

Polytopal complexes: introduction, examples and some associated results

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Outline

- 1 Polytopal complexes
 - Complexes: what? why?
 - Polytopal meshes: what? why?
- 2 The Discrete De Rham method
 - DDR: generic principles
 - Discrete $H(\mathbf{grad}, \Omega)$ space and gradient/potential reconstructions
 - The DDR complex
- 3 Properties of the DDR complex
- 4 Example of application: Stokes equations in curl-curl formulation

Slides



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Maxwell equations: preservation of constraint

- Maxwell equations in the void:

Evolution

$$\partial_t E = \mathbf{curl} B$$

$$\partial_t B = -\mathbf{curl} E$$

Constraints

$$\operatorname{div} E = 0$$

$$\operatorname{div} B = 0$$

Maxwell equations: preservation of constraint

- Maxwell equations in the void:

Evolution

$$\partial_t E = \mathbf{curl} B$$

$$\partial_t B = -\mathbf{curl} E$$

Constraints

$$\operatorname{div} E = 0$$

$$\operatorname{div} B = 0$$

- Constraint preservation:

$$\partial_t(\operatorname{div} E) = \operatorname{div}(\partial_t E) = \operatorname{div} \mathbf{curl} B = 0$$

since

$$\operatorname{div} \mathbf{curl} = 0$$

Stokes in “standard” formulation

- Ω domain, $\nu > 0$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Find $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ and $p : \Omega \rightarrow \mathbb{R}$ s.t. $\int_{\Omega} p = 0$ and

$$-\nu \Delta \mathbf{u} + \mathbf{grad} p = \mathbf{f} \quad \text{in } \Omega, \quad (\text{momentum conservation})$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (\text{mass conservation})$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (\text{boundary condition})$$

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- Weak formulation: Find $(\mathbf{u}, p) \in H_0(\mathbf{grad}, \Omega)^d \times L^2(\Omega)$ s.t. $\int_{\Omega} p = 0$ and

$$\nu(\mathbf{grad} \mathbf{u}, \mathbf{grad} \mathbf{v})_{L^2} - (p, \operatorname{div} \mathbf{v})_{L^2} = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in H_0(\mathbf{grad}, \Omega)^d,$$

$$(\operatorname{div} \mathbf{u}, q)_{L^2} = 0 \quad \forall q \in L^2(\Omega)$$

Note: for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \operatorname{div}\}$,

$$H(\bullet, \Omega) := \{z \in L^2(\Omega) : \bullet z \in L^2(\Omega)\}$$

and 0 indicates zero boundary conditions.

Stokes in “standard” formulation

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- A priori estimates require: ($a \lesssim b$ means $a \leq Cb$ with C independent of a, b .)

- Poincaré inequality: $\|\cdot\|_{L^2} \lesssim \|\mathbf{grad} \cdot\|_{L^2}$ on $H_0(\mathbf{grad}, \Omega)$,

- inf-sup $\sup_{\mathbf{v} \in H_0(\mathbf{grad}, \Omega)} \frac{(p, \operatorname{div} \mathbf{v})_{L^2}}{\|\mathbf{v}\|_{H_0(\mathbf{grad}, \Omega)}} \geq C\|p\|_{L^2}$, equivalent to $\operatorname{Im} \operatorname{div} = L^2(\Omega)$.

Stokes in curl-curl formulation: weak form

- Recasting the Stokes equations:

$$\begin{aligned} \overbrace{\nu(\mathbf{curl\,curl}\, \mathbf{u} - \mathbf{grad\,div}\, \mathbf{u})}^{-\nu\Delta\mathbf{u}} + \mathbf{grad}\, p &= \mathbf{f} && \text{in } \Omega, && \text{(momentum conservation)} \\ \mathbf{div}\, \mathbf{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)} \\ \mathbf{curl}\, \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ and } \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)} \\ \int_{\Omega} p &= 0 \end{aligned}$$

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- Weak formulation: Find $(\mathbf{u}, p) \in \mathbf{H}(\mathbf{curl}; \Omega) \times H(\mathbf{grad}, \Omega)$ such that $\int_{\Omega} p = 0$ and

$$\begin{aligned} \nu(\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{L^2} + (\mathbf{grad} p, \mathbf{v})_{L^2} &= (\mathbf{f}, \mathbf{v})_{L^2} && \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ -(\mathbf{u}, \mathbf{grad} q)_{L^2} &= 0 && \forall q \in H(\mathbf{grad}, \Omega). \end{aligned}$$

Stokes equations in curl-curl formulation: stability

$$\begin{aligned}\nu(\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{L^2} + (\mathbf{grad} p, \mathbf{v})_{L^2} &= (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ -(\mathbf{u}, \mathbf{grad} q)_{L^2} &= 0 \quad \forall q \in H(\mathbf{grad}, \Omega)\end{aligned}$$

- Make $\mathbf{v} = \mathbf{grad} p$ to get $\|\mathbf{grad} p\|_{L^2} \leq \|\mathbf{f}\|_{L^2}$ since $\mathbf{curl grad} = 0$.
- Make $(\mathbf{v}, q) = (\mathbf{u}, p)$:

$$\nu \|\mathbf{curl} \mathbf{u}\|_{L^2}^2 \leq \|\mathbf{f}\|_{L^2} \|\mathbf{u}\|_{L^2}.$$

Stokes equations in curl-curl formulation: stability

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- Make $(\mathbf{v}, q) = (\mathbf{u}, p)$:

$$\nu \|\mathbf{curl} \mathbf{u}\|_{L^2}^2 \leq \|\mathbf{f}\|_{L^2} \|\mathbf{u}\|_{L^2}.$$

- If Ω does not have any tunnel,

$$\text{Im grad} = \text{Ker curl}.$$

The incompressibility gives $\mathbf{u} \perp \text{Im grad}$, so $\mathbf{u} \in (\text{Ker curl})^\perp$ and the

$$\text{Poincaré inequality: } \|\cdot\|_{L^2} \lesssim \|\mathbf{curl} \cdot\|_{L^2} \text{ on } (\text{Ker curl})^\perp$$

yields

$$\|\mathbf{u}\|_{L^2} \lesssim \|\mathbf{curl} \mathbf{u}\|_{L^2}.$$

Take-home message

Analysis of these models requires:

- Poincaré inequalities: $\|z\|_{L^2} \lesssim \|\bullet z\|_{L^2}$ for $z \in (\text{Ker } \bullet)^\perp$.
- Structural properties of the **de Rham complex**

$$H(\text{grad}, \Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

Take-home message

Analysis of these models requires:

- Poincaré inequalities: $\|z\|_{L^2} \lesssim \|\bullet z\|_{L^2}$ for $z \in (\text{Ker } \bullet)^\perp$.
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$$H(\text{grad}, \Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

Preserving those at the discrete level is essential to design robust schemes.

