Polynomial de Rham sequences of arbitrary degree on polyhedral meshes

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Discrete Functional Analysis: bridging pure and numerical mathematics

References

- Local DDR sequences [DP, Droniou and Rapetti, 2020]
- Global DDR sequences and stability [Di Pietro and Droniou, 2020a]
- Primal and dual consistency [DP and Droniou, ongoing]
- See [Di Pietro and Droniou, 2020b] for polytopal analysis tools

Outline

A (not so simple) model problem I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain that does not enclose any void
- Let a current density $\boldsymbol{J} \in \mathbf{curl}\,\mathbf{H}(\mathbf{curl};\Omega)$ be given
- We consider the problem: Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and the vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu\boldsymbol{H} - \mathbf{curl}\,\boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div}\boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

A (not so simple) model problem II

- In weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 & \forall \boldsymbol{\tau} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{A}\, \mathrm{div}\, \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} & \forall \boldsymbol{v} \in \mathbf{H}(\mathrm{div}; \Omega) \end{aligned}$$

- Well-posedness hinges on the exactness of the following portion of the de Rham sequence:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}; \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

Well-posedness analysis

Re-cast weak formulation as $\mathcal{A}((\boldsymbol{H}, \boldsymbol{A}), (\boldsymbol{\tau}, \boldsymbol{v})) = \ell(\boldsymbol{v})$ with

$$\mathcal{A}((\boldsymbol{H}, \boldsymbol{A}), (\boldsymbol{\tau}, \boldsymbol{v})) = \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} + \int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v}$$

Well-posedness analysis

$$\mathcal{A}((\boldsymbol{H},\boldsymbol{A}),(\boldsymbol{\tau},\boldsymbol{v})) = \int_\Omega \mu \boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} + \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \operatorname{div}\boldsymbol{A}\operatorname{div}\boldsymbol{v}$$

Proof of inf-sup property:

- Make $(\boldsymbol{\tau},\boldsymbol{v}) = (\boldsymbol{H},\boldsymbol{A})$ to estimate $\|\boldsymbol{H}\|_{L^2(\Omega)}$ and $\|\operatorname{div}\boldsymbol{A}\|_{L^2(\Omega)}$, then $(\boldsymbol{\tau},\boldsymbol{v}) = (\mathbf{0},\mathbf{curl}\,\boldsymbol{H})$ to estimate $\|\mathbf{curl}\,\boldsymbol{H}\|_{L^2(\Omega)}$.

Well-posedness analysis

$$\mathcal{A}((\boldsymbol{H},\boldsymbol{A}),(\boldsymbol{\tau},\boldsymbol{v})) = \int_\Omega \mu \boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} + \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \operatorname{div}\boldsymbol{A}\operatorname{div}\boldsymbol{v}$$

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- Write $\boldsymbol{A} = \boldsymbol{A}^\star + \boldsymbol{A}^\perp \in \operatorname{Ker}\operatorname{div} \oplus (\operatorname{Ker}\operatorname{div})^\perp$. Since $\operatorname{Im}\operatorname{div} = L^2(\Omega)$, we have an isomorphism $\operatorname{div} : (\operatorname{Ker}\operatorname{div})^\perp \to L^2(\Omega)$ and thus

$$\|\boldsymbol{A}^\perp\|_{L^2(\Omega)} \le C\|\operatorname{div}\boldsymbol{A}^\perp\|_{L^2(\Omega)} = C\|\operatorname{div}\boldsymbol{A}\|_{L^2(\Omega)}.$$

Well-posedness analysis

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- Use $\operatorname{Im}\mathbf{curl} = \operatorname{Ker}\operatorname{div}$ to see that $\mathbf{curl} : (\operatorname{Ker}\mathbf{curl})^\perp \to \operatorname{Ker}\operatorname{div}$ is an isomorphism and thus find $\boldsymbol{\tau} \in (\operatorname{Ker}\mathbf{curl})^\perp$ s.t. $\mathbf{curl}\,\boldsymbol{\tau} = -\boldsymbol{A}^\star$ and $\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\|\boldsymbol{A}^\star\|_{L^2(\Omega)}$.

  $\rightsquigarrow$ Use $(\boldsymbol{\tau},\boldsymbol{0})$ in $\mathcal{A}$ to estimate $\|\boldsymbol{A}^\star\|_{L^2(\Omega)}$.

