A polytopal exterior calculus framework

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References for this presentation

- Finite Element Exterior Calculus [Arnold, Falk, Winther, 2006, Arnold, 2018]
- Finite Element Systems [Christiansen and Gillette, 2016]
- Virtual element complexes [Beirão da Veiga et al., 2016, Beirão da Veiga et al., 2018]
- Discrete de Rham complexes [Di Pietro, Droniou, Rapetti, 2020], [Di Pietro and Droniou, 2023]
- Bridges VEM–DDR [Beirão da Veiga, Dassi, DP, Droniou, 2022]
- Polytopal Exterior Calculus [Bonaldi, Di Pietro, Droniou, Hu, 2023]
- C++ open-source implementation available in HArDCore3D

Outline

1 Motivation

2 The Discrete de Rham construction

3 The Virtual Element construction

The de Rham complex – scalar/vector-valued functions

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- Complex: image of an operator included in kernel of the next one.

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- Complex: image of an operator included in kernel of the next one.
- Key properties, depending on the topology of Ω and providing stability of PDE models:

$$\text{no "tunnels" } (b_1 = 0) \implies \mathrm{Im}\,\mathbf{grad} = \mathrm{Ker}\,\mathbf{curl} \quad \text{(Stokes in curl-curl)}$$

$$\text{no "voids" } (b_2 = 0) \implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics, Stokes)}$$

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$$\text{no “voids” } (b_2 = 0) \implies \mathrm{Im}\,\mathbf{curl} = \mathrm{Ker}\,\mathrm{div} \quad \text{(magnetostatics)}$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega) \quad \text{(magnetostatics, Stokes)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathrm{Ker}\,\mathbf{curl}/\mathrm{Im}\,\mathbf{grad} \quad \text{and} \quad \mathrm{Ker}\,\mathrm{div}/\mathrm{Im}\,\mathbf{curl}$$

- **Emulating these properties is key for stable discretizations**

The de Rham complex – differential forms

- Differential forms:
  - k-forms: mappings ω on Ω s.t. ω_x is a k-alternate linear form on $\mathbb{R}^n$.
  - d^k: exterior derivative of k-forms.
  - $H\Lambda^k(\Omega)$: k-forms $\omega \in L^2$ s.t. $\mathrm{d}^k\omega \in L^2$.
- The de Rham complex with differential forms:

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

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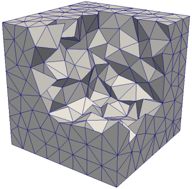
- For $n = 3$, the following links are established through vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

The Finite Element way

Global complex



$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh.

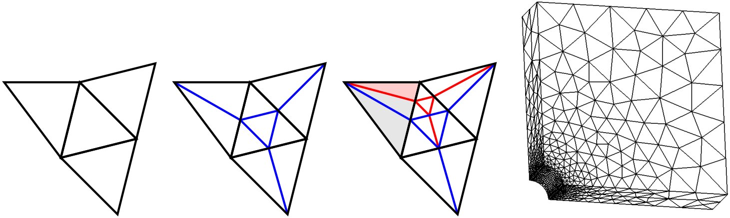
- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccc} V_h^0 & \xrightarrow{\mathrm{d}^0} & V_h^1 & \xrightarrow{\mathrm{d}^1} & V_h^2 & \xrightarrow{\mathrm{d}^2} & V_h^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) \end{array}$$

- Example: conforming $\mathcal{P}_k$–Nédélec–Raviart-Thomas spaces [Arnold, 2018].
- **Gluing only works on special meshes!**

The Finite Element way

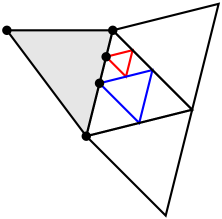
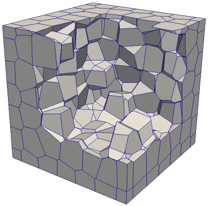
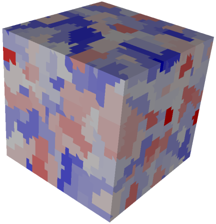
Shortcomings



- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

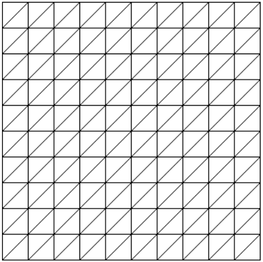
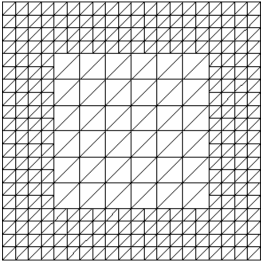
Polytopal meshes I

- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

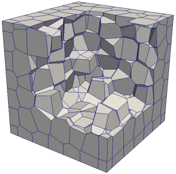
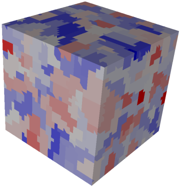
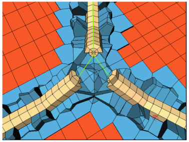
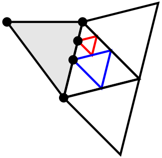
Polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathcal{P}_2$-$(\mathcal{P}_1 + \mathcal{B}^3)$ scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

Outline

Domain and polytopal mesh

- Assume $\Omega \subset \mathbb{R}^n$ polytopal (polygon if $n = 2$, polyhedron if $n = 3, \ldots$)
- We consider a polytopal mesh $\mathcal{M}_h$ with flat d-cells, $0 \leq d \leq n$
- d-cells in $\mathcal{M}_h$ are collected in $\Delta_d(\mathcal{M}_h)$.

  When $n = 3$:

  - $\Delta_0(\mathcal{M}_h) = \mathcal{V}_h$: set of vertices
  - $\Delta_1(\mathcal{M}_h) = \mathcal{E}_h$: set of edges
  - $\Delta_2(\mathcal{M}_h) = \mathcal{F}_h$: set of faces
  - $\Delta_3(\mathcal{M}_h) = \mathcal{T}_h$: set of elements

General ideas

- Discrete spaces with polynomial components attached to mesh entities, representing projections of traces of k-forms.
- Recursive and hierarchical construction on d-cells, $d = k+1, \ldots, n$, with enhancement strategy:
  - Discrete exterior derivative
$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f)$$
  - Discrete potential (playing the role of a k-form inside f), using the discrete exterior derivative
$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k}(f)$$
- Reconstructions mimic the Stokes formula: $\forall(\omega,\mu) \in \Lambda^\ell(f) \times \Lambda^{n-\ell-1}(f)$,

$$\int_f \mathrm{d}^\ell \omega \wedge \mu = (-1)^{\ell+1} \int_f \omega \wedge \mathrm{d}^{n-\ell-1}\mu + \int_{\partial f} \mathrm{tr}_{\partial f}\,\omega \wedge \mathrm{tr}_{\partial f}\,\mu$$

Trimmed polynomial spaces

- Let $f \in \Delta_d(\mathcal{M}_h)$, fix $\boldsymbol{x}_f \in f$, and define the Koszul complement

$$\mathcal{K}_r^\ell(f) \coloneqq \kappa \mathcal{P}_{r-1}\Lambda^{\ell+1}(f) \quad \text{with} \quad (\kappa\omega)_{\boldsymbol{x}}(\cdot, \ldots) \coloneqq \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \ldots)$$

- For $\ell \geq 1$ we define the trimmed polynomial spaces

$$\begin{aligned} \mathcal{P}_r^-\Lambda^0(f) &\coloneqq \mathcal{P}_r\Lambda^0(f), \\ \mathcal{P}_r^-\Lambda^\ell(f) &\coloneqq \mathrm{d}\mathcal{P}_r\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \text{if } \ell \geq 1 \end{aligned}$$

Remarks:

- ◊ *Full space* $\mathcal{P}_r\Lambda^\ell(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \quad \big(= \mathrm{d}\mathcal{K}_{r+1}^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f)\big)$.
- ◊ $\mathcal{P}_r^-\Lambda^0(f) = \mathcal{P}_r\Lambda^0(f), \quad \mathcal{P}_r^-\Lambda^d(f) = \mathcal{P}_{r-1}\Lambda^d(f)$.

