

Continuous and fully discrete analysis: principles, and application to Discrete De Rham scheme for Stokes equations

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Main references for this presentation

- *Key challenges and bridges among convergence analysis techniques for polytopal methods.* L. Beirão da Veiga, D. A. Di Pietro, J. Droniou. 2026. <https://arxiv.org/abs/2605.23405>.
- *Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes.* L. Beirão da Veiga, F. Dassi, D. A. Di Pietro, J. Droniou. *Comput. Methods Appl. Mech. Engrg.* 397, Paper No. 115061, 31p, 2022. doi: 10.1016/j.cma.2022.115061. <https://arxiv.org/abs/2112.09750>.

These slides:



Outline

- 1 Concept of adjoint consistency error
 - How it arises
 - Interpretation in the context of discrete complexes
- 2 Estimating the consistency error
 - Through virtual functions
 - Through conforming lifting
 - Discussion
- 3 Application to DDR and VEM schemes for Stokes equations
- 4 Numerical tests

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Slides:



Continuous problem

- V Hilbert space with norm $\|\cdot\|_V$.
- $a : V \times V \rightarrow \mathbb{R}$ continuous bilinear form, $\ell : V \rightarrow \mathbb{R}$ continuous linear form.
- Continuous problem: Find $u \in V$ s.t.

$$a(u, v) = \ell(v) \quad \forall v \in V.$$

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Discrete problem

- V_h Euclidean space with norm $\|\cdot\|_{V_h}$.
- $a_h : V_h \times V_h \rightarrow \mathbb{R}$ bilinear form, $\ell_h : V_h \rightarrow \mathbb{R}$ linear form
- Discrete problem: Find $u_h \in V_h$ s.t.

$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h.$$

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$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h.$$

May have $V_h \not\subset V$ (could even be $V = H_0^1(\Omega)$ and $V_h = \mathbb{R}^N$).

Abstract error estimate

- Assume coercivity of the discrete form:

$$\alpha \|w_h\|_{V_h}^2 \leq a_h(w_h, w_h).$$

- Take $I_h u \in V_h$ “interpolate” of the solution u to the continuous problem.
- Simple algebra:

$$\alpha \|u_h - I_h u\|_{V_h}^2 \leq a_h(u_h - I_h u, u_h - I_h u) = \ell_h(u_h - I_h u) - a_h(I_h u, u_h - I_h u).$$

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- Define the **consistency error**:

$$\mathcal{E}_h(v_h) := \ell_h(v_h) - a_h(I_h u, v_h) \quad \forall v_h \in V_h$$

so that

$$\alpha \|u_h - I_h u\|_{V_h}^2 \leq \mathcal{E}_h(u_h - I_h u) \leq \|\mathcal{E}_h\|_{V_h'} \|u_h - I_h u\|_{V_h}.$$

Related to similar concept for non-conforming FE, see [10].

Abstract error estimate

Consistency error: $\mathcal{E}_h(v_h) := \ell_h(v_h) - a_h(I_h u, v_h) \quad \forall v_h \in V_h.$

This is the error obtained when plugging the (interpolate of) exact solution into the scheme.

Theorem (Estimate of the approximation error)

If a_h is coercive with constant α then

$$\|u_h - I_h u\|_{V_h} \leq \alpha^{-1} \|\mathcal{E}_h\|_{V_h'}.$$

Remark (The conforming case)

*In the **conforming case** $V_h \subset V$, the difference $u - u_h$ makes sense and*

$$\|u - u_h\|_V \leq \|u - I_h u\|_V + \|u_h - I_h u\|_V \leq \|u - I_h u\|_V + \alpha^{-1} \|\mathcal{E}_h\|_{V'}.$$

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Complexes and Hodge–Laplace problem

Commuting diagram of continuous and discrete complexes:

$$\begin{array}{ccccc} V_I^{i-1} & \xrightarrow{d^{i-1}} & V_I^i & \xrightarrow{d^i} & V_I^{i+1} \\ \downarrow I_h^{i-1} & & \downarrow I_h^i & & \downarrow I_h^{i+1} \\ V_h^{i-1} & \xrightarrow{d_h^{i-1}} & V_h^i & \xrightarrow{d_h^i} & V_h^{i+1} \end{array}$$

with discrete inner product $(\cdot, \cdot)_{i,h}$ on V_h^i .

V_I^\bullet : subspaces of V^\bullet with enough regularity for the interpolations I_h^\bullet to be defined (or could use quasi-interpolators [12, 9, 1, 5]).

