Introduction to the Discrete De Rham complex

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(based on collaborations with D. Di Pietro, F. Rapetti, L. Beirão da Veiga, F. Dassi, S. Pitassi...)

Plan

The Stokes problem in curl-curl formulation

- Given Ω contractible, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:
  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: relevant Hilbert spaces:

$$
\begin{aligned}
H^1(\Omega) &:= \left\{q \in L^2(\Omega) \,:\, \mathbf{grad}\,q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3\right\},\\
\boldsymbol{H}(\mathbf{curl};\Omega) &:= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\right\},\\
\boldsymbol{H}(\mathrm{div};\Omega) &:= \left\{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \,:\, \mathrm{div}\,\boldsymbol{w} \in L^2(\Omega)\right\}
\end{aligned}
$$

The Stokes problem in curl-curl formulation

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\textstyle\int_\Omega p &= 0
\end{aligned}
$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
-\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

Sketch of stability analysis

$$\begin{aligned} \int_\Omega \nu \,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

1. Make $\boldsymbol{v} = \mathbf{grad}\,p \rightsquigarrow \|\mathbf{grad}\,p\| \le \|\boldsymbol{f}\|$ $\qquad$ ($\|\cdot\| = L^2$-norms).

Sketch of stability analysis

$$\int_\Omega \nu\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega),$$

$$-\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q = 0 \qquad \forall q \in H^1(\Omega)$$

1. Make $\boldsymbol{v} = \mathbf{grad}\, p \rightsquigarrow \|\mathbf{grad}\, p\| \leq \|\boldsymbol{f}\|$ $\quad$ ($\|\cdot\| = L^2$-norms).
2. Make $\boldsymbol{v} = \boldsymbol{u}$ and $q = p \rightsquigarrow \|\mathbf{curl}\, \boldsymbol{u}\|^2 \leq \nu^{-1} \|\boldsymbol{f}\| \|\boldsymbol{u}\|$.

Sketch of stability analysis

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3. Estimate $\|\boldsymbol{u}\|$:

Sketch of stability analysis

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3. Estimate $\|\boldsymbol{u}\|$:
   - Write $\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^\perp \in \ker \mathbf{curl} \oplus (\ker \mathbf{curl})^\perp$.

> Poincaré inequality: $\|\cdot\| \le C\|\operatorname{\mathbf{curl}} \cdot\|$ on $(\ker \mathbf{curl})^\perp$

   - So $\|\boldsymbol{u}^\perp\| \le C\|\operatorname{\mathbf{curl}} \boldsymbol{u}^\perp\| = C\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|$.

Sketch of stability analysis

$$\int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega),$$

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Poincaré inequality: $\|\cdot\| \le C\|\mathbf{curl} \cdot\|$ on $(\ker \mathbf{curl})^\perp$

- So $\|\boldsymbol{u}^\perp\| \le C\|\mathbf{curl}\, \boldsymbol{u}^\perp\| = C\|\mathbf{curl}\, \boldsymbol{u}\|$.

No tunnel in $\Omega \Rightarrow \ker \mathbf{curl} = \operatorname{Im} \mathbf{grad}$

- So $\boldsymbol{u}^0 = \mathbf{grad}\, q$ and thus $\|\boldsymbol{u}^0\| \le C\|\boldsymbol{u}^\perp\| \le C\|\mathbf{curl}\, \boldsymbol{u}\|$.
- Combine: $\|\boldsymbol{u}\| \le C\|\mathbf{curl}\, \boldsymbol{u}\|$.

De Rham complex

- The de Rham sequence is

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- It is a complex: the range of each operator is included in the kernel of the next one *(i.e.* $\mathbf{grad}\, i_\Omega = 0$, $\mathbf{curl}\,\mathbf{grad} = 0$, $\mathrm{div}\,\mathbf{curl} = 0$*)*.

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- It is exact (inclusions $\rightsquigarrow$ equalities) if Ω has a trivial topology:

$$\mathbb{R} = \ker \mathbf{grad}\,,\;\; \mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}\,,\;\; \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,,\;\; \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

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$$\mathbb{R} = \ker\mathbf{grad}\,,\ \boxed{\mathrm{Im}\,\mathbf{grad} = \ker\mathbf{curl}}\,,\ \mathrm{Im}\,\mathbf{curl} = \ker\mathrm{div}\,,\ \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

- Exactness $\Rightarrow$ well-posedness of the Stokes problem in curl–curl form *(same for the Stokes problem in Δ form...)*.

Reproducing this exactness at the discrete level is instrumental to designing stable numerical approximations.

Plan

- Let $T \subset \mathbb{R}^3$ be a tetrahedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) := \{\text{restrictions of 3-variate polynomials of degree} \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by $\boldsymbol{x}_T$ a point inside T,

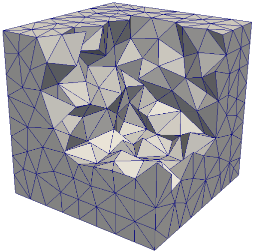
$$\begin{aligned}\mathcal{P}^k(T)^3 &= \overbrace{\mathbf{grad}\,\mathcal{P}^{k+1}(T)}^{\mathcal{G}^k(T)} \oplus \overbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathcal{P}^{k-1}(T)^3}^{\mathcal{G}^{\mathrm{c},k}(T)} \\ &= \underbrace{\mathbf{curl}\,\mathcal{P}^{k+1}(T)^3}_{\mathcal{R}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{k-1}(T)}_{\mathcal{R}^{\mathrm{c},k}(T)}\end{aligned}$$

- Define the trimmed spaces that sit between $\mathcal{P}^k(T)^3$ and $\mathcal{P}^{k+1}(T)^3$:

$$\mathcal{N}^{k+1}(T) := \mathcal{G}^k(T) \oplus \mathcal{G}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\mathcal{RT}^{k+1}(T) := \mathcal{R}^k(T) \oplus \mathcal{R}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

