Serendipity discrete de Rham method

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(joint work with D. Di Pietro [mostly])

Plan

The Stokes problem in curl-curl formulation

- Given Ω contractible, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
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$$\boxed{\boldsymbol{u} \perp \mathbf{grad}\,H^1(\Omega)}$$

De Rham complex

- The de Rham sequence is

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- It is a complex: the range of each operator is included in the kernel of the next one *(i.e.* $\mathbf{grad}\, i_\Omega = 0$, $\mathbf{curl}\,\mathbf{grad} = 0$, $\mathrm{div}\,\mathbf{curl} = 0$*)*.
- It is exact (inclusions $\rightsquigarrow$ equalities) if Ω has a trivial topology:

$$\mathbb{R} = \ker \mathbf{grad}\,,\ \ \mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}\,,\ \ \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,,\ \ \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

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- Exactness $\Rightarrow$ well-posedness of the Stokes problem in curl–curl form *(same for the Stokes problem in Δ form...)*.

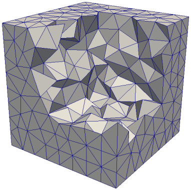
Reproducing this exactness at the discrete level is instrumental to designing stable numerical approximations.

Plan

1. Motivation for continuous and discrete complexes
2. Finite Elements approach, and its limitations
3. Overview of the Discrete De Rham complexes
   - Principles guiding arbitrary-order polytopal complexes
   - DDR – regular version
   - DDR – serendipity version
4. Numerical illustration

The Finite Element way

Global complex



$\mathcal{T}_h = \{T\}$ conforming tetrahedral/hexahedral mesh of Ω.

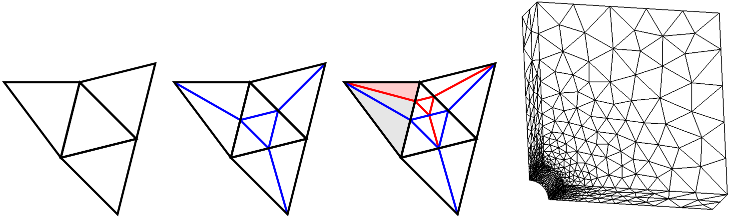
- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & V_{h,\mathbf{grad}} & \xrightarrow{\mathbf{grad}} & V_{h,\mathbf{curl}} & \xrightarrow{\mathbf{curl}} & V_{h,\mathrm{div}} & \xrightarrow{\mathrm{div}} & V_{h,L^2} & \xrightarrow{0} & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}$$

- Example: conforming $\mathcal{P}^k$–Nédélec–Raviart-Thomas spaces (see [Arnold, 2018] for a generic approach).
- **Gluing only works on special meshes!**

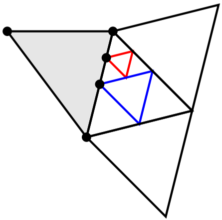
The Finite Element way

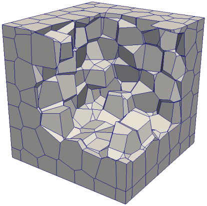
Shortcomings

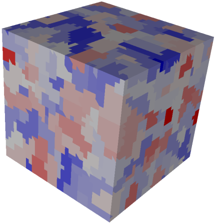


- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - $\Longrightarrow$ significant increase of DOFs on hexahedral elements

Polytopal meshes I



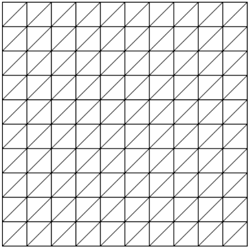
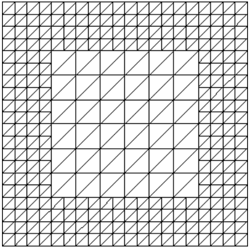




- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

Polytopal meshes II

Example of efficiency: Reissner–Mindlin plate problem.

