Serendipity discrete de Rham method

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(joint work with D. Di Pietro [mostly])
Plan

1. Motivation for continuous and discrete complexes

2. Finite Elements approach, and its limitations

3. Overview of the Discrete De Rham complexes
   - Principles guiding arbitrary-order polytopal complexes
   - DDR – regular version
   - DDR – serendipity version

4. Numerical illustration
The Stokes problem in curl-curl formulation

- Given $\Omega$ contractible, $\nu > 0$ and $f \in L^2(\Omega)$, the Stokes problem reads:

Find the velocity $u : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

\[
\begin{align*}
-\nu \Delta u & \quad \nu(\text{curl} \text{ curl } u - \text{grad} \text{ div } u) + \text{grad } p = f \quad \text{in } \Omega, \quad \text{(momentum conservation)} \\
\text{div } u & = 0 \quad \text{in } \Omega, \quad \text{(mass conservation)} \\
\text{curl } u \times n & = 0 \quad \text{and } u \cdot n = 0 \quad \text{on } \partial \Omega, \quad \text{(boundary conditions)} \\
\int_{\Omega} p & = 0
\end{align*}
\]

- Weak formulation: Find $(u, p) \in H(\text{curl}; \Omega) \times H^1(\Omega)$ s.t. $\int_{\Omega} p = 0$ and

\[
\begin{align*}
\int_{\Omega} \nu \text{curl } u \cdot \text{curl } v + \int_{\Omega} \text{grad } p \cdot v & = \int_{\Omega} f \cdot v \quad \forall v \in H(\text{curl}; \Omega), \\
- \int_{\Omega} u \cdot \text{grad } q & = 0 \quad \forall q \in H^1(\Omega)
\end{align*}
\]
The Stokes problem in curl-curl formulation

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  \]
  controls \( \text{curl } u \) only...
  \[
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The Stokes problem in curl-curl formulation

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$$

controls $\text{curl } u$ only...

$$
\mathbf{u} \perp \text{grad } H^1(\Omega)
$$
The de Rham sequence is

\[ \mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\} \]

- It is a complex: the range of each operator is included in the kernel of the next one (i.e. \( \text{grad} i_\Omega = 0 \), \( \text{curl} \, \text{grad} = 0 \), \( \text{div} \, \text{curl} = 0 \)).
- It is exact (inclusions \( \leadsto \) equalities) if \( \Omega \) has a trivial topology:

\[ \mathbb{R} = \ker \text{grad}, \; \text{Im} \, \text{grad} = \ker \text{curl}, \; \text{Im} \, \text{curl} = \ker \text{div}, \; \text{Im} \, \text{div} = L^2(\Omega). \]
De Rham complex

- The de Rham sequence is

\[
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\mathbb{R} = \ker \text{grad}, \ [\text{Im} \text{grad} = \ker \text{curl}], \ \text{Im} \text{curl} = \ker \text{div}, \ \text{Im} \text{div} = L^2(\Omega).
\]

- Exactness \( \Rightarrow \) well-posedness of the Stokes problem in curl–curl form (same for the Stokes problem in \( \Delta \) form...).

Reproducing this exactness at the discrete level is instrumental to designing stable numerical approximations.
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The Finite Element way

Global complex

\[ \mathcal{T}_h = \{T\} \] conforming tetrahedral/hexahedral mesh of \( \Omega \).

- Define local polynomial spaces on each element, and glue them together to form a sub-complex of the de Rham complex:

\[
\begin{align*}
\mathbb{R} & \xrightarrow{\text{grad}} V_{h,\text{grad}} \xrightarrow{\text{curl}} V_{h,\text{curl}} \xrightarrow{\text{div}} V_{h,\text{div}} \xrightarrow{\text{0}} V_{h,L^2} \xrightarrow{\{0\}} \\
\mathbb{R} & \xrightarrow{\text{grad}} H^1(\Omega) \xrightarrow{\text{curl}} H(\text{curl}; \Omega) \xrightarrow{\text{div}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\{0\}}
\end{align*}
\]


- Gluing only works on special meshes!
The Finite Element way

Shortcomings

- Approach limited to conforming meshes with standard elements
  - local refinement requires to trade mesh size for mesh quality
  - complex geometries may require a large number of elements
  - the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - significant increase of DOFs on hexahedral elements
Local refinement (to capture geometry or solution features) is **seamless**, and can preserve mesh regularity.

