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Introduction

It's a mystery to me — the game commences
For the usual fee — plus expenses
Confidential information — it's in a diary
This is my investigation — it's not a public inquiry.

Nous étudions dans ce document différents problèmes d'équations aux dérivées partielles. La majeure partie de ce document concerne des équations de type elliptique (parties I et II, annexe A et une partie du chapitre 9), mais nous abordons aussi certaines questions concernant les systèmes et équations de type hyperboliques (partie III ainsi qu'une partie du chapitre 9) et les équations de type parabolique (chapitre 10).

Partie I

La première partie de ce travail consiste en une étude d'équations elliptiques non coercitives.

On sait depuis longtemps qu'une bonne manière d'aborder les équations aux dérivées partielles de la forme

$$\begin{cases} -\mathrm{div}(A\nabla u) + \mathrm{div}(\mathbf{v}u) + bu = L & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega \end{cases} \tag{0.1}$$

(avec Ω ouvert borné de $\mathbb{R}^N$) est de considérer une formulation dite "variationnelle" (ou, pour être plus exact, faible) de ces problèmes, faisant intervenir la forme bilinéaire

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v - \int_\Omega u\mathbf{v} \cdot \nabla v + \int_\Omega buv \tag{0.2}$$

sur l'espace de Hilbert $H^1_0(\Omega)$ (on impose des hypothèses d'intégrabilité *ad-hoc* sur les données pour que tous les termes de (0.2) aient un sens lorsque $(u,v) \in H^1_0(\Omega)$). Le théorème de Lax-Milgram (un outil linéaire) donne alors, sous l'hypothèse que a est "coercitive" (i.e. $a(u,u) \geq \alpha ||u||^2_{H^1_0}$ pour un certain $\alpha > 0$), l'existence et l'unicité de la solution, en un sens faible, à (0.1). Il existe un autre outil, le théorème de Leray-Lions (principalement utilisé pour les équations non-linéaires), qui donne, sinon l'unicité, du moins l'existence d'une solution sous une hypothèse de coercitivité sur a plus faible (à savoir: $\frac{a(u,u)}{||u||_{H^1_0}} \to \infty$ lorsque $||u||_{H^1_0} \to \infty$) ([1]).

[1]Le théorème de Leray-Lions demande aussi une propriété de monotonie de l'opérateur.

On sait bien que ce genre d'hypothèse n'est pas totalement dispensable: par exemple, il est impossible, à cause de l'existence de valeurs propres pour l'opérateur $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, d'espérer pouvoir résoudre, dans le cadre fonctionnel $H_0^1(\Omega)$, l'équation $-\Delta u + bu = f$ pour tout second membre f (même très régulier) et tout $b \in \mathbb{R}$.
Il est alors généralement imposé des hypothèses structurelles de la forme: A est une fonction matricielle bornée uniformément définie positive et $\frac{1}{2}\mathrm{div}(\mathbf{v}) + b \geq 0$. Sous ces conditions, la forme bilinéaire a précédemment définie satisfait les hypothèses du théorème de Lax-Milgram et on a donc un cadre d'existence et d'unicité de la solution de (0.1).

Chapitre 1

Nous prouvons, dans le chapitre 1, que l'on peut se passer de condition structurelle sur la partie convective de (0.1) (i.e. d'une hypothèse sur $\mathrm{div}(\mathbf{v})$). En considérant des hypothèses de coercitivité uniquement sur A (A est uniformément définie positive) et sur b (b est positive), la forme bilinéaire (0.2) n'est alors plus, en général, coercitive (ni au sens de Lax-Milgram, ni au sens de Leray-Lions), mais nous obtenons néanmoins existence et unicité d'une solution à (0.1) dans un cadre variationnel, i.e. une unique solution dans $H_0^1(\Omega)$. En fait, nous ne nous limitons pas au cas de conditions au bord de type Dirichlet: nous considérons des conditions au bord mixtes Dirichlet/Fourier. De même, nous ne traitons pas uniquement des termes convectifs sous forme conservative: grâce au caractère linéaire de l'équation, nous prouvons aussi l'existence et l'unicité d'une solution faible lorsque le terme convectif est sous forme non conservative.