Stabilised $\mathcal{P}^2$-($\mathcal{P}^1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

Plan

Plan

Key ideas for a discrete complex

- Finite-dimensional spaces of vectors made of polynomials on relevant mesh entities: vertices, edges, faces, elements.
- Interpolators give meaning to these polynomials/DOFs.
- Bespoke operators between the spaces, that represent $\mathbf{grad}$, $\mathbf{curl}$, div.

$$
\begin{array}{lccccccccccc}
(\mathrm{PC}): & \mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\mathbf{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\mathbf{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & X^k_{h,L^2} & \xrightarrow{0} & \{0\} \\
& & & \uparrow{\scriptstyle \underline{I}^k_{\mathbf{grad},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}} & & \uparrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},h}} & & \uparrow{\scriptstyle I^k_{L^2,h}} & & \\
(\mathrm{dR}): & \mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

Note: interpolators are actually well-defined on smaller (more regular) subspaces.

Key ideas for a discrete complex

$$
\begin{array}{ccccccccccc}
\text{(PC)}: \ \mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\mathbf{grad}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\mathbf{curl}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{\mathrm{div}^k_h} & X^k_{h,L^2} & \xrightarrow{0} & \{0\} \\
 & & \uparrow \underline{I}^k_{\mathbf{grad},h} & & \uparrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \uparrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \uparrow I^k_{L^2,h} & & \\
\text{(dR)}: \ \mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

Main properties:

- Cohomology of (PC) $\simeq$ cohomology of (dR), on generic topologies [Di Pietro et al., 2022].
- Local polynomial consistency: for $\mathcal{D} \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$ and $T \in \mathcal{T}_h$, there is $\boldsymbol{P}^k_{\mathcal{D},T} : \underline{X}^k_{\mathcal{D},T} \to \mathcal{P}^k(T)$ s.t.

$$\boldsymbol{P}^k_{\mathcal{D},T}\underline{I}^k_{\mathcal{D},T}\omega = \omega\,, \quad \mathcal{D}^k_T\underline{I}^k_{\mathcal{D},T}\omega = \mathcal{D}\omega \qquad \forall \omega \in \mathcal{P}^k(T).$$

Plan

Degrees of freedom and interpolator

- Degrees of freedom: (trimmed) polynomials on vertices, edges, faces, elements.

Space	V	E	F	T
$\underline{\boldsymbol{X}}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^{k}(E)$	$\boldsymbol{\mathcal{R}}^{k-1}(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F)$	$\boldsymbol{\mathcal{R}}^{k-1}(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^{k}(F)$	$\boldsymbol{\mathcal{G}}^{k-1}(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

$$\boldsymbol{\mathcal{R}}^{k-1}(\mathsf{P}) = \mathbf{curl}\,\mathcal{P}^k(\mathsf{P}), \quad \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(\mathsf{P}) = (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}^{k-1}(\mathsf{P}) \quad (\mathsf{P} = F, T)$$
$$\boldsymbol{\mathcal{G}}^{k-1}(T) = \mathbf{grad}\,\mathcal{P}^k(T), \quad \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) = (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}^{k-1}(T).$$

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$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^{k}(F)$	$\mathcal{G}^{k-1}(T) \oplus \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

- Interpolators: L^2-projections of (traces) of functions. E.g.:

$$\forall \boldsymbol{v} \in C^0(\overline{\Omega})^3, \quad \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = \big((\pi^k_{\mathcal{P},E}(\boldsymbol{v}\cdot\boldsymbol{t}_E))_{E\in\mathcal{E}_h}, \\ (\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}(\boldsymbol{v}_{\mathrm{t},F}), \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}(\boldsymbol{v}_{\mathrm{t},F}))_{F\in\mathcal{F}_h}, \\ (\boldsymbol{\pi}^{k-1}_{\mathcal{R},T}\boldsymbol{v}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T}\boldsymbol{v})_{T\in\mathcal{T}_h}\big).$$

Potential reconstructions and discrete operators

- Hierarchical constructions: from lowest-dimensional mesh entity to higher-dimensional entities.
- Enhancement: Discrete operator first (based on IBP, polynomially consistent), then used for potential reconstruction (also based on IBP, and also polynomially consistent).

