Interplay between diffusion anisotropy and mesh skewness in Hybrid High-Order schemes

J. Droniou (Monash University)

FINITE VOLUMES FOR COMPLEX APPLICATIONS IX
(Bergen, and all across the world)

Co.llaborators on HHO: D. A. Di Pietro, L. Botti, A. Ern, D. Anderson...

Australian Government
Australian Research Council

Discrete Functional Analysis: bridging pure and numerical mathematics

Plan

1. **Motivation**
2. **Hybrid High-Order method**
   - Model and framework for discretisation
   - Notations, local spaces and operators
   - Presentation of HHO scheme
   - Error analysis on skewed meshes
3. **Numerical tests**
   - HArDCore library
   - Anisotropic heterogeneous diffusion on isotropic regular mesh
   - Isotropic diffusion and skewed mesh
   - Interplay between diffusion and skewness

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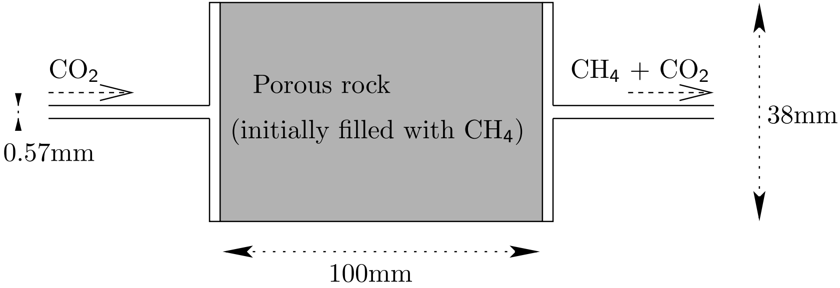
Complex geometries require skewed elements

Experimental device to measure rock permeability



Source: T. Hughes, Fac. of Engineering, Monash.

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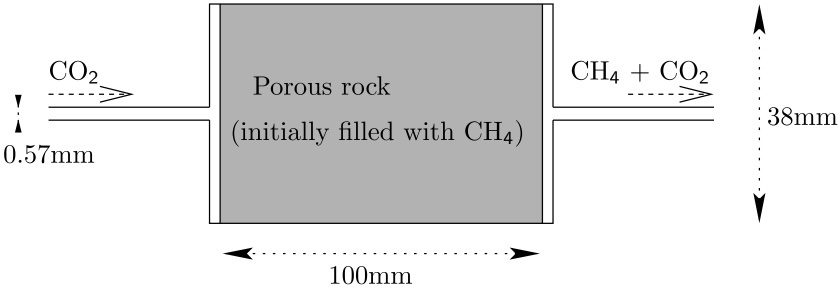

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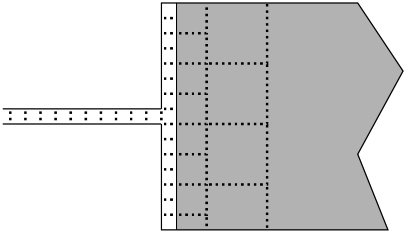
- Numerical simulation of the flow $\rightsquigarrow$ meshing the domain

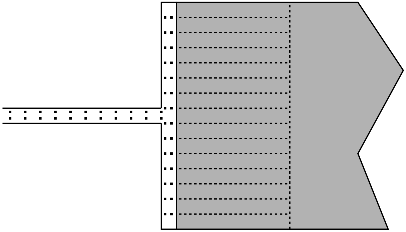
Complex geometries require skewed elements

Meshing junction free flow/porous flow: with **regular** elements, requires specific, ad-hoc strategy.



Complex geometries require skewed elements

Meshing junction free flow/porous flow: with **elongated** elements, more straightforward and flexible.



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Model: anisotropic heterogeneous diffusion equation

$$
\begin{aligned}
-\mathrm{div}(\boldsymbol{K}\boldsymbol{\nabla} u) &= f \text{ in } \Omega,\\
u &= 0 \text{ on } \partial\Omega.
\end{aligned}
$$

- Ω polytopal domain in $\mathbb{R}^d$,
- $\boldsymbol{K} : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ uniformly elliptic and piecewise constant,
- $f \in L^2(\Omega)$.

Weak formulation and approximation framework

Weak formulation: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in H_0^1(\Omega),$$

where $a(u, v) = (K\nabla u, \nabla v)_\Omega$ and $\ell(v) = (f, v)_\Omega$.

