Partie III

La Condition d'Hyperbolicité pour les Systèmes Linéaires du Premier Ordre

Chapitre 6

A New Approach for the Hyperbolicity Condition of First Order Systems

J. Droniou.

Abstract We study here first order linear systems of partial differential equations with constant coefficients. We prove a necessary and sufficient condition for such systems to have solutions when we consider initial conditions of Riemann type. We also prove that this condition is necessary when dealing with more regular initial conditions.

6.1 Introduction

We study systems of the form

$$\begin{cases} u_t(x,t) + \sum_{i=1}^{N} A_i u_{x_i}(x,t) = 0, & x \in \Omega\,,\ t \in]0,T[, \\ u(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{6.1}$$

where Ω is an open set of $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$, $u = (u_1, \ldots, u_l) : \Omega \times [0,T[\to \mathbb{R}^l$, $(A_1, \ldots, A_N)$ are $l \times l$ real matrices and $u^0 = (u_1^0, \ldots, u_l^0) : \Omega \to \mathbb{R}^l$ is an initial condition. We will not handle the boundary conditions, which are anyway unnecessary to obtain the hyperbolicity condition.

When $\Omega = \mathbb{R}^N$, a classical way to study this kind of problem is to use the Fourier Transform. In this case, the natural space of solutions is built on $L^2(\mathbb{R}^N)$ (for example $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^N))$); when we ask for Problem (6.1) to be well-posed in this space in the sense of Hadamard, we can then prove that the matrices $(A_j)_{j\in[1,N]}$ must satisfy: $\sup_{\xi\in\mathbb{R}^N} ||\exp(i\sum_{j=1}^N \xi_j A_j)|| < +\infty$, see e.g. [67] (this condition implies, but is not equivalent to, the diagonalizability of $\sum_{j=1}^N \xi_j A_j$ for all $\xi \in \mathbb{R}^N$; see [48] and item ii) in Remark 6.2). Notice that the well-known Friedrichs systems (that is to say systems of the kind (6.1) such that there exists a symmetric definite positive matrix S satisfying: for all $j \in [1,N]$, SA_j is symmetric) always satisfy this condition (see [67]).
Because of the use of the Fourier transform, this method is limited to the case $\Omega = \mathbb{R}^N$ and demands that (6.1) be well-posed in $L^2(\mathbb{R}^N)$.

When we search for classical $\mathcal{C}^\infty$ global ($\Omega = \mathbb{R}^N$) solutions, the Lax-Mizohata theorem (see [18]) gives a result of the same kind. This theorem however demands that (6.1) be well-posed (in $\mathcal{C}^\infty(\mathbb{R}^N \times [0,\infty[)$) not only for a null right-hand side, but for any $\mathcal{C}^\infty$ right-hand side; in this case, it states that, for any $\xi \in \mathbb{R}^N$, the matrix $\sum_{i=1}^N \xi_i A_i$ must have real eigenvalues (the Lax-Mizohata theorem is not limited to constant-coefficient problems of the first order); in fact, since our system has constant coefficients, it is even here a sufficient condition for the system to be well-posed in $\mathcal{C}^\infty$ (this is the Garding theorem). Notice however that the important part in the proof of the Lax-Mizohata theorem is not the initial condition, but the non-null right hand side (see [18]), which we do not need here; it is also very important that the datas be $\mathcal{C}^\infty$ regular. Indeed, we prove here that, when we take less regular datas, the matrix $\sum_{i=1}^N \xi_i A_i$ must not only have real eigenvalues, but must also be diagonalizable on $\mathbb{R}$.

Our aim here is to study (6.1) when the functions u^0 and u are neither in L^2, nor regular; in fact, we will consider very specific initial conditions (of Riemann type) and solutions in the largest possible space of functions (the largest space of functions which is endowed in the space of distributions), that is to say $L^1_{\rm loc}$. The study of the Riemann problem coming from (6.1) is quite interesting (see item ii) in Remark 6.3).
We will prove, using methods that seem new, that, when we ask for (6.1) to have a local weak solution in $(L^1_{\rm loc}(\Omega\times[0,T[))^l$ for any initial condition of Riemann type, then the matrices $(A_i)_{i\in[1,N]}$ must satisfy a so-called hyperbolicity condition; this condition will also appear to be a sufficient one. We will not need to suppose the well-posedness of the system in the sense of Hadamard, neither to consider boundary conditions: the mere existence of a local solution to the equation inside the open set of study is enough to obtain the hyperbolicity condition.

6.2 Definitions, remarks and results

Definition 6.1 *(local weak solution) Let $u^0\in(L^1_{\rm loc}(\Omega))^l$ and $T>0$. A function $u\in(L^1_{\rm loc}(\Omega\times[0,T[))^l$ is a local weak solution on $\Omega\times[0,T[$ to (6.1) if, for all $\varphi\in\mathcal{C}_c^\infty(\Omega\times[0,T[)$,*

$$\int_0^T\int_\Omega u(x,t)\varphi_t(x,t)\,dx\,dt+\int_0^T\int_\Omega\sum_{i=1}^N A_i u(x,t)\varphi_{x_i}(x,t)\,dx\,dt+\int_\Omega u^0(x)\varphi(x,0)\,dx=0. \tag{6.2}$$

Remark 6.1 *i) Of course, a solution to (6.1) in the sense of Definition 6.1 is also a solution in the sense of the distributions on $\Omega\times]0,T[$.*

ii) Notice that (6.2) is an equality in $\mathbb{R}^l$. Moreover, to handle the initial condition, we take test functions φ which do not necessary vanish at $t=0$ ($\mathcal{C}_c^\infty(\Omega\times[0,T[)$ is the space of the restrictions to $\Omega\times[0,T[$ of functions in $\mathcal{C}_c^\infty(\Omega\times]-\infty,T[)$); since we do not take into account boundary conditions, these test functions vanish on the boundary of Ω.

iii) The technique of Hölmgren associated to a simple form of the Cauchy-Kowalewska theorem gives a finite speed propagation result on the local weak solutions of (6.1) (see Corollary 6.2). In particular, when $\Omega=\mathbb{R}^N$, the local weak solution to (6.1) is unique.

Definition 6.2 *The system (6.1) is hyperbolic if, for all $\xi\in\mathbb{R}^N$, $\sum_{i=1}^N\xi_i A_i$ is diagonalizable on $\mathbb{R}$.*

Remark 6.2 *i) There are numerous definitions of "hyperbolicity" for a first order linear system of equations; some only ask for the matrix $\sum_{i=1}^N\xi_i A_i$ to have real eigenvalues ([18], [46]). The definition we have chosen here is the one that appears in [37]; it is stronger than in [18] but weaker than in [67] (see item ii) below).*

ii) The hyperbolicity condition in [67], namely $\sup_{\xi\in\mathbb{R}^N}||\exp(i\sum_{j=1}^N\xi_jA_j)||<+\infty$, is equivalent to the existence of $P:S^{N-1}\to \mathrm{GL}(l;\mathbb{R})$ such that, for all $\xi\in S^{N-1}$, $P(\xi)^{-1}\sum_{j=1}^N\xi_jA_jP(\xi)$ is diagonal and $\sup_{\xi\in S^{N-1}}||P(\xi)||\,||P(\xi)^{-1}||<+\infty$ (see [48]). When $N=1$ or $N=l=2$, this definition is equivalent to Definition 6.2 (in fact, in these cases, a system which is hyperbolic in the sense *of Definition 6.2 is a Friedrichs system); however, as soon as $N \geq 2$ and $l \geq 3$, the definition of hyperbolicity in [67] is strictly stronger than our definition. For example, when $N = 2$ and $l = 3$, the system defined by*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

is hyperbolic in the sense of Definition 6.2 but not in the sense given in [67] (when we take matrices $P(\xi)$ that diagonalize $\xi_1 A_1 + \xi_2 A_2$, we notice that $||P(\xi)||\,||P(\xi)^{-1}||$ must explode when $\xi \to (1,0)$ with $\xi_2 \neq 0$).

When Ω is an open set of $\mathbb{R}^N$ and $a \in \Omega$, we define $E(\Omega, a)$ as the set of functions $f : \Omega \to \mathbb{R}^l$ for which there exists $\xi \in \mathbb{R}^N \backslash \{0\}$ and $f^* \in \mathbb{R}^l$ such that, for all $x \in \Omega$, $f(x) = f^* \mathbf{1}_{\mathbb{R}^-}((x-a)\cdot\xi)$, where $\mathbf{1}_E$ denotes the characteristic function of a set $E \subset \mathbb{R}$ and $X \cdot Y$ is the usual Euclidean product of two vectors $(X,Y) \in \mathbb{R}^N \times \mathbb{R}^N$ (that is to say, $f = 0$ on one of the half-space defined by the affine hyperplane passing by a and having ξ as normal vector, and $f = f^*$ on the other half-space).
The main result of our paper is the following:

Theorem 6.1 *Let Ω be an open set of $\mathbb{R}^N$ and $a \in \Omega$. There is equivalence between:*

1) *For all $u_0 \in E(\Omega, a)$, there exists $T > 0$ and a local weak solution $u \in (L^1_{\mathrm{loc}}(\Omega \times [0,T[))^l$ on $\Omega \times [0,T[$ to (6.1).*

2) *The system (6.1) is hyperbolic.*

Remark 6.3 *i) In fact, we will see that, when the system is hyperbolic, the local weak solutions we obtain, for initial conditions of Riemann type, exist for all $T > 0$.*

ii) Some important numerical schemes to compute the solutions of non-linear systems of the kind

$$\begin{cases} u_t + \sum_{i=1}^{N} (f_i(u))_{x_i} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{6.3}$$

reduce to solving linear Riemann problems in one space dimension (for exemple, the Roe and VFRoe schemes, see [37]). In fact, the linear Riemann problems that occur in the Roe and VFRoes schemes are often given by matrices of the kind $\sum_{i=1}^N \xi_i f_i'(\overline{u})$, for some $\xi \in S^{N-1}$ and $\overline{u} \in D \subset \mathbb{R}^l$ (where D is the admissible values of the solution u to (6.3)); thus, thanks to Theorem 6.1, we see that, to use such schemes, a necessary and sufficient condition is that $\sum_{i=1}^N \xi_i f_i'(\overline{u})$ be diagonalizable on $\mathbb{R}$ for all $\xi \in S^{N-1}$ and all $\overline{u} \in D$: this is what [37] calls the hyperbolicity condition for non-linear systems (and we notice that this is consistant with Definition 6.2 in the case of linear systems).

iii) In [67], as we have seen in item ii) of Remark 6.2, the hyperbolicity condition obtained on the matrices $(A_i)_{i\in[1,N]}$ is in general stronger than our hyperbolicity condition; it is however not so surprising since, to obtain this condition, [67] asks for (6.1) to be well-posed in the sense of Hadamard in $L^2(\mathbb{R}^N)$, which is a stronger hypothesis (more initial conditions, more properties on the solutions) than the one of 1) $\Longrightarrow$ 2) in Theorem 6.1.

Definition 6.3 *The system (6.1) is solvable on an open set* Ω *of* $\mathbb{R}^N$ *if, for any initial condition* $u^0 \in (L^\infty(\Omega))^l$, *there exists* $T > 0$ *and at least one local weak solution* $u \in (L^1_{\text{loc}}(\Omega \times [0,T[))^l$ *on* $\Omega \times [0,T[$ *to (6.1).*

Remark 6.4 *i) Problem (6.1) can also be studied with more regular initial conditions, for example* $u^0 \in (\mathcal{C}^k(\Omega))^l$ *(with* $k \in \mathbb{N}$*). In this case, the definition of "solvability" of (6.1) would also demand that the corresponding solution be as regular in space as* u^0 *(since (6.1) is a first order linear system), that is to say in* $(L^1([0,T];\mathcal{C}^k(\Omega))^l$. *We also study the "solvability" of (6.1) under this definition.*

ii) Notice that our definition of "solvable" is much weaker than the classical (Hadamard's) definition of "well-posed"; indeed, this last definition demands the uniqueness of the solution to (6.1) (which, in our case, would oblige us to handle some boundary conditions) as well as the continuous dependence of this solution with respect to the initial and boundary conditions. In fact, for systems of the kind of (6.1), there is some sort of uniqueness result: a finite speed propagation result; since we will need this result, we will talk about it later on.

An immediate consequence (since the functions of $E(\Omega, a)$ are in $(L^\infty(\Omega))^l$) of Theorem 6.1 is the following corollary.

Corollary 6.1 *If the system (6.1) is solvable on an open set of* $\mathbb{R}^N$ *in the sense of Definition 6.3, then it is hyperbolic.*

Remark 6.5 *i) It is however not sure that the hyperbolicity condition is a sufficient property for (6.1) to be solvable. This comes from the fact that, as we will see in the proof of Theorem 6.1, when we solve an hyperbolic system of the kind (6.1) with an initial condition of Riemann type, the bound we can obtain on the solution with respect to the initial condition depends on the norm of the matrices* $(P(\xi), P(\xi)^{-1})$ *which diagonalize* $\sum_{i=1}^N \xi_i A_i$; *and as we have seen in item ii) of Remark 6.2, there are examples of hyperbolic systems such that the norm of the matrices* $P(\xi)$ *or* $P(\xi)^{-1}$ *explodes when* ξ *tends to some* ξ_0.

ii) We will see in Section 6.3 that, in the 1-dimensional case ($N = 1$*), the hyperbolicity of (6.1) is in fact equivalent to its solvability (in the sense of Definition 6.3 or in the sense of item i) in Remark 6.4).*

As said in item i) of Remark 6.4, we also have a result concerning regular initial conditions.

Theorem 6.2 *Let* Ω *be an open set of* $\mathbb{R}^N$. *If, for any* $u_0 \in (\mathcal{C}_c^k(\Omega))^l$, *there exists* $T > 0$ *and a local weak solution* $u \in (L^1([0,T];\mathcal{C}^k(\Omega))^l$ *on* $\Omega \times [0,T[$ *to (6.1), then this system is hyperbolic.*

Remark 6.6 *i)* $\mathcal{C}_c^k(I)$ *denotes the space of functions* $I \to \mathbb{R}$ *which have a compact support in* I *and are* k *times continuously derivable on* I.

ii) In the course of the proof of this theorem, we will find back the Lax-Mizohata result (without the need of a non-null right-hand side) in our particular case: when we ask for (6.1) to have a $\mathcal{C}^\infty$ *solution for any* $\mathcal{C}^\infty$ *initial condition, then* $\sum_{i=1}^N \xi_i A_i$ *must have real eigenvalues.*

6.2.1 A preliminary result

In all the sequel, $|\cdot|_\infty$ denotes the norm of the suppremum on $\mathbb{R}^N$ and "dist" the associated distance. e is the vector $(1,\ldots,1)$ of $\mathbb{R}^N$.

Lemma 6.1 *Let Ω be an open set of $\mathbb{R}^N$, $P \in \mathrm{GL}(l,\mathbb{R})$, $x^0 \in \mathbb{R}^N$, $(\lambda,\mu) \in \mathbb{R}\times\mathbb{R}^*$ and O be an open set relatively compact in $\Omega_{x^0,\mu} = x^0 + \mu\Omega$. If $u^0 \in (L^1_{\mathrm{loc}}(\Omega))^l$ and $u \in (L^1_{\mathrm{loc}}(\Omega\times[0,T[))^l$ is a local weak solution on $\Omega\times[0,T[$ to (6.1), then, by denoting $T_O = \inf\left(T, \frac{1}{|\lambda|}\mathrm{dist}(O,\mathbb{R}^N\backslash\Omega_{x^0,\mu})\right)$, the function $v \in (L^1_{\mathrm{loc}}(O\times[0,T_O[))^l$ defined by*

$$v(x,t) = Pu\left(\frac{1}{\mu}(x-x^0) + \frac{\lambda}{\mu}te, t\right), \qquad (x,t)\in O\times[0,T_O[, \tag{6.4}$$

is a local weak solution on $O\times[0,T_O[$ to the system (6.1) with $\mu PA_iP^{-1} - \lambda Id$ instead of A_i ($i\in[1,N]$) and $v^0(x) = Pu^0((x-x^0)/\mu) \in (L^1(O))^l$ instead of u_0.

The proof of this lemma is quite straightforward and we leave it to the reader.

Remark 6.7 *i) When $\lambda = 0$, we can take any open set $O \subset \Omega_{x^0,\mu}$ and $T_O = T$.*

ii) We thus deduce that, for $a\in\Omega$, if the system (6.1) has a local weak solution for any initial condition in $E(\Omega,a)$, then the system (6.1) with $\mu PA_iP^{-1} - \lambda Id$ instead of A_i ($i\in[1,N]$) has a local weak solution for any initial condition in $E(O, x^0+\mu a)$. Of course, one can notice that the hyperbolicity condition is invariant with respect to the transformations up above of the system.

6.3 The 1-dimensional case

We study here the 1-dimensional case, i.e. $N=1$. The system (6.1) is thus reduced to

$$\begin{cases} u_t(x,t) + Au_x(x,t) = 0, & x\in\Omega\,,\ t\in]0,T[, \\ u(x,0) = u^0(x), & x\in\Omega, \end{cases} \tag{6.5}$$

and Ω is an open set in $\mathbb{R}$.