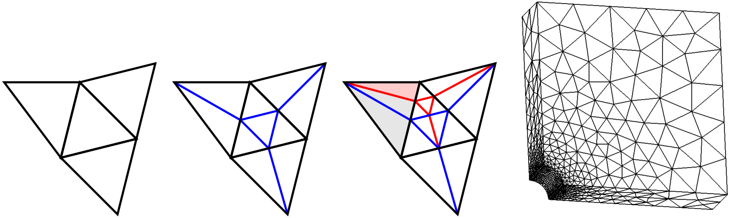
- See also [Arnold, 2018]



- Let $\mathcal{T}_h = \{T\}$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form a global FE complex:

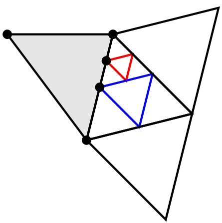
$$\begin{array}{ccccccccccc}
\mathbb{R} & \hookrightarrow & \mathcal{P}_c^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{N}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) & \xrightarrow{0} & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \hookrightarrow & H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \xrightarrow{0} & \{0\}
\end{array}$$

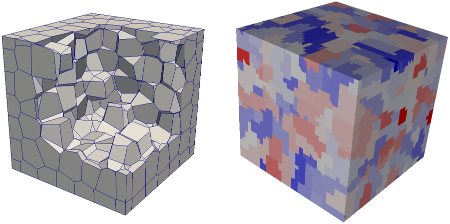
- **Gluing only works on conforming meshes!**



- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

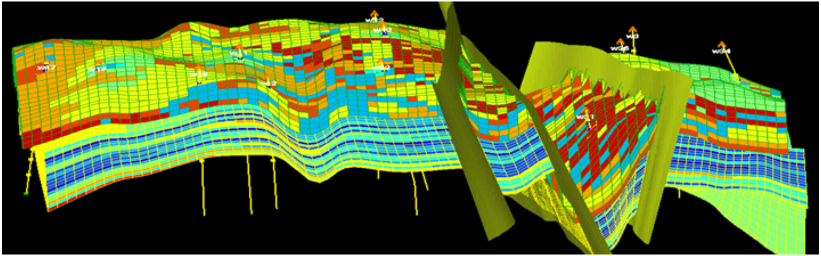
Polytopal meshes I





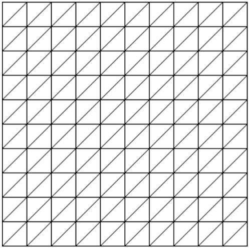
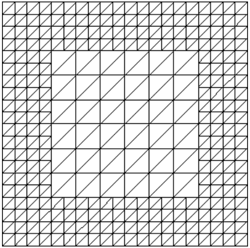
- Local refinement (to capture geometry or solution features) is seamless, and can preserve mesh regularity.
- Agglomerated elements are also easy to handle (and useful, e.g., in multi-grid methods).
- High-level approach can lead to leaner methods (fewer DOFs).

Polytopal meshes II



A practical example of efficency of a polytopal approach

Reissner–Mindlin plate problem.

Stabilised $\mathcal{P}^2$-($\mathcal{P}^1 + \mathcal{B}^3$) scheme		DDR scheme	
			
nb. DOFs	Error	nb. DOFs	Error
2403	0.138	550	0.161
9603	6.82e-2	2121	6.77e-2
38402	3.40e-2	8329	3.1e-2

High-level approach of polytopal methods: gains in DOFs

deg.	Disc. H^1	Disc. $\boldsymbol{H}(\mathbf{curl};\Omega)$	Disc. $\boldsymbol{H}(\mathrm{div};\Omega)$	Disc. L^2
0	4 (4)	6 (6)	4 (4)	1 (1)
1	15 (10)	28 (20)	18 (15)	4 (4)
2	32 (20)	65 (45)	44 (36)	10 (10)

Table: Comparisons of DOFs between DDR and Raviart–Thomas–Nédélec (in brackets) spaces: tetrahedras.

deg.	Disc. H^1	Disc. $\boldsymbol{H}(\mathbf{curl};\Omega)$	Disc. $\boldsymbol{H}(\mathrm{div};\Omega)$	Disc. L^2
0	8 (8)	12 (12)	6 (6)	1 (1)
1	27 (27)	46 (54)	24 (36)	4 (8)
2	54 (64)	99 (144)	56 (108)	10 (27)

Table: Comparisons of DOFs between DDR and Raviart–Thomas–Nédélec (in brackets) spaces: hexahedras.

Can actually be reduced further by serendipity DDR.

Plan

Plan

The 2D de Rham complex

2D de Rham complex on a domain $\Omega \subset \mathbb{R}^2$:

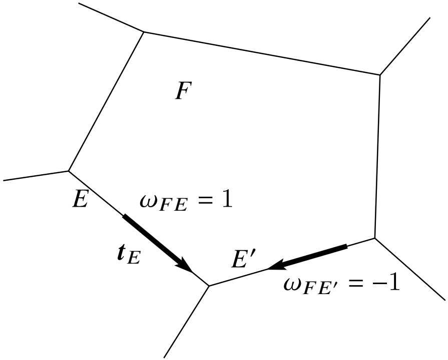
$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathrm{rot};\Omega) \xrightarrow{\mathrm{rot}} L^2(\Omega) \xrightarrow{0} \{0\}$$

where

$$\begin{aligned} \mathrm{rot}\,\boldsymbol{v} &= \mathrm{div}(\rho_{-\pi/2}\boldsymbol{v}), \\ \boldsymbol{H}(\mathrm{rot};\Omega) &= \{\boldsymbol{v} \in L^2(\Omega)^2 \ : \mathrm{rot}\,\boldsymbol{v} \in L^2(\Omega)\}. \end{aligned}$$

Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of faces/edges/vertices (edges are oriented).