- $(\cdot, \cdot)_X$: *inner product on* $L^2(X)$.

Weak formulation and approximation framework

Weak formulation: find $u \in H^1_0(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in H^1_0(\Omega),$$

where $a(u, v) = (K\nabla u, \nabla v)_\Omega$ and $\ell(v) = (f, v)_\Omega$.

- $(\cdot, \cdot)_X$: *inner product on* $L^2(X)$.

Numerical scheme requires:

Discretisation method

- V_h finite dimensional space,
- $I_h : V \to V_h$ interpolator.

Discretised operators

- $\mathrm{a}_h : V_h \times V_h \to \mathbb{R}$ bilinear,
- $\ell_h : V_h \to \mathbb{R}$ linear.

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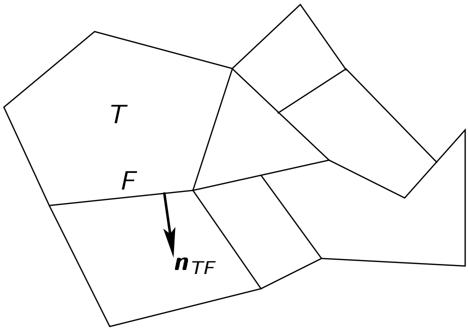
Scheme: find $u_h \in V_h$ such that

$$\mathrm{a}_h(u_h, v_h) = \ell_h(v_h) \qquad \forall v_h \in V_h.$$

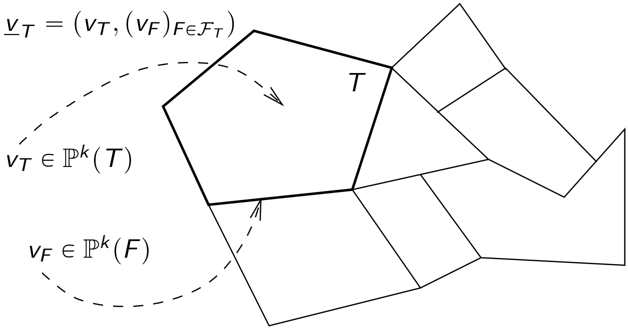
Plan

Mesh



- T: element (cell) / set of elements: $\mathcal{T}_h$;
- F: face / set of faces: $\mathcal{F}_h$;
- $\boldsymbol{n}_{TF}$: outer normal to T on F,
- $\mathcal{F}_T$: set of faces of T,
- $\boldsymbol{K}$ constant on each element, value $\boldsymbol{K}_T$.

Local unknowns



- $\mathbb{P}^k(X)$: polynomials of degree $\leqslant k$ on $X = T, F$;
- $\pi_X^{0,k} : L^1(X) \to \mathbb{P}^k(X)$ polynomial $L^2(X)$-projector;
- Space: $\underline{\boldsymbol{U}}_T^k = \{\underline{v}_T = (v_T, (v_F)_{F\in\mathcal{F}_T}) \,:\, v_T \in \mathbb{P}^k(T)\,,\ v_F \in \mathbb{P}^k(F)\}$;
- Interpolator: $\underline{\boldsymbol{I}}_T^k : H^1(T) \to \underline{\boldsymbol{U}}_T^k$ such that $\underline{\boldsymbol{I}}_T^k w = (\pi_T^{0,k} w, (\pi_F^{0,k} w)_{F\in\mathcal{F}_T})$.

Diffusion-dependent potential reconstruction

Higher-order potential reconstruction: $\mathrm{p}_{\boldsymbol{K},T}^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T)$ defined by:

For all $\underline{v}_T \in \underline{\boldsymbol{U}}_T^k$ and $q \in \mathbb{P}^{k+1}(T)$,

$$(\boldsymbol{K}_T \nabla \mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T, \nabla q)_T = (\boldsymbol{K}_T \nabla v_T, \nabla q)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \boldsymbol{K}_T \nabla q \cdot \boldsymbol{n}_{TF})_F \,,$$

$$(\mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T, 1)_T = (v_T, 1)_T.$$

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$$(\mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T, 1)_T = (v_T, 1)_T.$$

Note:

- Oblique elliptic projector $\pi_{\boldsymbol{K},T}^{1,k+1} : H^1(T) \to \mathbb{P}^{k+1}(T)$ defined by:
  For all $w \in H^1(T)$ and all $q \in \mathbb{P}^{k+1}(T)$,

$$(\boldsymbol{K}_T \nabla \pi_{\boldsymbol{K},T}^{1,k+1} w, \nabla q)_T = (\boldsymbol{K}_T \nabla w, \nabla q)_T ,$$
$$(\pi_{\boldsymbol{K},T}^{1,k+1} w, 1)_T = (w, 1)_T.$$