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Example with curl:

- Face curl: $C_F^k \underline{\boldsymbol{v}}_F \in \mathcal{P}^k(F)$ such that

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \, r_F = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \qquad \forall r_F \in \mathcal{P}^k(F).$$

- Tangential trace reconstruction: $\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{P}}^k(F)$ such that

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, r_F + \boldsymbol{w}_F) = \int_F C_F^k \underline{\boldsymbol{v}}_F \, r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F + \int_F \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}} \cdot \boldsymbol{w}_F$$
$$\forall (r_F, \boldsymbol{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F).$$

DDR complex

Patch local spaces (no continuity between elements), project local (polynomial) operators on DOFs:

$$
\begin{array}{lccccccccccc}
\text{(DDR)}: & \mathbb{R} & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
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\text{(dR)}: & \mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}
$$

Properties for the design and analysis of stable numerical schemes:

- Cohomology and polynomial consistency.
- Analytical properties: Poincaré inequalities, primal and adjoint consistency, commutation properties of interpolators and differential operators, etc.

DDR scheme for Stokes in curl-curl formulation

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned}\int_\Omega \nu\, \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q &= 0 && \forall q \in H^1(\Omega)\end{aligned}$$

- Set

$$\underline{X}^k_{\mathbf{grad},h,0} \coloneqq \left\{ \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{q}_h, \underline{I}^k_{\mathbf{grad},h} 1)_{\mathbf{grad},h} = 0 \right\}.$$

- DDR scheme: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{p}_h \in \underline{X}^k_{\mathbf{grad},h,0}$ such that

$$\begin{aligned}\nu(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\ -(\underline{\boldsymbol{G}}^k_h \underline{q}_h, \underline{\boldsymbol{u}}_h)_{\mathbf{curl},h} &= 0 && \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}.\end{aligned}$$

Plan

Key ideas to eliminate some DOFs

Lemma

Let $k \geq 0$, take T a polyhedron with η_T faces that are not pairwise parallel, and set $\ell_T = k + 1 - \eta_T$.
Any $q \in \mathcal{P}^{k+1}(T)$ is entirely determined by $(q_{|F})_{F \in \mathcal{F}_T}$ and $\pi^{\ell_T}_{\mathcal{P},T} q$, and

$$\|q\|_{L^2(T)} \lesssim \|\pi^{\ell_T}_{\mathcal{P},T} q\|_{L^2(T)} + h_T^{\frac{1}{2}} \sum_{F \in \mathcal{F}_T} \|q_{|F}\|_{L^2(F)}.$$

- If $\eta_T > k + 1$, then $\pi^{\ell_T}_{\mathcal{P},T} q = 0$ (only face values are required).
- Also works with $T \rightsquigarrow F$ polygon.

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- If $\eta_T > k + 1$, then $\pi^{\ell_T}_{\mathcal{P},T} q = 0$ (only face values are required).
- Also works with $T \rightsquigarrow F$ polygon.
- DDR spaces already fully encode traces on faces/elements boundaries $\rightsquigarrow$ polynomial consistency should not require that much information *inside* the faces/elements.

Serendipity DDR: degrees of freedom

Space	V	E	F	T
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\boldsymbol{\mathcal{R}}^{k-1}(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F)$	$\boldsymbol{\mathcal{R}}^{k-1}(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{G}}^{k-1}(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

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$\underline{\widehat{\boldsymbol{X}}}^{k}_{\mathbf{curl},T}$		$\mathcal{P}^{k}(E)$	$\boldsymbol{\mathcal{R}}^{k-1}(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell_F+1}(F)$	$\boldsymbol{\mathcal{R}}^{k-1}(T) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell_T+1}(T)$
$\underline{\boldsymbol{X}}^{k}_{\mathrm{div},T}$			$\mathcal{P}^{k}(F)$	$\boldsymbol{\mathcal{G}}^{k-1}(T) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)$
$\mathcal{P}^{k}(T)$				$\mathcal{P}^{k}(T)$

- Note that $\ell_F < k-1$ and $\ell_T < k-1$.