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3 Properties of the DDR complex

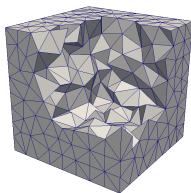
4 Example of application: Stokes equations in curl-curl formulation

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The Finite Element way

Global complex



$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

- Define **local polynomial spaces** on each element, and **glue them together** to form a sub-complex of the de Rham complex:

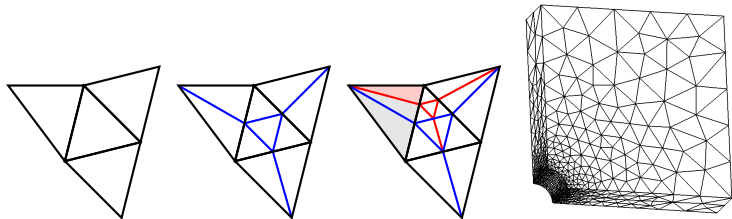
$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H(\text{grad}, \Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \end{array}$$

Example: conforming $\mathcal{P}^k$ -Nédélec-Raviart-Thomas spaces [Arnold, 2018].

- Gluing only works on special meshes...**

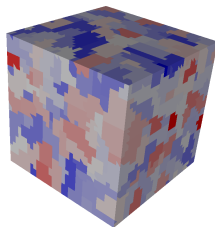
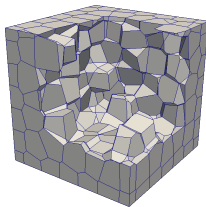
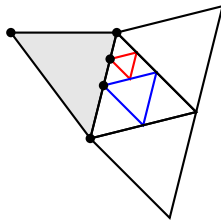
The Finite Element way

Shortcomings



- Approach limited to **conforming meshes** with **standard elements**
 - ⇒ local refinement requires to **trade mesh size for mesh quality**
 - ⇒ complex geometries may require a **large number of elements**
 - ⇒ the element shape cannot be seamlessly **adapted to the solution** (e.g. hexahedra in boundary layers + tetrahedra in the bulk for CFD simulations)
- Need for (global) basis functions
 - ⇒ significant increase of DOFs on hexahedral elements

Benefits of polytopal meshes

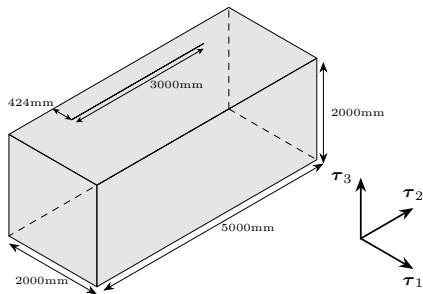


- **Local refinement** is easy, and preserves mesh regularity.
- **Agglomeration of elements** (e.g., for multigrid methods) is seamless.
- High-level approach can lead to **leaner methods** (fewer DOFs).
- Can be **combined with standard Finite Elements** on hybrid meshes (made of tetrahedra/hexahedra + polyhedral elements).

A practical example from CEA-CESTA

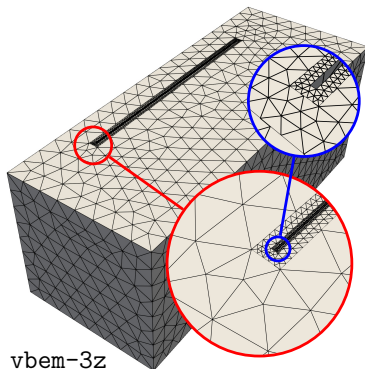
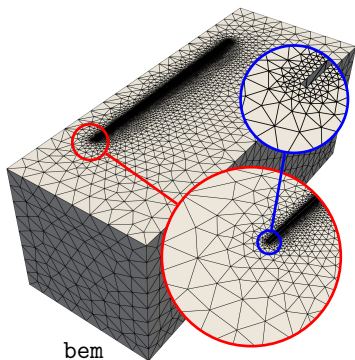
[Touzalin, 2025]

Problem: use a boundary element method to analyse the shielding effectiveness of a perfectly conductive box with a very small slit.



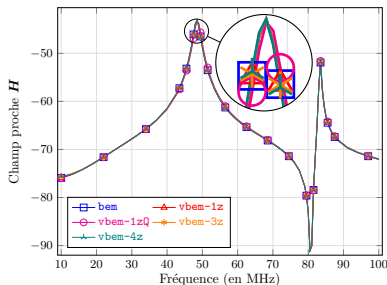
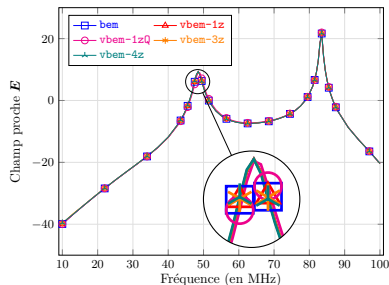
A practical example from CEA-CESTA

Meshes: conforming triangular for finite-element boundary method (bem), non-conforming triangular (polygonal) for virtual element boundary method (vbem-3z).



A practical example from CEA-CESTA

Accuracy: comparison of modulus of reflected near fields at the top.



Computational cost

<i>Method</i>	<i>Assembly</i>	<i>Resolution</i>
bem	813s	125s
vbem-3z	321s	19s

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Overview of the Discrete De Rham (DDR) complex

$$\mathbb{R} \longrightarrow \underline{X}_{\mathbf{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\mathbf{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\mathbf{div},h}^k \xrightarrow{D_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0$$

- **Fully discrete complex** (not sub-complex) of bespoke finite-dimensional spaces and operators.
- Discrete spaces **not made of functions** but:
 - $\underline{X}_{\bullet,h}^k$ made of vectors of **polynomials on vertices, edges, faces, elements**.
 - **Discrete operators** (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.

