Partie IV

Autres Travaux

Chapitre 8

A Density Result in Sobolev Spaces

J. Droniou.

Abstract We prove, when $1 \leq p < \infty$ and Ω is a polygonal or regular open set in $\mathbb{R}^N$, the density in $W^{1,p}(\Omega)$ of a space of regular functions satisfying a Neumann condition on $\partial\Omega$. We also give some applications of this result and a generalization concerning mixed Dirichlet-Neumann boundary conditions.

8.1 Introduction

8.1.1 Definitions

N is an integer greater than or equal to 2. The usual Euclidean scalar product of two vectors (x, y) of $\mathbb{R}^N$ is denoted by $x \cdot y$; $|\cdot|$ is the induced norm and "dist" the associated distance. For $\delta > 0$, $B_N(\delta)$ is the Euclidean ball in $\mathbb{R}^N$ of center 0 and radius δ. When E is a measurable subset of $\mathbb{R}^N$, $|E|$ is the Lebesgue measure of E. For $x \in \mathbb{R}^N$, we denote $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.
If Ω is an open set of $\mathbb{R}^N$ and $p \in [1, \infty[$, $W^{1,p}(\Omega)$ is the usual Sobolev space, endowed with the norm $||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\,|\nabla u|\,||_{L^p(\Omega)}$.

Definition 8.1 *Let $k \in \mathbb{N}$ and U be an open set of $\mathbb{R}^l$ ($l \geq 1$) or of a submanifold of $\mathbb{R}^l$. A function $\varphi : U \to \mathbb{R}$ is $\mathcal{C}^{k,1}$-continuous on U if φ is k times continuously differentiable on U, if the k first derivatives of φ are bounded on U and if the k^{th} derivative of φ is Lipschitz continuous on U. A function is $\mathcal{C}^{\infty,1}$-continuous on U if it is $\mathcal{C}^{k,1}$-continuous on U for all $k \in \mathbb{N}$. When a function takes its values into $\mathbb{R}^m$ for a $m \geq 1$, it is $\mathcal{C}^{k,1}$-continuous if each of its component is $\mathcal{C}^{k,1}$-continuous.*

We denote by $\mathcal{C}^{k,1}(U)$ the set of $\mathcal{C}^{k,1}$-continuous functions on U and by $\mathcal{C}_c^{k,1}(U)$ the set of functions in $\mathcal{C}^{k,1}(U)$ which have a compact support in U. Notice that, for all $k \in \mathbb{N} \cup \{\infty\}$, if $\varphi \in (\mathcal{C}^{k,1}(U))^m$ and $f \in \mathcal{C}^{k,1}(\mathbb{R}^m)$, then $f \circ \varphi \in \mathcal{C}^{k,1}(U)$.

Definition 8.2 *Let Ω be an open bounded set of $\mathbb{R}^N$ ($N \geq 2$) and $k \in \mathbb{N} \cup \{\infty\}$.*

i) Ω has a $\mathcal{C}^{k,1}$-continuous boundary if, for all $a \in \partial\Omega$, there exists an orthonormal coordinate system $\mathcal{R}$ centered at a, an open set V of $\mathbb{R}^N$ containing a, such that $V = V' \times]-\alpha, \alpha[$ in $\mathcal{R}$, and a $\mathcal{C}^{k,1}$-continuous function $\eta : V' \to]-\alpha, \alpha[$ such that, in $\mathcal{R}$, $\partial\Omega \cap V = \{(y', \eta(y'))\,,\ y' \in V'\}$ and $\Omega \cap V = \{(y', y_N) \in V \mid y_N > \eta(y')\}$.

ii) Ω has a Lipschitz continuous boundary if it has a $\mathcal{C}^{0,1}$-continuous boundary.

iii) Ω is polygonal if it has a Lipschitz continuous boundary and if its boundary is included in a finite union of affine hyperplanes.

In the sequel, the open sets Ω we consider have at least a Lipschitz continuous boundary. We can then define a $(N-1)$-dimensional measure σ on $\partial\Omega$ and a unit normal $\mathbf{n} \in (L^\infty(\partial\Omega))^N$ to $\partial\Omega$ outward to Ω (when Ω has a $\mathcal{C}^{k,1}$-continuous boundary, $\mathbf{n} \in (\mathcal{C}^{k-1,1}(\partial\Omega))^N$).
For $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $a \in \partial\Omega$, we denote, when such a quantity exists,

$$\frac{\partial\varphi}{\partial\mathbf{n}}(a) = \lim_{t\to 0} \frac{\varphi(a+t\mathbf{n}(a)) - \varphi(a)}{t}.$$

If φ is $\mathcal{C}^1$-continuous, this limit exists and is equal to $\nabla\varphi(a)\cdot\mathbf{n}(a)$.

Definition 8.3 *Let $k \in \mathbb{N}\cup\{\infty\}$. We define $E^k(\Omega)$ as the space of the restrictions to Ω of functions $\varphi \in \mathcal{C}_c^{k,1}(\mathbb{R}^N)$ satisfying, for σ-a.e. $a \in \partial\Omega$, $\frac{\partial\varphi}{\partial\mathbf{n}}(a) = 0$.*

8.1.2 Main Results

Theorem 8.1 *Let $p \in [1,\infty[$. If $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary, then $E^{k-1}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Remark 8.1 *i) There is, in [55], an alternate proof of this result to the one we present here. However, this proof (which relies on the idea of transporting the problem with well-chosen diffeomorphisms) can only be applied to open sets with at least $\mathcal{C}^{2,1}$-continuous boundaries.*

ii) We will in fact prove a more general result than Theorem 8.1, not asking for Ω to have a $\mathcal{C}^{k,1}$-continuous boundary "everywhere" (see Theorem 8.3).

Theorem 8.2 *Let $p \in [1,\infty[$. If Ω is a polygonal open set of $\mathbb{R}^N$, then $E^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Remark 8.2 *i) We will see in Section 8.4 that there exists open sets Ω with a Lipschitz continuous boundary such that the space of the restrictions to Ω of functions in $\mathcal{C}^1(\mathbb{R}^N)$ satisfying a Neumann boundary condition on $\partial\Omega$ is not dense in $W^{1,p}(\Omega)$.*

ii) (Thierry Gallouët [40]) There is an alternate result to Theorem 8.1 which avoid the loss of a derivative (with respect to the regularity of the open set): if Ω has a $C^{1,1}$-continuous boundary or is polygonal convex, then for all $u \in H^1(\Omega)$, there exists $(u_n)_{n\geq 1} \in H^2(\Omega)$ satisfying, for all $n \geq 1$, $\nabla u_n \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$ and such that $u_n \to u$ in $H^1(\Omega)$.

The idea is to solve the following Neumann problem

$$\begin{cases} v_\varepsilon - \varepsilon\Delta v_\varepsilon = u & \text{in } \Omega, \\ \nabla v_\varepsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

Ω having a $\mathcal{C}^{1,1}$-continuous boundary or being polygonal convex, the variational solution to this problem is in $H^2(\Omega)$; by multiplying the equation by Δv_ε, we notice that $(v_\varepsilon)_{\varepsilon>0}$ is bounded in $H^1(\Omega)$ and that it converges weakly in this space to u; by Mazur's lemma, a convex combination of the $(v_\varepsilon)_{\varepsilon>0}$ converges strongly to u.

If this technique avoids the loss of a derivative (we get the density of H^2 functions when Ω has a $\mathcal{C}^{1,1}$-continuous boundary), in contrary to Theorem 8.1 (density of $\mathcal{C}^{0,1}$-continuous functions under the same hypothesis), the derivatives are however far less regular than in Theorem 8.1 (in L^2 instead of L^∞). Moreover, in the case of a polygonal open set, Theorem 8.2 gives a far better result than the method up above.

8.2 Theorem 8.1 and a generalization

As said before, we will prove a more general result than Theorem 8.1. To state this result, we need a generalization of Definition 8.2: when $\mathcal{K}$ is a compact subset of $\mathbb{R}^N$ and $k \in \mathbb{N} \cup \{\infty\}$, we say that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$ if it satisfies item i) of Definition 8.2 not for all $a \in \partial\Omega$, but for all $a \in \partial\Omega \cap \mathcal{K}$.

With this definition, the generalization of Theorem 8.1 is the following.

Theorem 8.3 *Let $p \in [1,\infty[$ and Ω be an open set of $\mathbb{R}^N$ with a Lipschitz continuous boundary. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$, $k \in \mathbb{N}\backslash\{0\}$ or $k=\infty$ and suppose that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$. If K is a compact subset of $\mathbb{R}^N$ and $u \in W^{1,p}(\Omega)$ has a compact support in the interior of K([1]), there exists a sequence of functions $(u_n)_{n\geq 1} \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with supports in the interior of K such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and, for all $n \geq 1$ and all $a \in \partial\Omega \cap \mathcal{K}$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$.*

Remark 8.3 *i) The global hypothesis on the boundary of Ω (i.e. the Lipschitz continuity of $\partial\Omega$) is only used, in the proof, to ensure that the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ are dense in $W^{1,p}(\Omega)$; thus, we could replace this hypothesis by a weaker one (for example asking that Ω satisfies the segment property, see [1]).*

ii) Notice that the Neumann boundary condition is satisfied for all $a \in \partial\Omega \cap \mathcal{K}$, *even when $k=1$ (in which case it is even not obvious that $\frac{\partial u_n}{\partial \mathbf{n}}(a)$ is defined for σ-a.e. $a \in \partial\Omega \cap \mathcal{K}$, let alone for all $a \in \partial\Omega \cap \mathcal{K}$).*

This Theorem is an easy consequence of the following proposition (which states in fact the result of Theorem 8.3 when u is regular and $K = \mathcal{K}$).