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$\mathcal{P}^{k}(T)$				$\mathcal{P}^{k}(T)$

- Note that $\ell_F < k-1$ and $\ell_T < k-1$.
- Why can't we reduce more (e.g. the $\mathcal{R}^{k-1}$ components, or $\underline{\boldsymbol{X}}^{k}_{\mathrm{div},h}$)? Due to the constraints of preserving the complex properties...

DDR vs. SDDR vs. Raviart-Thomas-Nédélec

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	$4 \diamond 4 \diamond 4$	$15 \diamond 10 \diamond 10$	$32 \diamond 20 \diamond 20$
$\boldsymbol{H}(\mathbf{curl}; T)$	$6 \diamond 6 \diamond 6$	$28 \diamond 23 \diamond 20$	$65 \diamond 53 \diamond 45$
$\boldsymbol{H}(\mathrm{div}; T)$	$4 \diamond 4 \diamond 4$	$18 \diamond 18 \diamond 15$	$44 \diamond 44 \diamond 36$
$L^2(T)$	$1 \diamond 1 \diamond 1$	$4 \diamond 4 \diamond 4$	$10 \diamond 10 \diamond 10$

Table: Tetrahedron: dimensions of the local spaces in the DDR $\diamond$ SDDR $\diamond$ RTN.

Discrete space	$k = 0$	$k = 1$	$k = 2$
$H^1(T)$	$8 \diamond 8 \diamond 8$	$27 \diamond 20 \diamond 27$	$54 \diamond 32 \diamond 64$
$\boldsymbol{H}(\mathbf{curl}; T)$	$12 \diamond 12 \diamond 12$	$46 \diamond 39 \diamond 54$	$99 \diamond 77 \diamond 144$
$\boldsymbol{H}(\mathrm{div}; T)$	$6 \diamond 6 \diamond 6$	$24 \diamond 24 \diamond 36$	$56 \diamond 56 \diamond 108$
$L^2(T)$	$1 \diamond 1 \diamond 1$	$4 \diamond 4 \diamond 8$	$10 \diamond 10 \diamond 27$

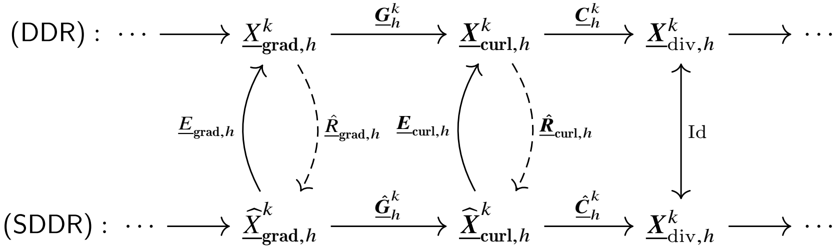
Table: Hexahedron: dimensions of the local spaces in the DDR $\diamond$ SDDR $\diamond$ RTN.

SDDR complex

$$\mathbb{R} \longrightarrow \underline{\widehat{X}}^k_{\mathbf{grad},h} \xrightarrow{\underline{\hat{\boldsymbol{G}}}^k_h} \underline{\widehat{\boldsymbol{X}}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\hat{\boldsymbol{C}}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- SDDR scheme: exactly as a DDR scheme, substituting spaces and operators with those above.

SDDR: analysis



- Extensions and reduction link the two complexes.