Overview of the Discrete De Rham (DDR) complex

$$\begin{array}{ccccccccccc}
 \mathbb{R} & \hookrightarrow & C^\infty(\overline{\Omega}) & \xrightarrow{\text{grad}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\text{curl}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\text{div}} & C^\infty(\overline{\Omega}) & \xrightarrow{0} & 0 \\
 & & \downarrow \underline{I}_{\text{grad},h}^k & & \downarrow \underline{I}_{\text{curl},h}^k & & \downarrow \underline{I}_{\text{div},h}^k & & \downarrow \underline{I}_{L^2,h}^k & & \\
 \mathbb{R} & \longrightarrow & \underline{X}_{\text{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\text{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\text{div},h}^k & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & 0
 \end{array}$$

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 - **Discrete operators** (differential and function reconstructions) built from these DOFs via integration-by-parts formulas.
 - **Interpolators** $\underline{I}_{\bullet,h}^k$ give meaning to these polynomials/DOFs as moments.

Guiding principles for the construction

Joint work with D. Di Pietro and F. Rapetti.

(Ref: [Di Pietro et al., 2020], [Di Pietro and Droniou, 2023].)

- **Hierarchical** construction: from vertices, to edges, to faces, to elements.
- **Enhancement**: on each (relevant) mesh entity,
 - **discrete differential operator** first,
 - **potential reconstruction** using the discrete differential operator.
(both polynomially consistent, both based on IBP formulas.)
- The definition of the **spaces (DOFs)** also guided by these IBP formulas.

Same guiding principles as the Hybrid High-Order (HHO) method
[Di Pietro et al., 2014], [Di Pietro and Droniou, 2020].

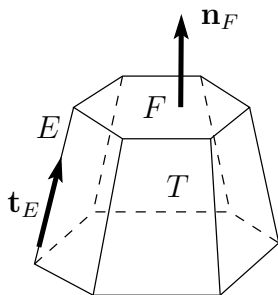
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Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements (T), faces (F), edges (E), vertices (V), with intrinsic orientations (tangent, normal).
- $\mathcal{P}^\ell(X)$ polynomial of degree $\leq \ell$ on $X = T, F, E$.



$\mathcal{P}^k$ -consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = - \int_E q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\mathbf{x}_{V_2})r(\mathbf{x}_{V_2}) - q(\mathbf{x}_{V_1})r(\mathbf{x}_{V_1}) \quad \forall r \in \mathcal{P}^k(E)$$

with derivatives in the direction $\mathbf{t}_E$.

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$$\int_E q' r = - \int_E \pi_{\mathcal{P}, E}^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\mathbf{x}_{V_2})r(\mathbf{x}_{V_2}) - q(\mathbf{x}_{V_1})r(\mathbf{x}_{V_1}) \quad \forall r \in \mathcal{P}^k(E)$$

with $\pi_{\mathcal{P}, E}^{k-1}$ the L^2 -projection on $\mathcal{P}^{k-1}(E)$.

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- Space and interpolator:

$$\underline{X}_{\text{grad}, E}^k = \left\{ \underline{q}_E = (\mathbf{q}_E, (q_V)_{V \in \mathcal{V}_E}) : q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\},$$

$$\underline{I}_{\text{grad}, E}^k q = (\pi_{\mathcal{P}, E}^{k-1} q, (q(\mathbf{x}_V))_{V \in \mathcal{V}_E}) \quad \forall q \in C(\overline{E}).$$

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- Space: $\underline{X}_{\text{grad}, E}^k = \left\{ \underline{q}_E = (\underline{q}_E, (q_V)_{V \in \mathcal{V}_E}) : q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\}$.
- Edge gradient $G_E^k \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = - \int_E q_E r' + q_{V_2} r(\mathbf{x}_{V_2}) - q_{V_1} r(\mathbf{x}_{V_1}) \quad \forall r \in \mathcal{P}^k(E).$$

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- Potential reconstruction $\gamma_E^{k+1} \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma_E^{k+1} \underline{q}_E) z' = - \int_E (G_E^k \underline{q}_E) z + q_{V_2} z(\mathbf{x}_{V_2}) - q_{V_1} z(\mathbf{x}_{V_1}) \quad \forall z \in \mathcal{P}^{k+2}(E).$$

$\mathcal{P}^k$ -consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ then

$$\int_E q' r = - \int_E \pi_{\mathcal{P}, E}^{k-1} q r' + q(\mathbf{x}_{V_2}) r(\mathbf{x}_{V_2}) - q(\mathbf{x}_{V_1}) r(\mathbf{x}_{V_1}) \quad \forall r \in \mathcal{P}^k(E)$$

- Space: $\underline{X}_{\text{grad}, E}^k = \left\{ \underline{q}_E = (\textcolor{red}{q}_E, (q_V)_{V \in \mathcal{V}_E}) : q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\}$.
- Edge gradient** $G_E^k \underline{q}_E \in \mathcal{P}^k(E)$ s.t.

$$\int_E (G_E^k \underline{q}_E) r = - \int_E q_E r' + q_{V_2} r(\mathbf{x}_{V_2}) - q_{V_1} r(\mathbf{x}_{V_1}) \quad \forall r \in \mathcal{P}^k(E).$$

- Potential reconstruction** $\gamma_E^{k+1} \underline{q}_E \in \mathcal{P}^{k+1}(E)$ s.t.

$$\int_E (\gamma_E^{k+1} \underline{q}_E) z' = - \int_E (\textcolor{blue}{G}_E^k \underline{q}_E) z + q_{V_2} z(\mathbf{x}_{V_2}) - q_{V_1} z(\mathbf{x}_{V_1}) \quad \forall z \in \mathcal{P}^{k+2}(E).$$

Polynomially consistent: $G_E^k \underline{I}_{\text{grad}, E}^k q = q'$ and $\gamma_E^{k+1} \underline{I}_{\text{grad}, E}^k q = q$ for all $q \in \mathcal{P}^{k+1}(E)$.