Discrete Hodge–Laplace: find $(u_h, p_h) \in V_h^{i-1} \times V_h^i$ such that

$$\begin{aligned} (u_h, v_h)_{i-1,h} - (p_h, d_h^{i-1}v)_{i,h} &= 0 & \forall v_h \in V_h^{i-1}, \\ (d_h^{i-1}u_h, q_h)_{i,h} + (d_h^i p_h, d_h^i q_h)_{i+1,h} &= \ell_h(q_h) & \forall q_h \in V_h^i. \end{aligned}$$

Complexes and Hodge–Laplace problem

Discrete Hodge–Laplace: find $(u_h, p_h) \in V_h^{i-1} \times V_h^i$ such that

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Equivalent formulation: find $(u_h, p_h) \in V_h^{i-1} \times V_h^i$ such that

$$a_h((u_h, p_h), (v_h, q_h)) = \ell_h(q_h) \quad \forall (v_h, q_h) \in V_h^{i-1} \times V_h^i,$$

where

$$\begin{aligned}a_h((u_h, p_h), (v_h, q_h)) &:= (u_h, v_h)_{i-1,h} - (p_h, d_h^{i-1}v_h)_{i,h} \\ &\quad + (d_h^{i-1}u_h, q_h)_{i,h} + (d_h^i p_h, d_h^i q_h)_{i+1,h}.\end{aligned}$$

Interpretation of the adjoint consistency error

- Assume $\ell_h(q_h) = (I_h^i g, q_h)_{i,h}$ and recall $g = d^{i-1}u + \delta^{i+1}d^i p$.
- Use the commutation $I_h d = d_h I_h$ to write:

$$\begin{aligned}\mathcal{E}_h(v_h, q_h) &= \left[\cancel{(I_h^i d^{i-1} u, q_h)_{i,h}} + (I_h^i \delta^{i+1} (d^i p), q_h)_{i,h} \right] \\ &\quad - (I_h^{i-1} u, v_h)_{i-1,h} + (I_h^i p, d_h^{i-1} v_h)_{i,h} \\ &\quad - \cancel{(d_h^{i-1} (I_h^{i-1} u), q_h)_{i,h}} - (I_h^{i+1} (d^i p), d_h^i q_h)_{i+1,h} \\ &= (I_h^i \delta^{i+1} (d^i p), q_h)_{i,h} - (I_h^{i+1} (d^i p), d_h^i q_h)_{i+1,h} \\ &\quad + (I_h^i p, d_h^{i-1} v_h)_{i,h} - (I_h^{i-1} \delta^i p, v_h)_{i-1,h} \\ &= \tilde{\mathcal{E}}_h^i(d^i p; q_h) - \tilde{\mathcal{E}}_h^{i-1}(p; v_h).\end{aligned}$$

$$\tilde{\mathcal{E}}_h^j(\omega; w_h) := (I_h^j \delta^{j+1} \omega, w_h)_{j,h} - (I_h^{j+1} \omega, d_h^j w_h)_{j+1,h}$$

↪ Defect of the **adjointness** of δ^{j+1} and d_h^j through the discrete inner products.

Quantities identified in Daniele's talk.

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Virtual framework

From hereon, $\ell(v) = (f, v)_L$ for some $f \in L$, with $L \hookrightarrow V$.

- Virtual element space $V_h \subset V$ on a mesh \mathcal{T}_h .
- Bilinear (symmetric) form $a_h(w_h, v_h) := \sum_{T \in \mathcal{T}_h} a_T(w_T, v_T)$ with:
 - Polynomial consistency: for some $\mathcal{P}_T \subset V_T$,

$$a_T(w, v_T) = a|_T(w, v_T) \quad \forall (w, v_T) \in \mathcal{P}_T \times V_T.$$

- Coercivity and local boundedness:

$$a_h(v_h, v_h) \gtrsim \|v_h\|_V^2, \quad a_T(v_T, v_T) \lesssim \|v_T\|_{V|_T}^2 \quad \forall T \in \mathcal{T}_h.$$

- Linear form: For some $\tilde{\mathcal{P}}_T \subset \mathcal{P}_T$ and Π_T^L the L -orthogonal projection on $\tilde{\mathcal{P}}_T$ (assumed computable on V_T):

$$\ell_h(v_h) = \sum_{T \in \mathcal{T}_h} (f, \Pi_T^L v_T)_{L|_T}.$$

Finite elements: take $\tilde{\mathcal{P}}_T = \mathcal{P}_T = V_T$.

Estimate of consistency error

- Introduce $-\ell(v_h) + a(u, v_h) = 0$ to decompose the consistency error:

$$\mathcal{E}_h(v_h) = \underbrace{\ell_h(v_h) - \ell(v_h)}_{=:\mathcal{E}_{\ell,h}} + \underbrace{a(u, v_h) - a_h(I_h u, v_h)}_{=:\mathcal{E}_{a,h}}.$$

- For $w \in \mathcal{P}_T$, properties of a_T :

$$\begin{aligned} a_{|T}(u, v_T) - a_T(I_T u, v_T) &\stackrel{\text{Consistency}}{=} a_{|T}(u - w, v_T) + a_T(w - I_T u, v_T) \\ &\stackrel{\text{Boundedness}}{\lesssim} \left(\|u - w\|_{V|_T}^2 + \|w - I_T u\|_{V|_T}^2 \right)^{\frac{1}{2}} \|v_T\|_{V|_T}, \end{aligned}$$

