Pressure-Robust Discrete De Rham and Virtual Element Schemes for the Stokes Problem

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joint work with
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Plan

The Stokes problem in curl-curl formulation

- Given Ω contractible, $\nu > 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem reads:

  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned}
\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(momentum conservation)}\\
\mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(mass conservation)}\\
\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, && \text{(boundary conditions)}\\
\int_\Omega p &= 0
\end{aligned}$$

- Weak formulation: relevant Hilbert spaces:

$$\begin{aligned}
H^1(\Omega) &:= \left\{q \in L^2(\Omega) \;:\; \mathbf{grad}\,q \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^3\right\},\\
\boldsymbol{H}(\mathbf{curl};\Omega) &:= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;:\; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\right\},\\
\boldsymbol{H}(\mathrm{div};\Omega) &:= \left\{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \;:\; \mathrm{div}\,\boldsymbol{w} \in L^2(\Omega)\right\}
\end{aligned}$$

The Stokes problem in curl-curl formulation

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  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$\overbrace{\nu(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u})}^{-\nu\Delta\boldsymbol{u}} + \mathbf{grad}\,p = \boldsymbol{f} \quad \text{in } \Omega, \qquad \text{(momentum conservation)}$$

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \text{(mass conservation)}$$

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \qquad \text{(boundary conditions)}$$

$$\int_\Omega p = 0$$

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),$$

$$-\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q = 0 \qquad \forall q \in H^1(\Omega)$$

Sketch of stability analysis

$$\begin{aligned} \int_\Omega \nu \,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

1. Make $\boldsymbol{v} = \boldsymbol{u}$ and $q = p \rightsquigarrow \|\mathbf{curl}\,\boldsymbol{u}\|^2 \leq \nu^{-1}\|\boldsymbol{f}\|\|\boldsymbol{u}\|$.

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2. Estimate $\|\boldsymbol{u}\|$:

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2. Estimate $\|\boldsymbol{u}\|$:
   - Write $\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^\perp \in \ker\mathbf{curl} \oplus (\ker\mathbf{curl})^\perp$.

Poincaré inequality: $\|\cdot\| \le C\|\mathbf{curl}\cdot\|$ on $(\ker\mathbf{curl})^\perp$

   - So $\|\boldsymbol{u}^\perp\| \le C\|\mathbf{curl}\,\boldsymbol{u}^\perp\| = C\|\mathbf{curl}\,\boldsymbol{u}\|$.

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No tunnel in $\Omega \Rightarrow \ker \mathbf{curl} = \operatorname{Im} \mathbf{grad}$

   - So $\boldsymbol{u}^0 = \mathbf{grad}\, q$ and thus $\|\boldsymbol{u}^0\| \leq C\|\boldsymbol{u}^\perp\| \leq C\|\mathbf{curl}\, \boldsymbol{u}\|$.
   - Combine: $\|\boldsymbol{u}\| \leq C\|\mathbf{curl}\, \boldsymbol{u}\|$.

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   - So $\boldsymbol{u}^0 = \mathbf{grad}\, q$ and thus $\|\boldsymbol{u}^0\| \le C\|\boldsymbol{u}^\perp\| \le C\|\mathbf{curl}\, \boldsymbol{u}\|$.
   - Combine: $\|\boldsymbol{u}\| \le C\|\mathbf{curl}\, \boldsymbol{u}\|$.
3. Make $\boldsymbol{v} = \mathbf{grad}\, p \rightsquigarrow \|\mathbf{grad}\, p\| \le \|\boldsymbol{f}\|$.

De Rham complex

- The de Rham sequence is

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- It is a complex: the range of each operator is included in the kernel of the next one.

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- Exact complex (inclusions $\rightsquigarrow$ equalities) if Ω has a trivial topology:

$$\mathbb{R} = \ker \mathbf{grad}\,,\ \ \mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}\,,\ \ \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,,\ \ \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

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$$\mathbb{R} = \ker \mathbf{grad}\,,\ \boxed{\mathrm{Im}\,\mathbf{grad} = \ker \mathbf{curl}}\,,\ \mathrm{Im}\,\mathbf{curl} = \ker \mathrm{div}\,,\ \mathrm{Im}\,\mathrm{div} = L^2(\Omega).$$

- Exactness $\Rightarrow$ well-posedness of the Stokes problem in curl–curl form *(same for the Stokes problem in Δ form...)*.