$\mathcal{P}^k$ -consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F q) \cdot \mathbf{v} = - \int_F q \underbrace{\operatorname{div}_F \mathbf{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$ -consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F q) \cdot \mathbf{v} = - \int_F \pi_{\mathcal{P}, F}^{k-1} q \underbrace{\operatorname{div}_F \mathbf{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2$$

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- Space and interpolator:

$$\underline{X}_{\mathbf{grad}, F}^k = \left\{ \underline{q}_F = (q_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) : \right.$$

$$\left. q_F \in \mathcal{P}^{k-1}(F), q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\},$$

$$\underline{I}_{\mathbf{grad}, F}^k q = (\pi_{\mathcal{P}, F}^{k-1} q, (\pi_{\mathcal{P}, E}^{k-1} q|_E)_{E \in \mathcal{E}_F}, (q(\mathbf{x}_V))_{V \in \mathcal{V}_F}) \quad \forall q \in C(\overline{F}).$$

$\mathcal{P}^k$ -consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$,

$$\int_F (\mathbf{grad}_F q) \cdot \mathbf{v} = - \int_F \pi_{\mathcal{P}, F}^{k-1} q \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2.$$

- Space: $\underline{X}_{\mathbf{grad}, F}^k = \left\{ \underline{q}_F = (\mathbf{q}_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) : \right.$

$$\left. q_F \in \mathcal{P}^{k-1}(F), q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\},$$

- Face gradient $\mathbf{G}_F^k : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{v} = - \int_F q_F \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma_E^{k+1} \underline{q}_E) \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2.$$

$\mathcal{P}^k$ -consistent gradient

Face F

- Space: $\underline{X}_{\text{grad},F}^k = \left\{ \underline{q}_F = (\mathbf{q}_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) : \right.$

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$$\int_F (\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{v} = - \int_F q_F \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma_E^{k+1} \underline{q}_E) \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2.$$

- Potential reconstruction $\gamma_F^{k+1} \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma_F^{k+1} \underline{q}_F) \operatorname{div}_F \mathbf{z} = - \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{z} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\gamma_E^{k+1} \underline{q}_E) \mathbf{z} \cdot \mathbf{n}_{FE}$$
$$\forall \mathbf{z} \in \mathcal{R}^{c,k+2}(F) := (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{k+1}(F).$$

$\mathcal{P}^k$ -consistent gradient

Face F

- Space: $\underline{X}_{\text{grad},F}^k = \left\{ \underline{q}_F = (\underline{q}_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) : \right.$

$$\left. q_F \in \mathcal{P}^{k-1}(F), q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\},$$

- Face gradient $\mathbf{G}_F^k : \underline{q}_F \in \mathcal{P}^k(F)^2$ s.t.

$$\int_F (\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{v} = - \int_F q_F \operatorname{div}_F \mathbf{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma_E^{k+1} \underline{q}_E) \mathbf{v} \cdot \mathbf{n}_{FE} \quad \forall \mathbf{v} \in \mathcal{P}^k(F)^2.$$

- Potential reconstruction $\gamma_F^{k+1} \underline{q}_F \in \mathcal{P}^{k+1}(F)$ s.t.

$$\int_F (\gamma_F^{k+1} \underline{q}_F) \operatorname{div}_F \mathbf{z} = - \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{z} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\gamma_E^{k+1} \underline{q}_E) \mathbf{z} \cdot \mathbf{n}_{FE}$$
$$\forall \mathbf{z} \in \mathcal{R}^{c,k+2}(F) := (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{k+1}(F).$$

Polynomially consistent: $\mathbf{G}_F^k \underline{I}_{\text{grad},E}^k q = \mathbf{grad} q$ and $\gamma_F^{k+1} \underline{I}_{\text{grad},E}^k q = q$
for all $q \in \mathcal{P}^{k+1}(F)$.

Same principle! Based on IBP we determine:

- An additional unknown ($q_T \in \mathcal{P}^{k-1}(T)$) to get the space $\underline{X}_{\text{grad},T}^k$, and its meaning (polynomial moment on T) to get the interpolator $\underline{I}_{\text{grad},T}^k$.
- A formula for the element gradient $\mathbf{G}_T^k : \underline{X}_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)^3$.
- A potential reconstruction $P_{\text{grad},T}^{k+1} : \underline{X}_{\text{grad},T}^k \rightarrow \mathcal{P}^{k+1}(T)$.

$\underline{X}_{\text{grad},h}^k$, the discrete $H(\text{grad}, \Omega)$ space

- Discrete $H(\text{grad}, \Omega)$ space:

$$\underline{X}_{\text{grad},h}^k := \left\{ \underline{q}_h = ((q_T)_{T \in \mathcal{T}_h}, (q_F)_{F \in \mathcal{F}_h}, (q_E)_{E \in \mathcal{E}_h}, (q_V)_{V \in \mathcal{V}_h}) : \right. \\ \left. q_T \in \mathcal{P}^{k-1}(T), q_F \in \mathcal{P}^{k-1}(F), q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\}.$$

- Interpolator: $\underline{I}_{\text{grad},h}^k q = \underline{q}_h$ with

$$q_V = q(\mathbf{x}_V),$$

$$q_X = L^2\text{-projection on } \mathcal{P}^{k-1}(X) \text{ of } q, \text{ for } X \in \{E, F, T\}.$$

- Operator reconstructions:**

- ★ edge gradient $\underline{G}_E^k \underline{q}_E \in \mathcal{P}^k(E) \rightsquigarrow$ edge trace $\gamma_E^{k+1} \underline{q}_E \in \mathcal{P}^{k+1}(E),$
- ★ face gradient $\underline{G}_F^k \underline{q}_F \in \mathcal{P}^k(F)^2 \rightsquigarrow$ face trace $\gamma_F^{k+1} \underline{q}_F \in \mathcal{P}^{k+1}(F),$
- ★ element gradient $\underline{G}_T^k \underline{q}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow$ el. potential $\underline{P}_{\text{grad},T}^{k+1} \underline{q}_T \in \mathcal{P}^{k+1}(T),$

$\underline{X}_{\text{grad},h}^k$, the discrete $H(\text{grad}, \Omega)$ space

- Discrete $H(\text{grad}, \Omega)$ space:

$$\underline{X}_{\text{grad},h}^k := \left\{ \underline{q}_h = ((q_T)_{T \in \mathcal{T}_h}, (q_F)_{F \in \mathcal{F}_h}, (q_E)_{E \in \mathcal{E}_h}, (q_V)_{V \in \mathcal{V}_h}) : \right. \\ \left. q_T \in \mathcal{P}^{k-1}(T), q_F \in \mathcal{P}^{k-1}(F), q_E \in \mathcal{P}^{k-1}(E), q_V \in \mathbb{R} \right\}.$$