**Agglomerated elements** are also easy to handle (and useful, e.g., in multi-grid methods).

High-level approach can lead to **leaner methods** (fewer DOFs).
Example of efficiency: Reissner–Mindlin plate problem.

<table>
<thead>
<tr>
<th>Stabilised $P^2-(P^1 + B^3)$ scheme</th>
<th>DDR scheme</th>
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<tbody>
<tr>
<td><img src="image1.png" alt="Mesh" /></td>
<td><img src="image2.png" alt="Mesh" /></td>
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<td>550</td>
<td>0.161</td>
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<tr>
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<td>6.77e-2</td>
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4. Numerical illustration
Key ideas for a discrete complex

- Finite-dimensional spaces of vectors made of polynomials on relevant mesh entities: vertices, edges, faces, elements.
- Interpolators give meaning to these polynomials/DOFs.
- Bespoke operators between the spaces, that represent $\text{grad}$, $\text{curl}$, $\text{div}$.

$$(\text{PC}): \mathbb{R} \rightarrow \mathbb{X}^k_{\text{grad}, h} \xrightarrow{\text{grad}^k_h} \mathbb{X}^k_{\text{curl}, h} \xrightarrow{\text{curl}^k_h} \mathbb{X}^k_{\text{div}, h} \xrightarrow{\text{div}^k_h} \mathbb{X}^k_{h, L^2} \xrightarrow{0} \{0\}$$

$$(\text{dR}): \mathbb{R} \xleftarrow{} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

Note: interpolators are actually well-defined on smaller (more regular) subspaces.
Key ideas for a discrete complex

(PC) :  \( \mathbb{R} \xrightarrow{\text{grad}} X^k_{\text{grad},h} \xrightarrow{\text{grad}_h^k} X^k_{\text{curl},h} \xrightarrow{\text{curl}_h^k} X^k_{\text{div},h} \xrightarrow{\text{div}_h^k} X^k_{h,L^2} \xrightarrow{0} \{0\} \)

(dR) :  \( \mathbb{R} \xleftarrow{\text{grad}} H^1(\Omega) \xrightarrow{\text{grad}} \mathcal{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \mathcal{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\} \)

Main properties:

- **Cohomology** of (PC) \( \simeq \) cohomology of (dR), on generic topologies [Di Pietro et al., 2022].

- **Local polynomial consistency**: for \( \mathcal{D} \in \{\text{grad, curl, div}\} \) and \( T \in \mathcal{T}_h \), there is \( P^k_{\mathcal{D},T} : X^k_{\mathcal{D},T} \to \mathcal{P}^k(T) \) s.t.

\[
P^k_{\mathcal{D},T} I^k_{\mathcal{D},T} \omega = \omega , \quad \mathcal{D}^k_{T} I^k_{\mathcal{D},T} \omega = \mathcal{D} \omega \quad \forall \omega \in \mathcal{P}^k(T).
\]
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4. Numerical illustration
Degrees of freedom and interpolator

- Degrees of freedom: (trimmed) polynomials on vertices, edges, faces, elements.

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$\mathcal{R}^{k-1}(P) = \text{curl } \mathcal{P}^k(P)$, \quad $\mathcal{R}^{c,k}(P) = (x - x_T)\mathcal{P}^{k-1}(P)$ \quad (P = F, T)

$\mathcal{G}^{k-1}(T) = \text{grad } \mathcal{P}^k(T)$, \quad $\mathcal{G}^{c,k}(T) = (x - x_T) \times \mathcal{P}^{k-1}(T)$. 
Degrees of freedom and interpolator

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- Interpolators: $L^2$-projections of (traces) of functions. E.g.:

$$\forall \mathbf{v} \in C^0(\bar{\Omega})^3, \quad I^k_{\text{curl}, h} \mathbf{v} = (\pi^k_{\mathcal{P}, E}(\mathbf{v} \cdot t_E))_{E \in \mathcal{E}_h},$$
$$\quad (\pi^{k-1}_{\mathcal{R}, F} (\mathbf{v}_t, F), \pi^{c,k}_{\mathcal{R}, F} (\mathbf{v}_t, F))_{F \in \mathcal{F}_h},$$
$$\quad (\pi^{k-1}_{\mathcal{R}, T} \mathbf{v}, \pi^{c,k}_{\mathcal{R}, T} \mathbf{v})_{T \in \mathcal{T}_h}.$$
Potential reconstructions and discrete operators

- **Hierarchical** constructions: from lowest-dimensional mesh entity to higher-dimensional entities.
- **Enhancement**: Discrete operator first (based on IBP, polynomially consistent), then used for potential reconstruction (also based on IBP, and also polynomially consistent).
Potential reconstructions and discrete operators

- **Hierarchical** constructions: from lowest-dimensional mesh entity to higher-dimensional entities.

