Chapitre 10

Parabolic Capacity and soft measures for nonlinear equations

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10.1 Introduction

Let Ω be a bounded, open subset of $\mathbf{R}^N$, T a positive number and $Q =]0,T[\times\Omega$. Let p be a real number, with $1 < p < \infty$, and let p' be its conjugate Hölder exponent (i.e. $1/p + 1/p' = 1$).
In this paper we deal with the parabolic initial boundary value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in }]0,T[\times\Omega, \\ u = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{10.1}$$

where A is a nonlinear monotone and coercive operator in divergence form which acts from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a model example, problem (10.1) includes the p–Laplace evolution equation:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu & \text{in }]0,T[\times\Omega, \\ u = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{10.2}$$

The main feature of our study is the presence of singular data μ and u_0, which are bounded measures (respectively on Q and on Ω). It is well known that, if $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, J. L. Lions [51] proved existence and uniqueness of a weak solution. Under the general assumption that μ and u_0 are bounded measures, the existence of a distributional solution was proved in [9], by approximating (10.1) with problems having regular data and using compactness arguments.
Unfortunately, due to the lack of regularity of the solutions, the distributional formulation is not strong enough to have uniqueness, as it can be proved by adapting the counterexample of J. Serrin for the stationary problem (see [68] and the refinement in [65]). In case of linear operators the difficulty can be overcome by defining the solution through the adjoint operator, this method is used in [70] for the stationary problem and yields a formulation having a unique solution. However, for nonlinear operators a new concept of solution needs to be defined to get a well–posed problem. In case of problem (10.1) with L^1 data, this was done independently in [5] and in [64] (see also [3]), where the notions of *renormalized* solution, and of *entropy* solution, respectively, were introduced. Both these approaches are able to obtain existence and uniqueness of solutions if $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$.
Our main goal here is to extend the result of existence and uniqueness to a larger class of measures which includes the L^1 case. Precisely, we prove (in the framework of renormalized solutions) that problem

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(10.1) has a unique solution for every measure μ which is zero on subsets of zero capacity, where the notion of capacity is suitably defined according to the operator $u_t + A(u)$. In fact, the importance of the measures not charging sets of null capacity was first observed in the stationary case in [12], where the authors prove existence and uniqueness of entropy solutions (as introduced in [4]) of the elliptic problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10.3}$$

if μ is a measure which does not charge the sets of zero p–capacity, i.e. the capacity defined from the Sobolev space $W_0^{1,p}(\Omega)$. Actually, this result relies on the fact that every such measure belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.
In order to use a similar approach in the evolution case, one needs to develop the theory of capacity related to the parabolic operator $u_t + A(u)$ and then investigate the relationships between time–space dependent measures and capacity. We introduce here the notion of capacity defined from the parabolic p–laplace equation in the same spirit of [62], where the standard notion of capacity constructed from the heat operator is presented in a useful functional approach (without any tool of potential theory or linear arguments). Indeed, letting

$$W = \left\{ u \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega)),\ u_t \in L^{p'}(0,T; (W_0^{1,p}(\Omega) \cap L^2(\Omega))') \right\} ,$$

we define the capacity of a set B as, roughly speaking, minimizing the norm of W for functions greater than 1 on B. This approach allows us to use the same arguments as in [20] and then to obtain a representation theorem for measures that are zero on subsets of Q that are of zero capacity (see Definition 10.5).
Thus our first main result extends the one in [12] for stationary measures and capacity.

Theorem 10.1 *Let μ be a bounded measure on Q which does not charge the sets of null capacity. Then there exist $g_1 \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$ and $h \in L^1(Q)$, such that*

$$\int_Q \varphi \, d\mu = \int_0^T \langle g_1, \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g_2 \rangle \, dt + \int_Q h \, \varphi \, dx dt \,, \tag{10.4}$$

for any $\varphi \in C_c^\infty([0,T] \times \Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $(W_0^{1,p}(\Omega) \cap L^2(\Omega))'$ and $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ ([3]).

Thanks to this decomposition result, for such class of measures (continuous with respect to capacity) we can still set our problem (10.1) in the framework of renormalized solutions. The idea is that, since μ can be splitted as in (10.4), problem (10.1) can be formally rewritten as $(u - g_2)_t + A(u) = g_1 + h$, and the renormalization argument can be applied to the difference $u - g_2$. We leave to Section 3 the precise definition of renormalized solution, let us state here our result of existence and uniqueness of renormalized solutions.

Theorem 10.2 *Let μ be a bounded measure on Q which is zero on subsets of Q that have zero capacity, and let $u_0 \in L^1(\Omega)$. Then there exists a unique renormalized solution u (see Definition 10.7) of (10.1). Moreover u satisfies the additional regularity: $u \in L^\infty(0,T; L^1(\Omega))$ and $T_k(u) = \max(-k, \min(k,u)) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$.*

Let us stress that, as far as the initial datum is concerned, the class of measure data which do not charge the parabolic capacity of the operator reduces to consider u_0 in $L^1(\Omega)$, so that no improvement can be obtained with respect to previous results. This is a consequence of the following lemma, which we prove in Section 2.

[3]Notice that, since $W^{-1,p'}(\Omega) \hookrightarrow (W_0^{1,p}(\Omega) \cap L^2(\Omega))'$, we have $g_1 \in L^{p'}(0,T; (W_0^{1,p}(\Omega) \cap L^2(\Omega))')$ so that the term involving g_1 in (10.4) is well defined.

Theorem 10.3 *Let B be a Borel set in Ω. Let $t_0 \in]0,T[$ fixed. One has*

$$\operatorname{cap}_p(\{t_0\} \times B) = 0 \quad \textit{if and only if} \quad \operatorname{meas}_\Omega(B) = 0.$$

A counterpart of Lemma 10.3 will also be proved (Theorem 10.6), stating that, for any interval $(t_0, t_1) \subset (0,T)$, $\operatorname{cap}_p((t_0,t_1) \times B) = 0$ if and only if the elliptic capacity (defined from $W_0^{1,p}(\Omega)$) of B is zero.
The plan of the paper is the following. In the next section, we give the definition and prove the basic properties of parabolic capacity, among which the existence of a unique cap–quasi continuous representative for functions in W. We also prove Theorem 10.3 as far as the restriction of capacity to sections $\{t\} \times \Omega$ is concerned. We investigate then the link between measures defined on the σ–algebra of borelians of Q and the previously defined capacity, and we prove the decomposition theorem stated above. In the third section we give first a result of existence and uniqueness for (10.1) if $\mu \in W'$, the dual space of W, which seems a natural extension of the classical result of J.L. Lions. Finally, we give the definition of renormalized solution and we prove existence and uniqueness.

In the sequel C will denote a constant that can change from line to line. For v a function of (t,x) and for k a real number, we will denote, for example, $\{v > k\}$ the set $\{(t,x) \in Q : v > k\}$, while χ_A denotes the characteristic function of a set A.

10.2 Parabolic capacity and measures

10.2.1 Capacity

The approach followed to define the capacity is in the same spirit of Pierre ([62]).

Definition 10.1 *Let us define $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$, and*

$$W = \left\{ u \in L^p(0,T;V), \ u_t \in L^{p'}(0,T;V') \right\},$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$. We will also use the non-homogeneous (when $p \neq 2$) quantity, linked to the energy estimates,

$$[u]_W = \|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;V')} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}. \tag{10.5}$$

Remark 10.1 *Since $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, we see that W is continuously embedded in $\mathcal{C}([0,T];L^2(\Omega))$ (cf [24]), which means that there exists $C > 0$ such that, for all $u \in W$, $\|u\|_{L^\infty(0,T;L^2(\Omega))} \le C\,\|u\|_W$. Thus one has, for all $u \in W$,*

$$[u]_W \le C \ \max\left\{ \|u\|_W^p, \|u\|_W^{p'} \right\}, \quad \|u\|_W \le C \ \max\left\{ [u]_W^{\frac{1}{p}}, [u]_W^{\frac{1}{p'}} \right\}. \tag{10.6}$$

Remark 10.2 *When $\theta \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^N)$ and $u \in W$, then $\theta u \in W$ and there exists $C(\theta)$ not depending on u such that $\|\theta u\|_W \le C(\theta)\|u\|_W$. Indeed, when $u \in L^p(0,T;V)$, it is quite obvious, by the regularity of θ, that $\theta u \in L^p(0,T;V)$ with $\|\theta u\|_{L^p(0,T;V)} \le C(\theta)\|u\|_{L^p(0,T;V)}$. For the time derivative, it is a little bit tricky; we have, in the sense of distributions, $(\theta u)_t = \theta_t u + \theta u_t$. The second term is not a problem: since $u_t \in L^{p'}(0,T;V')$, one has $\theta u_t \in L^{p'}(0,T;V')$ and $\|\theta u_t\|_{L^{p'}(0,T;V')} \le C(\theta)\|u_t\|_{L^{p'}(0,T;V')}$. For the first term, that is $\theta_t u$, we must use the injection of W in $\mathcal{C}([0,T];L^2(\Omega))$, thus also in $L^{p'}(0,T;L^2(\Omega))$; thanks to this injection, it is then easy to get $\theta_t u \in L^{p'}(0,T;L^2(\Omega))$ with $\|\theta_t u\|_{L^{p'}(0,T;L^2(\Omega))} \le C(\theta)\|u\|_W$; $L^2(\Omega)$ being injected in V', we have $L^{p'}(0,T;L^2(\Omega)) \hookrightarrow L^{p'}(0,T;V')$ which gives $\theta_t u \in L^{p'}(0,T;V')$ and $\|\theta_t u\|_{L^{p'}(0,T;V')} \le C(\theta)\|u\|_W$.*

Remark 10.3 *Since $L^{p'}(0,T;V') = (L^p(0,T;V))'$ (see [32], V is a separable reflexive space), and since $L^p(0,T;V) = L^p(0,T;W_0^{1,p}(\Omega)) \cap L^p(0,T;L^2(\Omega)) = E \cap F$, with $E \cap F$ being dense in E and F, we have $L^{p'}(0,T;V') = E' + F' = L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega))$ and the norms of these spaces are equivalent.*

In fact, the natural space that appears in the study of the p–laplacian parabolic operator is not W but $\widetilde{W} \subset W$, defined as follows.

Definition 10.2 *We define*

$$\widetilde{W} = \left\{ u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \right\}.$$

Remark 10.4 *$\widetilde{W}$ is continuously embedded in W.*

We will define the parabolic capacity using the space W, whereas a more natural definition would perhaps start from $\widetilde{W}$. However, using this space instead of W would entail some technical difficulties and since, as we will notice, the sets of null capacity with regards to W are the same than the sets of null capacity with regards to $\widetilde{W}$, there is no loss in working with W instead of $\widetilde{W}$ (see Remark 10.7).

Definition 10.3 *If $U \subset Q$ is an open set, we define*

$$\text{cap}_p(U) = \inf\{\|u\|_W : u \in W,\ u \geq \chi_U \text{ almost everywhere in } \quad Q\} \tag{10.7}$$

(we will use the convention that $\inf \emptyset = +\infty$), then for any borelian subset $B \subset Q$ the definition is extended by setting:

$$\text{cap}_p(B) = \inf\left\{\text{cap}_p(U),\ U \textit{ open subset of } Q,\ B \subset U\right\}. \tag{10.8}$$

Proposition 10.1 *The capacity previously defined satisfies the subadditivity property, that is*

$$\text{cap}_p\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \text{cap}_p(E_i)\,, \tag{10.9}$$

for every collection of borelian sets E_i.

Proof. Let, for all $i \geq 1$, U_i be an open set containing E_i such that $\text{cap}_p(U_i) \leq \text{cap}_p(E_i) + \frac{\varepsilon}{2^i}$, and let u_i be such that $u_i \geq \chi_{U_i}$ a. e. in Q and $\|u_i\|_W \leq \text{cap}_p(U_i) + \frac{\varepsilon}{2^i}$. Without loss of generality we can assume that $\sum_{i=1}^{\infty} \text{cap}_p(E_i) < \infty$ (otherwise (10.9) is trivial); this implies that $\sum_{i=1}^{\infty} u_i$ is strongly convergent in W. Let then $u = \sum_{i\geq 1} u_i$; clearly $u \geq \chi_U$ a.e. in Q where $U = \bigcup_{i=1}^{\infty} U_i$, so that, U being open,

$$\text{cap}_p(U) \leq \|u\|_W \leq \sum_{i=1}^{\infty} \|u_i\|_W \leq \sum_{i=1}^{\infty} \text{cap}_p(E_i) + 2\varepsilon\,.$$

Since $\bigcup_{i=1}^{\infty} E_i \subset U$ this implies (10.9). ∎

Remark 10.5 *As usual, the capacity defined above depends in fact of the open ambient set Q and we should have denoted $\text{cap}_p(B,Q)$ to stress on this dependance. However, Proposition 10.1, along with Remark 10.2, allows to see that, when B is a borel set of Q and $\text{cap}_p(B,Q) = 0$, then $\text{cap}_p(B,U) = 0$ for all open sets $U \subset Q$ containing B. Indeed, take a sequence of compacts $K_n \subset U$ with $U = \bigcup_{n\geq 1} K_n$, then we have $\text{cap}_p(B,U) = \text{cap}_p(\cup_{n\geq 1} B \cap K_n, U) \leq \sum_{n\geq 1} \text{cap}_p(B \cap K_n, U)$. Since K_n is a compact subset of U and since $\text{cap}_p(B \cap K_n, Q) = 0$, we can prove, using a nonnegative function $\zeta_n \in C_c^\infty(U)$ such that $\zeta_n \equiv 1$ on a neighborhood of K_n, that $\text{cap}_p(B \cap K_n, U) = 0$ for any n, which proves our assertion.*

The definition of capacity can be alternatively done starting from the compact sets in Q, as follows. We denote $\mathcal{C}_c^\infty([0,T]\times\Omega)$ the space of restrictions to Q of smooth functions in $\mathbb{R}\times\mathbb{R}^N$ with compact support in $\mathbb{R}\times\Omega$.

Definition 10.4 *Let K be a compact subset of Q. The p-capacity of K with respect to Q is defined as:*

$$\mathrm{CAP}(K) = \inf\{\|u\|_W : u \in \mathcal{C}_c^\infty([0,T]\times\Omega),\ u \geq \chi_K\}\,.$$

The p-capacity of any open subset U of Q is then defined by:

$$\mathrm{CAP}(U) = \sup\{\mathrm{CAP}(K),\ K \text{ compact},\ K \subset U\}\,,$$

and the p-capacity of any Borelian set $B \subset Q$ by

$$\mathrm{CAP}(B) = \inf\{\mathrm{CAP}(U),\ U \textit{ open subset of } Q,\ B \subset U\}\,.$$

This second definition of capacity given for compact subsets is motivated by the following theorem.

Theorem 10.4 *Let Ω be an open bounded set in $\mathbb{R}^N$ and $1 < p < \infty$. Then $\mathcal{C}_c^\infty([0,T]\times\Omega)$ is dense in W.*

The proof of this theorem will be given in the appendix.

Remark 10.6 *Notice also that, when $u \in W$ has a compact support in Q and $(\rho_n)_{n\geq 1}$ is a space-time regularizing kernel, then $u * \rho_n$ is well defined (at least for n large enough), is a function of $\mathcal{C}_c^\infty(Q)$ and $u * \rho_n \to u$ in W (see Lemma 10.7 in the appendix).*

Proposition 10.2 *The capacity* CAP *satisfies the subadditivity property.*

Proof. Let us first prove the subadditivity for finite unions of open sets, starting from compact sets. Indeed, let K_1, K_2 be compact subsets of Q, then there exist two functions u_1, $u_2 \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $u_i \geq \chi_{K_i}$ and $\|u_i\|_W \leq \mathrm{CAP}(K_i) + \varepsilon$, $i = 1, 2$. Since

$$u_1 + u_2 \in \mathcal{C}_c^\infty([0,T]\times\Omega)\,, u_1 + u_2 \geq \chi_{K_1\cup K_2}\,, \quad \|u_1+u_2\|_W \leq \|u_1\|_W + \|u_2\|_W\,,$$

it follows that $\mathrm{CAP}(K_1 \cup K_2) \leq \mathrm{CAP}(K_1) + \mathrm{CAP}(K_2)$. Let now A, B be open subsets of Q, and let K be a compact subset of $A\cup B$. It is easy to find compact subsets K_A, K_B such that $K = K_A \cup K_B$, with $K_A \subset A$ and $K_B \subset B$ (for instance, define $F = \{z \in A : \mathrm{dist}(z, A^c) \geq \frac{m}{2}\}$ where $m = \min_{z\in K}[\mathrm{dist}(z,A^c) + \mathrm{dist}(z,B^c)]$, then $K_A = K \cap F$ and $K_B = K \cap \overline{F^c}$ fit the requirement). Therefore we have $\mathrm{CAP}(K) \leq \mathrm{CAP}(K_A) + \mathrm{CAP}(K_B) \leq \mathrm{CAP}(A) + \mathrm{CAP}(B)$, and taking the supremum over $K \subset A\cup B$ we get

$$\mathrm{CAP}(A\cup B) \leq \mathrm{CAP}(A) + \mathrm{CAP}(B)\,, \quad \text{for all open sets} \quad A\,, B \subset Q\,. \tag{10.10}$$

Finally, let $\{E_i\}_{i\geq 1}$ be borelian subsets of Q, and let $E = \bigcup_{i\geq 1} E_i$. Assume that $\sum_{i\geq 1}\mathrm{CAP}(E_i) < \infty$ and let A_i be open sets such that $E_i \subset A_i$ and $\mathrm{CAP}(A_i) \leq \mathrm{CAP}(E_i) + \frac{\varepsilon}{2^i}$, so that $\sum_{i\geq 1}\mathrm{CAP}(A_i) \leq \sum_{i\geq 1}\mathrm{CAP}(E_i) + \varepsilon$. Let $A = \bigcup_{i\geq 1} A_i$, and take a compact subset $K \subset A$. Since the A_i are a covering of K, there exists a finite number l such that $K \subset \bigcup_{i=1}^l A_i$, hence using (10.10) we get

$$\mathrm{CAP}(K) \leq \mathrm{CAP}\left(\bigcup_{i=1}^l A_i\right) \leq \sum_{i=1}^l \mathrm{CAP}(A_i) \leq \sum_{i=1}^\infty \mathrm{CAP}(E_i) + \varepsilon\,.$$

Taking the supremum over $K \subset A$ and since $E \subset A$ we have

$$\mathrm{CAP}(E) \leq \mathrm{CAP}(A) \leq \sum_{i=1}^\infty \mathrm{CAP}(E_i) + \varepsilon\,,$$

which concludes the proof as ε tends to zero. ■

Note that, in the elliptic case, the two possible constructions of the capacity in the space $W_0^{1,p}(\Omega)$ (from the open sets of from the compacts) coincide. Here, we are not able to prove the same result (because of approximation difficulties), nevertheless we have that both capacities yield the same sets of zero capacity, which is in fact what matters.

Proposition 10.3 *Let B be a borelian subset of Q. Then one has* $\mathrm{CAP}(B) = 0$ *if and only if* $\mathrm{cap}_p(B) = 0$.

Proof. We first prove that $\mathrm{CAP}(B) \geq \mathrm{cap}_p(B)$ for every borelian set B, which will imply $\mathrm{cap}_p(B) = 0$ whenever $\mathrm{CAP}(B) = 0$.
Indeed, let A be open. Assume that $\mathrm{CAP}(A)$ is finite, and let $K_n = \{x \in A : \mathrm{dist}(x, \partial A) \geq \frac{1}{n}\}$. By definition there exists a sequence $\{\varphi_n\}$ of functions in $\mathcal{C}_c^\infty([0,T] \times \Omega)$ such that

$$\varphi_n \geq \chi_{K_n} \quad \text{in } Q, \qquad \|\varphi_n\|_W \leq \mathrm{CAP}(K_n) + \frac{1}{n} \leq \mathrm{CAP}(A) + \frac{1}{n}.$$

In particular we have that φ_n is a bounded sequence in W, which is a reflexive space, so that there exists a subsequence, not relabeled, and a function $\varphi \in W$ such that:

$$\begin{aligned} \varphi_n &\to \varphi && \text{weakly in } L^p(0,T;V), \\ (\varphi_n)_t &\to \varphi_t && \text{weakly in } L^{p'}(0,T;V'), \\ \varphi_n &\to \varphi && \text{almost everywhere in } Q. \end{aligned}$$

Last convergence is a consequence of standard compactness arguments (see [69]). Since $\varphi_n \geq \chi_{K_n}$ for every n, we deduce that $\varphi \in W$ and $\varphi \geq \chi_A$ almost everywhere in Q, so that φ can be used in Definition 10.3 above. By lower semicontinuity of the norm we get, as n tends to infinity:

$$\|\varphi\|_W \leq \mathrm{CAP}(A),$$

which yields that $\mathrm{cap}_p(A) \leq \mathrm{CAP}(A)$. This inequality being satisfied for all open sets A, we deduce from the definition that it is also true for all borelians of Q.
Now, let us obtain the reverse implication. We take B a borelian such that $\mathrm{cap}_p(B) = 0$. Since CAP is sub-additive, it is enough to prove that, for any compact $K \subset Q$, one has $\mathrm{CAP}(B \cap K) = 0$. We take thus K a compact subset of Q and $\zeta \in \mathcal{C}_c^\infty(Q)$ such that $\zeta = 1$ on an open set O which contains K.
Since $\mathrm{cap}_p(B) = 0$, there exists, for all $\varepsilon > 0$, an open set A_ε containing B such that $\mathrm{cap}_p(A_\varepsilon) < \varepsilon$; we can then take $u \in W$ such that

$$u \geq \chi_{A_\varepsilon} \quad \text{a.e. in } Q, \qquad \|u\|_W \leq 2\varepsilon.$$

We have $\zeta u \in W$ and $\|\zeta u\|_W \leq 2C(\zeta)\varepsilon$, with $C(\zeta)$ only depending on ζ (see Remark 10.2).
We will now estimate $\mathrm{CAP}(A_\varepsilon \cap O)$. Let L be a compact subset of $A_\varepsilon \cap O$ and $(\rho_n)_{n\geq 1}$ be a regularizing kernel in $\mathbb{R} \times \mathbb{R}^N$; since ζu has a compact support in Q, $(\zeta u) * \rho_n \in \mathcal{C}_c^\infty(Q)$ is well defined (at least for n large enough) and $(\zeta u) * \rho_n$ strongly converges to ζu in W (see Remark 10.6 and the appendix). We can thus fix $n(L,\varepsilon)$ such that $\|(\zeta u) * \rho_{n(L,\varepsilon)} - (\zeta u)\|_W \leq \varepsilon$ and $(\zeta u) * \rho_{n(L,\varepsilon)} \geq 1$ in L (recall that $\zeta u \geq 1$ on the open set $A_\varepsilon \cap O$ and that L is a compact subset of $A_\varepsilon \cap O$); with this choice of $n(L,\varepsilon)$, $v = (\zeta u) * \rho_{n(L,\varepsilon)} \in \mathcal{C}_c^\infty(Q) \subset \mathcal{C}_c^\infty([0,T] \times \Omega)$ and $v \geq \chi_L$. Thus, $\mathrm{CAP}(L) \leq \|v\|_W \leq \|v - \zeta u\|_W + \|\zeta u\|_W \leq (1 + 2C(\zeta))\varepsilon$. This being true for any compact subset L of the open set $A_\varepsilon \cap O$, we deduce that $\mathrm{CAP}(A_\varepsilon \cap O) \leq (1 + 2C(\zeta))\varepsilon$. But $B \cap K \subset A_\varepsilon \cap O$, so that $\mathrm{CAP}(B \cap K) \leq (1 + 2C(\zeta))\varepsilon$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we deduce that $\mathrm{CAP}(B \cap K) = 0$. ■

However, henceforth we will make use of Definition 10.3 of capacity, which can be handled more easily in many situations, as in the following result, where we give the characterization of sets of null capacity contained in the sections $\{t_0\} \times \Omega$ of the parabolic cylinder.