6.3.1 Particularities of the 1-dimensional case

The case $N=1$ is very particular. Indeed, we first notice that an hyperbolic system in one dimension is always solvable in the sense of Definition 6.3; in fact, in this case, (6.5) has a local weak solution for any initial condition $u^0 \in (L^1(\Omega))^l$ (or $(L^1_{\mathrm{loc}}(\Omega))^l$ if $\Omega = \mathbb{R}$) and this solution is as regular as u^0.
To see this, choose a basis $(e_1,\ldots,e_l)$ of $\mathbb{R}^l$ which is made of eigenvectors of A; if $u^0(\cdot) = u^0_1(\cdot)e_1 + \ldots + u^0_l(\cdot)e_l \in (L^1_{\mathrm{loc}}(\mathbb{R}))^l$ and λ_i is the eigenvalue of A associated to e_i, then $u(x,t) = u^0_1(x-\lambda_1 t)e_1 + \ldots + u^0_l(x-\lambda_l t)e_l \in (L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+))^l$ is a (in fact *the*) local weak solution on $\mathbb{R}\times\mathbb{R}^+$ to (6.5); if $v^0 \in (L^1(\Omega))^l$, then define $u^0 \in (L^1(\mathbb{R}))^l$ as the extension of v^0 by 0 outside Ω, take $u \in (L^1(\mathbb{R}\times\mathbb{R}^+))^l$ as defined above: $u_{|\Omega\times\mathbb{R}^+} \in (L^1(\Omega\times\mathbb{R}^+))^l$ is a local weak solution on $\Omega\times\mathbb{R}^+$ to the system (6.5) with v^0 instead of u^0.
This also proves 2) $\Longrightarrow$ 1) in Theorem 6.1 when $N=1$.

Remark 6.8 *Notice also that, in the one dimensional case and if $\Omega = \mathbb{R}$, our definition of solvability is equivalent to Hadamard's definition of well-posedness. Indeed, if (6.5) is solvable in the sense of Definition 6.3, Corollary 6.1 tells us that A is diagonalizable on $\mathbb{R}$; we know then (they have been computed up above) the expressions of the solutions to (6.5), and it is clear, with these expressions, that this system is well posed in the sense of Hadamard.*

We will see (Corollary 6.2) that, in any dimension N, we have a "local uniqueness" result for the solution of (hyperbolic or not) systems of the kind (6.1).
But an interesting consequence of the particularity of the 1-dimensional case is that the proof of this uniqueness result is, when the system is hyperbolic, far simplier than in the general case.

With Theorem 6.1 or 6.2, the following proposition tells us that the existence of solutions to (6.5) implies a local uniqueness of these solutions. The trick which allows us to prove Theorem 6.1 and 6.2 by induction on the size l of the system (6.5) is then the following: we split the system of size l (say S) in two systems, one of size $l-2$ (say S1) and the other of size 2 (say S2), in such a way that the existence of solutions to S implies the existence of solutions to S1; by induction (S1 is of size $l-2$) and Proposition 6.1, we get the uniqueness of the solution to S1 and we reduce thus the study of S to the study of the remaining system $S2$ (of size 2).
We state this very particular and simple case ($N=1$, A is diagonalizable) of Corollary 6.2 to see that, in dimension $N=1$ (in converse to the case $N \geq 2$), thanks to the trick described up above, we need no general result on the uniqueness of the solutions to (6.5) to obtain the hyperbolicity condition.

Proposition 6.1 *If A is diagonalizable on $\mathbb{R}$ then, for any open set O relatively compact in Ω, there exists $T_O > 0$ such that, when $u^0 = 0$ a.e. on Ω, any local weak solution on $\Omega \times [0,T[$ to (6.5) is null a.e. on $O \times]0, \inf(T_O, T)[$.*

To prove this classical finite speed propagation result, one juste need to take a basis of $\mathbb{R}^l$ in which A is diagonal; in this basis, each component of the solution satisfies a scalar transport equation and the result is then quite obvious; we leave the details to the reader.

6.3.2 Necessity of the Hyperbolicity Condition

Proof of Theorems 6.1 and 6.2 when $N=1$
We have already seen, at the beginning of Subsection 6.3.1, that (when $N=1$) 2) $\Longrightarrow$ 1) in Theorem 6.1 holds. It remains to prove Theorem 6.2 and 1) $\Longrightarrow$ 2) in Theorem 6.1.

This proof is made by contradiction. We will in fact prove by induction on the size l of A that, if A is not diagonalizable on $\mathbb{R}$, then

i) there exists $u^* \in \mathbb{R}^l$ such that, for any open interval I, for any $a \in I$ and for any $T > 0$, (6.5) has no local weak solution in $(L^1_{\mathrm{loc}}(I \times [0,T]))^l$ for the initial condition $u^0 = u^* \mathbf{1}_{I \cap]-\infty, a[}$.

ii) for any open interval I and for any $k \in \mathbb{N}$, there exists $u^0 \in (\mathcal{C}^k_c(I))^l$ such that, for any $T > 0$, (6.5) has no local weak solution in $(L^1([0,T]; \mathcal{C}^k(I)))^l$ for the initial condition u^0.

The case $l=1$ being trivial (every matrix of size 1 is diagonalizable on $\mathbb{R}$), we begin with the case $l=2$.

Step 1: $l=2$.
Let I be an open interval of $\mathbb{R}$ and $A \in M_2(\mathbb{R})$. Suppose that A is not diagonalizable on $\mathbb{R}$; two cases arise then: A has two different complex conjugate eigenvalues or A has only one real eigenvalue and is not in the space $\mathbb{R}Id$.

Step 1.1.
Let us handle the case when the eigenvalues of A are different complex conjugate; by denoting $\lambda + \frac{i}{\mu}$ and $\lambda - \frac{i}{\mu}$ ($(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^*$) these eigenvalues, there exists $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda & \frac{1}{\mu} \\ -\frac{1}{\mu} & \lambda \end{pmatrix} = \lambda Id + \frac{1}{\mu} A_0, \tag{6.6}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us take J an open interval, $T'>0$ and $v^0\in L^1(J)$ such that there exists $v=(v_1,v_2)\in(L^1_{\mathrm{loc}}(J\times[0,T'[))^2$ a local weak solution on $J\times[0,T'[$ to the system (6.5) with A_0 instead of A and $(v^0,0)$ as an initial condition. We have thus, for all $\psi\in\mathcal{C}_c^\infty(J\times[0,T'[)$,

$$\int_0^{T'}\int_J v_1(x,t)\psi_t(x,t)\,dx\,dt+\int_0^{T'}\int_J v_2(x,t)\psi_x(x,t)\,dx\,dt+\int_J v^0(x)\psi(x,0)\,dx=0, \tag{6.7}$$

$$\int_0^{T'}\int_J v_2(x,t)\psi_t(x,t)\,dx\,dt-\int_0^{T'}\int_J v_1(x,t)\psi_x(x,t)\,dx\,dt=0. \tag{6.8}$$

Let $\varphi\in\mathcal{C}_c^\infty(J\times[0,T'[)$. By applying (6.7) with $\psi=\varphi_t$ and (6.8) with $\psi=\varphi_x$, we find

$$\int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt+\int_J v^0(x)\varphi_t(x,0)\,dx=0. \tag{6.9}$$

Define $\widetilde{v_1}\in L^1_{\mathrm{loc}}(J\times]-T',T'[)$ by $\widetilde{v_1}(x,t)=v_1(x,t)$ if $t\geq 0$ and $\widetilde{v_1}(x,t)=v_1(x,-t)$ if $t<0$. If $\varphi\in\mathcal{D}(J\times]-T',T'[)$, a change of variable gives

$$\begin{aligned}\int_{-T'}^0\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt &= \int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\varphi(x,-t)\,dx\,dt\\ &= \int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\widetilde{\varphi}(x,t)\,dx\,dt,\end{aligned}$$

where $\widetilde{\varphi}\in\mathcal{C}_c^\infty(J\times[0,T'[)$ is defined by $\widetilde{\varphi}(x,t)=\varphi(x,-t)$. Using (6.9) with $\widetilde{\varphi}$ instead of φ, we get thus

$$\int_{-T'}^0\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt=-\int_J v^0(x)\widetilde{\varphi}_t(x,0)\,dx=\int_J v^0(x)\varphi_t(x,0)\,dx, \tag{6.10}$$

since $\widetilde{\varphi}_t(x,t)=-\varphi_t(x,-t)$.
The definition of $\widetilde{v_1}$, associated to (6.9) and (6.10), gives then, for all $\varphi\in\mathcal{D}(J\times]-T',T'[)$,

$$\int_{-T'}^{T'}\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt=0,$$

that is to say $\Delta\widetilde{v_1}=0$ in $\mathcal{D}'(J\times]-T',T'[)$; by Lemma 6.2, $\widetilde{v_1}$ is then in $\mathcal{C}^\infty(J\times]-T',T'[)$ (and even real analytic). We have thus $v_1\in\mathcal{C}^\infty(J\times[0,T'[)$ and $(\widetilde{v_1})_t(x,0)=0$. Thanks to some classical integrates by part in (6.9), we see then that v^0 is the restriction of $\widetilde{v_1}$ to $J\times\{0\}$ and must thus be in $\mathcal{C}^\infty(J)$ (and even real analytic).

To prove items i) and ii) in this case, we take

$u^*=P^{-1}\binom{1}{0}$, I an open interval of $\mathbb{R}$, $a\in I$, $u^0=u^*\mathbf{1}_{I\cap]-\infty,a[}$ and $T>0$ — in the case of item i),

I an open interval of $\mathbb{R}$, $w\in\mathcal{C}_c^k(I)\backslash\mathcal{C}^\infty(I)$, $a\in I$ such that w is not infinitely differentiable at a, $u^0=P^{-1}\binom{w}{0}$ and $T>0$ — in the case of item ii),

(notice that u^* does not depend on I, a or T) and we suppose that there exists a local weak solution on $I\times[0,T[$ to (6.5). By taking J an open interval, relatively compact in μI, which contains μa, Lemma 6.1 tells us that there exists a local weak solution on $J\times[0,T'[$ (for a $T'\in]0,T]$) to the system defined by A_0 with the initial condition $Pu^0(x/\mu)=(v^0(x),0)$; thanks to the preceding reasoning, this means that v^0 is $\mathcal{C}^\infty$ on J, which is a contradiction since $\mu a\in J$.
Since v^0 must even be real analytic on J, we also obtain a stronger form of the Lax-Mizohata result in our particular case: there exists $u^0\in(\mathcal{C}^\infty(I))^2$ such that (6.1) has no local weak solution in $(L^1_{\mathrm{loc}}(I\times[0,T[))^2$

(take u^0 which is not real analytic). Using the finite speed propagation result of Corollary 6.2, this result can be extented to the cases $l \geq 3$ and $N \geq 2$.

Step 1.2
Let us now treat the case when A has a unique real eigenvalue without being diagonalizable on $\mathbb{R}$. By denoting λ this eigenvalue, there exists $P \in \mathrm{GL}(2,\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda Id + A_1, \tag{6.11}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let J be an open interval, $T' > 0$ and $v^0 \in L^1(J)$ such that there exists a local weak solution $v = (v_1, v_2) \in (L^1_{\text{loc}}(J \times [0,T'[))^2$ to the system (6.5) with A_1 instead of A and $(0, v^0)$ as an initial condition. We thus have, for all $\varphi \in \mathcal{C}_c^\infty(J \times [0,T'[)$,

$$\int_0^{T'} \int_J v_1(x,t)\varphi_t(x,t)\, dx\, dt + \int_0^{T'} \int_J v_2(x,t)\varphi_x(x,t)\, dx\, dt = 0, \tag{6.12}$$

$$\int_0^{T'} \int_J v_2(x,t)\varphi_t(x,t) + \int_J v^0(x)\varphi(x,0)\, dx = 0. \tag{6.13}$$

By taking $\varphi \in \mathcal{D}(J\times]0,T'[)$, (6.13) tells us that $(v_2)_t = 0$ in $\mathcal{D}'(J\times]0,T'[)$, so that v_2 only depends on x: $v_2(x,t) = \widetilde{v_2}(x)$ a.e. on $J\times]0,T'[$. Take then $\varphi(x,t) = \zeta(x)\theta(t)$, where $\zeta \in \mathcal{D}(J)$ and $\theta \in \mathcal{C}_c^\infty([0,T'[)$, $\theta(0) = 1$. The same equation (6.13) for this φ tells us that

$$\int_J \widetilde{v_2}(x)\zeta(x)\, dx = \int_J v^0(x)\zeta(x)\, dx,$$

that is to say $\widetilde{v_2} = v^0$ a.e. on J.
Then, with $\varphi(x,t) = \zeta(x)\theta(t)$, where $\zeta \in \mathcal{D}(J)$ and $\theta \in \mathcal{D}(]0,T'[)$ is such that $\int_0^{T'} \theta(t)\, dt = 1$, (6.12) gives us

$$\int_J \zeta(x) \left(\int_0^{T'} v_1(x,t)\theta'(t)\, dt \right) = -\int_J \zeta'(x)v^0(x)\, dx,$$

which means that

$$(v^0)' = \int_0^{T'} v_1(\cdot,t)\theta'(t)\, dt \quad \text{in} \quad \mathcal{D}'(J). \tag{6.14}$$

We can now conclude this step: take

$u^* = P^{-1}\binom{0}{1}$, I open interval of $\mathbb{R}$, $a \in I$, $u^0 = u^*\mathbf{1}_{I\cap]-\infty,a[}$ and $T > 0$	in the case of item i),
I open interval of $\mathbb{R}$, $w \in \mathcal{C}_c^k(I)\backslash\mathcal{C}^{k+1}(I)$, $a \in I$ such that w is not $\mathcal{C}^{k+1}$ at a, $u^0 = P^{-1}\binom{0}{w}$ and $T > 0$	in the case of item ii),

(notice that u^* does not depend on I, a or T) and suppose that there exists a local weak solution

$u \in (L^1_{\text{loc}}(I \times [0,T[))^2$ in the case of item i),
$u \in (L^1([0,T];\mathcal{C}^k(I)))^2$ in the case of item ii)

on $I \times [0,T[$ to (6.5). Then by taking J an open interval, relatively compact in I, which contains a, Lemma 6.1 tells us that $v(x,t) = Pu(x + \lambda t, t)$ (defined on $J \times [0,T'[$ for a $T' \in]0,T]$) is a local weak solution on $J\times[0,T'[$ to the system (6.5) with A_1 instead of A and $Pu^0_{|J}=(0,v^0)\in(L^1(J))^2$ instead of u^0. In the case of item i), $v\in(L^1_{\rm loc}(J\times[0,T'[))^2$ and the right-hand side of (6.14) defines thus a function in $L^1_{\rm loc}(J)$; this means that $v^0=\mathbf{1}_{J\cap]-\infty,a[}\in W^{1,1}_{\rm loc}(J)$, which is impossible since $a\in J$. In the case of item ii), $v\in(L^1([0,T'];\mathcal{C}^k(J)))^2$ and (6.14) shows us that $(v^0)'\in\mathcal{C}^k(J)$, i.e. that $v^0=w\in\mathcal{C}^{k+1}(J)$, which is impossible since $a\in J$.

i) and ii) are thus fully proved in the cases $l=1$ and $l=2$. We prove the general case by induction.

Step 2: $l\geq 3$.
We suppose that i) and ii) are true for systems of size $l-1$ or less, and we prove, by contradiction, that they are true for systems of size l. Since the ideas of the proofs of both items are the same, we only detail the proof of item i).

Suppose that $A\in M_l(\mathbb{R})$ is not diagonalizable on $\mathbb{R}$ and that, nevertheless, we have: for every $u^*\in\mathbb{R}^l$, there exists an open interval I, $a\in I$ and $T>0$ such that (6.5) has a local weak solution $u\in(L^1_{\rm loc}(I\times[0,T[))^l$ for the initial condition $u^0=u^*\mathbf{1}_{I\cap]-\infty,a[}$.
Since A is not diagonalizable on $\mathbb{R}$, there exists $P\in{\rm GL}(l,\mathbb{R})$ such that

$$PAP^{-1}=\begin{pmatrix}\widetilde{A} & C\\ 0 & B\end{pmatrix},$$

where $C\in M_{2,l-2}(\mathbb{R})$, $B\in M_{l-2}(\mathbb{R})$, 0 is the zero of $M_{l-2,2}(\mathbb{R})$ and $\widetilde{A}\in M_2(\mathbb{R})$ is of type (6.6) or (6.11) (depending whether all the eigenvalue of A are real or not).