- **Enhancement:** Discrete operator first (based on IBP, polynomially consistent), then used for potential reconstruction (also based on IBP, and also polynomially consistent).

Example with curl:

- Face curl: $C^k_{F}v_F \in \mathcal{P}^k(F)$ such that

$$
\int_{F} C^k_{F}v_F r_f = \int_{F} v_{R,F} \cdot \text{rot}_F r_f - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_{E} v_{E} r_F \quad \forall r_f \in \mathcal{P}^k(F).
$$

- Tangential trace reconstruction: $\gamma^k_{t,F}v_F \in \mathcal{P}^k(F)$ such that

$$
\int_{F} \gamma^k_{t,F}v_F \cdot (\text{rot}_F r_f + w_F) = \int_{F} C^k_{F}v_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_{E} v_{E} r_F + \int_{F} v^c_{R,F} \cdot w_F \\
\forall (r_f, w_F) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F).
$$
Patch local spaces (no continuity between elements), project local (polynomial) operators on DOFs:

\[
\begin{align*}
\text{(DDR)}: & \quad \mathbb{R} & \quad \xrightarrow{X^k_{\text{grad},h}} & \quad X^k_{\text{curl},h} & \quad G_h^k & \quad \xrightarrow{X^k_{\text{div},h}} & \quad X^k_{\text{div},h} & \quad D_h^k & \quad \xrightarrow{P^k(T_h)} & \quad 0 & \quad \{0\} \\
& & \uparrow I_{\text{grad},h}^k & \quad \uparrow I_{\text{curl},h}^k & \quad \uparrow I_{\text{div},h}^k & \quad \uparrow \pi_{P,h}^k & & & & & \\
\text{(dR)}: & \quad \mathbb{R} & \quad \xleftarrow{H^1(\Omega)} & \quad \xrightarrow{\text{grad}} & \quad H(\text{curl}; \Omega) & \quad \xrightarrow{\text{curl}} & \quad H(\text{div}; \Omega) & \quad \xrightarrow{\text{div}} & \quad L^2(\Omega) & \quad 0 & \quad \{0\}
\end{align*}
\]

Properties for the design and analysis of stable numerical schemes:

- **Cohomology and polynomial consistency.**

- **Analytical properties:** Poincaré inequalities, primal and adjoint consistency, commutation properties of interpolators and differential operators, etc.
DDR scheme for Stokes in curl-curl formulation

- **Weak formulation**: Find \((u, p) \in H(\text{curl}; \Omega) \times H^1(\Omega)\) s.t. \(\int_{\Omega} p = 0\) and

\[
\int_{\Omega} \nu \text{curl} u \cdot \text{curl} v + \int_{\Omega} \text{grad} p \cdot v = \int_{\Omega} f \cdot v \quad \forall v \in H(\text{curl}; \Omega),
\]

\[
- \int_{\Omega} u \cdot \text{grad} q = 0 \quad \forall q \in H^1(\Omega)
\]

- **Set**

\[
X^k_{\text{grad},h,0} := \left\{ q_h \in X^k_{\text{grad},h} : (q_h, I^k_{\text{grad},h} 1)_{\text{grad},h} = 0 \right\}.
\]

- **DDR scheme**: Find \(u_h \in X^k_{\text{curl},h}\) and \(p_h \in X^k_{\text{grad},h,0}\) such that

\[
\nu(C^k_{h} u_h, C^k_{h} v_h)_{\text{div},h} + (G^k_{h} p_h, v_h)_{\text{curl},h} = (I^k_{\text{curl},h} f, v_h)_{\text{curl},h} \quad \forall v_h \in X^k_{\text{curl},h},
\]

\[
- (G^k_{h} q_h, u_h)_{\text{curl},h} = 0 \quad \forall q_h \in X^k_{\text{grad},h,0}.
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Key ideas to eliminate some DOFs

**Lemma**

Let $k \geq 0$, take $T$ a polyhedron with $\eta_T$ faces that are not pairwise parallel, and set $\ell_T = k + 1 - \eta_T$.