- Link with potential reconstruction:

$$\mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{\boldsymbol{I}}_T^k w = \pi_{\boldsymbol{K},T}^{1,k+1} w.$$

Plan

Local bilinear form

Consistent contribution plus stabilisation: $\mathrm{a}_{\boldsymbol{K},T} : \underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$ defined by

$$\mathrm{a}_{\boldsymbol{K},T}(\underline{v}_T, \underline{w}_T) = (\boldsymbol{K}_T \nabla \mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T, \nabla \mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{w}_T)_T + \mathrm{s}_{\boldsymbol{K},T}(\underline{v}_T, \underline{w}_T).$$

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(Example of) stabilisation:

$$\mathrm{s}_{\boldsymbol{K},T}(\underline{v}_T, \underline{w}_T) = \sum_{F\in\mathcal{F}_T} \frac{K_{TF}}{d_{TF}} \left(\delta_{\boldsymbol{K},TF}^k \underline{v}_T - \delta_{\boldsymbol{K},T}^k \underline{v}_T \,,\, \delta_{\boldsymbol{K},TF}^k \underline{w}_T - \delta_{\boldsymbol{K},T}^k \underline{w}_T\right)_F$$

with

- $K_{TF} = \boldsymbol{K}_T \boldsymbol{n}_{TF} \cdot \boldsymbol{n}_{TF}$,
- $d_{TF} = \frac{|T|_d}{|F|_{d-1}}$,
- Face and element difference operators:

$$\delta_{\boldsymbol{K},TF}^k \underline{v}_T = \pi_F^{0,k}(\mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T - v_F)\,, \quad \delta_{\boldsymbol{K},T}^k \underline{v}_T = \pi_T^{0,k}(\mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T - v_T).$$

Equivalently: $(\delta_{\boldsymbol{K},T}^k \underline{v}_T, (\delta_{\boldsymbol{K},TF}^k \underline{v}_T)_{F\in\mathcal{F}_T}) = \underline{\boldsymbol{I}}_T^k \mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T - \underline{v}_T$

HHO scheme

Discretisation method:

- Space

$$\underline{\boldsymbol{U}}_{h,0}^k = \{\underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \,:\, v_T \in \mathbb{P}^k(T)\,,\ v_F \in \mathbb{P}^k(F)\,,\\ v_F = 0 \text{ if } F \subset \partial\Omega\}$$

- Interpolator

$$\underline{\boldsymbol{I}}_h^k w = ((\pi_T^{0,k} w)_{T\in\mathcal{T}_h}, (\pi_F^{0,k} w)_{F\in\mathcal{F}_h}).$$

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Discretised operators:

- Bilinear form

$$\mathrm{a}_{\boldsymbol{K},h}(\underline{v}_h, \underline{w}_h) = \sum_{T\in\mathcal{T}_h} \mathrm{a}_{\boldsymbol{K},T}(\underline{v}_T, \underline{w}_T)$$

- Linear form

$$\ell_h(\underline{w}_h) = (f, w_h)_\Omega \quad \text{where } (w_h)_{|T} = w_T \text{ for all } T \in \mathcal{T}_h.$$

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HHO scheme: find $\underline{u}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ such that

$$\mathrm{a}_{\boldsymbol{K},h}(\underline{u}_h, \underline{w}_h) = \ell_h(\underline{w}_h) \qquad \forall \underline{w}_h \in \underline{\boldsymbol{U}}_{h,0}^k,$$

Finite volume presentation

Face residuals: $R_{\partial T} = (R_{TF})_{F\in\mathcal{F}_T} : \underline{\boldsymbol{U}}_T^k \to \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F)$ defined by

$$-\sum_{F\in\mathcal{F}_T} (R_{TF}\underline{v}_T, q_F)_F = \mathrm{s}_{\boldsymbol{\kappa},T}(\underline{v}_T, (0, (q_F)_{F\in\mathcal{F}_T})) \quad \forall (q_F)_{F\in\mathcal{F}_T} \in \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F).$$