> Reproducing this exactness at the discrete level is instrumental to designing stable numerical approximations.

Plan

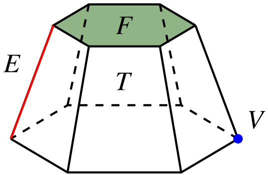
Mesh notations

- Mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h)$ where:
  - $\mathcal{T}_h$ set of elements,
  - $\mathcal{F}_h$ set of faces, with $\mathcal{F}_T$ faces attached to $T \in \mathcal{T}_h$, and each $F \in \mathcal{F}_h$ oriented by the choice of a normal $\boldsymbol{n}_F$,
  - $\mathcal{E}_h$ set of edges, with $\mathcal{E}_F$ edges attached to $F \in \mathcal{F}_h$, and each $E \in \mathcal{E}_h$ oriented by the choice of a tangent $\boldsymbol{t}_E$,
  - $\mathcal{V}_h$ set of vertices.
- $\omega_{TF} \in \{+1, -1\}$ orientation of $F \in \mathcal{F}_T$ w.r.t. T, such that $\omega_{TF}\boldsymbol{n}_F$ outer normal to T.
- $\omega_{FE} \in \{+1, -1\}$ orientation of $E \in \mathcal{E}_F$ w.r.t. F, such that $\omega_{FE}\boldsymbol{t}_E$ clockwise on F.

Overview: Discrete de Rham

Principle:

- Replace continuous spaces by fully discrete ones made of vectors of polynomials,
- Polynomials attached to geometric entities to emulate expected continuity properties of each space,
- Create discrete operators between them.



DDR complex:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

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Example: For $\mathsf{P} = F, T$, set

$$\mathcal{R}^{k-1}(\mathsf{P}) = \mathbf{curl}\,\mathcal{P}^k(\mathsf{P})\,, \qquad \mathcal{R}^{\mathrm{c},k}(\mathsf{P}) = (\boldsymbol{x} - \boldsymbol{x}_{\mathsf{P}})\mathcal{P}^{k-1}(\mathsf{P}).$$

Discrete $\boldsymbol{H}(\mathbf{curl};\Omega)$ space:

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} := \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ :& \\ \boldsymbol{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} \in \mathcal{R}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h,& \\ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \mathcal{R}^{\mathrm{c},k}(F) \text{ for all } F \in \mathcal{F}_h,& \\ \text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h \Big\},&\end{aligned}$$

- $v_E \sim$ projection of tangential component to E.
- $\boldsymbol{v}_{\mathcal{R},\mathsf{P}}\,,\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},\mathsf{P}} \sim$ projections on $\mathcal{R}^{k-1}(\mathsf{P})\,,\ \mathcal{R}^{\mathrm{c},k}(\mathsf{P})$.

Restriction to a face:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\rm c}_{\mathcal{R},F}, (v_E)_{E\in\mathcal{E}_F} \big) \ : \ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}^{\rm c}_{\mathcal{R},F} \in \mathcal{R}^{{\rm c},k}(F), \text{ and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_F \Big\},$$

- Integration-by-parts: for a smooth $\boldsymbol{v}$,

$$\int_F (\mathrm{rot}_F\, \boldsymbol{v})\ r_F = \int_F \boldsymbol{v} \cdot \mathbf{rot}_F\, r_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v}\cdot \boldsymbol{t}_E) r_F\,.$$

Overview: Discrete de Rham

Restriction to a face:

$$\underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} \coloneqq \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}_{\mathcal{R},F}^{\rm c}, (v_E)_{E\in\mathcal{E}_F}\big) \ : \ \boldsymbol{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \boldsymbol{v}_{\mathcal{R},F}^{\rm c} \in \mathcal{R}^{{\rm c},k}(F), \text{ and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_F \Big\},$$

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- $\rightsquigarrow$ discrete face curl $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^{k} \to \mathcal{P}^k(F)$ such that

$$\int_F C_F^k \underline{\boldsymbol{v}}_F r_F = \int_F \boldsymbol{v}_{\mathcal{R},F}\cdot\mathrm{rot}_F\, r_F - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E v_E r_F\,.$$

C_F^k = face component of $\underline{\boldsymbol{C}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{curl},h}^{k} \to \underline{\boldsymbol{X}}_{\mathrm{div},h}^{k}$

Overview: Discrete de Rham

- Potential reconstructions for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$:
  - tangent trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \boldsymbol{\mathcal{P}}^k(F)$,
  - element potential $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \boldsymbol{\mathcal{P}}^k(T)$.

