Chapitre 9

Convergence of a finite volume - mixed finite element method for a system of a hyperbolic and an elliptic equations

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Abstract : This paper gives a proof of convergence for the approximate solution of a system of an elliptic equation and of a hyperbolic equation, describing the conservation of two immiscible incompressible phases flowing in a porous medium. The approximate solution is obtained by a mixed finite element method on a large class of meshes for the elliptic equation and a finite volume method for the hyperbolic equation. Since the considered meshes are not necessarily structured, the proof uses a weak total variation inequality, which cannot yield a BV-estimate. We thus prove, under an L^∞ estimate, the weak convergence of the finite volume approximation. The strong convergence proof is then sketched under regularity assumptions which ensure that the fluxes are Lipschitz-continuous.

9.1 Introduction

The purpose of oil reservoir simulation implies to account for several phenomena such as chemical reactions, thermodynamical equilibrium and polyphasic flows. Since the full model is complex too much, a simplified model, describing the flow of two incompressible immiscible fluids through a porous medium, has been extensively studied. In this simplified model, two fluid phases, oil and water, flow through the pores of some possibly heterogeneous and anisotropic porous medium; water is injected through injection wells in order to displace the oil towards production wells. Here we neglect the gravity effects as well as the capillary pressure, which leads to the study of a first order conservation law for the saturation of one of the phases coupled with an elliptic equation for the pressure. Assuming the total mobility of the two phases to be constant and the mobility of water to be linear, the conservation equations of the two phases in a domain Ω yield the following system of equations.

$$u_t(x,t) - \operatorname{div}(u(x,t)\mathbf{\Lambda}(x)\nabla p(x)) = s(x,t)f^+(x) - u(x,t)f^-(x),$$

$$(1-u)_t(x,t) - \operatorname{div}((1-u(x,t))\mathbf{\Lambda}(x)\nabla p(x)) = (1-s(x,t))f^+(x) - (1-u(x,t))f^-(x),$$

for $(x,t) \in \Omega\times\mathbb{R}^+$. In the above equations, the saturation of the water phase is denoted by u, the common pressure of both phases is denoted by p. The absolute permeability $\mathbf{\Lambda}$ is a symmetric definite positive matrix (in anisotropic media, the eigenvalues of the matrix $\mathbf{\Lambda}$ are not all identical) which depends on the space variable in heterogeneous media (the symmetry hypothesis has no influence on the mathematical

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study of the problem). The function f represents the internal source terms, corresponding to the presence of wells drilled into the reservoir (f^+ and f^- denote the positive and negative parts of f). A positive source term corresponds to an injection well, a negative one corresponds to a production well. The function s represents the fraction of the water phase in the injected source term, and the saturation u of the water in place is the fraction of water in the produced source term. This problem, completed with initial and boundary conditions, is rewritten as follows.

$$u_t(x,t) + \mathrm{div}(u\mathbf{q})(x,t) + u(x,t)f^-(x) = s(x,t)f^+(x) \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}^+, \tag{9.1}$$

$$\mathbf{\Lambda}^{-1}(x)\mathbf{q}(x) + \nabla p(x) = 0 \text{ for a.e. } x \in \Omega, \tag{9.2}$$

$$\mathrm{div}\ \mathbf{q}(x) = f(x) \text{ for a.e. } x \in \Omega, \tag{9.3}$$

$$\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \text{ for a.e. } x \in \partial\Omega, \tag{9.4}$$

$$u(x,t) = \overline{u}(x,t) \text{ for a.e. } (x,t) \in \partial\Omega^- \times \mathbb{R}^+, \tag{9.5}$$

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \Omega, \tag{9.6}$$

Notice that the boundary condition for the saturation is only given on the part $\partial\Omega^-$ of the boundary where the flow enters into the domain, that means $\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \leq 0$.
In Eqs (9.1)-(9.6) (refered in the following as Problem (P)) the following hypotheses (refered in the following as Hypotheses (H)) are used.

Hypotheses (H):

1. *Ω is an open bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3 in pratical) such that, locally, Ω either has a $\mathcal{C}^{1,1}$ regular boundary or is convex.*

2. *$\mathbf{\Lambda}(x)$ is a measurable application from Ω to the set of symmetric real $d \times d$ matrices, such that there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfying $\lambda_1|z| \leq |\mathbf{\Lambda}(x)z| \leq \lambda_2|z|$ for almost every $x \in \Omega$ and all $z \in \mathbb{R}^d$.*

3. *$f \in L^2(\Omega)$.*

4. *$g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ for some $\mathbf{q}_0 \in (H^1(\Omega))^d$ and*

$$\int_\Omega f(x)\,dx - \int_{\partial\Omega} g(x)\,d\gamma(x) = 0.$$

5. *$\overline{u} \in L^\infty(\partial\Omega^- \times \mathbb{R}^+)$ where $\partial\Omega^- = \{x \in \partial\Omega, g(x) \leq 0\}$.*

6. *$u_0 \in L^\infty(\Omega)$.*

7. *$s \in L^\infty(\Omega \times \mathbb{R}^+)$.*

Here and in the following, when U is an open subset of $\mathbb{R}^d$ with a sufficient regular boundary (see Definition 9.2), we denote by $\mathbf{n}_{\partial U}$ the unit normal to ∂U outward to U and by γ the $(d-1)$-dimensional measure on ∂U. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ and $x \cdot y$ denotes the Euclidean scalar product of $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. When X is a subset of $\mathbb{R}^d$, $\delta(X)$ denotes the diameter of X, that is to say $\delta(X) = \sup_{(x,y)\in X^2} |x-y|$. $B(z,r)$ denotes the Euclidean ball of center $z \in \mathbb{R}^d$ and radius $r > 0$.

Remark 9.1 *Since we allow Ω to have a non-regular boundary, there is no convenient way to characterize the regularity condition on g. Indeed, if Ω has a $\mathcal{C}^{1,1}$-regular boundary, it is easy to see that $g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ if and only if $g \in H^{1/2}(\partial\Omega)$, but on the non-regular parts of $\partial\Omega$, this condition is not necessary and it is not even obvious that it is sufficient. For example, take $\Omega =]0,1[^2$, $g = 1$ on $(\{0\}\times]0,1[) \cup (\{1\}\times]0,1[)$ and $g = 0$ on $(]0,1[\times\{0\}) \cup (]0,1[\times\{1\})$; then g does not belong to $H^{1/2}(\partial\Omega)$, but g can be written as $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ with $\mathbf{q}_0(x,y) = (-1+2x, 0) \in (H^1(\Omega))^2$.*

A weak solution of Problem (P) is defined by the following sense.

Definition 9.1 *Under Hypotheses (H), a weak solution of (P) is given by $(u, p, \mathbf{q}) \in L^\infty(\Omega \times \mathbb{R}^+) \times L^2(\Omega) \times H_g(\mathrm{div}, \Omega)$ such that*

$$\int_{\mathbb{R}^+}\int_\Omega u(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + \mathbf{q}(x)\cdot\nabla\phi(x,t) - \phi(x,t)f^-(x)\right)dx\,dt =$$
$$-\int_\Omega u_0(x)\phi(x,0)\,dx + \int_{\mathbb{R}^+}\int_{\partial\Omega^-}\overline{u}(x,t)\phi(x,t)g(x)\,d\gamma(x)\,dt - \int_{\mathbb{R}^+}\int_\Omega \phi(x,t)s(x,t)f^+(x)\,dx\,dt, \tag{9.7}$$

$\forall\phi \in C^1_c(\mathbb{R}^d \times \mathbb{R})$ *such that* $\phi = 0$ *on* $\partial\Omega^+ \times \mathbb{R}^+ = (\partial\Omega \setminus \partial\Omega^-) \times \mathbb{R}^+$,

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{q}(x)\,dx - \int_\Omega p(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = 0,\ \forall\mathbf{y} \in H_0(\mathrm{div},\Omega), \tag{9.8}$$

$$\int_\Omega v(x)\mathrm{div}\ \mathbf{q}(x)\,dx = \int_\Omega f(x)v(x)\,dx,\ \forall v \in L^2(\Omega), \tag{9.9}$$

and

$$\int_\Omega p(x)\,dx = 0. \tag{9.10}$$

where the function spaces $H(\mathrm{div},\Omega)$, $H_0(\mathrm{div},\Omega)$ and $H_g(\mathrm{div},\Omega)$ are defined by

$$H(\mathrm{div},\Omega) = \{\mathbf{q} \in (L^2(\Omega))^d\,,\ \mathrm{div}\ \mathbf{q} \in L^2(\Omega)\}\,,\quad H_0(\mathrm{div},\Omega) = \{\mathbf{q} \in H(\mathrm{div},\Omega)\,,\ \mathbf{q}\cdot\mathbf{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$$
$$\text{and}\quad H_g(\mathrm{div},\Omega) = \{\mathbf{q} \in H(\mathrm{div},\Omega)\,,\ \mathbf{q}\cdot\mathbf{n}_{\partial\Omega} = g \text{ on } \partial\Omega\}.$$

A number of numerical schemes for this problem in the case $\mathbf{\Lambda} = Id$ have already been discussed in the literature. Nevertheless, the numerical schemes used to approximate the solution of this simplified model, which is a system of an elliptic equation and a scalar hyperbolic equation, have only recently be studied from a convergence point of view. In particular Eymard and Gallouët [34] have proven the convergence of a numerical scheme involving a finite volume method for the computation of the saturation u and a standard finite element for the computation of the pressure p whereas Vignal [73] presents a convergence proof for a finite volume method for the discretization of both equations. Here we also discretize the conservation law for the saturation by means of a finite volume method but apply the mixed finite element method to discretize the elliptic equation. Error estimates have been derived by Jaffré and Roberts [47] for a semi-discretized problem in the simulation of miscible displacements involving an elliptic equation for the pressure coupled to a parabolic equation for saturation. For the numerical discretization they combine the mixed finite element method with an upstream weighting scheme. More recently Ölberger [60] has derived error estimates in the case that the finite volume method is applied for the discretization of a parabolic equation instead of the first order conservation law (9.1).

Here we deal with a mixed finite element method with an original basis for the elliptic equation. On a partition of the domain, the hypotheses on which are very large, we define the generalization of the

Raviart-Thomas space. The proof of the “inf-sup” condition and the proof that the interpolation error of regular functions tends to zero with the space step make use of Lipschitz-continuous homeomorphisms with Lipschitz-continuous inverse mappings and of some trace inequalities, for which the constants are given as functions of the size of the domain (the classical proofs of the trace inequalities with null average, by the way of contradiction, being unable to yield the relation of such constants with the domain). Then the hyperbolic equation is discretized by the classical upstream weighting scheme, and the convergence proof of the scheme is obtained, thanks to some ”weak BV” inequalities. Such inequalities have only recently be introduced and proved for the proof of convergence of finite volume schemes on unstructured meshes for hyperbolic equations, since strong BV estimates have not been actually obtained on the discrete approximation. Thus this paper completes a lot of previous numerical works in which this scheme has been used on particular meshes (generally triangular meshes).
The organization of this paper is as follows. In Section 9.2, we present the numerical scheme that we use. In Section 9.3, we prove a convergence result for the mixed finite element method. In Section 9.4, we deal with the finite volume scheme, concluding to the weak convergence of a subsequence without additional regularity hypotheses on the data, and to the strong convergence otherwise.

9.2 The discretization

9.2.1 Admissible discretizations

In order to define the scheme, a notion of admissible discretization is given, which is used below in the definition of approximate discrete solutions.