- Estimate on $\mathcal{E}_{a,h}$:

$$\mathcal{E}_{a,h} \lesssim \left(\|u - I_h u\|_V^2 + \sum_{T \in \mathcal{T}_h} \inf_{w \in \mathcal{P}_T} \|u - w\|_{V|_T}^2 \right)^{\frac{1}{2}} \|v_h\|_V.$$

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- Estimate on $\mathcal{E}_{a, h}$:

$$\mathcal{E}_{a, h} \lesssim \left(\|u - \mathbb{I}_h u\|_V^2 + \sum_{T \in \mathcal{T}_h} \inf_{w \in \mathcal{P}_T} \|u - w\|_{V|_T}^2 \right)^{\frac{1}{2}} \|v_h\|_V.$$

- Estimate on $\mathcal{E}_{\ell, h}$:

$$\begin{aligned} \mathcal{E}_{\ell, h} &= \sum_{T \in \mathcal{T}_h} (f, \Pi_T^L v_T - v_T)_{L|_T} = \sum_{T \in \mathcal{T}_h} (\Pi_T^L f - f, v_T)_{L|_T} \\ &\leq \left(\sum_{T \in \mathcal{T}_h} \|f - \Pi_T^L f\|_{L|_T}^2 \right)^{\frac{1}{2}} \|v_h\|_V. \end{aligned}$$

Theorem (Error estimate for the VEM scheme)

Assume the *polynomial consistency*, *continuity* and *coercivity* of the bilinear form a_h . Then,

$$\|u_h - I_h u\|_V \lesssim \|u - I_h u\|_V + \left[\sum_{T \in \mathcal{T}_h} \left(\inf_{w \in \mathcal{P}_T} \|u - w\|_{V|_T}^2 + \|f - \Pi_T^L f\|_{L|_T}^2 \right) \right]^{\frac{1}{2}}.$$

- Translates into rates of convergence, by approximation properties of Π_T^L and \mathcal{P}_T .
- Also requires **approximation properties of the interpolator** to estimate $\|u - I_h u\|_V$.
Can be quite challenging to establish (especially for curl and div-based problems)...

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Fully discrete setting

- **Discrete space** \mathbb{V}_h not necessarily subspace of V , could be made of vector of DOFs.
- **Bilinear (symmetric) form** $\mathfrak{a}_h(\mathbf{w}_h, \mathbf{v}_h) := \sum_{T \in \mathcal{T}_h} \mathfrak{a}_T(\mathbf{w}_T, \mathbf{v}_T)$ with:
 - **Coercivity** and **local boundedness**:

$$\mathfrak{a}_h(\mathbf{v}_h, \mathbf{v}_h) \gtrsim \|\mathbf{v}_h\|_{\mathbb{V}_h}^2, \quad \mathfrak{a}_T(\mathbf{v}_T, \mathbf{v}_T) \lesssim \|\mathbf{v}_T\|_{\mathbb{V}_T}^2 \quad \forall T \in \mathcal{T}_h.$$

- **Linear form**: Assume a reconstruction $R_T^L : \mathbb{V}_T \rightarrow L|_T$ and set

$$\ell_h(\mathbf{v}_h) = \sum_{T \in \mathcal{T}_h} (f, R_T^L \mathbf{v}_T)_{L|_T}.$$

Conforming lifting

We assume the existence of $L_h : \mathbb{V}_h \rightarrow V$ that satisfies:

- **Polynomial consistency.** For all $T \in \mathcal{T}_h$,

$$a_T(\mathbb{I}_T w, \mathbb{v}_T) = a|_T(w, (L_h \mathbb{v}_h)|_T) \quad \forall (w, \mathbb{v}_h) \in \mathcal{P}_T \times \mathbb{V}_h;$$

- **Projection on $\tilde{\mathcal{P}}_T$.** For all $T \in \mathcal{T}_h$,

$$\Pi_T^L(L_h \mathbb{v}_h)|_T = R_T^L \mathbb{v}_T \quad \forall \mathbb{v}_T \in \mathbb{V}_T;$$

- **Boundedness.** For all $\mathbb{v}_h \in \mathbb{V}_h$,

$$\|L_h \mathbb{v}_h\|_V \lesssim \|\mathbb{v}_h\|_{\mathbb{V}_h}.$$

Remarks: [8]

- ◇ Lifting can be **explicit** (e.g., in a FE space on a submesh).
- ◇ Can be (relatively) easy to construct for problem based on **grad**, much more challenging for problems based on **curl**, **div**.

Estimate of the consistency error

Similar to the virtual approach, with lifting replacing virtual functions.