Theorem 10.5 *Let B be a borelian set in Ω. Let $t_0 \in]0,T[$ fixed. One has*

$$\mathrm{cap}_p(\{t_0\} \times B) = 0 \quad \textit{if and only if} \quad \mathrm{meas}_{\Omega}(B) = 0.$$

Proof. Assume first that $\mathrm{cap}_p(\{t_0\} \times B) = 0$ and let K be any compact set contained in B, so that $\mathrm{cap}_p(\{t_0\} \times K) = 0$. Since, by Proposition 10.3, we also have that $\mathrm{CAP}(\{t_0\} \times K) = 0$, then, for all $\delta > 0$ there exists a function $\psi_\delta \in \mathcal{C}_c^\infty([0,T] \times \Omega)$ such that $\|\psi_\delta\|_W \le \delta$ and $\psi_\delta(t_0) \ge 1$ on K. Since W is continuously imbedded in $\mathcal{C}([0,T], L^2(\Omega))$, we have

$$\mathrm{meas}_{\Omega}(K) \le \int_K |\psi_\delta(t_0)|^2\,dx \le \|\psi_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\|\psi_\delta\|^2_W \le C\delta^2,$$

so we deduce that $\mathrm{meas}_{\Omega}(K) \le C\delta^2$, and from the arbitrariness of δ then $\mathrm{meas}_{\Omega}(K) = 0$. Since this is true for any compact subset contained in B, by regularity of the Lebesgue measure we conclude that $\mathrm{meas}_{\Omega}(B) = 0$.

Conversely, if $\mathrm{meas}_{\Omega}(B) = 0$ then there exists, for all $\delta > 0$, an open set A_δ such that $B \subset A_\delta$ and $\mathrm{meas}_{\Omega}(A_\delta) < \delta$. Let us consider δ fixed in the following, and let K_n be a sequence of compact sets contained in A_δ such that $K_n \subset K_{n+1} \subset \ldots$, $\bigcup_{n=1}^\infty K_n = A_\delta$. Let $\varphi_n \in C_c(A_\delta)$ be such that $0 \le \varphi_n \le 1$, $\varphi_n \equiv 1$ on K_n and $\varphi_n \le \varphi_{n+1}$. Then we solve for $t \in [t_0, T]$,

$$\begin{cases} (\psi_n)_t - \mathrm{div}(|\nabla\psi_n|^{p-2}\nabla\psi_n) = 0 & \text{in }]t_0,T[\times\Omega, \\ \psi_n = 0 & \text{on }]t_0,T[\times\partial\Omega, \\ \psi_n(t_0) = \varphi_n & \text{in } \Omega. \end{cases} \tag{10.11}$$

Clearly we have that $\psi_n \in L^p(t_0,T;W_0^{1,p}(\Omega)) \cap L^\infty(t_0,T;L^2(\Omega))$ and $(\psi_n)_t \in L^{p'}(t_0,T;W^{-1,p'}(\Omega))$. Let us construct a function $\widetilde{\psi}_n$ defined on $[0,T]$, by setting

$$\begin{cases} \widetilde{\psi}_n = \psi_n & \text{in }]t_0,T] \times \Omega, \\ \widetilde{\psi}_\delta = \psi_n\left(T - \dfrac{t(T-t_0)}{t_0}\right) & \text{in } [0,t_0] \times \Omega. \end{cases}$$

It is not difficult to see that $\widetilde{\psi}_n$ belongs to W and by the energy estimates obtained from (10.11) by using ψ_n itself as test function we have (recall the notation in (10.5)) :

$$[\widetilde{\psi}_n]_W \le C\|\varphi_n\|^2_{L^2(\Omega)} \le C\mathrm{meas}(A_\delta) \le C\delta\,. \tag{10.12}$$

By regularity results on the p–laplacian evolution equation (see [27]) we have that ψ_n is continuous in $[t_0,T] \times \Omega$, hence $\widetilde{\psi}_n \in \mathcal{C}([0,T] \times \Omega)$. Thus we can define the open set $U_n := \{\widetilde{\psi}_n > \frac{1}{2}\}$. Since U_n is open and $2\widetilde{\psi}_n \ge \chi_{U_n}$ we have

$$\mathrm{cap}_p(U_n) \le 2\|\widetilde{\psi}_n\|_W \le C\max(\delta^{\frac{1}{p}}, \delta^{\frac{1}{p'}})\,. \tag{10.13}$$

Since $\{\varphi_n\}$ is nondecreasing we have that $\{\widetilde{\psi}_n\}$ is nondecreasing as well, hence $U_n \subset U_{n+1}$, and $\mathrm{cap}_p(U_n)$ is also a nondecreasing sequence, and bounded too. Setting $U_\infty = \bigcup_{n=1}^\infty U_n$, we have that

$$\mathrm{cap}_p(U_\infty) = \lim_{n\to\infty} \mathrm{cap}_p(U_n)\,. \tag{10.14}$$

Indeed, since $U_n \subset U_\infty$ we have $\lim\limits_{n\to\infty} \mathrm{cap}_p(U_n) \le \mathrm{cap}_p(U_\infty)$. On the other hand, let $u_n \in W$ be such that

$$u_n \ge \chi_{U_n} \quad \text{a.e. in } Q \qquad \text{and} \qquad \|u_n\|_W \le \mathrm{cap}_p(U_n) + \frac{1}{n}\,,$$

(in fact, it can also be chosen u_n such that $\|u_n\|_W = \mathrm{cap}_p(U_n)$, but this is not essential). It follows from (10.13) that u_n is a bounded sequence in W, hence there exists a function $u \in W$ such that, up to a subsequence,

$$u_n \to u \qquad \text{weakly in } W \text{ and a.e. in } Q.$$

The almost everywhere convergence of this subsequence and the fact that $(U_n)_{n\geq 1}$ is nondecreasing imply that $u \geq \chi_{U_\infty}$ almost everywhere in Q; since U_∞ is open, we get

$$\mathrm{cap}_p(U_\infty) \leq \|u\|_W \leq \liminf_{n\to\infty} \|u_n\|_W \leq \lim_{n\to\infty} \mathrm{cap}_p(U_n)\,,$$

so that (10.14) is proved. Since $\varphi_n = 1$ on K_n for each n and $A_\delta = \bigcup_{n=1}^{\infty} K_n$, we have that U_∞ is an open set which contains $\{t_0\} \times A_\delta \supset \{t_0\} \times B$, so that we conclude from (10.14) and (10.13)

$$\mathrm{cap}_p(\{t_0\} \times B) \leq \mathrm{cap}_p(U_\infty) \leq C \max(\delta^{\frac{1}{p}}, \delta^{\frac{1}{p'}})\,,$$

which implies that $\mathrm{cap}_p(\{t_0\} \times B) = 0$. ■

The following result can be considered a counterpart of the previous result, since we consider subsets $(0,T) \times B$, $B \subset \Omega$.

Theorem 10.6 *Let $B \subset \Omega$ be a borelian set, and $0 \leq t_0 < t_1 \leq T$. Then we have*

$$\mathrm{cap}_p((t_0,t_1) \times B) = 0 \quad \textit{if and only if} \quad \mathrm{cap}_p^e(B) = 0,$$

where cap_p^e denotes the elliptic capacity defined from $W_0^{1,p}(\Omega)$.

Proof.
If $\mathrm{cap}_p^e(B) = 0$, then there exists, for all $0 < \delta < 1$, an open set U_δ with $B \subset U_\delta$ such that $\mathrm{cap}_p^e(U_\delta) < \delta$. It is then a well-known result of the elliptic capacity that we can choose $v_\delta \in W_0^{1,p}(\Omega)$ with $1 \geq v_\delta \geq \chi_{U_\delta}$ a.e. in Ω and $||v_\delta||_{W_0^{1,p}(\Omega)} \leq \delta$. If $p \geq 2$, this also gives $||v_\delta||_{L^2(\Omega)} \leq C\delta$ (C not depending on δ); if $p < 2$, since $|v_\delta| \leq 1$, we have $\int_\Omega |v_\delta|^2 \leq \int_\Omega |v_\delta|^p \leq C\delta^p$ (C still not depending on δ), that is to say $||v_\delta||_{L^2(\Omega)} \leq \sqrt{C}\delta^{p/2}$. In either case, we have thus $v_\delta \geq \chi_{U_\delta}$ such that $||v_\delta||_{W_0^{1,p}(\Omega)} + ||v_\delta||_{L^2(\Omega)} \leq C \max(\delta, \delta^{p/2})$. Using then $u(t,x) = v_\delta(x)$ in (10.7) for the definition of the parabolic capacity of $(t_0,t_1) \times U_\delta$ we deduce that $\mathrm{cap}_p((t_0,t_1) \times U_\delta) \leq C \max(\delta, \delta^{p/2})$, and then as δ goes to zero we get $\mathrm{cap}_p((t_0,t_1) \times B) = 0$.
Conversely, assume that $\mathrm{cap}_p((t_0,t_1) \times B) = 0$ and take $t_0 < t_0' < t_1' < t_1$. Since $\mathrm{cap}_p((t_0,t_1) \times B) = 0$, for all $\delta > 0$, there exists an open set A_δ such that $((t_0,t_1) \times B) \subset A_\delta$ and $\mathrm{cap}_p(A_\delta) < \delta$. For every fixed $x \in B$, since the compact set $[t_0', t_1'] \times \{x\}$ is contained in the open set A_δ, there exists an open set $U_x \subset \Omega$ such that $(t_0', t_1') \times \{x\} \subset (t_0', t_1') \times U_x \subset A_\delta$. Hence, setting $U = \bigcup_{x\in B} U_x$ we have that $B \subset U \subset \Omega$, U is an open set and $(t_0', t_1') \times U \subset A_\delta$, so that $\mathrm{cap}_p((t_0', t_1') \times U) \leq \mathrm{cap}_p(A_\delta) < \delta$. Let then $u_\delta \in W$ be such that $u_\delta \geq \chi_{(t_0',t_1')\times U}$ and $\|u_\delta\|_W \leq \delta$. Defining

$$v_\delta = \frac{1}{t_1' - t_0'} \int_{t_0'}^{t_1'} u_\delta \, dt\,,$$

we easily check that $v_\delta \in W_0^{1,p}(\Omega)$, $v_\delta \geq \chi_U$ almost everywhere in Ω and

$$\|v_\delta\|_{W_0^{1,p}(\Omega)} \leq \frac{1}{t_1' - t_0'} \int_{t_0'}^{t_1'} \|u_\delta\|_V \, dxdt \leq \frac{T^{\frac{p-1}{p}}}{t_1' - t_0'} \|u_\delta\|_W \leq \frac{T^{\frac{p-1}{p}}}{t_1' - t_0'}\, \delta\,.$$

Since U is open and contains B, the arbitrariness of δ implies $\mathrm{cap}_p^e(B) = 0$.

■

10.2.2 Quasicontinuous functions

Let us recall that a function u is called cap–quasi continuous if for every $\varepsilon > 0$ there exists an open set F_ε, with $\mathrm{cap}_p(F_\varepsilon) \leq \varepsilon$, and such that $u_{|Q\setminus F_\varepsilon}$ (the restriction of u to $Q \setminus F_\varepsilon$) is continuous in $Q \setminus F_\varepsilon$. As usual, a property will be said to hold cap–quasi everywhere if it holds everywhere except on a set of zero capacity. The following lemma is essential to prove the existence of a cap–quasicontinuous representative in W. In fact, remark that if $u \in W$, one may have $|u| \notin W$, since the time derivative may lack of regularity. To overcome this obstacle we use some ideas contained in [62].

Lemma 10.1 *(i) Let u belong to W; then there exists a function z in $\widetilde{W}$ (see Definition 10.2) such that $|u| \le z$ and*

$$\|z\|_{\widetilde{W}} \le C \max\left\{\|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}}\right\}. \tag{10.15}$$

(ii) If u belongs to $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ and u_t is in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ then there exists $z \in \widetilde{W}$ such that $|u| \le z$ and:

$$\begin{aligned}[z]_W \le C\Big(&\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\ &+\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Big).\end{aligned}$$

Remark 10.7 *In case (i), notice that, when $\|u\|_W$ is small, so is $\|z\|_{\widetilde{W}}$; this allows to prove that the sets of null capacity coming from W are the same than the sets of null capacity coming from $\widetilde{W}$.*
The case (ii) of Lemma 10.1 will not be useful to us, but we state and prove it because it allows to see that, if u is as in this case, then u has a unique cap–quasi continuous representative (see also Remark 10.12).

Proof. We divide the proof in two steps. We will denote $\Delta_p(u_\varepsilon) = \mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$.
Step 1. Let us consider the penalizing problem

$$\begin{cases} (u_\varepsilon)_t - \Delta_p(u_\varepsilon) = \frac{1}{\varepsilon}(u_\varepsilon - u)^- & \text{in }]0,T[\times\Omega, \\ u_\varepsilon = 0 & \text{on }]0,T[\times\partial\Omega, \\ u_\varepsilon(0) = u^+(0) & \text{in } \Omega, \end{cases} \tag{10.16}$$

which admits a nonnegative solution u_ε in $\mathcal{C}([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ by results in [51]. Choosing $u_\varepsilon - u$ as test function in (10.16) we get, for every t in $[0,T]$:

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon - u|^2(t)}{2}\,dx + \int_0^t\!\!\int_\Omega |\nabla u_\varepsilon|^p\,dxdt \le& \int_0^t\!\!\int_\Omega |\nabla u|\,|\nabla u_\varepsilon|^{p-1}\,dxdt \\ &+ \frac{1}{\varepsilon}\int_0^t\!\!\int_\Omega (u_\varepsilon - u)(u_\varepsilon - u)^- \\ &- \int_0^t \langle u_t, u_\varepsilon - u\rangle\,dt + \frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

which yields, using also Young's inequality, and $(u_\varepsilon - u)(u_\varepsilon - u)^- \le 0$,

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon - u|^2(t)}{2}\,dx + \frac{1}{2}\int_0^t\!\!\int_\Omega |\nabla u_\varepsilon|^p\,dxdt \le& C\int_Q |\nabla u|^p\,dxdt \\ &- \int_0^t \langle u_t, u_\varepsilon - u\rangle\,dt + \frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}.\end{aligned} \tag{10.17}$$

If we are in case (i), u is in W and we have

$$\begin{aligned}&\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle dt\right| \\ \le\;& \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_V \\ \le\;& \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_{W_0^{1,p}(\Omega)} + \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_{L^2(\Omega)} \\ \le\;& \|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^1(0,T;V')}\,\|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \\ \le\;& \|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))}\end{aligned}$$

so that we easily deduce from (10.17), using Young's inequality:

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le C \max\left\{\|u\|^p_W, \|u\|^{p'}_W\right\} . \tag{10.18}$$

If we are in case (ii), then the duality product $\int_0^t \langle u_t, u_\varepsilon - u\rangle$ in (10.17) is between the spaces

$$L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q) \quad \text{ and } \quad L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q),$$

and we need to prove an $L^\infty(Q)$ estimate on u_ε. This can be easily achieved by choosing $G_k(u_\varepsilon) = (u_\varepsilon - k)^+$ (let us recall that $u_\varepsilon \ge 0$) as test function in (10.16), with $k = \|u\|_{L^\infty(Q)}$: since $G'_k = \chi_{]k,\infty[} = (G'_k)^p$, we have

$$\int_Q |\nabla G_k(u_\varepsilon)|^p \, dxdt = \int_Q G'_k(u_\varepsilon)|\nabla u_\varepsilon|^p \, dxdt \le \frac{1}{\varepsilon}\int_Q (u_\varepsilon - u)^- \, G_k(u_\varepsilon)\, dxdt \,,$$

and since $(u_\varepsilon - u)^- G_k(u_\varepsilon) = 0$ for $k = \|u\|_{L^\infty(Q)}$, we deduce that $\|u_\varepsilon\|_{L^\infty(Q)} \le \|u\|_{L^\infty(Q)}$. Thus, writing $u_t = u_t^1 + u_t^2$ with $u_t^1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_t^2 \in L^1(Q)$ such that $\|u_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|u_t^2\|_{L^1(Q)} \le 2\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$,

$$\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle \, dt\right| \le \int_0^T \|u_t^1\|_{W^{-1,p'}(\Omega)} \|u_\varepsilon - u\|_{W_0^{1,p}(\Omega)}\, dt + \|u_t^2\|_{L^1(Q)} \|u_\varepsilon - u\|_{L^\infty(Q)}$$
$$\le C\|u_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \frac{1}{4}\|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u\|_{L^\infty(Q)}\|u_t^2\|_{L^1(Q)} \,.$$

Then

$$\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle \, dt\right| \le C\|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$$
$$+ \frac{1}{4}\|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega)))} + C\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega)))} + C\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \,.$$

We deduce from (10.17) that, for all $t \in [0,T]$,

$$\int_\Omega |u_\varepsilon - u|^2(t)\, dx + \int_0^t\int_\Omega |\nabla u_\varepsilon|^p \, dxdt \le C\left(\int_Q |\nabla u|^p \, dxdt + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}\right.$$
$$\left. + \|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\right) \,,$$

which implies

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))}$$
$$\le C\left(\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}\right.$$
$$\left. + \|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\right) \,. \tag{10.19}$$

From (10.18) or (10.19) we deduce that there exists a nonnegative function w in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ such that (up to subsequences)

$$u_\varepsilon \to w \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Note also that if $\varepsilon < \eta$ then $u_\varepsilon \ge u_\eta$; indeed, we have

$$-\int_0^t \langle (u_\varepsilon - u_\eta)_t, (u_\varepsilon - u_\eta)^-\rangle \, dt - \int_0^t\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u_\eta|^{p-2}\nabla u_\eta)\nabla(u_\varepsilon - u_\eta)^- \, dxdt$$
$$= -\int_0^t\int_\Omega \left(\frac{1}{\varepsilon}(u_\varepsilon - u)^- - \frac{1}{\eta}(u_\eta - u)^-\right)(u_\varepsilon - u_\eta)^- \, dxdt \,,$$

which yields, using the fact that the second term of the last equation is non negative and integrating by parts,

$$\frac{1}{2}\int_\Omega |(u_\varepsilon-u_\eta)^-(t)|^2\,dx \le \int_0^t\int_\Omega (u_\varepsilon-u_\eta)^-\left(\frac{1}{\eta}(u_\eta-u)^- - \frac{1}{\varepsilon}(u_\varepsilon-u)^-\right)dxdt$$
$$\le \int_0^t\int_\Omega (u_\varepsilon-u_\eta)^-(u_\eta-u)^-\left(\frac{1}{\eta}-\frac{1}{\varepsilon}\right)dxdt \le 0\,,$$

for every t in $]0,T[$. Thus $(u_\varepsilon)_\varepsilon$ is a non negative sequence bounded in $L^1(Q)$, moreover it is increasing as ε tends to zero, hence thanks to the monotone convergence theorem, u_ε converges to w in $L^1(Q)$ and almost everywhere in Q. We have, choosing $(u_\varepsilon-u)^-$ as test function in (10.16),

$$\int_0^T \langle (u_\varepsilon)_t,(u_\varepsilon-u)^-\rangle\,dt + \int_0^T |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon-u)^-\,dxdt = \frac{1}{\varepsilon}\int_Q |(u_\varepsilon-u)^-|^2\,dxdt\,,$$

which implies

$$\frac{1}{\varepsilon}\int_Q |(u_\varepsilon-u)^-|^2\,dxdt + \int_\Omega \frac{|(u_\varepsilon-u)^-|^2(T)}{2}\,dx = \int_0^T \langle u_t,(u_\varepsilon-u)^-\rangle\,dt$$
$$+\int_Q |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\,\nabla(u_\varepsilon-u)^-\,dxdt\,.$$

Using either (10.18) in case (i) or (10.19) and the L^∞ estimate in case (ii) we deduce:

$$\frac{1}{\varepsilon}\int_Q |(u_\varepsilon-u)^-|^2\,dxdt \le C\,, \tag{10.20}$$

which implies, by Fatou's lemma, that $w\ge u$, and $w\ge u^+$ since $w\ge 0$.
Step 2: Let us now replace u_ε by a sequence converging in $\widetilde{W}$. Precisely, we define z_ε the solution of the following parabolic problem:

$$\begin{cases} -z^\varepsilon_t - \Delta_p z^\varepsilon = -2\Delta_p u_\varepsilon & \text{in }]0,T[\times\Omega,\\ z^\varepsilon = 0 & \text{on }]0,T[\times\partial\Omega,\\ z^\varepsilon(T) = u_\varepsilon(T) & \text{in } \Omega. \end{cases} \tag{10.21}$$

Since $-2\Delta_p u_\varepsilon \ge -(u_\varepsilon)_t - \Delta_p(u_\varepsilon)$ in distributional sense, we can easily deduce from (10.21) that $z^\varepsilon \ge u_\varepsilon$. Moreover using z^ε itself as test function and integrating between t and T, we have the following energy estimates:

$$\begin{aligned} &\|z^\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z^\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} \le C(\|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))})\\ &\|z^\varepsilon_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C(\|z^\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))})\,. \end{aligned} \tag{10.22}$$

In virtue of (10.22), we get that z^ε is bounded in $\widetilde{W}$, hence there exists a function $z\in L^p(0,T;W^{1,p}_0(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and a function $w\in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that (up to subsequences) $z^\varepsilon\to z$ weakly in $L^p(0,T;W^{1,p}_0(\Omega))$ and weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and $z^\varepsilon_t\to w$ weakly-$*$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$; it is then quite easy to see that $z_t=w$, so that z is in fact in $\widetilde{W}$. The classical compactness argument contained in [69] implies that z^ε is also compact in $L^1(Q)$. Thus we deduce, up to subsequences, that z^ε almost everywhere converges to z in Q, and since $z^\varepsilon\ge u_\varepsilon$ passing to the limit we obtain that:

$$z\ge w\ge u^+ \qquad \text{a.e. in } Q.$$

Moreover, using either (10.18) or (10.19) and (10.22), we deduce that, if u is in W, then

$$\|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|z_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C\max\left\{\|u\|^p_W, \|u\|^{p'}_W\right\}\,,$$

which implies (10.15), and if u is in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and u_t belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, then

$$[z]_W \le C(\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\ +\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}) .$$

A similar construction can be made for the negative part u^-, so the conclusion of the lemma follows by writing $|u| = u^+ + u^-$. ■

The previous lemma has the following important consequence.

Proposition 10.4 *If u is cap–quasi continuous and belongs to W, then, for all $t>0$,*

$$\operatorname{cap}_p(\{|u|>t\}) \le \frac{C}{t}\max\left\{\|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}}\right\}. \tag{10.23}$$

Proof. Let us first handle a simple case, that is to say $u \in \mathcal{C}_c^\infty([0,T]\times\Omega)$; then the set $\{|u|>t\}$ is open and its capacity can be computed according to (10.7). By Lemma 10.1 there exists a function $z \ge |u|$ satisfying (10.15). Since $\frac{z}{t} \ge 1$ on the set $\{|u|>t\}$ we have:

$$\operatorname{cap}_p(\{|u|>t\}) \le \frac{\|z\|_W}{t} \le \frac{C}{t}\max\left\{\|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}}\right\}$$

Let us now prove the general case: u is cap–quasi continuous and belongs to W. Let $\varepsilon>0$ and A_ε be an open set such that $\operatorname{cap}_p(A_\varepsilon)\le\varepsilon$ and $u_{|Q\setminus A_\varepsilon}$ is continuous in $Q\setminus A_\varepsilon$; by definition, this implies that $\{|u_{|Q\setminus A_\varepsilon}|>t\}\cap(Q\setminus A_\varepsilon)$ is an open set of $Q\setminus A_\varepsilon$, i.e. that there exists an open set U of $\mathbb{R}^N$ such that $\{|u_{|Q\setminus A_\varepsilon}|>t\}\cap(Q\setminus A_\varepsilon) = U\cap(Q\setminus A_\varepsilon)$. Thus,

$$\{|u|>t\}\cup A_\varepsilon = \left(\{|u_{|Q\setminus A_\varepsilon}|>t\}\cap(Q\setminus A_\varepsilon)\right)\cup A_\varepsilon = (U\cup A_\varepsilon)\cap Q$$

is an open set. Let then $z\in W$ be such that $z\ge|u|$ and (10.15) holds; let $w\in W$ be such that $\|w\|_W \le \operatorname{cap}_p(A_\varepsilon)+\varepsilon \le 2\varepsilon$ and $w\ge\chi_{A_\varepsilon}$; we have $w+\frac{z}{t}\ge 1$ almost everywhere on $\{\,|u|>t\}\cup A_\varepsilon$, hence

$$\operatorname{cap}_p(\{|u|>t\}\cup A_\varepsilon) \le \|w\|_W + \frac{1}{t}\|z\|_W \le 2\varepsilon + \frac{1}{t}\|z\|_W .$$

Thus we get

$$\operatorname{cap}_p(\{|u|>t\}) \le 2\varepsilon + \frac{1}{t}\|z\|_W ,$$

which implies again (10.23). ■

We can now prove the result on quasicontinuity, whose proof follows the standard approach with the help of Lemma 10.1.