Definition 9.2 (Admissible discretization of Ω) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary. An admissible discretization $\mathcal{D}$ of Ω is given by a finite set $\mathcal{M}$ of open subsets $K \subset \Omega$ with weakly Lipschitz-continuous boundaries and a finite set $\mathcal{A}$ of disjoint subsets $a \subset \overline{\Omega}$ such that:*

(i) $\cup_{K\in\mathcal{M}}\overline{K} = \overline{\Omega}$,

(ii) For all $K \in \mathcal{M}$, there exists a Lipschitz-continuous homeomorphism $\mathcal{L}_K$ from $\overline{K}$ to $\overline{B(0,\delta(K))}$ such that the inverse mapping is Lipschitz-continuous as well. One denotes by C_K the maximum value of both Lipschitz constants and by m_K the Lebesgue measure of K.

(iii) For all $(K,L) \in \mathcal{M}^2$ with $K \neq L$, one has $K \cap L = \emptyset$.

(iv) For all $a \in \mathcal{A}$, there exists $K \in \mathcal{M}$ such that a is a non-empty open subset of ∂K. By denoting $\mathcal{A}_K = \{a \in \mathcal{A} \mid a \subset \partial K\}$, we assume that $\partial K = \cup_{a\in\mathcal{A}_K}\overline{a}$. We denote by m_a the $(d-1)$-dimensional measure of a.

(v) The sets $\mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{A}_e \subset \mathcal{A}$ are defined by $A_i = \{a \in \mathcal{A}, \exists (K,L) \in \mathcal{M}^2,\ K \neq L,\ a \subset \partial K \cap \partial L\}$ and $A_e = \{a \in \mathcal{A}, \exists K \in \mathcal{M}, a \subset \partial K \cap \partial\Omega\}$ ([4]). One assumes that $(\mathcal{A}_i, \mathcal{A}_e)$ forms a partition of $\mathcal{A}$.

(vi) For all $a \in \mathcal{A}_i$, one between the two different $(K,L) \in \mathcal{M}^2$ such that $a \subset \partial K \cap \partial L$ is selected. Then we denote $K(a) = K$ and $L(a) = L$, and we set $\varepsilon_{K,a} = 1$ and $\varepsilon_{L,a} = -1$. The normal vector $\mathbf{n}_a(x)$ to a at $x \in a$ is defined by $\mathbf{n}_a(x) = \mathbf{n}_{\partial K}(x) = -\mathbf{n}_{\partial L}(x)$([5]). For all $a \in \mathcal{A}_e$, let K be the unique element of $\mathcal{M}$ such that $a \subset \partial K \cap \partial\Omega$. Then one denotes $K(a) = K$ and $\varepsilon_{K,a} = 1$. The normal vector $\mathbf{n}_a(x)$ to a at $x \in a$ is defined by $\mathbf{n}_a(x) = \mathbf{n}_{\partial\Omega}(x) = \mathbf{n}_{\partial K}(x)$ ([6]).

(vii) For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one assumes that there exists $x_{K,a} \in a$ and $\zeta_{K,a} > 0$ such that $a \supset \partial K \cap B(x_{K,a}, \zeta_{K,a}\delta(K))$.

[4] One can then show that, when $a \in \mathcal{A}_i$, the $\{K,L\} \subset \mathcal{M}$ such that $K \neq L$ and $a \subset \partial K \cap \partial L$ are unique; this is the same, when $a \in A_e$, for the $K \in \mathcal{M}$ such that $a \subset \partial K \cap \partial\Omega$.

[5] We can indeed show that, in such a situation, we have $\mathbf{n}_{\partial K} = -\mathbf{n}_{\partial L}$ on a.

[6] As for the preceding case, this equality between $\mathbf{n}_{\partial\Omega}$ and $\mathbf{n}_{\partial K}$ is not supposed, it can be proved.

By denoting $\overline{\mathbf{n}_a}$ the mean value of $\mathbf{n}_a$ on a, the thinness of the discretization $\mathcal{D}$ (controling the size of $\mathcal{D}$ and the behaviour of the edges of $\mathcal{D}$) is defined by

$$\operatorname{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}} \left(\delta(K), \max_{a\in\mathcal{A}_K} \left(\frac{1}{\sqrt{m_a}} ||\mathbf{n}_a - \overline{\mathbf{n}_a}||_{L^2(a)} \right) \right) \tag{9.11}$$

and a geometrical factor, linked to the regularity of the discretization, is defined by

$$\operatorname{regul}(\mathcal{D}) = \max_{K\in\mathcal{M}} \left(C_K, \max_{a\in\mathcal{A}_K} \left(\frac{1}{\zeta_{K,a}} \right) \right). \tag{9.12}$$

Remark 9.2 *The definition of an open set with weakly Lipschitz-continuous boundary is given in [31] or in [42] under the name "d-dimensional Lipschitz-continuous submanifold of $\mathbb{R}^d$". It is far weaker than the definition of Lipschitz-continuous boundary given in [58].*

Remark 9.3 *The above definition is easily satisfied for a large variety of meshes. In the case $d = 2$, subsets K such that ∂K is defined in polar coordinates from an origin $M_K \in K$ by a 2π- periodic continuous piecewise C^1 function satisfy condition (ii). That is the case for convex polyhedra, such as triangles or parallelograms for example.*

Remark 9.4 *In the above definition, one cannot define edges by the sets $\partial K \cap \partial L$ or $\partial K \cap \partial\Omega$; indeed, $\operatorname{thin}(\mathcal{D})$ is destined to tend to 0 (in order to obtain the convergence results), which can lead to share the sets $\partial K \cap \partial L$ in different edges. In fact, $\operatorname{thin}(\mathcal{D}) \to 0$ means that the size of the discretization tends to 0 and that the edges become more and more planar.*
Notice that if Ω is polyhedral and the edges are planes, then $\operatorname{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}} \delta(K)$ is simply the size of the discretization.

Remark 9.5 *Hypothesis (vii) is only used for the study of the convergence of the finite volume scheme to the solution of the hyperbolic equation. It is not used in the proof of convergence of the mixed finite element method. Notice that this hypothesis, along with Hypothesis (ii) and Lemma 9.11, implies $m_a \geq C\delta(K)^{d-1}$, where C only depends on d, C_K and $\zeta_{K,a}$.*

9.2.2 Discrete function spaces

One now defines the set of basis functions for the mixed finite element method, which is a generalization of the Raviart-Thomas space $RT_0^0(\mathcal{M})$ (see [14], [72] or [59]).

Definition 9.3 (Discrete function spaces) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2. For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one denotes by $w_{K,a} \in H^1(K)$ the unique variational solution with $\int_K w_{K,a}(x)\,dx = 0$ of the Neumann problem*

$$\Delta w_{K,a}(x) = \frac{m_a}{m_K} \text{ for a.e. } x \in K,$$

and

$$\begin{aligned} \nabla w_{K,a}(x)\cdot \mathbf{n}_{\partial K}(x) &= 1 \quad \text{for a.e. } x \in a,\\ \nabla w_{K,a}(x)\cdot \mathbf{n}_{\partial K}(x) &= 0 \quad \text{for a.e. } x \in \partial K \setminus a. \end{aligned}$$

One then defines the function $\mathbf{w}_{K,a}$ from Ω to $\mathbb{R}^d$ by $\mathbf{w}_{K,a}(x) = \nabla w_{K,a}(x)$ for a.e. $x \in K$ and $\mathbf{w}_{K,a}(x) = 0$ for all $x \in \Omega \setminus K$.
One defines, for all $a \in \mathcal{A}_i$, $\mathbf{w}_a = \mathbf{w}_{K(a),a} - \mathbf{w}_{L(a),a}$ and, for all $a \in \mathcal{A}_e$, $\mathbf{w}_a = \mathbf{w}_{K(a),a}$. Then one gets $\mathbf{w}_a \in H(\operatorname{div},\Omega)$. The set $\mathbf{Q}_\mathcal{D} \subset H(\operatorname{div},\Omega)$ is the space generated by the functions $(\mathbf{w}_a)_{a\in\mathcal{A}}$, the set

$\mathbf{Q}_{\mathcal{D},0} \subset H_0(\mathrm{div},\Omega)$ *is the space generated by the functions* $(\mathbf{w}_a)_{a\in\mathcal{A}_i}$*, and for any* $b \in L^2(\partial\Omega)$*, the set* $\mathbf{Q}_{\mathcal{D},b} \subset \mathbf{Q}_{\mathcal{D}}$ *is the space* $\left\{\mathbf{q} + \sum_{a\in\mathcal{A}_e} \frac{1}{m_a}\int_a b(x)\,d\gamma(x)\ \mathbf{w}_a\,,\ \mathbf{q}\in\mathbf{Q}_{\mathcal{D},0}\right\}$.

$V_{\mathcal{D}} \in L^2(\Omega)$ *is the space of functions* $f = \sum_{K\in\mathcal{M}} \alpha_K\ \chi_K$ *(where, for all* $K \in \mathcal{M}$*,* $\alpha_K \in \mathbb{R}$ *and* χ_K *is the characteristic function of* K*) such that* $\int_\Omega f(x)\,dx = \sum_{K\in\mathcal{M}} m_K\alpha_K = 0$.

9.2.3 The mixed finite element scheme

The mixed finite element approximate of (9.2)-(9.4) is a pair of functions

$$(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g},$$

solution of

$$\int_\Omega v(x)\mathrm{div}\ \mathbf{q}_{\mathcal{D}}(x)\,dx = \int_\Omega f(x)v(x)\,dx, \forall v \in V_{\mathcal{D}}, \tag{9.13}$$

and

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{q}_{\mathcal{D}}(x)\,dx - \int_\Omega p_{\mathcal{D}}(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = 0,\ \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}. \tag{9.14}$$

The unknown functions can be written as

$$\mathbf{q}_{\mathcal{D}} = \sum_{a\in\mathcal{A}} q_a\mathbf{w}_a$$

and

$$p_{\mathcal{D}} = \sum_{K\in\mathcal{M}} p_K\ \chi_K.$$

Then equations (9.13) and (9.14) lead to the following system of linear equations, with unknowns $(q_a)_{a\in\mathcal{A}}$ and $(p_K)_{K\in\mathcal{M}}$:

$$\begin{gathered}\sum_{a'\in\mathcal{A}} q_{a'}\int_\Omega \mathbf{w}_a(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{w}_{a'}(x)\,dx - m_a(p_{K(a)} - p_{L(a)}) = 0,\ \forall a\in\mathcal{A}_i,\\ q_a = g_a,\ \forall a\in\mathcal{A}_e,\\ \sum_{a\in\mathcal{A}_K} m_a q_a\ \varepsilon_{K,a} = f_K,\ \forall K\in\mathcal{M},\\ \sum_{K\in\mathcal{M}} m_K p_K = 0,\end{gathered} \tag{9.15}$$

where we denote

$$f_K = \int_K f(x)\,dx,\ \forall K\in\mathcal{M}, \tag{9.16}$$

and

$$g_a = \frac{1}{m_a}\int_a g(x)\,d\gamma(x),\ \forall a\in\mathcal{A}_e. \tag{9.17}$$

The existence and uniqueness of a solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}})$ to system (9.13)-(9.14) is stated in the following lemma.