- Introduce $-\ell(L_h \mathbb{v}_h) + a(u, L_h \mathbb{v}_h) = 0$ to decompose the consistency error:

$$\mathcal{E}_h(\mathbb{v}_h) = \underbrace{\ell_h(\mathbb{v}_h) - \ell(L_h \mathbb{v}_h)}_{=:\mathcal{E}_{\ell,h}} + \underbrace{a(u, L_h \mathbb{v}_h) - \mathfrak{a}_h(\mathbb{I}_h u, \mathbb{v}_h)}_{=:\mathcal{E}_{a,h}}.$$

- For $w \in \mathcal{P}_T$,

$$\begin{aligned} a|_T(u, L_h \mathbb{v}_h) - \mathfrak{a}_T(\mathbb{I}_T u, \mathbb{v}_T) &\stackrel{\text{Consistency } L_h}{=} a|_T(u - w, L_h \mathbb{v}_h) + \mathfrak{a}_T(\mathbb{I}_T(w - u), \mathbb{v}_T) \\ &\stackrel{\text{Boundedness } \mathfrak{a}_T}{\lesssim} \|u - w\|_{V|_T} \|L_h \mathbb{v}_h\|_{V|_T} + \|\mathbb{I}_T(w - u)\|_{\mathbb{V}_T} \|\mathbb{v}_T\|_{\mathbb{V}_T}. \end{aligned}$$

- Estimate on $\mathcal{E}_{a,h}$: by boundedness of L_h ,

$$\mathcal{E}_{a,h} \lesssim \left(\sum_{T \in \mathcal{T}_h} \inf_{w \in \mathcal{P}_T} \left(\|u - w\|_{V|_T}^2 + \|\mathbb{I}_T(u - w)\|_{\mathbb{V}_T}^2 \right) \right)^{\frac{1}{2}} \|\mathbb{v}_h\|_V.$$

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$$\mathcal{E}_h(\mathbb{v}_h) = \underbrace{\ell_h(\mathbb{v}_h) - \ell(L_h \mathbb{v}_h)}_{=: \mathcal{E}_{\ell,h}} + \underbrace{a(u, L_h \mathbb{v}_h) - \mathfrak{a}_h(\mathbb{I}_h u, \mathbb{v}_h)}_{=: \mathcal{E}_{a,h}}.$$

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- Estimate on $\mathcal{E}_{\ell,h}$:

$$\begin{aligned} \mathcal{E}_{\ell,h} &= \sum_{T \in \mathcal{T}_h} (f, R_T^L \mathbb{v}_T - L_h \mathbb{v}_T)_{L|_T} \stackrel{\text{Proj. on } \tilde{P}_T}{=} \sum_{T \in \mathcal{T}_h} (f, \Pi_T^L(L_h \mathbb{v}_h) - L_h \mathbb{v}_h)_{L|_T} \\ &\lesssim \left(\sum_{T \in \mathcal{T}_h} \|f - \Pi_T^L f\|_{L|_T}^2 \right)^{\frac{1}{2}} \|\mathbb{v}_h\|_V. \end{aligned}$$

Theorem (Error estimate for the fully discrete scheme)

Assume the *coercivity* and *continuity* of a_h , and the existence of an adapted *conforming lifting*. Then,

$$\begin{aligned} & \| \mathbb{w}_h - \mathbb{I}_h u \|_{V_h} \\ & \lesssim \left[\sum_{T \in \mathcal{T}_h} \left(\inf_{w \in \mathcal{P}_T} \left(\| u - w \|_{V|_T}^2 + \| \mathbb{I}_T(u - w) \|_{V_T}^2 \right) + \| f - \Pi_T^L f \|_{L|_T}^2 \right) \right]^{\frac{1}{2}}. \end{aligned}$$

- No need for approximation properties of interpolator in V -norm, just pure polynomial approximation properties.

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Algebraic equivalence between schemes

- In virtual elements, there is a **DOFs map isomorphism** $\sigma_h : \mathbb{V}_h \rightarrow \mathbb{V}_h$.
- Both schemes are therefore **strictly equivalent** if

$$\mathfrak{a}_h(\cdot, \cdot) = \mathfrak{a}_h(\sigma_h \cdot, \sigma_h \cdot), \quad \Pi_T^L = R_T^L \circ \sigma_T.$$

*Expresses that \mathfrak{a}_h and Π_T^L are **computable** from the DOFs.*

- $L_h = \sigma_h^{-1}$ can be used as lifting, but more explicit approaches (e.g., FE lifting on submesh) can be far easier to design and analyse.

Continuous and discrete norms

Assume $a_h(\cdot, \cdot) = a_h(\sigma_h \cdot, \sigma_h \cdot)$.

- Two norms on the virtual space V_h :

continuous norm $\|\cdot\|_V$, and **discrete** norm $\|\sigma_h \cdot\|_{V_h}$ (computable).

- Virtual analysis based on $\|\cdot\|_V$, fully discrete analysis based on $\|\sigma_h \cdot\|_{V_h}$.