- L^2-like inner product on $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$:

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T)_{\mathbf{curl},T} = \int_T \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \cdot \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T),$$

($\mathrm{s}_{\mathbf{curl},T}$ penalises differences on the boundary between element and face potentials).

Overview: Virtual Element Method

Principle:

- Replace continuous spaces by finite-dimensional subspaces of functions that are not fully known,
- Select spaces and DOFs to ensure that key quantities (projections) are computable,
- These quantities should allow the computation of the relevant DOFs of the $\mathbf{grad}, \mathbf{curl}, \mathrm{div}$ of the functions.

VEM complex:

$$\mathbb{R} \xrightarrow{\;\mathrm{i}\;} V^{\mathrm{n}}_{k+1,0} \xrightarrow{\;\mathbf{grad}\;} V^{\mathrm{e}}_{k} \xrightarrow{\;\mathbf{curl}\;} V^{\mathrm{f}}_{k} \xrightarrow{\;\mathrm{div}\;} V^{\mathrm{v}}_{k} \xrightarrow{\;0\;} 0,$$

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Example:

$$V^{\mathrm{e}}_{k} \coloneqq \left\{ \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \text{ such that } \boldsymbol{v}_{|T} \in V^{\mathrm{e}}_{k}(T) \quad \forall T \in \mathcal{T}_h \right\},$$

with

$$V^{\mathrm{e}}_{k}(T) \coloneqq \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(T) \ : \mathrm{div}\,\boldsymbol{v} \in \mathcal{P}^{k-1}(T),\ \mathbf{curl}(\mathbf{curl}\,\boldsymbol{v}) \in \mathcal{P}^{k}(T),$$
$$\boldsymbol{v}_{\mathrm{t},F} \in V^{\mathrm{e}}_{k}(F) \quad \forall F \in \mathcal{F}_T,\ \boldsymbol{v}\cdot\boldsymbol{t}_E \text{ single valued on } E \in \mathcal{E}_T \Big\},$$

$$V^{\mathrm{e}}_{k}(F) \coloneqq \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(F) \ :\ \mathrm{div}_F\,\boldsymbol{v} \in \mathcal{P}^{k}(F),\ \mathrm{rot}_F\,\boldsymbol{v} \in \mathcal{P}^{k}(F), \Big\}$$
$$\boldsymbol{v}\cdot\boldsymbol{t}_E \in \mathcal{P}^{k}(E) \quad \forall E \in \mathcal{E}_F.$$

Overview: Virtual Element Method

$$V_k^{\mathrm{e}}(T) := \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(T) \ : \mathrm{div}\, \boldsymbol{v} \in \mathcal{P}^{k-1}(T),\ \mathbf{curl}(\mathbf{curl}\, \boldsymbol{v}) \in \mathcal{P}^k(T),$$
$$\boldsymbol{v}_{\mathrm{t},F} \in V_k^{\mathrm{e}}(F) \quad \forall F \in \mathcal{F}_T,\ \boldsymbol{v} \cdot \boldsymbol{t}_E \text{ single valued on } E \in \mathcal{E}_T \Big\},$$

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$$\boldsymbol{v} \cdot \boldsymbol{t}_E \in \mathcal{P}^k(E) \quad \forall E \in \mathcal{E}_F.$$

- Degrees of freedom (DOFs) – known quantities on $\boldsymbol{v} \in V_k^{\mathrm{e}}(T)$:
  - $\int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) p$ for all $p \in \mathcal{P}^k(E)$, for all $E \in \mathcal{E}_T$,
  - $\int_F \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_F) p$ for all $p \in \mathcal{P}^k(F)$, for all $F \in \mathcal{F}_T$,
  - $\int_F \mathrm{rot}_F\, \boldsymbol{v}\ p$ for all $p \in \mathcal{P}^k(F)$ with $\int_F p = 0$, for all $F \in \mathcal{F}_T$.
  - $\int_T \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_T) p$ for all $p \in \mathcal{P}^{k-1}(T)$,
  - $\int_T (\mathbf{curl}\, \boldsymbol{v}) \cdot ((\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{P}^k(T)$.