SDDR: analysis

$$
\begin{array}{lccccc}
(\mathrm{DDR}):\ \cdots \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}_h^k} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}_h^k} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \longrightarrow \cdots \\
 & \underline{E}_{\mathbf{grad},h}\uparrow\ \downarrow\underline{\hat{R}}_{\mathbf{grad},h} & & \underline{\boldsymbol{E}}_{\mathbf{curl},h}\uparrow\ \downarrow\underline{\hat{\boldsymbol{R}}}_{\mathbf{curl},h} & & \updownarrow \mathrm{Id} \\
(\mathrm{SDDR}):\ \cdots \longrightarrow & \underline{\widehat{X}}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\hat{\boldsymbol{G}}}_h^k} & \underline{\widehat{\boldsymbol{X}}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\hat{\boldsymbol{C}}}_h^k} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \longrightarrow \cdots
\end{array}
$$

- Extensions and reduction link the two complexes.
- Designed to ensure transfer of homological and analytical properties: *isomorphism of cohomologies, Poincaré inequalities, primal and adjoint consistency, etc.*

SDDR: analysis

$$
\begin{array}{ccccccccc}
\text{(DDR)} : \cdots & \longrightarrow & \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \longrightarrow & \cdots \\
 & & \underline{E}_{\mathbf{grad},h} \uparrow \; \downarrow \underline{\hat{R}}_{\mathbf{grad},h} & & \underline{\boldsymbol{E}}_{\mathbf{curl},h} \uparrow \; \downarrow \underline{\hat{\boldsymbol{R}}}_{\mathbf{curl},h} & & \updownarrow \mathrm{Id} & & \\
\text{(SDDR)} : \cdots & \longrightarrow & \underline{\widehat{X}}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\hat{\boldsymbol{G}}}^k_h} & \underline{\widehat{\boldsymbol{X}}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\hat{\boldsymbol{C}}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \longrightarrow & \cdots
\end{array}
$$

- Extensions and reduction link the two complexes.
- Designed to ensure transfer of homological and analytical properties: *isomorphism of cohomologies, Poincaré inequalities, primal and adjoint consistency, etc.*
- Generic blueprint that is applicable in many circumstances *(e.g.: analysis of cohomology of various discrete complexes, etc.)*.

Plan

Convergence of (S)DDR scheme for the Stokes problem in curl-curl formulation

Theorem (Pressure-robust estimates [Beirão da Veiga et al., 2022])

Setting the graph norms

$$\|\cdot\|^2_{\mathbf{curl},1,h} = \|\cdot\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \cdot\|^2_{\mathrm{div},h} \quad \text{on } \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},$$
$$\|\cdot\|^2_{\mathbf{grad},1,h} = \|\cdot\|^2_{\mathbf{grad},h} + \|\underline{\boldsymbol{G}}^k_h \cdot\|_{\mathbf{curl},h} \quad \text{on } \underline{X}^k_{\mathbf{grad},h},$$

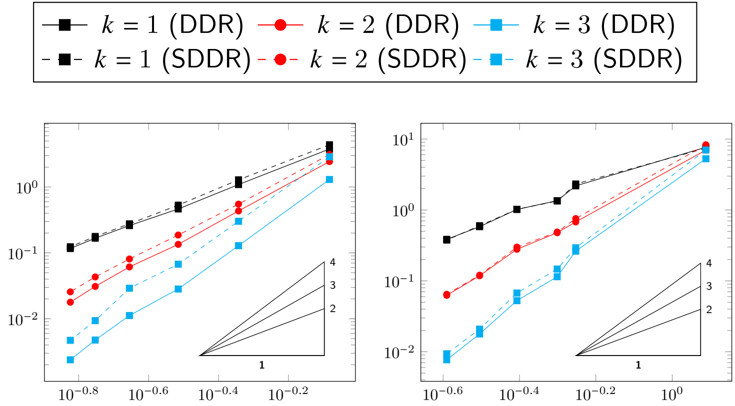
we have:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},1,h} + \|\underline{p}_h - \underline{I}^k_{\mathbf{grad},h} p\|_{\mathbf{grad},1,h} \lesssim C_1(\boldsymbol{u}) h^{k+1}.$$

with $C_1(\boldsymbol{u})$ depending $\boldsymbol{u}$ and some of its derivatives, but not p.