Lemma 10.2 *Any element v of W has a cap–quasi continuous representative $\widetilde{v}$ which is cap–quasi everywhere unique, in the sense that two cap–quasi continuous representatives of v are equal except on a set of null capacity.*

Proof. By density of $\mathcal{C}_c^\infty([0,T]\times\Omega)$ in W, there exists a sequence $(v^m)\subset\mathcal{C}_c^\infty([0,T]\times\Omega)$ such that (v^m) converges to v in W. We can also construct (v_m) such that

$$\sum_{m=1}^{\infty} 2^m \max\left\{\|v^{m+1}-v^m\|_W^{\frac{p}{p'}}, \|v^{m+1}-v^m\|_W^{\frac{p'}{p}}\right\} < +\infty.$$

Let then define:

$$\omega^m = \{|v^{m+1}-v^m| > 2^{-m}\}, \qquad \Omega^r = \bigcup_{m\ge r}\omega^m .$$

Since $v^{m+1}-v^m$ is continuous, ω^m is an open set; moreover, by Proposition 10.4, we have

$$\mathrm{cap}_p(\omega^m) \le C2^m \max\left\{ \|v^{m+1}-v^m\|_W^{\frac{p}{p'}}, \|v^{m+1}-v^m\|_W^{\frac{p'}{p}} \right\}.$$

Thus we get:

$$\mathrm{cap}_p(\Omega^r) \le C \sum_{m\ge r} 2^m \max\left\{ \|v^{m+1}-v^m\|_W^{\frac{p}{p'}}, \|v^{m+1}-v^m\|_W^{\frac{p'}{p}} \right\}.$$

This proves that $\lim_{r\to\infty} \mathrm{cap}_p(\Omega^r)=0$. Moreover for any r:

$$\forall z \notin \Omega^r, \quad \forall m \ge r, \quad |v^{m+1}-v^m|(z) \le 2^{-m},$$

hence (v^m) converges uniformly on the complement of each Ω^r and pointwise in the complement of $\bigcap_{r=1}^{\infty} \Omega^r$. Since

$$\mathrm{cap}_p\left(\bigcap_{r=1}^{\infty} \Omega^r\right) \le \mathrm{cap}_p(\Omega^r) \to 0 \qquad \text{as } r \text{ tends to infinity,}$$

we have that $\mathrm{cap}_p(\bigcap_{r=1}^{\infty}\Omega^r)=0$. Therefore the limit of v^m is defined cap–quasi everywhere and is cap–quasi continuous. Let us call $\tilde v$ this cap–quasi continuous representative of v, and assume that there exists another representative z of v which is cap–quasi continuous and coincides with v almost everywhere in Q. Then we have, thanks to Proposition 10.4:

$$\mathrm{cap}_p\left\{ |\tilde v - z| > \frac{1}{n} \right\} \le Cn \max\left\{ \|\tilde v - z\|_W^{\frac{p}{p'}}, \|\tilde v - z\|_W^{\frac{p'}{p}} \right\} = 0,$$

since $\tilde v - z = 0$ almost everywhere. This being true for any n, we obtain that $z=\tilde v$ cap–quasi everywhere, so that the cap–quasi continuous representative of v is unique up to sets of zero capacity. ■

We can also prove the following result.

Lemma 10.3 *Let (v_n) be a sequence in W which converges to v in W, then there exists a subsequence of $(\tilde v_n)$ which converges to $\tilde v$ cap–quasi everywhere.*

Proof. Let us extract a subsequence of (v_n) such that

$$\sum_{n=1}^{\infty} 2^n \max\left\{ \|v_n - v\|_W^{\frac{p}{p'}}, \|v_n - v\|_W^{\frac{p'}{p}} \right\} < +\infty.$$

Thanks to Proposition 10.4 we have

$$\mathrm{cap}_p\{ |\tilde v_n - \tilde v| > 2^{-n}\} \le C2^n \max\left\{ \|v_n - v\|_W^{\frac{p}{p'}}, \|v_n - v\|_W^{\frac{p'}{p}} \right\}. \tag{10.24}$$

Using (10.24) we can repeat the proof of Lemma 10.2, which proves that $\tilde v_n$ converges to $\tilde v$ cap–quasi everywhere. ■

10.2.3 Measures

In the following, we denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the σ–algebra of borelian subsets of Q, and $\mathcal{M}_b^+(Q)$ will denote the subsets of nonnegative measures of $\mathcal{M}_b(Q)$.

Definition 10.5 *We define*

$$\mathcal{M}_0(Q) = \{\mu \in \mathcal{M}_b(Q) \,:\, \mu(E)=0 \quad \textit{for every subset } E\subset Q \textit{ such that } \mathrm{cap}_p(E)=0\}.$$

The nonnegative measures in $\mathcal{M}_0(Q)$ will be said to belong to $\mathcal{M}_0^+(Q)$.

We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality between W' and W. $W'\cap\mathcal{M}_b(Q)$ denotes the set of elements $\gamma\in W'$ such that there exists $C>0$ satisfying, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, $|\langle\langle\gamma,\varphi\rangle\rangle|\le C\|\varphi\|_{L^\infty(Q)}$; in such a case, by the Riesz representation theorem there exists a unique $\gamma^{\text{meas}}\in\mathcal{M}_b(Q)$ such that, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, $\langle\langle\gamma,\varphi\rangle\rangle=\int_Q\varphi\,d\gamma^{\text{meas}}$ (notice however that, if the knowledge of $\gamma\in W'$ entirely defines $\gamma^{\text{meas}}\in\mathcal{M}_b(Q)$, the converse is not true). We denote by $W'\cap\mathcal{M}_b^+(Q)$ the set of $\gamma\in W'\cap\mathcal{M}_b(Q)$ such that $\gamma^{\text{meas}}\in\mathcal{M}_b^+(Q)$. Now we investigate the link between measures in Q and the notion of capacity defined above. The main theorem in this sense can be obtained from the result on the "elliptic capacity"contained in [20], which also applies to this context of parabolic spaces. We rewrite thus, with the necessary adaptations to the parabolic case, the proof of G. Dal Maso.

Theorem 10.7 *Let μ belong to $\mathcal{M}_0^+(Q)$. Then there exists $\gamma\in W'\cap\mathcal{M}_b^+(Q)$ and a nonnegative function $f\in L^1(Q,d\gamma^{meas})$ such that $\mu=f\gamma^{meas}$.*

Proof. Let $\mu\in\mathcal{M}_0^+(Q)$. For any u in W, let $\tilde u$ be the cap–quasi continuous representative of u, which exists by Lemma 10.2. Since $\tilde u$ is uniquely defined up to sets of zero capacity we can define the functional $F:W\to[0,\infty]$ by

$$F(u)=\int_Q\tilde u^+\,d\mu$$

(indeed, this definition does not depend on the cap–quasi continuous representative of u, since two cap–quasi continuous representatives are equal except on a set of null capacity, that is to say μ-a.e.). Clearly F is convex, and it is also lower semicontinuous in W thanks to Lemma 10.3 and Fatou's lemma. By the separability of W', there exists then a sequence $\{a_n\}$ of real numbers and a sequence $\{\lambda_n\}$ in W' such that:

$$F(u)=\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle+a_n\}.$$

Since, for any positive t, $tF(u)=F(tu)\ge t\langle\langle\lambda_n,u\rangle\rangle+a_n$ for every n, dividing by t and letting t tend to infinity we get $F(u)\ge\langle\langle\lambda_n,u\rangle\rangle$ for all u in W. For $u=0$, we deduce that $a_n\le0$, hence

$$F(u)\ge\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle\}\ge\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle+a_n\}=F(u)\,. \tag{10.25}$$

By (10.25) and the definition of F, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, we have

$$\langle\langle\lambda_n,\varphi\rangle\rangle\le\int_Q\varphi^+\,d\mu\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}, \tag{10.26}$$

thus, applying this inequality to φ and $-\varphi$, we get $|\langle\langle\lambda_n,\varphi\rangle\rangle|\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}$, which implies that $\lambda_n\in W'\cap\mathcal{M}_b(Q)$; moreover, since $F(-\varphi)=0$ for any nonnegative $\varphi\in\mathcal{C}_c^\infty(Q)$, we have $0\le\langle\langle\lambda_n,\varphi\rangle\rangle=\int_Q\varphi\,d\lambda_n^{\text{meas}}$ for all such φ, which implies $\lambda_n^{\text{meas}}\in\mathcal{M}_b^+(Q)$ (that is to say $\lambda_n\in W'\cap\mathcal{M}_b^+(Q)$) and, applying once again (10.26) to any nonnegative $\varphi\in\mathcal{C}_c^\infty(Q)$,

$$\lambda_n^{\text{meas}}\le\mu. \tag{10.27}$$

We have thus, in particular, $\|\lambda_n^{\text{meas}}\|_{\mathcal{M}_b(Q)}\le\|\mu\|_{\mathcal{M}_b(Q)}$.
Thus the series

$$\gamma=\sum_{n=1}^\infty\frac{\lambda_n}{2^n(\|\lambda_n\|_{W'}+1)} \tag{10.28}$$

is absolutely convergent in W' and we have, for all $\varphi\in\mathcal{C}_c^\infty(Q)$,

$$\begin{aligned}|\langle\langle\gamma,\varphi\rangle\rangle| &= \left|\sum_{n=1}^\infty\frac{\langle\langle\lambda_n,\varphi\rangle\rangle}{2^n(\|\lambda_n\|_{W'}+1)}\right|\\ &\le \sum_{n=1}^\infty\frac{\|\lambda_n^{\text{meas}}\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}}{2^n}\\ &\le \|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)},\end{aligned}$$

so that $\gamma \in W' \cap \mathcal{M}_b(Q)$. The series $\sum_{n=1}^{\infty} \frac{\lambda_n^{\text{meas}}}{2^n(\|\lambda_n\|_{W'}+1)}$ strongly converges in $\mathcal{M}_b(Q)$, and we can see, applying (10.28) to functions of $\mathcal{C}_c^\infty(Q)$, that

$$\gamma^{\text{meas}} = \sum_{n=1}^{\infty} \frac{\lambda_n^{\text{meas}}}{2^n(\|\lambda_n\|_{W'}+1)}.$$

In particular, γ^{meas} is a nonnegative measure (each λ_n^{meas} is nonnegative).
Since $\lambda_n^{\text{meas}} << \gamma^{\text{meas}}$, there exists a nonnegative function $f_n \in L^1(Q, d\gamma^{\text{meas}})$ such that $\lambda_n^{\text{meas}} = f_n \gamma^{\text{meas}}$, thus (10.25) implies:

$$\int_Q \varphi \, d\mu = \sup_n \int_Q f_n \, \varphi \, d\gamma^{\text{meas}} , \tag{10.29}$$

for any nonnegative φ in $\mathcal{C}_c^\infty(Q)$. We also have, by (10.27), $f_n \gamma^{\text{meas}} \le \mu$, that is

$$\int_B f_n \, d\gamma^{\text{meas}} \le \mu(B) ,$$

for any borelian subset B in Q and every n. In particular, we have

$$\int_B \sup\{f_1, f_2, \dots, f_k\} \, d\gamma^{\text{meas}} \le \mu(B) ,$$

for any borelian subset B in Q and any $k \ge 1$. Letting k tend to infinity we deduce by the monotone convergence theorem:

$$\int_B f \, d\gamma^{\text{meas}} \le \mu(B) ,$$

where $f = \sup\limits_n f_n$. Then we conclude, using (10.29):

$$\int_Q \varphi \, d\mu = \sup_n \int_Q f_n \, \varphi \, d\gamma^{\text{meas}} \le \int_Q f \, \varphi \, d\gamma^{\text{meas}} \le \int_Q \varphi \, d\mu ,$$

for any positive $\varphi \in \mathcal{C}_c^\infty(Q)$, which yields that $\mu = f\gamma^{\text{meas}}$, and since $\mu(Q) < +\infty$ it follows that $f \in L^1(Q, d\gamma^{\text{meas}})$. ∎

In order to better specify the nature of a measure in $\mathcal{M}_0(Q)$, we need then to detail the structure of the dual space W'.

Lemma 10.4 *Let $g \in W'$. Then there exist $g_1 \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and $g_3 \in L^{p'}(0,T; L^2(\Omega))$ such that*

$$\langle\langle g, u \rangle\rangle = \int_0^T \langle g_1, u\rangle + \int_0^T \langle u_t, g_2\rangle + \int_Q g_3 u \, dxdt \qquad \forall\, u \in W .$$

Moreover, we can choose (g_1, g_2, g_3) such that

$$\|g_1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2\|_{L^p(0,T;V)} + \|g_3\|_{L^{p'}(0,T;L^2(\Omega))} \le C\|g\|_{W'}, \tag{10.30}$$

with C not depending on g.

Proof. Let $E = L^p(0,T;V) \times L^{p'}(0,T;V')$ and $T : W \mapsto E$ such that $T(u) = (u, u_t)$. If we endow E with the norm

$$\|(v_1, v_2)\|_E = \|v_1\|_{L^p(0,T;V)} + \|v_2\|_{L^{p'}(0,T;V')},$$

then T is isometric from W to E. Let $G = T(W)$, with the norm of E, thus T^{-1} is defined on G. Let $g \in W'$ and let $\Phi : G \mapsto \mathbb{R}$, $\Phi(v_1, v_2) = \langle\langle g, T^{-1}(v_1, v_2)\rangle\rangle$, then Φ is a continuous linear form on G.

Hence thanks to the Hahn-Banach theorem, it can be extended to a continuous linear form on E, also denoted Φ, with $\|\Phi\|_{E'} = \|g\|_{W'}$ (since T^{-1} is isometric). There exists thus $h_1 \in (L^p(0,T;V))'$ and $h_2 \in (L^{p'}(0,T;V'))'$ such that

$$\Phi(v_1, v_2) = \langle h_1, v_1\rangle_{(L^p(0,T;V))', L^p(0,T;V)} + \langle h_2, v_2\rangle_{(L^{p'}(0,T;V'))', L^{p'}(0,T;V')}$$

and $\|h_1\|_{(L^p(0,T;V))'} + \|h_2\|_{(L^{p'}(0,T;V'))'} \leq C\|\Phi\|_{E'}$. But $L^p(0,T;V)$ is reflexive and $(L^p(0,T;V))' = L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega))$ (with equivalent norms), so that we can find $g_2 \in L^p(0,T;V)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $g_3 \in L^{p'}(0,T;L^2(\Omega))$ satisfying

$$\Phi(v_1, v_2) = \int_0^T \langle g_1, v_1\rangle + \int_0^T \langle v_2, g_2\rangle + \int_Q g_3 v_1$$

and $\|g_1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2\|_{L^p(0,T;V)} + \|g_3\|_{L^{p'}(0,T;L^2(\Omega))} \leq C(\|h_1\|_{(L^p(0,T;V))'} + \|h_2\|_{(L^{p'}(0,T;V'))'}) \leq C\|g\|_{W'}$.
Hence for all $u \in W$, $\langle\langle g, u\rangle\rangle = \Phi(T(u)) = \int_0^T \langle g_1, u\rangle + \int_0^T \langle u_t, g_2\rangle + \int_Q g_3 u$, which concludes the proof.
■

We will need, in the following, to construct suitable smooth approximations of elements $\nu \in W' \cap \mathcal{M}_b(Q)$ which at the same time converge strongly in W' and weakly-$*$ in $\mathcal{M}_b(Q)$. As usual, we would like to start with measures ν having compact support. To this purpose, note that when θ is a regular function, since the multiplication $\varphi \to \theta\varphi$ is linear continuous from W to W, we can define the multiplication of an element $\nu \in W'$ by θ thanks to a duality method: $\theta\nu \in W'$ is defined by $\langle\langle \theta\nu, \varphi\rangle\rangle = \langle\langle \nu, \theta\varphi\rangle\rangle$.

Lemma 10.5 *Let $\nu \in W' \cap \mathcal{M}_b(Q)$ and $\theta \in \mathcal{C}_c^\infty(Q)$. We take $(\rho_n)_{n\geq 1}$ a sequence of symmetric ([4]) mollifiers in $\mathbb{R} \times \mathbb{R}^N$ and $\mu = \theta\nu \in W'$. Then $\mu \in W' \cap \mathcal{M}_b(Q)$, $\mu^{meas} = \theta\nu^{meas}$, μ^{meas} has a compact support in Q and*

$$\|\mu^{meas} * \rho_n\|_{L^1(Q)} \leq \|\mu^{meas}\|_{\mathcal{M}_b(Q)}, \qquad \mu^{meas} * \rho_n \to \mu \quad \text{in } W'. \tag{10.31}$$

Proof.
The fact that $\mu \in W' \cap \mathcal{M}_b(\Omega)$ is quite obvious since, for all $\varphi \in \mathcal{C}_c^\infty(Q)$, $|\langle\langle \mu, \varphi\rangle\rangle| = |\langle\langle \nu, \theta\varphi\rangle\rangle| \leq C||\theta\varphi||_{L^\infty(Q)} \leq C||\theta||_{L^\infty(Q)}||\varphi||_{L^\infty(Q)}$. Moreover, by definition, one has, for all $\varphi \in \mathcal{C}_c^\infty(Q)$,

$$\int_Q \varphi\, d\mu^{\text{meas}} = \langle\langle \mu, \varphi\rangle\rangle = \langle\langle \nu, \theta\varphi\rangle\rangle = \int_Q \theta\varphi\, d\nu^{\text{meas}},$$

so that $\mu^{\text{meas}} = \theta\nu^{\text{meas}}$; thus, the measure μ^{meas} has indeed a compact support and $\mu^{\text{meas}} * \rho_n$ is well defined and is, for n large enough, a function in $\mathcal{C}_c^\infty(Q)$. By a classical result of convolution of measures, one has $\|\mu^{\text{meas}} * \rho_n\|_{L^1(Q)} \leq \|\mu^{\text{meas}}\|_{\mathcal{M}_b(Q)}$.
Let now $(g_1, g_2, g_3) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \times L^p(0,T;V) \times L^{p'}(0,T;L^2(\Omega))$ be a decomposition of ν according to Lemma 10.4. Then, for all $\varphi \in W$, one has

$$\begin{aligned}\langle\langle \mu, \varphi\rangle\rangle &= \int_0^T \langle g_1, \theta\varphi\rangle + \int_0^T \langle(\theta\varphi)_t, g_2\rangle + \int_Q g_3\theta\varphi \\ &= \int_0^T \langle \theta g_1, \varphi\rangle + \int_0^T \langle \varphi_t, \theta g_2\rangle + \int_0^T \langle \theta_t\varphi, g_2\rangle + \int_Q \theta g_3\varphi.\end{aligned}$$

Since $\theta_t\varphi \in L^{p'}(0,T;L^2(\Omega))$ (see Remark 10.2), the term $\int_0^T \langle \theta_t\varphi, g_2\rangle$ is in fact $\int_Q \theta_t\varphi g_2$. Moreover, since $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, there exists $G_1 \in (L^{p'}(Q))^N$ such that $g_1 = \text{div}(G_1)$, so that

$$\int_0^T \langle \theta g_1, \varphi\rangle = \int_0^T \langle \text{div}(\theta G_1), \varphi\rangle - \int_0^T \langle G_1 \cdot \nabla\theta, \varphi\rangle.$$

[4] That is to say $\rho_n(-\cdot) = \rho_n(\cdot)$.

$G_1 \cdot \nabla\theta \in L^{p'}(Q)$ and we have thus in fact

$$\int_0^T \langle \theta g_1, \varphi \rangle = \int_0^T \langle \mathrm{div}(\theta G_1), \varphi \rangle - \int_Q G_1 \cdot \nabla\theta\varphi.$$

Thus, for all $\varphi \in W$, one has

$$\langle\langle \mu, \varphi \rangle\rangle = \int_0^T \langle \mathrm{div}(\theta G_1), \varphi \rangle + \int_0^T \langle \varphi_t, \theta g_2 \rangle + \int_Q \theta g_3 \varphi - \int_Q G_1 \cdot \nabla\theta\varphi + \int_Q \theta_t g_2 \varphi. \tag{10.32}$$

From now on, we take n large enough so that $\mathrm{Supp}(\theta) + \mathrm{Supp}(\rho_n)$ be included in a fixed compact subset K of Q. The support of $\mu^{\mathrm{meas}} * \rho_n = (\theta\nu^{\mathrm{meas}}) * \rho_n$ is then also contained in K; we take $\zeta \in \mathcal{C}_c^\infty(Q)$ such that $\zeta \equiv 1$ on a neighborhood of K. We also take n large enough so that $\mathrm{Supp}(\zeta) + \mathrm{Supp}(\rho_n)$ is a compact subset of Q.
By definition of the natural injection $\mathcal{C}_c^\infty(Q) \subset W'$, we have, for all $\varphi \in W$,

$$\langle\langle \mu^{\mathrm{meas}} * \rho_n, \varphi \rangle\rangle = \int_Q \varphi \mu^{\mathrm{meas}} * \rho_n.$$

For all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$, we have then

$$\langle\langle \mu^{\mathrm{meas}} * \rho_n, \varphi \rangle\rangle = \int_Q \zeta\varphi \mu^{\mathrm{meas}} * \rho_n = \int_Q (\zeta\varphi) * \rho_n \, d\mu^{\mathrm{meas}},$$

since n has been chosen large enough so that the support of $(\zeta\varphi) * \rho_n$ is a compact subset of Q; but $(\zeta\varphi) * \rho_n \in \mathcal{C}_c^\infty(Q)$, so that, by definition and (10.32),

$$\begin{aligned}
&\langle\langle \mu^{\mathrm{meas}} * \rho_n, \varphi \rangle\rangle \\
&= \langle\langle \mu, (\zeta\varphi) * \rho_n \rangle\rangle \\
&= \int_0^T \langle \mathrm{div}(\theta G_1), (\zeta\varphi) * \rho_n \rangle + \int_0^T \langle ((\zeta\varphi) * \rho_n)_t, \theta g_2 \rangle + \int_Q \theta g_3 (\zeta\varphi) * \rho_n \\
&\quad - \int_Q G_1 \cdot \nabla\theta (\zeta\varphi) * \rho_n + \int_Q \theta_t g_2 (\zeta\varphi) * \rho_n.
\end{aligned}$$

We have chosen n large enough according to the supports of θ and ζ to allow us to write

$$\begin{aligned}
&\langle\langle \mu^{\mathrm{meas}} * \rho_n, \varphi \rangle\rangle \\
&= \int_0^T \langle \mathrm{div}((\theta G_1) * \rho_n), \zeta\varphi \rangle + \int_0^T \langle (\zeta\varphi)_t, (\theta g_2) * \rho_n \rangle + \int_Q (\theta g_3) * \rho_n \zeta\varphi \\
&\quad - \int_Q (G_1 \cdot \nabla\theta) * \rho_n \zeta\varphi + \int_Q (\theta_t g_2) * \rho_n \zeta\varphi.
\end{aligned}$$

But $\zeta \equiv 1$ on a neighborhood of $\mathrm{Supp}(\theta) + \mathrm{Supp}(\rho_n)$, so that

$$\begin{aligned}
&\langle\langle \mu^{\mathrm{meas}} * \rho_n, \varphi \rangle\rangle \\
&= \int_0^T \langle \mathrm{div}((\theta G_1) * \rho_n), \varphi \rangle + \int_0^T \langle \varphi_t, (\theta g_2) * \rho_n \rangle + \int_Q (\theta g_3) * \rho_n \varphi \\
&\quad - \int_Q (G_1 \cdot \nabla\theta) * \rho_n \varphi + \int_Q (\theta_t g_2) * \rho_n \varphi.
\end{aligned} \tag{10.33}$$

This equality has only been established for $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$, but since this space is dense in W and both sides are continuous with respect to the norm of W, this equality is still valid for all $\varphi \in W$.