Well-posedness analysis

$$\mathcal{A}((\boldsymbol{H},\boldsymbol{A}),(\boldsymbol{\tau},\boldsymbol{v})) = \int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{\tau} + \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{v} + \int_\Omega \operatorname{div}\boldsymbol{A}\operatorname{div}\boldsymbol{v}$$

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  $\rightsquigarrow$ Use $(\boldsymbol{\tau},\boldsymbol{0})$ in $\mathcal{A}$ to estimate $\|\boldsymbol{A}^\star\|_{L^2(\Omega)}$.

The exactness property is also essential at the discrete level!

- **Key idea:** define subspaces that form exact sequence
- Let $T \subset \mathbb{R}^3$ be a polyhedron and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) := \{\text{restrictions of 3-variate polynomials of degree} \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, taking $\boldsymbol{x}_T \in T$,

$$\begin{aligned}\mathcal{P}^k(T)^3 &= \underbrace{\mathbf{grad}\,\mathcal{P}^{k+1}(T)}_{\mathcal{G}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathcal{P}^{k-1}(T)^3}_{\mathcal{G}^{\mathrm{c},k}(T)}\\ &= \underbrace{\mathbf{curl}\,\mathcal{P}^{k+1}(T)^3}_{\mathcal{R}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{k-1}(T)}_{\mathcal{R}^{\mathrm{c},k}(T)}\end{aligned}$$

- Define the trimmed spaces

$$\mathcal{N}^{k+1}(T) := \mathcal{G}^k(T) \oplus \mathcal{G}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\mathcal{RT}^{k+1}(T) := \mathcal{R}^k(T) \oplus \mathcal{R}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

The Finite Element way

Global FE sequence

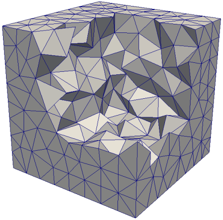
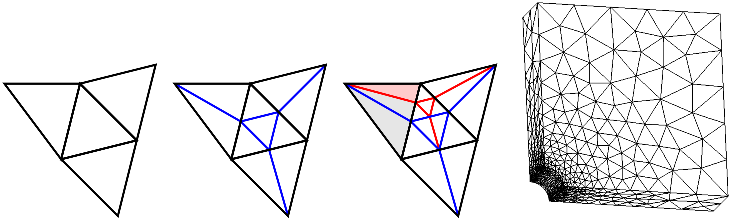


Figure: Conforming tetrahedral mesh of the unit cube (clip)

- Let $\mathcal{T}_h = \{T\}$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form the global FE sequence

$$\mathbb{R} \xrightarrow{i_\Omega} \mathcal{P}_c^{k+1}(\mathcal{T}_h) \xrightarrow{\textbf{grad}} \boldsymbol{N}^k(\mathcal{T}_h) \xrightarrow{\textbf{curl}} \boldsymbol{\mathcal{RT}}^k(\mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- **This procedure only works on conforming meshes!**
- See [Arnold, 2018] for a unified approach (FE Exterior Calculus)



- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ complex geometries may require a large number of elements
  - $\Longrightarrow$ the element shape cannot be adapted to the solution
- The implementation of high-order versions may be tricky
- ...

The Virtual Elements way

$$\mathbb{R} \xrightarrow{i_\Omega} V^k_{\text{vert}} \xrightarrow{\textbf{grad}} V^{k-1}_{\text{edge}} \xrightarrow{\textbf{curl}} V^{k-2}_{\text{face}} \xrightarrow{\text{div}} \mathcal{P}^{k-3}(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- The good:
  - Conforming: each discrete space is a subspace of the corresponding continuous space.
  - Applicable to generic meshes with polyhedral elements

- The bad:
  - Degree decreases by one at each application of differential operator
  - Functions not fully known, only certain moments or values are accessible
  - Exactness not usable in a scheme due to the variational crime (only certain projections of the functions are used in the scheme)