La méthode employée pour obtenir ce résultat fait appel au degré topologique de Leray-Schauder. Grâce à cet outil, prouver l'existence d'une solution à (0.1) se ramène à obtenir des estimations *a priori* sur les solutions de cette équation.
Une astuce de Boccardo nous permet tout d'abord d'obtenir une estimation sur $\ln(1 + |u|)$; cette estimation n'est utile que par le contrôle qu'elle implique sur la mesure de Lebesgue des ensembles $E_k = \{x \in \Omega \mid |u(x)| > k\}$. La principale originalité de ce travail réside dans l'estimation de $S_k(u) = u - \max(-k, \min(u, k))$, qui utilise le contrôle précédemment trouvé sur la mesure de E_k afin d'éliminer le terme quadratique gênant en $S_k(u)$. L'estimation sur $T_k(u) = u - S_k(u)$ est, de son côté, assez évidente.
L'unicité de la solution de (0.1) s'obtient simplement en prouvant l'existence d'une solution pour le problème dual, ce qui revient à considérer (0.1) dans lequel le terme convectif sous forme conservative $\mathrm{div}(\mathbf{v}u)$ est remplacé par un terme convectif sous forme non conservative $\mathbf{v} \cdot \nabla u$.

Dans [25], De Giorgi prouve la continuité höldérienne sur les compacts de Ω des fonctions $u \in H^1(\Omega)$ vérifiant $-\mathrm{div}(A\nabla u) = 0$ dans $\mathcal{D}'(\Omega)$. Dans [70], Stampacchia prouve la continuité höldérienne sur $\overline{\Omega}$ des solutions variationnelles de problèmes elliptiques avec seconds membres non nuls (mais assez réguliers) et conditions au bord de type Dirichlet([2]); les méthodes employées par Stampacchia sont aussi sensiblement différentes de celles employées par De Giorgi (et font appel à plus de pré-requis).
Dans l'annexe A, nous montrons, grâce à des techniques de transport et réflexion, que les résultats de [70] concernant la continuité höldérienne sur $\overline{\Omega}$ des solutions de problèmes elliptiques sont aussi valables lorsque l'on considère certaines conditions au bord mixtes. Nous avons aussi écrit une preuve du résultat de [70] concernant la continuité sur les compacts de Ω en utilisant les méthodes, beaucoup plus abordables à notre goût, de [25]. Enfin, la manière dont les différentes estimations obtenues dépendent des données a été détaillée, ce qui est essentiel pour établir les résultats de stabilité de la section 2.4.

Munis de l'annexe A, nous prouvons alors la régularité höldérienne des solutions d'équations non coercitives de la forme (0.1), ce qui permet ensuite, par la méthode de dualité de [70], d'obtenir l'existence et l'unicité (en un certain sens) de solutions pour des problèmes elliptiques non coercitifs à seconds membres mesures.

[2] La preuve dans [70] de la continuité au bord ne nous parait cependant pas claire...

Chapitre 2

Le chapitre 2 contient quelques extensions, remarques et résultats supplémentaires à propos des thèmes abordés dans le premier chapitre.

Nous commençons par parler du cas non-linéaire non-coercitif. Comme nous le signalons en section 1.2.2, l'outil que nous employons dans le chapitre 1, pour prouver l'existence de solutions à des problèmes variationnels linéaires non-coercitif, est non-linéaire et permet de traiter certains termes de convections non-linéaires. Cependant, la méthode générale employée dans le cas linéaire (i.e. prouver l'existence d'un point fixe à une application en utilisant le degré topologique) n'est pas très adaptée pour traiter les équations dont les parties non-convectives sont non-linéaires, i.e. de la forme

$$\left\{ \begin{array}{ll} -\mathrm{div}(a(x,u,\nabla u)) + \mathrm{div}(\Phi(x,u)) + b|u|^{p-2}u = L & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega \end{array} \right. \tag{0.3}$$

(car on ne peut, en général, définir comme dans le cas linéaire l'application dont nous pourrions chercher un point fixe). Nous prouvons cependant que, associées à une méthode d'approximation déjà employée dans [8], le même genre d'estimations que dans le cas linéaire permet de prouver l'existence d'une solution “variationnelle” (i.e. lorsque L est dans le dual de l'espace d'energie $W_0^{1,p}(\Omega)$ associé à l'opérateur $\mathrm{div}(a(x,u,\nabla u))$) à (0.3).