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Fluxes:

$$\Phi_{TF}(\underline{v}_T) = -\boldsymbol{K}_T \boldsymbol{\nabla} \mathrm{p}_{\boldsymbol{K},T}^{k+1} \underline{v}_T \cdot \boldsymbol{n}_{TF} + R_{TF}\underline{v}_T.$$

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Local balances and conservativity: HHO scheme equivalent to

$$(\boldsymbol{K}_T \boldsymbol{\nabla} \mathrm{p}_{\boldsymbol{K},T}^{k+1}\underline{u}_T, \boldsymbol{\nabla} v_T)_T + \boxed{\sum_{F\in\mathcal{F}_T} (\Phi_{TF}(\underline{u}_T), v_T)_F} = (f, v_T)_T \quad \forall T \in \mathcal{T}_h,\ \forall v_T \in \mathbb{P}^k(T),$$

$$\boxed{\Phi_{T_1F}(\underline{u}_{T_1}) + \Phi_{T_2F}(\underline{u}_{T_2}) = 0} \qquad \forall F \in \mathcal{F}_h \text{ interface between } T_1 \text{ and } T_2.$$

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Standard regular meshes

- *h_X: diameter of $X = T, F$.*

Standard regularity assumptions for polytopal methods: isotropic elements.

- Each T star-shaped with respect to a ball of diameter $\sim h_T$.
- Each F star-shaped with respect to a disc of diameter $\sim h_F$, and $h_F \sim h_T$ if $F \in \mathcal{F}_T$.

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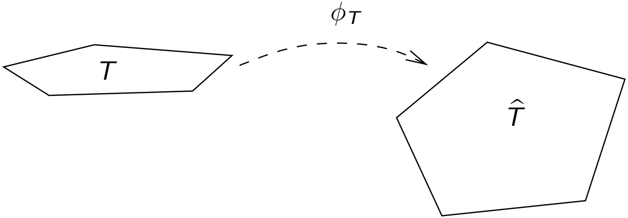
Notable variations (but still mostly isotropic elements):

- HHO: T not star-shaped, just existence of regular sub-division into simplices.
- (conforming) VEM: can deal with small faces ($h_F \ll h_T$ for $F \in \mathcal{F}_T$), and large number of faces in each element.
  - *But not a FV method...*

Skewed element

- Polytope T that can be transported by linear map onto isotropic polytope $\widehat{T}$, such that $h_{\widehat{T}} \sim h_T$.



Regular skewed mesh sequence

Definition: $(\mathcal{M}_h)_h = (\mathcal{T}_h, \mathcal{F}_h)_h$ where each element $T \in \mathcal{M}_h$ is skewed, with $\widehat{T}$ isotropic uniformly w.r.t. h (as in standard regular mesh).

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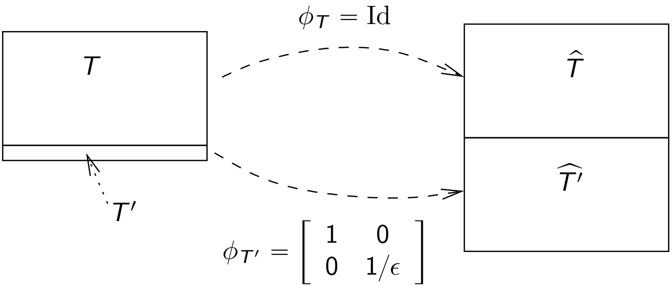
- No direct relation between mappings ϕ_T and $\phi_{T'}$ of two neighbouring elements.



- *Not ok for conforming FE/VEM [Weißer, 2019], [Antonietti et al, 2019]*

Regular skewed mesh sequence

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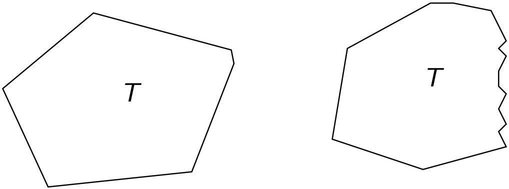
- Does not accept small faces in otherwise isotropic elements.