Proposition 8.1 *Let $p \in [1,\infty[$, $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$ and Ω be an open set of $\mathbb{R}^N$. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ has its support in the interior of $\mathcal{K}$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, there exists a sequence of functions $(u_n)_{n\geq 1} \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with supports in the interior of $\mathcal{K}$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$ and, for all $n \geq 1$ and all $a \in \partial\Omega$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$.*

Proof of Theorem 8.3
Thanks to the definition of "Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$", we see that there exists a compact set $\mathcal{K}'$ of $\mathbb{R}^N$ containing $\mathcal{K}$ in its interior such that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}'$; let $\theta \in \mathcal{C}_c^\infty(\mathrm{int}(\mathcal{K}'))$ such that $\theta \equiv 1$ on the neighborhood of $\mathcal{K}$.

Ω having a Lipschitz continuous boundary, there exists $(\varphi_n)_{n\geq 1} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\Omega)$.
Let $\Theta \in \mathcal{C}_c^\infty(\mathrm{int}(K))$ such that $\Theta \equiv 1$ on the neighborhood of $\mathrm{supp}(u)$ and define $v_n = \Theta\theta\varphi_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. $v_n \to \Theta\theta u = \theta u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and the support of v_n is included in the interior of $\mathcal{K}' \cap K$; by Proposition 8.1, there exists thus $w_n \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}' \cap K$ such that $||v_n - w_n||_{W^{1,p}(\Omega)} \leq 1/n$ and, for all $a \in \partial\Omega$, $\frac{\partial w_n}{\partial \mathbf{n}}(a) = 0$.
Let $u_n = w_n + (1-\theta)\Theta\varphi_n \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$; the support of u_n is a compact subset of the interior of K (since $\mathrm{supp}(w_n) \cup \mathrm{supp}(\Theta) \subset \mathrm{int}(K)$). Since $1-\theta \equiv 0$ on the neighborhood of $\mathcal{K}$, one has $\frac{\partial((1-\theta)\Theta\varphi_n)}{\partial \mathbf{n}} \equiv 0$ on $\mathcal{K} \cap \partial\Omega$, so that, for all $a \in \mathcal{K} \cap \partial\Omega$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$. Since $w_n \to \theta u$ and $(1-\theta)\Theta\varphi_n \to (1-\theta)\Theta u = (1-\theta)u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, we have $u_n \to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$, and this concludes the proof of the theorem.
■

It remains now to prove Proposition 8.1.

[1] That is to say, the extension of u to $\mathbb{R}^N$ by 0 outside Ω has a compact support in $\mathrm{int}(K)$

The idea is the following: the function $g = \nabla u \cdot \mathbf{n}$ is $\mathcal{C}^{k-1,1}$-continuous on the neighborhood of $\partial\Omega \cap \mathcal{K}$; we construct a sequence $(\gamma_n)_{n\geq 1} \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$ which converges to 0 in $W^{1,p}(\Omega)$ and such that, for all $n \geq 1$, $\frac{\partial\gamma_n}{\partial\mathbf{n}} = g$ on $\partial\Omega \cap \mathcal{K}$; the sequence $u_n = u - \gamma_n$ converges then to u in $W^{1,p}(\Omega)$ and satisfies the Neumann boundary condition on $\partial\Omega \cap \mathcal{K}$.
The main difficulty of this proof is to construct the sequence $(\gamma_n)_{n\geq 1}$.

The first lemma is quite classical when $\partial\Omega$ is a regular submanifold of $\mathbb{R}^N$. We however prove it completely because, when $\partial\Omega$ is only $\mathcal{C}^{1,1}$-continuous, the main tool of the proof is not so common.

Lemma 8.1 *Let Ω be an open set of $\mathbb{R}^N$. If $\mathcal{K}$ is a compact subset of $\mathbb{R}^N$, $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, there exists an open set U of $\mathbb{R}^N$ containing $\partial\Omega \cap \mathcal{K}$ and a $\mathcal{C}^{k-1,1}$-continuous application $P : U \to \partial\Omega$ such that, for all $y \in U$, $P(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$. Moreover, for all $a \in \mathcal{K} \cap \partial\Omega$, there exists $t_a > 0$ such that, for all $|t| < t_a$, $P(a + t\mathbf{n}(a)) = a$.*

Remark 8.4 *We will also see, in the course of the proof, that U can be chosen so that*

$$\forall y \in U\backslash\partial\Omega\,,\ \mathbf{n}(P(y)) \cdot (y - P(y)) \neq 0. \tag{8.2}$$

Proof of Lemma 8.1
Step 1: local construction.
We prove in this step that, for all $a \in \partial\Omega \cap \mathcal{K}$, there exists an open set U_a of $\mathbb{R}^N$ containing a and a $\mathcal{C}^{k-1,1}$-continuous application $P_a : U_a \to \partial\Omega$ such that, for all $y \in U_a$, $P_a(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$.

Let $a \in \partial\Omega \cap \mathcal{K}$ and $\mathcal{R}$, $V = V' \times]-\alpha, \alpha[$ and $\eta : V' \to]-\alpha, \alpha[$ given for a by the definition of "Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$". From now on, all the coordinates are taken in $\mathcal{R}$ (notice that the norm and the distance are not modified by this change of coordinates).

Let us first study, for a given $y = (y', y_N)$, the solutions x' to $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
$F(x', y) = x' - y' + (\eta(x') - y_N)\nabla\eta(x')$ is $\mathcal{C}^{k-1,1}$-continuous on $V' \times \mathbb{R}^N$ and is null at $(x', y) = (0, 0)$. Moreover, when it exists, $\frac{\partial F}{\partial x'}(x', y) = Id + \nabla\eta(x')\nabla\eta(x')^T + (\eta(x') - y_N)\eta''(x')$ (where $\eta''(x')$ is confused with the Hessian matrix of η).
If $k \geq 2$, then F being $\mathcal{C}^{k-1}$-continuous and $\frac{\partial F}{\partial x'}(0,0) = Id + \nabla\eta(0)\nabla\eta(0)^T$ being definite positive, thus invertible, the classical implicit function theorem gives an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a $\mathcal{C}^{k-1}$-continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$. Moreover, since $f'(y) = -\left(\frac{\partial F}{\partial x'}(f(y), y)\right)^{-1} \circ \frac{\partial F}{\partial y}(f(y), y)$ and F is $\mathcal{C}^{k-1,1}$-continuous, f is in fact $\mathcal{C}^{k-1,1}$-continuous (even if it means to reduce U).
If $k = 1$, then $\nabla\eta$ is Lipschitz continuous on V'; there exists thus $C > 0$ such that, for every $x' \in V'$, if $\eta''(x')$ exists, then $||\eta''(x')|| \leq C$ ($||\cdot||$ denotes a norm on the space of $(N-1) \times (N-1)$ matrices). Thus, for all $\xi \in \mathbb{R}^{N-1}$, if (x', y) is such that $\frac{\partial F}{\partial x'}(x', y)$ exists, we have

$$\frac{\partial F}{\partial x'}(x', y)\xi \cdot \xi \geq |\xi|^2 + (\nabla\eta(x')^T\xi)^2 - C|\eta(x') - y_N||\xi|^2.$$

Supposing that $(x', y) \to (0, 0)$ and that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ exists, passing to the limit in this inequality lets us see that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ is a 1-coercive matrix (that is to say a $(N-1) \times (N-1)$ matrix A such that, for all $\xi \in \mathbb{R}^{N-1}$, $A\xi \cdot \xi \geq |\xi|^2$). Thus, any convex combination that can be made with such limits is also 1-coercive; this implies that, by denoting S the set of $(x', y) \in V' \times \mathbb{R}^N$ such that F is differentiable with respect to x' at (x', y), the set

$$\mathrm{co}\left\{\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)\,,\ (x', y) \in S\right\}$$

is made of invertible matrices. The Lipschitz implicit function theorem of [19] gives then an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a Lipschitz continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$.

Let $\beta > 0$ such that $B_N(\beta) \subset V$, $B_{N-1}(\beta) \subset W$ and $B_N(\beta/2) \subset U$; let $y \in B_N(0, \beta/2)$. By compacity of $\partial\Omega$, there exists some points in $\partial\Omega$ that are at distance $\mathrm{dist}(y, \partial\Omega)$ of y. Moreover, since $0 \in \partial\Omega$, when x is such a point, we have $|x| \le |y| + |x - y| \le |y| + |y - 0| < \beta$, that is to say $x \in B_N(\beta) \subset V$.
x can thus be written as $(x', \eta(x'))$ for a $x' \in B_{N-1}(\beta) \subset W$; x' is then a minimum of the $\mathcal{C}^1$-continuous function $|\cdot - y'|^2 + |\eta(\cdot) - y_N|^2$ on V' and we deduce that $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
Since $(x', y) \in W \times U$, x' is unique and $x' = f(y)$ (f has been constructed up above).
There can thus be only one projection of y on $\partial\Omega$; it is given by a function of y which is $\mathcal{C}^{k-1,1}$-continuous on $B_N(\delta/2)$. This concludes this step (with $U_a = B_N(\delta/2)$ and $P_a(y) = (f(y), \eta(f(y)))$ in $\mathcal{R}$).