- Interpolator: $\underline{I}_{\text{grad},h}^k q = \underline{q}_h$ with

$$q_V = q(\mathbf{x}_V),$$

$$q_X = L^2\text{-projection on } \mathcal{P}^{k-1}(X) \text{ of } q, \text{ for } X \in \{E, F, T\}.$$

- Operator reconstructions:**

$$\star \text{ edge gradient } \mathbf{G}_E^k \underline{q}_E \in \mathcal{P}^k(E) \rightsquigarrow \text{edge trace } \gamma_E^{k+1} \underline{q}_E \in \mathcal{P}^{k+1}(E),$$

$$\star \text{ face gradient } \mathbf{G}_F^k \underline{q}_F \in \mathcal{P}^k(F)^2 \rightsquigarrow \text{face trace } \gamma_F^{k+1} \underline{q}_F \in \mathcal{P}^{k+1}(F),$$

$$\star \text{ element gradient } \mathbf{G}_T^k \underline{q}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow \text{el. potential } \mathbf{P}_{\text{grad},T}^{k+1} \underline{q}_T \in \mathcal{P}^{k+1}(T),$$

We want:

$$\mathbb{R} \xrightarrow{\underline{I}_{\text{grad},h}^k} \underline{X}_{\text{grad},h}^k \xrightarrow{\underline{\mathbf{G}}_h^k} \underline{\mathbf{X}}_{\text{curl},h}^k \xrightarrow{\underline{\mathbf{C}}_h^k} \underline{\mathbf{X}}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

$\underline{X}_{\text{curl},h}^k$, the discrete $H(\text{curl}; \Omega)$ space

- Discrete $H(\text{curl}; \Omega)$ space:

$$\underline{X}_{\text{curl},h}^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}, (v_E)_{E \in \mathcal{E}_h}) : \right. \\ \left. v_T \in \mathcal{RT}^k(T), v_F \in \mathcal{RT}^k(F), v_E \in \mathcal{P}^k(E) \right\},$$

with $\mathcal{RT}^k(X) = \mathcal{P}^{k-1}(X) \oplus (\mathbf{x} - \mathbf{x}_X) \mathcal{P}^{k-1}(X)$.

- Interpolator: $\underline{I}_{\text{curl},h}^k \mathbf{v} = \underline{v}_h$ with

$$v_E = L^2\text{-projection on } \mathcal{P}^k(E) \text{ of } \mathbf{v} \cdot \mathbf{t}_E,$$

$$v_F = L^2\text{-projection on } \mathcal{RT}^k(F) \text{ of } v_{\mathbf{t},F} \text{ (tangential projection),}$$

$$v_T = L^2\text{-projection on } \mathcal{RT}^k(T) \text{ of } v.$$

- Operator reconstructions:

$$\star \text{ face (scalar) curl } C_F^k \underline{v}_F \in \mathcal{P}^k(F) \rightsquigarrow \text{ tangent trace } \gamma_{\mathbf{t},F}^k \underline{v}_F \in \mathcal{P}^k(F)^2,$$

$$\star \text{ element curl } C_T^k \underline{v}_T \in \mathcal{P}^k(T)^3 \rightsquigarrow \text{ element potential } P_{\text{curl},T}^k \underline{v}_T \in \mathcal{P}^k(T)^3.$$

Discrete gradient

- We want $\underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k$.
- We have: $\underline{G}_E^k \underline{q}_E \in \mathcal{P}^k(E)$, $\underline{G}_F^k \underline{q}_F \in \mathcal{P}^k(F)^2$, $\underline{G}_T^k \underline{q}_T \in \mathcal{P}^k(T)^3$ and

$$\underline{X}_{\text{curl},h}^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}, (v_E)_{E \in \mathcal{E}_h}) : \right. \\ \left. v_T \in \mathcal{RT}^k(T), v_F \in \mathcal{RT}^k(F), v_E \in \mathcal{P}^k(E) \right\}.$$

$\rightsquigarrow$ Define $\underline{G}_h^k : \underline{X}_{\text{grad},h}^k \rightarrow \underline{X}_{\text{curl},h}^k$ by projecting edge, face and element reconstructed gradients onto the corresponding components:

$$\underline{G}_h^k \underline{q}_h = ((\pi_{\mathcal{RT},T}^k \underline{G}_T^k \underline{q}_h)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{RT},F}^k \underline{G}_F^k \underline{q}_h)_{F \in \mathcal{F}_h}, (G_E^k \underline{q}_h)_{E \in \mathcal{E}_h}).$$

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 - Polytopal meshes: what? why?
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DDR complex: summary

$$\mathbb{R} \xrightarrow{I_{\text{grad},h}^k} \underline{X}_{\text{grad},h}^k \xrightarrow{G_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{C_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

- No basis functions, **Fully discrete spaces** not made of functions, but of vectors of polynomials (DOFs) attached to **geometric entities** (emulates continuity properties of each space).

Space	V	E	F	T
$\underline{X}_{\text{grad},T}^k$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}_{\text{curl},T}^k$		$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{X}_{\text{div},T}^k$			$\mathcal{P}^k(F)$	$\mathcal{N}^k(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

- Polynomial reconstructions** of differential operator and potential by mimicking IBPs.

L^2 -like inner products

- Local L^2 -like inner product on the DDR spaces:

for $\bullet \in \{\text{grad}, \text{curl}, \text{div}\}$ and $k_{\text{grad}} = k + 1$, $k_{\text{curl}} = k_{\text{div}} = k$,

$$(x_T, y_T)_{\bullet, T} = \int_T P_{\bullet, T}^{k_{\bullet}} x_T \cdot P_{\bullet, T}^{k_{\bullet}} y_T + s_{\bullet, T}(x_T, y_T) \quad \forall x_T, y_T \in \underline{X}_{\bullet}^k(T),$$

($s_{\bullet, T}$ penalises differences on the boundary between element and face/edge potentials).

- Global L^2 -like product $(\cdot, \cdot)_{\bullet, h}$ by standard assembly of local ones.