Let us first see that B is diagonalizable on $\mathbb{R}$.
Take $v^*=(v_1^*,\ldots,v_{l-2}^*)\in\mathbb{R}^{l-2}$ and denote $u^*=P^{-1}(0,0,v_1^*,\ldots,v_{l-2}^*)\in\mathbb{R}^l$; by hypothesis on A, there exists an open interval I, $a\in I$, $T>0$ and a local weak solution $u\in(L^1_{\rm loc}(I\times[0,T[))^l$ on $I\times[0,T[$ to the system (6.5) with $u^*\mathbf{1}_{I\cap]-\infty,a[}$ instead of u^0. Thus, Pu is a local weak solution on $I\times[0,T[$ to the system (6.5) with PAP^{-1} instead of A and $(0,0,v_1^*,\ldots,v_{l-2}^*)\mathbf{1}_{I\cap]-\infty,a[}$ instead of u^0.
The $l-2$ last equations of the system satisfied by $Pu=(u_1,u_2,v_1,\ldots,v_{l-2})\in(L^1_{\rm loc}(I\times[0,T[))^l$ let us see that $v=(v_1,\ldots,v_{l-2})\in(L^1_{\rm loc}(I\times[0,T[))^{l-2}$ is a local weak solution to the system defined by B with an initial condition $v^*\mathbf{1}_{I\cap]-\infty,a[}$.
By induction, the result is true for systems of size $l-2$ (the size of B) and B is thus diagonalizable on $\mathbb{R}$.

Since the result is true for systems of size 2 (Step 1), and since $\widetilde{A}$ is not diagonalizable on $\mathbb{R}$, there exists $(u_1^*,u_2^*)\in\mathbb{R}^2$ such that, for any open interval K, any $b\in K$ and any $T'>0$, the system defined by $\widetilde{A}$ has no local weak solution in $(L^1_{\rm loc}(K\times[0,T'[))^2$ for the initial condition $(u_1^*,u_2^*)\mathbf{1}_{K\cap]-\infty,b[}$.
Take $u^*=(u_1^*,u_2^*,0,\ldots,0)\in\mathbb{R}^l$; by hypothesis on A, there exists an open interval I, $a\in I$ and $T>0$ such that the system defined by A has a local weak solution $\widetilde{u}\in(L^1_{\rm loc}(I\times[0,T[))^l$ for the initial condition $u^0=P^{-1}u^*\mathbf{1}_{I\cap]-\infty,a[}$.
The $l-2$ last equations of the system satisfied by $P\widetilde{u}=u=(u_1,\ldots,u_l)$ let us see that $(u_3,\ldots,u_l)$ is a local weak solution on $I\times[0,T[$ of the system defined by B for a null initial condition; thus, since B is diagonalizable on $\mathbb{R}$, Proposition 6.1 let us see that, if J is a relatively compact open interval of I, there exists $T_J\in]0,T]$ such that $(u_3,\ldots,u_l)=0$ a.e. on $J\times]0,T_J[$.
Taking J which contains a and returning to the first two equations of the system satisfied by u, we see that $(u_1,u_2)\in(L^1_{\rm loc}(J\times[0,T_J[))^2$ is a local weak solution on $J\times[0,T_J[$ of the system defined by $\widetilde{A}$ for the initial conditon $(u_1^*,u_2^*)\mathbf{1}_{J\cap]-\infty,a[}$: this is a contradiction with the choice of (u_1^*,u_2^*). ∎

Remark 6.9 *i) Denote by $\mathcal{C}^{-k}(\Omega)$ the space of the distributions on Ω having order k; it is endowed with the weak-* topology of the distributions on Ω. With very few changes (some integrals changed into duality products), the preceding proof also shows that, if Ω is an open set of $\mathbb{R}^N$ and if we ask* *for (6.5) to have a local weak solution in $(\mathcal{C}([0,T];\mathcal{C}^{-k}(\Omega)))^l$ for any initial condition in $(\mathcal{C}^{-k}(\Omega))^l$, then (6.5) must be hyperbolic.*

ii) In fact, we could also prove, with the same method, that if A has a non-trivial Jordan block of size $m \geq 2$, there exists a vector $u^* \in \mathbb{R}^l$ such that, if (6.5) has a local solution $u \in (L^1([0,T];\mathcal{C}^k(\Omega)))^l$ with $u_0 = wu^*$ for some $w \in L^1(\Omega)$, then w must be in $\mathcal{C}^{k+m-1}(\Omega)$.

6.4 The multi-dimensional case

We now study the case $N \geq 2$. The proof of Theorems 6.1 (1) $\Longrightarrow$ 2)) and 6.2 in this case uses the result for $N = 1$, by taking particular initial conditions that only depends on x_1; but to use the result for $N = 1$, we need to know that the corresponding solutions also only depend on x_1. This is why we need the following uniqueness results.

6.4.1 On the uniqueness of solutions to (6.1)

Here, $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^N$. $||\cdot||$ is any norm on $\mathbb{R}^l$ and, when $B \in M_l(\mathbb{R})$, $||B||$ denotes the induced norm. The following proposition is a form of the Hölmgren theorem, but we state and prove it to give some estimates on the space-time range of uniqueness of the solution to (6.1).

Proposition 6.2 *(Finite Speed Propagation) Let Ω be an open set in $\mathbb{R}^N$, $u_0 \in (L^1_{\rm loc}(\Omega))^l$ and $u \in (L^1_{\rm loc}(\Omega \times [0,T[))^l$ a local weak solution on $\Omega \times [0,T[$ to (6.1). Let $M = \sum_{i=1}^N ||A_i^T||$, $x^0 \in \Omega$ and $B(x^0,r)$ a ball (for the Euclidean norm) of center x^0 and radius r relatively compact in Ω. If $\varepsilon < \inf(1/4Mr(N+1), T/r^2)$ and $u^0 = 0$ a.e. on $B(x^0,r)$ then, by denoting $\mathcal{U} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid x \in B(x^0,r)\,,\ 0 < t < \varepsilon(r^2 - |x-x^0|^2)\}$, we have $u = 0$ a.e. on $\mathcal{U}$.*

Remark 6.10 *The condition $\varepsilon r^2 < T$ ensures that $\mathcal{U} \subset \Omega \times [0,T[$.*

Proof of Proposition 6.2
Thanks to Lemma 6.1, we see that $v(x,t) = u(x^0 + x, t)$ defines a function $v \in (L^1_{\rm loc}(B(0,r) \times [0,T[))^l$ which is a local weak solution to

$$\begin{cases} v_t(x,t) + \sum\limits_{i=1}^N A_i v_{x_i}(x,t) = 0, & (x,t) \in B(0,r)\times]0,T[, \\ v(x,0) = 0, & x \in B(0,r). \end{cases} \tag{6.15}$$

Proving that $u = 0$ a.e. on $\mathcal{U}$ is equivalent to proving that $v = 0$ a.e. on $\mathcal{V} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r\,,\ 0 < t < \varepsilon(r^2 - |x|^2)\} \subset B(0,r)\times]0,T[$.

Let $\varphi = (\varphi_1, \ldots, \varphi_l) \in (\mathcal{C}^\infty_c(B(0,r) \times [0,T[))^l$; by adding, for $j = 1$ to l, the $j^{\rm th}$ component of the equation satisfied by v when we use φ_j as a test function, and by denoting $X \cdot Y$ the scalar product of two vectors $(X,Y) \in \mathbb{R}^l$, we find

$$\int_{B(0,r)\times]0,T[} v(x,t) \cdot \varphi_t(x,t) + \sum_{i=1}^N A_i v(x,t) \cdot \varphi_{x_i}(x,t)\, dx\, dt = 0,$$

that is to say, with $B_i = A_i^T$,

$$\int_{B(0,r)\times]0,T[} v(x,t) \cdot \left(\varphi_t(x,t) + \sum_{i=1}^N B_i \varphi_{x_i}(x,t)\right) dx\, dt = 0, \quad \forall \varphi \in (\mathcal{C}^\infty_c(B(0,r) \times [0,T[))^l. \tag{6.16}$$

We will see that, thanks to Lemma 6.3, (6.16) implies $v = 0$ a.e. on $\mathcal{V}$.

Since $4Mr(N+1)\varepsilon < 1$ (with $M = \sum_{i=1}^N ||B_i||$), thanks to Lemma 6.3, there exists, for all $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^l$ polynomial function, $f \in (\mathcal{C}^\infty(\overline{V}))^l$ (with $V = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r\,,\ |t| < \varepsilon(r^2 - |x|^2)\}$) solution to (6.27).
Suppose that we can use $\varphi = f\mathbf{1}_{\mathcal{V}}$ as a test function in (6.16); we would then obtain

$$\int_{\mathcal{V}} v(x,t) \cdot F(x,t)\, dx\, dt = 0 \tag{6.17}$$

for all polynomial function F; since these functions are dense in $(\mathcal{C}(\overline{\mathcal{V}}))^l$ and since $v \in (L^1(\mathcal{V}))^l$, this equation would also be true for all $F \in (\mathcal{C}(\overline{\mathcal{V}}))^l$, which is enough to see that $v = 0$ a.e. on $\mathcal{V}$.

Our aim now is to prove that, by approximating f in a precise way, (6.17) is true, which will conclude the proof of this proposition.
It is, in fact, fairly simple thanks to Lemma 6.4: with the sequence $(\gamma_n)_{n\geq 1}$ given in this lemma, we notice that, for all $n \geq 1$, $f_n = (\gamma_n f)_{|B(0,r)\times[0,T[} \in (\mathcal{C}_c^\infty(B(0,r) \times [0,T[))^l$ (because $f \in (\mathcal{C}^\infty(\overline{V}))^l$ and $\mathrm{supp}(\gamma_n)$ is a compact set of $V \subset B(0,r) \times]-\infty, T[$); thus, $\varphi = f_n$ is valid in (6.16) and we get

$$\int_{\mathcal{V}} v(x,t) \cdot \left((f_n)_t(x,t) + \sum_{i=1}^N B_i (f_n)_{x_i}(x,t) \right) dx\, dt = 0, \quad \forall n \geq 1. \tag{6.18}$$

Moreover, for $X \in \{x_1, \ldots, x_N, t\}$, we have

$$\frac{\partial f_n}{\partial X} = \frac{\partial \gamma_n}{\partial X} f_n + \gamma_n \frac{\partial f}{\partial X}.$$

The properties of γ_n gives us $\partial_X \gamma_n(x,t) = 0$ when $|x| < r - 1/n$ and $0 \leq t < \varepsilon(r^2 - |x|^2) - K/n$ (with K given in Lemma 6.4), and $\gamma_n(x,t) \to 1$ for all $(x,t) \in \mathcal{V}$. Thus, $\partial_X f_n(x,t) \to \partial_X f(x,t)$ for all $(x,t) \in \mathcal{V}$. We want now to prove that $\partial_X f_n$ is bounded in $(L^\infty(\mathcal{V}))^l$. Let $n > 1/r$.

- If $|x| < r - 1/n$ and $0 \leq t < \varepsilon(r^2 - |x|^2) - K/n$, then

$$\left|\left| \frac{\partial f_n}{\partial X}(x,t) \right|\right| \leq |\gamma_n(x,t)| \left|\left| \frac{\partial f}{\partial X}(x,t) \right|\right| \leq \left|\left| \frac{\partial f}{\partial X} \right|\right|_{(L^\infty(\mathcal{V}))^l}. \tag{6.19}$$

- If $(x,t) \in \overline{\mathcal{V}}$ are such that $r - 1/n \leq |x|$ or $\varepsilon(r^2 - |x|^2) - K/n \leq t$, then there exists $x' \in \overline{B}(0,r)$ such that $|x - x'| \leq 1/n$ and $|t - \varepsilon(r^2 - |x'|^2)| \leq \sup(2\varepsilon r, K)/n$ (take $x' = rx/|x|$ if $t < \varepsilon(r^2 - |x|^2) - K/n$ and $x' = x$ else).

  Since f is a Lipschitz continuous function on $\overline{V}$ (it is in $(\mathcal{C}^\infty(\overline{V}))^l$), and since $f(x', \varepsilon(r^2 - |x'|^2) = 0$ for all $|x'| \leq r$, there exists thus $C_0 > 0$ such that, for all $n \geq 1$, $||f(x,t)|| \leq C_0/n$ as soon as $r - 1/n \leq |x| \leq r$ or $\varepsilon(r^2 - |x|^2) - K/n \leq t \leq \varepsilon(r^2 - |x|^2)$. This gives then, with the properties of the derivatives of γ_n, for all such (x,t),

$$\left|\left| \frac{\partial f_n}{\partial X}(x,t) \right|\right| \leq \left| \frac{\partial \gamma_n}{\partial X}(x,t) \right| ||f(x,t)|| + |\gamma_n(x,t)| \left|\left| \frac{\partial f}{\partial X}(x,t) \right|\right| \leq CC_0 + \left|\left| \frac{\partial f}{\partial X} \right|\right|_{(L^\infty(\mathcal{V}))^l}. \tag{6.20}$$

The derivatives of f_n converge thus on $\mathcal{V}$ to the derivatives of f, and, thanks to (6.19) and (6.20), are bounded in $(L^\infty(\mathcal{V}))^l$; using the dominated convergence theorem, we can pass to the limit $n \to \infty$ in (6.18) to obtain

$$\int_{\mathcal{V}} v(x,t) \cdot \left(f_t(x,t) + \sum_{i=1}^N B_i f_{x_i}(x,t) \right) dx\, dt = \int_{\mathcal{V}} v(x,t) \cdot F(x,t)\, dx\, dt = 0,$$

i.e. what we wanted. ■

Corollary 6.2 *Let* $\Omega =]-3\nu\sqrt{N}, 3\nu\sqrt{N}[^N$ *(*$\nu \in \mathbb{R}_*^+$*),* $(A_1,\dots,A_N) \in M_l(\mathbb{R})$ *and* $T>0$*. There exists* $T_0 \in]0,T]$ *such that, if* $u \in (L^1_{\mathrm{loc}}(\Omega\times[0,T[))^l$ *is a local weak solution on* $\Omega\times[0,T[$ *to (6.1) for an initial condition which is null a.e. on* $]-2\nu\sqrt{N}, 2\nu\sqrt{N}[^N$*, then* $u=0$ *a.e. on* $]-\nu,\nu[^N\times]0,T_0[$*.*

Proof of Corollary 6.2
Let $r = 2\nu\sqrt{N}$ and choose ε satisfying the hypothesis of Proposition 6.2 (notice that ε only depends on N, r, $\sum_{i=1}^N ||A_i^T||$ and T). Let $T_0 = \varepsilon(r^2-(r/2)^2) = 3\varepsilon r^2/4$ (which only depends on N, r, $(A_1,\dots,A_N)$, a and T).
By applying Proposition 6.2 to u and $x^0=0$ (in which case $B(0,r)\subset]-2\nu\sqrt{N},2\nu\sqrt{N}[^N$ and the initial condition corresponding to u is null a.e. on $B(0,r)$), we get $u=0$ a.e. on $\mathcal{U} = \{(x,t)\in\mathbb{R}^N\times\mathbb{R} \mid |x| < 2\nu\sqrt{N},\ 0<t<\varepsilon(r^2-|x|^2)\}$.
For all $x\in]-\nu,\nu[^N$, one has $|x|<\nu\sqrt{N} = r/2$ so that $\varepsilon(r^2-|x|^2) > T_0$; thus, $]-\nu,\nu[^N\times]0,T_0[\subset\mathcal{U}$ and the corollary is proved. ■

6.4.2 Proof of the main result

We can now prove Theorems 6.1 and 6.2 when $N\geq 2$. The proof of Theorem 6.2 being very similar to the proof of 1) $\Longrightarrow$ 2) in Theorem 6.1, we only detail the proof of Theorem 6.1.

Proof of Theorem 6.1 when $N \geq 2$
We prove 1) $\Longrightarrow$ 2) in the first two steps, and 2) $\Longrightarrow$ 1) in the third step.

Step 1: the case $\xi = (1,0,\dots,0)$.
We suppose that Ω is an open set of $\mathbb{R}^N$, that a belongs to Ω and that, for any initial condition in $E(\Omega,a)$, (6.1) has a local weak solution on $\Omega\times[0,T[$ (for some $T>0$ depending on the initial condition), and we will prove that, under these hypotheses, A_1 is diagonalizable on $\mathbb{R}$.
Thanks to item ii) of Remark 6.7, we can suppose (by taking an hypercube $O =]-6\nu\sqrt{N}, 6\nu\sqrt{N}[$ in $-a+\Omega$, $P=Id$, $\mu=1$ and $x^0=-a$) that (6.1) has a local weak solution for any initial condition in $E(O,0)$.
To show that A_1 is diagonalizable on $\mathbb{R}$, we will prove that the system (6.5) with A_1 instead of A has, for any initial condition in $E(]-\nu,\nu[,0)$, a local weak solution on $]-\nu,\nu[\times[0,T[$ (for a $T>0$ depending on the initial condition); once this result is proved, and since we have already proven Theorem 6.1 when $N=1$, this will allow us to conclude that A_1 is indeed diagonalizable on $\mathbb{R}$.

Let $v^0 \in E(]\nu,\nu[,0)$; we have, for some $\xi_1\in\mathbb{R}^*$ and some $v^*\in\mathbb{R}^l$, $v^0(s) = v^*\mathbf{1}_{\mathbb{R}^-}(s\xi_1)$. Define $u^0\in E(O,0)$ by $u^0(x) = v^*\mathbf{1}_{\mathbb{R}^-}(x\cdot\xi)$, where $\xi=(\xi_1,0,\dots,0)$, that is to say $u^0(x) = v^*\mathbf{1}_{\mathbb{R}^-}(x_1\xi_1)$.
By hypothesis, there exists $T>0$ and a local weak solution $u\in(L^1_{\mathrm{loc}}(O\times[0,T[))^l$ on $O\times[0,T[$ to (6.1).