Lemma 9.1 (Existence and uniqueness of the discrete approximation) *Let us assume hypotheses (H). Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2. Then system (9.13)-(9.14) defines one and only one approximate solution $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$.*

Proof. Since Lemma 9.4 (which is proved below) shows that the only solution of a linear system with the same matrix as (9.13)-(9.14) and a null right hand side is null, this matrix is invertible. This proves the lemma.

9.2.4 The finite volume scheme

One denotes, for all $K \in \mathcal{M}$ and $a \in \mathcal{A}_K$, $F_{K,a} = m_a q_a \varepsilon_{K,a}$ (then $F_{K(a),a} + F_{L(a),a} = 0$ holds for all $a \in \mathcal{A}_i$).

One then discretizes the hyperbolic problem. Let $\Delta t > 0$ be a constant time step. One defines a discrete source term

$$s_K^n = \frac{1}{\Delta t m_K} \int_{n\Delta t}^{(n+1)\Delta t} \int_K s(x,t)\, dx\, dt, \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}. \tag{9.18}$$

Prolonging by 0 the function $\overline{u}$ on $\partial\Omega^+ \times \mathbb{R}_+$, one defines

$$\overline{u}_a^n = \frac{1}{\Delta t m_a} \int_{n\Delta t}^{(n+1)\Delta t} \int_a \overline{u}(x)\, d\gamma(x)\, dt, \ \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}. \tag{9.19}$$

The discretization of the initial value (Eq. (9.6)) is given by

$$u_K^0 = \frac{1}{m_K} \int_K u_0(x) dx, \ \forall K \in \mathcal{M}. \tag{9.20}$$

The finite volume scheme discretization of equation (9.1) is written:

$$m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} + \sum_{a \in \mathcal{A}_K} u_a^n F_{K,a} = s_K^n f_K^+ - u_K^n f_K^-, \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}, \tag{9.21}$$

where u_a^n is defined by :

$$\begin{array}{l} u_a^n = u_{K(a)}^n \text{ if } q_a > 0, \text{ else } u_a^n = u_{L(a)}^n, \ \forall a \in \mathcal{A}_i, \ \forall n \in \mathbb{N} \\ u_a^n = u_{K(a)}^n \text{ if } q_a > 0, \text{ else } u_a^n = \overline{u}_a^n, \ \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}. \end{array} \tag{9.22}$$

For a given discretization $\mathcal{D}$ and a time step Δt, we can define the approximate solution by:

$$u_{\mathcal{D},\Delta t}(x,t) = u_K^n, \text{ for a.e. } (x,t) \in K \times [n\Delta t, (n+1)\Delta t), \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}. \tag{9.23}$$

9.3 The convergence of the mixed method

One has the following result.

Theorem 9.1 (Convergence of the mixed finite element scheme) *Under Hypotheses (H), let ξ be a fixed positive real value and let $\mathcal{D}$ be a discretization of Ω in the sense of definition 9.2 such that* $\text{regul}(\mathcal{D}) \le \xi$*. Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10) and $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13)-(9.14).*
Then

$$\begin{array}{rcl} \lim\limits_{\text{thin}(\mathcal{D}) \to 0} \|\mathbf{q} - \mathbf{q}_\mathcal{D}\|_{H(\text{div},\Omega)} &=& 0, \\ \lim\limits_{\text{thin}(\mathcal{D}) \to 0} \|p - p_\mathcal{D}\|_{L^2(\Omega)} &=& 0. \end{array} \tag{9.24}$$

In order to prove Theorem 9.1, some lemmata must be previously shown. The next lemma deals with an interpolation result for regular functions.

Lemma 9.2 (Interpolation of regular functions) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \text{regul}(\mathcal{D})$. Let $\mathbf{q} \in (H^1(\Omega))^d$. Let $\mathbf{y} \in H(\text{div}, \Omega)$ be defined by*

$$\mathbf{y} = \sum_{a \in \mathcal{A}} \frac{1}{m_a} \int_a \mathbf{q}(x) \cdot \mathbf{n}_a(x)\, d\gamma(x)\ \mathbf{w}_a.$$

Then we have $\text{div}\ \mathbf{y} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K \text{div}\ \mathbf{q}(x)\, dx\ \chi_K$ *and there exists $C_1 > 0$ which only depends on d and ξ such that*

$$\|\mathbf{q} - \mathbf{y}\|_{L^2(\Omega)} \leq C_1\, \text{thin}(\mathcal{D}) \|\mathbf{q}\|_{(H^1(\Omega))^d}. \tag{9.25}$$

One can notice then that, when $\text{thin}(\mathcal{D}) \to 0$, the function y such defined tends to q in $H(\text{div}, \Omega)$.

Proof. In the following proof, C_i denotes different positive real values which only depend on ξ and d. The proof of $\text{div}\ \mathbf{y} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K \text{div}\ \mathbf{q}(x)\, dx\ \chi_K$ is straightforward, since $\text{div}\ \mathbf{w}_a = 0$ on K if $K \notin \{K(a), L(a)\}$ and $\text{div}\ \mathbf{w}_a = \varepsilon_{K,a} \frac{m_a}{m_K}$ on K if $K \in \{K(a), L(a)\}$.
Let $K \in \mathcal{M}$. Let us define the function $w \in H^1(K)$ by

$$w = \sum_{a \in \mathcal{A}_K} \left(\frac{1}{m_a} \int_a \mathbf{q}(x) \cdot \mathbf{n}_{\partial K}(x)\, d\gamma(x) \right) w_{K,a},$$

which is such that $\nabla w(x) = \mathbf{y}(x)$ for a.e. $x \in K$. Similarly, denoting $\tilde{\mathbf{q}} = \frac{1}{m_K} \int_K \mathbf{q}(x)\, dx$, we define $\tilde{w} \in H^1(K)$ by

$$\tilde{w} = \sum_{a \in \mathcal{A}_K} \left(\frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x)\, d\gamma(x) \right) w_{K,a}.$$

We get

$$\|\mathbf{q} - \mathbf{y}\|^2_{L^2(K)} \leq 3\|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)} + 3\|\tilde{\mathbf{q}} - \nabla \tilde{w}\|^2_{L^2(K)} + 3\|\nabla \tilde{w} - \nabla w\|^2_{L^2(K)}.$$

Let us first deal with $A = \|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)}$. Thanks to the Cauchy-Schwarz inequality, one has

$$A \leq \frac{1}{m_K} \int_K \int_K |\mathbf{q}(x) - \mathbf{q}(y)|^2\, dx\, dy,$$

which yields, using (9.63) proved in Lemma 9.13,

$$\|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)} \leq C_2\, \delta(K)^2 \|\mathbf{q}\|^2_{(H^1(K))^d}. \tag{9.26}$$

We now turn to the study of $B = \|\tilde{\mathbf{q}} - \nabla \tilde{w}\|^2_{L^2(K)}$. One defines the function $h \in H^2(K)$ by $h(x) = \tilde{\mathbf{q}} \cdot x - \frac{1}{m_K} \int_K (\tilde{\mathbf{q}} \cdot y)\, dy$. This function thus satisfies $\nabla h = \tilde{\mathbf{q}}$ and $\int_K h(x)\, dx = 0$. Since $h - \tilde{w}$ is the variational solution of a Neumann problem on K with null average and $\Delta(h - \tilde{w})$ constant, one gets

$$B = \sum_{a \in \mathcal{A}_K} \int_a (h(x) - \tilde{w}(x)) \left(\tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x) - \frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(y)\, d\gamma(y) \right) d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, we deduce

$$B^2 \le B' \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - h(x))^2 \, d\gamma(x),$$

where

$$B' = \sum_{a\in\mathcal{A}_K} \int_a \left(\tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x) - \frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(y)\, d\gamma(y) \right)^2 d\gamma(x).$$

We use (9.56) proved in Lemma 9.12. It yields $\sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - h(x))^2 \, d\gamma(x) \le C_3\, \delta(K) B$, thus we obtain

$$B \le C_3\, \delta(K) B'. \tag{9.27}$$

We have, by definition of thin$(\mathcal{D})$,

$$\begin{aligned}
\delta(K)B' &\le \delta(K)|\tilde{\mathbf{q}}|^2 \sum_{a\in\mathcal{A}_K} \int_a (\mathbf{n}_a(x) - \overline{\mathbf{n}_a})^2 \, d\gamma(x) \\
&\le \frac{\delta(K)}{m_K} \int_K |\mathbf{q}(x)|^2 \, dx \sum_{a\in\mathcal{A}_K} \text{thin}(\mathcal{D})^2 m_a \\
&\le C_4 \,\text{thin}(\mathcal{D})^2 \int_K |\mathbf{q}(x)|^2 \, dx \times \frac{\delta(K) m_{\partial K}}{m_K}.
\end{aligned}$$

Using $m_K \ge C_5\, \delta(K)^d$ and $m_{\partial K} \le C_6\, \delta(K)^{d-1}$ (hypothesis (ii) of Definition 9.2 and Lemma 9.11), relation (9.27) gives

$$\|\tilde{\mathbf{q}} - \nabla \tilde{w}\|^2_{L^2(K)} \le C_7 \,\text{thin}(\mathcal{D})^2 \|\mathbf{q}\|^2_{L^2(K)}. \tag{9.28}$$

We finaly study the term $C = \|\nabla \tilde{w} - \nabla w\|^2_{L^2(K)}$. We have

$$C = \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x)) \left(\frac{1}{m_a} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y)) \cdot \mathbf{n}_{\partial K}(y)\, d\gamma(y) \right) d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, one has

$$C^2 \le C' \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x))^2 \, d\gamma(x),$$

where

$$C' = \sum_{a\in\mathcal{A}_K} \int_a \left(\frac{1}{m_a} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y)) \cdot \mathbf{n}_{\partial K}(y)\, d\gamma(y) \right)^2 d\gamma(x)$$

Thanks again to (9.56) given by Lemma 9.12, we get $\sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x))^2 \, d\gamma(x) \le C_3\, \delta(K) C$, which leads to

$$C \le C_3\, \delta(K) C'. \tag{9.29}$$

Turning to the study of C', and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} C' &\leq \sum_{a\in\mathcal{A}_K} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y))^2 \, d\gamma(y) = \int_{\partial K} (\tilde{\mathbf{q}} - \mathbf{q}(y))^2 \, d\gamma(y) \\ &\leq \int_{\partial K} \frac{1}{m_K} \int_K (\mathbf{q}(z) - \mathbf{q}(y))^2 \, dz \, d\gamma(y). \end{aligned} \tag{9.30}$$

Thanks again to Lemma 9.12, we get

$$C' \leq C_2 \, \delta(K) \|\mathbf{q}\|^2_{(H^1(K))^d},$$

and therefore, thanks to (9.29) and (9.30), there exists $C_8 > 0$ such that

$$\|\nabla\tilde{w} - \nabla w\|^2_{L^2(K)} \leq C_8 \, \delta(K)^2 \|\mathbf{q}\|^2_{(H^1(K))^d}. \tag{9.31}$$

Summing relations (9.26), (9.28) and (9.31) on $K \in \mathcal{M}$ gives (9.25).

Lemma 9.3 *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $v \in V_{\mathcal{D}}$ and let $h \in H^2(\Omega)$ be the variational solution of $-\Delta h = v$ on Ω, with a homogeneous Neumann boundary condition and $\int_\Omega h(x)\,dx = 0$ (the existence of such a function resulting from the regularity hypotheses on Ω, see [42]). Let us define $\mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}$ by*

$$\mathbf{y} = \sum_{a\in\mathcal{A}} \left(\frac{1}{m_a} \int_a \nabla h(x) \cdot \mathbf{n}_a \, d\gamma(x) dx \right) \mathbf{w}_a. \tag{9.32}$$

Then there exists C_9, only depending on Ω, d and ξ such that $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq C_9 \, \|v\|_{L^2(\Omega)}$.