- Vector proxies:
  - $\mathcal{P}_r^-\Lambda^1 \sim$ local Nédélec space (face or elements).
  - $\mathcal{P}_r^-\Lambda^2 \sim$ local Raviart–Thomas space (elements).

Discrete $H\Lambda^k(\Omega)$ spaces and interpolator

$$\underline{X}^k_{r,h} := \bigtimes_{d=k}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \mathcal{P}^-_r\Lambda^{d-k}(f)$$

For $\omega \in \Lambda^k(\Omega)$, $\quad \underline{I}^k_{r,h}\omega = \left(\pi^{-,d-k}_{r,f}(\star\,\mathrm{tr}_f\,\omega)\right)_{f\in\Delta_{[k\dots n]}(\mathcal{M}_h)} \in \underline{X}^k_{r,h}$

Space	$f_0 \equiv V$	$f_1 \equiv E$	$f_2 \equiv F$	$f_3 \equiv T$
$\underline{X}^0_{r,h}$	$\mathbb{R} = \mathcal{P}_r\Lambda^0(f_0)$	$\mathcal{P}^-_r\Lambda^1(f_1)$	$\mathcal{P}^-_r\Lambda^2(f_2)$	$\mathcal{P}^-_r\Lambda^3(f_3)$
$\underline{X}^1_{r,h}$		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{P}^-_r\Lambda^1(f_2)$	$\mathcal{P}^-_r\Lambda^2(f_3)$
$\underline{X}^2_{r,h}$			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{P}^-_r\Lambda^1(f_3)$
$\underline{X}^3_{r,h}$				$\mathcal{P}_r\Lambda^0(f_3)$
$\underline{X}^r_{\mathbf{grad},h}$	$\mathbb{R} = \mathcal{P}_r(V)$	$\mathcal{P}_{r-1}(E)$	$\mathcal{P}_{r-1}(F)$	$\mathcal{P}_{r-1}(T)$
$\underline{\boldsymbol{X}}^r_{\mathbf{curl},h}$		$\mathcal{P}_r(E)$	$\mathcal{RT}_r(F)$	$\mathcal{RT}_r(T)$
$\underline{\boldsymbol{X}}^r_{\mathrm{div},h}$			$\mathcal{P}_r(F)$	$\boldsymbol{\mathcal{N}}_r(T)$
$\mathcal{P}_r(\mathcal{T}_h)$				$\mathcal{P}_r(T)$

Discrete local potential and exterior derivative

For $d = k, \ldots, n$, all $f \in \Delta_d(\mathcal{M}_h)$, and all $\underline{\omega}_f = (\omega_{f'})_{f' \in \Delta_{[k \ldots d]}(f)} \in \underline{X}^k_{r,f}$:

- If $d = k$, we let

$$P^k_{r,f}\underline{\omega}_f := \star^{-1}\omega_f \in \mathcal{P}_r\Lambda^d(f)$$

- If $d \geq k+1$:
  - Define $\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^{k+1}(f)$ such that, for all $\mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu.$$

  - Define $P^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^k(f)$ using $\mathcal{P}_r\Lambda^{d-k}(f) = \mathrm{d}\mathcal{K}^{d-k-1}_{r+1}(f) \oplus \mathcal{K}^{d-k}_r(f)$:
    - For all $\mu \in \mathcal{K}^{d-k-1}_{r+1}(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