We have $(\theta G_1) * \rho_n \to \theta G_1$ in $(L^{p'}(Q))^N$, $(\theta g_2) * \rho_n \to \theta g_2$ in $L^p(0,T;V)$, $(\theta g_3) * \rho_n \to \theta g_3$ in $L^{p'}(0,T;L^2(\Omega))$, $(G_1 \cdot \nabla\theta) * \rho_n \to G_1 \cdot \nabla\theta$ in $L^{p'}(Q)$ and $(\theta_t g_2) * \rho_n \to \theta_t g_2$ in $L^p(0,T;L^2(\Omega))$. Substracting (10.32) and (10.33), we have, for all $\varphi \in W$,

$$\begin{aligned}
&\langle\langle \mu^{\text{meas}} * \rho_n - \mu, \varphi\rangle\rangle \\
= &\int_0^T \langle \operatorname{div}((\theta G_1) * \rho_n - \theta G_1), \varphi\rangle + \int_0^T \langle \varphi_t, (\theta g_2) * \rho_n - \theta g_2\rangle + \int_Q ((\theta g_3) * \rho_n - \theta g_3)\varphi \\
&+ \int_Q (G_1 \cdot \nabla\theta - (G_1 \cdot \nabla\theta) * \rho_n)\varphi + \int_Q ((\theta_t g_2) * \rho_n - \theta_t g_2)\varphi \\
\leq\ & \|(\theta G_1) * \rho_n - \theta G_1\|_{(L^{p'}(Q))^N} \|\nabla\varphi\|_{L^p(Q)} + \|(\theta g_2) * \rho_n - \theta g_2\|_{L^p(0,T;V)} \|\varphi_t\|_{L^{p'}(0,T;V')} \\
&+ \|(\theta g_3) * \rho_n - \theta g_3\|_{L^{p'}(0,T;L^2(\Omega))} \|\varphi\|_{L^p(0,T;L^2(\Omega))} + \|G_1 \cdot \nabla\theta - (G_1 \cdot \nabla\theta) * \rho_n\|_{L^{p'}(Q)} \|\varphi\|_{L^p(Q)} \\
&+ \|(\theta_t g_2) * \rho_n - \theta_t g_2\|_{L^p(0,T;L^2(\Omega))} \|\varphi\|_{L^{p'}(0,T;L^2(\Omega))} \\
\leq\ & C\Big(\|(\theta G_1) * \rho_n - \theta G_1\|_{(L^{p'}(Q))^N} + \|(\theta g_2) * \rho_n - \theta g_2\|_{L^p(0,T;V)} + \|(\theta g_3) * \rho_n - \theta g_3\|_{L^{p'}(0,T;L^2(\Omega))} \\
&+ \|G_1 \cdot \nabla\theta - (G_1 \cdot \nabla\theta) * \rho_n\|_{L^{p'}(Q)} + \|(\theta_t g_2) * \rho_n - \theta_t g_2\|_{L^p(0,T;L^2(\Omega))} \Big) \|\varphi\|_W
\end{aligned}$$

which proves the convergence of $\mu^{\text{meas}} * \rho_n$ to μ in W'. ∎

Before stating and proving the decomposition theorem for elements of $\mathcal{M}_0(Q)$, let us first make a remark on the preceding proof, that will be useful to approximate elements of $\mathcal{M}_0(Q)$ in a suitable way.

Remark 10.8 *When $L \in W'$, we say that $(G_1, g_2, g_3, h_1, h_2)$ is a pseudo-decomposition of L if $G_1 \in (L^{p'}(Q))^N$, $g_2 \in L^p(0,T;V)$, $g_3 \in L^{p'}(0,T;L^2(\Omega))$, $h_1 \in L^{p'}(Q)$, $h_2 \in L^p(0,T;L^2(\Omega))$ and, for all $\varphi \in W$,*

$$\langle\langle L, \varphi\rangle\rangle = \int_0^T \langle \operatorname{div}(G_1), \varphi\rangle + \int_0^T \langle \varphi_t, g_2\rangle + \int_Q g_3\varphi + \int_Q h_1\varphi + \int_Q h_2\varphi.$$

The proof of Lemma 10.5 states the following: if $(\operatorname{div}(G_1), g_2, g_3)$ is a decomposition of ν according to Lemma 10.4, then $(\theta G_1, \theta g_2, \theta g_3, -G_1 \cdot \nabla\theta, \theta_t g_2)$ is a pseudo-decomposition of $\mu = \theta\nu$ (see (10.32)) and $((\theta G_1) * \rho_n, (\theta g_2) * \rho_n, (\theta g_3) * \rho_n, (-G_1 \cdot \nabla\theta) * \rho_n, (\theta_t g_2) * \rho_n)$ is a pseudo-decomposition of $\mu^{\text{meas}} * \rho_n$ (see (10.33)).
Thus, we have proven that a pseudo-decomposition of $\mu^{\text{meas}} * \rho_n$ converges to a pseudo-decomposition of μ. It is a weaker result than the one that would state that a decomposition of $\mu^{\text{meas}} * \rho_n$ (i.e. according to Lemma 10.4) converges to a decomposition of μ, but this last result is not clear. Indeed, to compute the elements of a decomposition of $\mu^{\text{meas}} * \rho_n$ we need to start from a decomposition of μ such that each term of the decomposition has a compact support; to obtain such a property, we need to introduce the cut-off function θ (because, in Lemma 10.4, it is not clear at all that, when g has a "compact support" — notice that, in fact, this expression has no sense since g is not a distribution —, we can take (g_1, g_2, g_3) with compact supports too), and the introduction of this cut-off function θ entails the apparition of the additional term $\theta_t g_2$, which cannot in general (if $p < 2$) be put in one of the terms of a decomposition of μ according to Lemma 10.4. Moreover, when we want to represent the term in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ of a decomposition of μ as the divergence of an element of $(L^{p'}(Q))^N$ with compact support (to manipulate this term using the convolution, we need such an hypothesis on the support), the introduction of the cut-off function creates the additional term $-G_1 \cdot \nabla\theta$, and finally leads to a pseudo-decomposition of μ as defined above.
Notice however that, if $p \geq 2$, the term $\theta_t g_2 \in L^p(0,T;L^2(\Omega))$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$ but the term $G_1 \cdot \nabla\theta \in L^{p'}(Q)$ remains; if $p \leq 2$, the term $G_1 \cdot \nabla\theta$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$, but the term $\theta_t g_2$ remains. In the special case $p = 2$, both terms $G_1 \cdot \nabla\theta$ and $\theta_t g_2$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$ and, in this case, we have in fact proven that there exists a decomposition of $\mu^{\text{meas}} * \rho_n \in W'$ (in the sense of Lemma 10.4) which converges to a decomposition of $\mu \in W'$.

Let us now prove a decomposition result as in [12].

Theorem 10.8 *If $\mu \in \mathcal{M}_0(Q)$, then there exist $g \in W'$ and $h \in L^1(Q)$, such that $\mu = g + h$, in the sense that*

$$\int_Q \varphi\, d\mu = \langle\langle g, \varphi\rangle\rangle + \int_Q h\,\varphi\, dxdt\,, \tag{10.34}$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$.

Proof. We follow the proof of [12]. First of all, using the Hahn decomposition of μ, if $\mu \in \mathcal{M}_0(Q)$ also μ^+, $\mu^- \in \mathcal{M}_0(Q)$, hence we can assume that μ is nonnegative. Applying Theorem 10.7 above there exists $\gamma \in W' \cap \mathcal{M}_b^+(Q)$ and a nonnegative Borel function $f \in L^1(Q, d\gamma^{\text{meas}})$, such that

$$\mu(B) = \int_B f\, d\gamma^{\text{meas}}$$

for every Borel set B in Q. Now let us replace μ with a compactly supported measure. To this end, it is enough to use the fact that $\mathcal{C}_c^\infty(Q)$ is dense in $L^1(Q, d\gamma^{\text{meas}})$ since γ^{meas} is a regular measure; there exists thus a sequence $\{f_n\} \in \mathcal{C}_c^\infty(Q)$ such that f_n strongly converges to f in $L^1(Q, d\gamma^{\text{meas}})$. Without loss of generality we can assume that $\sum_0^\infty \|f_n - f_{n-1}\|_{L^1(Q,d\gamma^{\text{meas}})} < \infty$, so that, defining $\nu_n = (f_n - f_{n-1})\gamma \in W'$, we have, by Lemma 10.5, $\nu_n \in W' \cap \mathcal{M}_b(Q)$ and $\sum_0^\infty \nu_n^{\text{meas}} = \sum_0^\infty (f_n - f_{n-1})\gamma^{\text{meas}} = \mu$ converges in the strong topology of measures.
The convergence result of Lemma 10.5 applied to ν_n implies that $\rho_l * \nu_n^{\text{meas}}$ strongly converges to ν_n in W' as l tends to infinity. We can therefore extract a subsequence l_n such that $\|\rho_{l_n} * \nu_n^{\text{meas}} - \nu_n\|_{W'} \le \frac{1}{2^n}$. We have then

$$\sum_{k=0}^n \nu_k^{\text{meas}} = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}} + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})\,. \tag{10.35}$$

The first member involved in this equality, denoted hereafter by m_n, is a measure with compact support. The second term, denoted by h_n, is a function in $\mathcal{C}_c^\infty(Q)$. By letting $g_n = \sum_{k=0}^n (\nu_k - \rho_{l_k} * \nu_k^{\text{meas}}) \in W' \cap \mathcal{M}_b(Q)$, the third term of (10.35) is g_n^{meas}; moreover, we can write $g_n = \theta_n g_n$ with $\theta_n \in \mathcal{C}_c^\infty(Q)$ (indeed, take $\theta_n \equiv 1$ on a neighborhood of $\text{Supp}(f_0) \cup \cdots \cup \text{Supp}(f_n)$ and on the neighborhood of the support of the $\mathcal{C}_c^\infty(Q)$ function $\sum_{k=0}^n \rho_{k_l} * \nu_k$). (10.35) is an equality in $\mathcal{M}_b(Q)$, i.e. involving g_n^{meas} and that can be applied only with test functions in $\mathcal{C}_c^\infty(Q)$; but thanks to the preceding remarks concerning the support of the elements involved in this equality, we can in fact deduce that, for all $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$, we have

$$\int_Q \varphi\, dm_n = \int_Q \varphi h_n + \langle\langle g_n, \varphi\rangle\rangle. \tag{10.36}$$

Indeed, the measures in (10.35) having compact supports, this formula can be applied to functions in $\mathcal{C}^\infty(Q)$ and since

$$\langle g_n^{\text{meas}}, \varphi\rangle = \langle g_n^{\text{meas}}, \theta_n\varphi\rangle = \langle\langle g_n, \theta_n\varphi\rangle\rangle = \langle\langle g_n, \varphi\rangle\rangle,$$

this gives (10.36).
Now, m_n is strongly convergent in $\mathcal{M}_b(Q)$ to μ. h_n strongly converges in $L^1(Q)$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}}\|_{L^1(Q)} \le \|\nu_k^{\text{meas}}\|_{\mathcal{M}_b(Q)}$ and $\sum_{k=0}^\infty \nu_k^{\text{meas}}$ is totally convergent in $\mathcal{M}_b(Q)$); we denote by h its limit. We also have that g_n is strongly convergent in W' (because $\|\rho_{l_k} * \nu_k^{\text{meas}} - \nu_k\|_{W'} \le \frac{1}{2^k}$), denoting by g its limit we get, for every $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$,

$$\langle\langle g_n, \varphi\rangle\rangle \to \langle\langle g, \varphi\rangle\rangle. \tag{10.37}$$

By convergence of h_n to h in $L^1(Q)$, and since φ is bounded, we also have

$$\int_Q h_n\varphi \to \int_Q h\varphi. \tag{10.38}$$

To prove the convergence of $\int_Q \varphi\, dm_n$ to $\int_Q \varphi\, d\mu$, we just recall that there is a natural injection

$$\begin{cases} \mathcal{M}_b(Q) \longrightarrow (\mathcal{C}_b(Q))' \\ \qquad m \longrightarrow \widetilde{m} \quad \text{defined by} \quad \widetilde{m}(f) = \int_Q f\, dm \end{cases}$$

which is linear and continuous. Thus, since m_n strongly converges in $\mathcal{M}_b(Q)$ to μ, $\widetilde{m_n}$ strongly converges in $(\mathcal{C}_b(Q))'$ to $\widetilde{\mu}$ and, since $\varphi \in \mathcal{C}_b(Q)$,

$$\int_Q \varphi\, dm_n = \widetilde{m_n}(\varphi) \to \widetilde{\mu}(\varphi) = \int_Q \varphi\, d\mu. \tag{10.39}$$

Gathering (10.36), (10.37), (10.38) and (10.39), we get (10.34). ∎

Combining Theorem 10.8 and Lemma 10.4 we deduce the following.

Theorem 10.9 *Let $\mu \in \mathcal{M}_0(Q)$, then there exists a decomposition (f, g_1, g_2) of μ in the sense that $f \in L^1(Q)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and*

$$\int_Q \varphi\, d\mu = \int_Q f\,\varphi\, dxdt + \int_0^T \langle g_1, \varphi\rangle\, dt - \int_0^T \langle \varphi_t, g_2\rangle\, dt\,, \qquad \forall \varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)\,.$$

Of course, there are infinitely many possible different decompositions of the same measure $\mu \in \mathcal{M}_0(Q)$, so the following lemma will be useful for further purposes.

Lemma 10.6 *Let $\mu \in \mathcal{M}_0(Q)$, and let (f, g_1, g_2) and $(\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ be two different decompositions of μ according to Theorem 10.9. Then we have $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$ in distributional sense, $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T];L^1(\Omega))$ and $(g_2 - \tilde{g}_2)(0) = 0$.*

Proof. By assumption we have :

$$\int_Q (\tilde{f} - f)\varphi\, dxdt + \int_0^T \langle(\tilde{g}_1 - g_1), \varphi\rangle\, dt = -\int_0^T \langle \varphi_t, g_2 - \tilde{g}_2\rangle\, dt \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)\,, \tag{10.40}$$

which implies, in particular, that $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$ in distributional sense. Thus $g_2 - \tilde{g}_2 \in L^p(0,T;W_0^{1,p}(\Omega))$ and $(g_2 - \tilde{g}_2)_t \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. By Theorem 1.1 in [63] it follows that $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T];L^1(\Omega))$. Since

$$\int_0^T \langle \varphi_t, g_2 - \tilde{g}_2\rangle\, dt + \int_0^T \langle (g_2 - \tilde{g}_2)_t, \varphi\rangle\, dt = -\int_\Omega (g_2 - \tilde{g}_2)(0)\varphi(0)\, dx$$

for all $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $\varphi(T) = 0$, we deduce from (10.40) (since $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$) that

$$\int_\Omega (g_2 - \tilde{g}_2)(0)\varphi(0)\, dx = 0$$

for all $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $\varphi(T) = 0$. Choosing $\varphi = (T-t)\psi$, with $\psi \in \mathcal{C}_c^\infty(\Omega)$ implies that $(g_2 - \tilde{g}_2)(0) = 0$. ∎

We will now state and prove, thanks to what has been done in the proof of Theorem 10.8 and Remark 10.8, an approximation result concerning elements of $\mathcal{M}_0(Q)$, which will allow us to obtain additional regularity results on the renormalized solution of (10.1).

Proposition 10.5 *Let $\mu \in \mathcal{M}_0(Q)$. Then there exist a decomposition $(f, \mathrm{div}(G_1), g_2)$ of μ in the sense of Theorem 10.9 and an approximation μ_n of μ satisfying:*

$$\begin{aligned}
&\mu_n \in \mathcal{C}_c^\infty(Q)\,, \quad \|\mu_n\|_{\mathcal{M}_b(Q)} \le C\,,\\
&\int_Q \mu_n\varphi\, dxdt = \int_Q \varphi\, f_n\, dxdt + \int_0^T \langle \mathrm{div}(G_1^n), \varphi\rangle\, dt - \int_0^T \langle \varphi_t, g_2^n\rangle\, dt \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)\,,\\
&f_n \in \mathcal{C}_c^\infty(Q)\,, \quad f_n \to f \qquad \textit{strongly in } L^1(Q),\\
&G_1^n \in (\mathcal{C}_c^\infty(Q))^N\,, \quad G_1^n \to G_1 \qquad \textit{strongly in } (L^{p'}(Q))^N,\\
&g_2^n \in \mathcal{C}_c^\infty(Q)\,, \quad g_2^n \to g_2 \qquad \textit{strongly in } L^p(0,T;V).
\end{aligned}$$

Proof. We will prove that there exists a decomposition $(f, \operatorname{div}(G_1), g_2)$ of μ such that, for all $\varepsilon > 0$, we can find $\mu_\varepsilon \in \mathcal{C}_c^\infty(Q)$ satisfying $\|\mu_\varepsilon\|_{L^1(Q)} \le C$,

$$\int_Q \mu_\varepsilon \varphi \, dxdt = \int_Q \varphi \, f_\varepsilon \, dxdt + \int_0^T \langle \operatorname{div}(G_1^\varepsilon), \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g_2^\varepsilon \rangle \, dt \,, \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega) \,,$$

with $f_\varepsilon \in \mathcal{C}_c^\infty(Q)$ such that $\|f_\varepsilon - f\|_{L^1(Q)} \le C\varepsilon$, $G_1^\varepsilon \in (\mathcal{C}_c^\infty(Q))^N$ such that $\|G_1^\varepsilon - G_1\|_{(L^{p'}(Q))^N} \le C\varepsilon$ and $g_2^\varepsilon \in \mathcal{C}_c^\infty(Q)$ such that $\|g_2^\varepsilon - g_2\|_{L^p(0,T;V)} \le C\varepsilon$ (with C not depending on ε).
We use the notations of the proof of Theorem 10.8.
Recalling that $\nu_k = (f_k - f_{k-1})\gamma$, we take $\zeta_k \in \mathcal{C}_c^\infty(Q)$ such that $\zeta_k \equiv 1$ on a neighborhood of $\mathrm{Supp}(f_k - f_{k-1})$; there exists $C(\zeta_k)$ only depending on ζ_k such that,

if $E \in \{(L^{p'}(Q))^N \,,\ L^p(0,T;V) \,,\ L^{p'}(0,T;L^2(\Omega))\}$ and $h \in E$, then $\|\zeta_k h\|_E \le C(\zeta_k)\|h\|_E$,
if $H \in (L^{p'}(Q))^N$ then $\|H \cdot \nabla\zeta_k\|_{L^{p'}(Q)} \le C(\zeta_k)\|H\|_{(L^{p'}(Q))^N}$,
if $h \in L^p(0,T,L^2(\Omega))$, then $\|(\zeta_k)_t h\|_{L^p(0,T;L^2(\Omega))} \le C(\zeta_k)\|h\|_{L^p(0,T;L^2(\Omega))}$.

Instead of the l_k chosen in the proof of Theorem 10.8, we take here l_k such that $\|\rho_{l_k} * \nu_k^{\mathrm{meas}} - \nu_k\|_{W'} \le 1/(2^k(C(\zeta_k)+1))$ and $\zeta_k \equiv 1$ on a neighborhood of $\mathrm{Supp}(\rho_{l_k} * \nu_k^{\mathrm{meas}})$. With this choice and taking $(\operatorname{div}(B_1^k), b_2^k, b_3^k)$ a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$ as in Lemma 10.4, satisfying moreover

$$\|B_1^k\|_{(L^{p'}(Q))^N} + \|b_2^k\|_{L^p(0,T;V)} + \|b_3^k\|_{L^{p'}(0,T;L^2(\Omega))} \le C\|\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}\|_{W'}$$

with C not depending on k (this is possible thanks to (10.30)), we notice that

$$\begin{array}{l} \sum_{k\ge1} \zeta_k B_1^k \text{ converges in } (L^{p'}(Q))^N \,,\ \sum_{k\ge1} \zeta_k b_2^k \text{ converges in } L^p(0,T;V) \,, \\ \sum_{k\ge1} \zeta_k b_3^k \text{ converges in } L^{p'}(0,T;L^2(\Omega)) \,,\ \sum_{k\ge1} B_1^k \cdot \nabla\zeta_k \text{ converges in } L^{p'}(Q) \,, \\ \sum_{k\ge1} (\zeta_k)_t b_2^k \text{ converges in } L^p(0,T;L^2(\Omega)). \end{array} \tag{10.41}$$

We denote by G_1, $-g_2$, f_1, f_2 and f_3 the respective limits of these terms; notice that the last three convergences imply in particular the convergence in $L^1(Q)$.
Since $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}})$ in W' (by choice of ζ_k and l_k) and $(\operatorname{div}(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$, $(\zeta_k B_1^k, \zeta_k b_2^k, \zeta_k b_3^k, -B_1^k \cdot \nabla\zeta_k, (\zeta_k)_t b_2^k)$ is a pseudo-decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$ (see Remark 10.8).
Thus, by (10.36), for all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$,

$$\begin{aligned} \int_Q \varphi \, dm_n &= \int_Q \varphi h_n + \int_0^T \langle \operatorname{div}\left(\sum_{k=0}^n \zeta_k B_1^k\right), \varphi \rangle + \int_0^T \langle \varphi_t, \sum_{k=0}^n \zeta_k b_2^k \rangle \\ &\quad + \int_0^T \sum_{k=0}^n \zeta_k b_3^k \varphi + \int_Q \sum_{k=0}^n (-B_1^k \cdot \nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (\zeta_k)_t b_2^k \varphi, \end{aligned}$$

and, by the convergences of m_n to μ, of h_n to h and (10.41), we deduce that

$$\int_Q \varphi \, d\mu = \int_Q (h + f_1 - f_2 + f_3)\varphi + \int_0^T \langle \operatorname{div}(G_1), \varphi \rangle - \int_0^T \langle \varphi_t, g_2 \rangle,$$

i.e. that $(f = h + f_1 - f_2 + f_3, \operatorname{div}(G_1), g_2)$ is a decomposition of μ in the sense of Theorem 10.9.
We fix now $\varepsilon > 0$ and take n large enough (in fact $n = n_\varepsilon$ is fixed in dependence of ε hereafter) so that

$$\left\|\sum_{k=0}^n \zeta_k B_1^k - G_1\right\|_{(L^{p'}(Q))^N} \le \varepsilon \,, \tag{10.42}$$

$$\left\|\sum_{k=0}^n \zeta_k b_2^k + g_2\right\|_{L^p(0,T;V)} \le \varepsilon \,, \tag{10.43}$$

$$\left\|h_n + \sum_{k=0}^n \zeta_k b_3^k - \sum_{k=0}^n (B_1^k \cdot \nabla\zeta_k) + \sum_{k=0}^n (\zeta_k)_t b_2^k - f\right\|_{L^1(Q)} \le \varepsilon \,. \tag{10.44}$$

Since $\nu_k - \rho_{l_k} * \nu_k^{\text{meas}} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$ and $(\text{div}(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\text{meas}}$, we also know that, for j large enough, $((\zeta_k B_1^k) * \rho_j, (\zeta_k b_2^k) * \rho_j, (\zeta_k b_3^k) * \rho_j, (-B_1^k \cdot \nabla\zeta_k) * \rho_j, ((\zeta_k)_t b_2^k) * \rho_j)$ is a pseudo-decomposition of $(\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_j \in \mathcal{C}_c^\infty(Q)$ (see Remark 10.8). We take j_n such that, for all $k \in [0, n]$,

$$\|(\zeta_k B_1^k) * \rho_{j_n} - \zeta_k B_1^k\|_{(L^{p'}(Q))^N} \leq \frac{\varepsilon}{n+1}, \tag{10.45}$$

$$\|(\zeta_k b_2^k) * \rho_{j_n} - \zeta_k b_2^k\|_{L^p(0,T;V)} \leq \frac{\varepsilon}{n+1}, \tag{10.46}$$

$$\begin{aligned} &\|(\zeta_k b_3^k) * \rho_{j_n} - \zeta_k b_3^k\|_{L^1(Q)} + \|(B_1^k \cdot \nabla\zeta_k) * \rho_{j_n} - B_1^k \cdot \nabla\zeta_k\|_{L^1(Q)} \\ &\quad + \|((\zeta_k)_t b_2^k) * \rho_{j_n} - (\zeta_k)_t b_2^k\|_{L^1(Q)} \leq \frac{\varepsilon}{n+1} \end{aligned} \tag{10.47}$$