Let $h=(0,h_2,\dots,h_N)\in]-3\nu\sqrt{N},3\nu\sqrt{N}[^N$ and define, on $]-3\nu\sqrt{N},3\nu\sqrt{N}[$, $u_h(x) = u(x+h)$. Thanks to Lemma 6.1 and the first item of Remark 6.7 (with $P=Id$, $\lambda=0$, $\mu=1$, $x^0=-h$ and $]-3\nu\sqrt{N},3\nu\sqrt{N}[^N$ as the open set included in x^0+O), we see that u_h is a local weak solution on $]-3\nu\sqrt{N},3\nu\sqrt{N}[\times[0,T[$ to the system (6.1) with an initial condition ${u^0}_h(x) = u^0(x+h) = u^0(x)$ (by definition of u^0 and h). Thus, u_h-u is a local weak solution on $]-3\nu\sqrt{N},3\nu\sqrt{N}[\times[0,T[$ to the system (6.1) with a null initial condition.
Corollary 6.2 tells us that there exists $T_0\in]0,T]$ *not depending on* h such that $u_h-u=0$ a.e. on $]-\nu,\nu[^N\times]0,T_0[$.
Thus, for a.e. $(x,t)\in]-\nu,\nu[^N\times]0,T_0[$ and any $h=(0,h_2,\dots,h_N)\in]-2\nu,2\nu[^N\subset]-3\nu\sqrt{N},3\nu\sqrt{N}[^N$, we have $u(x+h,t)=u(x,t)$; this means that $u_{|]-\nu,\nu[^N\times]0,T_0[}$ only depends on the first space variable x_1 and on t, i.e. that there exists $v\in(L^1_{\mathrm{loc}}(]-\nu,\nu[\times[0,T_0[))^l$ such that, for a.e. $(x,t)\in]-\nu,\nu[^N\times[0,T_0]$, $u(x,t)=v(x_1,t)$.

Since u is a local weak solution to (6.1) on $O\times[0,T[$, it is also a local weak solution of the same system on $]-\nu,\nu[^N\times[0,T_0[$, that is to say: for all $\varphi\in\mathcal{C}_c^\infty(]-\nu,\nu[^N\times[0,T_0[)$,

$$\int_{]-\nu,\nu[^N\times]0,T_0[} u(x,t)\varphi_t(x,t)+\sum_{i=1}^N A_iu(x,t)\varphi_{x_i}(x,t)\,dx\,dt+\int_{]-\nu,\nu[^N} v^0(x_1)\varphi(x,0)\,dx=0. \tag{6.21}$$

But, for all $\varphi\in\mathcal{C}_c^\infty(]-\nu,\nu[^N\times[0,T_0[)$, the Fubini theorem gives, if $i\geq 2$,

$$\begin{aligned}&\int_0^{T_0}\int_{]-\nu,\nu[^N} A_iu(x,t)\varphi_{x_i}(x,t)\,dx\,dt\\ &\quad=\int_0^{T_0}\int_{]-\nu,\nu[^{N-1}} A_iv(x_1,t)\left(\int_{-\nu}^{\nu}\varphi_{x_i}(x,t)\,dx_i\right)dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_N\,dt=0.\end{aligned} \tag{6.22}$$

Take now $\psi\in\mathcal{C}_c^\infty(]-\nu,\nu[\times[0,T_0[)$ and $\varphi(x,t)=\psi(x_1,t)\gamma(x_2,\ldots,x_N)$, with $\gamma\in\mathcal{C}_c^\infty(]-\nu,\nu[^{N-1})$ the integral of which is equal to 1. (6.21) and (6.22) gives us

$$\int_{]-\nu,\nu[\times]0,T_0[} v(x_1,t)\psi_t(x_1,t)+A_1v(x_1,t)\psi_{x_1}(x_1,t)\,dx_1\,dt+\int_{]-\nu,\nu[} v^0(x_1)\psi(x_1,0)\,dx_1=0,$$

which exactly means that $v\in(L^1_{\rm loc}(]-\nu,\nu[^N\times[0,T_0[))^l$ is a local weak solution on $]-\nu,\nu[\times[0,T_0[$ to (6.5) with A_1 instead of A and v^0 instead of u^0. Since v^0 was an arbitrary function in $E(]-\nu,\nu[,0)$, this concludes this step.

Step 2: the general case.
We suppose that Ω is an open set of $\mathbb{R}^N$, that a belongs to Ω and that, for any initial condition in $E(\Omega,a)$, (6.1) has a local weak solution on $\Omega\times[0,T[$ (for some $T>0$ depending on the initial condition).
Take $\xi\in\mathbb{R}^N\backslash\{0\}$ (the case $\xi=0$ being trivial). We want to show that $\sum_{i=1}^N A_i\xi_i$ is diagonalizable on $\mathbb{R}$; obviously, by dividing this matrix by $|\xi|$, we can consider that $|\xi|=1$.

We notice first an easy property of (6.1) with respect to orthogonal changes of variables.
Let $Q=((q_{i,j}))_{(i,j)\in[1,N]^2}$ be an orthogonal $N\times N$ matrix, and denote $\Omega'=Q^{-1}(\Omega)$.
Take $u^0\in(L^1_{\rm loc}(\Omega'))^l$ and $u\in(L^1_{\rm loc}(\Omega'\times[0,T[))^l$ (for some $T>0$) and define $v^0\in(L^1_{\rm loc}(\Omega))^l$ and $v\in(L^1_{\rm loc}(\Omega\times[0,T[))^l$ by $v^0(x)=u^0(Q^{-1}x)$ and $v(x,t)=u(Q^{-1}x,t)$.
Then one has, for any $\varphi\in\mathcal{C}_c^\infty(\Omega'\times[0,T[)$, by denoting $\psi(x,t)=\varphi(Q^{-1}x,t)\in\mathcal{C}_c^\infty(\Omega\times[0,T[)$ (which satisfies $\psi_t(x,t)=\varphi_t(Q^{-1}x,t)$ and $\nabla_x\psi(x,t)=(Q^{-1})^T\nabla_x\varphi(Q^{-1}x,t)=Q\nabla_x\varphi(Q^{-1}x,t)$),

$$\begin{aligned}&\int_0^T\int_\Omega v(x,t)\psi_t(x,t)+\sum_{i=1}^N A_iv(x,t)\psi_{x_i}(x,t)\,dx\,dt+\int_\Omega v^0(x)\psi(x,0)\,dx\\
&\quad=\int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t)+\sum_{i=1}^N A_iu(x,t)\left(Q\nabla_x\varphi\right)_i(x,t)\,dx\,dt+\int_{\Omega'} u^0(x)\varphi(x,0)\,dx\\
&\quad=\int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t)+\sum_{i=1}^N A_iu(x,t)\sum_{j=1}^N q_{i,j}\varphi_{x_j}(x,t)\,dx\,dt+\int_{\Omega'} u^0(x)\varphi(x,0)\,dx\\
&\quad=\int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t)+\sum_{j=1}^N\left(\sum_{i=1}^N q_{i,j}A_i\right)u(x,t)\varphi_{x_j}(x,t)\,dx\,dt+\int_{\Omega'} u^0(x)\varphi(x,0)\,dx.\end{aligned}$$

Thus, if v is a local weak solution on $\Omega\times[0,T[$ to (6.1) with v^0 as initial condition, then u is a local weak solution on $\Omega'\times[0,T[$ to (6.1) with $\mathcal{A}_j=\sum_{i=1}^N q_{i,j}A_i$ instead of A_j ($j\in[1,N]$) and u^0 as initial condition.

Take now an orthogonal matrix Q such that $q_{i,1} = \xi_i$ for all $i \in [1, N]$ (this is possible since $|\xi| = 1$). We still denote $\Omega' = Q^{-1}(\Omega)$.
Let $u^0 \in E(\Omega', Q^{-1}a)$; u^0 can be written as $u^*\mathbf{1}_{\mathbb{R}^-}((\cdot - Q^{-1}a)\cdot\eta)$ for some $u^* \in \mathbb{R}^l$ and $\eta \in \mathbb{R}^N\backslash\{0\}$. $v^0 = u^0 \circ Q^{-1}$ is then $v^0(x) = u^*\mathbf{1}_{\mathbb{R}^-}((Q^{-1}x - Q^{-1}a)\cdot\eta) = u^*\mathbf{1}_{\mathbb{R}^-}((x-a)\cdot Q\eta)$ (because $Q^{-1} = Q^T$ since Q is an orthogonal matrix), and we have then $v^0 \in E(\Omega, a)$; by hypothesis, there exists thus $T > 0$ and a local weak solution $v \in (L^1_{\rm loc}(\Omega \times [0,T[))^l$ on $\Omega \times [0, T[$ to (6.1) for the initial condition v^0.
Thus, by the preceding calculus, $u(x,t) = v(Qx, t)$ is a local weak solution on $\Omega' \times [0, T[$ to (6.1) with $\mathcal{A}_j = \sum_{i=1}^N q_{i,j}A_i$ instead of A_j ($j \in [1, N]$) and u^0 as initial condition. Since u^0 is an arbitrary function of $E(\Omega', Q^{-1}a)$, the reasoning of Step 1 shows that $\mathcal{A}_1 = \sum_{i=1}^N \xi_i A_i$ is diagonalizable on $\mathbb{R}$.

Step 3: proof of 2) $\Longrightarrow$ 1).
Let $u^0 = u^*\mathbf{1}_{\mathbb{R}^-}((\cdot - a)\cdot\xi) \in E(\Omega, a)$. We can always suppose that $|\xi| = 1$ (because $\mathbf{1}_{\mathbb{R}^-}((\cdot - a)\cdot\xi) = \mathbf{1}_{\mathbb{R}^-}((\cdot - a)\cdot\frac{\xi}{|\xi|})$).

The idea is of course (as indicated by the first step of this proof) to search for a solution u only depending on the coordinate along ξ, that is to say $u(x,t) = v((x-a)\cdot\xi, t)$. We see then that v must satisfy

$$\left\{\begin{array}{ll} v_t(z,t) + \left(\sum_{i=1}^N \xi_i A_i\right) v_z(z,t) = 0 & z \in \mathbb{R},\ t > 0 \\ v(z,0) = u^*\mathbf{1}_{\mathbb{R}^-}(z) & z \in \mathbb{R}. \end{array}\right. \tag{6.23}$$

This problem is a 1-dimensional one and, since the matrix $\sum_{i=1}^N \xi_i A_i$ is diagonalizable on $\mathbb{R}$, we can solve it.

Take $(e_1, \ldots, e_l)$ a basis of $\mathbb{R}^l$ made of eigenvectors of $\sum_{i=1}^N \xi_i A_i$ and denote by λ_α the eigenvalue of $\sum_{i=1}^N \xi_i A_i$ associated to e_α ($\alpha \in [1, l]$). We write $u^* = \sum_{\alpha=1}^l u^*_\alpha e_\alpha$.
Define $u \in (L^1_{\rm loc}(\mathbb{R}^N \times [0,\infty[))^l$ by $u(x,t) = \sum_{\alpha=1}^l u^*_\alpha \mathbf{1}_{\mathbb{R}^-}((x-a)\cdot\xi - \lambda_\alpha t)e_\alpha$. Then, using the calculus made in Step 2 (to reduce to the case $\xi = (1, 0, \ldots, 0)$) and some simple changes of variables, one can see that u is a weak solution on $\mathbb{R}^N \times [0, \infty[$ to (6.1) with u^0 as initial condition. ■

6.5 About the linearization of a particular non-hyperbolic problem

Recall that a non-linear problem

$$\left\{\begin{array}{l} u_t + (f(u))_x = 0, \\ u(x,0) = u^0(x) \end{array}\right. \tag{6.24}$$

is (unconditionaly) hyperbolic if, for every $\overline{u} \in \mathbb{R}^l$, the matrix $f'(\overline{u})$ is diagonalizable on $\mathbb{R}$ (i.e. the linearized problem around any state $\overline{u}$ is hyperbolic).
We have already noticed (see item ii) of Remark 6.3) that the hyperbolicity of a non-linear system of the kind (6.24) is interesting, for example when we want to apply classical Finite Volumes Schemes to the problem.

Consider the 2×2 following classical system (this is the pressureless gas system):

$$\left\{\begin{array}{ll} \begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in \mathbb{R},\ t \in \mathbb{R}^+_*, \\ \begin{pmatrix} \rho \\ u \end{pmatrix}(x,0) = \begin{pmatrix} \rho^0 \\ u^0 \end{pmatrix}(x), & x \in \mathbb{R}, \end{array}\right.$$

This system is equivalent, for regular solutions such that $\rho \neq 0$, to

$$\left\{\begin{array}{ll} \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ u^2/2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in \mathbb{R},\ t \in \mathbb{R}_*^+, \\ \begin{pmatrix} \rho \\ u \end{pmatrix}(x,0) = \begin{pmatrix} \rho^0 \\ u^0 \end{pmatrix}(x), & x \in \mathbb{R}, \end{array}\right. \tag{6.25}$$

Take now $u^0(x) = -1$ if $x < 0$, $u^0(x) = +1$ if $x > 0$ and $\rho^0 \in L^\infty(\mathbb{R})$.
A straightforward computation lets us see that the functions

$$\rho(x,t) = \left\{\begin{array}{ll} \rho^0(x-t) & \text{if } x < -t, \\ 0 & \text{if } -t \leq x \leq t, \\ \rho^0(x+t) & \text{if } t < x \end{array}\right.$$

and

$$u(x,t) = \left\{\begin{array}{ll} -1 & \text{if } x < -t, \\ \dfrac{x}{t} & \text{if } -t \leq x \leq t, \\ +1 & \text{if } t < x, \end{array}\right. \quad \text{(that is to say the entropy solution of the Burgers problem)}$$

define a (space and time global) weak solution to (6.25) in the following natural sense: $(\rho, u) \in (L^\infty(\mathbb{R} \times [0,+\infty[))^2$ and, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times [0,+\infty[)$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \begin{pmatrix} \rho \\ u \end{pmatrix}(x,t)\varphi_t(x,t)\,dx\,dt + \int_0^{+\infty}\int_{\mathbb{R}} \begin{pmatrix} \rho u \\ u^2/2 \end{pmatrix}(x,t)\varphi_x(x,t)\,dx\,dt + \int_{\mathbb{R}} \begin{pmatrix} \rho^0 \\ u^0 \end{pmatrix}(x)\varphi(x,0)\,dx = 0.$$

However, the linearization of (6.25) around a state $(\overline{\rho}, \overline{u}) \in \mathbb{R}^2$ is the linear system given by the matrix

$$A(\overline{\rho}, \overline{u}) = \begin{pmatrix} \overline{u} & \overline{\rho} \\ 0 & \overline{u} \end{pmatrix}.$$

This matrix being diagonalizable only if $\overline{\rho} = 0$, the system (6.25) is not unconditionaly hyperbolic (it is conditionaly hyperbolic in the sense that the hyperbolicity of the linearized system depends on the state around which the linearization has been made).
Moreover, a close examination of the step 1.2 of the proof of Theorem 6.1 when $N = 1$ shows that the initial condition (ρ^0, u^0) we have chosen here is precisely the kind of initial condition for which the system defined by $A(\overline{\rho}, \overline{u})$ has no local weak solution (when $\overline{\rho} \neq 0$, which is for example the case when we choose $\rho^0 \equiv 1$ and we linearize around a state in the image of the initial condition).
On this example, we have shown that a non-linear system can have a weak solution for a particular initial datum, although the corresponding linearized system has no solution for this initial datum.

Notice however that, in our example, the solution we have found partly pass, for any $t > 0$, in the hyperbolicity zone of (6.25) (i.e. $\{(\overline{\rho}, \overline{u}) \in \mathbb{R}^2 \mid \overline{\rho} = 0\}$).

6.6 Appendix

6.6.1 Harmonic functions

The following lemma is a very classical result, but we give here a simple and self-contained proof.

Lemma 6.2 *Let Ω be an open set of $\mathbb{R}^N$, with $N \geq 1$. If $u \in L^1_{\text{loc}}(\Omega)$ satisfies $\Delta u = 0$ in $\mathcal{D}'(\Omega)$, then $u \in \mathcal{C}^\infty(\Omega)$.*

Proof of Lemma 6.2

Let us first introduce (or recall) some notations. $|.|$ designates the Euclidean norm in $\mathbb{R}^N$; when $r \in \mathbb{R}^+$, B_r is the Euclidean ball in $\mathbb{R}^N$, of center 0 and radius r and, for $x \in \mathbb{R}^N$, $B(x,r) = x + B_r$. When E is a borelian set of $\mathbb{R}^N$, $|E|$ stands for the Lebesgue measure of E.