    - For all $\nu \in \mathcal{K}^{d-k}_r(f)$:

$$\int_f \star^{-1} P^k_{r,f}\underline{\omega}_f \wedge \nu = \int_f \star^{-1}\omega_f \wedge \nu.$$

Polynomial consistency

Theorem (Polynomial consistency)

For all integers $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$, *it holds*

$$P^k_{r,f} \underline{I}^k_{r,f} \omega = \omega \qquad \forall \omega \in \mathcal{P}_r \Lambda^k(f),$$

and, if $d \ge k+1$,

$$\mathrm{d}^k_{r,f} \underline{I}^k_{r,f} \omega = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}^-_{r+1} \Lambda^k(f).$$

Global discrete exterior derivative and DDR complex

- Global discrete exterior derivative $\underline{\mathrm{d}}^k_{r,h} : \underline{X}^k_{r,h} \to \underline{X}^{k+1}_{r,h}$ s.t.

$$\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h \coloneqq \left(\pi^{-,d-k-1}_{r,f}(\star \mathrm{d}^k_{r,f}\underline{\omega}_f)\right)_{f\in\Delta_{[k+1\ldots n]}(\mathcal{M}_h)}$$

- The DDR sequence then reads

$$\underline{X}^0_{r,h} \xrightarrow{\underline{\mathrm{d}}^0_{r,h}} \underline{X}^1_{r,h} \longrightarrow \cdots \longrightarrow \underline{X}^{n-1}_{r,h} \xrightarrow{\underline{\mathrm{d}}^{n-1}_{r,h}} \underline{X}^n_{r,h} \longrightarrow \{0\}$$

Cohomology

Theorem (Cohomology of the Discrete de Rham complex)

The DDR sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all k,

$$\operatorname{Ker} \underline{\mathrm{d}}_{r,h}^{k} / \operatorname{Im} \underline{\mathrm{d}}_{r,h}^{k-1} \cong \operatorname{Ker} \mathrm{d}^{k} / \operatorname{Im} \mathrm{d}^{k-1}.$$

Discrete L^2-products

- We can define on $\underline{X}^k_{r,h}$ a discrete L^2-product $(\cdot,\cdot)_{k,h} : \underline{X}^k_{r,h} \times \underline{X}^k_{r,h} \to \mathbb{R}$:

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} \left(\int_f P^k_{r,f} \underline{\omega}_f \wedge \star P^k_{r,f} \underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \right)$$

- Above, $s_{k,f} : \underline{X}^k_{r,f} \times \underline{X}^k_{r,f} \to \mathbb{R}$ is a stabilisation that satisfies

$$s_{k,f}(\underline{I}^k_{r,f}\omega, \underline{\mu}_f) = 0 \qquad \forall \omega \in \mathcal{P}_r\Lambda^k(f)$$

- Numerical schemes are obtained replacing spaces, differential operators, and L^2-products with their discrete counterparts. Yield stable schemes, with $\mathcal{O}(h^{k+1})$ rates of convergence in energy norm. [Di Pietro and Droniou, 2021, Beirão da Veiga et al., 2022, Droniou and Qian, 2023, Di Pietro and Droniou, 2023, Di Pietro and Droniou, 2022]

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General ideas

- It is actually a fully discrete construction (virtual spaces are not required, but could be identified).
- Discrete spaces still made of polynomial components, but some of them represent traces of exterior derivatives.
- Components on k- and $(k+1)$-cells play a different role to the other ones.
- Construction does not intertwine discrete exterior derivative and potential, and leads to larger spaces.