Main error estimate

Theorem (HHO error estimates on skewed meshes)

With u solution to the PDE, $\underline{u}_h$ solution of HHO scheme, $r \in \{0, \ldots, k\}$:

$$\|\underline{\boldsymbol{I}}_h^k u - \underline{u}_h\|_{\mathrm{a},\boldsymbol{K},h} \leqslant C A_{\boldsymbol{K},h} h^{r+1} |u|_{H^{r+2}(\mathcal{T}_h)}$$

$$\text{with } A_{\boldsymbol{K},h} := \max_{T\in\mathcal{T}_h} \frac{\overline{K}_{\phi,T}^2}{\underline{K}_{\phi,T}},$$

- $\|\cdot\|_{\mathrm{a},\boldsymbol{K},h}$: *norm associated to* $\mathrm{a}_{\boldsymbol{K},h}(\cdot,\cdot)$,
- $\overline{K}_{\phi,T}$ *and* $\underline{K}_{\phi,T}$ *largest and smallest eigenvalues of the transported diffusion* $\boxed{\boldsymbol{K}_{\phi,\hat{T}} = \phi_T \boldsymbol{K}_T \phi_T^t}$.

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▸ Estimate based on transport relations. As sharp as for standard regular meshes.

Interplay between mesh anisotropy and diffusion

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Interplay between mesh anisotropy and diffusion

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- Assumption: for each T, in some local orthonormal basis,

$$\boldsymbol{K}_T = \begin{bmatrix} \lambda_T & 0 \\ 0 & 1 \end{bmatrix} \qquad \phi_T = \begin{bmatrix} a_T & 0 \\ 0 & b_T \end{bmatrix}.$$

- Then:

$$A_{\boldsymbol{K},h} = \max_{T\in\mathcal{T}_h} \left[\max(a_T\lambda_T^{\frac{1}{2}}, b_T) \max\left(\frac{a_T\lambda_T^{\frac{1}{2}}}{b_T}, \frac{b_T}{a_T\lambda_T^{\frac{1}{2}}} \right) \right].$$

Interplay between mesh anisotropy and diffusion

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- Then:

$$A_{K,h} = \max_{T\in\mathcal{T}_h} \left[\max(a_T\lambda_T^{\frac{1}{2}}, b_T) \max\left(\frac{a_T\lambda_T^{\frac{1}{2}}}{b_T}, \frac{b_T}{a_T\lambda_T^{\frac{1}{2}}} \right) \right].$$

- Consider $K = \mathrm{diag}(\lambda, 1)$ with $\lambda \gg 1$.

All T elongated in direction of strong diffusion ($a_T = 1 \ll b_T \sim \lambda^{\frac{1}{2}}$)	Isotropic mesh ($a_T = b_T = 1$)
$A_{K,h} \sim \lambda^{\frac{1}{2}}$	$A_{K,h} \sim \lambda$

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HArDCore: Open-source C++ library for discretisations of PDEs.

- Sources: `https://github.com/jdroniou/HArDCore`

Contributors (direct or indirect): D. Anderson, L. Botti, H. M. Cheng, D. Di Pietro, J. Droniou, L. Grose, T. Lemaitre, L. Yemm.

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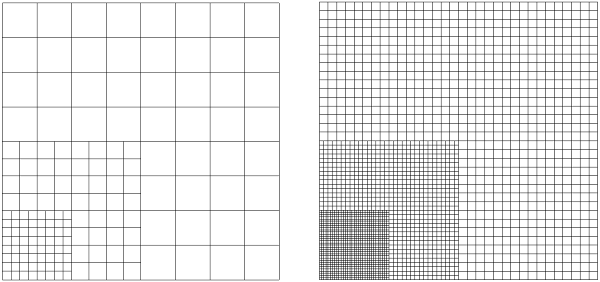
- For HHO (diffusion, advection...), an any other polytopal method (e.g. magnetostatic with Discrete De Rham method).
- Almost straightforward transition of code 2D $\leftrightarrow$ 3D.
- Some level of optimisation in the procedures (mutli-threading, etc.)

Plan

Description of test

Mesh: locally refined non-conforming, but isotropic.



Solution and diffusion: $u(x,y) = \cos(\pi x)\cos(\pi y)$ and, with $\lambda \in \{1, 10^{-6}, 10^{6}\}$,

$$\boldsymbol{K}(x,y) = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \quad \text{if } y < 0.5, \qquad \boldsymbol{K}(x,y) = \mathrm{Id} \quad \text{if } y \geqslant 0.5.$$