Proof.
Using $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq \|\mathbf{y}-\nabla h\|_{(L^2(\Omega))^d} + \|\nabla h\|_{(L^2(\Omega))^d}$, one applies Lemma 9.2 for $\mathbf{q} = \nabla h$, since $h \in H^2(\Omega)$ implies $\nabla h \in (H^1(\Omega))^d$. We thus get $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq (C_1 \,\mathrm{thin}(\mathcal{D}) + 1)\|h\|_{H^2(\Omega)}$. By hypothesis (H), one has $\|h\|_{H^2(\Omega)} \leq C_\Omega \|v\|_{L^2(\Omega)}$, which concludes the proof since $\mathrm{thin}(\mathcal{D}) \leq \max(\delta(\Omega), 2)$.

By noticing that the $\mathbf{y}$ defined by (9.32) satisfies div $\mathbf{y} = -v$, this lemma can also be stated in terms of an "inf-sup" condition.

Corollary 9.1 (Discrete "inf-sup" condition) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \mathrm{regul}(\mathcal{D})$. Then there exists $C_9 > 0$, only depending on Ω, d and ξ such that*

$$\inf_{v\in V_{\mathcal{D}}} \sup_{\mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}} \frac{\displaystyle\int_\Omega v(x) \ \mathrm{div}\ \mathbf{y}(x) \, dx}{\|v\|_{L^2(\Omega)} \ \|\mathbf{y}\|_{(L^2(\Omega))^d}} \geq \frac{1}{C_9}.$$

The following lemmata express the classical proof of the convergence of mixed finite element methods under an "inf-sup" condition and an interpolation result (detailed in [14] or [59] for example). We prove them for the sake of completeness, thus verifying that our hypotheses are sufficient to apply this convergence proof.

Lemma 9.4 (Estimate on the discrete approximations) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $h \in L^2(\Omega)$ and $\mathbf{r} \in (L^2(\Omega))^d$ be given.*
Then, there exists one and only one $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},0}$ solution of

$$\int_\Omega \mathrm{div}\ \mathbf{q}_{\mathcal{D}}(x) \ v(x) \, dx = \int_\Omega h(x) \ v(x) \, dx, \ \forall v \in V_{\mathcal{D}}, \tag{9.33}$$

and

$$\int_\Omega \mathbf{y}(x)\cdot \mathbf{\Lambda}^{-1}(x)\mathbf{q}_\mathcal{D}(x)\,dx - \int_\Omega p_\mathcal{D}(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = \int_\Omega \mathbf{r}(x)\cdot\mathbf{y}(x)\,dx,\ \forall \mathbf{y}\in \mathbf{Q}_{\mathcal{D},0}, \tag{9.34}$$

and there exists C_{10}*, only depending on* Ω*,* d*,* ξ*,* λ_1 *and* λ_2 *such that*

$$\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} + \|p_\mathcal{D}\|^2_{L^2(\Omega)} \le C_{10}\,(\|\mathbf{r}\|^2_{(L^2(\Omega))^d} + \|h\|^2_{L^2(\Omega)}). \tag{9.35}$$

Proof. We first remark that proving (9.35) for any solution $(p_\mathcal{D},\mathbf{q}_\mathcal{D})\in V_\mathcal{D}\times\mathbf{Q}_{\mathcal{D},0}$ to (9.33)-(9.34) is sufficient to prove that for a null right hand side, the discrete unknowns are null, and therefore that the linear system is invertible. For the proof of (9.35), one chooses, in (9.34), $\mathbf{y}=\mathbf{q}_\mathcal{D}$, and in (9.33), $v=p_\mathcal{D}$. It leads to

$$\frac{1}{\lambda_2}\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} \le \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d} + \|h\|_{L^2(\Omega)}\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{9.36}$$

One then applies Lemma 9.3, which gives the existence of $\mathbf{y}_0\in\mathbf{Q}_{\mathcal{D},0}$ such that div $\mathbf{y}_0 = p_\mathcal{D}$ a.e. in Ω and

$$\|\mathbf{y}_0\|_{(L^2(\Omega))^d} \le C_9\,\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{9.37}$$

Introducing $\mathbf{y}_0$ in (9.34), one gets

$$\|p_\mathcal{D}\|^2_{L^2(\Omega)} \le \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d},$$

which gives, thanks to (9.37),

$$\|p_\mathcal{D}\|_{L^2(\Omega)} \le C_9\left(\|\mathbf{r}\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\right). \tag{9.38}$$

Thanks to (9.36) and (9.38), one gets (9.35).

Lemma 9.5 (Bound on the approximation error by the interpolation error)
Under Hypotheses (H), let $\xi>0$ *and* $\mathcal{D}$ *be a discretization of* Ω *in the sense of definition 9.2 such that* $\mathrm{regul}(\mathcal{D})\le\xi$*. Let* $(p,\mathbf{q})\in L^2(\Omega)\times H_g(\mathrm{div},\Omega)$ *be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10) and* $(p_\mathcal{D},\mathbf{q}_\mathcal{D})\in V_\mathcal{D}\times\mathbf{Q}_{\mathcal{D},g}$ *be given by (9.13) and (9.14). Let* $\tilde{\mathbf{q}}_\mathcal{D}\in\mathbf{Q}_{\mathcal{D},g}$ *be given and let* $\tilde{p}_\mathcal{D}\in V_\mathcal{D}$ *be defined by* $\tilde{p}_\mathcal{D} = \sum_{K\in\mathcal{M}} \frac{1}{m_K}\int_K p(x)\,dx\ \chi_K$*.*

Then there exists C_{11}*, only depending on* Ω*,* d*,* ξ*,* λ_1 *and* λ_2 *such that*

$$\|\mathbf{q}-\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} + \|p-p_\mathcal{D}\|^2_{L^2(\Omega)} \le C_{11}\,(\|\mathbf{q}-\tilde{\mathbf{q}}_\mathcal{D}\|^2_{H(\mathrm{div},\Omega)} + \|p-\tilde{p}_\mathcal{D}\|^2_{L^2(\Omega)}). \tag{9.39}$$

Proof. One gets, using the variational formulations (9.8)-(9.9) and (9.13)-(9.14):

$$\int_\Omega \mathrm{div}(\mathbf{q}_\mathcal{D}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))v(x)\,dx = \int_\Omega \mathrm{div}(\mathbf{q}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))v(x)\,dx,\ \forall v\in V_\mathcal{D},$$

and

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)(\mathbf{q}_\mathcal{D}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))\,dx - \int_\Omega (p_\mathcal{D}(x)-\tilde{p}_\mathcal{D}(x))\mathrm{div}\ \mathbf{y}(x)\,dx =$$
$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)(\mathbf{q}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))\,dx - \int_\Omega (p(x)-\tilde{p}_\mathcal{D}(x))\mathrm{div}\ \mathbf{y}(x)\,dx,\ \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}.$$

For all $\mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}$, thanks to the definition of $\tilde{p}_{\mathcal{D}}$, one gets $\displaystyle\int_\Omega (p(x) - \tilde{p}_{\mathcal{D}}(x))\text{div } \mathbf{y}(x)\,dx = 0$. Thus $(p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}})$ is the solution of (9.33) and (9.34) with $\mathbf{r} = \mathbf{\Lambda}^{-1}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$ and $h = \text{div}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$. Applying Lemma 9.4 yields

$$\|\mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)} \le C_{10}\left(\frac{1}{\lambda_1}\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|\text{div } \mathbf{q} - \text{div } \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)}\right).$$

Using the Cauchy-Schwarz inequality, this leads to

$$\begin{aligned}\|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p - p_{\mathcal{D}}\|^2_{L^2(\Omega)} \;\; &\le \;\; 2\left(\frac{C_{10}}{\lambda_1} + 1\right)\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + 2C_{10}\,\|\text{div } \mathbf{q} - \text{div } \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)} \\ &\quad + 2\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)},\end{aligned}$$

which gives (9.39).

Proof of Theorem 9.1. We apply Lemma 9.5. On the one hand, thanks again to (9.64) proved in Lemma 9.13, the following inequality holds:

$$\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)} \le C_2\,\text{thin}(\mathcal{D})^2\|\nabla p\|^2_{L^2(\Omega)},$$

(notice that, when $p \in L^2(\Omega)$ satisfies (9.8), we have in fact $p \in H^1(\Omega)$) and therefore $\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}$ tends to 0 as $\text{thin}(\mathcal{D})$ tends to 0. On the other hand, it suffices to prove that one can choose $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ such that $\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|_{H(\text{div},\Omega)}$ is as small as desired. Notice that, in general, the property $\mathbf{q} \in (H^1(\Omega))^d \cap H_g(\text{div}, \Omega)$ is wrong. Therefore, one takes $\mathbf{q}_0 \in (H^1(\Omega))^d$ such that $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega} = g$; then, $\mathbf{q} - \mathbf{q}_0 \in H_0(\text{div}, \Omega)$ and since Hypotheses (H) are sufficient to prove that Ω is locally star-shaped, we can approximate $\mathbf{q} - \mathbf{q}_0$ in $H_0(\text{div}, \Omega)$ by regular functions with compact support in Ω (see [71]); thus, $\mathbf{q}$ can be approximated in $H_g(\text{div}, \Omega)$ by $\tilde{\mathbf{q}} \in (H^1(\Omega))^d \cap H_g(\text{div}, \Omega)$. Then, applying Lemma 9.2, one can approximate $\tilde{\mathbf{q}}$ by $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ as close as demanded by letting $\text{thin}(\mathcal{D})$ tend to zero.

9.4 The convergence of the finite volume method

We now show the following theorem.

Theorem 9.2 (Convergence of the finite volume scheme) *Under Hypotheses (H), let ξ and $\alpha \in (0, 1)$ be fixed positive real values. Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of definition 9.2 such that for all $m \in \mathbb{N}$, $\text{regul}(\mathcal{D}_m) \le \xi$ and $\lim_{m\to+\infty} \text{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote $(p_m, \mathbf{q}_m)$ the solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ given by (9.13) and (9.14) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\delta t_m > 0$, denoted δt, such that the condition*

$$\delta t \le (1 - \alpha) \inf_{K\in\mathcal{M}} \frac{m_K}{\displaystyle\sum_{a\in\mathcal{A}_K} m_a(q_a\varepsilon_{K,a})^+ + f_K^-}, \tag{9.40}$$

holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\delta t}$ defined by (9.18)-(9.23).

*Then there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, still denoted $(u_m)_{m\in\mathbb{N}}$, which converges for the weak * topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+)$ solution of (9.7).*

If we add some hypotheses giving that $\mathbf{q}$ is Lipschitz continuous on $\overline{\Omega}$ (for example, $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2) then
- the function u is unique
- the whole sequence $(u_m)_{m\in\mathbb{N}}$ converges to u in $L^p(\Omega\times]0, T[)$ for all $p \in [1, \infty)$ and all $T > 0$.

The proof of Theorem (9.2) is classical, and has been developped for various choices of the discretization of the velocity field $\mathbf{q}$ (see [16], [34] and [73]). The originality of this proof is the use of the technical lemma 9.14, which is non standard.

9.4.1 L^∞ estimate

The purpose of this section is to prove the following result.