Discrete $H\Lambda^k(\Omega)$ spaces and interpolator

$$\underline{V}_{r,h}^k \coloneqq \bigtimes_{f\in\Delta_k(\mathcal{M}_h)} \mathcal{P}_r\Lambda^0(f) \times \bigtimes_{f\in\Delta_{k+1}(\mathcal{M}_h)} \mathcal{K}_{r+1}^1(f)\times\mathcal{K}_{r}^0(f)$$
$$\times \bigtimes_{d=k+2}^{n}\bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \mathcal{K}_{r+1}^{d-k}(f)\times\mathcal{K}_{r+1}^{d-k-1}(f).$$

$$\underline{I}_{r,f}^k\omega = \Big(\big(\pi_{r,f'}^0(\star\operatorname{tr}_{f'}\omega)\big)_{f'\in\Delta_k(f)},$$
$$\big(\pi_{r+1,f'}^{\mathcal{K},1}(\star\operatorname{tr}_{f'}\omega),\pi_{r,f'}^{\mathcal{K},0}(\star\operatorname{tr}_{f'}\mathrm{d}\omega)\big)_{f'\in\Delta_{k+1}(f)}\Big),$$
$$\big(\pi_{r+1,f'}^{\mathcal{K},d'-k}(\star\operatorname{tr}_{f'}\omega),\pi_{r+1,f'}^{\mathcal{K},d'-k-1}(\star\operatorname{tr}_{f'}\mathrm{d}\omega)\big)_{f'\in\Delta_{[k+2\dots d]}(f)}\Big).$$

Space	$f_0\equiv V$	$f_1\equiv E$	$f_2\equiv F$	$f_3\equiv T$
$\underline{V}_{r,h}^0$	$\mathbb{R}=\mathcal{P}_r\Lambda^0(f_0)$	$\{0\}\times\mathcal{K}_r^0(f_1)$	$\{0\}\times\mathcal{K}_{r+1}^1(f_2)$	$\{0\}\times\mathcal{K}_{r+1}^2(f_3)$
$\underline{V}_{r,h}^1$		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{K}_{r+1}^1(f_2)\times\mathcal{K}_r^0(f_2)$	$\mathcal{K}_{r+1}^2(f_3)\times\mathcal{K}_{r+1}^1(f_3)$
$\underline{V}_{r,h}^2$			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{K}_{r+1}^1(f_3)\times\mathcal{K}_r^0(f_3)$
$\underline{V}_{r,h}^3$				$\mathcal{P}_r\Lambda^0(f_3)$

Discrete $H\Lambda^k(\Omega)$ spaces and interpolator

$$\underline{V}^k_{r,h} \coloneqq \bigtimes_{f\in\Delta_k(\mathcal{M}_h)} \mathcal{P}_r\Lambda^0(f) \times \bigtimes_{f\in\Delta_{k+1}(\mathcal{M}_h)} \mathcal{K}^1_{r+1}(f)\times\mathcal{K}^0_{r}(f)$$
$$\times \bigtimes_{d=k+2}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \mathcal{K}^{d-k}_{r+1}(f)\times\mathcal{K}^{d-k-1}_{r+1}(f).$$

$$\underline{\omega}_h = \Big((\omega_f)_{f\in\Delta_k(\mathcal{M}_h)}, (\omega_f, D_{\omega,f})_{f\in\Delta_{k+1}(\mathcal{M}_h)},$$
$$(\omega_f, D_{\omega,f})_{f\in\Delta_{[k+2\ldots n]}(\mathcal{M}_h)}\Big).$$

Space	$f_0\equiv V$	$f_1\equiv E$	$f_2\equiv F$	$f_3\equiv T$
$\underline{V}^0_{r,h}$	$\mathbb{R}=\mathcal{P}_r\Lambda^0(f_0)$	$\{0\}\times\mathcal{K}^0_r(f_1)$	$\{0\}\times\mathcal{K}^1_{r+1}(f_2)$	$\{0\}\times\mathcal{K}^2_{r+1}(f_3)$
$\underline{V}^1_{r,h}$		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{K}^1_{r+1}(f_2)\times\mathcal{K}^0_r(f_2)$	$\mathcal{K}^2_{r+1}(f_3)\times\mathcal{K}^1_{r+1}(f_3)$
$\underline{V}^2_{r,h}$			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{K}^1_{r+1}(f_3)\times\mathcal{K}^0_r(f_3)$
$\underline{V}^3_{r,h}$				$\mathcal{P}_r\Lambda^0(f_3)$