Let us take $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ whose support is contained in B_1 and which satisfies

$$\rho \geq 0\,, \quad \int_{\mathbb{R}^N} \rho(x)\, dx = 1 \text{ and } \rho(x) = \rho(y) \text{ whenever } |x| = |y|.$$

Define $\rho_n(x) = n^N \rho(nx)$. The function

$$u_n(x) = \int_\Omega u(t)\rho_n(x-t)\, dt = \int_{B_{1/n}} u(x+t)\rho_n(t)\, dt \qquad (\text{since } \rho_n(-\cdot) = \rho_n(\cdot))$$

is well defined and $\mathcal{C}^\infty$ on the open set $\Omega_n = \{x \in \Omega \mid d(x, \Omega^c) > 1/n\}$; moreover, one can verify that $\Delta u = 0$ in $\mathcal{D}'(\Omega)$ implies $\Delta u_n = 0$ in $\mathcal{D}'(\Omega_n)$ (thus in a classical way since u_n is regular).
u_n being regular and harmonic on Ω_n, it is well known that, for all $x \in \Omega_n$ and all $r < d(x, \Omega_n^c)$,

$$u_n(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u_n(y)\, dy. \tag{6.26}$$

Since $u \in L^1_{\text{loc}}(\Omega)$, we know that u_n (extented by 0 outside Ω_n) converges, as $n \to \infty$, to u in $L^1_{\text{loc}}(\Omega)$, thus a.e. up to a subsequence. For a point $x \in \Omega$ where $u_n(x) \to u(x)$, i.e. for a.e. $x \in \Omega$, and for $r < d(x, \Omega^c)$, passing to the limit $n \to \infty$ in (6.26) (which is satisfied as soon as $n > 1/(\text{dist}(x, \Omega^c) - r)$) gives us

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy = \frac{1}{|B_r|} \int_\Omega u(y) \mathbf{1}_{B(x,r)}(y)\, dy.$$

By the dominated convergence theorem, we see from this formula that u is (a.e. equal to) a continuous function on Ω; moreover, taking $s < r < d(x, \Omega^c)$, we obtain

$$\begin{aligned}
\frac{1}{|B(x,r)\backslash B(x,s)|} \int_{B(x,r)\backslash B(x,s)} u(y)\, dy &= \frac{1}{|B(x,r)| - |B(x,s)|} \int_{B(x,r)} u(y)\, dy \\
&\quad - \frac{1}{|B(x,r)| - |B(x,s)|} \int_{B(x,s)} u(y)\, dy \\
&= \frac{|B(x,r)|}{|B(x,r)| - |B(x,s)|} u(x) - \frac{|B(x,s)|}{|B(x,r)| - |B(x,s)|} u(x) = u(x).
\end{aligned}$$

Using the polar coordinates and passing to the limit $s \to r$ gives then, since u is continuous,

$$u(x) = \frac{1}{\sigma(S_1)} \int_{S_1} u(x+ry)\, d\sigma(y) \qquad \text{for all } x \in \Omega \text{ and } r < d(x, \Omega^c),$$

where $S_1 = \{y \in \mathbb{R}^N \mid |y| = 1\} = \partial B_1$ and σ is the $(N-1)$-dimensional measure on S_1.
Using the polar coordinates once again, and the fact that ρ_n is spherically invariant (that is to say, $\rho_n(x) = \theta_n(|x|)$ for a $\theta_n : \mathbb{R}^+ \to \mathbb{R}^+$), we get, for all $n \geq 1$ and all $x \in \Omega_n$,

$$\begin{aligned}
u_n(x) &= \int_0^{1/n} \left(\int_{S_1} u(x+ry)\, d\sigma(y) \right) r^{N-1} \theta_n(r)\, dr \\
&= u(x) \int_0^{1/n} \sigma(S_1) r^{N-1} \theta_n(r)\, dr \\
&= u(x) \int_0^{1/n} \int_{S_1} \rho_n(ry) r^{N-1}\, d\sigma(y)\, dr = u(x) \int_{B_{1/n}} \rho_n(t)\, dt = u(x),
\end{aligned}$$

which implies that $u \in \mathcal{C}^\infty(\Omega_n)$ for all $n \geq 1$, that is to say $u \in \mathcal{C}^\infty(\Omega)$. ∎

6.6.2 Lemmas for Proposition 6.2

Recall that $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$, $||\cdot||$ is any norm on $\mathbb{R}^l$ and, when $B \in M_l(\mathbb{R})$, $||B||$ denotes the induced norm.
The following result is a (very simple) particular case of the Cauchy-Kowalewska Theorem (see [33]). We however state and prove it, because we need the precise estimates on the time-space range of existence of the "local" solution given by the Cauchy-Kowalewska Theorem.

Lemma 6.3 *Let $(B_1, \ldots, B_N)$ be $l \times l$ real matrices; we denote $M = \sum_{i=1}^N ||B_i||$. Let $r \in \mathbb{R}_*^+$, and $\varepsilon > 0$ such that $4Mr(N+1)\varepsilon < 1$ and denote $V = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r\,,\ |t| < \varepsilon(r^2 - |x|^2)\}$. Then, for any polynomial function $F : \mathbb{R}^{N+1} \to \mathbb{R}^l$, there exists $f \in (C^\infty(\overline{V}))^l$ such that*

$$\begin{cases} f_t(x,t) + \sum_{i=1}^N B_i f_{x_i}(x,t) = F(x,t), & (x,t) \in V, \\ f(x, \varepsilon(r^2 - |x|^2)) = 0, & |x| \le r. \end{cases} \tag{6.27}$$

Proof of Lemma 6.3
If $M = 0$, then $B_i = 0$ for all $i \in [1,N]$ and the result is quite simple: the function $f(x,t) = \int_{\varepsilon(r^2-|x|^2)}^t F(x,s)\,ds$ defined and $\mathcal{C}^\infty$ on $\mathbb{R}^N \times \mathbb{R}$ is a solution to (6.27).
We suppose thus from now on that $M > 0$.

Let $\theta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be the $\mathcal{C}^\infty$-diffeomorphism $\theta(y,s) = (2My, s + \varepsilon(r^2 - |2My|^2))$. Denote $G = F \circ \theta$ (G is also a polynomial function), $U = \{(y,s) \in \mathbb{R}^{N+1} \mid |y| < r/2M\,,\ |s| < r/2M\}$ and suppose we have found $g \in (\mathcal{C}^\infty(\overline{U}))^l$ solution of

$$\begin{cases} g_s(y,s) + 4M\varepsilon \left(\sum_{i=1}^N y_i B_i\right) g_s(y,s) + \frac{1}{2M} \sum_{i=1}^N B_i g_{y_i}(y,s) = G(y,s), & (y,s) \in U, \\ g(y,0) = 0, & |y| \le r/2M. \end{cases} \tag{6.28}$$

Then, $f = g \circ \theta^{-1} \in (\mathcal{C}^\infty(\overline{\theta(U)}))^l$ and, since $\theta(U) \supset V$ (because $4Mr\varepsilon < 1/(N+1) < 1$), $f \in (\mathcal{C}^\infty(\overline{V}))^l$; moreover, $f(x,t) = g(x/2M, t - \varepsilon(r^2 - |x|^2))$ and we have thus, for all $(x,t) \in V$,

$$\begin{aligned} f_t(x,t) + \sum_{i=1}^N B_i f_{x_i}(x,t) &= g_s(x/2M, t - \varepsilon(r^2 - |x|^2))) + 2\varepsilon \sum_{i=1}^N 2M \frac{x_i}{2M} B_i g_s(x/2M, t - \varepsilon(r^2 - |x|^2)) \\ &\quad + \frac{1}{2M} \sum_{i=1}^N B_i g_{y_i}(x/2M, t - \varepsilon(r^2 - |x|^2)) \\ &= G(x/2M, t - \varepsilon(r^2 - |x|^2)) = F(x,t), \end{aligned}$$

with $f(x, \varepsilon(r^2 - |x|^2)) = g(x/2M, 0) = 0$ as soon as $|x| \le r$. Thus, solving (6.28) is enough to prove the lemma.

We search a solution to (6.28) under the form of a power series:

$$g(y,s) = \sum_{\alpha \in \mathbb{N}^N,\ k \in \mathbb{N}} g_{\alpha,k} y^\alpha s^k$$

(where $y^\alpha = y_1^{\alpha_1} \cdots y_N^{\alpha_N}$) with $g_{\alpha,k} \in \mathbb{R}^l$. It is then well known that, by denoting $D_y^\alpha = D_{y_1}^{\alpha_1} \cdots D_{y_N}^{\alpha_N}$ and $\alpha! = \alpha_1! \ldots \alpha_N!$, we must have $g_{\alpha,k} = D_y^\alpha D_s^k g(0,0)/\alpha! k!$. We will now compute these coefficients by supposing that g satisfies (6.28) and, then, show that the obtained coefficients define a $\mathcal{C}^\infty$ function on $\overline{U}$.

Using the second equation of (6.28), we see that $D_y^\alpha g(0,0) = 0$ for all $\alpha \in \mathbb{N}^N$. By applying, for $(\alpha, k) = (\alpha_1, \dots, \alpha_N, k) \in \mathbb{N}^N \times \mathbb{N}$, $D_y^\alpha D_s^k$ to the first equation of (6.28) and using Leibniz' formula, we find

$$\begin{aligned} D_y^\alpha D_s^{k+1} g_s(y,s) \quad &+ \quad 4M\varepsilon \sum_{i=1}^N (y_i B_i D_y^\alpha D_s^{k+1} g(y,s) + \alpha_i B_i D_y^{\alpha - e_i} D_s^{k+1} g(y,s)) \\ &+ \quad \frac{1}{2M} \sum_{i=1}^N B_i D_y^{\alpha+e_i} D_s^k g(y,s) = D_y^\alpha D_s^k G(y,s), \end{aligned}$$

where e_i is the element of $\mathbb{N}^N$ which has 1 on the i^{th} position and 0 on the other positions ($D_y^{\alpha - e_i}$ is the null operator if $\alpha_i = 0$). Thus,

$$D_y^\alpha D_s^{k+1} g(0,0) = D_y^\alpha D_s^k G(0,0) - 4M\varepsilon \sum_{i=1}^N \alpha_i B_i D_y^{\alpha-e_i} D_s^{k+1} g(0,0) - \frac{1}{2M} \sum_{i=1}^N B_i D_y^{\alpha+e_i} D_s^k g(0,0).$$

Define $a_{\alpha,0} = 0$ (for all $\alpha \in \mathbb{N}^N$) and, by induction on k and $|\alpha| = \sum_{i=1}^N \alpha_i$, for $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$,

$$a_{\alpha,k+1} = D_y^\alpha D_s^k G(0,0) - 4M\varepsilon \sum_{i=1}^N \alpha_i B_i a_{\alpha-e_i,k+1} - \frac{1}{2M} \sum_{i=1}^N B_i a_{\alpha+e_i,k} \tag{6.29}$$

(no matter the definition of $a_{\alpha-e_i,k+1}$ when $\alpha_i = 0$: since these coefficients are always multiplied by α_i, they can be omited in the sums up above).
We will now obtain an estimate on these $a_{\alpha,k}$. By denoting $K_n = \frac{1}{n!} \sup\{||a_{\alpha,k}||\,,\ |\alpha| + k = n\}$, (6.29) gives us, for all $n \geq 0$ and with $(\alpha, k) \in \mathbb{N}^{N+1}$ such that $|\alpha| + k = n$,

$$\begin{aligned} ||a_{\alpha,k+1}|| \quad &\leq \quad ||D_y^\alpha D_s^k G(0,0)|| + 4M\varepsilon \sum_{i=1}^N ||B_i|| \alpha_i n! K_n + \frac{1}{2M} \sum_{i=1}^N ||B_i|| (n+1)! K_{n+1} \\ &\leq \quad ||D_y^\alpha D_s^k G(0,0)|| + 4M^2 \varepsilon (n+1)! K_n + \frac{1}{2}(n+1)! K_{n+1}. \end{aligned}$$

Taking the suppremum of this on all $(\alpha, k) \in \mathbb{N}^{N+1}$ with $|\alpha| + k = n$ (recall that $a_{\alpha,0} = 0$ for all $\alpha \in \mathbb{N}^N$) and diving by $(n+1)!$, we get

$$K_{n+1} \leq \frac{\sup_{|\alpha|+k=n} ||D_y^\alpha D_s^k G(0,0)||}{(n+1)!} + 4M^2 \varepsilon K_n + \frac{1}{2} K_{n+1},$$

that is to say, with $C_n = \frac{2}{(n+1)!} \sup_{|\alpha|+k=n} ||D_y^\alpha D_s^k G(0,0)||$,

$$K_{n+1} \leq C_n + 8M^2 \varepsilon K_n \qquad \text{for all } n \geq 0.$$

We deduce from this the estimate that, for all $n \geq 1$ (recall that $K_0 = 0$),

$$K_n \leq C_{n-1} + (8M^2\varepsilon) C_{n-2} + \dots + (8M^2\varepsilon)^l C_{n-1-l} + \dots + (8M^2\varepsilon)^{n-1} C_0.$$

But, by denoting n_0 the degree of the polynomial G, we have $C_{n-1-l} = 0$ as soon as $n - 1 - l > n_0$; by taking $C = \sup(C_0, \dots, C_{n_0})$ and $n \geq n_0 + 1$, the preceding estimate is reduced to

$$K_n \quad \leq \quad C(8M^2\varepsilon)^{n-1-n_0}(1 + 8M^2\varepsilon + \ldots + (8M^2\varepsilon)^{n_0}).$$

Thanks to the hypothesis on ε, there exists thus $R > r/2M$ such that $8M^2\varepsilon < 1/(N+1)R$; we can thus find $\overline{K} \in \mathbb{R}^+$ such that, for all $n \geq 0$, $K_n \leq \overline{K}/((N+1)R)^n$, that is to say:

$$\text{for all } (\alpha, k) \in \mathbb{N}^N,\ ||a_{\alpha,k}|| \leq \frac{\overline{K}(|\alpha| + k)!}{((N+1)R)^{|\alpha|+k}}. \tag{6.30}$$

Using the fact that, as soon as $\sup_{i\in[1,N]}|y_i|<R$ and $|s|<R$, the series

$$\sum_{\alpha\in\mathbb{N}^N\,,\,k\in\mathbb{N}}\frac{(|\alpha|+k)!}{\alpha!k!}\frac{|y_1|^{\alpha_1}\cdots|y_N|^{\alpha_N}|s|^k}{((N+1)R)^{|\alpha|+k}}$$
$$=\sum_{n\geq 1}\sum_{|\alpha|+k=n}\frac{(|\alpha|+k)!}{\alpha!k!}\left(\frac{|y_1|}{(N+1)R}\right)^{\alpha_1}\cdots\left(\frac{|y_N|}{(N+1)R}\right)^{\alpha_N}\left(\frac{|s|}{(N+1)R}\right)^k$$
$$=\sum_{n\geq 0}\left(\frac{|y_1|}{(N+1)R}+\ldots+\frac{|y_N|}{(N+1)R}+\frac{|s|}{(N+1)R}\right)^n$$
$$=\frac{1}{1-(|y_1|/(N+1)R+\ldots+|y_N|/(N+1)R+|s|/(N+1)R)}$$

converges (because $(|y_1|+\ldots+|y_N|+|s|)/((N+1)R)<1$), we can prove, thanks to (6.30), that

$$g(y,s)=\sum_{\alpha\in\mathbb{N}^N\,,\,k\in\mathbb{N}}\frac{a_{\alpha,k}}{\alpha!k!}y^\alpha s^k$$

converges on $W=\{(y,s)\in\mathbb{R}^N\times\mathbb{R}\mid\sup_{i\in[1,N]}|y_i|<R,\ |s|<R\}$ and defines a function which is $\mathcal{C}^\infty$ on W (the derivative of g being obtained as the series of the derivatives of $\frac{a_{\alpha,k}}{\alpha!k!}y^\alpha s^k$).
Since $a_{\alpha,0}=0$ for all $\alpha\in\mathbb{N}^N$, we have $g(y,0)=0$ as soon as $(y,0)\in W$. Moreover, $S(y,s)=g_s(y,s)+4M\varepsilon(\sum_{i=1}^N y_iB_i)g_s(y,s)+\frac{1}{2M}\sum_{i=1}^N B_i g_{y_i}(y,s)-G(y,s)$ is a power series on W, the derivatives of which all vanish at $(y,s)=(0,0)$ (the $a_{\alpha,k}$ have been constructed to satisfy this property): thus, $S\equiv 0$ on W.
But $W\supset\{(y,s)\in\mathbb{R}^N\times\mathbb{R}\mid|y|<R,\ |s|<R\}\supset\overline{U}$ (recall that $R>r/2M$), so that $g\in(C^\infty(\overline{U}))^l$ satisfies (6.28). ■

Lemma 6.4 *Let $(r,\varepsilon)\in(\mathbb{R}_*^+)^2$ and $K>1+2\varepsilon r+\varepsilon$. There exists $C>0$ and $(\gamma_n)_{n\geq 1}\in\mathcal{C}_c^\infty(\mathbb{R}^N\times\mathbb{R})$ such that, by denoting $V_n=\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<r-1/n\,,\ |t|<\varepsilon(r^2-|x|^2)-K/n\}$, we have, for all $n\geq 1$:*

$$\begin{aligned}&\mathrm{supp}(\gamma_n)\subset V=\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<r\,,\ |t|<\varepsilon(r^2-|x|^2)\},\\&||\gamma_n||_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq 1,\\&\gamma_n\equiv 1\ \text{on } V_n,\\&||\nabla_{(x,t)}\gamma_n||_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq Cn.\end{aligned}$$