Another approach: Virtual Elements

- Construct a sub-complex of **finite-dimensional but not piecewise polynomial** spaces:

$$\begin{array}{ccccccccc} \mathbb{R} & \longrightarrow & V_{k+1}^n & \xrightarrow{\text{grad}} & V_k^e & \xrightarrow{\text{curl}} & V_k^f & \xrightarrow{\text{div}} & V_k^v & \xrightarrow{0} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \xrightarrow{0} & 0 \end{array}$$

- Functions in each $V_h^\bullet$ are not **explicitly described**, but:
 - $V_h^\bullet$ have DOFs that are **polynomials moments on the vertices, edges, faces or elements**.
 - Some **polynomial projections** of the functions/their derivatives can be computed from these DOFs.

Ref: [Beirão da Veiga et al., 2017], [Beirão da Veiga et al., 2018].

Virtual Elements: the big secret

VEM can be “fully-discretised” (which makes it closer to actual implementations). [Beirão da Veiga et al., 2022]

$$\begin{array}{ccccccccccc}
 \mathbb{R} & \longrightarrow & \underline{V}_{k+1}^n & \xrightarrow{\underline{\text{grad}}_h} & \underline{V}_k^e & \xrightarrow{\underline{\text{curl}}_h} & \underline{V}_k^f & \xrightarrow{\underline{\text{div}}_h} & \underline{V}_k^v & \xrightarrow{0} & 0 \\
 & & \uparrow \underline{I}_h^n & & \uparrow \underline{I}_h^e & & \uparrow \underline{I}_h^f & & \uparrow \underline{I}_h^v & & \\
 \mathbb{R} & \longrightarrow & V_{k+1}^n & \xrightarrow{\text{grad}} & V_k^e & \xrightarrow{\text{curl}} & V_k^f & \xrightarrow{\text{div}} & V_k^v & \xrightarrow{0} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \xrightarrow{0} & 0
 \end{array}$$

Note: DDR can also be “virtualised”.

Outline

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 - Complexes: what? why?
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The DDR complex and its properties

$$\mathbb{R} \xrightarrow{I_{\text{grad},h}^k} \underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} 0.$$

- Complex:

$$\underline{C}_h^k \circ \underline{G}_h^k = 0, \quad D_h^k \circ \underline{C}_h^k = 0.$$

- Same cohomology as the continuous de Rham complex; in particular, if Ω has a trivial topology,

$$\text{Im } \underline{G}_h^k = \ker \underline{C}_h^k, \quad \text{Im } \underline{C}_h^k = \ker D_h^k.$$

- Applicable on generic polytopal meshes.

[Di Pietro et al., 2020], [Di Pietro et al., 2023]

The DDR complex and its properties

- Poincaré inequalities.
- Consistency (both primal and adjoint).
- Commutation properties between the interpolators and the continuous/discrete operators.

[Di Pietro and Droniou, 2021a], [Di Pietro and Hanot, 2024],
[Di Pietro et al., 2025a]

↪ optimally-convergent schemes (error in $\mathcal{O}(h^{k+1})$) for a range of models: magnetostatics, Stokes & Navier–Stokes, plates problems, Yang–Mills geometric equations, etc.

↪ robust error estimates with respect to some physical parameters.

[Di Pietro and Droniou, 2021b], [Beirão da Veiga et al., 2022]

Primal and adjoint consistencies

Primal consistency: reconstruction composed with interpolator have **optimal approximation properties**, e.g.:

$$\begin{aligned} \|P_{\mathbf{grad},T}^k \underline{I}_{\mathbf{grad},T}^k q - q\|_{L^2(T)} &\lesssim h_T^{k+2} |q|_{H^{k+2}(T)}, \\ \|(\mathbf{P}_{\mathbf{curl},T}^k \underline{\mathbf{G}}_h^k) \underline{I}_{\mathbf{grad},T}^k q - \mathbf{grad} q\|_{L^2(T)} &\lesssim h_T^{k+1} |\mathbf{grad} q|_{H^{(k+1,2)}(T)}. \end{aligned}$$

where (to make sense of the interpolation)

$$|z|_{H^{(k+1,2)}(T)} = \begin{cases} |z|_{H^1(T)} + h_T |z|_{H^2(T)} & \text{if } k = 0, \\ |z|_{H^{k+1}(T)} & \text{if } k \geq 1. \end{cases}$$

Adjoint consistency: measure defect in **discrete integration-by-parts formulae**, e.g., if $\mathbf{v} \cdot \mathbf{n}_\Omega = 0$,

$$\begin{aligned} \left| (\underline{\mathbf{I}}_{\mathbf{curl},h}^k \mathbf{v}, \underline{\mathbf{G}}_h^k \underline{q}_h)_{\mathbf{curl},h} + \int_\Omega \operatorname{div} \mathbf{v} P_{\mathbf{grad},h}^k \underline{q}_h \right| \\ \lesssim h^{k+1} |\mathbf{v}|_{H^{(k+1,2)}(\mathcal{T}_h)} \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h}. \end{aligned}$$

Note: proof uses a **conforming lifting**. [Di Pietro and Droniou, 2023],
[Di Pietro et al., 2025b]

Further properties: commuting quasi-interpolators

Standard interpolators: require additional smoothness of the functions (for traces).

$$\begin{array}{ccccccc}
 C^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{\Omega})^3 & \xrightarrow{\mathbf{div}} & C^\infty(\overline{\Omega}) \\
 \downarrow \underline{I}_{\mathbf{grad},h}^k & & \downarrow \underline{I}_{\mathbf{curl},h}^k & & \downarrow \underline{I}_{\mathbf{div},h}^k & & \downarrow \pi_{\mathcal{P},h}^k \\
 \underline{X}_{\mathbf{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\mathbf{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\mathbf{div},h}^k & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h)
 \end{array}$$

Further properties: commuting quasi-interpolators

Standard interpolators: require additional smoothness of the functions (for traces).

$$\begin{array}{ccccccc} H^2(\Omega) & & H^2(\Omega)^3 & & H^1(\Omega)^3 & & L^2(\Omega) \\ \downarrow \underline{I}_{\text{grad},h}^k & & \downarrow \underline{I}_{\text{curl},h}^k & & \downarrow \underline{I}_{\text{div},h}^k & & \downarrow \pi_{\mathcal{P},h}^k \\ \underline{X}_{\text{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\text{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\text{div},h}^k & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h) \end{array}$$