- If $r \in \mathcal{P}^1$ and $E \in \mathcal{E}_h$,

$$\mathbf{grad}\, r \cdot \boldsymbol{t}_E = \frac{r(\boldsymbol{x}_{V_2}) - r(\boldsymbol{x}_{V_1})}{|E|}$$

with V_1, V_2 vertices of E oriented in the direction $\boldsymbol{t}_E$.

$\mathcal{P}^1$-consistent approximation of differential operators

Gradient

- If $r \in \mathcal{P}^1$ and $E \in \mathcal{E}_h$,

$$\mathbf{grad}\, r \cdot \boldsymbol{t}_E = \frac{r(\boldsymbol{x}_{V_2}) - r(\boldsymbol{x}_{V_1})}{|E|}$$

  with V_1, V_2 vertices of E oriented in the direction $\boldsymbol{t}_E$.

- Discrete $H^1(\Omega)$ space:

$$\underline{X}^0_{\mathbf{grad},h} = \left\{ \underline{q}_h = (q_V)_{V \in \mathcal{V}_h} \ : \ q_V \in \mathbb{R} \right\}$$

  and interpolator $\underline{I}^0_{\mathbf{grad},h} : C^0(\overline{\Omega}) \to \underline{X}^0_{\mathbf{grad},h}$ s.t.

$$\underline{I}^0_{\mathbf{grad},h} f = (f(\boldsymbol{x}_V))_{V \in \mathcal{V}_h}.$$

- If $r \in \mathcal{P}^1$ and $E \in \mathcal{E}_h$,

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$$\underline{I}^0_{\mathbf{grad},h} f = (f(\boldsymbol{x}_V))_{V \in \mathcal{V}_h}.$$

- Discrete gradient: $\underline{\boldsymbol{G}}^0_h = (G^0_E)_{E \in \mathcal{E}_h}$ with $G^0_E : \underline{X}^0_{\mathbf{grad},h} \to \mathbb{R}$ s.t.

$$G^0_E \underline{q}_h = \frac{q_{V_2} - q_{V_1}}{|E|}.$$

- If $\boldsymbol{v} \in (\mathcal{P}^1)^2$ and $F \in \mathcal{F}_h$,

$$\int_F \operatorname{rot} \boldsymbol{v} = -\sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{v} \cdot \boldsymbol{t}_E.$$

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- Discrete $\boldsymbol{H}(\mathrm{rot};\Omega)$ space:

$$\underline{\boldsymbol{X}}^0_{\mathrm{rot},h} = \left\{ \underline{\boldsymbol{v}}_h = (v_E)_{E \in \mathcal{E}_h} \ : \ v_E \in \mathbb{R} \right\}$$

and interpolator $\underline{\boldsymbol{I}}^0_{\mathrm{rot},h} : C^0(\overline{\Omega})^2 \to \underline{\boldsymbol{X}}^0_{\mathrm{rot},h}$ s.t.

$$\underline{\boldsymbol{I}}^0_{\mathrm{rot},h} \boldsymbol{f} = (\boldsymbol{\pi}^0_{\mathcal{P},E}(\boldsymbol{f} \cdot \boldsymbol{t}_E))_{E \in \mathcal{V}_h}.$$

with $\boldsymbol{\pi}^0_{\mathcal{P},E}$ orthogonal projector onto $\mathbb{P}^0(E)$ (*i.e., average*).

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$$\underline{\boldsymbol{I}}^0_{\mathrm{rot},h}\boldsymbol{f} = (\boldsymbol{\pi}^0_{\mathcal{P},E}(\boldsymbol{f}\cdot\boldsymbol{t}_E))_{E\in\mathcal{V}_h}.$$

with $\boldsymbol{\pi}^0_{\mathcal{P},E}$ orthogonal projector onto $\mathbb{P}^0(E)$ (*i.e., average*).

- Discrete rot: $\underline{\boldsymbol{C}}^0_h = (C^0_F)_{F\in\mathcal{F}_h}$ with $C^0_F : \underline{\boldsymbol{X}}^0_{\mathrm{rot},h} \to \mathbb{R}$ s.t.

$$C^0_F\underline{\boldsymbol{v}}_h = -\frac{1}{|F|}\sum_{E\in\mathcal{E}_F} \omega_{FE}|E|v_E.$$

Discrete De Rham complex in 2D

Continuous de Rham:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathrm{rot};\Omega) \xrightarrow{\mathrm{rot}} L^2(\Omega) \xrightarrow{0} \{0\}$$

Discrete De Rham:

$$\mathbb{R} \xrightarrow{\underline{\boldsymbol{I}}^0_{\mathbf{grad},h}} \underline{X}^0_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^0_h} \underline{\boldsymbol{X}}^0_{\mathrm{rot},h} \xrightarrow{\underline{\boldsymbol{C}}^0_h} \mathcal{P}^0(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

where $\mathcal{P}^0(\mathcal{T}_h)$ space of piecewise constant functions on $\mathcal{F}_h$.

Discrete De Rham complex in 2D

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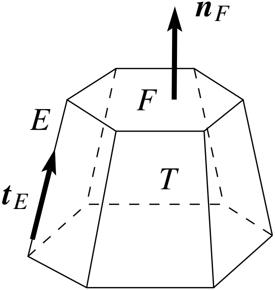
where $\mathcal{P}^0(\mathcal{T}_h)$ space of piecewise constant functions on $\mathcal{F}_h$.

- This is a complex.
- If Ω is connected without holes, it is exact.

Plan

Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ of elements/faces/edges/vertices, with intrinsic orientations (tangent, normal).
  - $\omega_{TF} \in \{+1, -1\}$ such that $\omega_{TF}\boldsymbol{n}_F$ outer normal to T.
  - $\omega_{FE} \in \{+1, -1\}$ such that $\omega_{FE}\boldsymbol{t}_E$ clockwise on F.