Global discrete exterior derivative

- For $f \in \Delta_{k+1}(\mathcal{M}_h)$, define $\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^{k+1}(f)$ by:

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge (\mu + \nu) = \int_{\partial f} \star^{-1}\omega_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu + \int_f \star^{-1} D_{\omega,f} \wedge \nu$$
$$\forall(\mu,\nu) \in \mathcal{P}_0\Lambda^0(f) \times \mathcal{K}^0_r(f),$$

- VEM sequence:

$$\underline{V}^0_{r,h} \xrightarrow{\underline{\mathrm{d}}^0_{r,h}} \underline{V}^1_{r,h} \longrightarrow \cdots \longrightarrow \underline{V}^{n-1}_{r,h} \xrightarrow{\underline{\mathrm{d}}^{n-1}_{r,h}} \underline{V}^n_{r,h} \longrightarrow \{0\}.$$

with

$$\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h := \left((\star\mathrm{d}^k_{r,f}\underline{\omega}_f)_{f\in\Delta_{k+1}(\mathcal{M}_h)}, (D_{\omega,f}, 0)_{f\in\Delta_{[k+2\ldots n]}(\mathcal{M}_h)}\right)$$

Space	$f_0 \equiv V$	$f_1 \equiv E$	$f_2 \equiv F$	$f_3 \equiv T$
$\underline{V}^0_{r,h}$	$\mathbb{R} = \mathcal{P}_r\Lambda^0(f_0)$	$\{0\} \times \mathcal{K}^0_r(f_1)$	$\{0\} \times \mathcal{K}^1_{r+1}(f_2)$	$\{0\} \times \mathcal{K}^2_{r+1}(f_3)$
$\underline{V}^1_{r,h}$		$\mathcal{P}_r\Lambda^0(f_1)$	$\mathcal{K}^1_{r+1}(f_2) \times \mathcal{K}^0_r(f_2)$	$\mathcal{K}^2_{r+1}(f_3) \times \mathcal{K}^1_{r+1}(f_3)$
$\underline{V}^2_{r,h}$			$\mathcal{P}_r\Lambda^0(f_2)$	$\mathcal{K}^1_{r+1}(f_3) \times \mathcal{K}^0_r(f_3)$
$\underline{V}^3_{r,h}$				$\mathcal{P}_r\Lambda^0(f_3)$

Polynomial consistency

- Discrete potentials $P^k_{r,f} : \underline{V}^k_{r,f} \to \mathcal{P}^-_{r+1}\Lambda^k(f)$ can be reconstructed from Stokes' formula.
- Discrete exterior derivative $\mathrm{d}^{k+1}_{r,f} : \underline{V}^k_{r,f} \to \mathcal{P}^-_{r+1}\Lambda^{k+1}(f)$ also, for any $f \in \Delta_d(\mathcal{M}_h)$ with $d \geq k+1$ (not just $d = k+1$).

Theorem (Polynomial consistency)

For all integers $0 \leq k \leq d \leq n$ and all $f \in \Delta_d(\mathcal{M}_h)$, it holds

$$P^k_{r,f}\underline{I}^k_{r,f}\omega = \omega \qquad \forall \omega \in \mathcal{P}^-_{r+1}\Lambda^k(f),$$

and, if $d \geq k+1$,

$$\mathrm{d}^k_{r,f}\underline{I}^k_{r,f}\omega = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}^-_{r+1}\Lambda^k(f).$$

Cohomology

Theorem (Cohomology of the Discrete de Rham complex)

The VEM sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all k,

$$\mathrm{Ker}\,\underline{\mathrm{d}}_{r,h}^{k}/\mathrm{Im}\,\underline{\mathrm{d}}_{r,h}^{k-1} \cong \mathrm{Ker}\,\mathrm{d}^{k}/\mathrm{Im}\,\mathrm{d}^{k-1}.$$