Further properties: commuting quasi-interpolators

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$$\begin{array}{ccccccc}
 H^2(\Omega) & & H^2(\Omega)^3 & & H^1(\Omega)^3 & & L^2(\Omega) \\
 \downarrow \underline{I}_{\text{grad},h}^k & & \downarrow \underline{I}_{\text{curl},h}^k & & \downarrow \underline{I}_{\text{div},h}^k & & \downarrow \pi_{\mathcal{P},h}^k \\
 \underline{X}_{\text{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\text{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\text{div},h}^k & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h)
 \end{array}$$

Quasi-interpolators: applicable on minimal-regularity spaces
 ([Chaumont-Frelet et al., 2025], based on [Chaumont-Frelet and Vohralík, 2024]).

$$\begin{array}{ccccccc}
 H(\text{grad}, \Omega) & \xrightarrow{\text{grad}} & H(\text{curl}; \Omega) & \xrightarrow{\text{curl}} & H(\text{div}; \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
 \downarrow \hat{\underline{I}}_{\text{grad},h}^k & & \downarrow \hat{\underline{I}}_{\text{curl},h}^k & & \downarrow \hat{\underline{I}}_{\text{div},h}^k & & \downarrow I_{L^2,h}^k \\
 \underline{X}_{\text{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\text{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\text{div},h}^k & \xrightarrow{D_h} & \mathcal{P}^k(\mathcal{T}_h)
 \end{array}$$

Usefulness: minimal-regularity convergence/error estimates, and proof of the **Maxwell compactness**.

Further properties: Maxwell compactness

Rellich compactness: relatively easy to prove.

Theorem

If $(q_n)_n$ is bounded in $H(\mathbf{grad}, \Omega)$ then $(q_n)_n$ is relatively compact in $L^2(\Omega)$.

Transfer to the discrete setting:

- Trivial for conforming finite elements.
- Now well established for non-conforming methods
[Eymard et al., 2000], [Di Pietro and Ern, 2010], [Li et al., 2015],
[Droniou et al., 2018].

Further properties: Maxwell compactness

Maxwell compactness: much more challenging to prove.

Theorem

If $(\mathbf{v}_n)_n$ is bounded in $\mathbf{H}(\text{curl}; \Omega)$ and

$$\int_{\Omega} \mathbf{v}_n \cdot \mathbf{grad} z = 0 \quad \forall z \in H(\mathbf{grad}, \Omega),$$

then $(\mathbf{v}_n)_n$ is relatively compact in $L^2(\Omega)^3$.

Transfer to the discrete setting:

- For Nédélec elements: [Kikuchi, 1987], [Weber, 1980], [Jochmann, 1997].
- For a polytopal method, not in a discrete complex: [Lemaire and Pitassi, 2025].
- For a **polytopal complex**: very recent first result [Chaumont-Frelet et al., 2025].

Usefulness: convergence for nonlinear problems or problems with rough coefficients, eigenvalue problems.

Outline

- 1 Polytopal complexes
 - Complexes: what? why?
 - Polytopal meshes: what? why?
- 2 The Discrete De Rham method
 - DDR: generic principles
 - Discrete $H(\mathbf{grad}, \Omega)$ space and gradient/potential reconstructions
 - The DDR complex
- 3 Properties of the DDR complex
- 4 Example of application: Stokes equations in curl-curl formulation

Slides



DDR scheme for Stokes in curl-curl formulation

- Weak formulation: Find $(\mathbf{u}, p) \in \mathbf{H}(\mathbf{curl}; \Omega) \times H_{\star}^1(\Omega)$ s.t.

$$\begin{aligned} \nu(\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{L^2} + (\mathbf{grad} p, \mathbf{v})_{L^2} &= (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ -(\mathbf{u}, \mathbf{grad} q)_{L^2} &= 0 \quad \forall q \in H_{\star}^1(\Omega). \end{aligned}$$

- DDR scheme: Find $(\underline{\mathbf{u}}_h, \underline{p}_h) \in \underline{\mathbf{X}}_{\mathbf{curl},h}^k \times \underline{X}_{\mathbf{grad},h,\star}^k$ such that

$$\begin{aligned} \nu(\underline{\mathbf{C}}_h^k \underline{\mathbf{u}}_h, \underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h)_{\mathbf{div},h} + (\underline{\mathbf{G}}_h^k \underline{p}_h, \underline{\mathbf{v}}_h)_{\mathbf{curl},h} &= (\underline{\mathbf{I}}_{\mathbf{curl},h}^k \mathbf{f}, \underline{\mathbf{v}}_h)_{\mathbf{curl},h} \quad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\mathbf{curl},h}^k, \\ -(\underline{\mathbf{u}}_h, \underline{\mathbf{G}}_h^k \underline{q}_h)_{\mathbf{curl},h} &= 0 \quad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h,\star}^k \end{aligned}$$

where $\underline{X}_{\mathbf{grad},h,\star}^k := \{\underline{r}_h \in \underline{X}_{\mathbf{grad},h}^k : (\underline{I}_{\mathbf{grad},h}^k 1, \underline{r}_h)_{\mathbf{grad},h} = 0\}$.

Convergence result

Theorem (Error estimates [Beirão da Veiga et al., 2022])

We have

$$\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{u}\|_{\text{curl},h} + \|\underline{\mathbf{C}}_h^k(\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{u})\|_{\text{div},h} \lesssim C(\mathbf{u})h^{k+1}$$

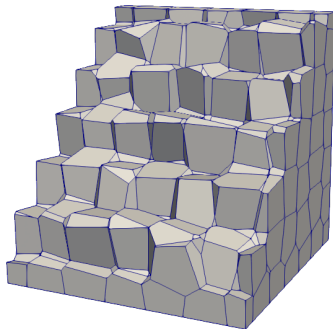
and

$$\|\underline{\mathbf{G}}_h^k(\underline{p}_h - \underline{I}_{\text{grad},h}^k p)\|_{\text{curl},h} \lesssim C(\mathbf{u})h^{k+1}.$$

- **Robust** estimate with respect to the pressure (RHS does not depend on p).
- Based on **discrete Poincaré inequalities** and **primal/adjoint consistencies**.