$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ and $r \in \mathcal{P}^k(E)$,

$$\int_E q' r = -\int_E q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1})$$

with derivatives in the direction $\boldsymbol{t}_E$.

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ and $r \in \mathcal{P}^k(E)$,

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1})$$

with $\pi_{\mathcal{P},E}^{k-1}$ the L^2-projection on $\mathcal{P}^{k-1}(E)$.

$\mathcal{P}^k$-consistent gradient
Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ and $r \in \mathcal{P}^k(E)$,

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1})$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \, : \, q_E \in \mathcal{P}^{k-1}(E), \; q_V \in \mathbb{R} \right\},$$

$$\underline{I}_{\mathbf{grad},E}^k q = (\pi_{\mathcal{P},E}^{k-1} q, (q(\boldsymbol{x}_V))_{V \in \mathcal{V}_E}) \qquad \forall q \in C(\overline{E}).$$

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ and $r \in \mathcal{P}^k(E)$,

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1})$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \,:\, q_E \in \mathcal{P}^{k-1}(E)\,,\ q_V \in \mathbb{R} \right\},$$

- Edge gradient $G_E^k : \underline{X}_{\mathbf{grad},E}^k \to \mathcal{P}^k(E)$ s.t., for all $r \in \mathcal{P}^k(E)$,

$$\int_E G_E^k \underline{q}_E r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}).$$

$\mathcal{P}^k$-consistent gradient

Edge E

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(E)$ and $r \in \mathcal{P}^k(E)$,

$$\int_E q' r = -\int_E \pi_{\mathcal{P},E}^{k-1} q \underbrace{r'}_{\in \mathcal{P}^{k-1}(E)} + q(\boldsymbol{x}_{V_2}) r(\boldsymbol{x}_{V_2}) - q(\boldsymbol{x}_{V_1}) r(\boldsymbol{x}_{V_1})$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},E}^k = \left\{ \underline{q}_E = (q_E, (q_V)_{V \in \mathcal{V}_E}) \,:\, q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \right\},$$

- Edge gradient $G_E^k : \underline{X}_{\mathbf{grad},E}^k \to \mathcal{P}^k(E)$ s.t., for all $r \in \mathcal{P}^k(E)$,

$$\int_E G_E^k \underline{q}_E r = -\int_E q_E r' + q_{V_2} r(\boldsymbol{x}_{V_2}) - q_{V_1} r(\boldsymbol{x}_{V_1}).$$

- Potential reconstruction $\gamma_E^{k+1} : \underline{X}_{\mathbf{grad},E}^k \to \mathcal{P}^{k+1}(E)$ s.t., for all $z \in \mathcal{P}^{k+2}(E)$ with $\int_E z = 0$,

$$\int_E \gamma_E^{k+1} \underline{q}_E z' = -\int_E G_E^k \underline{q}_E z' + q_{V_2} z(\boldsymbol{x}_{V_2}) - q_{V_1} z(\boldsymbol{x}_{V_1}).$$

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$ and $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\mathrm{div}_F\, \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$ and $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q\boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$ and $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi^{k-1}_{\mathcal{P},F} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

- Space and interpolator:

$$\underline{X}^k_{\mathbf{grad},F} = \Big\{ \underline{q}_F = (q_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) \,:\, q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \Big\},$$

$$\underline{I}^k_{\mathbf{grad},F} q = (\pi^{k-1}_{\mathcal{P},F} q, (\pi^{k-1}_{\mathcal{P},E} q_{|E})_{E \in \mathcal{E}_F}, (q(\boldsymbol{x}_V))_{V \in \mathcal{V}_F}) \qquad \forall q \in C(\overline{F}).$$

$\mathcal{P}^k$-consistent gradient

Face F

- IBP is the starting point: if $q \in \mathcal{P}^{k+1}(F)$ and $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F \pi_{\mathcal{P},F}^{k-1} q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},F}^k = \Big\{ \underline{q}_F = (q_F, (q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F}) \ : \\ q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R} \Big\},$$

- Face gradient $\boldsymbol{G}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^k(F)^2$ s.t., for all $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \boldsymbol{G}_F^k \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E \boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

$\mathcal{P}^k$-consistent gradient

Face F

- Space and interpolator:

$$\underline{X}_{\mathbf{grad},F}^k = \Big\{\underline{q}_F = (q_F, (q_E)_{E\in\mathcal{E}_F}, (q_V)_{V\in\mathcal{V}_F}) \ : \ q_F \in \mathcal{P}^{k-1}(F),\ q_E \in \mathcal{P}^{k-1}(E),\ q_V \in \mathbb{R}\Big\},$$

- Face gradient $\boldsymbol{G}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^k(F)^2$ s.t., for all $\boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\int_F \boldsymbol{G}_F^k \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E \boldsymbol{v} \cdot \boldsymbol{n}_{FE}.$$

- Potential reconstruction $\gamma_F^{k+1} : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^{k+1}(F)$ s.t., for all $\boldsymbol{z} \in \mathcal{R}^{\mathrm{c},k+2}(F) := (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k+1}(F)$,

$$\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \boldsymbol{z} = -\int_F \boldsymbol{G}_F^k \underline{q}_F \cdot \boldsymbol{z} + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_F \gamma_E^{k+1} \underline{q}_E \boldsymbol{z} \cdot \boldsymbol{n}_{FE}$$

($\operatorname{div}_F : \mathcal{R}^{\mathrm{c},k+2}(F) \to \mathcal{P}^{k+1}(F)$ *is an isomorphism.*)

$\mathcal{P}^k$-consistent gradient

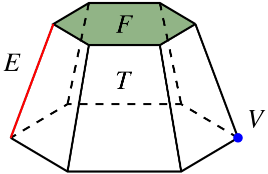
Element T

Same principle! Based on IBP we determine:

- An additional unknown ($q_T \in \mathcal{P}^{k-1}(T)$) to get the space $\underline{X}^k_{\mathbf{grad},T}$, and its meaning to get the interpolator $\underline{I}^k_{\mathbf{grad},T}$.
- A formula for the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3$.
- A potential reconstruction $P^{k+1}_{\mathbf{grad},T} : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T)$.