Step 2: we cover the compact set $\mathcal{K} \cap \partial\Omega$ by a finite number of U_{a_i}, $i = 1, \ldots, l$, constructed in step 1. $\cup_{i=1}^l U_{a_i}$ being an open set of $\mathbb{R}^N$ containing $\mathcal{K} \cap \partial\Omega$, there exists an open set U of $\mathbb{R}^N$ containing $\mathcal{K} \cap \partial\Omega$ and relatively compact in $\cup_{i=1}^l U_{a_i}$. Define $P : U \to \partial\Omega$ by: $\forall y \in U$, $P(y)$ is the unique point of $\partial\Omega$ at distance $\mathrm{dist}(y, \partial\Omega)$ of y (since $y \in U_{a_i}$ for a certain $i \in [1, l]$, we know that this point exists and is unique).
By construction of P and of the $(P_{a_i})_{i\in[1,l]}$, and by uniqueness of the point at distance $\mathrm{dist}(y, \partial\Omega)$ of y when $y \in U$, we have $P = P_{a_i}$ on U_{a_i}. P is thus $\mathcal{C}^{k-1,1}$-continuous on U.

Let us now check that, for all $a \in \mathcal{K} \cap \partial\Omega$ and t small enough, we have $P(a + t\mathbf{n}(a)) = a$. Since the projection of a point of U on $\partial\Omega$ is unique, we have, on the neighborhood of a, $P = P_a$. By the study made in step 1, and using the notations introduced on the neighborhood of a (in which case the expression of $\mathbf{n}(a)$ is $(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}(\nabla\eta(0), -1)^T)$, we see that, for t small enough, $P(a + t\mathbf{n}(a)) = (x', \eta(x'))$ where x' is the unique solution on the neighborhood of 0 to $x' - t(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}\nabla\eta(0) + (\eta(x') + t(\sqrt{1 + |\nabla\eta(0)|^2})^{-1})\nabla\eta(x') = 0$; but $x' = 0$ is a solution to this equation. This means that $P(a+t\mathbf{n}(a)) = (0, \eta(0)) = 0$ in $\mathcal{R}$, that is to say $P(a + t\mathbf{n}(a)) = a$.

To conclude this proof, we see that the open set U given above satisfies (8.2).
Let $y \in U$; there exists $i \in [1, l]$ such that $y \in U_{a_i}$; by the study made in step 1, and with the notations of this step, we have $P(y) = (x', \eta(x'))$ where $x' \in V'$ satisfies $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$. If $\mathbf{n}(P(y)) \cdot (y - P(y)) = 0$, then $(\nabla\eta(x'), -1)^T \cdot (y' - x', y_N - \eta(x'))^T = 0$ (because $\mathbf{n}(P(y)) = (\sqrt{1 + |\nabla\eta(x')|^2})^{-1}(\nabla\eta(x'), -1))$, so that $(y' - x') \cdot \nabla\eta(x') - (y_N - \eta(x')) = 0$. By using the equation satisfied by x', we deduce that $(\eta(x') - y_N)(|\nabla\eta(x')|^2 + 1) = 0$, that is to say $y_N = \eta(x')$ and, once again thanks to the equation satisfied by x', $x' = y'$. This gives $y = P(y) \in \partial\Omega$.
Thus, if $y \in U\backslash\partial\Omega$, we have $\mathbf{n}(P(y)) \cdot (y - P(y)) \neq 0$.■

The following lemma gives the existence of the $(\gamma_n)_{n\ge1}$ needed in the proof of Proposition 8.1.

Lemma 8.2 *Let $p \in [1, +\infty[$, Ω be an open set of $\mathbb{R}^N$ and $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$. If $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$, $g \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ has its support in the interior of $\mathcal{K}$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, then for all $\varepsilon > 0$, there exists $\gamma \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$ such that $||\gamma||_{W^{1,p}(\Omega)} < \varepsilon$ and, for all $a \in \partial\Omega$, $\frac{\partial\gamma}{\partial\mathbf{n}}(a) = g(a)$.*

Proof of Lemma 8.2
Let U and P given for $\mathcal{K}$ by Lemma 8.1; we can suppose that U is bounded and satisfies (8.2). Let $\theta \in \mathcal{C}_c^\infty(\mathrm{int}(\mathcal{K}) \cap U)$ such that $\theta \equiv 1$ on the neighborhood of $\mathrm{supp}(g) \cap \partial\Omega$.
Let $h \in \mathcal{C}_c^\infty(]-1, 1[)$ such that $h(0) = 0$ and $h'(0) = 1$; when $\delta > 0$, we take $h_\delta(x) = \delta h(x/\delta)$.
Define $\gamma_\delta(y) = \theta(y)g(P(y))h_\delta(\mathbf{n}(P(y)) \cdot (y - P(y)))$; this function is well defined and $\mathcal{C}^{k-1,1}$-continuous on U; since its support is a compact subset of $\mathrm{int}(\mathcal{K}) \cap U$, its extension to $\mathbb{R}^N$ by 0 outside U is in $\mathcal{C}^{k-1,1}(\mathbb{R}^N)$ and has a compact support in the interior of $\mathcal{K}$.

Let us first check that, for all $a \in \partial\Omega$, $\frac{\partial \gamma_\delta}{\partial \mathbf{n}}(a)$ exists and is equal to $g(a)$. We study different cases, depending on the position of a on $\partial\Omega$.
When $a \in \partial\Omega \backslash \mathcal{K}$, it is quite clear because, for t small enough, $(a, a + t\mathbf{n}(a)) \notin \mathrm{supp}(\theta)$ so that $\gamma_\delta(a + t\mathbf{n}(a)) = \gamma_\delta(a) = 0 = g(a)$.
When $a \in \partial\Omega \cap (\mathcal{K}\backslash \mathrm{supp}(g))$, we have, by Lemma 8.1, $P(a + t\mathbf{n}(a)) = a$ for t small enough, so that $g(P(a + t\mathbf{n}(a))) = g(a) = 0$; this implies $\gamma_\delta(a + t\mathbf{n}(a)) = \gamma_\delta(a) = 0 = g(a)$.
When $a \in \partial\Omega \cap \mathrm{supp}(g)$, then, for t small enough, $\theta(a + t\mathbf{n}(a)) = \theta(a) = 1$ ($\theta \equiv 1$ on the neighborhood of $\mathrm{supp}(g) \cap \partial\Omega$) and $P(a + t\mathbf{n}(a)) = a$, so that $\gamma_\delta(a + t\mathbf{n}(a)) - \gamma_\delta(a) = g(a)h_\delta(\mathbf{n}(a) \cdot (a + t\mathbf{n}(a) - a)) - g(a)h_\delta(\mathbf{n}(a) \cdot (a - a)) = g(a)h_\delta(t)$; since $h_\delta(0) = 0$ and $h'_\delta(0) = 1$, we deduce that $\frac{\partial \gamma_\delta}{\partial \mathbf{n}}(a)$ exists and is equal to $g(a)$.

Let us now prove that $\gamma_\delta \to 0$ in $W^{1,p}(\Omega)$ as $\delta \to 0$; this will conclude the proof of the lemma (by taking $\gamma = \gamma_\delta$ for δ small enough).
Notice first that, for all $x \in \Omega$, $|\gamma_\delta(x)| \le \delta ||h||_{L^\infty(\mathbb{R})} ||\theta||_{L^\infty(\mathbb{R}^N)} ||g||_{L^\infty(\mathbb{R}^N)}$; thus, when $\delta \to 0$, $\gamma_\delta \to 0$ in $L^\infty(\Omega)$, and so in $L^p(\Omega)$.
Since h_δ is regular and $\theta g \circ P$, $n \circ P \cdot (Id - P)$ are Lipschitz continuous, we have, on U,

$$\begin{aligned} \nabla \gamma_\delta &= h_\delta(\mathbf{n} \circ P \cdot (Id - P)) \nabla(\theta g \circ P) \\ &\quad + \theta g \circ P h'_\delta(\mathbf{n} \circ P \cdot (Id - P)) \nabla(\mathbf{n} \circ P \cdot (Id - P)). \end{aligned}$$

But $||h_\delta(\mathbf{n} \circ P \cdot (Id - P)) \nabla(\theta g \circ P)||_{L^\infty(\Omega)} \le \delta ||h||_{L^\infty(\mathbb{R})} ||\nabla(\theta g \circ P)||_{L^\infty(\Omega)} \to 0$ as $\delta \to 0$. Moreover, by (8.2), for all $y \in U \cap \Omega$, $\mathbf{n}(P(y)) \cdot (y - P(y)) \ne 0$, so that $h'_\delta(n(P(y)) \cdot (y - P(y))) \to 0$ as $\delta \to 0$ (the support of h'_δ is included in $]-\delta, \delta[$); thus, $\theta g \circ P h'_\delta(\mathbf{n} \circ P \cdot (Id - P)) \nabla(\mathbf{n} \circ P \cdot (Id - P)) \to 0$ on Ω; since $||h'_\delta||_{L^\infty(\mathbb{R})} \le ||h'||_{L^\infty(\mathbb{R})}$, we deduce, by the dominated convergence theorem, that $\theta g \circ P h'_\delta(\mathbf{n} \circ P \cdot (Id - P)) \nabla(\mathbf{n} \circ P \cdot (Id - P)) \to 0$ in $L^p(\Omega)$ ($n \circ P \cdot (Id - P)$ is Lipschitz continuous on U, thus its gradient is essentially bounded on U). ■

Proof of Proposition 8.1
Step 1: we prove that $\nabla u \cdot \mathbf{n} : \partial\Omega \to \mathbb{R}$ has an extension $g \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$.