Numerical results

- $\Omega = (0, 1)^3$.
- Voronoi mesh families (similar results on tetrahedral meshes):



Numerical results

- Exact solution: for some $\lambda \geq 0$,

$$p(x, y, z) = \lambda \sin(2\pi x) \sin(2\pi y) \sin(2\pi z),$$
$$\mathbf{u}(x, y, z) = \begin{bmatrix} \frac{1}{2} \sin(2\pi x) \cos(2\pi y) \cos(2\pi z) \\ \frac{1}{2} \cos(2\pi x) \sin(2\pi y) \cos(2\pi z) \\ -\cos(2\pi x) \cos(2\pi y) \sin(2\pi z) \end{bmatrix}.$$

- Measured errors:
 - E_u^d and E_p^d in discrete norms between the approximate solutions and the *interpolates* of the exact solution (as in the theorems).
E.g: $E_u^d = \|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{u}\|_{\text{curl},1,h}.$
 - E_u^c and E_p^c in continuous norms between reconstructed potentials/projections of the approximate solutions and the exact solution.

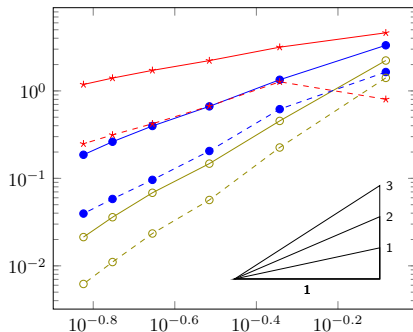
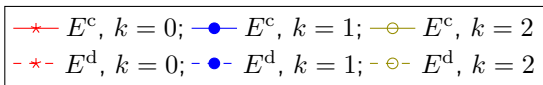
$$E.g: E_p^c = \|P_{\text{grad},h}^k p_h - p\|_{H^1(\Omega)}.$$

- DDR (and VEM) implemented in the HArDCore3D library¹.

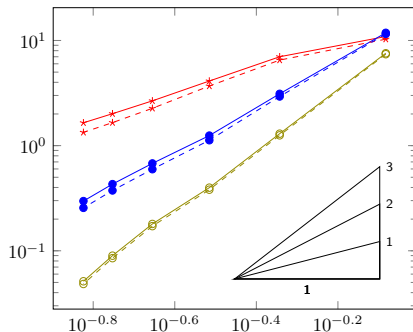
*The HArDCore3D library also includes a **serendipity version for DDR**, which leads to a reduction of more than 50% of the solving time.*

¹<https://github.com/jdroniou/HArDCore>

Results: $\lambda = 1$



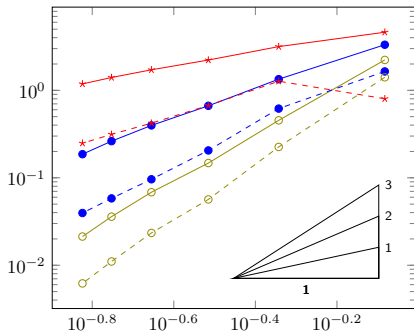
(a) Errors on u



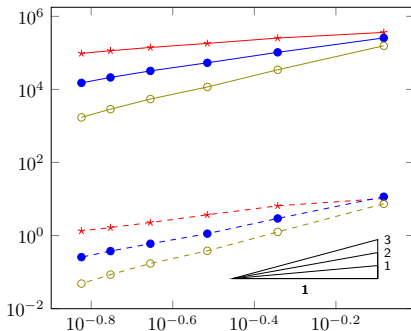
(b) Errors on p

Results; $\lambda = 10^5$

$-\star- E^c, k=0$; $-\bullet- E^c, k=1$; $-\circ- E^c, k=2$
 $- \star - E^d, k=0$; $- \bullet - E^d, k=1$; $- \circ - E^d, k=2$



(a) Errors on u



(b) Errors on p

Navier–Stokes flow in cavity with mixed BCs

In the unit cube $\Omega = (0, 1)^3$:

- **Essential BCs** (pressure and tangential velocity):

$$p(x, y, z) = -z \quad \text{and} \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}$$

on the bottom corner $\{0\} \times (0, 0.25) \times (0, 0.25)$ of the face $x = 0$.

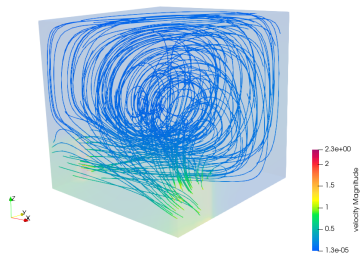
- **Natural BCs** (tangential vorticity and flux):

$$\mathbf{curl} \, \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 1$$

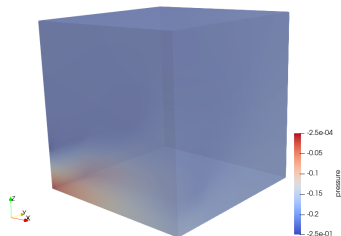
on the bottom corner $\{1\} \times (0, 0.25) \times (0, 0.25)$ of the face $x = 1$,

- Homogeneous natural BCs elsewhere.

Navier–Stokes flow in cavity with mixed BCs



(a) Velocity



(b) Pressure

Figure: Velocity streamlines and pressure

Conclusions

- **Polytopal complexes** essential to design stable schemes for certain PDE models on generic meshes (flexibility for local refinement etc.).
- Construction is **hierarchical** on mesh entities (vertices $\rightarrow$ edges $\rightarrow$ faces $\rightarrow$ cells).
- DDR: based on suitably chosen degrees of freedom and reconstruction operators inspired by **integration-by-parts formula**.
- **Range of results** for analysis of schemes: Poincaré inequalities, primal/adjoint consistency, commutation properties, compactness, etc.
- **Fully discrete approach** already extended to other complexes (e.g., div-div complex).

~> presentation by [Arax Leroy](#)
(today 13:00 in the Numerical Analysis session).

<https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html>



COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**

- Short description (in french)
- Summary of notations and formulas
- Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
- Part 1, second course: Low order case, 29/09/22 (video)
- Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
- Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
- Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)



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New generation
methods for numerical
simulations

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Thank you for your attention!

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