Comparison DDR–VEM–RTN

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	4 ⋄ 9 ⋄ 4	15 ⋄ 26 ⋄ 10	32 ⋄ 50 ⋄ 20
$\boldsymbol{H}(\mathbf{curl};T)$	6 ⋄ 14 ⋄ 6	28 ⋄ 47 ⋄ 20	65 ⋄ 98 ⋄ 45
$\boldsymbol{H}(\mathrm{div};T)$	4 ⋄ 7 ⋄ 4	18 ⋄ 26 ⋄ 15	44 ⋄ 59 ⋄ 36
$L^2(T)$	1 ⋄ 1 ⋄ 1	4 ⋄ 4 ⋄ 4	10 ⋄ 10 ⋄ 10

Table: Tetrahedron: dimensions of the local spaces in the DDR ⋄ VEM ⋄ RTN.

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	8 ⋄ 15 ⋄ 8	27 ⋄ 42 ⋄ 27	54 ⋄ 78 ⋄ 64
$\boldsymbol{H}(\mathbf{curl};T)$	12 ⋄ 22 ⋄ 12	46 ⋄ 69 ⋄ 54	99 ⋄ 138 ⋄ 144
$\boldsymbol{H}(\mathrm{div};T)$	6 ⋄ 9 ⋄ 6	24 ⋄ 32 ⋄ 36	56 ⋄ 71 ⋄ 108
$L^2(T)$	1 ⋄ 1 ⋄ 1	4 ⋄ 4 ⋄ 8	10 ⋄ 10 ⋄ 27

Table: Hexahedron: dimensions of the local spaces in the DDR ⋄ VEM ⋄ RTN.

Comparison of *serendipity* DDR–VEM vs. RTN

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	4 ⋄ 4 ⋄ 4	10 ⋄ 10 ⋄ 10	20 ⋄ 20 ⋄ 20
$\boldsymbol{H}(\mathbf{curl};T)$	6 ⋄ 9 ⋄ 6	23 ⋄ 31 ⋄ 20	53 ⋄ 68 ⋄ 45
$\boldsymbol{H}(\mathrm{div};T)$	4 ⋄ 7 ⋄ 4	18 ⋄ 26 ⋄ 15	44 ⋄ 59 ⋄ 36
$L^2(T)$	1 ⋄ 1 ⋄ 1	4 ⋄ 4 ⋄ 4	10 ⋄ 10 ⋄ 10

Table: Tetrahedron: dimensions of the local spaces in the sDDR ⋄ sVEM ⋄ RTN.

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	8 ⋄ 8 ⋄ 8	20 ⋄ 20 ⋄ 27	32 ⋄ 32 ⋄ 64
$\boldsymbol{H}(\mathbf{curl};T)$	12 ⋄ 15 ⋄ 12	39 ⋄ 47 ⋄ 54	77 ⋄ 92 ⋄ 144
$\boldsymbol{H}(\mathrm{div};T)$	6 ⋄ 9 ⋄ 6	24 ⋄ 32 ⋄ 36	56 ⋄ 71 ⋄ 108
$L^2(T)$	1 ⋄ 1 ⋄ 1	4 ⋄ 4 ⋄ 8	10 ⋄ 10 ⋄ 27

Table: Hexahedron: dimensions of the local spaces in the sDDR ⋄ sVEM ⋄ RTN.

Conclusion

- Polytopal exterior calculus: framework for discrete polytopal complexes of arbitrary order, in the langage of differential forms.

  Unifies the analysis of all operators.

- Polynomial consistency and same cohomology as the continuous de Rham complex.

  Ensures accuracy and robustness of schemes.

- Ongoing work: Poincaré inequalities, analysis tools (adjoint consistency, etc.).

Conclusion

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Thanks!

- Notes and series of introductory lectures to DDR (vector proxy form):

`https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html`

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

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