Cover $\mathcal{K} \cap \partial\Omega$ by a finite number of open sets $(V_i)_{i \in [1,l]}$ of $\mathbb{R}^N$ such that, for all $i \in [1, l]$, there exists an orthonormal coordinate system $\mathcal{R}_i$ in which $V_i = V'_i \times]-\alpha_i, \alpha_i[$ and a $\mathcal{C}^{k,1}$-continuous application $\eta_i : V'_i \to]-\alpha_i, \alpha_i[$ satisfying, in $\mathcal{R}_i$, $\Omega \cap V_i = \{(y', y_N) \in V_i \mid y_N > \eta_i(y')\}$ and $\partial\Omega \cap V_i = \{(y', \eta_i(y')),\ y' \in V'_i\}$. Take $(\theta_i)_{i \in [1,l]}$ such that, for all $i \in [1, l]$, $\theta_i \in \mathcal{C}_c^\infty(V_i)$ and $\sum_{i=1}^l \theta_i \equiv 1$ on $\mathcal{K} \cap \partial\Omega$.
Let $i \in [1, l]$. Using the coordinates $\mathcal{R}_i$, we have, at a point $(y', \eta_i(y')) \in \partial\Omega \cap V_i$, $\mathbf{n}(y', \eta_i(y')) = (\sqrt{1 + |\nabla \eta_i(y')|^2})^{-1} (\nabla \eta_i(y'), -1)^T$; define then, for $(y', y_N) \in V_i$,

$$g_i(y', y_N) = \theta_i(y', \eta_i(y')) \nabla u(y', \eta_i(y')) \cdot ((\sqrt{1 + |\nabla \eta_i(y')|^2})^{-1} (\nabla \eta_i(y'), -1)^T)$$

(i.e. g_i does not depend on y_N). g_i is $\mathcal{C}^{k-1,1}$-continuous on V_i and has a compact support in V_i; its extension, still denoted g_i, to $\mathbb{R}^N$ by 0 outside V_i is thus in $\mathcal{C}^{k-1,1}(\mathbb{R}^N)$.

Let $\Theta \in \mathcal{C}_c^\infty(\mathrm{int}(\mathcal{K}))$ such that $\Theta \equiv 1$ on the neighborhood of $\mathrm{supp}(u)$ and $g = \Theta \sum_{i=1}^l g_i$. $g \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$ has a compact support in the interior of $\mathcal{K}$. On $\partial\Omega$, one has $g_i = \theta_i \nabla u \cdot \mathbf{n}$, so that $g = \Theta \sum_{i=1}^l \theta_i \nabla u \cdot \mathbf{n} = (\sum_{i=1}^l \theta_i) \Theta \nabla u \cdot \mathbf{n}$; but $\Theta = \sum_{i=1}^l \theta_i \equiv 1$ on $\mathrm{supp}(\nabla u \cdot \mathbf{n})$ (because $\mathrm{supp}(\nabla u \cdot \mathbf{n}) \subset \mathcal{K} \cap \partial\Omega$ and $\sum_{i=1}^l \theta_i \equiv 1$ on $\mathcal{K} \cap \partial\Omega$), and we have thus $g = \nabla u \cdot \mathbf{n}$ on $\partial\Omega$.

Step 2: conclusion.
By Lemma 8.2, we can find, for all $n \ge 1$, $\gamma_n \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$ such that $||\gamma_n||_{W^{1,p}(\Omega)} \le 1/n$ and $\frac{\partial \gamma_n}{\partial \mathbf{n}} = g = \nabla u \cdot \mathbf{n}$ on $\partial\Omega$. The sequence $(u - \gamma_n)_{n \ge 1}$ satisfies thus the conclusion of the proposition. ■

8.3 Polygonal open set

The idea of the proof of Theorem 8.2 is to approximate any regular function by functions that, on the neighborhood of each affine part of $\partial\Omega$, only depend on the coordinates along this affine part (for example, on the neighborhood of a vertex of $\partial\Omega$, the approximating functions will be constant; on the neighborhood of an edge of $\partial\Omega$, it will only depend on the 1-dimensional coordinate along this edge; etc...).
We first introduce some notations and then prove a lemma that entails Theorem 8.2.

Let Ω be a polygonal open set of $\mathbb{R}^N$ and $H_1,\dots,H_{r_1}$ some affine hyperplanes, the union of which contains $\partial\Omega$ (we also choose these hyperplanes pairwise distincts and such that, for all $i\in[1,r_1]$, $H_i\cap\partial\Omega\neq\emptyset$).
For $i\in[1,r_1]$, let $F_i^1=H_i\cap\partial\Omega$. Define $q=\sup\{s\geq 1\mid \exists J\subset[1,r_1]$ of cardinal s such that $\cap_{i\in J}F_i^1\neq\emptyset\}$; for $d\in[1,q]$, we denote $J_1^d,\dots,J_{r_d}^d$ the subsets of cardinal d of $[1,r_1]$ such that, for all $i\in[1,r_d]$, $F_i^d=\cap_{l\in J_i^d}F_l^1\neq\emptyset$.
When $d\in[1,q]$ and $i\in[1,r_d]$, $A(F_i^d)$ denotes the affine space of minimal dimension containing F_i^d (2) and $V(F_i^d)$ the vector space parallel to $A(F_i^d)$. The orthogonal projection on $V(F_i^d)$ is denoted by $\widetilde{P}_i^d$; we also take $f_i^d\in A(F_i^d)$, so that the orthogonal projection of a point x on $A(F_i^d)$ is $P_i^d(x)=f_i^d+\widetilde{P}_i^d(x-f_i^d)$.

When $d\in[1,q+1]$, we say that a function $u\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_d$ if, for all $m\in[d,q]$ and for all $i\in[1,r_m]$, $u=u\circ P_i^m$ on the neighborhood of F_i^m (i.e., on the neighborhood of F_i^m, u only depends on the coordinates along F_i^m). Notice that any function in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_{q+1}$ (there exists no $m\in[q+1,q]$).

Lemma 8.3 *Let $p\in[1,\infty[$, $d\in[2,q+1]$ and K be a compact subset of $\mathbb{R}^N$. If $u\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ has its support in the interior of K and satisfies $\mathcal{B}_d$, there exists a sequence of fuctions $u_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ with supports in the interior of K such that $u_n\to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$ and, for all $n\geq 1$, u_n satisfies $\mathcal{B}_{d-1}$.*

Proof of Lemma 8.3
Before beginning the proof itself, let us make some remarks:

$$\text{If } i\neq j,\ F_i^{d-1}\cap F_j^{d-1} \text{ is either empty or equal to a certain } F_k^m, \text{ for a } m\in[d,q] \text{ and a } k\in[1,r_m]. \tag{8.3}$$

$$\text{If } F_j^m\subset F_i^d \text{ and } x\in F_j^m+B_N(\delta), \text{ then } P_i^d(x)\in F_j^m+B_N(\delta). \tag{8.4}$$

$$\text{If } F_j^m\subset F_i^d, \text{ then } P_j^m\circ P_i^d=P_i^d\circ P_j^m=P_j^m. \tag{8.5}$$