Proposition (Continuous and discrete norm equivalence on virtual space)

Consider the properties (locally or globally)

a_h is coercive and bounded for the V -norm, (Vcb)

a_h is coercive and bounded for the V_h -norm, (FDcb)

Continuous and discrete norms are equivalent. (E)

We have:

- If (Vcb) and (FDcb) hold, then (E) holds.
- If (E) holds, then (FDcb) and (Vcb) are equivalent.

Continuous and discrete norms

a_h is continuous and coercive for the V -norm, (Vcb)
 a_h is continuous and coercive for the \mathbb{V}_h -norm, (FDcb)
Continuous and discrete norms are equivalent. (E)

- (E) can be very challenging to obtain, because virtual functions are not polynomial functions (*no obvious trace/inverse inequality, implicitly defined by PDEs in the elements*).
- (E) usually used to prove the continuity of the interpolator I_h on virtual space (defined through σ_h), and thus its **approximation properties**.
- (Vcb) very often **implicitly obtained through** (FDcb) and (E).

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The Stokes problem in curl-curl formulation

- Given Ω contractible, $\nu > 0$ constant and $\mathbf{f} \in L^2(\Omega)$, the Stokes problem reads:

Find the **velocity** $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ and **pressure** $p : \Omega \rightarrow \mathbb{R}$ s.t.

$$\overbrace{\nu(\mathbf{curl} \mathbf{curl} \mathbf{u} - \mathbf{grad} \operatorname{div} \mathbf{u})}^{-\nu \Delta \mathbf{u}} + \mathbf{grad} p = \mathbf{f} \quad \text{in } \Omega, \quad (\text{momentum conservation})$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (\text{mass conservation})$$

$$\mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (\text{boundary conditions})$$

$$\int_{\Omega} p = 0$$

- Weak formulation:** Find $(\mathbf{u}, p) \in \mathbf{H}(\mathbf{curl}; \Omega) \times H^1(\Omega)$ such that

$$\nu(\mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v})_{L^2} + (\mathbf{grad} p, \mathbf{v})_{L^2} = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega),$$

$$-(\mathbf{u}, \mathbf{grad} q)_{L^2} = 0 \quad \forall q \in H^1(\Omega),$$

$$(p, 1)_{L^2} = 0.$$

- Weak formulation: Find $(\mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times H^1(\Omega)$ s.t.

$$\begin{aligned} \nu(\text{curl } \mathbf{u}, \text{curl } \mathbf{v})_{L^2} + (\text{grad } p, \mathbf{v})_{L^2} &= (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in \mathbf{H}(\text{curl}; \Omega), \\ -(\mathbf{u}, \text{grad } q)_{L^2} &= 0 \quad \forall q \in H^1(\Omega), \\ (p, 1)_{L^2} &= 0. \end{aligned}$$

- Scheme from the **DDR method** of [6]: set $\mathbb{V}_h := \underline{X}_{\text{curl},h}^r \times \underline{X}_{\text{grad},h}^r$ and find $(\underline{\mathbf{u}}_h, \underline{p}_h) \in \mathbb{V}_h$ s.t., for all $(\underline{\mathbf{v}}_h, \underline{q}_h) \in \mathbb{V}_h$,

$$\begin{aligned} \nu(\underline{\mathbf{C}}_h^r \underline{\mathbf{u}}_h, \underline{\mathbf{C}}_h^r \underline{\mathbf{v}}_h)_{\text{div},h} + (\underline{\mathbf{G}}_h^k \underline{p}_h, \underline{\mathbf{v}}_h)_{\text{curl},h} &= (\underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{f}, \underline{\mathbf{v}}_h)_{\text{curl},h}, \\ -(\underline{\mathbf{G}}_h^k \underline{q}_h, \underline{\mathbf{u}}_h)_{\text{curl},h} &= 0, \\ (\underline{p}_h, \underline{\mathbf{I}}_{\text{grad},h}^k 1)_{\text{grad},h} &= 0. \end{aligned}$$

- Weak formulation: Find $(\mathbf{u}, p) \in \mathbf{H}(\text{curl}; \Omega) \times H^1(\Omega)$ s.t.

$$\begin{aligned} \nu(\text{curl } \mathbf{u}, \text{curl } \mathbf{v})_{L^2} + (\text{grad } p, \mathbf{v})_{L^2} &= (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in \mathbf{H}(\text{curl}; \Omega), \\ -(\mathbf{u}, \text{grad } q)_{L^2} &= 0 \quad \forall q \in H^1(\Omega), \\ (p, 1)_{L^2} &= 0. \end{aligned}$$

- Scheme from the **VEM method** of [2]: set $V_h := V_r^e \times V_{r+1}^n$ and find $(\mathbf{u}_h, p_h) \in V_h$ s.t., for all $(\mathbf{v}_h, q_h) \in V_h$,

$$\begin{aligned} [\text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h]_{V_r^f} + [\text{grad } p_h, \mathbf{v}_h]_{V_r^e} &= [\mathcal{I}_r^e \mathbf{f}, \mathbf{v}_h]_{V_r^e}, \\ -[\text{grad } q_h, \mathbf{u}_h]_{V_r^e} &= 0, \\ (p_h, 1)_{L^2} &= 0. \end{aligned}$$

Stability and pressure-robustness

- Stability follows from **exactness of the discrete complexes**, together with **Poincaré inequalities** for **grad** and **curl** [3, 4, 6, 7]

Same argument as for the analysis of the continuous model.