Any $q \in \mathcal{P}^{k+1}(T)$ is **entirely determined by** $(q|_F)_{F \in \mathcal{T}_T}$ and $\pi_{\mathcal{P},T}^{\ell_T}q$, and

\[
\|q\|_{L^2(T)} \lesssim \|\pi_{\mathcal{P},T}^{\ell_T}q\|_{L^2(T)} + h_T^{\frac{1}{2}} \sum_{F \in \mathcal{T}_T} \|q|_F\|_{L^2(F)}.
\]

- If $\eta_T > k + 1$, then $\pi_{\mathcal{P},T}^{\ell_T}q = 0$ (only face values are required).
- Also works with $T \sim F$ polygon.
Key ideas to eliminate some DOFs

Lemma

Let $k \geq 0$, take $T$ a polyhedron with $\eta_T$ faces that are not pairwise parallel, and set $\ell_T = k + 1 - \eta_T$. Any $q \in \mathcal{P}^{k+1}(T)$ is entirely determined by $(q|_F)_{F \in \mathcal{F}_T}$ and $\pi_{\mathcal{P},T}^{\ell_T}q$, and

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$$

- If $\eta_T > k + 1$, then $\pi_{\mathcal{P},T}^{\ell_T}q = 0$ (only face values are required).
- Also works with $T \sim F$ polygon.
- DDR spaces already fully encode traces on faces/elements boundaries $\sim$ polynomial consistency should not require that much information inside the faces/elements.
Serendipity DDR: degrees of freedom

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Note that $\ell_F < k - 1$ and $\ell_T < k - 1$. 
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<tr>
<td>$\widetilde{X}_k^{\text{curl},T}$</td>
<td>$\mathcal{P}^{k}(E)$</td>
<td>$\mathcal{R}^{k-1}(F) \oplus \mathcal{R}^{c,\ell_F+1}(F)$</td>
<td>$\mathcal{R}^{k-1}(T) \oplus \mathcal{R}^{c,\ell_T+1}(T)$</td>
<td></td>
</tr>
<tr>
<td>$X_k^{\text{div},T}$</td>
<td>$\mathcal{P}^{k}(F)$</td>
<td>$\mathcal{G}^{k-1}(T) \oplus \mathcal{G}^{c,k}(T)$</td>
<td>$\mathcal{P}^{k}(T)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}^{k}(T)$</td>
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<td>$\mathcal{P}^{k}(T)$</td>
<td></td>
</tr>
</tbody>
</table>

- Note that $\ell_F < k - 1$ and $\ell_T < k - 1$.
- Why can’t we reduce more (e.g. the $\mathcal{R}^{k-1}$ components, or $X_k^{\text{div},T}$)? Due to the constraints of preserving the complex properties...
### DDR vs. SDDR vs. Raviart-Thomas-Nédélec

<table>
<thead>
<tr>
<th>Discrete space</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^1(T)$</td>
<td>4 ♣ 4 ♣ 4 ♣ 4</td>
<td>15 ♣ 10 ♣ 10 ♣ 10</td>
<td>32 ♣ 20 ♣ 20 ♣ 20</td>
</tr>
<tr>
<td>$H(\text{curl}; T)$</td>
<td>6 ♣ 6 ♣ 6 ♣ 6</td>
<td>28 ♣ 23 ♣ 20 ♣ 20</td>
<td>65 ♣ 53 ♣ 45 ♣ 45</td>
</tr>
<tr>
<td>$H(\text{div}; T)$</td>
<td>4 ♣ 4 ♣ 4 ♣ 4</td>
<td>18 ♣ 18 ♣ 15 ♣ 15</td>
<td>44 ♣ 44 ♣ 36 ♣ 36</td>
</tr>
<tr>
<td>$L^2(T)$</td>
<td>1 ♣ 1 ♣ 1 ♣ 1</td>
<td>4 ♣ 4 ♣ 4 ♣ 4</td>
<td>10 ♣ 10 ♣ 10 ♣ 10</td>
</tr>
</tbody>
</table>