• *Proof of (8.3)*: by definition, $F_i^{d-1}=\cap_{l\in J_i^{d-1}}F_l^1$ and $F_j^{d-1}=\cap_{l\in J_j^{d-1}}F_l^1$, with J_i^{d-1} and J_j^{d-1} distinct (since $i\neq j$) subsets of cardinal $d-1$ of $[1,r_1]$. Thus, $J_i^{d-1}\cup J_j^{d-1}$ has a cardinal $m\geq d$; if $F_i^{d-1}\cap F_j^{d-1}\neq\emptyset$, then $\cap_{l\in(J_i^{d-1}\cup J_j^{d-1})}F_l^1=F_i^{d-1}\cap F_j^{d-1}$ is not empty, so that, by definition, $m\leq q$ and $J_i^{d-1}\cup J_j^{d-1}$ is a certain J_k^m, for a $k\in[1,r_m]$; we get then $F_k^m=\cap_{l\in J_k^m}F_l^1=F_i^{d-1}\cap F_j^{d-1}$.
• *Proof of (8.4)*: we have $x=z+h$, where $z\in F_j^m\subset A(F_i^d)$ and $|h|<\delta$; since z belongs to $A(F_i^d)$, it is equal to its projection on this affine space, so that $P_i^d(x)=f_i^d+\widetilde{P}_i^d(z-f_i^d)+\widetilde{P}_i^d(h)=z+\widetilde{P}_i^d(h)$; $\widetilde{P}_i^d$ being an orthogonal projection, we have $|\widetilde{P}_i^d(h)|\leq|h|<\delta$, and this proves the result.
• *Proof of (8.5)*: we first notice that the range of P_j^m is included in $A(F_j^m)$, thus in $A(F_i^d)$; since P_i^d is equal to the identity mapping on $A(F_i^d)$, we deduce that $P_i^d\circ P_j^m=P_j^m$; it remains to prove the second equality. Let $x\in\mathbb{R}^N$; $P_j^m(x)$ is the unique $z\in A(F_j^m)$ such that $(x-z)\perp V(F_j^m)$; but $x-P_j^m(P_i^d(x))=x-P_i^d(x)+P_i^d(x)-P_j^m(P_i^d(x))$ with $x-P_i^d(x)$ orthogonal to $V(F_i^d)$ (by definition of P_i^d), thus also to $V(F_j^m)\subset V(F_i^d)$, and $P_i^d(x)-P_j^m(P_i^d(x))$ orthogonal to $V(F_j^m)$ (by definition of P_j^m); thus, $P_j^m(P_i^d(x))$ is in $A(F_j^m)$ and satisfies $(x-P_j^m(P_i^d(x)))\perp V(F_j^m)$, which implies $P_j^m(P_i^d(x))=P_j^m(x)$.

2 We could also have taken $A(F_i^d)=\cap_{l\in J_i^d}H_l$.

Step 1: we define a sequence of functions $(v_n)_{n\geq 1}$.
Let $\delta > 0$ such that, for all $m \in [d,q]$ and all $i \in [1,r_m]$, $u = u \circ P_i^m$ on $F_i^m + B_N(\delta)$.

When $i \in [1, r_{d-1}]$, we define $L_i = \{(m,k) \in \mathbb{N}^2 \mid d \leq m \leq q\,,\ 1 \leq k \leq r_m\,,\ F_k^m \subset F_i^{d-1}\}$. $G_i^{d-1} = F_i^{d-1} \backslash (\cup_{(m,k)\in L_i} (F_k^m + B_N(\delta/2)))$ is a compact set which does not intersect the compact set $\cup_{j\neq i} F_j^{d-1}$; indeed, suppose that the intersection of these compact sets contains a point: this point is then in $F_i^{d-1} \cap F_j^{d-1}$ for a $j \neq i$, thus, by (8.3), in some F_k^m for a $m \in [d,q]$ and a $k \in [1,r_m]$ such that $F_k^m \subset F_i^{d-1}$; this point can then not belong to G_i^{d-1}.
Thus, $\delta_0 = \inf(\delta, \inf\{\mathrm{dist}(G_i^{d-1}, \cup_{j\neq i} F_j^{d-1})\,,\ i \in [1, r_{d-1}]\})$ is positive. Take then, for all $i \in [1,r_{d-1}]$, $\theta_i \in \mathcal{C}_c^\infty\left(F_i^{d-1} + B_N(\delta_0/2)\right)$ such that $\theta_i \equiv 1$ on $F_i^{d-1} + B_N(\delta_0/4)$.
Let $\gamma \in \mathcal{C}_c^\infty(]-2,2[)$ such that $\gamma \equiv 1$ on $]-1,1[$; we denote $\gamma_n(t) = \gamma(nt)$ (notice that $\gamma_n(|\cdot|)$ is $\mathcal{C}^\infty$-continuous, since γ_n is constant on a neighborhood of 0); by denoting Id the identity mapping of $\mathbb{R}^N$, we let

$$v_n = \sum_{i=1}^{r_{d-1}} \theta_i \gamma_n\left(\left|Id - P_i^{d-1}\right|\right)\left(u - u\circ P_i^{d-1}\right) \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

Step 2: we prove that $v_n \to 0$ in $W^{1,p}(\Omega)$ as $n \to \infty$.
We have, as $n\to\infty$, for all $i \in [1,r_{d-1}]$, $\gamma_n(|x - P_i^{d-1}(x)|) \to 0$ when $x \neq P_i^{d-1}(x)$, that is to say when $x \notin A(F_i^{d-1})$. The sets $(A(F_i^{d-1}))_{i\in[1,r_{d-1}]}$ being of null measure (they are included in some hyperplanes), this means that $v_n \to 0$ a.e. on Ω. By the dominated convergence theorem, since Ω is bounded and

$$||v_n||_{L^\infty(\Omega)} \leq 2\sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\mathbb{R}^N)} ||\gamma||_{L^\infty(\mathbb{R})} ||u||_{L^\infty(\mathbb{R}^N)},$$

v_n tends thus to 0 in $L^p(\Omega)$ as $n\to\infty$.
Let us now study the gradient of v_n. It is the sum of

$$\sum_{i=1}^{r_{d-1}} \gamma_n\left(\left|Id - P_i^{d-1}\right|\right) \nabla\left(\theta_i\left(u - u\circ P_i^{d-1}\right)\right) \tag{8.6}$$

and

$$\sum_{i=1}^{r_{d-1}} \theta_i\left(u - u\circ P_i^{d-1}\right)\gamma_n'\left(\left|Id - P_i^{d-1}\right|\right)\zeta_i \tag{8.7}$$

where $\zeta_i = \nabla(|Id - P_i^{d-1}|) = (Id - \widetilde{P}_i^{d-1})\frac{Id - P_i^{d-1}}{|Id - P_i^{d-1}|}$ (because $I - \widetilde{P}_i^{d-1}$ is symmetric).
By the same argument than before, the term (8.6) tends to 0 in $L^p(\Omega)$ as $n\to\infty$.
$\widetilde{P}_i^{d-1}$ being an orthogonal projection, we have, for a.e. $x \in \mathbb{R}^N$ (for all $x \notin A(F_i^{d-1})$),

$$|\zeta_i(x)| \leq \left|\frac{x - P_i^{d-1}(x)}{|x - P_i^{d-1}(x)|}\right| \leq 1.$$

Thus, the norm in $L^p(\Omega)$ of (8.7) is bounded by

$$\sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\Omega)} \left|\left|(u - u\circ P_i^{d-1})\gamma_n'\left(\left|Id - P_i^{d-1}\right|\right)\right|\right|_{L^p(\Omega)}.$$

But, $\gamma_n'(|x - P_i^{d-1}(x)|) = 0$ when $|x - P_i^{d-1}(x)| \geq 2/n$, that is to say when $x \notin A(F_i^{d-1}) + B_N(2/n)$ (recall that $|x - P_i^{d-1}(x)|$ is the distance between x and $A(F_i^{d-1})$); thus, using the Lipschitz continuity of u and the estimate $||\gamma_n'||_{L^\infty(\mathbb{R})} \leq n||\gamma'||_{L^\infty(\mathbb{R})}$, we can bound the norm in $L^p(\Omega)$ of (8.7) by

$$2||\gamma'||_{L^\infty(\mathbb{R})}\mathrm{lip}(u) \sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\Omega)} \left|\Omega \cap \left(A(F_i^{d-1}) + B_N(2/n)\right)\right|^{1/p}. \tag{8.8}$$

Since Ω is of finite measure and $\cap_{n\geq 1}(A(F_i^{d-1})+B_N(2/n))=A(F_i^{d-1})$ is a non-increasing intersection of null measure, (8.8) tends to 0 as $n\to\infty$.
Both terms (8.6) and (8.7) going to 0 in $L^p(\Omega)$ as $n\to\infty$, we deduce that $v_n\to 0$ in $W^{1,p}(\Omega)$ as $n\to\infty$.

Step 3: study of v_n on the neighborhood of a F_i^{d-1}.
Let $i\in[1,r_{d-1}]$ and $U_{i,n}$ be the open set $F_i^{d-1}+B_N(\inf(\delta_0/4,1/n))$. If $x\in U_{i,n}$, then $|x-P_i^{d-1}(x)|=\mathrm{dist}(x,A(F_i^{d-1}))\leq \mathrm{dist}(x,F_i^{d-1})<1/n$, so that $\gamma_n(|x-P_i^{d-1}(x)|)=1$. Thus, on $U_{i,n}$,

$$v_n=u-u\circ P_i^{d-1}+\sum_{j\neq i}\theta_j\gamma_n(|Id-P_j^{d-1}|)(u-u\circ P_j^{d-1}).$$

Let $j\neq i$ and $x\in U_{i,n}$ such that $\theta_j(x)\neq 0$. We have then $x\in(F_i^{d-1}+B_N(\delta_0/4))\cap(F_j^{d-1}+B_N(\delta_0/2))$; by writing $x=z+h$ with $z\in F_j^{d-1}$ and $|h|<\delta_0/2$, we have $z\in F_j^{d-1}\cap(F_i^{d-1}+B_N(3\delta_0/4))$, thus $z\in\cup_{(m,k)\in L_j}(F_k^m+B_N(\delta/2))$ by definition of δ_0 (z cannot belong to G_j^{d-1} since the distance between G_j^{d-1} and F_i^{d-1} is greater than or equal to δ_0); since $|x-z|<\delta_0/2\leq\delta/2$, we deduce that $x\in F_k^m+B_N(\delta)$ for a $m\in[d,q]$ and a $k\in[1,r_m]$ such that $F_k^m\subset F_j^{d-1}$. By (8.4), we get then $(x,P_j^{d-1}(x))\in(F_k^m+B_N(\delta))^2$, which gives, by definition of δ and by (8.5), $u(x)=u(P_k^m(x))$ and $u(P_j^{d-1}(x))=u(P_k^m(P_j^{d-1}(x)))=u(P_k^m(x))$, which implies $u(x)-u(P_j^{d-1}(x))=0$.
We have thus, on $U_{i,n}$, $v_n=u-u\circ P_i^{d-1}$.