- Pressure-robustness achieved owing to a **commutation property**, which leads to pressure invariance:

DDR

$$\underline{\mathbf{G}}_h^k(\underline{\mathbf{I}}_{\text{grad},h}^k \psi) = \underline{\mathbf{I}}_{\text{curl},h}^k(\text{grad } \psi)$$

VEM

$$\text{grad}(\mathcal{I}_r^n \psi) = \mathcal{I}_r^e(\text{grad } \psi)$$

Usual FE approaches for robustness: $\mathbf{H}(\text{div}; \Omega)$ -conforming reconstruction of the test functions. [11]

Theorem (DDR)

Setting

$$\begin{aligned}\|\underline{\mathbf{v}}_h\|_{\text{curl},1,h}^2 &= \|\underline{\mathbf{v}}_h\|_{\text{curl},h}^2 + \|\underline{\mathbf{C}}_h^r \underline{\mathbf{v}}_h\|_{\text{div},h}^2, \\ \|\underline{q}_h\|_{\text{grad},1,h}^2 &= \|\underline{q}_h\|_{\text{grad},h}^2 + \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\text{curl},h}^2,\end{aligned}$$

we have:

$$\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{u}\|_{\text{curl},1,h} + \|\underline{p}_h - \underline{\mathbf{I}}_{\text{grad},h}^k p\|_{\text{grad},1,h} \lesssim C_1(\mathbf{u}) h^{r+1}.$$

with $C_1(\mathbf{u})$ depending \mathbf{u} and some of its derivatives, but not p .

Theorem (VEM)

We have:

$$\|\mathbf{u}_h - \mathcal{I}_r^e \mathbf{u}\|_{\mathbf{H}(\text{curl};\Omega)} + \|p_h - \mathcal{I}_r^n p\|_{H^1(\Omega)} \lesssim C_2(\mathbf{u}) h^{r+1}.$$

with $C_2(\mathbf{u})$ depending \mathbf{u} and some of its derivatives, but not p .

Adjoint consistency for the curl (*global Stokes formula*)

- Measure of defect in discrete integration-by-parts: for $\omega \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ and $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\mathbf{curl},h}^r$,

$$\mathcal{E}_{\mathbf{curl}}(\omega, \underline{\mathbf{w}}_h) = \int_{\Omega} \omega \cdot \mathbf{C}_h^r \underline{\mathbf{w}}_h + \int_{\Omega} \mathbf{curl} \omega \cdot \mathbf{P}_{\mathbf{curl},h}^r \underline{\mathbf{w}}_h$$

- **Daniele's claim:**

$$|\mathcal{E}_{\mathbf{curl}}(\omega, \underline{\mathbf{w}}_h)| \lesssim h^{r+1} (\|\omega\|_{H^{r+1}(\Omega)} \|\mathbf{C}_h^r \underline{\mathbf{w}}_h\|_{\text{div},h} + \|\mathbf{curl} \omega\|_{H^{r+1}(\Omega)} \|\underline{\mathbf{w}}_h\|_{\mathbf{curl},h})$$

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- Proof by **conforming lifing**: with $L_h : \underline{\mathbf{X}}_{\mathbf{curl},h}^r \rightarrow \mathbf{H}(\mathbf{curl}; \Omega)$,
 - Subtracting

$$\int_{\Omega} \omega \cdot \mathbf{curl}(L_h \underline{\mathbf{w}}_h) + \int_{\Omega} \mathbf{curl} \omega \cdot (L_h \underline{\mathbf{w}}_h) \stackrel{\text{IBP}}{=} 0,$$

we have

$$\mathcal{E}_{\mathbf{curl}}(\omega, \underline{\mathbf{w}}_h) = \int_{\Omega} \omega \cdot [\mathbf{C}_h^r \underline{\mathbf{w}}_h - \mathbf{curl}(L_h \underline{\mathbf{w}}_h)] + \int_{\Omega} \mathbf{curl} \omega \cdot [\mathbf{P}_{\mathbf{curl},h}^r \underline{\mathbf{w}}_h - L_h \underline{\mathbf{w}}_h].$$