Lemma 9.6 (L^∞ stability of the finite volume scheme) *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ an admissible discretization in the sense of definition 9.2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\Delta t > 0$ such that*

$$\Delta t \leq \inf_{K\in\mathcal{M}} \frac{m_K}{\sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^+ + f_K^-}. \tag{9.41}$$

Then the approximate solution $u_{\mathcal{D},\Delta t}$ given by (9.18)-(9.23) is such that

$$\|u_{\mathcal{D},\Delta t}\|_{L^\infty(\Omega\times\mathbb{R}^+)} \leq \max(\|u_0\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\partial\Omega^-\times\mathbb{R}^+)}, \|s\|_{L^\infty(\Omega\times\mathbb{R}^+)}). \tag{9.42}$$

Proof. According to the scheme (9.21), we have

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} u_a^n F_{K,a} + u_K^n f_K^- - s_K^n f_K^+\right),$$

which gives

$$u_K^{n+1} = u_K^n \left(1 - \frac{\Delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^+ + f_K^-\right)\right) + \frac{\Delta t}{m_K}\sum_{a\in\mathcal{A}_K} F_{K,a}^- u_a^n + \frac{\Delta t}{m_K} f_K^+ s_K^n, \tag{9.43}$$

Thanks to the stability condition (9.41), equation (9.43) expresses u_K^{n+1} as a convex combination of the values u_K^n, $\overline{u}_a^n$, s_K^n. An easy proof by induction concludes the proof of the lemma.

Remark 9.6 *If the data are regular enough, the term $\sum_{a\in\mathcal{A}_K} m_a|q_a|$ behaves like size$(\mathcal{D})^{d-1}$ as size$(\mathcal{D})$ tends to 0, and the condition (9.41) takes the form $\Delta t \leq C$size$(\mathcal{D})$ (where size$(\mathcal{D}) = \max_{K\in\mathcal{M}} \delta(K)$).*

9.4.2 A weak inequality on the spatial variations

Lemma 9.7 (Weak spatial variations inequality) *Under hypotheses (H), let $\xi > 0$, $\alpha \in (0,1)$, $T > 0$, and let $\mathcal{D}$ be an admissible discretization in the sense of definition 9.2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\Delta t > 0$ such that the condition (9.40) holds. Let N_T be such that $N_T\Delta t \leq T < (N_T+1)\Delta t$ and let $(u_K^n)_{K\in\mathcal{M},\ n\in\mathbb{N}}$, $(u_a^n)_{a\in\mathcal{A},\ n\in\mathbb{N}}$ be defined by (9.18)-(9.22).*

Then there exists C_{12}, which only depends on d, Ω, T, ξ, α, f, s, g, $\overline{u}$ and u_0 (but not on $\mathcal{D}$ or Δt), such that

$$\sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \left(\sum_{a\in\mathcal{A}_K} m_a (q_a\varepsilon_{K,a})^- (u_a^n - u_K^n)^2\right) \leq C_{12}\,. \tag{9.44}$$

Remark 9.7 *In references [16], [34] and [73], a weak BV-estimate is obtained from (9.44). We do not do so here, since in the convergence proof, the use of Lemma 9.14 takes advantage of a local bound of the diameter of each control volume. Otherwise, we should assume the existence of some $\beta > 0$ with*

$$\delta(K) \;\geq\; \beta \; size(\mathcal{D}), \;\; \forall K \in \mathcal{M}.$$

Proof. First, the discrete elliptic scheme (9.15) is used to get

$$\sum_{a\in\mathcal{A}_K} F^+_{K,a} + f^-_K = \sum_{a\in\mathcal{A}_K} F^-_{K,a} + f^+_K, \tag{9.45}$$

and therefore the scheme (9.21) also writes

$$m_K(u_K^{n+1} - u_K^n) + \Delta t \left(\sum_{a\in\mathcal{A}_K} F^-_{K,a}(u_K^n - u_a^n) + f^+_K(u_K^n - s_K^n) \right) = 0, \; \forall K \in \mathcal{M}, \; \forall n \in \mathbb{N}. \tag{9.46}$$

For all $n \in \mathbb{N}$ and $K \in \mathcal{M}$, let us multiply the equation (9.46) by u_K^n and sum the result on $K \in \mathcal{M}$ and $n = 0, \ldots, N_T$. It gives $T_1 + T_2 = 0$ with

$$T_1 = \sum_{n=0}^{N_T} \sum_{K\in\mathcal{M}} m_K(u_K^{n+1} - u_K^n) u_K^n$$

and

$$T_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \left(\sum_{a\in\mathcal{A}_K} F^-_{K,a}(u_K^n - u_a^n) u_K^n + f^+_K(u_K^n - s_K^n) u_K^n \right).$$

Writing $u_K^{n+1} u_K^n = -\frac{1}{2}(u_K^{n+1} - u_K^n)^2 + \frac{1}{2}(u_K^{n+1})^2 + \frac{1}{2}(u_K^n)^2$, one gets

$$T_1 = T_{11} + T_{12},$$

where

$$T_{11} = -\frac{1}{2} \sum_{n=0}^{N_T} \sum_{K\in\mathcal{M}} m_K(u_K^{n+1} - u_K^n)^2.$$

and

$$T_{12} = \frac{1}{2} \left(\sum_{K\in\mathcal{M}} m_K((u_K^{N_T+1})^2 - (u_K^0)^2) \right).$$

Using (9.46) and the Cauchy-Schwarz inequality gives

$$m_K^2(u_K^{n+1} - u_K^n)^2 \leq \Delta t \left(\sum_{a\in\mathcal{A}_K} F^-_{K,a} + f^+_K \right) \left(\Delta t \sum_{a\in\mathcal{A}_K} F^-_{K,a}(u_a^n - u_K^n)^2 + f^+_K(s_K^n - u_K^n)^2 \right),$$
$$\forall K \in \mathcal{M}, \; \forall n \in \mathbb{N}.$$

Using condition (9.40) and equation (9.45), one gets

$$m_K(u_K^{n+1}-u_K^n)^2 \le (1-\alpha)\left(\Delta t \sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n-u_K^n)^2 + f_K^+(s_K^n-u_K^n)^2\right), \quad \forall K\in\mathcal{M},\ \forall n\in\mathbb{N}. \tag{9.47}$$

One computes T_2. One gets $T_2 = T_{21}+T_{22}$ with

$$T_{21} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n-u_K^n)^2 + f_K^+(s_K^n-u_K^n)^2\right)$$

and

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-((u_K^n)^2-(u_a^n)^2) + f_K^+((u_K^n)^2-(s_K^n)^2)\right).$$

We get thus, thanks to (9.47),

$$T_{11}+T_{21}\ge \alpha T_{21}.$$

Thanks to (9.45), the term T_{22} can be rewritten as

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}(u_a^n)^2 + f_K^-(u_K^n)^2 - f_K^+(s_K^n)^2\right).$$

Thus, gathering by edges, one gets

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\left(\sum_{a\in\mathcal{A}_e} m_a g_a (u_a^n)^2 + \sum_{K\in\mathcal{M}}(f_K^-(u_K^n)^2 - f_K^+(s_K^n)^2)\right).$$

Since terms T_{12} and T_{22} can easily be bounded, using Lemma 9.6 (since condition (9.41) is weaker than (9.40)), we thus get (9.44).

9.4.3 The proof of the convergence theorem 9.2

We first notice that Lemma 9.6 gives the existence of a subsequence u_m and of a function $u\in L^\infty(\Omega\times\mathbb{R}^+)$ such that u_m converges to u for the weak * topology of $L^\infty(\Omega\times\mathbb{R}^+)$ as $m\to+\infty$. Recall that we have proved above (Theorem 9.1) that $\mathbf{q}_m$ tends to $\mathbf{q}$ in $H(\mathrm{div},\Omega)$ as $m\to+\infty$. This section is devoted to the proof that u satisfies (9.7) (the uniqueness part of the proof being studied in the next section).

Let $\phi\in C_c^1(\mathbb{R}^d\times\mathbb{R})$ be such that $\phi=0$ on $\partial\Omega\backslash\partial\Omega^-\times\mathbb{R}^+$. Let $T>0$ be such that

$$\phi = 0 \quad \text{on} \quad \mathbb{R}^d\times[T,+\infty[. \tag{9.48}$$

In this proof, we denote by C_i various positive real values which only depend on d, Ω, ϕ, T, ξ, α, s, f, g, $\overline{u}$, u_0 and not on $\mathcal{D}$ or Δt.

In the following, we use the notations $\mathcal{D}=\mathcal{D}_m$ and $\Delta t=\Delta t_m$. Let us denote N_T the integer value such that $N_T\Delta t\le T<(N_T+1)\Delta t$. Setting

$$\phi_K^n = \frac{1}{\Delta t\ m_K}\int_K\int_{n\Delta t}^{(n+1)\Delta t}\phi(x,t)\,dx\,dt,\ \forall K\in\mathcal{M},\ \forall n\in\mathbb{N},$$

one multiplies the equality (9.46) by ϕ_K^n and sum on $K\in\mathcal{M}$ and $n\in\mathbb{N}$. One obtains $E_1+E_2=0$ with

$$E_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{M}} m_K (u_K^{n+1} - u_K^n) \phi_K^n,$$

and

$$E_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{M}} \left(\sum_{a \in \mathcal{A}_K} F_{K,a}^- (u_K^n - u_a^n) \phi_K^n + f_K^+ (u_K^n - s_K^n) \phi_K^n \right).$$

We also define

$$\phi_a^n = \frac{1}{\Delta t \ m_a} \int_a \int_{n\Delta t}^{(n+1)\Delta t} \phi(x,t) \, d\gamma(x) \, dt.$$

Let us study E_1. Thanks to (9.48), for all $K \in \mathcal{M}$, $\phi_K^{N_T+1} = 0$ holds and therefore

$$E_1 = \sum_{n=1}^{N_T+1} \sum_{K \in \mathcal{M}} m_K u_K^n (\phi_K^{n-1} - \phi_K^n) - \sum_{K \in \mathcal{M}} m_K u_K^0 \phi_K^0.$$

Using the weak * convergence of $(u_m)_{m \in \mathbb{N}}$ to u, we deduce the convergence of E_1 to

$$-\int_{\Omega \times \mathbb{R}^+} u(x,t) \frac{\partial \phi}{\partial t}(x,t) \, dx \, dt - \int_\Omega u_0(x) \phi(x,0) \, dx.$$