Principles of the Discrete de Rham method

- Contrary to FE, do not seek explicit (or any!) basis functions.
- Replace continuous spaces by fully discrete ones made of vectors of polynomials,
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Create discrete operators between them.



DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- For $\mathsf{P} = F, T$, set

$$\mathcal{R}^{k-1}(\mathsf{P}) = \begin{cases} \mathbf{rot}\,\mathcal{P}^k(F)^2 \\ \mathbf{curl}\,\mathcal{P}^k(T)^3 \end{cases}, \qquad \mathcal{R}^{\mathrm{c},k}(\mathsf{P}) = (\boldsymbol{x} - \boldsymbol{x}_{\mathsf{P}})\mathcal{P}^{k-1}(\mathsf{P}).$$

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big) :\\ \boldsymbol{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} \in \mathcal{R}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h,\\ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \mathcal{R}^{\mathrm{c},k}(F) \text{ for all } F \in \mathcal{F}_h,\\ \text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h\Big\},\end{aligned}$$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- For $\mathsf{P} = F, T$, set

$$\mathcal{R}^{k-1}(\mathsf{P}) = \begin{cases} \mathbf{rot}\,\mathcal{P}^k(F)^2 \\ \mathbf{curl}\,\mathcal{P}^k(T)^3 \end{cases}, \qquad \mathcal{R}^{\mathrm{c},k}(\mathsf{P}) = (\boldsymbol{x} - \boldsymbol{x}_{\mathsf{P}})\mathcal{P}^{k-1}(\mathsf{P}).$$

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \coloneqq \Big\{\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big) \;:\; \\ \boldsymbol{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} \in \mathcal{R}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h, \\ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \mathcal{R}^{\mathrm{c},k}(F) \text{ for all } F \in \mathcal{F}_h, \\ \text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h\Big\},\end{aligned}$$

- Interpolator: $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = \underline{\boldsymbol{v}}_h$ with

$$\begin{aligned} v_E &= L^2\text{-projection on } \mathcal{P}^k(E) \text{ of } \boldsymbol{v}\cdot\boldsymbol{t}_E, \\ \boldsymbol{v}_{\mathcal{R},F},\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} &= L^2\text{-projections on } \mathcal{R}^{k-1}(F),\ \mathcal{R}^{\mathrm{c},k}(F) \text{ of } \boldsymbol{v}_{\mathrm{t},F}, \\ \boldsymbol{v}_{\mathcal{R},T},\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} &= L^2\text{-projections on } \mathcal{R}^{k-1}(T),\ \mathcal{R}^{\mathrm{c},k}(T) \text{ of } \boldsymbol{v}. \end{aligned}$$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, the discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space

- For $\mathsf{P} = F, T$, set

$$\mathcal{R}^{k-1}(\mathsf{P}) = \begin{cases} \mathbf{rot}\,\mathcal{P}^k(F)^2 \\ \mathbf{curl}\,\mathcal{P}^k(T)^3 \end{cases}, \qquad \mathcal{R}^{\mathrm{c},k}(\mathsf{P}) = (\boldsymbol{x}-\boldsymbol{x}_{\mathsf{P}})\mathcal{P}^{k-1}(\mathsf{P}).$$

- Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{\underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big) :\\ \boldsymbol{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} \in \mathcal{R}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h,\\ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \mathcal{R}^{\mathrm{c},k}(F) \text{ for all } F \in \mathcal{F}_h,\\ \text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h\Big\},\end{aligned}$$

- Potential reconstructions for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$:
  - tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$,
  - element potential $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3$.

Discrete gradient

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete gradient: project the face/element/edge gradients

$$\mathsf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3, \qquad \mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2,$$
$$G^k_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$$

onto the proper spaces.

Discrete gradient

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete gradient: project the face/element/edge gradients

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$$G^k_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$$

onto the proper spaces. A vector $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ is

$$\underline{\boldsymbol{v}}_h = \Big((\underbrace{\boldsymbol{v}_{\mathcal{R},T}}_{\in \mathcal{R}^{k-1}(T)} , \underbrace{\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T}}_{\in \mathcal{R}^{\mathrm{c},k}(T)})_{T\in\mathcal{T}_h}, (\underbrace{\boldsymbol{v}_{\mathcal{R},F}}_{\in \mathcal{R}^{k-1}(F)} , \underbrace{\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}}_{\in \mathcal{R}^{\mathrm{c},k}(F)})_{F\in\mathcal{F}_h}, (\underbrace{v_E}_{\in \mathcal{P}^k(E)})_{E\in\mathcal{E}_h} \Big)$$

Discrete gradient

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Discrete gradient: project the face/element/edge gradients

$$\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^k(T)^3, \qquad \mathbf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2,$$
$$G^k_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$$

onto the proper spaces. A vector $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ is

$$\underline{\boldsymbol{v}}_h = \Big((\underbrace{\boldsymbol{v}_{\mathcal{R},T}}_{\in \mathcal{R}^{k-1}(T)} , \underbrace{\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T}}_{\in \mathcal{R}^{\mathrm{c},k}(T)})_{T\in\mathcal{T}_h}, (\underbrace{\boldsymbol{v}_{\mathcal{R},F}}_{\in \mathcal{R}^{k-1}(F)} , \underbrace{\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}}_{\in \mathcal{R}^{\mathrm{c},k}(F)})_{F\in\mathcal{F}_h}, (\underbrace{v_E}_{\in \mathcal{P}^k(E)})_{E\in\mathcal{E}_h} \Big)$$

So:

$$\underline{\boldsymbol{G}}^k_h \underline{q}_h = \Big((\boldsymbol{\pi}^{k-1}_{\mathcal{R},T} \mathbf{G}^k_T \underline{q}_T, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T} \mathbf{G}^k_T \underline{q}_T)_{T\in\mathcal{T}_h},$$
$$(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F} \mathbf{G}^k_F \underline{q}_F, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F} \mathbf{G}^k_F \underline{q}_F)_{F\in\mathcal{F}_h}, (G^k_E \underline{q}_E)_{E\in\mathcal{E}_h} q \Big).$$

Plan

Algebraic properties

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

- It is a complex.