Pour établir, dans l'annexe A, les résultats de continuité höldérienne des solutions de problèmes elliptiques linéaires avec conditions au bord mixtes, nous imposons une hypothèse sur la manière dont ces conditions au bord se repartissent sur $\partial\Omega$ (hypothèse (1.42)). Nous prouvons dans la section 2.2 que l'on ne peut se passer d'une hypothèse de ce genre.

Les sections 2.3 et 2.4 établissent quelques résultats concernant les solutions par dualité de problèmes elliptiques à seconds membres mesures, définies dans le premier chapitre. La première de ces sections donne quelques éléments pour comprendre pourquoi ce que nous appelons “solution par dualité” est effectivement, en un sens, solution d'un problème elliptique; la seconde section établit un résultat de stabilité concernant ces solutions par dualité, résultat bien naturel (mais pas évident) puisque l'on a existence et unicité de cette notion de solution.

Chapitre 3

Dans le troisième et dernier chapitre de cette partie, nous abordons le problème de l'étude numérique, via un schéma de type volumes finis, des équations linéaires non coercitives dont nous parlons dans le premier chapitre. Essentiellement, cela consiste à adapter les méthodes d'estimation du cas continu au cas discret.
Cependant, au lieu de considérer une discrétisation de (0.1) avec $L \in L^2(\Omega)$, comme c'est le cas par exemple dans [37], nous discrétisons cette équation avec un second membre de la forme $L = f + \mathrm{div}(G)$ où $f \in L^2(\Omega)$ et $G \in (\mathcal{C}(\overline{\Omega}))^N$; cela n'entraine pas vraiment de difficulté supplémentaire par rapport au cas étudié dans [37], mais cela a l'énorme avantage de faire apparaitre les estimations d'erreur comme conséquences immédiates des estimations *a priori* sur les solutions du problème discrétisé.
Nous généralisons aussi [37] dans le sens ou nous prouvons la convergence du schéma volumes finis non pas uniquement quand la convection $\mathbf{v}$ est supposée $\mathcal{C}^1$ mais aussi quand elle n'est supposée que continue.

Partie II

La deuxième partie concerne l'unicité des solutions à des problèmes elliptiques non linéaires avec seconds membres mesures.

Comme nous l'avons signalé auparavant, Stampacchia a introduit dans [70] une notion de solution pour des problèmes elliptiques *linéaires* avec seconds membres mesures, notion qui donne existence et unicité de la solution. Dans le cas de problèmes elliptiques non-linéaires, Boccardo et Gallouët ont prouvé dans [10] (voir aussi [66]) l'existence d'une solution (en un sens faible assez naturel) par des méthodes d'approximation: en prenant f_n régulier qui converge vers la mesure μ du second membre et en notant u_n une solution du problème avec f_n comme second membre, on peut prouver que (à une sous-suite près) u_n converge vers une solution faible u du problème avec μ comme second membre (u est alors appelée "SOLA": solution obtenue comme limite d'approximations). Cependant, on sait (cf [65]) que, même dans le cas linéaire, cette solution faible n'est pas unique. Plusieurs notions de solutions ont alors été développées afin de récupérer l'unicité: SOLA (voir [22]), solutions entropiques (voir [4]) ou encore solutions renormalisées (voir [21]).

Chapitre 4

Dans le premier chapitre de cette partie, nous établissons l'unicité dans le cadre des SOLA des solutions d'équations elliptiques non-linéaires (définies par des opérateurs de Leray-Lions $\mathrm{div}(a(x,u,\nabla u))$ agissant sur $H^1(\Omega)$) avec des mesures générales comme second membre. Un résultat similaire avait été établi par Boccardo (voir [6]), mais avec des hypothèses plus restrictives sur la fonction a (a indépendant de u et de classe $\mathcal{C}^1$ par rapport à ∇u) et des conditions au bord de type Dirichlet. La principale originalité de ce chapitre est que nous acceptons une certaine dépendance de a par rapport à u, ainsi que des conditions au bord plus générales.