Choice of discrete source term & commutation properties of DDR operators ensures pressure robustness...

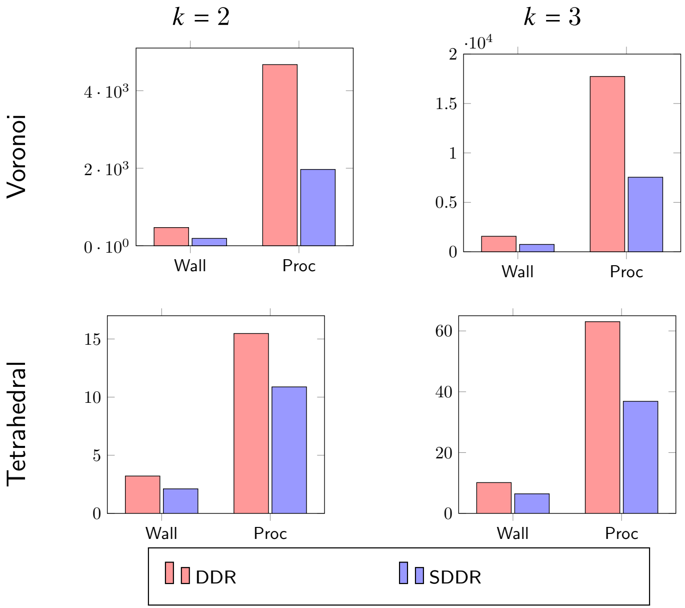
Convergence test



(a) Voronoi meshes (b) Tetrahedral meshes

Figure: Relative errors in discrete $\boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ norm vs. h: DDR (continuous lines) and SDDR (dashed lines).

Wall and CPU times on the finest meshes



Conclusion I

- DDR: discrete complex, of arbitrary degree of accuracy, applicable on polytopal meshes, yields stable schemes even for models with "incomplete" differential operators [Di Pietro et al., 2020][Di Pietro and Droniou, 2021a].

  Other approach: virtual element complexes [Beirão da Veiga et al., 2018][Beirão da Veiga et al., 2022]

- Systematic serendipity reduction of number of DOFs (on any polytopal mesh) [Di Pietro and Droniou, 2022b].

- Leaner complexes than FE approches on certain meshes (*and fully compatible with FE complexes on hybrid meshes*).

Conclusion II

- Full set of homological and analytical results: cohomology, Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.
  (Facilitated by a generic framework to transfer properties from one complex to another one.)

- Some other applications/complexes:
  - div-div plates complex and serendipity version [Di Pietro and Droniou, 2022a][Botti et al., 2023].
  - Magnetostatics equations [Di Pietro and Droniou, 2021b].
  - Yang–Mills equations [Droniou et al., 2023].
  - Stokes complex [Hanot, 2021].
  - Rot-rot complex [Di Pietro, 2023].

- Notes and series of introductory lectures to DDR:

https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

Thank you!

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Plan

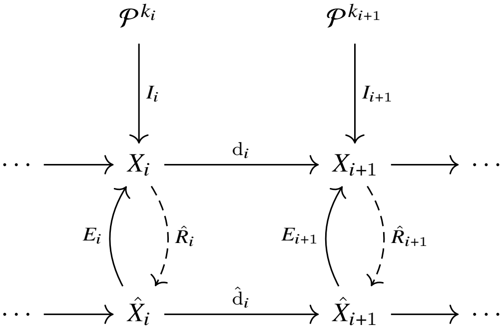
Setting I

Starting point: a complex with some analytical properties (including some polynomial consistency).