The discrete de Rham (DDR) approach I

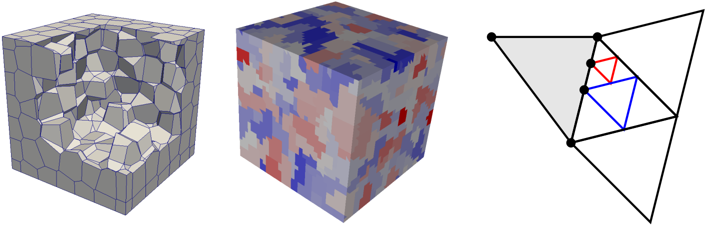


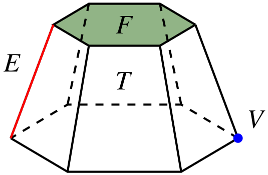
Figure: Examples of polytopal meshes supported by the DDR approach

- **Key idea:** replace spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of general polyhedral meshes and high-order
- Exactness proved at the discrete level (directly usable for stability)
- (Relatively) simple implementation of high-order versions

The discrete de Rham (DDR) approach II



- The fully discrete spaces are spanned by vectors of polynomials
- Polynomial components attached to geometric objects to mimic
  - full continuity for the approximation of $H^1(\Omega)$
  - continuity of tangential traces for the approximation of $\mathbf{H}(\mathbf{curl};\Omega)$
  - continuity of normal traces for the approximation of $\mathbf{H}(\mathrm{div};\Omega)$
- Selected so as to enable the reconstruction of consistent
  - discrete vector calculus operators
  - (scalar or vector) discrete potentials

Outline

- Let $F \subset \mathbb{R}^2$ be a simply connected polygon embedded in $\mathbb{R}^3$
- Let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q \coloneqq \varrho_{-\pi/2}(\mathbf{grad}_F\, q) \qquad \mathrm{rot}_F\, \boldsymbol{v} \coloneqq \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v})$$

- We derive a discrete counterpart of the exact local sequence:

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\mathbf{grad}_F} \mathbf{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

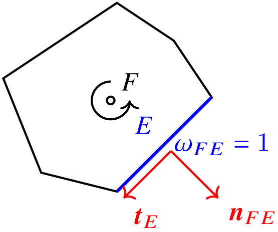
$$\begin{aligned}\mathcal{P}^k(F)^2 &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{G}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp \mathcal{P}^{k-1}(F)}_{\mathcal{G}^{\mathrm{c},k}(F)}\end{aligned}$$

- Denote by $\pi_{\mathcal{P},F}^{k-1}$ the L^2-orthogonal projector on $\mathcal{P}^{k-1}(F)$
- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

$$= -\int_F \pi_{\mathcal{P},F}^{k-1} q \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi_{\mathcal{P},F}^{k-1} q$ and $q_{|\partial F}$



- The discrete counterpart of $H^1(F)$ is

$$\boxed{\underline{X}^k_{\mathbf{grad},F} \coloneqq \left\{\underline{q}_F = (q_F, q_{\partial F}) \,:\, q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_F)\right\}}$$

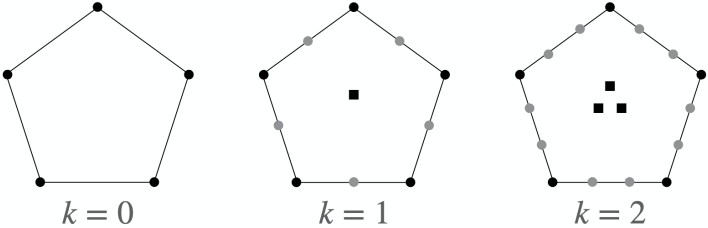


Figure: Number of degrees of freedom for $\underline{X}^k_{\mathbf{grad},F}$ for $k \in \{0, 1, 2\}$

- The interpolator $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ is s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q \coloneqq (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \;\forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \;\forall V \in \mathcal{V}_F$$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F \coloneq (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- Using another IBP, we reconstruct a face potential in $\mathcal{P}^{k+1}(F)$

- We reason starting from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F) := \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) \underbrace{q_{|E}}_{\in \mathcal{P}^k(E)} \qquad \forall q \in \mathcal{P}^k(F)$$

- This leads to the following discrete counterpart of $\mathbf{H}(\mathrm{rot}; F)$:

$$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}, (v_E)_{E \in \mathcal{E}_F}\big) : \boldsymbol{v}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$