Proof of Lemma 6.4
Take $\rho\in\mathcal{C}_c^\infty(\mathbb{R}^{N+1})$, the support of which is included in $\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<1\,,\ |t|<1\}$, $\rho\geq 0$ and $\int_{\mathbb{R}^{N+1}}\rho(x,t)\,dx\,dt=1$. Let $\rho_n(x,t)=n^{N+1}\rho(nx,nt)$; the support of ρ_n is included in $\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<1/n\,,\ |t|<1/n\}$
Define $\gamma_n=\mathbf{1}_{V_{2n}}*\rho_{2n}$; γ_n is in $\mathcal{C}_c^\infty(\mathbb{R}^N\times\mathbb{R})$ and is bounded by 1 in $L^\infty(\mathbb{R}^N\times\mathbb{R})$ (Young inequality).
For $X\in\{x_1,\ldots,x_N,t\}$, one has

$$\frac{\partial\gamma_n}{\partial X}=\mathbf{1}_{V_{2n}}*\frac{\partial\rho_{2n}}{\partial X},$$

and, since $\partial_X\rho_{2n}(x,t)=2n\times(2n)^{N+1}\partial_X\rho(2nx,2nt)$, we get, thanks to Young inequality,

$$\left|\left|\frac{\partial\gamma_n}{\partial X}\right|\right|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq\left|\left|\frac{\partial\rho_{2n}}{\partial X}\right|\right|_{L^1(\mathbb{R}^N\times\mathbb{R})}\leq 2n\left|\left|\frac{\partial\rho}{\partial X}\right|\right|_{L^1(\mathbb{R}^N\times\mathbb{R})}.$$

To conclude the proof of this lemma, we will need the following computations. Let $(p,q) \in \mathbb{N}^*$ and $(x,t) \in V_p$, $(x',t') \in \mathbb{R}^N \times \mathbb{R}$ such that $|x'| < 1/q$ and $|t'| < 1/q$; we notice first that

$$|x+x'| \leq |x| + |x'| < r - \frac{1}{p} + \frac{1}{q}. \tag{6.31}$$

Since $|x+x'|^2 = |x|^2 + 2x\cdot x' + |x'|^2 \leq |x|^2 + 2|x|/q + 1/q^2 \leq |x|^2 + 2r/q + 1/q$, we have

$$\begin{aligned} \varepsilon(r^2 - |x+x'|^2) &> |t| + \frac{K}{p} - \frac{2\varepsilon r}{q} - \frac{\varepsilon}{q} + |t'| - \frac{1}{q} \\ &\geq |t+t'| + \frac{K}{2p}\left(2 - \frac{4\varepsilon r p}{Kq} - \frac{2p\varepsilon}{Kq} - \frac{2p}{Kq}\right). \end{aligned} \tag{6.32}$$

Let us now check that $\mathrm{supp}(\gamma_n) \subset V$. We know that $\mathrm{supp}(\gamma_n) \subset V_{2n} + \mathrm{supp}(\rho_{2n})$; by taking $p = q = 2n$ in (6.31) and (6.32) we see, thanks to the hypothesis on K, that $V_{2n} + \mathrm{supp}(\rho_{2n}) \subset V$.
The last property on γ_n is also easy to check. If $(x,t) \in V_n$ then, with $(p,q) = (n,2n)$ in (6.31) and (6.32), one sees, thanks to the hypothesis on K, that $(x,t) - \mathrm{supp}(\rho_n) \subset V_{2n}$ so that

$$\gamma_n(x,t) = \int_{V_{2n}\cap(x,t)-\mathrm{supp}(\rho_{2n})} \rho_{2n}((x,t)-(y,s))\,dy\,ds = \int_{\mathrm{supp}(\rho_{2n})} \rho_{2n}(y,s)\,dy\,ds = 1.$$

The sequence $(\gamma_n)_{n\geq 1}$ satisfies thus the conclusions of the lemma. ∎

Chapitre 7

Un contre-exemple intéressant

7.1 Hyperbolique “Fourier” et Hyperbolique au sens de [37]

Lorsque l'on demande que le problème soit bien posé au sens de Hadamard dans $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^N))$, une petite étude de Fourier (voir [67]) donne la condition nécessaire suivante: $\sup_{\xi\in\mathbb{R}^N} ||e^{i\sum_{j=1}^N \xi_j A_j}|| < +\infty$, ce qui revient à dire (cf [48]) que les matrices $\sum_{j=1}^N \xi_j A_j$ doivent être uniformément diagonalisables lorsque $\xi \in \mathbb{R}^N$. Etant donné l'homogénéité de ces matrices, c'est équivalent à dire qu'il existe $P : S^{N-1} \to \mathrm{GL}(l;\mathbb{R})$ tel que $\sup_{\xi\in S^{N-1}} ||P(\xi)||\,||P(\xi)^{-1}|| < +\infty$ et, pour tout $\xi \in S^{N-1}$, $P(\xi)^{-1}\sum_{j=1}^N \xi_j A_j P(\xi)$ est diagonale. Cette condition semble plus forte que notre condition d'hyperbolicité (où l'on ne demande pas de borne sur $||P(\xi)||\,||P(\xi)^{-1}||$). En fait, en dimension $N = 1$, ces conditions sont trivialement équivalentes (car S^{N-1} est alors réduit à deux points, et il n'y a qu'une seule matrice à diagonaliser); lorsque $N = l = 2$, on peut prouver que tout système hyperbolique (en notre sens) est symétrisable, et donc hyperbolique au sens de [67]. Cependant, lorsque $N \geq 2$ et $l \geq 3$, les deux notions d'hyperbolicité diffèrent et celle de [67] est strictement plus forte, en général, que la notre (comme cela a déjà été signalé en remarque 6.2).

7.1.1 Cas $N = l = 2$ (d'après l'exercice 3.9 de [67])

On suppose donc que $N = l = 2$ et que le système est hyperbolique au sens de la définition 6.2; nous allons montrer que ce système est alors symétrisable (i.e. qu'il existe une matrice S symétrique définie positive telle que SA_1 et SA_2 soient symétriques), ce qui impliquera en particulier qu'il est hyperbolique au sens de [67].

Commençons par remarquer que l'on peut se ramener au cas où A_1 est diagonale; en effet, par hypothèse A_1 est diagonalisable, i.e. il existe P inversible telle que PA_1P^{-1} soit diagonale.
Supposons que nous ayons trouvé S symétrique définie positive telle que SPA_1P^{-1} et SPA_2P^{-1} soient diagonales; alors $S' = P^TSP$ est symétrique définie positive et, pour $i \in \{1,2\}$,

$$S'A_i = P^TSPA_i = P^T(SPA_iP^{-1})P$$

est symétrique.
Quitte à remplacer $(A_i)_{i\in[1,2]}$ par $(PA_iP^{-1})_{i\in[1,2]}$, ce qui ne change pas l'hyperbolicité du système, on peut donc supposer que A_1 est diagonale.

Si A_1 est proportionnelle à l'identité, disons aId, alors le résultat est évident; en prenant S symétrique définie positive telle que SA_2 est symétrique (un tel S existe toujours: il suffit de prendre une matrice P inversible telle que PA_2P^{-1} soit diagonale, puis de poser $S = P^TP$), on constate que $SA_1 = aS$ est aussi symétrique.
On peut donc maintenant supposer que $A_1 = \mathrm{diag}(\alpha,\beta)$ avec $\alpha \neq \beta$.

Ecrivons $A_2 = ((a_{i,j}))_{(i,j)\in[1,2]^2}$; pour tout $\lambda \in \mathbb{R}$, le polynôme caractéristique de $\lambda A_1 + A_2$, c'est à dire $X^2 - (a_{1,1} + \lambda\alpha + a_{2,2} + \lambda\beta)X + (a_{1,1} + \lambda\alpha)(a_{2,2} + \lambda\beta) - a_{1,2}a_{2,1}$ ne doit avoir que des racines réelles. Son discriminant

$$\begin{aligned}\Delta &= (a_{1,1} + \lambda\alpha + a_{2,2} + \lambda\beta)^2 + 4a_{1,2}a_{2,1} - 4(a_{1,1} + \lambda\alpha)(a_{2,2} + \lambda\beta)\\ &= (a_{1,1} + \lambda\alpha - (a_{2,2} + \lambda\beta))^2 + 4a_{1,2}a_{2,1}\\ &= (\alpha - \beta)^2\lambda^2 - 2(\alpha - \beta)(a_{1,1} - a_{2,2})\lambda + 4a_{1,2}a_{2,1} + (a_{1,1} - a_{2,2})^2\end{aligned}$$

doit donc toujours être positif ou nul. Ce discriminant étant lui-même un polynôme en λ, ses racines doivent donc être complexes non réelles ou réelles doubles; cela signifie que son propre discriminant en λ doit être négatif ou nul:

$$(\alpha - \beta)^2(a_{1,1} - a_{2,2})^2 - (4a_{1,2}a_{2,1} + (a_{1,1} - a_{2,2})^2)(\alpha - \beta)^2 \le 0.$$

Comme $\alpha \neq \beta$, cela implique donc $a_{1,2}a_{2,1} \ge 0$.
Si $a_{1,2} = 0$, alors $\lambda A_1 + A_2$ s'écrit

$$\begin{pmatrix} a_{1,1} + \lambda\alpha & a_{2,1} \\ 0 & a_{2,2} + \lambda\beta \end{pmatrix}.$$

Cette matrice devant être diagonalisable pour tout $\lambda \in \mathbb{R}$, en particulier pour $\lambda = (a_{2,2} - a_{1,1})/(\alpha - \beta)$ (λ pour lequel les éléments diagonaux sont égaux), on en déduit que $a_{2,1} = 0$. De même, si $a_{2,1} = 0$, on prouve que $a_{1,2} = 0$.
Ainsi, on a soit $a_{1,2} = a_{2,1} = 0$, soit $a_{1,2}a_{2,1} > 0$; dans le premier cas, A_2 est diagonale et A_1 et A_2 sont donc déjà symétriques. Dans le deuxième cas, $a_{1,2}$ et $a_{2,1}$ ayant le même signe,

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{a_{2,1}}{a_{1,2}} \end{pmatrix}$$

est symétrique définie positive et on constate par le calcul que SA_1 et SA_2 sont symétriques.

Le fait qu'un système symétrisable soit hyperbolique au sens de [67] est assez simple à voir.
En notant S une matrice symétrique définie positive telle que, pour tout $i \in [1, N]$, SA_i soit symétrique, on prend H la racine carrée symétrique définie positive de S et on constate que $HA_iH^{-1} = H^{-1}SA_iH^{-1}$ est symétrique; ainsi, pour tout $\xi \in \mathbb{R}^N$, $H\sum_{i=1}^N \xi_i A_i H^{-1}$ est symétrique, donc diagonalisable dans une base orthonormée: on peut trouver $\overline{P}(\xi)$ orthogonale telle que $\overline{P}(\xi)H\sum_{i=1}^N \xi_i A_i H^{-1}\overline{P}(\xi)^{-1}$ soit diagonale.
Ainsi, $P(\xi) = \overline{P}(\xi)H$ diagonalise $\sum_{i=1}^N \xi_i A_i$ et, puisque $\overline{P}(\xi)$ est orthogonale, $\sup_{\xi\in\mathbb{R}^N} ||P(\xi)|| < +\infty$ et $\sup_{\xi\in\mathbb{R}^N} ||P(\xi)^{-1}|| < +\infty$, ce qui suffit à voir que le système est hyperbolique au sens de [67].

7.1.2 Cas général

Comme nous l'avons dit, lorsque $N \ge 2$ et $l \ge 3$, les notions d'hyperbolicité de [67] et de la définition 6.2 diffèrent. Voici un exemple. Prenons $N = 2$, $l = 3$ et

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{et} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}. \tag{7.1}$$

A_1 est clairement diagonalisable (elle est diagonale!) et, pour $\lambda \in \mathbb{R}$, une étude du polynôme caractéristique de $\lambda A_1 + A_2$ donne pour valeurs propres 0, $(\lambda + \sqrt{\lambda^2 + 16})/2$ et $(\lambda - \sqrt{\lambda^2 + 16})/2$; ces valeurs propres étant deux à deux distinctes, $\lambda A_1 + A_2$ est diagonalisable.

Ainsi, pour tout $\xi \in \mathbb{R}^2$, $\xi_1 A_1 + \xi_2 A_2$ est diagonalisable: le système considéré est donc hyperbolique au sens de la définition 6.2 (mais non strictement hyperbolique, car A_1 a 0 comme valeur propre double). Nous allons cependant prouver que ce système n'est pas hyperbolique au sens de [67].

On voit assez simplement que, pour $\lambda \in \mathbb{R}$, les sous-espaces propres de $\lambda A_1 + A_2$ sont respectivement

$$\ker(\lambda A_1 + A_2 - 0Id) = \mathbb{R}(-4, 2-\lambda, 2)^T$$
$$\ker\left(\lambda A_1 + A_2 - \frac{\lambda + \sqrt{\lambda^2+16}}{2} Id\right) = \mathbb{R}(0, 4, \lambda + \sqrt{\lambda^2+16})^T$$
$$\ker\left(\lambda A_1 + A_2 - \frac{\lambda - \sqrt{\lambda^2+16}}{2} Id\right) = \mathbb{R}(0, 4, \lambda - \sqrt{\lambda^2+16})^T.$$

Lorsque $Q^{-1}(\lambda A_1 + A_2)Q$ est diagonale, les colonnes de la matrice Q sont des vecteurs propres de $\lambda A_1 + A_2$, donc des multiples des vecteurs générateurs des sous-espaces propres décrits ci-dessus (car tous ces sous-espaces propres sont de dimension 1); il existe donc une matrice de permutation $\mathcal{P}$ et trois réels non-nuls α_1, α_2 et α_3 tels que

$$Q = \begin{pmatrix} -4\alpha_1 & 0 & 0 \\ (2-\lambda)\alpha_1 & 4\alpha_2 & 4\alpha_3 \\ 2\alpha_1 & (\lambda + \sqrt{\lambda^2+16})\alpha_2 & (\lambda - \sqrt{\lambda^2+16})\alpha_3 \end{pmatrix} \mathcal{P}.$$

Notons

$$Q_\alpha = \begin{pmatrix} -4\alpha_1 & 0 & 0 \\ (2-\lambda)\alpha_1 & 4\alpha_2 & 4\alpha_3 \\ 2\alpha_1 & (\lambda + \sqrt{\lambda^2+16})\alpha_2 & (\lambda - \sqrt{\lambda^2+16})\alpha_3 \end{pmatrix}.$$

Un simple calcul par co-matrice donne

$$Q_\alpha^{-1} = \begin{pmatrix} \frac{-1}{4\alpha_1} & 0 & 0 \\ \frac{8 - (2-\lambda)(\lambda - \sqrt{\lambda^2+16})}{32\alpha_2\sqrt{\lambda^2+16}} & \frac{\sqrt{\lambda^2+16} - \lambda}{8\alpha_2\sqrt{\lambda^2+16}} & \frac{1}{2\alpha_2\sqrt{\lambda^2+16}} \\ \frac{(2-\lambda)(\lambda + \sqrt{\lambda^2+16}) - 8}{32\alpha_3\sqrt{\lambda^2+16}} & \frac{\lambda + \sqrt{\lambda^2+16}}{8\alpha_3\sqrt{\lambda^2+16}} & \frac{-1}{2\alpha_3\sqrt{\lambda^2+16}} \end{pmatrix}.$$

Soit N la norme sur $M_l(\mathbb{R})$ du suppremum des coefficients; cette norme étant équivalente à $||\cdot||$, il existe $C > 0$ tel que $N(\cdot) \le C||\cdot||$.
Puisque $Q_\alpha = Q\mathcal{P}^{-1}$ et $Q_\alpha^{-1} = \mathcal{P}Q^{-1}$, avec $\mathcal{P}$ et $\mathcal{P}^{-1}$ des matrices de permutation, on a $N(Q) = N(Q_\alpha)$ et $N(Q^{-1}) = N(Q_\alpha^{-1})$, donc $N(Q_\alpha)N(Q_\alpha^{-1}) \le C^2||Q||\,||Q^{-1}||$.
Le coefficient $(2,1)$ de Q_α nous dit que $N(Q_\alpha) \ge |2-\lambda||\alpha_1|$, et le coefficient $(1,1)$ de Q_α^{-1} donne $N(Q_\alpha^{-1}) \ge 1/4|\alpha_1|$; ainsi, $N(Q_\alpha)N(Q_\alpha^{-1}) \ge |2-\lambda|/4$.
On a donc prouvé que, pour toute matrice Q diagonalisant $\lambda A_1 + A_2$, on a $||Q||\,||Q||^{-1} \ge |2-\lambda|/4C^2$.

