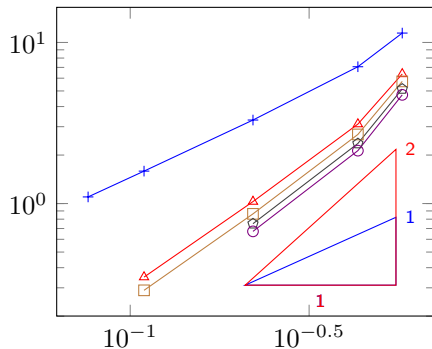
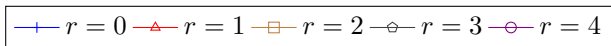


(a) Total energy $E^2 + B^2$.

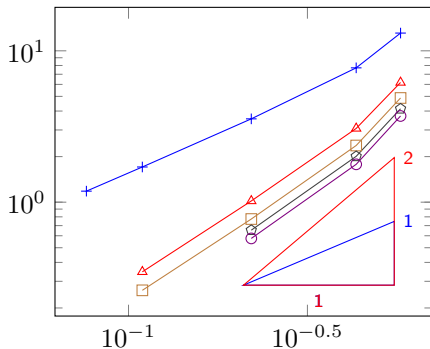


(b) Error on the discrete constraint.

Numerical results: piecewise smooth C^0 solution



(a) Error on E .



(b) Error on B .

Conclusion

- **Abstrat setting** for polynomial spaces on polytopal meshes on manifolds, identifies the key properties for developing discrete de Rham complexes.
- Practical construction from charts in special cases (include curved meshes).
- Arbitrary-order **polytopal exterior calculus on manifolds**.
- Set of **algebraic** results, and **consistency** (without exact polynomial consistency).
- **To do**: better/more general construction of local polynomial spaces; Poincaré inequalities (ongoing); adjoint consistency.



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Thank you for your attention!

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