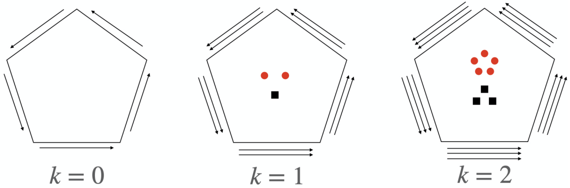


Figure: Number of degrees of freedom for $\underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ for $k \in \{0, 1, 2\}$

- The face curl operator $C^k_F : \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C^k_F \underline{\boldsymbol{v}}_F \; q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \; q \quad \forall q \in \mathcal{P}^k(F)}$$

- Define the interpolator $\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} : H^1(F)^2 \to \underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ s.t., $\forall \boldsymbol{v} \in H^1(F)^2$,

$$\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v} \coloneqq \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F} \boldsymbol{v}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F} \boldsymbol{v}, \left(\pi^k_{\mathcal{P},E} (\boldsymbol{v}_{|E} \cdot \boldsymbol{t}_E) \right)_{E \in \mathcal{E}_F} \right).$$

- C^k_F is polynomially consistent by construction:

$$C^k_F \left(\underline{\boldsymbol{I}}^k_{\mathrm{rot},F} \boldsymbol{v} \right) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{N}^{k+1}(F)$$

- By another IBP, we reconstruct a vector potential in $\mathcal{P}^k(F)^2$

Theorem (Exactness of the two-dimensional local DDR sequence)

If F is simply connected, the following local sequence is exact:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\},$$

where $\underline{\boldsymbol{G}}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$ is the discrete gradient s.t., $\forall \underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F := \left(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), (G^k_E \underline{q}_F)_{E\in\mathcal{E}_F} \right)$$

The two-dimensional case

Summary

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathrm{rot},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (polygon)
$\underline{X}^k_{\mathbf{grad},F}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}^k_{\mathrm{rot},F}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

Table: Polynomial components for the two-dimensional spaces

- Interpolators = component-wise L^2-projections
- Discrete operators = L^2-projections of full operator reconstructions

The three-dimensional case I

Exact sequence

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F (face)	T (polyhedron)
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Table: Polynomial components for the three-dimensional spaces

Theorem (Exactness of the three-dimensional local DDR sequence)

If the polyhedron T has a trivial topology, this sequence is exact.

The three-dimensional case II

Exact sequence

Lemma (Commutative diagram with the sequence of trimmed spaces)

The following commutative diagram holds, expressing the polynomial consistency of the discrete vector calculus operators:

$$
\begin{array}{ccccccc}
\mathcal{P}^{k+1}(T) & \xrightarrow{\mathbf{grad}} & \boldsymbol{\mathcal{N}}^{k+1}(T) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}^{k+1}(T) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{k}(T) \\
\Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathbf{grad},T} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathbf{curl},T} & & \Big\downarrow \underline{\boldsymbol{I}}^{k}_{\mathrm{div},T} & & \Big\downarrow i_T \\
\underline{X}^{k}_{\mathbf{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^{k}_{T}} & \underline{\boldsymbol{X}}^{k}_{\mathbf{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^{k}_{T}} & \underline{\boldsymbol{X}}^{k}_{\mathrm{div},T} & \xrightarrow{D^{k}_{T}} & \mathcal{P}^{k}(T)
\end{array}
$$

The three-dimensional case
Local discrete L^2-products

- Emulating integration by part formulas, define the local potentials

$$
\begin{aligned}
P^{k+1}_{\mathbf{grad},T} &: \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T),\\
\boldsymbol{P}^k_{\mathbf{curl},T} &: \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3,\\
\boldsymbol{P}^k_{\mathrm{div},T} &: \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \mathcal{P}^k(T)^3
\end{aligned}
$$

- Based on these potentials, we construct local discrete L^2-products

$$
(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}
$$

- The L^2-products are polynomially exact

The three-dimensional case

Global sequence

- Let $\Omega \subset \mathbb{R}^3$ as before and let $\mathcal{T}_h$ be a polyhedral mesh
- Global DDR spaces are defined gluing boundary components:

$$\underline{\boldsymbol{X}}^k_{\mathbf{grad},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}^k_h, \qquad \underline{\boldsymbol{C}}^k_h, \qquad D^k_h$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise
- The global DDR sequence is

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Outline

Exactness

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Theorem (Exactness properties of the global DDR sequence)

Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain.