Overview: Virtual Element Method

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  - $\int_F \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_F) p$ for all $p \in \mathcal{P}^k(F)$, for all $F \in \mathcal{F}_T$,
  - $\int_F \operatorname{rot}_F \boldsymbol{v} \;\; p$ for all $p \in \mathcal{P}^k(F)$ with $\int_F p = 0$, for all $F \in \mathcal{F}_T$.
  - $\int_T \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_T) p$ for all $p \in \mathcal{P}^{k-1}(T)$,
  - $\int_T (\mathbf{curl}\, \boldsymbol{v}) \cdot ((\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{p})$ for all $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}^k(T)$.

- Computable projection: $\boldsymbol{\pi}_{\mathcal{P},F}^{k+1} \boldsymbol{v}_{\mathrm{t},F} \in \boldsymbol{\mathcal{P}}^{k+1}(F)$ given by

$$\int_F \pi_{\mathcal{P},F}^{k+1} \boldsymbol{v}_{\mathrm{t},F} \cdot (\mathbf{rot}_F\, r + (\boldsymbol{x} - \boldsymbol{x}_F) p)$$

$$= \int_F \operatorname{rot}_F \boldsymbol{v} \;\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) r + \int_F \boldsymbol{v} \cdot ((\boldsymbol{x} - \boldsymbol{x}_F) p)$$

$$\forall r \in \mathcal{P}^{k+2}(F)\,, \;\forall p \in \mathcal{P}^k(F).$$

Overview: Virtual Element Method

- Also computable: $\pi^k_{\mathcal{P},T}\boldsymbol{v}$ and all the DOFs of $\mathbf{curl}\,\boldsymbol{v}$ in V^{f}_k.
- L^2-like inner product on $V^{\mathrm{e}}_k(T)$, computable from the DOFs:

$$[\boldsymbol{v}_T, \boldsymbol{w}_T]_{V^{\mathrm{e}}_k(T)} = \int_T \boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{v}_T \cdot \boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{w}_T + s^T((I-\boldsymbol{\pi}^k_{\mathcal{P},T})\underline{\boldsymbol{v}}_T, (I-\boldsymbol{\pi}^k_{\mathcal{P},T})\underline{\boldsymbol{w}}_T),$$

for suitable s^T.

Two complementary visions I

Virtualising DDR: there exists

$$X_{k+1}^{\mathrm{n}} \subset H^1(\Omega)\,, \quad X_k^{\mathrm{e}} \subset \boldsymbol{H}(\mathbf{curl};\Omega)\,, \quad X_k^{\mathrm{f}} \subset \boldsymbol{H}(\mathrm{div};\Omega)$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
X_{k+1}^{\mathrm{n}} & \xrightarrow{\mathbf{grad}} & X_k^{\mathrm{e}} & \xrightarrow{\mathbf{curl}} & X_k^{\mathrm{f}} & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(\mathcal{T}_h) \\
\downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \wr\ (\mathrm{DoF}) & & \downarrow \mathrm{Id} \\
\underline{X}_{\mathbf{grad},h}^k & \xrightarrow{\underline{\boldsymbol{G}}_h^k} & \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k & \xrightarrow{\underline{\boldsymbol{C}}_h^k} & \underline{\boldsymbol{X}}_{\mathrm{div},h}^k & \xrightarrow{D_h^k} & \mathcal{P}^k(\mathcal{T}_h)
\end{array}$$

Two complementary visions II

Fully discretising VEM: let

$$\underline{V}^{\mathrm{n}}_{k+1}\,,\ \underline{V}^{\mathrm{e}}_{k}\,,\ \underline{V}^{\mathrm{f}}_{k} = \text{ spaces gathering the DOFs of } V^{\mathrm{n}}_{k+1}\,,\ V^{\mathrm{e}}_{k}\,,\ V^{\mathrm{f}}_{k}.$$