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○ Design L_h with L^2 -projection properties on $\mathcal{P}^r(\mathcal{T}_h)^3$:

$$\mathbf{C}_h^r \underline{\mathbf{w}}_h = \pi_{\mathcal{T}_h}^r(\mathbf{curl}(L_h \underline{\mathbf{w}}_h)), \quad \mathbf{P}_{\text{curl},h}^r \underline{\mathbf{w}}_h = \pi_{\mathcal{T}_h}^r(L_h \underline{\mathbf{w}}_h).$$

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○ Substitute and use self-adjoint properties of $\pi_{\mathcal{T}_h}^r - \text{Id}$:

$$\mathcal{E}_{\text{curl}}(\omega, \underline{\mathbf{w}}_h) = \int_{\Omega} [(\pi_{\mathcal{T}_h}^r - \text{Id})\omega] \cdot \mathbf{curl}(L_h \underline{\mathbf{w}}_h) + \int_{\Omega} [(\pi_{\mathcal{T}_h}^r - \text{Id})\mathbf{curl} \omega] \cdot L_h \underline{\mathbf{w}}_h.$$

Et voilà!

Adjoint consistency for the divergence

Sometimes, direct manipulation can be much easier...

- Adjoint consistency error: for $q \in H_0(\mathbf{grad}; \Omega)$ and $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{div},h}^r$,

$$\mathcal{E}_{\text{div}}(q, \underline{\mathbf{v}}_h) = \int_{\Omega} q D_h^r \underline{\mathbf{v}}_h + \int_{\Omega} \mathbf{grad} q \cdot \mathbf{P}_{\text{div},h}^r \underline{\mathbf{v}}_h.$$

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- Definition of $\mathbf{P}_{\text{div},h}^r$: for all $w \in \mathcal{P}^{r+1}(\mathcal{T}_h)$,

$$\int_{\Omega} \mathbf{P}_{\text{div},h}^r \underline{v}_h \cdot \mathbf{grad}_h w = - \int_{\Omega} w D_h^r \underline{v}_h + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \nu_F w|_T.$$

- Subtract and use $\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \nu_F q = 0$ to get

$$\begin{aligned} \mathcal{E}_{\text{div}}(q, \underline{v}_h) &= \int_{\Omega} (q - w) D_h^r \underline{v}_h + \int_{\Omega} \mathbf{grad}(q - w) \cdot \mathbf{P}_{\text{div},h}^r \underline{v}_h \\ &\quad + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \nu_F (r|_T - w). \end{aligned}$$

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Make $w = \pi_{\mathcal{T}_h}^{r+1} q \dots$ et voilà!

Outline

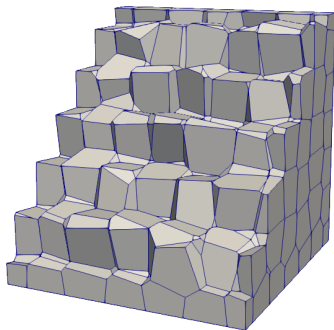
- 1 Concept of adjoint consistency error
 - How it arises
 - Interpretation in the context of discrete complexes
- 2 Estimating the consistency error
 - Through virtual functions
 - Through conforming lifting
 - Discussion
- 3 Application to DDR and VEM schemes for Stokes equations
- 4 Numerical tests

Slides:



Setting

- $\Omega = (0, 1)^3$.
- Voronoi mesh families (similar results on tetrahedral meshes):



(a) Voronoi mesh

- Exact solution: for some $\lambda \geq 0$,

$$p(x, y, z) = \lambda \sin(2\pi x) \sin(2\pi y) \sin(2\pi z),$$
$$\mathbf{u}(x, y, z) = \begin{bmatrix} \frac{1}{2} \sin(2\pi x) \cos(2\pi y) \cos(2\pi z) \\ \frac{1}{2} \cos(2\pi x) \sin(2\pi y) \cos(2\pi z) \\ -\cos(2\pi x) \cos(2\pi y) \sin(2\pi z) \end{bmatrix}.$$

- Measured errors:

- **Discrete**, using interpolates of exact solutions, e.g.:

$$E_{\mathbf{u}}^d = \|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\text{curl},h}^k \mathbf{u}\|_{\text{curl},1,h}, \quad E_p^d = \|\mathbf{grad} p_h - \mathbf{grad}(\mathcal{I}_r^n p)\|_{L^2(\Omega)}$$

From the theorems, expected $\lesssim C(\mathbf{u})h^{r+1}$.

- **Continuous**, using reconstructions from approximate solutions, e.g.:

$$E_{\mathbf{u}}^c = \|C_h^r \underline{\mathbf{u}}_h - \mathbf{curl} \mathbf{u}\|_{L^2(\Omega)}, \quad E_p^d = \|\mathbf{grad} p_h - \mathbf{grad} p\|_{L^2(\Omega)}$$

- DDR and VEM implemented in the HArDCore3D library¹, using the Intel MKL Pardiso solver².