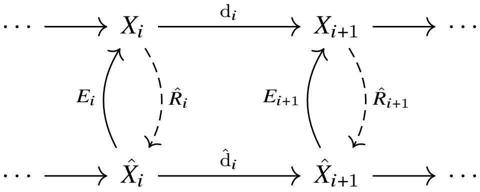
$$
\begin{array}{ccccccc}
 & & \mathcal{P}^{k_i} & & \mathcal{P}^{k_{i+1}} & & \\
 & & \big\downarrow I_i & & \big\downarrow I_{i+1} & & \\
\cdots & \longrightarrow & X_i & \xrightarrow{\ \mathrm{d}_i\ } & X_{i+1} & \longrightarrow & \cdots
\end{array}
$$

Setting II

Objective: link a second complex, to ensure that homological and analytical properties are also satisfied by this second complex.



Homological properties



Theorem

If the extensions and reductions satisfy

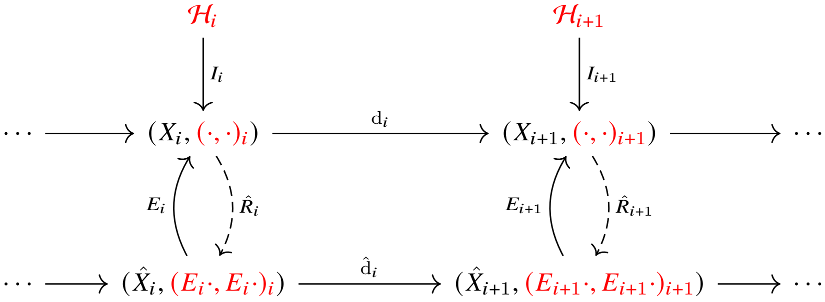
- $\hat{R}_i E_i = \mathrm{Id}$ *for all* i,
- $(E_{i+1}\hat{R}_{i+1} - \mathrm{Id})(\ker \mathrm{d}_{i+1}) \subset \mathrm{Im}\,\mathrm{d}_i$ *for all* i,
- E *and* $\hat{R}$ *are cochain maps (each one commutes with the differential operators),*

then:

- ⋆ *the cohomologies of* $(X_i, \mathrm{d}_i)_i$ *and* $(\hat{X}_i, \hat{\mathrm{d}}_i)_i$ *are isomorphic.*

Analytical properties I

Take $\mathcal{H}_i \supset \mathcal{P}^{k_i}$ (Sobolev-like space) and $(\cdot,\cdot)_i$ L^2-like inner products on the first complex.



Analytical properties II

Theorem

If the extensions and reductions satisfy

- $\hat{R}_i : X_i \longrightarrow \hat{X}_i$ *is continuous,*
- *(polynomial consistency)* $E_i \hat{R}_i I_i = I_i$,
- $I_i : \mathcal{H}_i \longrightarrow X_i$ *is continuous,*

then the following properties are transferred from $(X_i, \mathrm{d}_i)_i$ *to* $(\hat{X}_i, \hat{\mathrm{d}}_i)_i$*:*

- ⋆ *Poincaré inequalities,*
- ⋆ *Consistency of the inner product (polynomial and in* L^2*-like norm on* $\mathcal{H}_i$*),*
- ⋆ *Consistency of potential reconstruction,*

and, under cochain map property:

- ⋆ *Commutation property and consistency of differential operators,*
- ⋆ *Adjoint consistency (controls error in discrete IBP).*

Application to the SDDR complex

- Construction of extensions and reductions for the SDDR complex $\rightsquigarrow$ limits on reduction of number of DOFs.
- Gives all the properties required for analysis of schemes based on SDDR:
  - SDDR has the same cohomology as the de Rham complex.
  - SDDR satisfies all analytical properties: Poincaré inequalities, primal and adjoint consistency, commutation properties of interpolator and differential operators...