Step 4: conclusion.
Let $\Theta\in\mathcal{C}_c^\infty(\mathrm{int}(K))$ and $\varepsilon>0$ such that $\Theta\equiv 1$ on $\mathrm{supp}(u)+B_N(\varepsilon)$.
Define $u_n=u-\Theta v_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$; $u_n\to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$, the support of u_n is included in the interior of K and, for all $i\in[1,r_{d-1}]$, we have, on $U_{i,n}$, $u_n=u-\Theta u+\Theta(u\circ P_i^{d-1})=(1-\Theta)u+\Theta(u\circ P_i^{d-1})=\Theta(u\circ P_i^{d-1})$ (because $1-\Theta\equiv 0$ on the neighborhood of $\mathrm{supp}(u)$).
Let $\mathcal{U}_{i,n}=F_i^{d-1}+B_N(\inf(\delta_0/4,1/n,\varepsilon/2))\subset U_{i,n}$. If $x\in\mathcal{U}_{i,n}\cap(\mathrm{supp}(u)+B_N(\varepsilon))$, then $u_n(x)=\Theta(x)u(P_i^{d-1}(x))=u(P_i^{d-1}(x))$ because $\Theta\equiv 1$ on $\mathrm{supp}(u)+B_N(\varepsilon)$. If $x\in\mathcal{U}_{i,n}\backslash(\mathrm{supp}(u)+B_N(\varepsilon))$, then $x=z+h$ with $z\in F_i^{d-1}$ and $|h|<\varepsilon/2$, so that ($z\in A(F_i^{d-1})$ is equal to its projection on this space) $P_i^{d-1}(x)=z+\widetilde{P}_i^{d-1}(h)$, thus $|P_i^{d-1}(x)-x|\leq|h|+|\widetilde{P}_i^{d-1}(h)|<\varepsilon/2+\varepsilon/2$ (because $\widetilde{P}_i^{d-1}$ is an orthogonal projection on a vector space and satisfies thus $|\widetilde{P}_i^{d-1}(h)|\leq|h|$); we get then $P_i^{d-1}(x)\notin\mathrm{supp}(u)$ (because $\mathrm{dist}(x,\mathrm{supp}(u))\geq\varepsilon$), which gives $u_n(x)=\Theta(x)u(P_i^{d-1}(x))=0=u(P_i^{d-1}(x))$.
Thus, for all $i\in[1,r_{d-1}]$,

$$u_n=u\circ P_i^{d-1}\quad\text{on}\quad\mathcal{U}_{i,n}=F_i^{d-1}+B_N(\inf(\delta_0/4,1/n,\varepsilon/2)).\tag{8.9}$$

If $x\in\mathcal{U}_{i,n}$, then by (8.4), $P_i^{d-1}(x)\in\mathcal{U}_{i,n}$, so that, by (8.9) and (8.5),

$$u_n(P_i^{d-1}(x))=u(P_i^{d-1}(P_i^{d-1}(x)))=u(P_i^{d-1}(x))=u_n(x);$$

thus, $u_n=u_n\circ P_i^{d-1}$ on the neighborhood of F_i^{d-1}, for all $i\in[1,r_{d-1}]$.
It remains to prove that u_n satisfies $\mathcal{B}_d$. Let $m\in[d,q]$ and $i\in[1,r_m]$. There exists $j\in[1,r_{d-1}]$ such that $F_i^m\subset F_j^{d-1}$ ([3]); let $W=F_i^m+B_N(\inf(\delta_0/4,1/n,\varepsilon/2))\subset\mathcal{U}_{j,n}$.
When $x\in W$, by (8.9), $u_n(x)=u(P_j^{d-1}(x))$. But, by (8.4), $P_j^{d-1}(x)\in W\subset F_i^m+B_N(\delta)$; the definition of δ and (8.5) give thus

$$u_n(x)=u(P_j^{d-1}(x))=u(P_i^m(P_j^{d-1}(x)))=u(P_i^m(x)).\tag{8.10}$$

Moreover, by (8.4), $P_i^m(x)\in W\subset\mathcal{U}_{j,n}$, which gives, thanks to (8.9) and (8.5),

$$u_n(P_i^m(x))=u(P_j^{d-1}(P_i^m(x)))=u(P_i^m(x)).\tag{8.11}$$

[3] Indeed, one just need to take $J\subset J_i^m$ of cardinal $d-1$ and to notice that $\cap_{l\in J}F_l^1\supset\cap_{l\in J_i^m}F_l^1=F_i^m\neq\emptyset$, so that J is a J_j^{d-1} for a $j\in[1,r_{d-1}]$ and $F_j^{d-1}=\cap_{l\in J}F_l^1\supset F_i^m$.

(8.10) and (8.11) give $u_n = u_n \circ P_i^m$ on W, neighborhood of F_i^m.
u_n satisfies thus $\mathcal{B}_{d-1}$, and this concludes the proof of the lemma. ■

Proof of Theorem 8.2
Denote, for $d \in [1, q+1]$, $\mathcal{E}_d$ the space of the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfying $\mathcal{B}_d$.
Ω having a Lipschitz continuous boundary, $\mathcal{E}_{q+1}$ (the space of the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$) is dense in $W^{1,p}(\Omega)$. Lemma 8.3 allows us to see that $\mathcal{E}_q$ is then dense in $W^{1,p}(\Omega)$ and, by induction, that $\mathcal{E}_1$ is dense in $W^{1,p}(\Omega)$. We will prove that $\mathcal{E}_1 \subset E^\infty(\Omega)$, which will conclude the proof of this theorem.

Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfying $\mathcal{B}_1$, $d \in [1, q]$ and $i \in [1, r_d]$. On the neighborhood of F_i^d, we have $u = u \circ P_i^d$, so that $\nabla u = (\widetilde{P}_i^d)^T \nabla u \circ P_i^d = \widetilde{P}_i^d \nabla u \circ P_i^d$ (an orthogonal projection is always symmetric); thus, by denoting, for $l \in [1, r_1]$, $\widetilde{H}_l$ the vector hyperspace parallel to H_l, we have

$$\nabla u \in V(F_i^d) \subset \bigcap_{l \in J_i^d} \widetilde{H}_l \text{ on the neighborhood of } F_i^d. \tag{8.12}$$

Let $x \in \partial\Omega$ and $J_x = \{l \in [1, r_1] \mid x \in H_l\}$; since $\cap_{l \in J_x} F_l^1$ is not empty (it contains x), J_x is a J_i^d for a $d \in [1, q]$ and a $i \in [1, r_d]$.
On the neighborhood of x, the only hyperplanes $(H_l)_l$ that intersect $\partial\Omega$ are $(H_l)_{l \in J_i^d}$ (because, when $l \notin J_i^d = J_x$, the compact sets $\{x\}$ and $F_l^1 = \partial\Omega \cap H_l$ are disjoint); thus, for σ-a.e. $y \in \partial\Omega$ on the neighborhood of x, there exists $l \in J_i^d$ such that $\mathbf{n}(y)$ is orthogonal to $\widetilde{H}_l$. Since $x \in F_i^d$, we deduce from (8.12) that $\nabla u \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$ on the neighborhood of x.
We have thus proven that $\nabla u \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$, that is to say $u_{|\Omega} \in E^\infty(\Omega)$. ■

Remark 8.5 *One can of course prove a similar result for open sets with singularities on the boundary that are of the same kind than the singularities on the boundary of polygonal open sets. For example, if we can transform locally, by a $\mathcal{C}^{r,1}$-diffeomorphism ($r \geq 1$) that preserves the outer normal ([4]), the singularities of an open set Ω into the singularities of a polygonal open set, we obtain the density of $E^r(\Omega)$ in $W^{1,p}(\Omega)$. This gives in fact another proof (the one in [55]) of Theorem 8.1, but only for $k \geq 2$.*
A crucial example of this is $\Omega = O \times]0, T[$ where O is an open set of $\mathbb{R}^{N-1}$ with a $\mathcal{C}^{r+1,1}$-continuous boundary. Though Ω has a boundary which is only Lipschitz continuous, the singularities of this boundary are, up to a $\mathcal{C}^{r,1}$-diffeomorphism, equivalent to the singularities of a polygonal open set.