- *If Ω is connected, it holds* $\boxed{\mathrm{Im}\,\underline{I}^k_{\mathrm{grad},h} = \mathrm{Ker}\,\underline{\boldsymbol{G}}^k_h}$.
- *If Ω is simply connected, it holds* $\boxed{\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h}$.
- *If Ω does not enclose any void, it holds* $\boxed{\mathrm{Im}\,\underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\,D^k_h}$.
- *It holds* $\boxed{\mathrm{Im}\,D^k_h = \mathcal{P}^k(\mathcal{T}_h)}$.

Exactnes: elements of proof I

- $\operatorname{Im} D_h^k = \mathcal{P}^k(\mathcal{T}_h)$ follows from the classical Fortin's argument
- The inclusions $\operatorname{Im} \underline{\boldsymbol{C}}_h^k \subset \operatorname{Ker} D_h^k$ and $\operatorname{Im} \underline{\boldsymbol{I}}_{\mathbf{grad},h}^k \subset \operatorname{Ker} \underline{\boldsymbol{G}}_h^k$ result from local exactness
- The inclusion $\operatorname{Ker} \underline{\boldsymbol{C}}_h^k \subset \operatorname{Im} \underline{\boldsymbol{G}}_h^k$ starts by creating a continuous polynomial on the skeleton, integrating over edge paths the edge components of $\underline{\boldsymbol{v}}_h \in \operatorname{Ker} \underline{\boldsymbol{C}}_h^k$.

Exactnes: elements of proof II

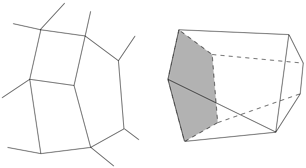
Most challenging: $\operatorname{Ker} D_h^k \subset \operatorname{Im} \underline{\boldsymbol{C}}_h^k$.

In two steps: if $\underline{\boldsymbol{v}}_h \in \operatorname{Ker} D_h^k$, then:

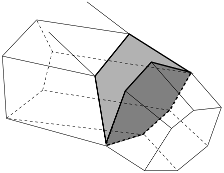
- Local exactness gives $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$ s.t. $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{\tau}}_T$ for all $T \in \mathcal{T}_h$
- The local vectors are then glued together, by topological assembly of the mesh using a succession of the following operations.

Exactnes: elements of proof III

1. Add a new element by glueing one ot its faces to an element in the mesh



2. Glue together two faces of elements in the mesh s.t. the edges along which the faces are already glued together form a connected path



This is only possible since Ω does not enclose any void!

Consistency and Poincaré inequalities

- Primal consistency: for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, optimal local L^2-approximation properties of $P_{\bullet,T} I_{\bullet,T}$, e.g.

$$\|P^{k+1}_{\mathbf{grad},T} \underline{I}^k_{\mathbf{grad},T} q - q\|_{L^2(T)} \leq C h_T^{k+2} |q|_{H^{k+2}(T)}.$$

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- Dual consistency: estimate error in discrete global integration-by-parts, e.g.

$$\left|\sum_{T\in\mathcal{T}_h}\left[(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\,\boldsymbol{w}_{|T}, \underline{\boldsymbol{C}}^k_T\,\underline{\boldsymbol{v}}_T)_{\mathrm{div},T} - \int_T \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{P}^k_{\mathbf{curl},T}\,\underline{\boldsymbol{v}}_T\right]\right| \le C h^{k+1} |\boldsymbol{w}|_{H^{k+2}(\mathcal{T}_h)^3}\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right).$$

Consistency and Poincaré inequalities

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- Poincaré inequalities: for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, bound the norm of $\underline{\boldsymbol{v}}_h \in (\mathrm{Ker}\,\bullet)^\perp$ in terms of the norm of $\bullet\underline{\boldsymbol{v}}_h$, e.g.,

$$\|\boldsymbol{v}_h\|_{\mathrm{div},h} \le C\|D^k_h \underline{\boldsymbol{v}}_h\|_{L^2(\Omega)} \quad \forall \underline{\boldsymbol{v}}_h \in (\mathrm{Ker}\, D^k_h)^\perp.$$

Outline

- Continuous weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$ s.t.

$$\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} = 0 \qquad \forall \boldsymbol{\tau} \in \mathbf{H}(\mathbf{curl}; \Omega),$$
$$\int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{A}\, \mathrm{div}\, \boldsymbol{v} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \mathbf{H}(\mathrm{div}; \Omega)$$