Next we consider the term E_2. It can be written, using (9.22) and gathering by edges, as

$$\begin{aligned} E_2 \quad = \quad & \sum_{n=0}^{N_T} \Delta t \sum_{a \in \mathcal{A}_i} m_a q_a \phi_{K_d(a)}^n (u_{L(a)}^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{a \in \mathcal{A}_e} m_a q_a \phi_{K_d(a)}^n (u_a^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{M}} f_K^+ \phi_K^n (u_K^n - s_K^n), \end{aligned} \tag{9.49}$$

where we define, for all $a \in \mathcal{A}_i$, $K_d(a)$ (the "downstream" control volume) by $K_d(a) = K(a)$ if $q_a \leq 0$, else $K_d(a) = L(a)$, and for all $a \in \mathcal{A}_e$, $K_d(a) = K(a)$. We set

$$\begin{aligned} f_\mathcal{D}(x) = \frac{1}{m_K} f_K, &\quad \text{for a.e.} \quad x \in K, \ \forall K \in \mathcal{M}, \\ s_{\mathcal{D},\Delta t}(x,t) = s_K^n, &\quad \text{for a.e.} \quad (x,t) \in K \times [n\Delta t, (n+1)\Delta t), \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}, \\ \overline{u}_{\mathcal{D},\Delta t}(x,t) = \overline{u}_a^n, &\quad \text{for a.e.} \quad (x,t) \in a \times [n\Delta t, (n+1)\Delta t), \ \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}, \\ g_\mathcal{D}(x) = g_a, &\quad \text{for a.e.} \quad x \in a, \ \forall a \in \mathcal{A}_e, \end{aligned}$$

where f_K, g_a, s_K^n and $\overline{u}_a^n$ are respectively defined by (9.16), (9.17), (9.18) and (9.19). We define E_3 by

$$\begin{aligned} E_3 \quad = \quad & -\int_{\Omega \times \mathbb{R}^+} u_{\mathcal{D},\Delta t}(x,t) \mathbf{q}_\mathcal{D}(x) \cdot \nabla \phi(x,t) \, dx \, dt \\ & + \int_{\partial\Omega \times \mathbb{R}^+} \overline{u}_{\mathcal{D},\Delta t}(x,t) g_\mathcal{D}(x) \phi(x,t) \, d\gamma(x) \, dt \\ & + \int_{\Omega \times \mathbb{R}^+} (u_{\mathcal{D},\Delta t}(x,t) f_\mathcal{D}^-(x) - s_{\mathcal{D},\Delta t}(x,t) f_\mathcal{D}^+(x)) \phi(x,t) \, dx \, dt \end{aligned}$$

Since $u_{\mathcal{D},\Delta t}$ converges to u for the weak * topology of $L^\infty(\Omega \times \mathbb{R}^+)$ and since $\mathbf{q}_\mathcal{D}$ converges strongly to $\mathbf{q}$ in $L^2(\Omega)$ as $m \to +\infty$, in view of the definitions of $\overline{u}_{\mathcal{D},\Delta t}$ and $g_\mathcal{D}$, we deduce the convergence of E_3 to $-\int_{\Omega\times\mathbb{R}^+} u(x,t)\mathbf{q}(x)\cdot\nabla\phi(x,t)\,dx\,dt + \int_{\partial\Omega^-\times\mathbb{R}^+} \overline{u}(x,t)g(x)\phi(x,t)\,d\gamma(x)\,dt + \int_{\Omega\times\mathbb{R}^+}(u(x,t)f^-(x) - s(x,t)f^+(x))\phi(x,t)\,dx\,dt$ as $m \to +\infty$.
Using (9.15) and the definition of $\mathbf{w}_{K,a}$, one can rewrite E_3 as

$$\begin{aligned} E_3 &= \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_i} m_a q_a \phi_a^n (u^n_{L(a)} - u^n_{K(a)}) \\ &\quad + \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a q_a \phi_a^n (\overline{u}^n_a - u^n_{K(a)}) \\ &\quad + \sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}} (u^n_K - s^n_K) f^+_K \phi^n_K. \end{aligned} \tag{9.50}$$

From (9.49) and (9.50), we can deduce that

$$|E_3 - E_2| \le E_4 + E_5,$$

with

$$E_4 = \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a |q_a||\phi^n_a||\overline{u}^n_a - u^n_a|$$

and

$$E_5 = \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_i} m_a|q_a|\;|\phi^n_a - \phi^n_{K_d(a)}|\;|u^n_{K(a)} - u^n_{L(a)}| + \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a|q_a|\;|\phi^n_a - \phi^n_{K_d(a)}|\;|u^n_a - u^n_{K(a)}|.$$

Let us first study E_4. Since, for all $a \in \mathcal{A}_e$, relation (9.22) implies $u^n_a = \overline{u}^n_a$ when $q_a \le 0$, we can write

$$E_4 = \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e\,,\,q_a>0} m_a|q_a||\phi^n_a||\overline{u}^n_a - u^n_a|.$$

For all $a \in \mathcal{A}_e$ such that $q_a = m_a^{-1}\int_a g(x)\,d\gamma(x) > 0$, one has $\partial\Omega^+ \cap a \ne \emptyset$ (recall that $\partial\Omega^+ = \{x \in \partial\Omega \mid g(x) > 0\}$); thus, since $\phi = 0$ on $\partial\Omega^+ \times \mathbb{R}^+$, there exists $x \in a$ such that $\phi(x,t) = 0$ for all $t \ge 0$. By denoting C_{13} the Lipschitz constant of ϕ, one has then $|\phi(y,t)| \le C_{13}\,\delta(a)$ for all $y \in a$ and $t \ge 0$, which implies $|\phi^n_a| \le C_{13}\,\delta(a)$. Using (9.42), one gets then

$$\begin{aligned} E_4 &\le C_{14}\,\mathrm{thin}(\mathcal{D})\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a|q_a| \\ &\le C_{14}\,\mathrm{thin}(\mathcal{D})(T+\Delta t)\sum_{a\in\mathcal{A}_e}\int_a |g(x)|\,d\gamma(x) \\ &= C_{14}\,\mathrm{thin}(\mathcal{D})(T+\Delta t)\int_{\partial\Omega}|g(x)|\,d\gamma(x), \end{aligned}$$

which shows that E_4 tends to 0 as $m \to +\infty$.
We turn now to the study of E_5. Thanks to the Cauchy-Schwarz inequality, we obtain

$$E_5^2 \le C_{13}{}^2 \left(\sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}} m_a |q_a| \delta(K_d(a))^2\right) \left(\sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^- (u_a^n - u_K^n)^2\right).$$

This gives, using Lemma 9.7 and the Cauchy-Schwarz inequality,

$$E_5^2 \le C_{15}\, \mathrm{thin}(\mathcal{D}) \left(\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a))\right)^{1/2} \left(\sum_{a\in\mathcal{A}} m_a \delta(K_d(a))\right)^{1/2}.$$

One can then apply Lemma 9.14, which yields

$$\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a)) \le C_{16} \sum_{a\in\mathcal{A}} \left(\int_{K_d(a)} \mathbf{q}_\mathcal{D}^2(x)\,dx + \delta(K_d(a))^2 \int_{K_d(a)} (\mathrm{div}\mathbf{q}_\mathcal{D}(x))^2\,dx\right).$$

Under Hypotheses (H) (and in particular item (vii)), one gets that $\mathrm{card}\mathcal{A}_K \le C_{17}$. Therefore, since $\mathbf{q}_\mathcal{D}$ converges to $\mathbf{q}$ in $H(\mathrm{div},\Omega)$, it is bounded and

$$\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a)) \le C_{18}.$$

Using item (ii) of Hypotheses (H), one gets

$$\sum_{a\in\mathcal{A}} m_a \delta(K_d(a)) \le C_{19}.$$

Therefore, one can conclude

$$E_5 \le C_{20} \sqrt{\mathrm{thin}(\mathcal{D})},$$

which shows that E_2 tends to $-\int_{\Omega\times\mathbb{R}^+} u(x,t)\mathbf{q}(x)\cdot\nabla\phi(x,t)\,dx\,dt + \int_{\partial\Omega^-\times\mathbb{R}^+} \overline{u}(x,t)g(x)\phi(x,t)\,d\gamma(x)\,dt + \int_{\Omega\times\mathbb{R}^+} (u(x,t)f^-(x) - s(x,t)f^+(x))\phi(x,t)\,dx\,dt$ as $m\to+\infty$, and concludes the proof of Theorem 9.2.

9.5 Uniqueness of the weak solution under regularity on the data

We do not handle in details this part, since it does not involve the particular discrete frame we have developed in this paper. Some details can be found in [35], [15], [37] for example. We first state the following discrete result.

Lemma 9.8 *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ be an admissible discretization in the sense of definition 9.2 such that $\xi \ge \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\Delta t > 0$ such that the CFL condition (9.41) holds.*

Then the approximate solution $u_{\mathcal{D},\Delta t}$ given by (9.18)-(9.23) is such that

$$m_K(\eta(u_K^{n+1}) - \eta(u_K^n)) + \Delta t \left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-(\eta(u_K^n) - \eta(u_a^n)) + f_K^+ \eta'(u_K^n)(u_K^n - s_K^n)\right) \le 0,$$
$$\forall K\in\mathcal{M},\ \forall n\in\mathbb{N},\ \forall \eta\in C^1(\mathbb{R},\mathbb{R}) \text{ with } \eta'' \ge 0.$$

The proof of this lemma is easy, starting from the discrete relation (9.46) and multiplying it by $\eta'(u_K^n)$. From this lemma, one gets, letting $\text{thin}(\mathcal{D}) \to 0$, the following result, which proves the convergence of the scheme to a solution of the hyperbolic problem in a very weak sense ([36], [28]).

Lemma 9.9 (Convergence of the finite volume scheme to an entropy process solution) *Under Hypotheses (H), let $\xi > 0$ and $\alpha \in (0,1)$ be fixed real values. Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of definition 9.2 such that for all $m \in \mathbb{N}$, $\text{regul}(\mathcal{D}_m) \le \xi$ and $\lim_{m\to+\infty} \text{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote by $(p_m, \mathbf{q}_m)$ the solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ given by (9.13) and (9.14) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\Delta t_m > 0$, denoted Δt, such that the CFL condition (9.40) holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\Delta t}$ defined by (9.18)-(9.23).*

*Then there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted $(u_m)_{m\in\mathbb{N}}$, which converges for the nonlinear weak * topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$, solution of*

$$\int_{\mathbb{R}^+}\int_\Omega\int_0^1 \left(\begin{array}{l} \eta(u(x,t,\alpha))\dfrac{\partial \phi}{\partial t}(x,t) + \eta(u(x,t,\alpha))\text{div}(\phi(x,t)\mathbf{q}(x)) + \\ \eta'(u(x,t,\alpha))\phi(x,t)f^+(x)(s(x,t) - u(x,t,\alpha)) \end{array} \right) d\alpha\, dx\, dt$$
$$+\int_\Omega \eta(u_0(x))\ \phi(x,0)\, dx - \int_{\mathbb{R}^+}\int_{\partial\Omega^-} \eta(\overline{u}(x,t))\ \phi(x,t) g(x)\, d\gamma(x)\, dt \ge 0, \tag{9.51}$$

$\forall \phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^+)$ *such that* $\phi = 0$ *on* $\partial\Omega^+ \times \mathbb{R}^+ = (\partial\Omega \setminus \partial\Omega^-) \times \mathbb{R}^+$,
$\forall \eta \in C^1(\mathbb{R}, \mathbb{R})$ *with* $\eta'' \ge 0$.

The proof of the above lemma is fully similar to the one which is given in section 9.4.3. Using the classical "doubling variable technique" and Krushkov entropies [49] lead to a result of uniqueness, under sufficient hypotheses on the data giving that $\mathbf{q}$ is Lipschitz-continuous (see [61] or [74] for the particular problem of handling the boundary conditions).

Lemma 9.10 (Uniqueness of the entropy process solution) *Under Hypotheses (H), and the additional hypotheses $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2 (for example), let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10).*

Then $\mathbf{q}$ is Lipschitz-continuous in $\overline{\Omega}$, there exists one and only one function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$, solution of (9.51), and there exists one and only one $\tilde{u} \in L^\infty(\Omega \times \mathbb{R}^+)$, solution of (9.7), such that, for a.e. $(x,t,\alpha) \in \Omega \times \mathbb{R}^+ \times (0,1)$, $u(x,t,\alpha) = \tilde{u}(x,t)$.