**Table:** Tetrahedron: dimensions of the local spaces in the DDR ♣ SDDR ♣ RTN.

<table>
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<th>$k = 0$</th>
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<th>$k = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^1(T)$</td>
<td>8 ♣ 8 ♣ 8 ♣ 8</td>
<td>27 ♣ 20 ♣ 27 ♣ 27</td>
<td>54 ♣ 32 ♣ 64 ♣ 64</td>
</tr>
<tr>
<td>$H(\text{curl}; T)$</td>
<td>12 ♣ 12 ♣ 12 ♣ 12</td>
<td>46 ♣ 39 ♣ 54 ♣ 54</td>
<td>99 ♣ 77 ♣ 144 ♣ 144</td>
</tr>
<tr>
<td>$H(\text{div}; T)$</td>
<td>6 ♣ 6 ♣ 6 ♣ 6</td>
<td>24 ♣ 24 ♣ 36 ♣ 36</td>
<td>56 ♣ 56 ♣ 108 ♣ 108</td>
</tr>
<tr>
<td>$L^2(T)$</td>
<td>1 ♣ 1 ♣ 1 ♣ 1</td>
<td>4 ♣ 4 ♣ 8 ♣ 8</td>
<td>10 ♣ 10 ♣ 27 ♣ 27</td>
</tr>
</tbody>
</table>

**Table:** Hexahedron: dimensions of the local spaces in the DDR ♣ SDDR ♣ RTN.
SDDR scheme: exactly as a DDR scheme, substituting spaces and operators with those above.
SDDR: analysis

(DDR): \[ \cdots \rightarrow X^k_{\text{grad}, h} \xrightarrow{G^k_h} X^k_{\text{curl}, h} \xrightarrow{C^k_h} X^k_{\text{div}, h} \xrightarrow{} \cdots \]

\[ \downarrow \quad \hat{R}_{\text{grad}, h} \quad \downarrow \quad \hat{R}_{\text{curl}, h} \quad \downarrow \quad \text{Id} \]

(SDDR): \[ \cdots \rightarrow \hat{X}^k_{\text{grad}, h} \xrightarrow{\hat{G}^k_h} \hat{X}^k_{\text{curl}, h} \xrightarrow{\hat{C}^k_h} X^k_{\text{div}, h} \xrightarrow{} \cdots \]

- Extensions and reduction link the two complexes.
Extensions and reduction link the two complexes.

- Designed to ensure transfer of homological and analytical properties: isomorphism of cohomologies, Poincaré inequalities, primal and adjoint consistency, etc.
Extensions and reduction link the two complexes.

- Designed to ensure transfer of homological and analytical properties: *isomorphism of cohomologies, Poincaré inequalities, primal and adjoint consistency, etc.*

- Generic blueprint that is applicable in many circumstances (*e.g.*: analysis of cohomology of various discrete complexes, etc.).
1 Motivation for continuous and discrete complexes

2 Finite Elements approach, and its limitations

3 Overview of the Discrete De Rham complexes
   - Principles guiding arbitrary-order polytopal complexes
   - DDR – regular version
   - DDR – serendipity version

4 Numerical illustration


**Theorem (Pressure-robust estimates [Beirão da Veiga et al., 2022])**

Setting the graph norms

\[
\begin{align*}
\| \cdot \|_{\text{curl},1,h}^2 & = \| \cdot \|_{\text{curl},h}^2 + \| C_h^k \cdot \|_{\text{div},h}^2 & \text{on } & X_h^{k}, \\
\| \cdot \|_{\text{grad},1,h}^2 & = \| \cdot \|_{\text{grad},h}^2 + \| G_h^k \cdot \|_{\text{curl},h} & \text{on } & X_h^{k},
\end{align*}
\]

we have:

\[
\| u_h - I_h^k u \|_{\text{curl},1,h} + \| p_h - I_h^k p \|_{\text{grad},1,h} \lesssim C_1(u) h^{k+1}.
\]

with \( C_1(u) \) depending \( u \) and some of its derivatives, but not \( p \).

Choice of discrete source term & commutation properties of DDR operators ensures pressure robustness...
Convergence test

Figure: Relative errors in discrete $\mathbf{H}(\text{curl}; \Omega) \times H^1(\Omega)$ norm vs. $h$: DDR (continuous lines) and SDDR (dashed lines).
Wall and CPU times on the finest meshes

$k = 2$

$k = 3$

Tetrahedral

Voronoi
Conclusion I

- DDR: discrete complex, of arbitrary degree of accuracy, applicable on polytopal meshes, yields stable schemes even for models with “incomplete” differential operators [Di Pietro et al., 2020][Di Pietro and Droniou, 2021a].