8.4 Applications, Counter-example and Generalization

8.4.1 A new formulation for the Neumann problem

The classical variational formulation of the Neumann problem

$$\left\{ \begin{array}{ll} -\Delta u = L & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{8.13}$$

is the following:

$$\left\{ \begin{array}{l} u \in H^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi = \langle L, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}, \ \forall \varphi \in H^1(\Omega). \end{array} \right. \tag{8.14}$$

With Theorem 8.1 or 8.2 and an integrate by parts, we see that (8.14) is equivalent, when Ω has a $\mathcal{C}^{k+1,1}$-continuous boundary (with $k \in \mathbb{N} \backslash \{0\}$ or $k = \infty$) or is polygonal (in which case we take $k = \infty$

[4] Such diffeomorphisms can be constructed thanks to the flow of a vector field which is, on $\partial\Omega$, equal to the unit normal.

below),

$$\left\{\begin{array}{l} u \in H^1(\Omega), \\ -\displaystyle\int_\Omega u\Delta\varphi = \langle L, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)}\ ,\ \forall \varphi \in \mathcal{C}^{k,1}(\Omega) \text{ such that } \nabla\varphi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \end{array}\right. \tag{8.15}$$

This means that, exactly as for the Dirichlet problem, we have found a formulation of (8.13) — equivalent to the variational formulation, thus implying existence and uniqueness of a solution — that allows to put all the derivatives on the test functions.

This formulation can be useful, for example, to simplify the proof of the convergence of the finite volume discretization of the Neumann problem on polygonal open sets (see [37]): (8.15) allows to prove that the finite volume approximation converges to the variational solution without the need of a discrete trace theorem, with the same methods as in the Dirichlet case.

With Theorem 8.4 below, we can do the same for some mixed Dirichlet-Neumann problems.

8.4.2 Application to the convergence of a finite volume scheme

In [45], the authors prove the convergence of a finite volume scheme for a diffusion problem with mixed Dirichlet-Neumann-Signorini boundary conditions. It is classical, when studying finite volume schemes, to consider polygonal open sets of $\mathbb{R}^N$ (see [37]); in [45], the authors must however make an additional assumption on the open set: they must suppose that the open set is convex.

This restriction comes from the same restriction as in Remark 8.2: the authors need that an element of a Hodge decomposition be in H^2, which is ensured by the convexity of the open set (since this element comes from the resolution of a Neumann problem).

Theorem 8.2 allows us to see that the results of [45] are still true without the convexity hypothesis on the open set; moreover, it also simplifies quite a lot the proof of the result in [45] in which the Hodge decomposition was involved (with Theorem 8.2, the functions appearing in this proof are not only in H^2, but also $\mathcal{C}^\infty$-continuous, which makes the error estimates easier to obtain).

We will talk about another application of our results to finite volume scheme in item ii) of Remark 8.6.

8.4.3 Counter-example

Though polygonal open sets are not very regular, the singularities of their boundaries are of a kind that allows the density in $W^{1,p}$ of $\mathcal{C}^\infty$-continuous functions satisfying a Neumann boundary condition.

There is no similar result for general open sets with only Lipschiz continuous boundary; the loss of regularity noticed in Theorem 8.1 gives us the intuition of this (for open sets with $\mathcal{C}^{k,1}$-continuous boundary, we only get the density of $\mathcal{C}^{k-1,1}$-contiuous functions), and the following example gives us the proof of this intuition.

Let $(s_n)_{n\ge 1}$ be an enumeration of the rationals in $]-1,1[$ and $\eta(s) = \sum_{n\ge 1} 2^{-n}\sup(0, s-s_n) - c$ (where c is chosen so that $\eta(0)=0$); η is Lipschitz continuous on $]-1,1[$ and its derivative is $\eta'(s) = \sum_{n\ge 1} 2^{-n}\mathbf{1}_{]s_n,1[}(s) = \sum_{n\,|\,s>s_n} 2^{-n}$ ($\mathbf{1}_{]s_n,s[}$ is the characteristic function of the set $]s_n,s[$). Let Ω be an open set of $\mathbb{R}^2$ with a Lipschitz continuous boundary and such that $\Omega\cap]-1,1[\times]-1,1[=\{(s,t)\in]-1,1[\times]-1,1[\,|\,t>\eta(s)\}$; we will denote $\Lambda = \{(s,\eta(s))\,,\ s\in]-1,1[\}\subset\partial\Omega$.

Let $\varphi\in\mathcal{C}^1(\mathbb{R}^N)$ such that $\nabla\varphi\cdot\mathbf{n}=0$ on $\partial\Omega$. We have then, for a.e. $s\in]-1,1[$,

$$0 = (\sqrt{1+|\eta'(s)|^2})\nabla\varphi(s,\eta(s))\cdot\mathbf{n}(s,\eta(s)) = \eta'(s)\frac{\partial\varphi}{\partial x_1}(s,\eta(s)) - \frac{\partial\varphi}{\partial x_2}(s,\eta(s)). \tag{8.16}$$

Let $n\ge 1$; there exists two sequences $(s_k^{n,+})_{k\ge 1}$ and $(s_k^{n,-})_{k\ge 1}$ converging to s_n and such that $s_k^{n,+}>s_n$, $s_k^{n,-}<s_n$ and (8.16) is satisfied for all $s\in\{s_k^{n,+}, s_k^{n,-}\,,\ k\ge 1\}$. We have then $\eta'(s_k^{n,+})\to\sum_{m\,|\,s_n\ge s_m}2^{-m}$ and $\eta'(s_k^{n,-})\to\sum_{m\,|\,s_n>s_m}2^{-m}$ as $k\to\infty$.
By substracting (8.16) applied to $s_k^{n,-}$ to (8.16) applied to $s_k^{n,+}$, and then passing to the limit $k\to\infty$, we get $2^{-n}\frac{\partial\varphi}{\partial x_1}(s_n,\eta(s_n))=0$ for all $n\ge 1$; $(s_n)_{n\ge 1}$ being dense in $]-1,1[$, we deduce that the continuous function $\frac{\partial\varphi}{\partial x_1}(\cdot,\eta(\cdot))$ is null on $]-1,1[$ and, thanks to (8.16), that $\frac{\partial\varphi}{\partial x_2}(\cdot,\eta(\cdot))$ is also null on $]-1,1[$. Thus, $\varphi(\cdot,\eta(\cdot))$ is constant on $]-1,1[$: φ is constant on Λ.
Any limit in $W^{1,p}(\Omega)$ of functions $\varphi\in\mathcal{C}^1(\mathbb{R}^N)$ satisfying $\nabla\varphi\cdot\mathbf{n}=0$ on $\partial\Omega$ is thus constant σ-a.e. on Λ; since there exists functions in $W^{1,p}(\Omega)$ that are not constant σ-a.e. on Λ (for example, $u(x)=x_1$), we deduce that $\{\varphi_{|\Omega}\,,\ \varphi\in\mathcal{C}^1(\mathbb{R}^N)\,,\ \nabla\varphi\cdot\mathbf{n}=0 \text{ sur } \partial\Omega\}$ cannot be dense in $W^{1,p}(\Omega)$.

8.4.4 Mixed Dirichlet-Neumann boundary conditions

Let Γ be a measurable subset of $\partial\Omega$; we denote by $E^k_\Gamma(\Omega)$ the set of functions in $E^k(\Omega)$ the supports of which do not intersect Γ (such functions are, in particular, null on Γ).
$W^{1,p}_\Gamma(\Omega)$ is the space of functions in $W^{1,p}(\Omega)$ the trace of which is null σ-a.e. on Γ (if $u\in W^{1,p}(\Omega)$ is a limit in $W^{1,p}(\Omega)$ of a sequence in $E^k_\Gamma(\Omega)$, then we have $u\in W^{1,p}_\Gamma(\Omega)$). It is endowed with the same norm as $W^{1,p}(\Omega)$.

By denoting $B_+=\{(y',y_N)\in B_N(1)\mid y_N>0\}$, $D=\{(y',0)\in B_N(1)\}$, $B_{++}=\{(y'',y_{N-1},y_N)\in B_+\mid y_{N-1}>0\}$, $D_+=\{(y'',y_{N-1},0)\in D\mid y_{N-1}\ge 0\}$, we make the additional assumption:

$$\begin{array}{l}\Gamma \text{ is closed and, for all } a\in\Gamma, \text{ there exists an open } U \text{ of } \mathbb{R}^N \text{ containing } a\\ \text{and a Lipschitz continuous homeomorphism } \phi:U\to B_N(1) \text{ with a}\\ \text{Lipschitz continuous inverse mapping such that one of the following cases occurs:}\\ \text{i) } U\cap\Gamma=U\cap\partial\Omega\,,\ \phi(U\cap\Omega)=B_+ \text{ and } \phi(U\cap\partial\Omega)=D,\\ \text{ii) }\left\{\begin{array}{l}\phi(U\cap\Omega)=B_{++}\,,\ \phi(U\cap\partial\Omega)=D_+\cup\{(y'',0,y_N)\in B_N(1)\mid y_N>0\}\\ \text{and } \phi(U\cap\Gamma)=D_+\end{array}\right.\end{array}\tag{8.17}$$

An important example of a Γ satisfying this property is, when $\Omega=O\times]0,T[$ with O open set of $\mathbb{R}^{N-1}$ with a Lipschitz continuous boundary, $\Gamma=\overline{O}\times\{T\}$.