Define $G_1^\varepsilon = \sum_{k=0}^n (\zeta_k B_1^k) * \rho_{j_n} \in (\mathcal{C}_c^\infty(Q))^N$; we have, by (10.42) and (10.45), $\|G_1^\varepsilon - G_1\|_{(L^{p'}(Q))^N} \leq 2\varepsilon$. Let $g_2^\varepsilon = -\sum_{k=0}^n (\zeta_k b_2^k) * \rho_{j_n} \in \mathcal{C}_c^\infty(Q)$; we have, by (10.43) and (10.46), $\|g_2^\varepsilon - g_2\|_{L^p(0,T;V)} \leq 2\varepsilon$. If $f_\varepsilon = h_n + \sum_{k=0}^n (\zeta_k b_3^k) * \rho_{j_n} - \sum_{k=0}^n (B_1^k \cdot \nabla\zeta_k) * \rho_{j_n} + \sum_{k=0}^n ((\zeta_k)_t b_2^k) * \rho_{j_n} \in \mathcal{C}_c^\infty(Q)$, we have, by (10.44) and (10.47), $\|f_\varepsilon - f\|_{L^1(Q)} \leq 2\varepsilon$.
Define now $\mu_\varepsilon = f_\varepsilon + \text{div}(G_1^\varepsilon) + (g_2^\varepsilon)_t \in \mathcal{C}_c^\infty(Q)$; it remains to prove that $\|\mu_\varepsilon\|_{L^1(Q)} \leq C$ with C not depending on ε. To see this, we recall that $((\zeta_k B_1^k) * \rho_{j_n}, (\zeta_k b_2^k) * \rho_{j_n}, (\zeta_k b_3^k) * \rho_{j_n}, (-B_1^k \cdot \nabla\zeta_k) * \rho_{j_n}, ((\zeta_k)_t b_2^k) * \rho_{j_n})$ is a pseudo-decomposition of $(\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_{j_n}$ so that

$$\mu_\varepsilon = h_n + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_{j_n} = h_n + \left(\sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})\right) * \rho_{j_n} = h_n + g_n^{\text{meas}} * \rho_{j_n}.$$

Since, by (10.35), $g_n^{\text{meas}} = m_n - h_n$, we deduce that $\|\mu_\varepsilon\|_{L^1(Q)} \leq 2\|h_n\|_{L^1(Q)} + \|m_n\|_{\mathcal{M}_b(Q)}$. Since $\{h_n\}$ converges in $L^1(Q)$ and $\{m_n\}$ converges in $\mathcal{M}_b(Q)$, $\{\|h_n\|_{L^1(Q)}\}$ and $\{\|m_n\|_{\mathcal{M}_b(Q)}\}$ are bounded, which imply the desired majoration on $\|\mu_\varepsilon\|_{L^1(Q)}$. ■

10.3 The IBV problem with data in $\mathcal{M}_0(Q)$.

Let us turn to the study of initial boundary value problems with data taken in $\mathcal{M}_0(Q)$. We start by introducing the following nonlinear monotone operators.
Let $a :]0, T[\times\Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function (i.e., $a(\cdot, \cdot, \xi)$ is measurable on Q for every ξ in $\mathbf{R}^N$, and $a(t, x, \cdot)$ is continuous on $\mathbf{R}^N$ for almost every (t, x) in Q), such that the following holds:

$$a(t, x, \xi) \cdot \xi \geq \alpha\, |\xi|^p, \quad p > 1, \tag{10.48}$$

$$|a(t, x, \xi)| \leq \beta\, [b(t, x) + |\xi|^{p-1}], \tag{10.49}$$

$$[a(t, x, \xi) - a(t, x, \eta)] \cdot (\xi - \eta) > 0, \tag{10.50}$$

for almost every (t, x) in Q, for every ξ, η in $\mathbf{R}^N$, with $\xi \neq \eta$, where α and β are two positive constants, and b is a nonnegative function in $L^{p'}(Q)$.
Let us define the differential operator

$$A(u) = -\text{div}(a(t, x, \nabla u)), \qquad u \in L^p(0, T; W_0^{1,p}(\Omega)).$$

Under assumptions (10.48), (10.49) and (10.50), A is a coercive and pseudomonotone operator acting from the space $L^p(0, T; W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0, T; W^{-1,p'}(\Omega))$, hence for $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, (10.1) has a unique solution in $\widetilde{W}$ (see Definition 10.2) in the weak sense (see [51]).

10.3.1 Variational case

Let us justify the interest of W', giving the following existence and uniqueness theorem (this theorem could also be stated with right-hand sides in $\widetilde{W}'$ with no major modification in the proof).

Theorem 10.10 *Let g belong to W', and let $u_0 \in L^2(\Omega)$. Assume that (10.48)–(10.50) hold true. Then there exists a unique solution u of*

$$\begin{cases} u_t + A(u) = g & \text{in }]0,T[\times\Omega, \\ u = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{10.51}$$

in the sense that $u \in L^p(0,T;V)$ and satisfies

$$-\int_Q \langle \varphi_t, u\rangle\, dt - \int_\Omega u_0\varphi(0)\, dx + \int_Q a(t,x,\nabla u)\nabla\varphi\, dxdt = \langle\langle g,\varphi\rangle\rangle, \tag{10.52}$$

for all $\varphi \in W$ with $\varphi(T) = 0$.

Remark 10.9 *Since $g \in W'$, there exists*

$$g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))\,,\ g_2 \in L^p(0,T;V) \text{ and } g_3 \in L^{p'}(0,T;L^2(\Omega))$$

such that

$$\langle\langle g,\varphi\rangle\rangle = \int_0^T \langle g_1,\varphi\rangle - \int_0^T \langle \varphi_t, g_2\rangle + \int_Q g_3\varphi\,, \quad \forall\varphi\in W.$$

For any such decomposition, we deduce that u satisfying (10.52) is such that $(u-g_2)_t = -A(u)+g_1+g_3 \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$, so that $u - g_2 \in W \subset \mathcal{C}([0,T];L^2(\Omega))$ and, returning to (10.52), we find $(u-g_2)(0) = u_0$.
Moreover, for any two solutions u and v of (10.52), we have $u - v = u - g_2 - (v - g_2) \in W$ and $(u-v)(0) = 0$.

Proof of Theorem 10.10. We take $(g_1, -g_2, g_3)$ a decomposition of g according to Lemma 10.4. Let $(g_1^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_1 in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, $(g_2^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_2 in $L^p(0,T;V)$ and $(g_3^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_3 in $L^{p'}(0,T;L^2(\Omega))$ (the existence of such sequences is a consequence of Lemma 10.8 and Remark 10.17). Thanks to [51], there exists a solution u_n of

$$\begin{cases} u_t^n + A(u^n) = g_1^n + g_3^n + (g_2^n)_t & \text{in }]0,T[\times\Omega, \\ u^n = 0 & \text{on }]0,T[\times\partial\Omega, \\ u^n(0) = u_0 & \text{in } \Omega, \end{cases}$$

in the sense that $u^n \in \widetilde{W}$ and

$$\int_\Omega (u^n - g_2^n)(t)\varphi(t)\, dx - \int_0^t \langle\varphi_t, u^n - g_2^n\rangle ds - \int_\Omega u_0\varphi(0)\, dx \\ + \int_0^t\int_\Omega a(s,x,\nabla u^n)\nabla\varphi\, dxds = \int_0^t \langle g_1^n,\varphi\rangle ds + \int_0^t\int_\Omega g_3^n\varphi\, dxds$$

for all $\varphi \in W$ and $t \in [0,T]$. Note that since $g_2^n \in \mathcal{C}_c^\infty(Q)$, we have $(u^n - g_2^n)(0) = u^n(0) = u_0$. Using $u^n - g_2^n$ as test function, and integrating by parts, we find

$$\int_\Omega \frac{(u^n - g_2^n)^2(t)}{2} - \int_\Omega \frac{u_0^2}{2} + \int_0^t\int_\Omega a(s,x,\nabla u^n)\nabla(u^n - g_2^n)\, dxds \\ = \int_0^t \langle g_1^n, u^n - g_2^n\rangle ds + \int_0^t\int_\Omega g_3^n(u^n - g_2^n)\, dxds$$

thus, using (10.48), (10.49) and Young's inequality,

$$\begin{aligned}
&\int_\Omega \frac{(u^n-g_2^n)^2(t)}{2}\,dx + \int_0^t\int_\Omega |\nabla u_n|^p\,dxds\\
&\le C\Big(\|g_1^n\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|g_3^n\|^2_{L^{p'}(0,T;L^2(\Omega))} + \|u_0\|^2_{L^2(\Omega)}\\
&\quad +\|b\|^{p'}_{L^{p'}(Q)}\Big) + \frac{1}{4T^{\frac{2}{p}}}\|u^n-g_2^n\|^2_{L^p(0,T;L^2(\Omega))}\\
&\le C\Big(\|g_1^n\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|g_3^n\|^2_{L^{p'}(0,T;L^2(\Omega))} + \|u_0\|^2_{L^2(\Omega)}\\
&\quad +\|b\|^{p'}_{L^{p'}(Q)}\Big) + \frac{T^{\frac{2}{p}}}{4T^{\frac{2}{p}}}\|u^n-g_2^n\|^2_{L^\infty(0,T;L^2(\Omega))}
\end{aligned}$$

which implies

$$\|u^n-g_2^n\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le C\,. \tag{10.53}$$

Thanks to the equation, we deduce from this that $(u^n-g_2^n)_t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$ so that, in fact, $(u^n-g_2^n)$ is bounded in W. There exists thus $w\in W$ such that, up to a subsequence, $u^n-g_2^n\to w$ weakly in W. But, from (10.53), (u^n) is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and converges thus, up to a subsequence, weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to a function u. Since $g_2^n\to g_2$ in $L^p(0,T;W_0^{1,p}(\Omega))$, this implies that $u^n-g_2^n\to u-g_2$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ so that $w=u-g_2\in W\subset \mathcal{C}([0,T];L^2(\Omega))$; note also that, since $u-g_2\in W$ and $g_2\in L^p(0,T;V)$, one has $u\in L^p(0,T;V)$.

Moreover, $A(u^n)$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, thus (up to subsequences) it converges weakly to an element f in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Using the equation in the sense of the distributions, we have $(u-g_2)_t+f=g_1+g_3$, hence, since $u-g_2\in W$,

$$-\int_0^T\langle\varphi_t,u-g_2\rangle dt - \int_\Omega (u-g_2)(0)\varphi(0)\,dx = \int_0^T\langle g_1-f,\varphi\rangle dt + \int_Q g_3\varphi\,dxdt.$$

for all $\varphi\in W$ such that $\varphi(T)=0$. On the other hand the equation implies, passing to the limit in n, that, with $\varphi\in W$ such that $\varphi(T)=0$,

$$-\int_0^T\langle\varphi_t,u-g_2\rangle dt - \int_\Omega u_0\varphi(0)\,dx = \int_0^T\langle g_1-f,\varphi\rangle dt + \int_Q g_3\varphi\,dxdt$$

so that $(u-g_2)(0)=u_0$. Now using $(u^n-g_2^n)-(u-g_2)$ as test function (note that $((u^n-g_2^n)-(u-g_2))(0)=0$), one has

$$\begin{aligned}
&\int_\Omega \frac{((u^n-g_2^n)-(u-g_2))^2(T)}{2}\,dx + \int_0^T\langle (u-g_2)_t,(u^n-g_2^n)-(u-g_2)\rangle dt\\
&\quad+\int_Q [a(t,x,\nabla u^n)-a(t,x,\nabla u)]\nabla(u^n-u)\,dxdt + \int_Q a(t,x,\nabla u)\nabla(u^n-u)\,dxdt\\
&\quad+\int_Q a(t,x,\nabla u^n)\nabla(g_2-g_2^n)\,dxdt\\
&=\int_0^T\langle g_1^n,(u^n-g_2^n)-(u-g_2)\rangle dt + \int_Q g_3^n[(u^n-g_2^n)-(u-g_2)]\,dxdt\,.
\end{aligned}$$

Since the second term and last four terms converge to 0, thanks to the positivity of the first one and to (10.50), one gets

$$\lim_{n\to\infty}\int_Q [a(t,x,\nabla u^n)-a(t,x,\nabla u)]\nabla(u^n-u)\,dxdt = 0$$

hence, using the standard monotonicity argument (see Lemma 5 in [13]), one has the convergence almost everywhere of ∇u^n to ∇u and the strong convergence of $a(t,x,\nabla u^n)$ to $a(t,x,\nabla u)$ in $(L^{p'}(Q))^N$. This proves the existence of a solution.
For uniqueness, let us suppose there are two solutions u and v, thanks to Remark 10.9, $u-v \in W$ so that, subtracting the two equations, one can choose $u-v$ as test function, obtaining:

$$\int_\Omega \frac{(u-v)^2(t)}{2}\,dx + \int_0^t\int_\Omega [a(t,x,\nabla u) - a(t,x,\nabla v)]\nabla(u-v)\,dxdt = 0, \quad \forall t \in]0,T[,$$

thus $u = v$ using (10.50). ∎

10.3.2 Definition and properties of renormalized solutions

Now we want to deal with the general problem (10.1) when μ is a measure which does not charge sets of zero capacity. In virtue of Theorem 10.8, this means that we consider measure data which split in a term of W' and a term in $L^1(Q)$. It is then well known that, if dealing with L^1 data, the concept of solution in the sense of distributions of problems like (10.1) may turn out to be not convenient in order to prove uniqueness of solutions. Moreover, we will deal with functions that may not belong to Sobolev spaces, so that we need to give a suitable definition of "gradient" for functions that enjoy some properties. To this purpose, if $k > 0$, we define

$$T_k(s) = \max(-k, \min(k,s)), \qquad \forall s \in \mathbf{R},$$

the truncature at levels k and $-k$, and $\Theta_k(s) = \int_0^s T_k(t)\,dt$. One has $\Theta_k(s) \geq 0$.
The truncations will provide very useful for defining a good class of solutions, as in [4].

Definition 10.6 *Let u be a measurable function on Q such that $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. Then (see [4], Lemma 2.1) there exists a unique measurable function $v : Q \to \mathbf{R}^N$ such that*

$$\nabla T_k(u) = v\,\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } Q, \textit{ for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$. If u belongs to $L^1(0,T;W_0^{1,1}(\Omega))$, then this gradient coincides with the usual gradient in distributional sense.

Let us introduce the definition of renormalized solution of (10.1).

Definition 10.7 *Let $\mu \in \mathcal{M}_0(Q)$. A measurable function u is a renormalized solution of (10.1) if there exists a decomposition (f, g_1, g_2) of μ such that*

$$u - g_2 \in L^\infty(0,T;L^1(\Omega)),\ T_k(u-g_2) \in L^p(0,T;W_0^{1,p}(\Omega)) \textit{ for every } k > 0, \tag{10.54}$$

$$\lim_{n\to\infty} \int_{\{n\leq|u-g_2|\leq n+1\}} |\nabla u|^p\,dxdt = 0, \tag{10.55}$$

and, for every $S \in W^{2,\infty}(\mathbf{R})$ such that S' has compact support,

$$\begin{aligned}(S(u-g_2))_t - \mathrm{div}(a(t,x,\nabla u)S'(u-g_2)) + S''(u-g_2)a(t,x,\nabla u)\nabla(u-g_2) = \\ = S'(u-g_2)f + G_1 S''(u-g_2)\nabla(u-g_2) - \mathrm{div}(G_1 S'(u-g_2))\end{aligned} \tag{10.56}$$

in the sense of distributions (where $g_1 = -\mathrm{div}(G_1)$) and

$$S(u-g_2)(0) = S(u_0) \textit{ in } L^1(\Omega). \tag{10.57}$$

Remark 10.10 *Note that the distributional meaning of each term in (10.56) is well defined thanks to the fact that* $T_k(u-g_2)$ *belongs to* $L^p(0,T;W_0^{1,p}(\Omega))$ *for every* $k>0$ *and since* S' *has compact support. Indeed, by taking* M *such that* $\mathrm{Supp}(S')\subset]-M,M[$, *since* $S'(u-g_2)=S''(u-g_2)=0$ *as soon as* $|u-g_2|\geq M$, *we can replace, everywhere in (10.56),* $\nabla(u-g_2)$ *by* $\nabla(T_M(u-g_2))\in(L^p(Q))^N$ *and* ∇u *by* $\nabla(T_M(u-g_2))+\nabla g_2\in(L^p(Q))^N$.
We also have, for all S *as above,* $S(u-g_2)=S(T_M(u-g_2))\in L^p(0,T;W_0^{1,p}(\Omega))$*; thus, by the equation (10.56),* $(S(u-g_2))_t$ *belongs to the space* $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, *which implies that* $S(u-g_2)$ *belongs to* $\mathcal{C}([0,T];L^1(\Omega))$ *(again see [63]). Thus condition (10.57) makes sense. Furthermore, since* $(S(u-g_2))_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$ *we can use, as test functions in (10.56), not only functions in* $\mathcal{C}_c^\infty(Q)$ *but also functions in* $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$.
Finally, observe also that condition (10.55) is equivalent to

$$\lim_{n\to\infty}\int\limits_{\{n\leq|u-g_2|\leq n+1\}}|\nabla(u-g_2)|^p\,dxdt=0\,,$$

since $g_2\in L^p(0,T;W_0^{1,p}(\Omega))$ *and* $u-g_2$ *is almost everywhere finite.*

Remark 10.11 *The initial condition* $S(u-g_2)(0)=S(u_0)$ *is the renormalized version of the requirement that* $(u-g_2)(0)=u_0$. *However, it also expresses, in a weak sense, that* $u(0)=u_0$, *as written in (10.1). This is due to the fact that* μ *is a measure on the* σ*-algebra of the borelians of the open set* Q, *which implies that* μ *is taken in a way that it does not charge the sets at* $t=0$. *More precisely, if* $\xi_\varepsilon(t)=(\frac{\varepsilon-t}{\varepsilon})^+$, *for any* $\varphi\in\mathcal{C}_c^\infty(\Omega)$ *we have, by Lebesgue's theorem,*

$$\lim_{\varepsilon\to0}\int_Q\varphi\,\xi_\varepsilon\,d\mu=0\,.$$

It follows then for any decomposition of μ

$$\lim_{\varepsilon\to0}\int_Q f\xi_\varepsilon\,\varphi\,dxdt+\int_0^T\langle g_1,\varphi\rangle\xi_\varepsilon\,dt+\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega g_2\,\varphi\,dxdt=0\,,$$

which implies, by the time regularity of f *and* g_1,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega g_2\,\varphi\,dxdt=0\,,\quad\forall\varphi\in\mathcal{C}_c^\infty(\Omega)\,. \tag{10.58}$$

Note that (10.58) is a weak expression of the fact that $g_2(0)=0$, *so that* $(u-g_2)(0)=u(0)$ *in some weak sense thanks to the fact that the measure* μ *is defined in the interior of* Q.
On the other hand, it would also be possible to consider measures μ *on the* σ*-algebra of borelians of* $[0,T)\times\Omega$, *hence* μ *would charge the level* $t=0$. *However, this case easily reduces to the previous one. Indeed, we can split* $\mu=\mu_Q+\mu_i$, *where* $\mu_i=\mu_{|\{t=0\}}$ *is the restriction of* μ *to* $t=0$ *(i.e.* $\mu_i(E)=\mu(E\cap(\{t=0\}\times\Omega))$ *for any set* E*) and* μ_Q *is the restriction to the open set* Q. *In this case problem (10.1) is equivalent to problem*

$$\begin{cases} u_t+A(u)=\mu_Q & \text{in }]0,T[\times\Omega,\\ u=0 & \text{on }]0,T[\times\partial\Omega,\\ u(0)=u_0+\mu_i & \text{in } \Omega.\end{cases} \tag{10.59}$$

If μ *is a measure which does not charge sets of zero capacity we have by Theorem 10.5 that* $\mu_i\in L^1(\Omega)$, *and the study of (10.59) reduces to the one we can do for measures* μ *only defined on the interior of* Q.

Remark 10.12 *As we have already noticed, when* u *is a renormalized solution of (10.1) and* S *is as in Definition 10.7, we have* $S(u-g_2)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ *and* $(S(u-g_2))_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$*; this allows, thanks to (ii) in Lemma 10.1, to prove that* $S(u-g_2)$ *has a cap–quasi continuous representative.*

In order to deal with the renormalized formulation, we will often make use of the following auxiliary functions of real variable.

Definition 10.8 *We define:*

$$\theta_n(s) = T_1(s - T_n(s))\,, \quad h_n(s) = 1 - |\theta_n(s)|\,, \quad S_n(s) = \int_0^s h_n(r)dr\,, \qquad \forall s \in \mathbf{R}\,.$$

Let us first prove that the formulation of renormalized solution does not depend on the decomposition of μ. This essentially relies on Lemma 10.6 proved before.

Proposition 10.6 *Let u be a renormalized solution of (10.1). Then u satisfies (10.54), (10.55), (10.56) and (10.57) for every decomposition (f, g_1, g_2) of μ.*

Proof. Assume that u satisfies the conditions of Definition 10.7 for (f, g_1, g_2), and let $(\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ be a different decomposition of μ. In the following we write $\tilde{g}_1 = -\mathrm{div}(\tilde{G}_1)$. Note that since, by Lemma 10.6, $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T]; L^1(\Omega))$ we have $u - \tilde{g}_2 \in L^\infty(0,T; L^1(\Omega))$, hence it is also almost everywhere finite. First of all we prove that $T_k(u - \tilde{g}_2) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$. To do this, we let $S = S_n$ in (10.56), where S_n is defined in Definition 10.8, and we choose as test function $T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)$, which belongs to $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. Using Lemma 10.6 we have:

$$\begin{aligned}
&\int_0^T \langle (S_n(u - g_2) + g_2 - \tilde{g}_2)_t, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\rangle \,dt \\
&\quad + \int_Q S_n'(u - g_2)a(t,x,\nabla u)\nabla T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&= - \int_Q S_n''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2)\,T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&\quad + \int_Q ((S_n'(u - g_2) - 1)f + \tilde{f})\,T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&\quad + \int_Q ((S_n'(u - g_2) - 1)G_1 + \tilde{G}_1)\nabla T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&\quad + \int_Q S_n''(u - g_2)G_1\nabla(u - g_2)T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt\,.
\end{aligned} \tag{10.60}$$

Since by (10.49)

$$\begin{aligned}
&\Big| - \int_Q S_n''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2)\,T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&\quad + \int_Q S_n''(u - g_2)G_1\nabla(u - g_2)T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt\Big| \\
&\leq Ck \int_{\{n \leq |u - g_2| \leq n+1\}} (|\nabla u|^p + |\nabla g_2|^p + |G_1|^{p'} + |b|^{p'})\,dxdt\,,
\end{aligned}$$

thanks to (10.55) and the fact that $u - g_2$ is almost everywhere finite, we get

$$\begin{aligned}
\lim_{n\to\infty} &\Big| - \int_Q S_n''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2)\,T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt \\
&\quad + \int_Q S_n''(u - g_2)G_1\nabla(u - g_2)T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)\,dxdt\Big| = 0\,.
\end{aligned}$$

Let us denote by $\omega(n)$ quantities going to zero as n tends to infinity. Integrating the first term of (10.60) in time, using that $k|s| \geq \Theta_k(s) \geq 0$, $(g_2 - \tilde{g}_2)(0) = 0$ and $0 \leq S_n'(s) \leq 1$, we obtain

$$\begin{aligned}
&\int_Q S_n'(u-g_2) a(t,x,\nabla u) \nabla T_k(S_n(u-g_2) + g_2 - \tilde{g}_2)\, dxdt \\
&\quad \leq k\left(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right) \\
&\qquad + \int_Q \left((S_n'(u-g_2) - 1) G_1 + \tilde{G}_1\right) \nabla T_k(S_n(u-g_2) + g_2 - \tilde{g}_2)\, dxdt + \omega(n)\,.
\end{aligned}$$

Setting $E_n = \{|S_n(u-g_2) + g_2 - \tilde{g}_2| \leq k\}$ we have:

$$\begin{aligned}
&\int_{E_n} [S_n'(u-g_2)]^2 a(t,x,\nabla u) \nabla u\, dxdt \\
&\leq k\left(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right) + \int_{E_n} (|G_1| + |\tilde{G}_1|)\, S_n'(u-g_2) |\nabla u|\, dxdt \\
&\quad + \int_{E_n} S_n'(u-g_2) |a(t,x,\nabla u)| \left(|\nabla \tilde{g}_2| + |\nabla g_2|\right)\, dxdt \\
&\quad + \int_{E_n} [S_n'(u-g_2)]^2 |a(t,x,\nabla u)| |\nabla g_2|\, dxdt + 2\int_Q (|G_1| + |\tilde{G}_1|)\ (|\nabla \tilde{g}_2| + |\nabla g_2|)\ dxdt + \omega(n)\,.
\end{aligned}$$