Algebraic properties

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

- It is a complex.
- If Ω has a trivial topology, it is exact [Di Pietro et al., 2020, Di Pietro and Droniou, 2021a].

Algebraic properties

DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

- It is a complex.
- If Ω has a trivial topology, it is exact [Di Pietro et al., 2020, Di Pietro and Droniou, 2021a].
- For a general Ω, it has the same cohomology as the continuous de Rham complex [Di Pietro et al., 2022].

L^2-like inner products

- Local L^2-like inner product on the DDR spaces: for $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ or (div, k),

$$(x_T, y_T)_{\bullet,T} = \int_T \boldsymbol{P}^{\ell}_{\bullet,T} x_T \cdot \boldsymbol{P}^{\ell}_{\bullet,T} y_T + \mathrm{s}_{\bullet,T}(x_T, y_T) \qquad \forall x_T, y_T \in \underline{X}^k_{\bullet,T},$$

  ($\mathrm{s}_{\bullet,T}$ *penalises differences on the boundary between element and face potentials*).
- Global L^2-like product by standard assembly of local ones.

$\rightsquigarrow$ *schemes for PDEs by replacing continuous spaces/operators/inner products with the DDR spaces/operators/inner products.*

Analytical properties

$$
\begin{array}{ccccccc}
C^0(\overline{\Omega}) & & H^2(\Omega)^3 & & H^1(\Omega)^3 & & \\
\Big\downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{grad},h}} & & \Big\downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}} & & \Big\downarrow{\scriptstyle \underline{\boldsymbol{I}}^k_{\mathrm{div},h}} & & \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h) \\
(\cdot,\cdot)_{\mathbf{grad},h} & & (\cdot,\cdot)_{\mathbf{curl},h} & & (\cdot,\cdot)_{\mathrm{div},h} & & (\cdot,\cdot)_{L^2}
\end{array}
$$

- For stability, Poincaré inequalities: control of $x_h \in (\ker \mathrm{d}^k_h)^\perp$ by $\mathrm{d}^k_h x_h$ (for $\mathrm{d}^k_h = \underline{\boldsymbol{G}}^k_h, \underline{\boldsymbol{C}}^k_h, D^k_h$).
- For consistency:
  - Primal consistency for potentials: approximation properties of $\boldsymbol{P}^\ell_{\bullet,h} \circ \underline{I}^k_{\bullet,h}$; gives consistency of discrete L^2-inner products.
  - Primal consistency for differential operators: approximation properties of $\mathrm{d}^k_h \circ \underline{I}^k_{\bullet,h}$ (comes from commutation properties).
  - Adjoint consistency: control error in discrete integration-by-parts.

Poincaré inequalities

Notation: $a \lesssim b$ if $a \leq Cb$ for C only depending on k and the mesh regularity factor.

Theorem (Poincaré inequality for D_h^k and $\underline{\boldsymbol{C}}_h^k$ [Di Pietro and Droniou, 2021b, Di Pietro and Droniou, 2021a])

It holds:

$$\|\underline{q}_h\|_{\mathbf{grad},h} \lesssim \|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h} \qquad \forall \underline{q}_h \in (\operatorname{Ker} \underline{\boldsymbol{G}}_h^k)^\perp,$$

$$\|\underline{\boldsymbol{w}}_h\|_{\operatorname{div},h} \lesssim \|D_h^k \underline{\boldsymbol{w}}_h\|_{L^2(\Omega)} \quad \forall \underline{\boldsymbol{w}}_h \in (\operatorname{Ker} D_h^k)^\perp,$$

and, if Ω is simply connected and does not enclose any void,

$$\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\zeta}}_h\|_{\operatorname{div},h} \quad \forall \underline{\boldsymbol{\zeta}}_h \in (\operatorname{Ker} \underline{\boldsymbol{C}}_h^k)^\perp.$$

- Essential to use the complex exactness to get stability of numerical discretisations.

Primal consistency

Theorem (Consistency of potential reconstruction and stabilisation [Di Pietro and Droniou, 2021a])

It holds, for $(\bullet, \ell) = (\mathbf{grad}, k+1)$, $(\mathbf{curl}, k)$ *or* (div, k),

$$\|\boldsymbol{P}^{\ell}_{\bullet,T}\underline{I}^{k}_{\bullet,T} f - f\|_{L^2(T)} + \mathrm{s}_{\bullet,T}(\underline{I}^{k}_{\bullet,T} f, \underline{I}^{k}_{\bullet,T} f) \lesssim h_T^{\ell+1} |f|_{H^{\ell+1}(T)} \qquad \forall f \in H^{\ell+1}(T)$$

(caveat for $\bullet = \mathbf{curl}$*).*

- Comes from local polynomial consistency: $\boldsymbol{P}^{\ell}_{\bullet,T}\underline{I}^{k}_{\bullet,T} x_T = x_T$ and $\mathrm{s}_{\bullet,T}(\underline{I}^{k}_{\bullet,T} x_T, \cdot) = 0$ if $x_T \in \mathcal{P}^{\ell}(T)$.
- Gives consistency of discrete L^2 inner products.