Soit $P : S^1 \to \mathrm{GL}(l;\mathbb{R})$ telle que, pour tout $\xi \in S^1$, $P(\xi)^{-1}(\xi_1 A_1 + \xi_2 A_2)P(\xi)$ soit diagonale. Lorsque $\xi_2 \neq 0$, la matrice $P(\xi)$ diagonalise $\frac{\xi_1}{\xi_2}A_1 + A_2$ et on a donc $||P(\xi)||\,||P(\xi)^{-1}|| \ge |2 - \xi_1/\xi_2|/4C^2$, ce qui prouve que $||P(\xi)||\,||P(\xi)^{-1}||$ ne peut être borné sur S^1 (cette quantité tend, lorsque $\xi \to (1,0)$ avec $\xi_2 \neq 0$, vers l'infini).
Le système considéré n'est donc pas hyperbolique au sens de [67].

7.2 Non-résolubilités du système défini par (A_1, A_2)

Lorsque la condition initiale u^0 d'un système hyperbolique de la forme (6.1) ne dépend que d'une direction $\xi \in S^{N-1}$ (i.e. $u_0(x) = f(x\cdot\xi)$), on peut exprimer explicitement la solution correspondante en fonction de u^0.

Faisons-le, dans un cas particulier, pour le système défini par les matrices (7.1).

Soit $f \in L^1_{\text{loc}}(\mathbb{R})$ et $\lambda \in \mathbb{R}$. Considérons, en notant $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1) \in S^1$, la condition initiale

$$u^{0,(\lambda)}(x) = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} f(x \cdot \xi). \tag{7.2}$$

Puisque la décomposition du vecteur $(4,0,0)^T$ sur la base propre de $\xi_1 A_1 + \xi_2 A_2 = \frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ donnée dans la partie 7.1.2 est

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ \lambda - 2 \\ -2 \end{pmatrix} + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix}$$

avec

$$\alpha(\lambda) = \frac{2-\lambda}{8} + \frac{1}{\sqrt{\lambda^2+16}} + \frac{\lambda(\lambda-2)}{8\sqrt{\lambda^2+16}} = \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \text{lorsque } \lambda \to \infty$$

et

$$\beta(\lambda) = \frac{2-\lambda}{8} - \frac{1}{\sqrt{\lambda^2+16}} - \frac{\lambda(\lambda-2)}{8\sqrt{\lambda^2+16}} = -\frac{\lambda}{4}(1+\gamma(\lambda)) \quad \text{où } \gamma(\lambda) = \mathcal{O}(1/\lambda) \text{ lorsque } \lambda \to \infty,$$

la solution u du système définit par (A_1, A_2) (lorsque $\Omega = \mathbb{R}^2$) pour la condition initiale $u^{0,(\lambda)}$ est

$$\begin{aligned} u^{(\lambda)}(x,t) \quad = \quad & \begin{pmatrix} 4 \\ \lambda - 2 \\ 2 \end{pmatrix} f(x \cdot \xi - c_1(\lambda)t) + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} f(x \cdot \xi - c_2(\lambda)t) \\ & + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix} f(x \cdot \xi - c_3(\lambda)t) \end{aligned} \tag{7.3}$$

où $c_1(\lambda) = 0$ est la valeur propre associée au premier vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$, $c_2(\lambda) = \frac{\lambda+\sqrt{\lambda^2+16}}{2\sqrt{\lambda^2+1}} = 1 + \varepsilon(\lambda)$ est la valeur propre associée au deuxième vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ (on a $\varepsilon(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to \infty$) et $c_3(\lambda) = \frac{\lambda-\sqrt{\lambda^2+16}}{2\sqrt{\lambda^2+1}} = -\frac{4}{\lambda^2}(1+\eta(\lambda))$ est la valeur propre associée au troisième vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ (on a $\eta(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to \infty$).
En notant $\tau_h f(z) = f(z+h)$, on peut donc écrire $u^{(\lambda)}(x,t) = v^{(\lambda)}(x \cdot \xi, t)$ avec

$$\begin{aligned} v^{(\lambda)}(z,t) \quad = \quad & \begin{pmatrix} 4 \\ \lambda - 2 \\ 2 \end{pmatrix} f(z) + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} \tau_{-c_2(\lambda)t} f(z) \\ & + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix} \tau_{-c_3(\lambda)t} f(z). \end{aligned} \tag{7.4}$$

Un changement de variable donné par Q^T, où Q est une matrice orthognale telle que $Q\xi = (1,0)$ (ce qui est possible puisque $|\xi| = 1$), nous permet donc de voir que (puisque $]-1,1[^2 \subset B_2$ — rappelons que B_2 est la boule euclidienne de $\mathbb{R}^2$ de centre 0 et de rayon 2)

$$\int_0^1 \int_{B_2} |u^{(\lambda)}(x,t)| \, dx dt$$

$$
\begin{aligned}
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(x\cdot\xi,t)|\,dxdt \\
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(Q^Tx\cdot\xi,t)|\,dxdt \\
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(x_1,t)|\,dxdt \\
&\geq \int_0^1 \int_{-1}^1 \int_{-1}^1 |v^{(\lambda)}(x_1,t)|\,dx_1\,dx_2\,dt \\
&\geq \int_0^1 \int_{-1}^1 |v^{(\lambda)}(z,t)|\,dz\,dt \\
&\geq \int_0^1 \int_{-1}^1 |v_2^{(\lambda)}(z,t)|\,dz\,dt \\
&\geq \int_0^1 \int_{-1}^1 |(\lambda-2)f + 4\alpha(\lambda)\tau_{-c_2(\lambda)t}f + 4\beta(\lambda)\tau_{-c_3(\lambda)t}f|\,dz\,dt \\
&\geq \int_0^1 \int_{-1}^1 |(\lambda-2)f - \lambda(1+\gamma(\lambda))\tau_{-c_3(\lambda)t}f|\,dz\,dt - ||4\alpha(\lambda)\tau_{-c_2(\lambda)t}f||_{L^1(]-1,1[\times]0,1[)} \\
&\geq |\lambda| \int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t}f(z)|\,dz\,dt - 4|\alpha(\lambda)|\,||f||_{L^1(]-1-|c_2(\lambda)|,1+|c_2(\lambda)|[)} - 2||f||_{L^1(]-1,1[)} \\
&\quad -|\lambda\gamma(\lambda)|\,||f||_{L^1(]-1-|c_3(\lambda)|,1+|c_3(\lambda)|[)}.
\end{aligned}
$$

En notant

$$M_1 = \sup_{\lambda\in\mathbb{R}} |\alpha(\lambda)| < \infty\,, \quad M_2 = \sup_{\lambda\in\mathbb{R}} |\lambda\gamma(\lambda)| < \infty\,, \quad C_2 = \sup_{\lambda\in\mathbb{R}} |c_2(\lambda)| < \infty \quad \text{et } C_3 = \sup_{\lambda\in\mathbb{R}} |c_3(\lambda)| < \infty,$$

on en déduit

$$
\begin{aligned}
&|\lambda| \int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t}f(z)|\,dz\,dt \\
&\leq 4M_1||f||_{L^1(]-1-C_2,1+C_2[)} + 2||f||_{L^1(]-1,1[)} + M_2||f||_{L^1(]-1-C_3,1+C_3[)} \\
&\quad +||v^{(\lambda)}||_{(L^1(]-1,1[\times[0,1[))^3} \qquad (7.5) \\
&\leq 4M_1||f||_{L^1(]-1-C_2,1+C_2[)} + 2||f||_{L^1(]-1,1[)} + M_2||f||_{L^1(]-1-C_3,1+C_3[)} \\
&\quad +||u^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3}. \qquad (7.6)
\end{aligned}
$$

7.2.1 Une fonction f particulière

L'équation (7.6) nous dit que, si $u^{(\lambda)}$ est bornée dans $L^1(B_2\times[0,1[)^3$, alors

$$\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t}f(z)|\,dz\,dt = \mathcal{O}\left(\frac{1}{|\lambda|}\right).$$

Nous exhibons ici une fonction f, continue bornée sur $\mathbb{R}$, qui ne vérifie pas cette propriété.

Lemme 7.1 *Si $\alpha \in]0,1[$, il existe $f \in \mathcal{C}(\mathbb{R})$, $C > 0$ et $h_0 \in]0,1[$ tels que, pour tout $h \in]0,h_0[$, en notant $f_h(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt$, on a*

$$\int_0^1 |f(z) - f_h(z)|\,dz \geq Ch^\alpha.$$

Preuve du lemme 7.1
On cherche f sous la forme $f(z) = z^p \sin(z^{-n})$, où p et n sont des réels strictement positifs à déterminer.

Grâce à une intégration par parties, on a

$$\begin{aligned}\int f = \int z^p \sin(z^{-n})\,dz &= \frac{1}{n}\int z^{p+n+1}\left(nz^{-n-1}\sin(z^{-n})\right)\,dx \\ &= \frac{1}{n}\int z^{p+n+1}\frac{d}{dz}(\cos(z^{-n}))\,dz \\ &= \frac{1}{n}z^{p+n+1}\cos(z^{-n}) - \frac{n+p+1}{n}\int z^{p+n}\cos(z^{-n})\,dz.\end{aligned}$$

Fixons $\beta \in]0,\alpha[$; lorsque $h \leq 1$ et $0 \leq z \leq h^\beta - h$, on a donc

$$\begin{aligned}|f_h(z)| &= \left|\frac{1}{hn}\left((z+h)^{p+n+1}\cos((z+h)^{-n}) - z^{p+n+1}\cos(z^{-n})\right)\right. \\ &\qquad \left. -\frac{n+p+1}{hn}\int_z^{z+h} s^{p+n}\cos(s^{-n})\,ds\right| \\ &\leq \frac{2h^{(p+n+1)\beta-1}}{n} + \frac{(n+p+1)h^{(p+n)\beta}}{n} \\ &\leq C_1 h^{(p+n+1)\beta-1}\end{aligned}$$

où $C_1 < +\infty$ ne dépend que de n et p (rappelons que $\beta \in]0,1[$ et que $h \leq 1$, de sorte que $h^{1-\beta} \leq 1$). Ainsi,

$$\int_0^{h^\beta - h} |f_h(z)|\,dz \leq C_1 h^{(p+n+2)\beta-1}. \tag{7.7}$$

On veut maintenant minorer $\int_0^{h^\beta-h}|f(z)|\,dz = \int_0^{h^\beta-h} z^p|\sin(z^{-n})|\,dz$. Avec le changement de variable $y = z^{-n}$ et en notant $\mu_h = h^\beta - h$, on a

$$\begin{aligned}\int_0^{h^\beta-h}|f(z)|\,dz &= \frac{1}{n}\int_{\mu_h^{-n}}^{\infty} y^{-\frac{1}{n}-1}y^{-\frac{p}{n}}|\sin(y)|\,dy \\ &\geq \frac{1}{n}\sum_{k\geq k_h}\int_{2k\pi}^{2k\pi+\pi} y^{-\frac{p+n+1}{n}}\sin(y)\,dy,\end{aligned}$$

où $k_h = [\mu_h^{-n}/2\pi] + 1$ ($[\cdot]$ désigne la partie entière d'un réel). Comme $\sin \geq \sqrt{2}/2$ sur $[2k\pi + \pi/4, 2k\pi + 3\pi/4]$, on en déduit

$$\begin{aligned}&\int_0^{h^\beta-h}|f(z)|\,dz \\ &\geq \frac{\sqrt{2}}{2n}\sum_{k\geq k_h}\int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} y^{-\frac{p+n+1}{n}}\,dy \\ &\geq \frac{\sqrt{2}}{2n}\sum_{k\geq k_h}\frac{1}{\frac{p+n+1}{n}-1}\left(\left(2k\pi+\frac{\pi}{4}\right)^{1-\frac{p+n+1}{n}} - \left(2k\pi+\frac{3\pi}{4}\right)^{1-\frac{p+n+1}{n}}\right).\end{aligned}$$

Mais, par le théorème des accroissements finis, pour tout $k \in \mathbb{N}$, il existe $\theta \in [0,1]$ tel que

$$\left(2k\pi+\frac{\pi}{4}\right)^{1-\frac{p+n+1}{n}} - \left(2k\pi+\frac{3\pi}{4}\right)^{1-\frac{p+n+1}{n}} = \frac{\pi}{2}\times\left(\frac{p+n+1}{n}-1\right)\left(2k\pi+\frac{\pi}{4}+\theta\frac{\pi}{2}\right)^{-\frac{p+n+1}{n}}$$

$$\begin{aligned} &\geq C_2\,(2(k+1)\pi)^{-\frac{p+n+1}{n}} \\ &\geq C_3 k^{-\frac{p+n+1}{n}} \end{aligned}$$

où $C_2 > 0$ et $C_3 > 0$ ne dépendent que de n et p. Ainsi,

$$\int_0^{h^\beta - h} |f(z)|\,dz \quad \geq \quad C_4 \sum_{k \geq k_h} k^{-\frac{p+n+1}{n}}$$

avec $C_4 > 0$ ne dépendant que de n et p. La fonction $s \to s^{-(p+n+1)/n}$ étant décroissante, on a

$$\sum_{k \geq k_h} k^{-\frac{p+n+1}{n}} \geq \int_{k_h}^{\infty} s^{-\frac{p+n+1}{n}}\,ds = \frac{1}{\frac{n+p+1}{n} - 1} k_h^{1-\frac{n+p+1}{n}} = \frac{1}{\frac{n+p+1}{n} - 1} k_h^{-\frac{p+1}{n}}.$$

En constatant que $k_h = [\mu_h^{-n}/2\pi] + 1 \leq (\mu_h^{-n}/2\pi) + 2$, on en déduit

$$\int_0^{h^\beta - h} |f(z)|\,dz \quad \geq \quad C_5 \left(\frac{\mu_h^{-n}}{2\pi} + 2\right)^{-\frac{p+1}{n}} = C_5 \left(\frac{1}{2\pi} + 2\mu_h^n\right)^{-\frac{p+1}{n}} \mu_h^{p+1}$$

où $C_5 > 0$ ne dépend que de n et p. Puisque $\mu_h = h^\beta - h \in [h^\beta/2, 1]$ lorsque $h \leq (1/2)^{1/(1-\beta)}$, on a finalement, pour $h \leq (1/2)^{1/(1-\beta)}$,

$$\int_0^{h^\beta - h} |f(z)|\,dz \quad \geq \quad C_5 \left(\frac{1}{2\pi} + 2\right)^{-\frac{p+1}{n}} \frac{h^{(p+1)\beta}}{2^{p+1}} = C_6 h^{(p+1)\beta} \tag{7.8}$$

où $C_6 > 0$ ne dépend que de n et p.

De (7.7) et (7.8), on tire, lorsque $h \leq (1/2)^{1/(1-\beta)}$,

$$\begin{aligned} \|f - f_h\|_{L^1(]0,1[)} &\geq \|f - f_h\|_{L^1(]0,h^\beta - h[)} \\ &\geq \|f\|_{L^1(]0,h^\beta-h[)} - \|f_h\|_{L^1(]0,h^\beta-h[)} \\ &\geq C_6 h^{(p+1)\beta} - C_1 h^{(p+n+2)\beta - 1} \\ &\geq (C_6 - C_1 h^{(n+1)\beta-1}) h^{(p+1)\beta}. \end{aligned}$$

On choisit maintenant $p > 0$ tel que $(p+1)\beta = \alpha$ (ce choix est possible car $\beta < \alpha$) et $n > 0$ tel que $(n+1)\beta - 1 > 0$; il existe alors $h_0 \in]0, (1/2)^{1/(1-\beta)}[$ ne dépendant que de n et β tel que, pour tout $h \leq h_0$, $C_6 - C_1 h^{(n+1)\beta-1} \geq C_6/2$; on a alors, pour $h \leq h_0$,

$$\|f - f_h\|_{L^1(]0,1[)} \geq \frac{C_6}{2} h^\alpha,$$

ce qui conclut la preuve de ce lemme. ■

Prenons $\alpha = 1/4$ et $f \in \mathcal{C}(\mathbb{R})$, $C > 0$ et $h_0 \in]0,1[$ donnés par le lemme 7.1 pour cet α.
On a, pour $r > 0$,

$$f_r(z) = \frac{1}{r} \int_z^{z+r} f(t)\,dt = \int_0^1 f(z + rt)\,dt = \int_0^1 \tau_{rt} f(z)\,dt.$$

Comme $c_3(\lambda) = -\frac{4}{\lambda^2}(1 + \eta(\lambda))$ avec $\eta(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to +\infty$, il existe $\lambda_1 > 0$ tel que, pour tout $\lambda \geq \lambda_1$, $\lambda^{-2} \leq -c_3(\lambda) \leq h_0$; ainsi, pour tout $\lambda \geq \lambda_1$,

$$\int_{-1}^1 \int_0^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\,dtdz \quad \geq \quad \int_0^1 \left| \int_0^1 (f(z) - \tau_{-c_3(\lambda)t} f(z))\,dt \right| dz$$

$$\begin{aligned}
&\geq \int_0^1 \left| f(z) - \int_0^1 \tau_{-c_3(\lambda)t} f(z)\, dt \right| dz \\
&\geq \int_0^1 |f(z) - f_{-c_3(\lambda)}(z)|\, dz \\
&\geq C|c_3(\lambda)|^\alpha \geq C\lambda^{-\frac{1}{2}}. \qquad (7.9)
\end{aligned}$$

Cette relation ne faisant intervenir les valeurs de f que sur un compact de $\mathbb{R}$, on peut aussi supposer, quitte à changer f hors de ce compact, que $f \in \mathcal{C}_c(\mathbb{R})$.