Young's inequality then implies, using also (10.48), (10.49) and $S_n'(s) \leq S_n'(s)^2 + \chi_{\{n \leq |s| \leq n+1\}}$:

$$\begin{aligned}
&\int_{E_n} [S_n'(u-g_2)]^2 |\nabla u|^p\, dxdt \\
&\leq\ Ck(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) + C\int_Q (|G_1|^{p'} + |\tilde{G}_1|^{p'} + |\nabla \tilde{g}_2|^p + |\nabla g_2|^p + |b|^{p'})\, dxdt \\
&\quad + C \int_{\{n \leq |u-g_2| \leq n+1\}} |\nabla u|^p\, dxdt + \omega(n)\,.
\end{aligned}$$

Using $S_n'(s)^p \leq S_n'(s) \leq S_n'(s)^2 + \chi_{\{n \leq |s| \leq n+1\}}$ (because $0 \leq S_n' \leq 1$) and the fact that g_2 belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce from the preceding inequality that, for all $n \geq 1$,

$$\int_Q \chi_{E_n} |\nabla(S_n(u-g_2))|^p \leq C.$$

Since $\nabla(T_k(S_n(u-g_2) + g_2 - \tilde{g}_2)) = \chi_{E_n} \nabla(S_n(u-g_2) + g_2 - \tilde{g}_2)$ and since $g_2,\ \tilde{g}_2 \in L^p(0,T;W_0^{1,p}(\Omega))$, this implies that $v_n = T_k(S_n(u-g_2) + g_2 - \tilde{g}_2)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and converges, up to a subsequence, to v weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, thus also in $\mathcal{D}'(Q)$; but $v_n \to T_k(u - \tilde{g}_2)$ a.e. in Q and is bounded by k, so that $v_n \to T_k(u-\tilde{g}_2)$ in $\mathcal{D}'(Q)$. We have then $T_k(u-\tilde{g}_2) = v \in L^p(0,T;W_0^{1,p}(\Omega))$, for all $k > 0$.
Similarly we prove that (10.55) holds true for $\tilde{g}_2$ as well : we choose $S = S_n$ and test function $\theta_h(S_n(u - g_2) + g_2 - \tilde{g}_2)$ in (10.56). Reasoning as above we obtain, setting $F_n = \{h \leq |S_n(u-g_2) + g_2 - \tilde{g}_2| \leq h+1\}$:

$$\begin{aligned}
&\int_{F_n} [S_n'(u-g_2)]^2 a(t,x,\nabla u) \nabla u\, dxdt \\
&\leq \int_Q ((S_n'(u-g_2) - 1) f + \tilde{f}) \theta_h(S_n(u-g_2) + g_2 - \tilde{g}_2)\, dxdt + \int_\Omega \int_0^{S_n(u_0)} \theta_h(r)\, dr\, dx \\
&\quad + \int_{F_n} S_n'(u-g_2)(|G_1| + |\tilde{G}_1|) |\nabla u|\, dxdt + \int_{F_n} S_n'(u-g_2) |a(t,x,\nabla u)| \left(|\nabla \tilde{g}_2| + |\nabla g_2|\right)\, dxdt \\
&\quad + \int_{F_n} [S_n'(u-g_2)]^2 |a(t,x,\nabla u)| |\nabla g_2|\, dxdt + 2\int_{F_n} (|G_1| + |\tilde{G}_1|)\ (|\nabla \tilde{g}_2| + |\nabla g_2|)\ dxdt + \omega(n)\,.
\end{aligned}$$

As before, thanks to Young's inequality, (10.49) and by properties of S_n we get:

$$\begin{aligned}
&\int_{F_n} [S_n'(u-g_2)]^2 |\nabla u|^p \,dxdt \\
&\le C\int_Q (|f|+|\tilde f|)|\theta_h(S_n(u-g_2)+g_2-\tilde g_2)|\,dxdt + \int_\Omega\int_0^{S_n(u_0)} \theta_h(r)\,dr\,dx \\
&\quad +C\int_{F_n} (|G_1|^{p'}+|\tilde G_1|^{p'}+|\nabla \tilde g_2|^p+|\nabla g_2|^p+|b|^{p'})\,dxdt \\
&\quad +C\int_{\{n\le|u-g_2|\le n+1\}} |\nabla u|^p\,dxdt+\omega(n)\,.
\end{aligned}$$

Letting n tend to infinity, using (10.55) and since χ_{F_n} converges to $\chi_{\{h\le|u-\tilde g_2|\le h+1\}}$ we obtain:

$$\begin{aligned}
&\int_{\{h\le|u-\tilde g_2|\le h+1\}} |\nabla u|^p\,dxdt \le \int_{\{|u_0|>h\}} |u_0|\,dx \\
&\quad + \int_{\{|u-\tilde g_2|\ge h\}} \left(|f|+|\tilde f|+|G_1|^{p'}+|\tilde G_1|^{p'}+|\nabla\tilde g_2|^p+|\nabla g_2|^p+|b|^{p'}\right)\,dxdt\,,
\end{aligned}$$

which yields, as h tends to infinity (recall that $u-\tilde g_2$ is almost everywhere finite),

$$\lim_{h\to\infty} \int_{\{h\le|u-\tilde g_2|\le h+1\}} |\nabla u|^p\,dxdt = 0\,.$$

We are left with the proof that the renormalized equation (10.56) and the initial condition (10.57) hold with $\tilde g_2$ as well. To this aim, we take $S=S_n$ in (10.56), we choose a function S such that S' has compact support and we take $S'(S_n(u-g_2)+g_2-\tilde g_2)\varphi$ as test function in (10.56), with $\varphi\in\mathcal{C}_c^\infty(Q)$. By Lemma 10.6 we get:

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g_2)+g_2-\tilde g_2)_t, S'(S_n(u-g_2)+g_2-\tilde g_2)\varphi\rangle\,dt \\
&\quad +\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla\varphi\, S'(S_n(u-g_2)+g_2-\tilde g_2)\,dxdt \\
&\quad +\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla(S'(S_n(u-g_2)+g_2-\tilde g_2))\varphi\,dxdt \\
&\quad +\int_Q S_n''(u-g_2)\,a(t,x,\nabla u)\nabla(u-g_2)\,S'(S_n(u-g_2)+g_2-\tilde g_2)\,\varphi\,dxdt \\
&= \int_Q ((S_n'(u-g_2)-1)f+\tilde f)\,S'(S_n(u-g_2)+g_2-\tilde g_2)\,\varphi\,dxdt \\
&\quad +\int_Q ((S_n'(u-g_2)-1)G_1+\tilde G_1)\nabla\varphi\, S'(S_n(u-g_2)+g_2-\tilde g_2)\,dxdt \\
&\quad +\int_Q ((S_n'(u-g_2)-1)G_1+\tilde G_1)\nabla(S'(S_n(u-g_2)+g_2-\tilde g_2))\varphi\,dxdt \\
&\quad +\int_Q S_n''(u-g_2)\,G_1\nabla(u-g_2)S'(S_n(u-g_2)+g_2-\tilde g_2)\,\varphi\,dxdt\,.
\end{aligned} \tag{10.61}$$

We will now pass to the limit in each term of this equation.

To handle the first one, we write $\langle (S_n(u-g_2)+g_2-\tilde g_2)_t, S'(S_n(u-g_2)+g_2-\tilde g_2)\varphi\rangle = \langle (S(S_n(u-g_2)+g_2-\tilde g_2))_t,\varphi\rangle$, so that, by definition of the derivative in $\mathcal{D}'(Q)$, this term passes to the limit thanks to the dominated convergence theorem:

$$\int_0^T \langle (S(S_n(u-g_2)+g_2-\tilde g_2))_t,\varphi\rangle = -\int_0^T S(S_n(u-g_2)+g_2-\tilde g_2)\varphi_t \longrightarrow -\int_0^T S(u-\tilde g_2)\varphi_t.$$

To handle the other terms, we take M such that $\mathrm{Supp}(S')\subset[-M,M]$. Since $S_n(x)-1\le x\le S_n(x)+1$ for all $x\in[-n-1,n+1]$, one has

$$\mathrm{Supp}\,(S_n'(u-g_2)\,S'(S_n(u-g_2)+g_2-\tilde g_2))\subset\{|u-g_2|\le n+1, |u-\tilde g_2|\le M+1\};$$

thus, in each of the integrals on Q of (10.61), ∇u can be replaced by $V=\nabla(T_{M+1}(u-\tilde g_2)+\tilde g_2)\in(L^p(Q))^N$; we can then pass to the limit with the help of the dominated convergence theorem. Since $u-\tilde g_2 = T_{M+1}(u-\tilde g_2)$ whenever $S'(u-\tilde g_2)\neq 0$ or $S''(u-\tilde g_2)\neq 0$, we can then replace V by ∇u in each limit term.
Indeed, since $S_n'\to 1$ and is bounded by 1, we have

$$\begin{aligned}&\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla\varphi\,S'(S_n(u-g_2)+g_2-\tilde g_2)\,dxdt\\ &=\int_Q S_n'(u-g_2)\,a(t,x,V)\nabla\varphi\,S'(S_n(u-g_2)+g_2-\tilde g_2)\,dxdt\\ &\longrightarrow \int_Q a(t,x,V)\nabla\varphi\,S'(u-\tilde g_2)\,dxdt=\int_Q a(t,x,\nabla u)\nabla\varphi\,S'(u-\tilde g_2)\,dxdt.\end{aligned}$$

For the third term of (10.61), we write $\nabla(S'(S_n(u-g_2)+g_2-\tilde g_2)) = S''(S_n(u-g_2)+g_2-\tilde g_2)(S_n'(u-g_2)\nabla(u-g_2)+\nabla(g_2-\tilde g_2)) = S''(S_n(u-g_2)+g_2-\tilde g_2)(S_n'(u-g_2)(V-\nabla g_2)+\nabla(g_2-\tilde g_2))$ with V, ∇g_2, $\nabla\tilde g_2\in(L^p(Q))^N$ so that this term tends to

$$\int_Q S''(u-\tilde g_2)a(t,x,V)(V-\nabla\tilde g_2)\varphi\,dxdt=\int_Q S''(u-\tilde g_2)a(t,x,\nabla u)\nabla(u-\tilde g_2)\varphi\,dxdt.$$

The fourth term tends to 0, because $S_n''\to 0$ and, in this term, $a(t,x,\nabla u)\nabla(u-g_2)=a(t,x,V)(V-\nabla g_2)\in L^1(Q)$. A straight application of the dominated convergence theorem show that the fifth term tends to $\int_Q \tilde f S'(u-\tilde g_2)\varphi$ and that the sixth term tends to $\int_Q \tilde G_1\nabla\varphi S'(u-\tilde g_2)$.
To study the convergence of the seventh term, we write, as above, $\nabla(S'(S_n(u-g_2)+g_2-\tilde g_2) = S''(S_n(u-g_2)+g_2-\tilde g_2)(S_n'(u-g_2)(V-\nabla g_2)+\nabla(g_2-\tilde g_2))$ so that, again thanks to the dominated convergence theorem, the limit of this term is

$$\int_Q S''(u-\tilde g_2)\tilde G_1(V-\nabla\tilde g_2)\varphi\,dxdt=\int_Q S''(u-\tilde g_2)\tilde G_1\nabla(u-\tilde g_2)\varphi\,dxdt.$$

Since $\nabla(u-g_2)=V-\nabla g_2\in(L^p(Q))^N$ in the last term of (10.61), we see that this term tends to 0 as $n\to\infty$. Gathering all the preceding convergences, we see that u satisfies (10.56) with $\tilde g_2$ instead of g_2.
To get back the initial condition with $\tilde g_2$ instead of g_2, we take $\varphi=(T-t)\psi$ with $\psi\in\mathcal{C}_c^\infty(\Omega)$, and we use, as before, (10.56) with $S=S_n$ and the test function $S'(S_n(u-g_2)+g_2-\tilde g_2)\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$; this gives (10.61). Now, however, since $\varphi(0)\neq 0$, the integration by parts in time in the first term of (10.61) gives

$$\int_0^T\langle (S(S_n(u-g_2)+g_2-\tilde g_2))_t,\varphi\rangle = -\int_\Omega S(S_n(u-g_2)(0)+(g_2-\tilde g_2)(0))\varphi(0)-\int_Q S(S_n(u-g_2)+g_2-\tilde g_2)\varphi_t.$$

Since $S_n(u-g_2)(0)=S_n(u_0)$ and $(g_2-\tilde g_2)(0)=0$, we have $S(S_n(u-g_2)(0)+(g_2-\tilde g_2)(0))=S(S_n(u_0))$ so that the first term of (10.61) tends now to

$$-\int_\Omega S(u_0)\varphi(0)-\int_Q S(u-\tilde g_2)\varphi_t.$$

The other terms tend to the same limits as before and we get thus

$$\begin{aligned} &-\int_\Omega S(u_0)\varphi(0) - \int_Q S(u-\tilde g_2)\varphi_t + \int_Q a(t,x,\nabla u)\nabla\varphi\, S'(u-\tilde g_2)\,dxdt \\ &+ \int_Q S''(u-\tilde g_2)a(t,x,\nabla u)\nabla(u-\tilde g_2)\varphi\,dxdt \\ &= \int_Q \tilde f S'(u-\tilde g_2)\varphi + \int_Q \tilde G_1\nabla\varphi S'(u-\tilde g_2) + \int_Q S''(u-\tilde g_2)\tilde G_1\nabla(u-\tilde g_2)\varphi\,dxdt. \end{aligned} \tag{10.62}$$

On the other hand, since $S(u-\tilde g_2)\in L^p(0,T;W_0^{1,p}(\Omega))$ satisfies (10.56) (with $\tilde g_2$ instead of g_2), we have $(S(u-\tilde g_2))_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, so that $S(u-\tilde g_2)\in \mathcal{C}([0,T];L^1(\Omega))$ and we can use φ as a test function in (10.56) written with $\tilde g_2$; this gives

$$\begin{aligned} &-\int_\Omega S(u-\tilde g_2)(0)\varphi(0) - \int_Q S(u-\tilde g_2)\varphi_t + \int_Q a(t,x,\nabla u)\nabla\varphi\, S'(u-\tilde g_2)\,dxdt \\ &+ \int_Q S''(u-\tilde g_2)a(t,x,\nabla u)\nabla(u-\tilde g_2)\varphi\,dxdt \\ &= \int_Q \tilde f S'(u-\tilde g_2)\varphi + \int_Q \tilde G_1\nabla\varphi S'(u-\tilde g_2) + \int_Q S''(u-\tilde g_2)\tilde G_1\nabla(u-\tilde g_2)\varphi\,dxdt. \end{aligned} \tag{10.63}$$

From (10.62) and (10.63) we deduce that $\int_\Omega S(u-\tilde g_2)(0)\psi = \int_\Omega S(u_0)\psi$ for all $\psi\in\mathcal{C}_c^\infty(\Omega)$, that is to say $S(u-\tilde g_2)(0) = S(u_0)$. ∎

Remark 10.13 *It should be noted that the definition of renormalized solution is not restricted to the case that μ is a measure, since (10.54)–(10.57) make sense whenever $f\in L^1(Q)$, $g_1\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2\in L^p(0,T;V)$. Thus the definition of renormalized solution makes sense also if $\mu\in L^1(Q)+W'$, without being necessarily a measure. In this case (f,g_1,g_2) is a decomposition of μ in $L^1(Q)+W'$. Note also that the conclusion of Lemma 10.6 is still true if $\mu\in L^1(Q)+W'$, hence the result of Proposition 10.6 would remain true in this case too.*

10.3.3 Proof of existence and uniqueness theorems

We can now start the proof of the existence result for problem (10.1). Following a standard approach, we obtain the existence of a solution as limit of nonsingular approximating problems. To this purpose, let μ_n be an approximation of μ given by Proposition 10.5, and let $\{u_{0n}\}\in L^\infty(\Omega)$ converge to u_0 strongly in $L^1(\Omega)$. Then by classical results (see for instance [51]) there exists a unique solution u_n in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ of the Cauchy–Dirichlet problem:

$$\begin{cases} (u_n)_t - \operatorname{div}(a(t,x,\nabla u_n)) = \mu_n & \text{in }]0,T[\times\Omega, \\ u_n = 0 & \text{on }]0,T[\times\partial\Omega, \\ u_n(0) = u_{0n} & \text{in } \Omega. \end{cases} \tag{10.64}$$

Moreover, from Proposition 10.5, u_n satisfies:

$$\begin{aligned} &\int_0^t \langle (u_n - g_2^n)_t, \varphi\rangle\,ds + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla\varphi\,dxds = \int_0^t\int_\Omega f_n\varphi\,dxds \\ &+ \int_0^t \langle g_1^n,\varphi\rangle\,ds\,, \qquad \forall\varphi\in L^p(0,T;V)\,,\ \forall t\in[0,T]\,. \end{aligned} \tag{10.65}$$

Let us begin by getting *a priori* estimates on u_n.

Proposition 10.7 *Let u_n be the solution of (10.64). Then we have:*

$$\begin{aligned} &\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C\,, \\ &\int_Q |\nabla T_k(u_n)|^p\,dxdt \le C\,k\,, \\ &\|u_n - g_2^n\|_{L^\infty(0,T;L^1(\Omega))} \le C\,, \\ &\int_Q |\nabla T_k(u_n - g_2^n)|^p\,dxdt \le C\,(k+1)\,, \\ &\lim_{h\to\infty}\left(\sup_n \int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt\right) = 0\,, \quad \forall k>0\,. \end{aligned} \tag{10.66}$$

Moreover there exists a measurable function $u : Q \to \mathbf{R}$ such that $T_k(u)$ and $T_k(u-g_2)$ belong to $L^p(0,T;W_0^{1,p}(\Omega))$, u and $u-g_2$ belong to $L^\infty(0,T;L^1(\Omega))$ and, up to a subsequence, for any $k>0$:

$$\begin{aligned} &u_n \to u \qquad a.e.\ in\ Q, \\ &T_k(u_n - g_2^n) \to T_k(u-g_2) \qquad weakly\ in\ L^p(0,T;W_0^{1,p}(\Omega))\ and\ a.e.\ in\ Q. \end{aligned} \tag{10.67}$$

Finally, we have

$$\lim_{h\to\infty} \int_{\{h\le|u-g_2|\le h+k\}} |\nabla u|^p\,dxdt = 0\,, \quad \forall k>0\,. \tag{10.68}$$

Proof. First of all, we choose $T_k(u_n)$ as test function in (10.64) and we integrate in $]0,t[$ to get:

$$\int_\Omega \Theta_k(u_n)(t)\,dx + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla T_k(u_n)\,dxds = \int_0^t \mu_n\,T_k(u_n)\,dxds + \int_\Omega \Theta_k(u_{0n})\,dx\,,$$

which yields, from (10.48) and the fact that $\|u_{0n}\|_{L^1(\Omega)}$ and $\|\mu_n\|_{L^1(Q)}$ are bounded:

$$\int_\Omega \Theta_k(u_n)(t)\,dx + \int_0^t\int_\Omega |\nabla T_k(u_n)|^p\,dxds \le Ck\,.$$

Since $\Theta_k(s)\ge 0$, for $t=T$ we get that $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. If $k=1$, we also get:

$$\int_\Omega \Theta_1(u_n)(t) \le C \qquad \forall t\in[0,T]\,,$$

which implies, since $\Theta_1(s)\ge |s|-1$, that

$$\int_\Omega |u_n(t)|\,dx \le C \qquad \forall t\in[0,T]\,.$$

Taking the supremum on $]0,T[$ we obtain the estimate of u_n in $L^\infty(0,T;L^1(\Omega))$. Similarly we can get the estimates on $u_n - g_2^n$: let us choose $T_k(u_n - g_2^n)$ as test function in (10.65). Integrating by parts (recall that g_2^n has compact support, so that $u^n(0) - g_2^n(0) = u^n(0) = u_{0n}$) and using (10.48) this gives:

$$\begin{aligned} &\int_\Omega \Theta_k(u_n - g_2^n)(t)\,dx + \alpha\int_0^t\int_\Omega |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxds \\ &\le \int_\Omega \Theta_k(u_{0n})\,dx + \int_Q f_n\,T_k(u_n-g_2^n)\,dxdt + \int_0^t\int_\Omega G_1^n\nabla u_n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds \\ &\quad - \int_0^t\int_\Omega G_1^n\nabla g_2^n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla g_2^n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds\,. \end{aligned}$$

Using assumption (10.49) and by means of Young's inequality we obtain:

$$\int_\Omega \Theta_k(u_n - g_2^n)(t)\,dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxdt \le k\int_Q |f_n|\,dxdt$$
$$+\,C\int_Q |G_1^n|^{p'}\,dxdt + C\int_Q |\nabla g_2^n|^p\,dxdt + C\int_Q |b(t,x)|^{p'}\,dxdt + k\int_\Omega |u_{0n}|\,dx\,.$$

Since G_1^n is bounded in $L^{p'}(Q)$, g_2^n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f_n is bounded in $L^1(Q)$ and u_{0n} is bounded in $L^1(\Omega)$, we obtain

$$\int_\Omega \Theta_1(u_n - g_2^n)(t)\,dx \le C \qquad \forall t \in]0,T[\,,$$

which implies the estimate of $u_n - g_2^n$ in $L^\infty(0,T;L^1(\Omega))$, and also

$$\int_Q |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxdt \le C\,(k+1)\,,$$

which yields that $T_k(u_n - g_2^n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k > 0$ (recall that g_2^n itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). Now, let $\psi(s) = T_k(s - T_h(s))$ and take $\psi(u_n - g_2^n)$ as test function in (10.65). Reasoning as above, using that $\psi'(s) = \chi_{\{h\le|s|\le h+k\}}$ and applying Young's inequality we obtain:

$$\int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt \le Ck\int_{\{|u_{0n}|>h\}} |u_{0n}|\,dx + Ck\int_{\{|u_n-g_2^n|>h\}}|f_n|\,dxdt + C\int_{\{|u_n-g_2^n|>h\}}(|G_1^n|^{p'} + |\nabla g_2^n|^p + |b(x,t)|^{p'})\,dxdt\,.$$

Since $u_n - g_2^n$ is bounded in $L^\infty(0,T;L^1(\Omega))$ we have

$$\lim_{h\to\infty}\left(\sup_n \operatorname{meas}\{|u_n - g_2^n| > h\}\right) = 0\,,$$

then by means of the equi–integrability of the sequences f_n, $|G_1^n|^{p'}$ and $|\nabla g_2^n|^p$ in $L^1(Q)$ we deduce that:

$$\lim_{h\to\infty}\left(\sup_n \int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt\right) = 0\,, \tag{10.69}$$

for every $k > 0$.