  Other approach: virtual element complexes [Beirão da Veiga et al., 2018][Beirão da Veiga et al., 2022]

- Systematic serendipity reduction of number of DOFs (on any polytopal mesh) [Di Pietro and Droniou, 2022b].

- Leaner complexes than FE approaches on certain meshes (and fully compatible with FE complexes on hybrid meshes).
Full set of homological and analytical results: cohomology, Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.  

(Facilitated by a generic framework to transfer properties from one complex to another one.)

Some other applications/complexes:

- div-div plates complex and serendipity version
  [Di Pietro and Droniou, 2022a][Botti et al., 2023].
- Magnetostatics equations [Di Pietro and Droniou, 2021b].
- Yang–Mills equations [Droniou et al., 2023].
- Stokes complex [Hanot, 2021].
- Rot-rot complex [Di Pietro, 2023].
Notes and series of introductory lectures to DDR:
https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

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COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- Introduction to Discrete De Rham complexes
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)
Thank you!
*Finite Element Exterior Calculus.*  
SIAM.

A family of three-dimensional virtual elements with applications to magnetostatics.  

Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes.  

A serendipity fully discrete div-div complex on polygonal meshes.  
*Comptes Rendus Mécanique*, 351 (S1).

An arbitrary-order discrete rot-rot complex on polygonal meshes with application to a quad-rot problem.  
accepted for publication.
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency.  
Published online. DOI: 10.1007/s10208-021-09542-8.

An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence.  

A fully discrete plates complex on polygonal meshes with application to the Kirchhoff–Love problem.  

Homological- and analytical-preserving serendipity framework for polytopal complexes, with application to the DDR method.  
Submitted.

Cohomology of the discrete de rham complex on domains of general topology.  
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.

A polyhedral discrete de rham numerical scheme for the yang–mills equations.

An arbitrary-order fully discrete Stokes complex on general polygonal meshes.
Extension: transferring homological and analytical properties between two complexes
**Starting point**: a complex with some analytical properties (including some polynomial consistency).

\[ P_{k_i} \quad \cdots \quad \cdots \quad \cdots \quad P_{k_{i+1}} \]

\[ X_i \quad \overset{d_i}{\longrightarrow} \quad X_{i+1} \quad \cdots \]

\[ I_i \quad \downarrow \quad I_{i+1} \]
Objective: link a second complex, to ensure that homological and analytical properties are also satisfied by this second complex.
Theorem

If the extensions and reductions satisfy

1. $\hat{R}_i E_i = \text{Id}$ for all $i$,

2. $(E_{i+1} \hat{R}_{i+1} - \text{Id})(\ker d_{i+1}) \subset \text{Im} d_i$ for all $i$,

3. $E$ and $\hat{R}$ are cochain maps (each one commutes with the differential operators),

then:

- the cohomologies of $(X_i, d_i)_i$ and $(\hat{X}_i, \hat{d}_i)_i$ are isomorphic.
Take $\mathcal{H}_i \supset \mathcal{P}^k_i$ (Sobolev-like space) and $(\cdot, \cdot)_i L^2$-like inner products on the first complex.
Theorem

If the extensions and reductions satisfy

- \( \hat{R}_i : X_i \to \hat{X}_i \) is continuous,
- (polynomial consistency) \( E_i \hat{R}_i I_i = I_i \),
- \( I_i : \mathcal{H}_i \to X_i \) is continuous,

then the following properties are transferred from \((X_i, d_i)_i\) to \((\hat{X}_i, \hat{d}_i)_i\):

- Poincaré inequalities,
- Consistency of the inner product (polynomial and in \(L^2\)-like norm on \(\mathcal{H}_i\)),
- Consistency of potential reconstruction,

and, under cochain map property:

- Commutation property and consistency of differential operators,
- Adjoint consistency (controls error in discrete IBP).
Construction of extensions and reductions for the SDDR complex to limits on reduction of number of DOFs.

Gives all the properties required for analysis of schemes based on SDDR:

- SDDR has the same cohomology as the de Rham complex.
- SDDR satisfies all analytical properties: Poincaré inequalities, primal and adjoint consistency, commutation properties of interpolator and differential operators...