Discrete operators can be created such that the following diagram commutes:

$$\begin{array}{ccccccc}
V^{\mathrm{n}}_{k+1} & \xrightarrow{\ \mathbf{grad}\ } & V^{\mathrm{e}}_{k} & \xrightarrow{\ \mathbf{curl}\ } & V^{\mathrm{f}}_{k} & \xrightarrow{\ \mathrm{div}\ } & \mathcal{P}^k(\mathcal{T}_h) \\
\Big\downarrow \wr\ (\mathrm{DoF}) & & \Big\downarrow \wr\ (\mathrm{DoF}) & & \Big\downarrow \wr\ (\mathrm{DoF}) & & \Big\downarrow \mathrm{Id} \\
\underline{V}^{\mathrm{n}}_{k+1} & \xrightarrow{\ \underline{\boldsymbol{G}}^{\mathrm{n},k}_h\ } & \underline{V}^{\mathrm{e}}_{k} & \xrightarrow{\ \underline{\boldsymbol{C}}^{\mathrm{e},k}_h\ } & \underline{V}^{\mathrm{f}}_{k} & \xrightarrow{\ D^{\mathrm{f},k}_h\ } & \mathcal{P}^k(\mathcal{T}_h)
\end{array}$$

Plan

Discrete weak curl–curl formulations of Stokes

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned}\int_\Omega \nu\, \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\, p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q &= 0 && \forall q \in H^1(\Omega)\end{aligned}$$

- Set

$$\underline{X}^k_{\mathbf{grad},h,0} := \left\{\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{q}_h, \underline{I}^k_{\mathbf{grad},h} 1)_{\mathbf{grad},h} = 0\right\}.$$

- DDR scheme: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{p}_h \in \underline{X}^k_{\mathbf{grad},h,0}$ such that

$$\begin{aligned}\nu(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h \underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} &= (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} \quad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\ -(\underline{\boldsymbol{G}}^k_h \underline{q}_h, \underline{\boldsymbol{u}}_h)_{\mathbf{curl},h} &= 0 && \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}.\end{aligned}$$

Discrete weak curl–curl formulations of Stokes

- Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} \int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\ -\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\,q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

- Set

$$V^{\mathrm{n}}_{k+1,0} \coloneqq \left\{ q \in V^{\mathrm{n}}_{k+1} \ : \ \int_\Omega q = 0 \right\}.$$

- VEM scheme: Find $\boldsymbol{u}_h \in V^{\mathrm{e}}_k$ and $p_h \in V^{\mathrm{n}}_{k+1,0}$ such that

$$\begin{aligned} [\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}_h]_{V^{\mathrm{f}}_k} + [\mathbf{grad}\,p_h, \boldsymbol{v}_h]_{V^{\mathrm{e}}_k} &= [\mathcal{I}^{\mathrm{e}}_k \boldsymbol{f}, \boldsymbol{v}_h]_{V^{\mathrm{e}}_k} \quad && \forall \boldsymbol{v}_h \in V^{\mathrm{e}}_k, \\ -[\mathbf{grad}\,q_h, \boldsymbol{u}_h]_{V^{\mathrm{e}}_k} &= 0 && \forall q_h \in V^{\mathrm{n}}_{k+1,0}. \end{aligned}$$

Stability and pressure-robustness

- Stability follows from exactness of the discrete complexes, as for the analysis of the continuous model.

- Pressure-robustness achieved owing to a commutation property, which leads to pressure invariance (as in **Daniele's presentation**):

DDR	VEM
$\underline{\boldsymbol{G}}_h^k(\underline{I}_{\mathbf{grad},h}^k\psi) = \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k(\mathbf{grad}\,\psi)$	$\mathbf{grad}(\mathcal{I}_k^{\mathrm{n}}\psi) = \mathcal{I}_k^{\mathrm{e}}(\mathbf{grad}\,\psi)$

Usual FE approaches for robustness: $\boldsymbol{H}(\mathrm{div};\Omega)$*-conforming reconstruction of the test functions.*

Error estimates

Theorem (DDR)

Setting

$$\|\underline{\boldsymbol{v}}_h\|^2_{\mathbf{curl},1,h} = \|\underline{\boldsymbol{v}}_h\|^2_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|^2_{\mathrm{div},h},$$
$$\|\underline{q}_h\|^2_{\mathbf{grad},1,h} = \|\underline{q}_h\|^2_{\mathbf{grad},h} + \|\underline{\boldsymbol{G}}^k_h \underline{q}_h\|_{\mathbf{curl},h},$$

we have:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{u}\|_{\mathbf{curl},1,h} + \|\underline{p}_h - \underline{I}^k_{\mathbf{grad},h}p\|_{\mathbf{grad},1,h} \lesssim C_1(\boldsymbol{u})h^{k+1}.$$

with $C_1(\boldsymbol{u})$ *depending* $\boldsymbol{u}$ *and some of its derivatives, but not* p.