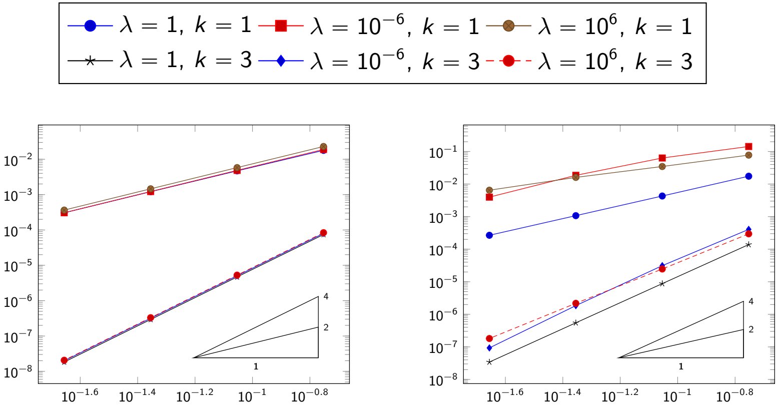
Rates of convergence w.r.t. h

- Errors measured:

$$E_{\mathrm{a},\boldsymbol{K},h} := \frac{\|\underline{\boldsymbol{I}}_h^k u - \underline{u}_h\|_{\mathrm{a},\boldsymbol{K},h}}{\|\underline{\boldsymbol{I}}_h^k u\|_{\mathrm{a},\boldsymbol{K},h}} \quad \text{and} \quad E_{1,h} := \frac{\|\underline{\boldsymbol{I}}_h^k u - \underline{u}_h\|_{1,h}}{\|\underline{\boldsymbol{I}}_h^k u\|_{1,h}},$$

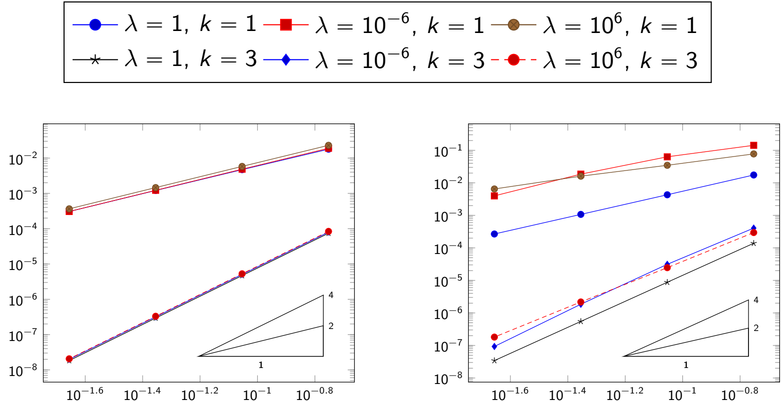
where $\|\cdot\|_{1,h}$ is a diffusion-independent discrete H^1-norm.

Rates of convergence w.r.t. h



(a) $E_{\mathrm{a},\boldsymbol{\kappa},h}$ vs. h.

(b) $E_{1,h}$ vs. h.



(a) $E_{\mathrm{a},\boldsymbol{K},h}$ vs. h.

(b) $E_{1,h}$ vs. h.

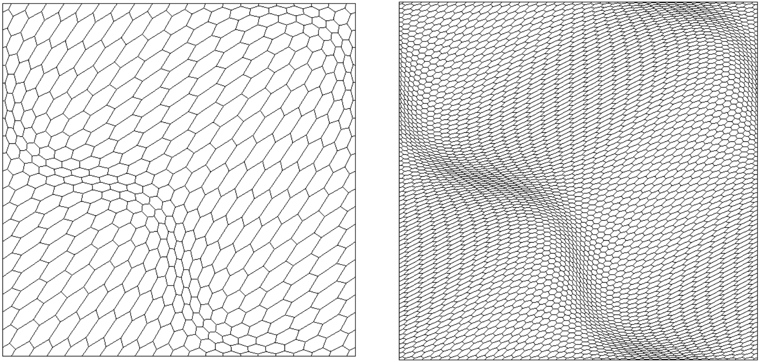
Behaviour:

- $E_{\mathrm{a},\boldsymbol{K},h}$ independent of λ.
- $E_{1,h}$ impacted by λ, more for low orders.

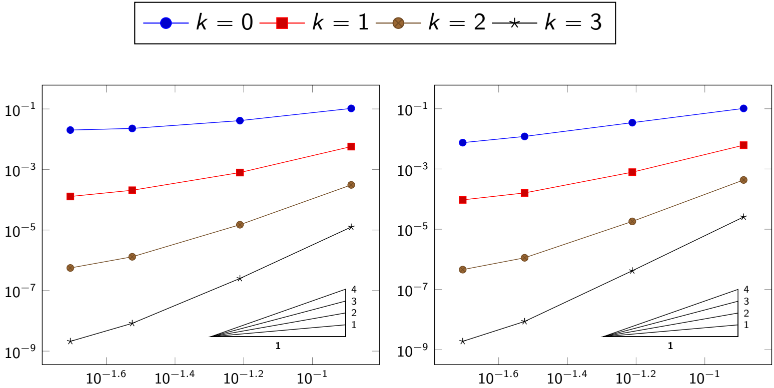
Plan

Description of test

Mesh: hexagonal stretched in x-direction as h is refined.