We are going to prove now that, up to subsequences, u_n converges almost everywhere in Q towards a measurable function u. To this aim, let $\mathcal{T}_k(s)$ be a $C^2(\mathbf{R})$, nondecreasing function such that $\mathcal{T}_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\mathcal{T}_k(s) = \operatorname{sgn}(s)k$ for $|s| > k$. If we multiply pointwise equation (10.64) by $\mathcal{T}_k'(u_n - g_2^n)$ (equivalently if we choose $\mathcal{T}_k'(u_n - g_2^n)\psi$ as test function in (10.65) with $\psi \in \mathcal{C}_c^\infty(Q)$) we get that:

$$\begin{aligned}
&(\mathcal{T}_k(u_n - g_2^n))_t - \operatorname{div}(a(t,x,\nabla u_n)\mathcal{T}_k'(u_n - g_2^n))\\
&\quad + a(t,x,\nabla u_n)\nabla(u_n - g_2^n)\mathcal{T}_k''(u_n - g_2^n)\\
&= \mathcal{T}_k'(u_n - g_2^n)f_n - \operatorname{div}(G_1^n\mathcal{T}_k'(u_n - g_2^n)) + \mathcal{T}_k''(u_n - g_2^n)G_1^n\nabla(u_n - g_2^n)\,.
\end{aligned} \tag{10.70}$$

Observe that thanks to the fact that $\mathcal{T}_k'$ has compact support and since $|\nabla u_n|^p\chi_{\{|u_n-g_2^n|\le k\}}$ is bounded in $L^1(Q)$ we deduce from (10.49) that $a(t,x,\nabla u_n)\nabla(u_n - g_2^n)\mathcal{T}_k''(u_n - g_2^n)$ is bounded in $L^1(Q)$ and so is $\mathcal{T}_k''(u_n-g_2^n)G_1^n\nabla(u_n-g_2^n)$ (since G_1^n is bounded in $(L^{p'}(Q))^N$). Similarly, we have that $a(t,x,\nabla u_n)\mathcal{T}_k'(u_n-g_2^n)$ as well as $G_1^n\mathcal{T}_k'(u_n-g_2^n)$ is bounded in $(L^{p'}(Q))^N$, so that we conclude from (10.70) that $(\mathcal{T}_k(u_n-g_2^n))_t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. Since we have just proven that $\mathcal{T}_k(u_n - g_2^n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ a classical compactness result (see [69]) allows us to deduce that $\mathcal{T}_k(u_n-g_2^n)$ is compact in $L^1(Q)$. Thus, for a subsequence, it also converges in measure. Let then $\sigma > 0$ and, given $\varepsilon > 0$, let us fix h such that, for every n, $\text{meas}\{|u_n - g_2^n| > \frac{h}{2}\} \leq \varepsilon$ (this is possible as a consequence of the estimate of $u_n - g_2^n$ in $L^\infty(0,T;L^1(\Omega))$). Since $\mathcal{T}_h(u_n - g_2^n)$ converges in measure, for n and m sufficiently large we have:

$$\text{meas}\{|\mathcal{T}_h(u_n - g_2^n) - \mathcal{T}_h(u_m - g_2^m)| > \sigma\} \leq \varepsilon\,.$$

On the other hand we have, by definition of $\mathcal{T}_k$:

$$\text{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\} \leq \text{meas}\{|u_n - g_2^n| > \frac{h}{2}\}$$
$$+ \text{meas}\{|u_m - g_2^m| > \frac{h}{2}\} + \text{meas}\{|\mathcal{T}_h(u_n - g_2^n) - \mathcal{T}_h(u_m - g_2^m)| > \sigma\}\,,$$

hence the choice of h implies, for n and m sufficiently large,

$$\text{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\} \leq 3\varepsilon\,,$$

so that $u_n - g_2^n$ is a Cauchy sequence in measure. Up to subsequences, we deduce that $u_n - g_2^n$ almost everywhere converges in Q, and since g_2^n strongly converges to g_2 in $L^p(0,T;W_0^{1,p}(\Omega))$, there exists a measurable function u such that u_n almost everywhere converges to u and $T_k(u_n - g_2^n)$ weakly converges to $T_k(u - g_2)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. The estimates previously obtained on u_n also imply that $u \in L^\infty(0,T;L^1(\Omega))$ (indeed, use Fatou's lemma on the estimate of (u_n) in $L^\infty(0,T;L^1(\Omega))$) and that $T_k(u_n)$ weakly converges to $T_k(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$.
Let us prove (10.68). Let $\psi(s) = T_k(s - T_h(s))$; one has

$$\int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt = \int_{\{h\leq|u_n - g_2^n|\leq h+k\}} |\nabla(u_n - g_2^n)|^p\,dxdt \leq \int_Q |\nabla T_{h+k}(u_n - g_2^n)|^p\,dxdt \leq C,$$

hence $\psi(u_n - g_2^n)$ converges (up to subsequences) weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and almost everywhere in Q to $\psi(u - g_2)$. Thus

$$\int_{\{h\leq|u-g_2|\leq h+k\}} |\nabla(u - g_2)|^p\,dxdt = \int_Q |\nabla\psi(u - g_2)|^p\,dxdt \leq \liminf_{n\to\infty} \int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt$$

Moreover

$$\int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt \leq C \int_{\{h\leq|u_n - g_2^n|\leq h+k\}} (|\nabla u_n|^p + |\nabla g_2^n|^p)\,dxdt$$

Hence, using (10.69), one gets

$$\lim_{h\to\infty} \int_{\{h\leq|u-g_2|\leq h+k\}} |\nabla(u - g_2)|^p\,dxdt = 0$$

as h tends to ∞, and (10.68) follows. ■

Remark 10.14 *In the proof of Proposition 10.7 we did not use the fact that the approximating sequence μ_n converging to μ is bounded in $L^1(Q)$ but for the first two estimates on u_n. The estimates concerning $u_n - g_2^n$ in (10.66) as well as (10.67) and (10.68) only needed the "separate" approximations of f, g_1, g_2 in the respective functional spaces. In particular, they hold true if μ belongs to $L^1(Q) + W'$, being (f, g_1, g_2) a decomposition of μ.*

Next we prove the strong convergence of $T_k(u_n - g_2^n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. To obtain this result, we use the same technique as in [63] adapted to the sequence $u_n - g_2^n$.
We need then to recall the following definition of a time–regularization of $T_k(u)$, which was first introduced in [50], then used in several papers afterwards (see particularly [23], [9]). Let z_ν be a sequence of functions such that:

$$z_\nu \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \qquad \|z_\nu\|_{L^\infty(\Omega)} \le k\,,$$
$$z_\nu \to T_k(u_0) \quad \text{a.e. in } \Omega \text{ as } \nu \text{ tends to infinity,}$$
$$\frac{1}{\nu}\|z_\nu\|^p_{W_0^{1,p}(\Omega)} \to 0 \qquad \text{as } \nu \text{ tends to infinity.}$$

Then, for fixed $k > 0$, and $\nu > 0$, we denote by $T_k(u)_\nu$ the unique solution of the problem

$$\begin{cases} \frac{\partial T_k(u)_\nu}{\partial t} = \nu(T_k(u) - T_k(u)_\nu) & \text{in the sense of distributions,}\\ T_k(u)_\nu(0) = z_\nu & \text{in } \Omega. \end{cases} \tag{10.71}$$

Then $T_k(u)_\nu$ belongs to $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ and $\frac{\partial T_k(u)_\nu}{\partial t}$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, and it can be proved (see also [50]) that, up to a subsequence,

$$\begin{aligned} &T_k(u)_\nu \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q,\\ &\|T_k(u)_\nu\|_{L^\infty(Q)} \le k \qquad \forall \nu > 0\,. \end{aligned} \tag{10.72}$$

Proposition 10.8 *Let u_n be the solution of (10.64), where μ_n is given by Proposition 10.5. Then there exists a measurable function u in Q and a subsequence, not relabeled, such that:*

$$T_k(u_n - g_2^n) \to T_k(u - g_2) \qquad \textit{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \textit{ for any } k > 0.$$

Proof.
We take a subsequence such that $u_n \to u$ almost everywhere in Q. Let us denote, throughout what follows, $v_n = u_n - g_2^n$, and $v = u - g_2$. By Proposition 10.7 we know that $v \in L^\infty(0,T;L^1(\Omega))$ (in particular it is almost everywhere finite), $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$ and

$$T_k(v_n) \to T_k(v) \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q \text{ for any } k > 0. \tag{10.73}$$

We take a subsequence of $T_k(v)_\nu$, the approximation of $T_k(v)$ defined in (10.71), such that $T_k(v)_\nu \to T_k(v)$ almost everywhere in Q (this subsequence only depends on v and k, i.e. quantities that will not vary in the following proof). For $h > 2k$, we then introduce the function

$$w_n = T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\,.$$

The use of w_n as test function to prove the strong convergence of truncations was first introduced in the stationary case in [52], then adapted to parabolic equations in [63]. The advantage in working with w_n is that, since

$$\nabla w_n = \nabla(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\chi_{E_n}\,,$$

with $E_n = \{|v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu| \le 2k\}$, in particular we have $\nabla w_n = 0$ if $|v_n| > h + 4k$. Thus the estimate on $T_k(v_n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ appearing in Proposition 10.7 implies that w_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, then by the almost everywhere convergence of v_n to v we deduce:

$$w_n \to T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q. \tag{10.74}$$

In the following we set $M = h+4k$, moreover we will denote by $\omega(n,\nu,h)$ all quantities (possibly different) such that

$$\lim_{h\to+\infty}\ \lim_{\nu\to+\infty}\ \limsup_{n\to+\infty} |\omega(n,\nu,h)| = 0\,, \tag{10.75}$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then ν, and finally h. Similarly we will write only $\omega(n)$, or $\omega(n,\nu)$, to mean that the limits are made only on the specified parameters. Choosing w_n as test function in (10.65) we have:

$$\int_0^T \langle (v_n)_t, w_n\rangle \, dt + \int_Q a(t,x,\nabla u_n)\nabla w_n \, dxdt = \int_Q f_n \, w_n \, dxdt + \int_0^T \langle g_1^n, w_n\rangle \, dt \,. \tag{10.76}$$

Then from (10.74) we obtain:

$$\lim_{n\to\infty} \int_Q f_n \, w_n \, dxdt = \int_Q f \, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \, dxdt \,,$$
$$\lim_{n\to\infty} \int_0^T \langle g_1^n, w_n\rangle \, dt = \int_0^T \langle g_1, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\rangle \, dt \,.$$

Moreover we have that $T_k(v)_\nu$ converges to $T_k(v)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ and almost everywhere in Q as ν tends to infinity, so that

$$\lim_{\nu\to\infty} \int_Q f \, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \, dxdt = \int_Q f \, T_{2k}(v - T_h(v)) \, dxdt \,,$$
$$\lim_{\nu\to\infty} \int_0^T \langle g_1, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\rangle \, dt = \int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle \, dt \,.$$

By means of Lebesgue's theorem we can conclude

$$\lim_{h\to\infty} \int_Q f \, T_{2k}(v - T_h(v)) \, dxdt = 0 \,.$$

Moreover, since

$$\int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle \, dt = \int_Q G_1 \nabla v \, \chi_{\{h\le |v|\le h+2k\}} \, dxdt \,,$$

Hölder's inequality implies

$$\left| \int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle \, dt \right| \le \|G_1\|_{(L^{p'}(Q))^N} \left(\int\limits_{\{h\le |u-g_2|\le h+2k\}} |\nabla (u - g_2)|^p \, dxdt \right)^{\frac{1}{p}} .$$

Then thanks to (10.68) we obtain:

$$\lim_{h\to\infty} \int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle \, dt = 0 \,.$$

Thus, recalling the notation introduced in (10.75), we have proven that

$$\int_Q f_n \, w_n \, dxdt + \int_0^T \langle g_1^n, w_n\rangle \, dt = \omega(n,\nu,h) \,. \tag{10.77}$$

Let us estimate the second term in (10.76). Since $\nabla w_n = 0$ if $|v_n| > M = h + 4k$ we have:

$$\int_Q a(t,x,\nabla u_n)\nabla w_n \, dxdt = \int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le M\}})\nabla w_n \,.$$

Next we split the integral in the sets $\{|v_n| \le k\}$ and $\{|v_n| > k\}$ so that we have, recalling that $E_n = \{|v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu| \le 2k\}$ and $h \ge 2k$:

$$\begin{aligned}\int_Q a(t,x,\nabla u_n)\nabla w_n \,dxdt &= \int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v)_\nu)\,dxdt \\ &+ \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt \\ &- \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)_\nu\,\chi_{E_n}\,dxdt\,.\end{aligned} \tag{10.78}$$

Let us denote (A), (B) and (C) the three terms of the right hand side in (10.78). Let us estimate (B). Clearly we have

$$\begin{aligned}&\left|\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt\right| \\ &\quad\le \int_{\{h\le|v_n|\le h+4k\}} |a(t,x,\nabla u_n)|\,|\nabla(u_n - g_2^n)|\,dxdt\,,\end{aligned}$$

and using (10.49) and Young's inequality we get:

$$\begin{aligned}&\int_{\{h\le|v_n|\le h+4k\}} |a(t,x,\nabla u_n)|\,|\nabla(u_n - g_2^n)|\,dxdt \\ &\le C\int_{\{h\le|v_n|\le h+4k\}} |\nabla u_n|^p\,dxdt + C\int_{\{h\le|v_n|\le h+4k\}} |\nabla g_2^n|^p\,dxdt + C\int_{\{h\le|v_n|\le h+4k\}} |b(x,t)|^{p'}\,dxdt\,.\end{aligned}$$

Thanks to the equi–integrability of $|\nabla g_2^n|^p$, using (10.66) and that meas$\{h \le |v_n| \le h+k\}$ converges to zero as h tends to infinity uniformly with respect to n we obtain:

$$\lim_{h\to\infty}\limsup_{n\to\infty}\left|\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt\right| = 0\,,$$

that is $(B) = \omega(n,h)$. For (C), let us remark that, since $\nabla u_n\chi_{\{|v_n|\le M\}}$ is bounded in $L^p(Q)$, (10.49) implies that $|a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})|$ is bounded in $L^{p'}(Q)$. The almost everywhere convergence of v_n to v implies that $|\nabla T_k(v)|\chi_{\{|v_n|>k\}}$ strongly converges to zero in $L^p(Q)$, so that we have

$$\lim_{n\to\infty}\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)\,\chi_{E_n}\,dxdt = 0\,.$$

Thus we get

$$\begin{aligned}&\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)_\nu\,\chi_{E_n}\,dxdt \\ &= \omega(n) + \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(T_k(v)_\nu - T_k(v))\,\chi_{E_n}\,dxdt\,.\end{aligned}$$

Using that $|a(t,x,\nabla u_n \chi_{\{|v_n|\le M\}})|$ is bounded in $L^{p'}(Q)$, applying Hölder's inequality and thanks to (10.72) we also have

$$\int_{\{|v_n|>k\}} a(t,x,\nabla u_n \chi_{\{|v_n|\le M\}}) \nabla (T_k(v)_\nu - T_k(v))\, \chi_{E_n}\, dxdt = \omega(n,\nu)\,,$$

therefore we conclude:

$$(C) = \int_{\{|v_n|>k\}} a(t,x,\nabla u_n \chi_{\{|v_n|\le M\}}) \nabla T_k(v)_\nu\, \chi_{E_n}\, dxdt = \omega(n,\nu)\,.$$

We have then obtained from (10.78), using that (B) and (C) converge to 0:

$$\int_Q a(t,x,\nabla u_n) \nabla w_n\, dxdt = \int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le k\}}) \nabla (v_n - T_k(v)_\nu)\, dxdt + \omega(n,\nu,h)\,. \tag{10.79}$$

Putting together (10.77), (10.79) and (10.76) we have:

$$\int_0^T \langle (v_n)_t, w_n\rangle\, dt + \int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le k\}}) \nabla (v_n - T_k(v)_\nu)\, dxdt = \omega(n,\nu,h)\,.$$

As far as the first term is concerned, that is

$$\int_0^T \langle (v_n)_t, T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\rangle\, dt\,,$$

we can apply Lemma 2.1 in [63] to the function v_n, using the fact that u_{0n} and z_ν strongly converge to u_0 and to $T_k(u_0)$ respectively in $L^1(\Omega)$. This lemma, based on the monotonicity properties of the time–regularization $T_k(v)_\nu$, gives that

$$\int_0^T \langle (v_n)_t, w_n\rangle\, dt \ge \omega(n,\nu,h)\,,$$

hence we finally have:

$$\int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le k\}}) \nabla (v_n - T_k(v)_\nu)\, dxdt \le \omega(n,\nu,h)\,. \tag{10.80}$$

Without loss of generality, we can assume that k is such that $\chi_{\{|v_n|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ (in fact this is true for almost every k, see also Lemma 3.2 in [9]). Then, the strong convergence of g_2^n in $L^p(0,T;W_0^{1,p}(\Omega))$ and (10.49) imply that $a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\le k\}})$ strongly converges to $a(t,x,\nabla(g_2 + T_k(v))\chi_{\{|v|\le k\}})$ in $L^{p'}(Q)^N$. Since

$$\begin{aligned}\int_Q a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\le k\}}) \nabla (v_n - T_k(v))\\ = \int_Q a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\le k\}}) \nabla (T_k(v_n) - T_k(v))\, dxdt\,,\end{aligned}$$

the weak convergence of $T_k(v_n)$ to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ allows to conclude that:

$$\lim_{n\to\infty} \int_Q a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\le k\}}) \nabla (v_n - T_k(v))\, dxdt = 0\,,$$

hence we obtain from (10.80), using also the strong convergence of $T_k(v)_\nu$ to $T_k(v)$ as ν tends to infinity:

$$\lim_{n\to\infty}\int_Q \big[a(t,x,\nabla u_n\chi_{\{|v_n|\le k\}}) \\ -a(t,x,\nabla(g_2^n+T_k(v))\chi_{\{|v_n|\le k\}})\big]\,(\nabla u_n-\nabla(g_2^n+T_k(v)))\,dxdt=0\,. \tag{10.81}$$

Using that $\chi_{\{|v_n|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ and that g_2^n strongly converges to g_2 in $L^p(0,T;W_0^{1,p}(\Omega))$, through the standard monotonicity argument which relies on (10.50) (see Lemma 5 in [13]) we can deduce from (10.81) that

$$\nabla u_n\chi_{\{|v_n|\le k\}}\to\nabla(g_2+T_k(v))\chi_{\{|v|\le k\}}=\nabla u\chi_{\{|v|\le k\}}\qquad\text{a.e. in } Q,$$

and then that $a(t,x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla u_n$ strongly converges to $a(t,x,\nabla u\chi_{\{|v|\le k\}})\nabla u$ in $L^1(Q)$. Finally, together with (10.48) this proves that the sequence $|\nabla u_n|^p\chi_{\{|u_n-g_2^n|\le k\}}$ is equi–integrable in Q, which as a consequence of Vitali's theorem and since g_2^n strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ yields

$$T_k(u_n-g_2^n)\to T_k(u-g_2)\qquad\text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

In fact, since we have proved it for almost every k the result holds true for any k as well. ∎

The proof of the existence of a renormalized solution will easily follow from the previous estimates and compactness results.

Theorem 10.11 *Assume that (10.48), (10.49), (10.50) hold true, and let $\mu\in\mathcal{M}_0(Q)$, $u_0\in L^1(\Omega)$. Then there exists a renormalized solution u of problem (10.1) in the sense of Definition 10.7. Moreover u belongs to $L^\infty(0,T;L^1(\Omega))$ and $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$.*

Remark 10.15 *We already remarked that the definition of renormalized solution does not make use of the fact that μ is a measure (only its decomposition in $L^1(Q)+W'$ is needed), in particular all the regularity asked on renormalized solutions concerns the difference $u-g_2$. However, due to the fact that μ is a measure (and can be approximated by sequences bounded in $L^1(Q)$) we have found a solution u with the additional regularity properties $u\in L^\infty(0,T;L^1(\Omega))$ and $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Last one in particular says that $|\nabla u|^p\chi_{\{|u|\le k\}}\in L^1(Q)$, which is not at all contained in the request $|\nabla u|^p\chi_{\{|u-g_2|\le k\}}\in L^1(Q)$ for renormalized solutions. Actually, this regularity result is consistent with the first existence result found in [10].*

Proof. Let u_n be the sequence of solutions of (10.64), where μ_n and u_{0n} approximate μ and u_0 respectively in the sense specified above, and let $u\in L^\infty(0,T;L^1(\Omega))$ be such that the results of Proposition 10.7 and Proposition 10.8 hold true. Then we have that

$$\begin{aligned}&u_n\to u\qquad\text{a.e. in } Q,\\ &T_k(u_n-g_2^n)\to T_k(u-g_2)\qquad\text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ for any } k>0 \text{ and a.e. in } Q.\end{aligned}\tag{10.82}$$

Let $S\in W^{2,\infty}(\mathbf{R})$ be such that S' has compact support, and take $S'(u_n-g_2^n)\varphi$ as test function in (10.65), with $\varphi\in\mathcal{C}_c^\infty(Q)$. Then we have:

$$\begin{aligned}&-\int_Q\varphi_t\,S(u_n-g_2^n)\,dxdt+\int_Q a(t,x,\nabla u_n)\nabla\varphi\,S'(u_n-g_2^n)\,dxdt\\ &+\int_Q S''(u_n-g_2^n)a(t,x,\nabla u_n)\nabla(u_n-g_2^n)\,\varphi\,dxdt=\int_Q f_nS'(u_n-g_2^n)\varphi\,dxdt\\ &+\int_Q G_1^n\nabla\varphi\,S'(u_n-g_2^n)\,dxdt+\int_Q S''(u_n-g_2^n)G_1^n\,\nabla(u_n-g_2^n)\,\varphi\,dxdt\,.\end{aligned}\tag{10.83}$$

Since Supp(S') is compact there exists $M > 0$ such that $a(t,x,\nabla u_n)S'(u_n - g_2^n) = a(t,x,\nabla T_M(u_n - g_2^n) + \nabla g_2^n)S'(u_n - g_2^n)$, so that (10.82), the strong convergence of g_2^n in $L^p(0,T;W_0^{1,p}(\Omega))$ and assumption (10.49) imply that

$$a(t,x,\nabla u_n)S'(u_n - g_2^n) \to a(t,x,\nabla u)S'(u - g_2) \qquad \text{strongly in } (L^{p'}(Q))^N.$$

Similarly we have that

$$S''(u_n - g_2^n)a(t,x,\nabla u_n)\nabla(u_n - g_2^n) \to S''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2) \qquad \text{strongly in } L^1(Q)$$

and

$$S''(u_n - g_2^n)\nabla(u_n - g_2^n) \to S''(u - g_2)\nabla(u - g_2) \qquad \text{strongly in } (L^p(Q))^N.$$

Therefore, by means of (10.82) and the dominated convergence theorem, we can pass to the limit in (10.83) as n tends to infinity obtaining:

$$\begin{aligned}
&-\int_Q \varphi_t S(u - g_2)\,dxdt + \int_Q a(t,x,\nabla u)\nabla\varphi S'(u - g_2)\,dxdt \\
&\quad + \int_Q S''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2)\,\varphi\,dxdt = \int_Q f S'(u - g_2)\varphi\,dxdt \\
&\quad + \int_Q G_1\nabla\varphi\, S'(u - g_2)\,dxdt + \int_Q S''(u - g_2)G_1\,\nabla(u - g_2)\,\varphi\,dxdt\,.
\end{aligned} \tag{10.84}$$

Thus u satisfies (10.56), while (10.55) is (10.68) with $k = 1$ and has been proved in Proposition 10.7. Finally, passing to the limit (thanks to (10.82)) in (10.83) written in distributional sense we have

$$(S(u_n - g_2^n))_t \quad \text{is strongly convergent in } L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q),$$

and since $S(u_n - g_2^n)$ strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ we deduce (see Theorem 1.1 in [63]) that

$$S(u_n - g_2^n) \to S(u - g_2) \qquad \text{strongly in } \mathcal{C}([0,T];L^1(\Omega)).$$

In particular, being $S(u_n - g_2^n)(0) = S(u_{0n})$ we get that $S(u - g_2)(0) = S(u_0)$ in $L^1(\Omega)$. This concludes the proof that u is a renormalized solution of (10.1).