Theorem (VEM)

We have:

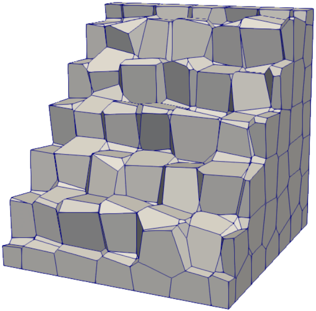
$$\|\boldsymbol{u}_h - \mathcal{I}^{\mathrm{e}}_k \boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|p_h - \mathcal{I}^n_k p\|_{H^1(\Omega)} \lesssim C_2(\boldsymbol{u})h^{k+1}.$$

with $C_2(\boldsymbol{u})$ *depending* $\boldsymbol{u}$ *and some of its derivatives, but not* p.

Plan

Setting

- $\Omega = (0,1)^3$.
- Voronoi mesh families (similar results on tetrahedral meshes):



(a) Voronoi mesh

Setting

- Exact solution: for some $\lambda \geq 0$,

$$p(x, y, z) = \lambda \sin(2\pi x) \sin(2\pi y) \sin(2\pi z),$$

$$\boldsymbol{u}(x, y, z) = \begin{bmatrix} \frac{1}{2} \sin(2\pi x) \cos(2\pi y) \cos(2\pi z) \\ \frac{1}{2} \cos(2\pi x) \sin(2\pi y) \cos(2\pi z) \\ - \cos(2\pi x) \cos(2\pi y) \sin(2\pi z) \end{bmatrix}.$$

- Measured errors:
  - $E_{\boldsymbol{u}}^{\mathrm{d}}$ and E_p^{d} in discrete norms between the approximate solutions and the *interpolates* of the exact solution (as in the theorems).
    E.g: $E_{\boldsymbol{u}}^{\mathrm{d}} = \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k \boldsymbol{u}\|_{\mathbf{curl},1,h}$.
  - $E_{\boldsymbol{u}}^{\mathrm{c}}$ and E_p^{c} in continuous norms between reconstructed potentials/projections of the approximate solutions and the exact solution.
    E.g: $E_p^{\mathrm{c}} = \|p_h - p\|_{H^1(\Omega)}$.

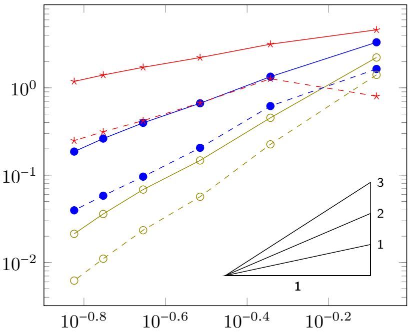
Setting

- DDR and VEM implemented in the `HArDCore3D` library[1], using the Intel MKL Pardiso solver[2].

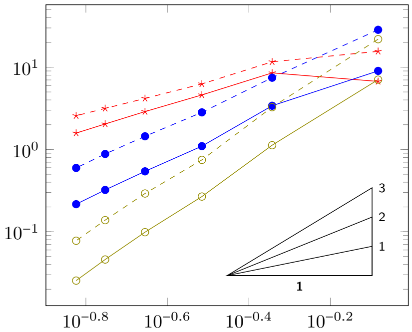
  The HArDCore3D library also includes a serendipity version for DDR, which leads to a reduction of more than 50% of the solving time.

[1] https://github.com/jdroniou/HArDCore
[2] https://software.intel.com/en-us/mkl