Solution and diffusion: $u(x,y) = \cos(\pi x)\cos(\pi y)$ and $\boldsymbol{K} = \mathrm{Id}$.



(a) $E_{\mathrm{a},\boldsymbol{\kappa},h}$ vs. h.

(b) $E_{1,h}$ vs. h.

Behaviour: loss of optimal rate of convergence.

Measure of mesh skewness effect

Element and mesh flatness: with ρ_T inradius of T,

$$\mathrm{fl}_T = \frac{h_T}{\rho_T}\,, \qquad \mathrm{fl}_h = \max_{T\in\mathcal{T}_h} \mathrm{fl}_T.$$

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$$\mathrm{fl}_T = \frac{h_T}{\rho_T}\,, \qquad \mathrm{fl}_h = \max_{T\in\mathcal{T}_h} \mathrm{fl}_T.$$

Predicted error: $\lambda_T = 1$, $a_T = 1$ and $b_T \sim \mathrm{fl}_T$ so $A_{\boldsymbol{\kappa},h} \sim \mathrm{fl}_h^2$ and

$$\frac{E_{\mathrm{a},\boldsymbol{\kappa},h}}{h^{k+1}} \sim \mathrm{fl}_h^{2}.$$

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$$\frac{E_{\mathrm{a},\boldsymbol{K},h}}{h^{k+1}} \sim \text{fl}_h^{2}.$$

Rates w.r.t. fl_h:

h	fl_h	$\frac{E_{\mathrm{a},\boldsymbol{K},h}}{h^{k+1}}$	rate
0.13	10	8e-01	–
0.06	22	6.7e-01	-0.2
0.03	46	7.6e-01	0.2
0.02	70	1e+00	0.7

$k = 0$

h	fl_h	$\frac{E_{\mathrm{a},\boldsymbol{K},h}}{h^{k+1}}$	rate
0.13	10	3.4e-01	–
0.06	22	2.1e-01	-0.6
0.03	46	2.3e-01	0.1
0.02	70	3.3e-01	0.8

$k = 1$

h	fl_h	$\frac{E_{\mathrm{a},\boldsymbol{K},h}}{h^{k+1}}$	rate
0.13	10	1.4e-01	–
0.06	22	6.4e-02	-1
0.03	46	4.9e-02	-0.4
0.02	70	7.3e-02	0.9

$k = 2$

h	fl_h	$\frac{E_{\mathrm{a},\boldsymbol{K},h}}{h^{k+1}}$	rate
0.13	10	4.4e-02	–
0.06	22	1.8e-02	-1.2
0.03	46	1.0e-02	-0.7
0.02	70	1.4e-02	0.7

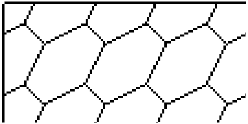
$k = 3$

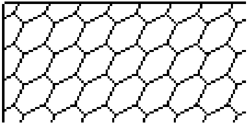
- Much lower impact of fl_h as expected.
- $E_{1,h}$ even less sensitive to fl_h.

Plan

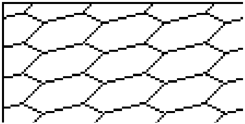
Meshes: hexagonal regular, and hexagonal stretched in x-direction (fl$_h$ doubling from one mesh to the next).

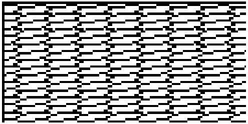
Regular:





Skewed:

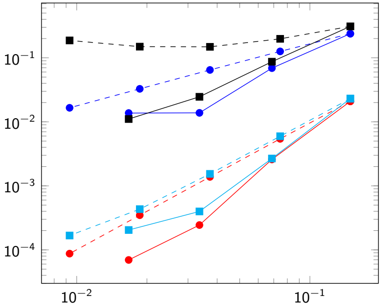




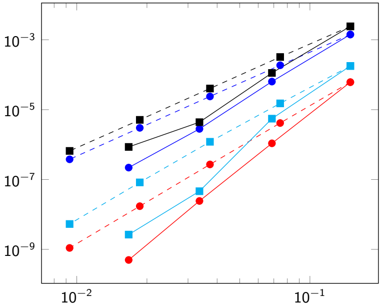
Solution and diffusion: $u(x,y) = \cos(\pi x)\cos(\pi y)$,

$$K = \begin{bmatrix} 10^6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Rates of convergence w.r.t. h



(a) Errors vs. h for $k = 0$ (top four plots) and $k = 1$ (bottom four plots).