- The global bilinear forms are approximated substituting

$$(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} \leftarrow \int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau}$$
$$(\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} \leftarrow \int_\Omega \mathbf{curl}\, \boldsymbol{\tau} \cdot \boldsymbol{v}$$
$$\int_\Omega D_h^k \underline{\boldsymbol{w}}_h \; D_h^k \underline{\boldsymbol{v}}_h \leftarrow \int_\Omega \mathrm{div}\, \boldsymbol{w} \;\; \mathrm{div}\, \boldsymbol{v}$$

- The current density linear form is l_h, defined similarly

- The DDR problem reads: Find $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned}
(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\
(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{H}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h \, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}
\end{aligned}$$

- Stability hinges on the exactness of the portion

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Theorem (Well-posedness)

Let $\Omega \subset \mathbb{R}^3$ be an open simply connected polyhedral domain that does not enclose any void. Then, $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ is unique and there exists $C > 0$ independent of h s.t.

$$\|\underline{\boldsymbol{H}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{H}}_h\|_{\mathrm{div},h} + \|\underline{\boldsymbol{u}}_h\|_{\mathrm{div},h} + \|D^k_h \underline{\boldsymbol{u}}_h\|_{L^2(\Omega)} \leq C \|\boldsymbol{J}\|_\Omega.$$

Proof.

Reproduce the proof of continuous case, replacing spaces/operators by discrete ones. □

- Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a regular polyhedral mesh sequence
- We consider a known solution $(\boldsymbol{H}, \boldsymbol{A})$ to assess convergence rate
- The error

$$(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h) \coloneqq (\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{H}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{A})$$

  is measured in the natural energy norm s.t.

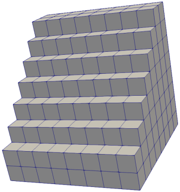
$$\|(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h)\|_{\mathrm{en},h} \coloneqq \left[(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{e}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{\varepsilon}}_h)_{\mathrm{div},h}\right]^{\frac{1}{2}}$$

- The implementation is based on the HArDCore3D C++ library[1]

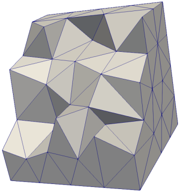
[1]See `https://tinyurl.com/HarDCore3D`

Numerical examples

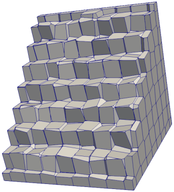
Meshes



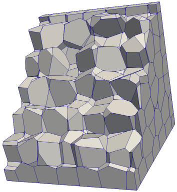
(a) Cubic-Cells



(b) Tetgen-Cube-0



(c) Voro-small-0



(d) Voro-small-1

Figure: Mesh families used in the numerical tests

Numerical examples

Convergence in the energy norm

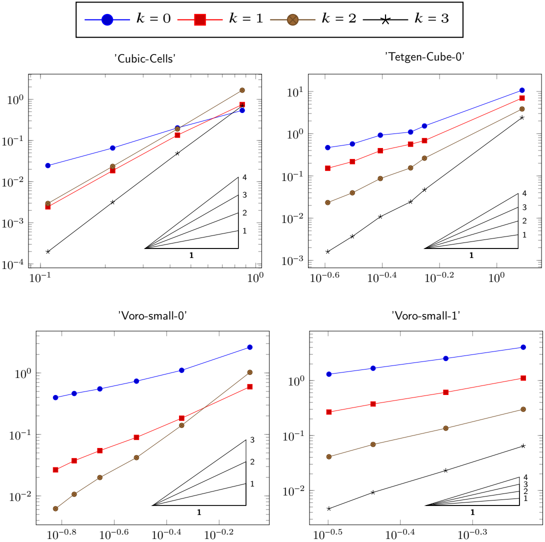


Figure: Energy error versus mesh size h. We have $\|(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h)\|_{\mathrm{en},h} \propto h^{k+1}$

Conclusions and perspectives

- A novel approach for the numerical solution of PDE problems
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to variable coefficients and nonlinearities

- Applications (electromagnetism, incompressible fluid mechanics,. . .)
- Formalization using differential forms (ongoing)
- Development of novel sequences (e.g., elasticity)
- . . .

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