Commutation properties

Theorem (Commutation properties of differential operators [Di Pietro and Droniou, 2021a])

We have

$$
\begin{aligned}
\underline{\boldsymbol{G}}_h^k(\underline{I}_{\mathbf{grad},h}^k r) &= \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k(\mathbf{grad}\, r) &\qquad \forall r \in C^1(\overline{\Omega}),\\
\underline{\boldsymbol{C}}_h^k(\underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{\tau}) &= \underline{\boldsymbol{I}}_{\mathrm{div},h}^k(\mathbf{curl}\, \boldsymbol{\tau}) &\qquad \forall \boldsymbol{\tau} \in \boldsymbol{H}^2(\Omega)^3,\\
D_h^k(\underline{\boldsymbol{I}}_{\mathrm{div},h}^k \boldsymbol{w}) &= \pi_{\mathcal{P},h}^k(\mathrm{div}\, \boldsymbol{w}) &\qquad \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)^3.
\end{aligned}
$$

$$
\begin{array}{ccc}
C^1(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^0(\overline{\Omega})\\
\Big\downarrow \underline{I}_{\mathbf{grad},h}^k & & \Big\downarrow \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k\\
\underline{X}_{\mathbf{grad},h}^k & \xrightarrow{\underline{\boldsymbol{G}}_h^k} & \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k
\end{array}
$$

Commutation properties

Theorem (Commutation properties of differential operators [Di Pietro and Droniou, 2021a])

We have

$$\begin{aligned}
\underline{\boldsymbol{G}}_h^k(\underline{I}_{\mathbf{grad},h}^k r) &= \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k(\mathbf{grad}\, r) && \forall r \in C^1(\overline{\Omega}),\\
\underline{\boldsymbol{C}}_h^k(\underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{\tau}) &= \underline{\boldsymbol{I}}_{\mathrm{div},h}^k(\mathbf{curl}\, \boldsymbol{\tau}) && \forall \boldsymbol{\tau} \in \boldsymbol{H}^2(\Omega)^3,\\
D_h^k(\underline{\boldsymbol{I}}_{\mathrm{div},h}^k \boldsymbol{w}) &= \pi_{\mathcal{P},h}^k(\mathrm{div}\, \boldsymbol{w}) && \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)^3.
\end{aligned}$$

- Together with the consistency of potential reconstruction, provides optimal approximation properties of the differential operators.
- Essential for robust approximations (e.g. pressure-robust for Stokes, locking-free for Reissner-Mindlin...).

Adjoint consistency

Theorem (Adjoint consistency for the discrete gradient [Di Pietro and Droniou, 2021a])

For all $\boldsymbol{v} \in C^0(\overline{\Omega}) \cap \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \boldsymbol{H}^{\max(k+1,2)}(\mathcal{T}_h)$ *and* $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$\left| (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v}, \underline{\boldsymbol{G}}^k_h \underline{q}_h)_{\mathbf{curl},h} + \int_\Omega \mathrm{div}\,\boldsymbol{v}\ P^{k+1}_{\mathbf{grad},h}\underline{q}_h \right| \lesssim h^{k+1} |\boldsymbol{v}|_{\boldsymbol{H}^{(k+1,2)}(\mathcal{T}_h)} \|\underline{\boldsymbol{G}}^k_h \underline{q}_h\|_{\mathbf{curl},h}.$$

- Similar adjoint consistencies for the curl, divergence.
- Essential for error estimates when IBP are involved in the weak formulations.

Plan

Discrete weak curl–curl formulations of Stokes

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} \int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

Discrete weak curl–curl formulations of Stokes

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- Set

$$\underline{X}^k_{\mathbf{grad},h,0} := \left\{ \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \;:\; (\underline{q}_h, \underline{I}^k_{\mathbf{grad},h} 1)_{\mathbf{grad},h} = 0 \right\}.$$

- DDR scheme: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{p}_h \in \underline{X}^k_{\mathbf{grad},h,0}$ such that

$$\begin{aligned} \nu(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\ -(\underline{\boldsymbol{G}}^k_h \underline{q}_h, \underline{\boldsymbol{u}}_h)_{\mathbf{curl},h} &= 0 && \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}. \end{aligned}$$

Choice of discrete source term & commutation properties of DDR operators ensures pressure robustness...

Error estimates

Theorem (Pressure-robust estimates [Beirão da Veiga et al., 2022])

Setting the graph norms

$$\|\cdot\|_{\mathbf{curl},1,h}^2 = \|\cdot\|_{\mathbf{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k \cdot\|_{\mathrm{div},h}^2 \quad \text{on } \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k,$$
$$\|\cdot\|_{\mathbf{grad},1,h}^2 = \|\cdot\|_{\mathbf{grad},h}^2 + \|\underline{\boldsymbol{G}}_h^k \cdot\|_{\mathbf{curl},h} \quad \text{on } \underline{X}_{\mathbf{grad},h}^k,$$

we have:

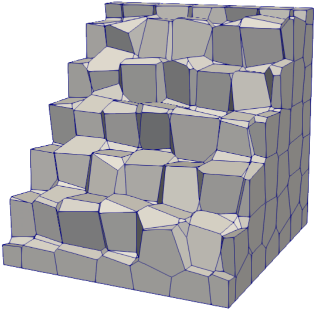
$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},1,h} + \|\underline{p}_h - \underline{I}_{\mathbf{grad},h}^k p\|_{\mathbf{grad},1,h} \lesssim C_1(\boldsymbol{u}) h^{k+1}.$$

with $C_1(\boldsymbol{u})$ depending $\boldsymbol{u}$ and some of its derivatives, but not p.