This result of uniqueness yields the convergence in $L^p(\Omega\times]0,T[)$, for all $p \in [1,\infty)$ and $T > 0$, of $(u_m)_{m\in\mathbb{N}}$ to the unique solution $\tilde{u}$ of the problem.

9.6 Appendix : technical lemmata

Lemma 9.11 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism ϕ from $Q_{\delta(K)} =]-\delta(K), \delta(K)[^d$ to K with Lipschitz-continuous inverse mapping; we denote by ξ an upper bound of the Lipschitz constants of ϕ and ϕ^{-1}. Then there exists $C_{21} > 0$ only depending on ξ and d such that, for all $f \in L^1(\partial K)$, $f \ge 0$,*

$${C_{21}}^{-1} \int_{\partial Q_{\delta(K)}} f \circ \phi(x)\, d\gamma(x) \le \int_{\partial K} f(x)\, d\gamma(x) \le C_{21} \int_{\partial Q_{\delta(K)}} f \circ \phi(x)\, d\gamma(x). \tag{9.52}$$

Notice that a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between two open sets has a unique extension as a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the closures of the open sets, and that this extension defines a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the boundaries of the open sets.

Remark 9.8 *The most useful inequality (and the easiest to obtain) in the following will be the second one of (9.52). We have also stated the first one in order that (9.52) allows to see that, when A is a measurable subset of ∂K, $\gamma(A)$ and $\gamma(\phi^{-1}(A))$ ([7]) are comparable, with constants only depending on an upper bound on the Lipschitz constants of ϕ and ϕ^{-1}.*

Proof.
We denote $\delta = \delta(K)$.
It is well known (see e.g. [31]) that the application

$$f \in L^1(\partial K) \to f \circ \phi \in L^1(\partial Q_\delta) \tag{9.53}$$

is an isomorphism; we want here to estimate the norm of this application (and of its inverse mapping) only in terms of ϕ and ϕ^{-1} (with bounds not depending on δ).
Let us first recall the definition of the integral on ∂K when K is an open set with weakly Lipschitz-continuous boundary: if V is an open set of $\mathbb{R}^d$ and $\tau :]-1,1[^{d-1} \to \partial K \cap V$ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping, then for $f \in L^1(\partial K)$, one has

$$\int_{\partial K \cap V} f(x)\, d\gamma(x) = \int_{]-1,1[^{d-1}} f \circ \tau(x) |\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau|(x)\, dx,$$

where $\partial_i \tau$ denotes the i-th partial derivative of τ (which is, by the Rademacher Theorem, a function in $(L^\infty(]-1,1[^{d-1}))^d$ and is essentially bounded by $\mathrm{lip}(\tau)$) and $\wedge$ is the inner product in $\mathbb{R}^d$.

With this definition, one can verify that the $(d-1)$-dimensional measure on ∂Q_δ is the $(d-1)$-Lebesgue measure on each piece of hyperplane the union of which is ∂Q_δ. One can also notice that $\partial Q_\delta = A \sqcup \sqcup_{i=1}^d (]-\delta,\delta[^{i-1} \times \{-\delta\} \times]-\delta,\delta[^{d-i} \sqcup]-\delta,\delta[^{i-1} \times \{\delta\} \times]-\delta,\delta[^{d-i})$ where $\gamma(A) = 0$ (A is made of sets of dimension $d-2$).
Since (9.53) is an isomorphism, the sets of null measure on ∂Q_δ are transported by ϕ on sets of null measure on ∂K. Thus, by denoting $H_{i,\pm} =]-\delta,\delta[^{i-1} \times \{\pm\delta\} \times]-\delta,\delta[^{d-i}$, one has, up to a set of null measure, $\partial K = \sqcup_{i=1}^d (\phi(H_{i,+}) \sqcup \phi(H_{i,-}))$. If $f \in L^1(\partial K)$, $f \geq 0$, the integral of f on ∂K can thus be estimated if we estimate all the integrals of f on $\phi(H_{i,\pm})$.
Let us do it for $H_{1,+}$, the other terms being studied the same way.

Define $\tau :]-1,1[^{d-1} \to \partial K \cap \phi(H_{1,+})$ by $\tau(x) = \phi(\delta, \delta x)$. τ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping; thus, by definition of the integral on ∂K,

$$\begin{aligned} \int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) &= \int_{]-1,1[^{d-1}} f \circ \tau(x) |\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau|(x)\, dx \qquad (9.54)\\ &= \int_{]-1,1[^{d-1}} f \circ \phi(\delta, \delta x) \delta^{d-1} \left| \frac{\partial \phi}{\partial y_2}(\delta, \delta x) \wedge \cdots \wedge \frac{\partial \phi}{\partial y_d}(\delta, \delta x) \right| (x)\, dx. \end{aligned}$$

Thus, by a change of variable,

$$\int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) = \int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y) \left| \frac{\partial \phi}{\partial y_2}(\delta, y) \wedge \cdots \wedge \frac{\partial \phi}{\partial y_d}(\delta, y) \right| (y)\, dy.$$

[7] Recall that γ denotes the $(d-1)$-dimensional measure on the boundary of any open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary.

Since ϕ is Lipschitz-continuous, we have, for all $i \in [2,d]$, $||\frac{\partial \phi}{\partial y_i}||_{L^\infty(H_{1,+})} \leq \mathrm{lip}(\phi)$ and there exists thus C_{22} only depending on ξ and d such that

$$\int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) \leq C_{22} \int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y)\, dy.$$

But, as we previously noticed, the $(d-1)$-dimensional measure on $H_{1,+}$ is the $(d-1)$-Lebesgue measure on this piece of hyperplane, and thus

$$\int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y)\, dy = \int_{H_{1,+}} f \circ \phi(x)\, d\gamma(x),$$

which proves the second inequality of (9.52).

The proof of the first inequality of (9.52) relies on a lemma (mainly algebraic) stating that there exists C_{23} only depending on d such that

$$|\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau| \geq C_{23}\, (\mathrm{lip}(\tau^{-1}))^{-(d-1)} \tag{9.55}$$

(see [31]). Since $\tau^{-1}(z) = \delta^{-1}((\phi^{-1}(z))_2, \ldots, (\phi^{-1}(z))_d)$, one has $\mathrm{lip}(\tau^{-1}) \leq \xi\delta^{-1}$; using this in (9.55) and returning to (9.54) we get, thanks again to a change of variable, the first inequality of (9.52).

Lemma 9.12 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote by m_K the measure of K. One assumes that there exists a Lipschitz-continuous homeomorphism $\mathcal{L}$ from K to $B(0,\delta(K))$ with Lipschitz-continuous inverse mapping. Let ξ be a real value greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g \in H^1(K)$. The trace of g on ∂K is still denoted by g.*
Then there exists $C_3 > 0$, only depending on ξ and d, such that

$$\frac{1}{m_K} \int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) \leq C_3\, \delta(K) \int_K (\nabla g(x))^2\, dx,$$

Thus, if $\int_K g(x)\, dx = 0$ holds, one has

$$\int_{\partial K} g(x)^2\, d\gamma(x) \leq C_3\, \delta(K) \int_K (\nabla g(x))^2\, dx. \tag{9.56}$$

Proof. In the following proof, C_i denotes real values which only depend on d and ξ; δ denotes $\delta(K)$.
The application $F : x \to (|x| / \sup_{i\in[1,d]} |x_i|)x$ is a Lipschitz-continuous homeomorphism with Lipschitz continuous inverse mapping between $B(0,\delta)$ and $Q =]-\delta,\delta[^d$; moreover, the Lipschitz constants of F and F^{-1} only depend on d. Thus, there exists a Lipschitz-continuous homeomorphism ϕ from Q to K, with Lipschitz continuous inverse mapping, such that the Lipschitz constants of ϕ and ϕ^{-1} are bounded by C_{24} only depending on d and ξ.
According to Lemma 9.11, there exists C_{25} only depending on d and ξ such that

$$\begin{aligned} \int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) &\leq C_{25} \int_{\partial Q} \int_K (g(\phi(y')) - g(x))^2\, dx\, d\gamma(y') \\ &= C_{25} \int_{\partial Q} \int_Q (g(\phi(y')) - g(\phi(x')))^2 J_{\phi,d}(x')\, dx'\, d\gamma(y'), \end{aligned}$$

where $J_{\phi,d}(x')$ is the absolute value of the jacobian in the change of variable ϕ. Setting $h = g \circ \phi$, one has $h \in H^1(Q)$. Then one gets the existence of $C_{26} > 0$, only depending on d and ξ, such that

$$\int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) \leq C_{26} \int_{\partial Q} \int_Q (h(y) - h(x))^2\, dx\, d\gamma(y).$$

The change of variable $x = \phi^{-1}(x')$ proves the existence of $C_{27} > 0$, only depending on d and ξ such that

$$\int_Q (\nabla h(x))^2 \, dx \le C_{27} \int_K (\nabla g(x'))^2 \, dx'. \tag{9.57}$$

Therefore, if one proves the existence of $C_{28} > 0$, only depending on d and ξ, such that

$$\int_{\partial Q} \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \le C_{28} \, \delta^{d+1} \int_Q (\nabla h(x))^2 \, dx, \tag{9.58}$$

one gets (9.12) from (9.57) and (9.58) and the fact that the existence of $\mathcal{L}$ ensures that there exists $C_5 > 0$ with $m_K \ge C_5 \, \delta^d$.
In order to prove (9.58), one may assume by a classical argument of density that $h \in C^1(\overline{Q})$. Since Q is a cube with $2d$ edges, it suffices to prove the existence of $C_{29} > 0$, only depending on d and ξ, such that

$$\int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \le C_{29} \, \delta^{d+1} \int_Q (\nabla h(x))^2 \, dx, \tag{9.59}$$

where $\sigma = \{-\delta\} \times [-\delta, \delta]^{d-1}$, to get (9.58) with $C_{28} = 2dC_{29}$. Let $H = [-\delta, \delta]^{d-1}$ and $Q^+ = [0, \delta] \times H$. We can now write, for all $z \in Q^+$,

$$\int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \le 2 \int_\sigma \int_Q (h(y) - h(z))^2 \, dx \, d\gamma(y) + 2 \int_\sigma \int_Q (h(z) - h(x))^2 \, dx \, d\gamma(y).$$

An integration with respect to $z \in Q^+$ leads to

$$2^{d-1}\delta^d \int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \le 2(2\delta)^d A + 2(2\delta)^{d-1} B, \tag{9.60}$$

with

$$A = \int_\sigma \int_{Q^+} (h(y) - h(z))^2 \, dz \, d\gamma(y),$$

and

$$B = \int_{Q^+} \int_Q (h(z) - h(x))^2 \, dx \, dz.$$

Let us first study A. By definition,

$$A = \int_H \int_H \int_0^\delta (h((-\delta, y)) - h((a, b)))^2 \, da \, db \, dy,$$

and therefore,

$$A = \int_H \int_H \int_0^\delta \left(\int_0^1 \nabla h((-\delta + \theta(a + \delta), y + \theta(b - y))) \cdot (a + \delta, b - y) \, d\theta \right)^2 \, da \, db \, dy.$$

Using the Cauchy-Schwarz inequality, one gets

$$A \le (2\delta)^2 d \int_H \int_H \int_0^\delta \int_0^1 (\nabla h((-\delta + \theta(a + \delta), y + \theta(b - y))))^2 \, d\theta \, da \, db \, dy.$$