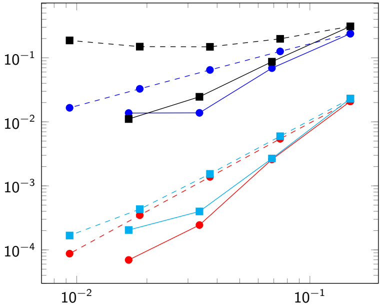


(b) Errors vs. h for $k = 2$ (top four plots) and $k = 3$ (bottom four plots).

Dashed lines: regular meshes; Continuous lines: skewed meshes.

Round markers: $E_{\mathrm{a},\boldsymbol{\kappa},h}$; *square markers:* $E_{1,h}$.

Rates of convergence w.r.t. h



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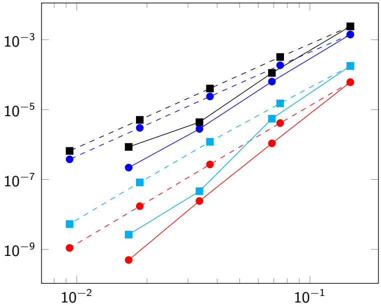

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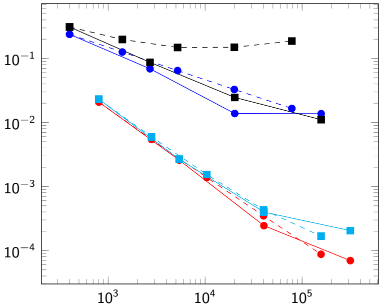
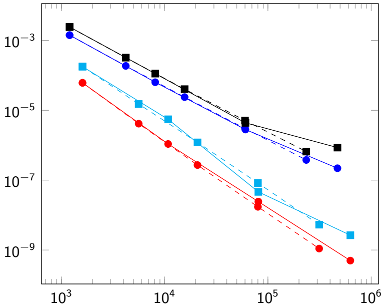
- Clear improvement when using meshes that are skewed in the direction of the diffusion.

Rates of convergence w.r.t. nb degrees of freedom



(a) Errors vs. nb DOFs for $k = 0$ (top four plots) and $k = 1$ (bottom four plots).



(b) Errors vs. nb DOFs for $k = 2$ (top four plots) and $k = 3$ (bottom four plots).

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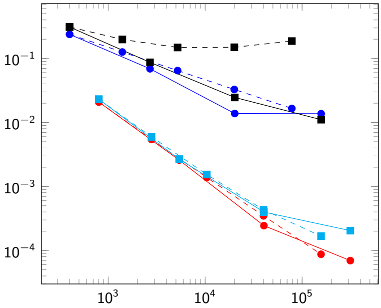

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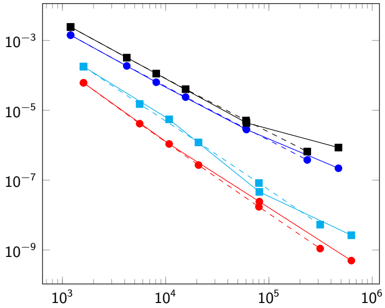

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Dashed lines: regular meshes; Continuous lines: skewed meshes.

Round markers: $E_{\mathrm{a},\boldsymbol{\kappa},h}$; square markers: $E_{1,h}$.

- Improvement less clear than w.r.t. h.

Meshes that are skewed "everywhere" have more edges than regular meshes.

Conclusions

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- No particular tweak to the method, standard HHO method.
- Interplay between mesh skewness and diffusion directions.
- Numerical results confirm interplay, but also show more robustness of HHO than in theoretical error estimate.
- Future work:
  - Make theoretical error estimate sharper *(only needs to be done for regular meshes)*.
  - Tweak HHO to make it robust uniformly with respect to mesh skewness.
  - Deal with small faces and/or lots of faces per element.

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Thanks.