7.2.2 Non-résolubilité $L^\infty - L^1_{\rm loc}$

Lorsque la condition initiale est de type Riemann, la forme très particulière des solutions permet de voir que l'on a une estimation sur la solution dans les espaces $(L^q_{\rm loc}(\mathbb{R}^2 \times]0,\infty[))^3$ en fonction de la norme de la condition initiale dans $(L^p_{\rm loc}(\mathbb{R}^2))^3$ (pour $q > 1$ assez petit et $p < \infty$ assez grand), estimation qui est indépendante de la direction de la condition initiale de type Riemann.
Cependant, cette estimation ne permet pas de résoudre le système considéré pour toute condition initiale dans $(L^\infty(\mathbb{R}^2))^3$ (ou même dans un espace plus petit).

Supposons en effet que, pour toute condition initiale $u^0 \in E = (\mathcal{C}_b(\mathbb{R}^2))^3$, on puisse trouver une solution $u \in F = (L^1_{\rm loc}(\mathbb{R}^2 \times [0,\infty[))^3$ au système défini par (A_1, A_2) pour la condition initiale u^0; par le corollaire 6.2, on a unicité de cette solution et l'application $L : u^0 \in E \to u \in F$ est alors bien définie; on voit immédiatement que L est linéaire.
Les espaces E et F étant métriques complets, L est continue si et seulement si son graphe est fermé. Or, si $(u^0_n)_{n\geq 1} \in E$ converge vers u^0 dans E et $(L(u^0_n))_{n\geq 1} = (u_n)_{n\geq 1} \in F$ converge vers u dans F, alors on constate en passant à la limite dans l'équation faible satisfaite par $(u^n)_{n\geq 1}$ que u est la solution du système défini par (A_1, A_2) pour la condition initiale u^0, c'est à dire que $u = L(u^0)$. Le graphe de L étant fermé, L est donc bien continue.

En particulier, en prenant la fonction $f \in \mathcal{C}_b(\mathbb{R})$ construite en 7.2.1, $\lambda \in \mathbb{R}$ et $u^{0,(\lambda)} \in E$ définie par (7.2) (rappelons que $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1)$), on a $||u^{(\lambda)}||_{(L^1(B_2\times[0,1]))^3} \leq C||u^{0,(\lambda)}||_E \leq 4C||f||_{\mathcal{C}_b(\mathbb{R})}$, où C ne dépend pas de λ (on a traduit la continuité de L et le fait que $u^{(\lambda)} = L(u^{0,(\lambda)})$). En injectant cette estimation dans (7.6), on en déduit qu'il existe $C' > 0$ tel que, pour tout $\lambda \in \mathbb{R}$,

$$\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\, dz\, dt \leq \frac{C'}{|\lambda|}.$$

f vérifiant (7.9), on en déduit que $C'\lambda^{-1} \geq C\lambda^{-1/2}$ pour tout λ assez grand, avec C et C' indépendants de λ, ce qui est une contradiction.

7.2.3 Non-résolubilité $BV_{\rm loc} - BV_{\rm loc}$

Supposons maintenant que, pour tout $u_0 \in (BV_{\rm loc}(\mathbb{R}^2))^3$, il existe une solution $u \in (L^1_{\rm loc}(\mathbb{R}^2 \times [0,\infty[))^3$ au système défini par (A_1, A_2), pour la condition initiale u^0, qui soit dans $(L^1_{\rm loc}([0,\infty[; BV(B_2)))^3$ (1).
Comme au début de la sous-section 7.2.2, par le théorème du graphe fermé, l'application $L : u^0 \in (BV_{\rm loc}(\mathbb{R}^2))^3 \to u \in (L^1_{\rm loc}(\mathbb{R}^2 \times [0,\infty[))^3 \cap (L^1_{\rm loc}([0,\infty[; BV(B_2))^3$ serait bien définie et linéaire continue. En particulier, il existerait $R > 0$ et $C > 0$ tel que, pour tout $u^0 \in (BV_{\rm loc}(\mathbb{R}^2))^3$,

$$\begin{aligned}
&||D_1 L(u^0)||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} + ||D_2 L(u^0)||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} \\
&\leq C||u^0||_{(L^1(B_R))^3} + C||D_1 u^0||_{(\mathcal{M}_b(B_R))^3} + C||D_2 u^0||_{(\mathcal{M}_b(B_R))^3}. \qquad (7.10)
\end{aligned}$$

(1) Rappelons que, pour $r > 0$, $BV(B_r)$ est le Banach formé des fonctions $f \in L^1(B_r)$ telles que $D_1 f$ et $D_2 f$ soient des mesures finies sur B_r, muni de la norme $||f||_{L^1(B_r)} + ||D_1 f||_{\mathcal{M}_b(B_r)} + ||D_2 f||_{\mathcal{M}_b(B_r)}$. $BV_{\rm loc}(\mathbb{R}^2)$ est l'espace métrique complet formé des fonctions $f \in L^1_{\rm loc}(\mathbb{R}^2)$ qui sont dans $BV(B_r)$ pour tout $r > 0$; il est muni de la famille dénombrable de semi-normes $||f||_{L^1(B_n)} + ||D_1 f||_{\mathcal{M}_b(B_n)} + ||D_2 f||_{\mathcal{M}_b(B_n)}$ $(n \geq 1)$.

Soit $f \in \mathcal{C}_b(\mathbb{R})$ et $F(z) = \int_0^z f(s)\,ds$; F est dans $\mathcal{C}^1(\mathbb{R})$. On se donne $\lambda \in \mathbb{R}$ et $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1)$.
Lorsque $U^{0,(\lambda)}(x) = (4,0,0)^T F(x\cdot\xi)$, on voit sur l'expression de la solution $U^{(\lambda)}$ du système défini par (A_1, A_2) pour la condition initiale $U^{0,(\lambda)}$ ($U^{(\lambda)}$ est donnée par (7.3) où l'on a changé f en F) que $U^{(\lambda)} \in (\mathcal{C}^1(\mathbb{R}^2 \times [0,\infty[))^3$; grâce à cette régularité, et puisque $|\xi| = 1$, on constate que $u^{(\lambda)}(x,t) = \xi_1 \partial_{x_1} U^{(\lambda)}(x,t) + \xi_2 \partial_{x_2} U^{(\lambda)}(x,t) \in (\mathcal{C}(\mathbb{R}^2 \times [0,\infty[))^3$ est solution du système défini par (A_1, A_2) avec la condition initiale $u^{0,(\lambda)}(x) = (4,0,0)^T f(x\cdot\xi)$.
f et $u^{(\lambda)}$ sont donc liés par (7.6). Mais, puisque $U^{(\lambda)}$ est de classe $\mathcal{C}^1$,

$$\begin{aligned} ||u^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3} &\le |\xi_1|\,||\partial_{x_1}U^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3} + |\xi_2|\,||\partial_{x_2}U^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3} \\ &\le ||D_1 U^{(\lambda)}||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} + ||D_2 U^{(\lambda)}||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3}. \end{aligned} \tag{7.11}$$

De plus, par un changement de variable défini par Q^T (où Q est une matrice orthogonale telle que $Q\xi = (1,0)$),

$$\begin{aligned} &||U^{0,(\lambda)}||_{(L^1(B_R))^3} + ||D_1 U^{0,(\lambda)}||_{(\mathcal{M}_b(B_R))^3} + ||D_2 U^{0,(\lambda)}||_{(\mathcal{M}_b(B_R))^3} \\ &\le 4\int_{B_R} |F(x_1)|\,dx_1\,dx_2 + 4|\xi_1| \int_{B_R} |f(x_1)|\,dx_1\,dx_2 + 4|\xi_2|\int_{B_R} |f(x_1)|\,dx_1\,dx_2 \\ &\le 4|B_R|R\,||f||_{\mathcal{C}_b(\mathbb{R})} + 8|B_R|\,||f||_{\mathcal{C}_b(\mathbb{R})}, \end{aligned} \tag{7.12}$$

car, lorsque $x \in B_R$, $|F(x_1)| \le |x_1|\,||f||_{\mathcal{C}_b(\mathbb{R})} \le R||f||_{\mathcal{C}_b(\mathbb{R})}$.
On déduit donc, de (7.6), (7.10) appliqué à U^0 au lieu de u^0 et de (7.11), (7.12) que

$$\begin{aligned} &|\lambda| \int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\,dz\,dt \\ &\le (4M_1(2+2C_2) + 4 + M_2(2+2C_3) + 4C|B_R|R + 8C|B_R|)||f||_{\mathcal{C}_b(\mathbb{R})}, \end{aligned}$$

ce qui nous conduit, grâce à (7.9) et comme en 7.2.2, à une contradiction.
On ne peut donc, pour tout $u^0 \in (BV_{\text{loc}}(\mathbb{R}^2))^3$, trouver une solution $u \in (L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ qui soit dans $(L^1_{\text{loc}}([0,\infty[; BV(B_2)))^3$.

7.3 Instabilité d'un système hyperbolique par rapport au flux

Considérons le problème

$$\begin{cases} (w_\varepsilon)_t(z,t) + (g_\varepsilon(w_\varepsilon))_z(z,t) = 0 & t > 0\,,\ z \in \mathbb{R}, \\ w_\varepsilon(z,0) = w^0(z) & z \in \mathbb{R}, \end{cases} \tag{7.13}$$

où $g_\varepsilon : \mathbb{R}^l \to \mathbb{R}^l$ et $w^0 : \mathbb{R} \to \mathbb{R}^l$. On suppose que g_ε converge (en un sens à préciser) vers une certaine fonction g lorsque $\varepsilon \to 0$ (w^0 reste fixé). La question naturelle est de savoir si "la" solution w_ε de (7.13) tend vers "la" solution w du système (7.13) avec g à la place de g_ε.

Lucier [54] a déjà apporté une réponse à cette question dans le cas scalaire (i.e. $l = 1$): si $g_\varepsilon \to g$ dans l'espace des fonctions lipschitziennes sur $\mathbb{R}$ et si $w^0 \in L^1_{\text{loc}}(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R})$, alors la solution entropique de (7.13) converge, lorsque $\varepsilon \to 0$, vers la solution entropique de (7.13) où g_ε est remplacé par g. En fait, grâce au principe de comparaison L^1 des solutions entropiques par rapport à leur condition initiale (cf. [67]), ce résultat reste vrai lorsque l'on suppose seulement $w^0 \in L^1_{\text{loc}}(\mathbb{R})$.

La question suivante serait de savoir si ce résultat se généralise au cas des systèmes. La première difficulté est bien sûr de définir "la" solution de (7.13): en général, lorsque g_ε est non linéaire, on n'a pas de théorème d'unicité pour (7.13).

On peut donc commencer par considérer le cas linéaire, pour lequel la solution faible est unique (corollaire 6.2). Prenons $l = 3$ et $g_\varepsilon(w) = B(\varepsilon)w$, avec $B(\varepsilon) \in M_3(\mathbb{R})$ diagonalisable sur $\mathbb{R}$ ([2]). Le système alors considéré est

$$\left\{\begin{array}{ll} (w_\varepsilon)_t(z,t) + B(\varepsilon)(w_\varepsilon)_z(z,t) = 0 & t > 0\,,\; z \in \mathbb{R}, \\ w_\varepsilon(z,0) = w^0(z) & z \in \mathbb{R}, \end{array}\right. \tag{7.14}$$

et on suppose que $B(\varepsilon) \to B$ dans $M_3(\mathbb{R})$ lorsque $\varepsilon \to 0$, avec B diagonalisable sur $\mathbb{R}$. On voudrait savoir si $w_\varepsilon \to w$, au moins dans $(\mathcal{M}_b(B_R \times [0,R[))^3$ faible-$*$ pour tout $R > 0$ (i.e. contre toute fonction de $\mathcal{C}_c(\mathbb{R}^2 \times [0,\infty[))$, avec w solution de (7.14) lorsque $B(\varepsilon)$ est remplacé par B (remarquons que, avec nos hypothèses, le problème (7.14) est, pour tout $\varepsilon \geq 0$, bien posé au sens de Hadamard dans $(L^2(\mathbb{R}))^3$).
Mais, contrairement au cas scalaire, même dans ce cas simple, nous allons voir que la réponse est négative en général. Et le contre-exemple que nous construisons se base encore une fois sur les matrices (A_1, A_2) données par (7.1).

En notant $\lambda = \varepsilon^{-1}$ on prend $B(\varepsilon) = \frac{\lambda}{\sqrt{\lambda^2+1}}A_1 + \frac{1}{\sqrt{\lambda^2+1}}A_2$ et $w^0(z) = (4,0,0)^T f(z)$ avec f construite en 7.2.1. $B(\varepsilon)$ converge, lorsque $\varepsilon \to 0$, vers A_1 (et toutes ces matrices sont bien diagonalisables sur $\mathbb{R}$).
La solution w_ε de (7.14) est alors $v^{(\lambda)}$ donnée par (7.4). Supposons que, pour un $\widetilde{\lambda} > 0$, $(v^{(\lambda)})_{\lambda \geq \widetilde{\lambda}}$ soit bornée dans $(L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ (ce qui est le cas si w_ε converge, lorsque $\varepsilon \to 0$, dans $(\mathcal{M}_b(B_R \times [0,R[))^3$ faible-$*$ pour tout $R > 0$); grâce à (7.5), on aurait, pour $\lambda \geq \widetilde{\lambda}$,

$$\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\, dz\, dt \leq \frac{C'}{|\lambda|}.$$

où C' ne dépend pas de λ. f vérifiant (7.9) pour λ assez grand, cela nous mène à une contradiction.
w_ε ne peut donc être bornée, pour ε assez petit, dans $(L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ et ne converge donc pas (au sens donné précédemment) vers w solution de (7.14) lorsque $B(\varepsilon)$ est remplacé par A_1.

La cause de cette non-convergence peut s'expliquer de la manière suivante.
Prenons $B : \mathbb{R} \to M_3(\mathbb{R})$ ([3]) continue telle que $B(s)$ est diagonalisable sur $\mathbb{R}$ pour tout $s \in \mathbb{R}$. Si $B(\cdot)$ a des valeurs propres de multiplicité constante (par rapport à s), alors on peut suivre continuement ces valeurs propres et, au moins au voisinage de $s = 0$, une base de $\mathbb{R}^3$ formée de vecteurs propres de $B(\cdot)$. On peut alors voir, sur la formule explicite des solutions décomposées sur ces bases de vecteurs propres, que la solution de (7.14) converge, lorsque $\varepsilon \to 0$, vers la solution de (7.14) avec $B(0)$ à la place de $B(\varepsilon)$.
Dans l'exemple précédent, la multiplicité des valeurs propres de $B(s)$ dépend de s (lorsque $s = 0$, 0 est valeur propre double de $B(0) = A_1$, tandis que pour $s \neq 0$, toutes les valeurs propres de $B(s)$ sont distinctes). De plus, lorsque l'on regarde le comportement des vecteurs propres normalisés

$$\frac{1}{\sqrt{4^2 + (\lambda-2)^2 + 2^2}} \begin{pmatrix} 4 \\ \lambda - 2 \\ -2 \end{pmatrix}$$

$$\frac{1}{\sqrt{4^2 + (\lambda - \sqrt{\lambda^2+16})^2}} \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix}$$

$$\frac{1}{\sqrt{4^2 + (\lambda + \sqrt{\lambda^2+16})^2}} \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix}$$

de $B(\varepsilon)$, on constate que deux de ces vecteurs propres tendent vers $(0,1,0)^T$, que le troisième tend vers $(0,0,1)^T$ et que la direction propre $(1,0,0)^T$ de $B(0) = A_1$ n'est jamais atteinte en passant à la limite; la condition initiale que nous avions choisie était justement portée par cette direction propre particulière.

[2] Dans le cas monodimensionnel ($N = 1$) considéré ici, cette condition est notre condition d'hyperbolicité, qui est équivalente à celle de [67].

[3] Ce que nous disons ici n'est pas limité au cas $l = 3$.

On peut aussi comprendre cette non-convergence au travers de la condition exprimant que (7.14) est bien posé dans $(L^2(\mathbb{R}))^3$: cette condition est $\sup_{\xi\in\mathbb{R}} ||e^{i\xi B(\varepsilon)}|| < \infty$. Pour $\varepsilon \geq 0$ fixé, elle est satisfaite (puisqu'elle est alors équivalente à la diagonalisabilité sur $\mathbb{R}$ de $B(\varepsilon)$). Mais elle n'est pas satisfaite uniformément lorsque $\varepsilon \to 0$, i.e. pour tout $\varepsilon_0 > 0$ on a $\sup_{0\leq\varepsilon\leq\varepsilon_0}(\sup_{\xi\in\mathbb{R}} ||e^{i\xi B(\varepsilon)}||) = +\infty$; lorsque cette condition est satisfaite uniformément pour $\varepsilon \to 0$ et $w^0 \in (L^2(\mathbb{R}))^3$, il n'est pas très dur de voir que la solution de (7.14) pour $\varepsilon > 0$ converge (au moins dans $(L^2(]0,T[\times\mathbb{R}))^3$) vers la solution de ce problème pour $\varepsilon = 0$.