Using the Fubini Theorem and the two changes of variable $b \to b' = b - y \in H_2 = [-2\delta, 2\delta]^{d-1}$, $y \to y' = y + \theta b' \in H$, we obtain then

$$A \le (2\delta)^2 d \int_{H_2} \int_0^\delta \int_0^1 \int_H \left(\nabla h((-\delta + \theta(a+\delta), y'))\right)^2 \, dy' \, d\theta \, da \, db'.$$

We now change the variable θ into $t = -\delta + \theta(a+\delta)$. This yields:

$$A \le (2\delta)^2 (4\delta)^{d-1} d \int_0^\delta \int_{-\delta}^a \int_H \left(\nabla h((t, y'))\right)^2 \frac{1}{a+\delta} \, dy' \, dt \, da.$$

Since, for all $a \in [0, \delta]$, $\frac{1}{a+\delta} \le \frac{1}{\delta}$, one gets, setting $x = (t, y)$,

$$A \le 2^{2d} \delta^{d+1} d \int_Q \left(\nabla h(x)\right)^2 \, dx. \tag{9.61}$$

Let us now study B. We have

$$B \le (2\delta)^2 d \int_{Q^+} \int_Q \int_0^1 \left(\nabla h(x + \theta(z - x))\right)^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the the two changes of variable $z \to z' = z - x \in Q_2 = [-2\delta, 2\delta]^d$, $x \to x' = x + \theta z' \in Q$, we get

$$B \le (2\delta)^2 d \int_{Q_2} \int_Q \left(\nabla h(x')\right)^2 \, dx' \, dz',$$

which gives

$$B \le 2^{2d+2} \delta^{d+2} d \int_Q \left(\nabla h(x')\right)^2 \, dx'. \tag{9.62}$$

Thus, using (9.60), (9.61) and (9.62), one concludes the proof of (9.59).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (9.56) is then a direct consequence of

$$\int_{\partial K} g(x)^2 \, d\gamma(x) = \int_{\partial K} \left(g(x)^2 - \frac{1}{m_K} \int_K g(y) \, dy \right) d\gamma(x) \le \frac{1}{m_K} \int_{\partial K} \int_K \left(g(x) - g(y)\right)^2 \, dx \, d\gamma(y).$$

Lemma 9.13 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote the measure of K by m_K. One assumes that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from $B(0, \delta(K))$ to K. Let ξ be a real value greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g \in H^1(K)$.*
Then there exists $C_2 > 0$, only depending on ξ and d, such that

$$\frac{1}{m_K} \int_K \int_K (g(y) - g(x))^2 \, dx \, dy \le C_2 \, \delta(K)^2 \int_K \left(\nabla g(x)\right)^2 \, dx. \tag{9.63}$$

Thus, if $\int_K g(x) \, dx = 0$ holds, one has

$$\int_K g^2(x) \, dx \le C_2 \, \delta(K)^2 \int_K (\nabla g(x))^2 \, dx. \tag{9.64}$$

Proof. We denote $\delta = \delta(K)$. Using the change of variables $x' = \mathcal{L}(x)$ and $y' = \mathcal{L}(y)$, and writing for simplicity of notations $B = B(0, \delta)$, one gets the existence of C_{30}, only depending on d and ξ, such that

$$\int_K \int_K (g(y) - g(x))^2 \, dx \, dy \le C_{30} \int_B \int_B (g(\mathcal{L}(y')) - g(\mathcal{L}(x')))^2 \, dx' \, dy'.$$

Setting $h = g \circ \mathcal{L}$, one has $h \in H^1(B)$. Then one gets the existence of $C_{31} > 0$, only depending on d and ξ, such that

$$\int_B (\nabla h(x))^2 \, dx \leq C_{31} \int_K (\nabla g(x'))^2 \, dx'. \tag{9.65}$$

Thus, if one proves the existence of $C_{32} > 0$, only depending on d and ξ, such that

$$\int_B \int_B (h(y) - h(x))^2 \, dx \, dy \leq C_{32} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx, \tag{9.66}$$

one gets (9.13) from (9.65), (9.66) and the fact that the existence of $\mathcal{L}$ ensures that there exists C_5 with $m_K \geq C_5 \, \delta^d$. In order to prove (9.66), one may assume by a classical argument of density that $h \in C^1(\overline{B})$. One sets

$$A = \int_B \int_B (h(z) - h(x))^2 \, dx \, dz.$$

Using the Cauchy-Schwarz inequality, we get

$$A \leq (2\delta)^2 d \int_B \int_B \int_0^1 (\nabla h(x + \theta(z - x)))^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the changes of variable $z \to z' = z - x \in B_2 := B(0, 2\delta)$, $x \to x' = x + \theta z'$, we get

$$A \leq (2\delta)^2 d \int_{B_2} \int_B (\nabla h(x))^2 \, dx' \, dz',$$

which gives the existence of some C_{33}, only depending on d, such that

$$A \leq C_{33} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx.$$

This concludes the proof of (9.66).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (9.64) follows, remarking that in such a case

$$\int_K g^2(x) \, dx = \int_K \left(g(x) - \frac{1}{m_K} \int_K g(y) \, dy \right)^2 dx \leq \frac{1}{m_K} \int_K \int_K (g(x) - g(y))^2 \, dx \, dy.$$

Lemma 9.14 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from K to $B(0, \delta(K))$. One denotes by ξ an upper bound of both Lipschitz constants. Let $a \subset \partial K$, such that there exists $x_0 \in a$ and $\zeta > 0$ with*

$$\partial K \cap B(x_0, \zeta \delta(K)) \subset a$$

Let m_a denote the $d-1$ Lebesque measure of a. Let $\mathbf{q} \in H(\mathrm{div}, K)$ such that $\mathbf{q} \cdot \mathbf{n}_{\partial K} \in L^2(\partial K)$ and there exists $q_a \in \mathbb{R}$ with $\mathbf{q}(x) \cdot \mathbf{n}_{\partial K}(x) = q_a$ for a.e. $x \in a$.
Then there exists C_{16}, only depending on d, ξ and ζ, such that

$$m_a q_a^2 \leq C_{16} \left(\frac{1}{\delta} \int_K \mathbf{q}^2(x) \, dx + \delta \int_K (\mathrm{div} \; \mathbf{q}(x))^2 \, dx \right) \tag{9.67}$$

Proof. Denoting $\delta = \delta(K)$, let $X \in \partial B(0,\delta)$ and $\eta \in (0,1]$. We have $\{Z \in \partial B(0,\delta) \mid Z \cdot X \geq (1-\eta)\delta^2\} = \partial B(0,\delta) \cap B(X, \sqrt{2\eta}\delta)$. Indeed, take $Z \in \partial B(0,\delta)$ and denote $h = Z - X$. One has, since $|Z|^2 = |X|^2 = \delta^2$, $|h|^2 = 2\delta^2 - 2Z \cdot X$; thus, $|h|^2 \leq 2\eta\delta^2$ if and only if $Z \cdot X \geq (1-\eta)\delta^2$.
Define

$$\mathcal{B}_\eta = \{y \in \partial K \mid \mathcal{L}(y) \cdot \mathcal{L}(x_0) \geq (1-\eta)\delta^2\} = \mathcal{L}^{-1}(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta)).$$

Let $F(x) = (|x|/\sup_{i\in[1,d]}|x_i|)x$. $\mathcal{L}^{-1} \circ F^{-1}$ is a Lipschitz continuous homeomorphism with Lipschitz-continuous inverse mapping between K and $Q_\delta =]-\delta,\delta[^d$; moreover, the Lipschitz constants of $\mathcal{L}^{-1} \circ F^{-1}$ and its inverse mapping are bounded by a real number only depending on d and ξ. Thus, by Lemma 9.11 applied to $f = \chi_{\mathcal{B}_\eta}$,

$$\gamma(\mathcal{B}_\eta) \geq C_{34}\, \gamma(F \circ \mathcal{L}(\mathcal{B}_\eta)) = C_{34}\, \gamma(F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))),$$

with C_{34} only depending on d and ξ. It is easy to see that $\gamma(F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))) \geq C_{35}\, \delta^{d-1}$, where C_{35} only depends on d and η (the set $F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))$ contains a significant part of a $(d-1)$-dimensional ball on ∂Q_δ with radius of order δ). Thus, one has

$$\gamma(B_\eta) \geq C_{36}\, \delta^{d-1}, \tag{9.68}$$

with C_{36} only depends on d, ξ and η.
Now, let $\eta_0 = \inf(1, (\zeta/\xi)^2/2) \in (0,1]$ (η_0 only depends on ζ and ξ); since $\mathcal{L}^{-1}$ is Lipschitz-continuous with constant ξ, one has

$$\mathcal{B}_{\eta_0} \subset \partial K \cap B(x_0, \zeta\delta) \subset a. \tag{9.69}$$

Let us define the function $v \in H^1(K)$ by

$$v(x) = \psi\left(\frac{\mathcal{L}(x) \cdot \mathcal{L}(x_0)}{\delta^2}\right),\ \forall x \in K,$$

where the function $\psi \in C([-1,1],[0,1])$ is defined by $\psi(s) = 0$ for all $s \in [-1, 1-\eta_0]$, $\psi(s) = \frac{2(s+\eta_0-1)}{\eta_0}$ for all $s \in [1-\eta_0, 1-\eta_0/2]$, $\psi(s) = 1$ for all $s \in [1-\eta_0/2, 1]$. One has therefore $v(x) \in [0,1]$ for all $x \in \overline{K}$, $v = 1$ on $\mathcal{B}_{\eta_0/2}$ and $v = 0$ on $\partial K \setminus \mathcal{B}_{\eta_0} \supset \partial K \setminus a$ and

$$\nabla v(x) = \frac{\psi'\left(\frac{\mathcal{L}(x)\cdot\mathcal{L}(x_0)}{\delta^2}\right)}{\delta^2}(D\mathcal{L}(x))^T \mathcal{L}(x_0).$$

Thus, since $|\mathcal{L}(x_0)| \leq \delta$, we have $\|\nabla v\|_{L^\infty(K)} \leq \frac{C_{37}}{\delta}$ where C_{37} only depends on d, ξ and ζ. For all $x \in \partial K \setminus a$, $v(x) = 0$, and therefore the following relation holds

$$\int_K \nabla v(x) \cdot \mathbf{q}(x)\, dx = -\int_K v(x) \operatorname{div} \mathbf{q}(x)\, dx + q_a \int_a v(x)\, d\gamma(x).$$

We have $\int_a v(x)\, d\gamma(x) \geq \gamma(\mathcal{B}_{\eta_0/2})$ (because v is non-negative and has value 1 on $\mathcal{B}_{\eta_0/2}$) and thus, by (9.68), $\int_a v(x)\, d\gamma(x) \geq C_{38}\, \delta^{d-1}$ with C_{38} only depending on d, ξ and ζ. Since $||\nabla v(x)||_{L^\infty(K)} \leq \frac{C_{37}}{\delta}$ and $m_K \leq C_{39}\, \delta^d$, one therefore gets

$$q_a^2 \leq C_{40}\left(\delta^{d-2-2(d-1)} \int_K \mathbf{q}(x)^2\, dx \ + \ \delta^{d-2(d-1)} \int_K (\operatorname{div} \mathbf{q}(x))^2\, dx\right),$$

which leads to (9.67), since $m_a \leq C_{41}\, \delta^{d-1}$.