Plan

Setting I

- $\Omega = (0, 1)^3$.
- Voronoi mesh families (similar results on tetrahedral meshes):



(a) Voronoi mesh

Setting II

- Exact solution: for some $\lambda \geq 0$,

$$p(x,y,z) = \lambda \sin(2\pi x)\sin(2\pi y)\sin(2\pi z)\,,$$
$$\boldsymbol{u}(x,y,z) = \begin{bmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) \\ \frac{1}{2}\cos(2\pi x)\sin(2\pi y)\cos(2\pi z) \\ -\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) \end{bmatrix}.$$

- Measured errors:
  - Discrete norms as in the theorem:

$$E_{\boldsymbol{u}}^{\rm d} = \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},1,h}\,,$$
$$E_p^{\rm d} = \|\underline{p}_h - \underline{I}_{\mathbf{grad},h}^k p\|_{\mathbf{grad},1,h}.$$

  - Continuous norms between reconstructed potentials and solutions:

$$E_{\boldsymbol{u}}^{\rm c} = \|\boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{L^2(\Omega)} + \|\mathbf{C}_h^k \underline{\boldsymbol{u}}_h - \mathbf{curl}\,\boldsymbol{u}\|_{L^2(\Omega)}\,,$$
$$E_p^{\rm c} = \|\mathbf{G}_h^k \underline{p}_h - \mathbf{grad}\,p\|_{L^2(\Omega)}.$$

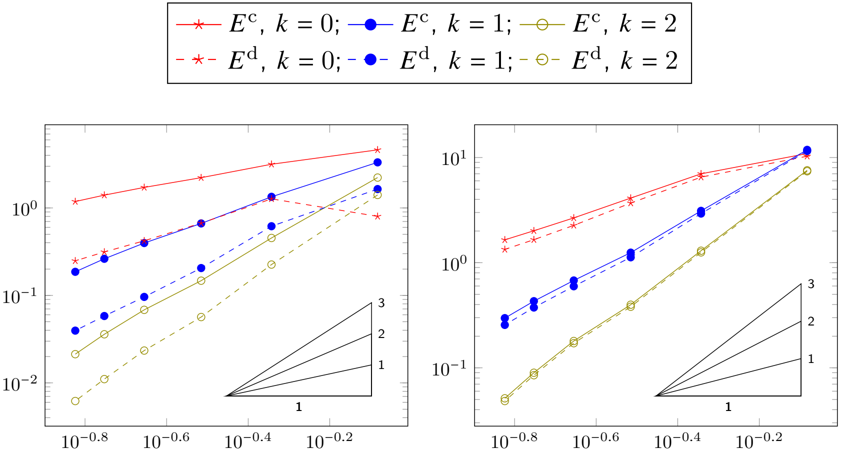
Setting III

- Implementation in the `HArDCore3D` library[1], using the Intel MKL Pardiso solver[2].

  The HArDCore3D library also includes a serendipity version for DDR [Di Pietro and Droniou, 2022b], which leads to a reduction of more than 50% of the solving time.

[1] https://github.com/jdroniou/HArDCore
[2] https://software.intel.com/en-us/mkl

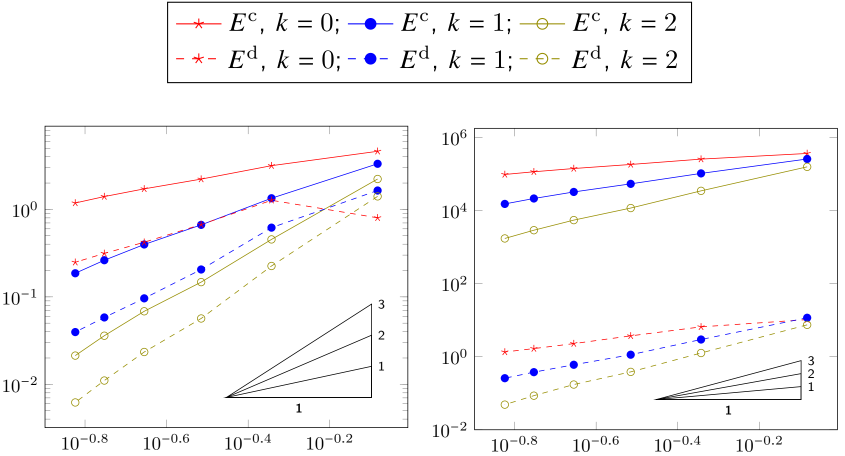
Results; $\lambda = 1$



(a) Errors on $\boldsymbol{u}$

(b) Errors on p

Results; $\lambda = 10^5$



(a) Errors on $\boldsymbol{u}$

(b) Errors on p

Conclusion: Discrete De Rham

- Discrete exact sequences yield stable schemes even for models with "incomplete" differential operators.
- Support of polytopal meshes and arbitrary degree of accuracy.
- Full set of analysis results: Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.
- Systematic serendipity reduction of number of DOFs (on any polytopal mesh).
- Analysis of cohomology for generic topologies (*future work: construct generators of cohomology groups*).

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- Support of polytopal meshes and arbitrary degree of accuracy.
- Full set of analysis results: Poincaré inequalities, primal and adjoint consistency, commutation properties, etc.
- Systematic serendipity reduction of number of DOFs (on any polytopal mesh).
- Analysis of cohomology for generic topologies (*future work: construct generators of cohomology groups*).
- Other applications/complexes:
  - Magnetostatics equations [Di Pietro and Droniou, 2021b].
  - Reissner–Mindlin plates [Di Pietro and Droniou, 2021c].
  - Yang–Mills equations [Droniou et al., 2022].
  - Stokes complex [Hanot, 2021].
  - Plates complex [Di Pietro and Droniou, 2022a].

- Notes and series of lectures:

https://math.unice.fr/~massonr/Cours-DDR/Cours-DDR.html

COURSE OF JEROME DRONIOU FROM MONASH UNIVERSITY, INVITED PROFESSOR AT UCA

- **Introduction to Discrete De Rham complexes**
  - Short description (in french)
  - Summary of notations and formulas
  - Part 1, first course: the de Rham complex and its usefulness in PDEs, 22/09/22 (video)
  - Part 1, second course: Low order case, 29/09/22 (video)
  - Part 1, third course: Design of the DDR complex in 2D, 07/10/22 (video)
  - Part 1, fourth course: Exactness of the DDR complex in 2D, 10/10/22 (video)
  - Part 2, fifth course: DDR in 3D, analysis tools, 17/11/22 (video)

Thank you!

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