Theorem 8.4 *Let $p\in[1,+\infty[$. If $k\in\mathbb{N}\backslash\{0\}$ or $k=\infty$, Ω is an open set of $\mathbb{R}^N$ with a $\mathcal{C}^{k,1}$-continuous boundary and $\Gamma\subset\partial\Omega$ satisfies (8.17), then $E^{k-1}_\Gamma(\Omega)$ is dense in $W^{1,p}_\Gamma(\Omega)$. If Ω is a polygonal open set of $\mathbb{R}^N$ and $\Gamma\subset\partial\Omega$ satisfies (8.17), then $E^\infty_\Gamma(\Omega)$ is dense in $W^{1,p}_\Gamma(\Omega)$.*

Remark 8.6 *i) Of course, we have the same kind of results when we can locally transform the open set with a diffeomorphism that preserves the outer normal (see Remark 8.5); for example, if $\Omega=O\times]0,T[$ with O open set of $\mathbb{R}^{N-1}$ with a $\mathcal{C}^{k+1,1}$-continuous boundary ($k\ge 1$) and $\Gamma=\overline{O}\times\{T\}$, we can prove the density of $E^k_\Gamma(\Omega)$ in $W^{1,p}_\Gamma(\Omega)$.*

ii) In [55], the author uses a similar result to prove the convergence of a finite volume scheme for a diffusion and non-instantaneous dissolution problem in porous medium, when the medium is represented by an open set with regular boundary (at least $\mathcal{C}^3$-continuous, see item i) of Remark 8.1). Theorem 8.4 allows to extend the results of [55] to polygonal open sets, which are quite natural when dealing with finite volume schemes (see [37]).

Proof of Theorem 8.4
Step 1: we prove that any function $u\in W^{1,p}_\Gamma(\Omega)$ can be approximated in $W^{1,p}(\Omega)$ by functions in $\mathcal{C}^\infty_c(\mathbb{R}^N)$ the supports of which do not intersect Γ.

Cover first the compact set Γ by a finite number of mappings $(U_i,\phi_i)_{i\in[1,l]}$ given by (8.17). We take, for $i\in[1,l]$, $\theta_i\in\mathcal{C}_c^\infty(U_i)$ such that $\sum_{i=1}^l\theta_i\equiv 1$ on the neighborhood of Γ.
Define, for $i\in[1,l]$, $u_i=\theta_i u$. The function $v_i=u_i\circ\phi_i^{-1}$ is in $W^{1,p}_{\phi_i(U_i\cap\Gamma)}(\phi_i(U_i\cap\Omega))$ and its support is relatively compact in $B_N(1)$ (it is included in the support of $\theta_i\circ\phi_i^{-1}$). We will now handle separately the cases when (U_i,ϕ_i) satisfies i) or ii) in (8.17).

- Case i): we have $v_i\in W^{1,p}_D(B_+)$; the extension w_i of v_i to $B_N(1)$ by 0 outside B_+ is then in $W^{1,p}(B_N(1))$ and its support is a compact subset of $B_N(1)$ included in $\overline{B_+}$; the extension $\widetilde{w_i}$ of w_i to $\mathbb{R}^N$ by 0 outside $B_N(1)$ is thus in $W^{1,p}(\mathbb{R}^N)$. Let $f_{i,n}(y)=\widetilde{w_i}(y',y_N-1/n)$; $f_{i,n}$ is in $W^{1,p}(\mathbb{R}^N)$ and the sequence $(f_{i,n})_{n\geq 1}$ converges to w_i in $W^{1,p}(B_N(1))$ (thus to v_i in $W^{1,p}(\phi_i(U_i\cap\Omega))$); moreover, for n large enough, $\mathrm{supp}(f_{i,n})\subset\mathrm{supp}(w_i)+(0,\ldots,0,1/n)$ is a compact subset of B_+ and does not intersect $\phi_i(U_i\cap\Gamma)=D$.

- Case ii): we have $v_i\in W^{1,p}_{D_+}(B_{++})$. The function $\widetilde{v_i}:B_+\to\mathbb{R}$ defined a.e. by $\widetilde{v_i}=v_i$ on B_{++} and $\widetilde{v_i}(y)=v_i(y'',-y_{N-1},y_N)$ if $y_{N-1}<0$ is in $W^{1,p}_D(B_+)$ and its support is relatively compact in $B_N(1)$. By the reasoning made in Case i), there exists thus $(f_{i,n})_{n\geq 1}\in W^{1,p}(\mathbb{R}^N)$ which converges to $\widetilde{v_i}$ in $W^{1,p}(B_+)$ (thus to v_i in $W^{1,p}(\phi_i(U_i\cap\Omega))$) and such that, for n large enough, $\mathrm{supp}(f_{i,n})$ is a compact subset of $B_N(1)$ that does not intersect $D\supset\phi_i(U_i\cap\Gamma)=D_+$.

In both cases, taking n large enough, the support of $g_{i,n}=f_{i,n}\circ\phi_i\in W^{1,p}(U_i)$ is compact in U_i, does not intersect $U_i\cap\Gamma$ and the sequence $(g_{i,n})_{n\geq 1}$ converges to u_i in $W^{1,p}(U_i\cap\Omega)$.
For n large enough, the extension $G_{i,n}$ of $g_{i,n}$ by 0 outside U_i is thus in $W^{1,p}(\mathbb{R}^N)$, has a compact support which does not intersect Γ and, since $\mathrm{supp}(u_i)$ is relatively compact in U_i, we have $G_{i,n}\to u_i$ in $W^{1,p}(\Omega)$ as $n\to\infty$.
Let, for n large enough, $\mathcal{U}_n=\sum_{i=1}^l G_{i,n}$; this function of $W^{1,p}(\mathbb{R}^N)$ has a compact support which does not intersect Γ and $\mathcal{U}_n\to(\sum_{i=1}^l\theta_i)u$ in $W^{1,p}(\Omega)$.
Since $\mathrm{supp}(\mathcal{U}_n)$ is a compact set that does not intersect Γ, there exists $\mathcal{V}_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ the support of which does not intersect Γ and such that $||\mathcal{U}_n-\mathcal{V}_n||_{W^{1,p}(\mathbb{R}^N)}\leq 1/n$ (take $(\rho_m)_{m\geq 1}$ a sequence of mollifiers such that $\mathrm{supp}(\rho_m)\subset B_N(1/m)$; since $\mathrm{supp}(\mathcal{U}_n*\rho_m)\subset\mathrm{supp}(\mathcal{U}_n)+B_N(1/m)$ and $\mathrm{supp}(\mathcal{U}_n)$ and Γ are disjoint compact sets, for m large enough, one has $\mathrm{supp}(\mathcal{U}_n*\rho_m)\cap\Gamma=\emptyset$; since $\mathcal{U}_n*\rho_m\to\mathcal{U}_n$ in $W^{1,p}(\mathbb{R}^N)$ as $m\to\infty$, one sees that, for a m large enough, $\mathcal{V}_n=\mathcal{U}_n*\rho_m$ is convenient).
Let $\mathcal{W}_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\mathcal{W}_n\to u$ in $W^{1,p}(\Omega)$.
The sequence of functions $\mathcal{V}_n+(1-\sum_{i=1}^l\theta_i)\mathcal{W}_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ converges to $(\sum_{i=1}^l\theta_i)u+(1-\sum_{i=1}^l\theta_i)u=u$ in $W^{1,p}(\Omega)$ and $\mathrm{supp}(\mathcal{V}_n+(1-\sum_{i=1}^l\theta_i)\mathcal{W}_n)\subset\mathrm{supp}(\mathcal{V}_n)\cup\mathrm{supp}(1-\sum_{i=1}^l\theta_i)$, that is to say a compact set that does not intersect Γ. This concludes Step 1.

Step 2: To prove the theorem, it is thus sufficient to approximate, in $W^{1,p}(\Omega)$, any function $u\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ the support of which does not intersect Γ by functions in $E^{k-1}(\Omega)$ the supports of which do not intersect Γ.

Let K be a compact subset of $\mathbb{R}^N$ containing $\mathrm{supp}(u)$ in its interior and such that $K\cap\Gamma=\emptyset$.
In the case when Ω has a $\mathcal{C}^{k,1}$-continuous boundary (for a $k\geq 1$), Theorem 8.3 applied to these u and K concludes the proof.
In the case when Ω is a polygonal open set, Lemma 8.3 allows to see, by induction, that there exists a sequence of functions $u_n\in\mathcal{C}_c^\infty(\mathrm{int}(K))$ satisfying $\mathcal{B}_1$ and converging to u in $W^{1,p}(\Omega)$. Since the restrictions to Ω of functions satisfying $\mathcal{B}_1$ are in $E^\infty(\Omega)$ (as seen in the proof of Theorem 8.2), this concludes the proof. ∎