Results; $\lambda = 1$, errors on $\boldsymbol{u}$

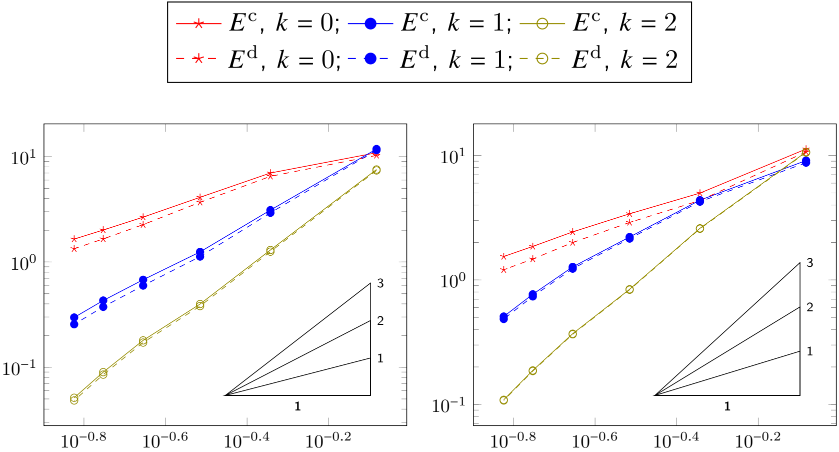


(a) DDR scheme



(b) VEM scheme

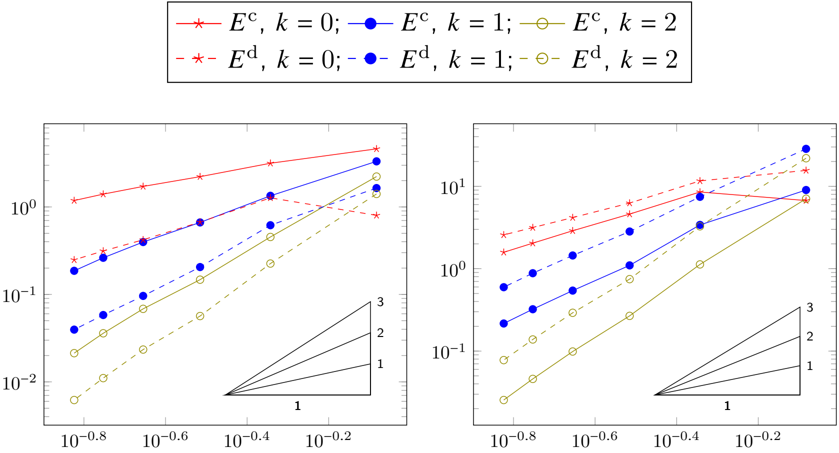
Results; $\lambda = 1$, errors on p



(a) DDR scheme

(b) VEM scheme

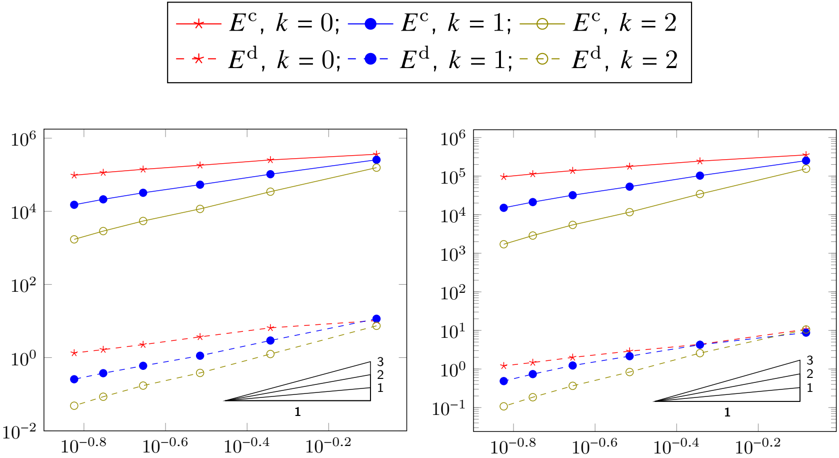
Results; $\lambda = 10^5$, errors on $\boldsymbol{u}$



(a) DDR scheme

(b) VEM scheme

Results; $\lambda = 10^5$, errors on p



(a) DDR scheme

(b) VEM scheme

Conclusion

- Discrete exact sequences yield stable schemes even for models with "incomplete" differential operators.
- DDR and VEM are two examples of discrete exact sequences applicable on polytopal meshes and of arbitrary degree of accuracy.
- The two approaches (fully discrete, and virtual) are complementary views.
- Commutation property key to obtaining pressure-independent estimates for Stokes.

Conclusion

- Discrete exact sequences yield stable schemes even for models with "incomplete" differential operators.
- DDR and VEM are two examples of discrete exact sequences applicable on polytopal meshes and of arbitrary degree of accuracy.
- The two approaches (fully discrete, and virtual) are complementary views.
- Commutation property key to obtaining pressure-independent estimates for Stokes.

Thank you!

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