■

Here we prove the uniqueness of the renormalized solution of (10.1)

Theorem 10.12 *Assume (10.48), (10.49), (10.50). Let $\mu \in \mathcal{M}_0(Q)$, then there exists a unique renormalized solution of (10.1).*

Proof. Let u_1, u_2 be two renormalized solutions of (10.1), let (f, g_1, g_2) be a decomposition of μ, so that u_1 and u_2 both satisfy (10.56). Note that the same decomposition of μ can be used for both equations of u_1 and u_2 thanks to Proposition 10.6. Let S_n be as defined in Definition 10.8, in particular we have that $S_n(u_1 - g_2)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ as well as $S_n(u_2 - g_2)$. We choose then $T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))$ as test function in both the equations solved by u_1 and u_2. In the following we write $v_1 = u_1 - g_2$

and $v_2 = u_2 - g_2$; subtracting the equations then we have:

$$\begin{aligned}
&\int_0^T \langle (S_n(v_1) - S_n(v_2))_t, T_k(S_n(v_1) - S_n(v_2))\rangle \, dt \\
&\quad + \int_Q [S_n'(v_1)a(t,x,\nabla u_1) - S_n'(v_2)a(t,x,\nabla u_2)] \cdot \nabla T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&= \int_Q f \, (S_n'(v_1) - S_n'(v_2)) \, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q G_1 \, (S_n'(v_1) - S_n'(v_2)) \, \nabla T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q [S_n''(v_1) G_1 \nabla v_1 - S_n''(v_2) G_1 \nabla v_2] \, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q [S_n''(v_2)a(t,x,\nabla u_2)\nabla v_2 - S_n''(v_1)a(t,x,\nabla u_1)\nabla v_1] \, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \, .
\end{aligned} \tag{10.85}$$

Let us denote by (A)–(F) the six integrals above, we study the behaviour of each as n tends to infinity. To this purpose, let us recall that by definition of S_n we have that $S_n'(s)$ converges to 1 for every s in $\mathbf{R}$. This is enough to conclude by means of Lebesgue's theorem that

$$\lim_{n\to\infty} (C) = 0 \, .$$

Let us study the limit of (E) now. We have $(E) = (E_1) + (E_2)$, where

$$(E_1) = \int_Q S_n''(v_1) G_1 \nabla v_1 \, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \, .$$

Since (E_2) has the same form of (E_1) with the roles of v_1 and v_2 interchanged, it is enough to deal with (E_1). Recalling that $S_n''(s) = -\text{sign}(s)\chi_{\{n\le|s|\le n+1\}}$, we have:

$$|(E_1)| \le k \int_{\{n\le|v_1|\le n+1\}} |G_1| \, |\nabla v_1| \, dxdt \, ,$$

so that, using Hölder's inequality we get:

$$|(E_1)| \le k \|G_1\|_{L^{p'}(Q)} \left(\int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla u_1 - \nabla g_2|^p \, dxdt \right)^{\frac{1}{p}} .$$

Thus by (10.55) written for u_1 we get that (E_1) converges to zero as n tends to infinity. The same is true for (E_2), hence we deduce:

$$\lim_{n\to\infty} (E) = 0 \, .$$

The term (F) can be dealt with in the same way. First we write $(F) = (F_1) + (F_2)$, with

$$(F_1) = \int_Q S_n''(v_2) a(t,x,\nabla u_2) \nabla v_2 \, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \, .$$

Clearly, by symmetry between (F_1) and (F_2) it is enough to prove that (F_1) tends to zero. To this goal, using again the properties of S_n'' and (10.49) we have:

$$|(F_1)| \le \beta \, k \int_{\{n\le|v_2|\le n+1\}} |\nabla v_2| \, (|b(x,t)| + |\nabla u_2|^{p-1}) \, dxdt \, ,$$

which yields, by Young's inequality:

$$|(F_1)| \leq C \left(\int\limits_{\{n \leq |u_2 - g_2| \leq n+1\}} (|\nabla g_2|^p + |b(x,t)|^{p'})\, dxdt + \int\limits_{\{n \leq |u_2 - g_2| \leq n+1\}} |\nabla u_2|^p \, dxdt \right).$$

Using that $u_2 - g_2$ is almost everywhere finite and thanks to (10.55) written for u_2 we conclude that (F_1) converges to zero, and (F_2) as well, so that

$$\lim_{n\to\infty} (F) = 0 .$$

As regards (D) note that, since $S_n'(v_1) - S_n'(v_2) = 0$ in $\{|v_1| \leq n \,, |v_2| \leq n\} \cup \{|v_1| > n+1 \,, |v_2| > n+1\}$ we can split the integral as follows:

$$\begin{aligned} &\int\limits_{\{|S_n(v_1) - S_n(v_2)| \leq k\}} G_1 \left(S_n'(v_1) - S_n'(v_2)\right) \nabla (S_n(v_1) - S_n(v_2)) \chi_{\{|v_1| \leq n\}} \, \chi_{\{|v_2| > n\}} \, dxdt \\ &+ \int\limits_{\{|S_n(v_1) - S_n(v_2)| \leq k\}} G_1 \left(S_n'(v_1) - S_n'(v_2)\right) \nabla (S_n(v_1) - S_n(v_2)) \chi_{\{n < |v_1| \leq n+1\}} \, dxdt \\ &+ \int\limits_{\{|S_n(v_1) - S_n(v_2)| \leq k\}} G_1 \left(S_n'(v_1) - S_n'(v_2)\right) \nabla (S_n(v_1) - S_n(v_2)) \chi_{\{|v_2| \leq n+1\}} \, \chi_{\{|v_1| > n+1\}} \, dxdt . \end{aligned} \tag{10.86}$$

We call (D_1)–(D_3) the three integrals in (10.86). Using the properties of S_n and S_n' (recall that $S_n(t) = t$ if $|t| \leq n$, that S_n is nondecreasing and $\mathrm{Supp}(S_n') \subset [-n-1, n+1]$) we have:

$$|(D_1)| \leq \int\limits_{\{n-k \leq |u_1 - g_2| \leq n\}} |G_1| \, |\nabla (u_1 - g_2)| \, dxdt + \int\limits_{\{n \leq |u_2 - g_2| \leq n+1\}} |G_1| \, |\nabla (u_2 - g_2)| \, dxdt .$$

Applying Hölder's inequality and using property (10.55) for renormalized solutions we easily get that (D_1) converges to zero as n tends to infinity. Similarly, since $|S_n(t)| > n-k$ implies $|t| > n-k$ we have:

$$|(D_2)| \leq \int\limits_{\{n \leq |u_1 - g_2| \leq n+1\}} |G_1| \, |\nabla (u_1 - g_2)| \, dxdt + \int\limits_{\{n-k \leq |u_2 - g_2| \leq n+1\}} |G_1| \, |\nabla (u_2 - g_2)| \, dxdt .$$

Again, Hölder 's inequality together with (10.55) allow to deduce that (D_2) converges to zero as well. The term (D_3) is dealt with in the same way (using that $S_n'(t) = 0$ if $|t| > n+1$), so that we finally get that

$$\lim_{n\to\infty} (D) = 0 .$$

We deal with (B) splitting it as below:

$$\begin{aligned} (B) = &\int\limits_{\{|v_1| \leq n, |v_2| \leq n\}} [a(t,x,\nabla u_1) - a(t,x,\nabla u_2)] \cdot \nabla T_k(u_1 - u_2) \, dxdt \\ &+ \int\limits_{\left\{ \substack{|S_n(v_1) - S_n(v_2)| \leq k \\ |v_1| \leq n \,,\, |v_2| > n} \right\}} [S_n'(v_1) a(t,x,\nabla u_1) - S_n'(v_2) a(t,x,\nabla u_2)] \cdot \nabla (S_n(v_1) - S_n(v_2)) \, dxdt \\ &+ \int\limits_{\left\{ \substack{|S_n(v_1) - S_n(v_2)| \leq k \\ |v_1| > n} \right\}} [S_n'(v_1) a(t,x,\nabla u_1) - S_n'(v_2) a(t,x,\nabla u_2)] \cdot \nabla (S_n(v_1) - S_n(v_2)) \, dxdt . \end{aligned}$$

Let us set (B_1)–(B_3) the three integrals above. Since $\{|S_n(v_1)-S_n(v_2)|\leq k\,,|v_1|>n\}\subset\{|v_1|>n\,,|v_2|>n-k\}$, we have, using that $S_n'(t)=0$ if $|t|>n+1$:

$$
\begin{aligned}
|(B_3)| \leq & \int_{\{n\leq|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_1)|\,|\nabla(u_1-g_2)|\,dxdt \\
& + \int_{\{n\leq|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_1)|\,|\nabla(u_2-g_2)|\chi_{\{n-k\leq|u_2-g_2|\leq n+1\}}\,dxdt \\
& + \int_{\{n\leq|u_1-g_2|\leq n+1\}} |a(t,x,\nabla u_2)|\,|\nabla(u_1-g_2)|\chi_{\{n-k\leq|u_2-g_2|\leq n+1\}}\,dxdt \\
& + \int_{\{n-k\leq|u_2-g_2|\leq n+1\}} |a(t,x,\nabla u_2)|\,|\nabla(u_2-g_2)|\,dxdt\,.
\end{aligned}
\tag{10.87}
$$

Using assumption (10.49), Young's inequality and the condition (10.55) for renormalized solutions, we can conclude as we did before that all the four terms in the right hand side of (10.87) converge to zero. Thus we get that (B_3) converges to zero. Changing the roles of u_1 and u_2, the same arguments prove that (B_2) also converges to zero as n tends to infinity. Thus we conclude, using Fatou's lemma in (B_1):

$$
\liminf_{n\to\infty}(B)\geq\int_Q(a(t,x,\nabla u_1)-a(t,x,\nabla u_2))\cdot\nabla T_k(u_1-u_2)\,dxdt\,.
$$

In the term (A) of (10.85) we can integrate using that $S_n(v_1)$ and $S_n(v_2)$ belong to $\mathcal{C}([0,T];L^1(\Omega))$ and $S_n(v_1)(0)=S_n(v_2)(0)=S_n(u_0)$. We then obtain:

$$
(A)=\int_\Omega\Theta_k(S_n(v_1)-S_n(v_2))(T)\,dx\,,
$$

where $\Theta_k(s)=\int_0^s T_k(t)\,dt$, and since Θ_k is nonnegative we conclude that $(A)\geq 0$. Putting together the results obtained on (A)–(F) we obtain from (10.85), as n tends to infinity:

$$
\int_{\{|u_1-u_2|\leq k\}}(a(t,x,\nabla u_1)-a(t,x,\nabla u_2))\cdot\nabla(u_1-u_2)\,dxdt\leq 0\,,
$$

and then, letting k tend to infinity:

$$
\int_Q(a(t,x,\nabla u_1)-a(t,x,\nabla u_2))\cdot\nabla(u_1-u_2)\,dxdt\leq 0\,.
$$

The strict monotonicity assumption (10.50) then implies that $u_1=u_2$ almost everywhere in Q. ■

Remark 10.16 *In fact, the proof of the uniqueness of renormalized solutions does not need the strict monotonicity assumption (10.50) but only that*

$$
(a(t,x,\xi)-a(t,x,\eta))\cdot(\xi-\eta)\geq 0\qquad\forall(\xi,\eta)\in\mathbb{R}^N\,.
$$

This can be seen performing the same proof as in Theorem 10.12 above in the interval $]0,t[$, *with* $t<T$. *Using that the term* (A) *is not only nonnegative as we already remarked but indeed*

$$
\liminf_{n\to\infty}(A)\geq\int_\Omega\Theta_k(u_1-u_2)(t)\,dx\,,
$$

we can obtain

$$
\int_\Omega\Theta_k(u_1-u_2)(t)\,dx\leq 0\qquad\forall t\in]0,T[\,,
$$

hence it follows that $u_1=u_2$.

10.3.4 Data in $L^1 + W'$.

It is possible to extend the result on existence and uniqueness of renormalized solutions to data which belong to $L^1 + W'$, without being necessarily measures. In fact, let $\mu \in L^1(Q) + W'$, then a renormalized solution of (10.1) is defined exactly as in Definition 10.7, where f, g_1, g_2 is a decomposition of μ in $L^1(Q) + W'$, moreover this definition does not depend on the decomposition of μ (see Remark 10.13). Then all the results proved in the previous section apply without any change except for the first two estimates of Proposition 10.7 for which we used the fact that μ was a bounded measure (see Remark 10.14). Thus, we obtain the following result.

Theorem 10.13 *Let $\mu \in L^1(Q) + W'$, and let $u_0 \in L^1(\Omega)$. Assume that hypotheses (10.48), (10.49), (10.50) hold true. Then there exists a unique renormalized solution of problem (10.1) in the sense of Definition 10.7.*

10.4 Appendix: Proof of the density theorem

10.4.1 The case of compactly supported functions

Lemma 10.7 *Let $u \in W$ have a compact support in Q and $(\rho_n)_{n\geq 1}$ be a smoothing kernel. Then, for n large enough (depending on the support of u), $u * \rho_n$ is well defined, is in $\mathcal{C}_c^\infty(Q)$ and $u * \rho_n \to u$ in W as $n \to \infty$.*

Proof. The fact that $u * \rho_n$ is well defined and is in $\mathcal{C}_c^\infty(Q)$ for n large enough is a classical convolution result. It is still classical, since $u \in L^p(Q) \cap L^p(0,T;L^2(\Omega))$, that $u * \rho_n \to u$ in $L^p(Q) \cap L^p(0,T;L^2(\Omega))$. Moreover, in the sense of distributions, $\nabla(u * \rho_n) = \nabla u * \rho_n$ so that, since $\nabla u \in (L^p(Q))^N$, one has $\nabla(u * \rho_n) \to \nabla u$ in $(L^p(Q))^N$.
Thus, $u * \rho_n \to u$ in $L^p(0,T;V)$ and it remains to prove the convergence of the time derivative.
To see this, we take $v_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $v_2 \in L^{p'}(0,T;L^2(\Omega))$ such that $u_t = v_1 + v_2$. We have $u = \theta u$ for some $\theta \in \mathcal{C}_c^\infty(Q)$ so that $u_t = \theta_t u + \theta u_t = \theta v_1 + (\theta v_2 + \theta_t u)$ with $\theta v_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\theta v_2 + \theta_t u \in L^{p'}(0,T;L^2(\Omega))$ (because, u being in W, it is also in $\mathcal{C}([0,T];L^2(\Omega))$); moreover, θv_1 and $\theta v_2 + \theta_t u$ have compact supports in Q. Denote $w_1 = \theta v_1$ and $w_2 = \theta v_2 + \theta_t u$.
We have then, in the sense of distributions, $(u * \rho_n)_t = u_t * \rho_n = w_1 * \rho_n + w_2 * \rho_n$ for n large enough. Since $w_2 \in L^{p'}(0,T;L^2(\Omega))$, we have $w_2 * \rho_n \to w_2$ in $L^{p'}(0,T;L^2(\Omega))$. For the convergence of $w_1 * \rho_n$, write $v_1 = \mathrm{div}(V_1)$ for some $V_1 \in (L^{p'}(Q))^N$; we have $w_1 = \mathrm{div}(\theta V_1) - V_1 \cdot \nabla\theta$ with $\theta V_1 \in (L^{p'}(Q))$ and $V_1 \cdot \nabla\theta \in L^{p'}(Q)$ having compact supports in Q, so that $w_1 * \rho_n = \mathrm{div}((\theta V_1) * \rho_n) - (V_1 \cdot \nabla\theta) * \rho_n$; since $(\theta V_1) * \rho_n \to \theta V_1$ in $(L^{p'}(Q))^N$ and $(V_1 \cdot \nabla\theta) * \rho_n \to V_1 \cdot \nabla\theta$ in $L^{p'}(Q)$, this gives the convergence of $w_1 * \rho_n$ to w_1 in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.
We have thus proven that $(u * \rho_n)_t \to w_1 + w_2 = u_t$ in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$, and this concludes the proof. ■

This technique of approximation is however limited to compactly supported elements of W; for general elements of W, we must find another kind of approximation by regular functions.

10.4.2 The general case

We prove the density of $\mathcal{C}_c^\infty([0,T] \times \Omega)$ in W, that is Theorem 10.4. To prove this density result, we will use two main tools: some results coming from the vector-valued integral and Sobolev space theory and the following theorem, which states a density result in spaces of functions on Ω. Let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$.

Theorem 10.14 *If Ω is a bounded open subset of $\mathbb{R}^N$ and $1 < p < \infty$, then $\mathcal{C}_c^\infty(\Omega)$ is dense in V.*

Proof of Theorem 10.14.
Let $u \in V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$.
Let $S \in \mathcal{C}^\infty(\mathbb{R})$ such that $S(s) = s$ when $|s| \leq 1$ and $S'(s) = 0$ when $|s| \geq 2$. We define, for $n \geq 1$, $S_n(s) = nS(\frac{s}{n})$; notice that $S_n(s) \to s$ and $S_n'(s) = S'(\frac{s}{n}) \to 1$ when $n \to \infty$; moreover, $|S_n(s)| \leq \|S_n'\|_{L^\infty(\mathbb{R})}|s|$ and $\|S_n'\|_{L^\infty(\mathbb{R})} \leq \|S'\|_{L^\infty(\mathbb{R})}$.
$S_n(u) \to u$ on Ω and is dominated by $\|S'\|_{L^\infty(\mathbb{R})}|u| \in L^p(\Omega)\cap L^2(\Omega)$; the convergence thus also happens in $L^p(\Omega)\cap L^2(\Omega)$. Moreover, $\nabla(S_n(u)) = S_n'(u)\nabla u \to \nabla u$ on Ω and is dominated by $\|S'\|_{L^\infty(\mathbb{R})}|\nabla u| \in L^p(\Omega)$, which proves that $\nabla(S_n(u)) \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$. Thus, $S_n(u) \to u$ in V as $n \to \infty$.
Let $(\varphi_m)_{m\geq 1} \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi_m \to u$ in $W_0^{1,p}(\Omega)$ (by definition of $W_0^{1,p}(\Omega)$, such a sequence exists); we can suppose, up to a subsequence, that $\varphi_m \to u$ and $\nabla\varphi_m \to \nabla u$ a.e. on Ω. We have, for all $n \geq 1$, $S_n(\varphi_m) \in \mathcal{C}_c^\infty(\Omega)$ and $S_n(\varphi_m) \to S_n(u)$ a.e. on Ω when $m \to \infty$; since $(S_n(\varphi_m))_{m\geq 1}$ is bounded in $L^\infty(\Omega)$ (by $\|S_n\|_{L^\infty(\mathbb{R})}$) and Ω is of finite measure, this implies that $S_n(\varphi_m) \to S_n(u)$ in $L^q(\Omega)$ for all $q < \infty$, and in particular in $L^p(\Omega)$ and in $L^2(\Omega)$. We also have $\nabla(S_n(\varphi_m)) = S_n'(\varphi_m)\nabla\varphi_m \to S_n'(u)\nabla u = \nabla(S_n(u))$ a.e. on Ω and $|\nabla(S_n(\varphi_m))| \leq \|S'\|_{L^\infty(\mathbb{R})}|\nabla\varphi_m|$; this last inequality tells us that $(\nabla(S_n(\varphi_m)))_{m\geq 1}$ is equi-integrable in $(L^p(\Omega))^N$ (because $(\nabla\varphi_m)_{m\geq 1}$ is equi-integrable in this space, since it converges) and thus that $\nabla(S_n(\varphi_m)) \to \nabla(S_n(u))$ in $(L^p(\Omega))^N$ as $m \to \infty$.
We have proven that $S_n(\varphi_m) \to S_n(u)$ in V as $m \to \infty$. Take then $m_n \geq 1$ such that $\|S_n(\varphi_{m_n}) - S_n(u)\|_V \leq 1/n$; since $S_n(u) \to u$ in V, we deduce that $S_n(\varphi_{m_n}) \to u$ in V and this concludes the proof of this theorem. ■

The results coming from the vector-valued integral and Sobolev space theory we will use here are, for the most part, very intuitive when one recalls the same results for scalar-valued integral and Sobolev spaces. We will thus only give the ideas of the reasoning that lead to the use of Theorem 10.14, and refer the interested reader to [32].
One of these results, however, is a little bit tricky; it comes from the density of simple functions in $L^{p'}(0,T;B)$, but it is not easy to explain without going further into the theory (and, especially, without explaining the concept of μ-mesurability, which we do not want here). We will thus state it, without proof, in the following lemma.

Lemma 10.8 *Let B be a Banach space and D be a dense subset in B. If $1 \leq q < \infty$, then the set*

$$S(D) = \left\{ \sum_{i=1}^n d_i\varphi_i\,,\ n \geq 1\,,\ d_i \in D\,,\ \varphi_i \in \mathcal{C}^\infty([0,T];\mathbb{R}) \right\}$$

is dense in $L^q(0,T;B)$.

Remark 10.17 *In fact, the result of this lemma is still true if we take the functions φ_i in $\mathcal{C}_c^\infty(]0,T[;\mathbb{R})$ (see [32]).*

Let us now give the ideas that lead from Lemma 10.8 and Theorem 10.14 to Theorem 10.4.

Proof of Theorem 10.4. Let $u \in W$, that is to say $u \in L^p(0,T;V)$ such that $u_t \in L^{p'}(0,T;V')$. We want to find $(v_n)_{n\geq 1} \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $v_n \to u$ in $L^p(0,T;V)$ and $(v_n)_t \to u_t$ in $L^{p'}(0,T;V')$.
Step 1: define $\widetilde{u} :]-T,2T[\to V$ almost everywhere by:

$$\widetilde{u}(t) = \begin{cases} u(-t) & \text{if } t \in]-T,0[, \\ u(t) & \text{if } t \in]0,T[, \\ u(2T-t) & \text{if } t \in]T,2T[. \end{cases}$$

One has $\widetilde{u} \in L^p(-T,2T;V)$. Moreover, since we have made two even reflections, it is easy (as for the classical Sobolev spaces) to see that $\widetilde{u}_t \in L^{p'}(-T,2T;V')$ with

$$\widetilde{u}_t(t) = \begin{cases} -u_t(-t) & \text{if } t \in]-T,0[, \\ u_t(t) & \text{if } t \in]0,T[, \\ -u_t(2T-t) & \text{if } t \in]T,2T[. \end{cases}$$

Define $\overline{u} \in L^p(\mathbb{R};V)$ as the extension of $\widetilde{u}$ by 0 outside $]-T,2T[$ and take $(\rho_n)_{n\geq 1}$ a smoothing kernel on $\mathbb{R}$ such that $\mathrm{Supp}(\rho_n) \subset]-T,T[$. Let $\overline{u}_n = \overline{u} * \rho_n \in L^p(\mathbb{R};V)$ (the convolution product is defined exactly as for scalar-valued integral, and the same results as in the scalar-valued case hold in the vector-valued case). One has $\overline{u}_n \in \mathcal{C}^\infty(\mathbb{R};V) \subset \mathcal{C}^\infty(\mathbb{R};V')$ (since $V \hookrightarrow V'$) and $\overline{u}_n \to \overline{u}$ in $L^p(\mathbb{R};V)$; thus, $u_n = (\overline{u}_n)_{|]0,T[} \in \mathcal{C}^\infty([0,T];V) \subset \mathcal{C}^\infty([0,T];V')$ and $u_n \to u$ in $L^p(0,T;V)$. Moreover, since $\widetilde{u}_t \in L^p(-T,2T;V')$, one can verify that, by defining $v \in L^{p'}(\mathbb{R};V)$ as the extension of $\widetilde{u}_t$ by 0 outside $]-T,2T[$, we have $(\overline{u}_n)_t = v * \rho_n$ in $\mathcal{C}^\infty(\mathbb{R};V')$. Thus, $(u_n)_t = (v * \rho_n)_{|]0,T[} \to v_{|]0,T[} = u_t$ in $L^{p'}(0,T;V')$.
We thus have found $(u_n)_{n\geq 1} \in \mathcal{C}^\infty([0,T];V)$ such that $u_n \to u$ in $L^p(0,T;V)$ and $(u_n)_t \to u_t$ in $L^{p'}(0,T;V')$.

Step 2: to approximate u in W, we thus just need to approximate in W a given function $v \in \mathcal{C}^\infty([0,T];V)$. Let v be such a function, and let $D = \mathcal{C}_c^\infty(\Omega)$. According to Theorem 10.14, D is a dense subset of V. Since $v' \in \mathcal{C}^\infty([0,T];V) \subset L^{p'}(0,T;V)$, using Lemma 10.8, there exists $(w_n)_{n\geq 1} \in S(D)$ which converges to v' in $L^{p'}(0,T;V)$, and thus also in $L^{p'}(0,T;V')$. Moreover, in V, one has $v(t) = v(0) + \int_0^t v'(s)\,ds$. Define $W_n(t) = \int_0^t w_n(s)\,ds$; since $w_n \to v'$ in $L^{p'}(0,T;V)$, one has $W_n \to \int_0^\cdot v'(s)\,ds = v - v(0)$ in $L^\infty(0,T;V)$, and thus in $L^p(0,T;V)$. Taking $(d_n)_{n\geq 1} \in D$ which converges to $v(0) \in V$ in V, the functions $v_n = d_n + W_n$ converge to v in $L^p(0,T;V)$ and the derivatives of these functions, $v_n' = W_n' = w_n$, converges to v' in $L^{p'}(0,T;V')$.
By noticing that $v_n(t) = d_n + \int_0^t w_n(s)\,ds \in S(D)$, we have proven that v is approximable in W by a sequence of functions in $S(D)$. Since $S(D) \subset \mathcal{C}_c^\infty([0,T]\times\Omega)$, this concludes the proof. ∎