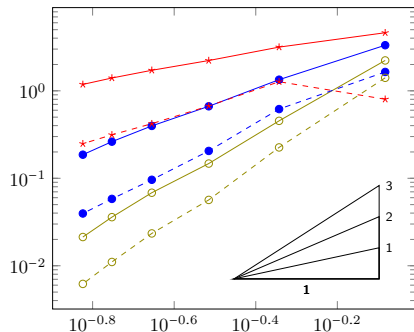
*The HArDCore3D library also includes a **serendipity version for DDR**, which leads to a reduction of more than 50% of the solving time.*

¹<https://github.com/jdroniou/HArDCore>

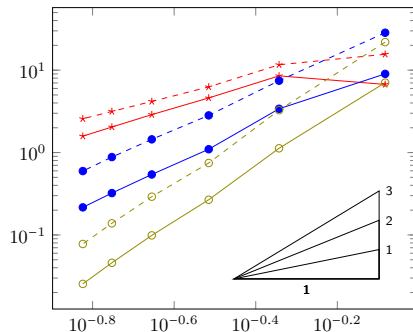
²<https://software.intel.com/en-us/mkl>

Results: $\lambda = 1$, errors on u

—*— $E^c, r = 0$; —●— $E^c, r = 1$; —○— $E^c, r = 2$
- - * - - $E^d, r = 0$; - - ● - - $E^d, r = 1$; - - ○ - - $E^d, r = 2$



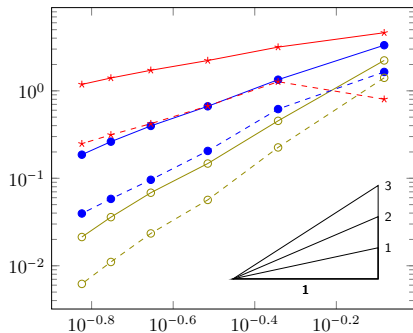
(a) DDR scheme



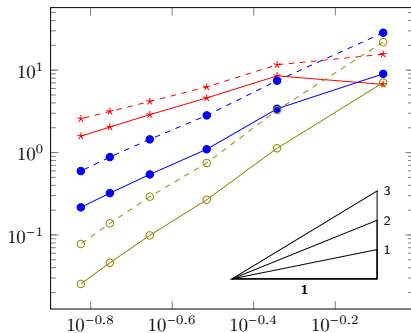
(b) VEM scheme

Results: $\lambda = 10^5$, errors on u

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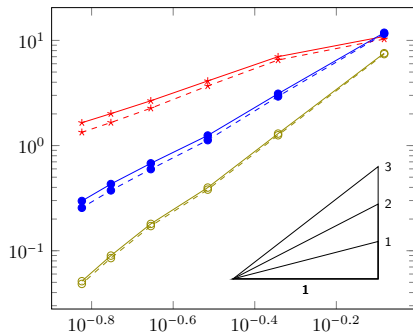
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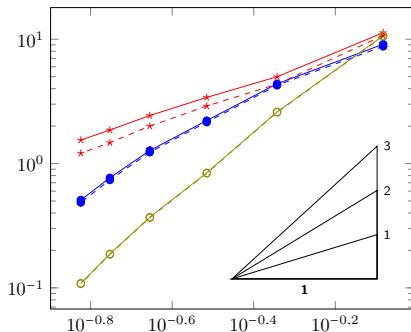
(b) VEM scheme

Results: $\lambda = 1$, errors on p

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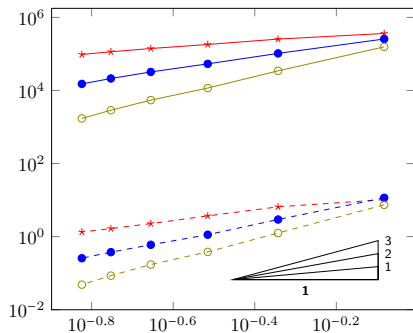
(a) DDR scheme



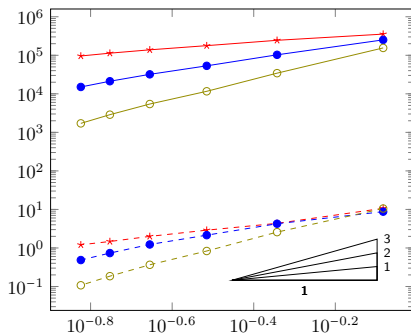
(b) VEM scheme

Results: $\lambda = 10^5$, errors on p

$\text{---} \star \text{---} E^c, r = 0$; $\text{---} \bullet \text{---} E^c, r = 1$; $\text{---} \circ \text{---} E^c, r = 2$
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(a) DDR scheme



(b) VEM scheme

Conclusions

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Proof of norm equivalence (or coercivity/boundedness of virtual forms) can be very tricky due to the non-polynomial nature of virtual functions.
- Estimates for adjoint consistency (DDR, VEM) provide **optimal convergence rates**.
- Suitable discretisation of source term leads to **pressure robustness**.



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Thank you for your attention!

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