La méthode employée pour prouver cette unicité est une méthode de dualité. En effet, bien que le problème considéré ne soit pas linéaire, notre résultat d'unicité se base sur l'existence (et la régularité) d'une solution à un problème "dual": si u et v sont deux SOLA pour un même second membre et $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ sont des approximations de u et v, alors on se rend compte que $u_n - v_n$ vérifie un problème elliptique linéaire (P_n) avec un second membre qui tend vers 0 dans $(\mathcal{C}(\overline{\Omega}))'$ faible-$*$; en utilisant la solution φ_n d'un problème dual de (P_n) comme fonction test dans (P_n), et grâce aux estimations höldériennes que l'on peut avoir sur φ_n, on peut passer à la limite([3]) et constater alors que $u = v$.

A la fin de ce même chapitre, nous établissons un résultat de stabilité sur la SOLA, beaucoup plus simple à obtenir (comme on peut s'y attendre, vu la définition même de ce qu'est une SOLA) que le résultat de stabilité des solutions par dualité dont nous parlons dans la partie I.

Chronologiquement parlant, ce chapitre a été établi avant les résultats (ou du moins une partie des résultats) de la partie I. En effet, lorsque l'on considère un opérateur non-linéaire ne dépendant pas de u, comme c'est le cas dans [6], le problème dual de (P_n) que l'on doit résoudre est un problème elliptique classique

$$\begin{cases} -\mathrm{div}(A_n \nabla \varphi_n) = f & \text{dans } \Omega, \\ \varphi_n = 0 & \text{sur } \partial\Omega. \end{cases} \tag{0.4}$$

Mais lorsque l'on accepte une dépendance de a par rapport à u, le problème dual de (P_n) devient

$$\begin{cases} -\mathrm{div}(A_n \nabla \varphi_n) + \mathbf{v}_n \cdot \nabla \varphi_n = f & \text{dans } \Omega, \\ \varphi_n = 0 & \text{sur } \partial\Omega, \end{cases} \tag{0.5}$$

où

$$\mathbf{v}_n = \frac{a(x,u_n,\nabla v_n) - a(x,v_n,\nabla v_n)}{u_n - v_n}.$$

On voit alors clairement qu'il est impossible d'imposer une condition sur $\mathrm{div}(\mathbf{v}_n)$ et que résoudre (0.5) revient donc à résoudre un problème linéaire non coercitif, avec un terme convectif sous forme non

[3] Car le second membre de (P_n) converge dans $(\mathcal{C}(\overline{\Omega}))'$ faible-$*$ et $(\varphi_n)_{n\geq 1}$, étant bornée dans un espace de Hölder, converge fortement à une sous-suite près dans $\mathcal{C}(\overline{\Omega})$.

conservative. C'est donc lors de l'étude de l'unicité de la SOLA que la résolution de problèmes linéaires non coercitifs s'est avérée nécessaire, et c'est pourquoi une étude sommaire (dans un cadre assez restreint) de ces problèmes apparait dans le chapitre 4. Si nous avons ensuite étudié plus à fond ces questions dans la partie I, c'est qu'elles nous paraissaient intéressantes en soi.

Chapitre 5

Le chapitre 5 clos cette deuxième partie. Nous y discutons tout d'abord brièvement de la manière dont on peut approcher les solutions de problèmes elliptiques à données mesures (et donc sur la définition même de SOLA).
Notre méthode pour prouver l'unicité de la SOLA repose essentiellement sur un résultat de continuité höldérienne des solutions de problèmes elliptiques linéaires; nous avons vu, dans la partie I, que pour obtenir un tel résultat lorsque l'on considère des conditions au bord mixtes, il est nécessaire de supposer une hypothèse de "bonne répartition" (hypothèse (1.42)) de ces conditions au bord. Nous prouvons à la fin du chapitre 5 que, sans une hypothèse de ce genre, le résultat d'unicité des SOLA n'est plus valable en général.

Partie III

Dans cette partie, nous nous intéressons aux systèmes linéaires d'équations du premier ordre à coefficients constants de la forme

$$\left\{\begin{array}{ll} u_t + \sum_{i=1}^N A_i u_{x_i} = 0 & \text{dans }]0,T[\times\Omega, \\ u(0) = u_0 & \text{dans } \Omega \end{array}\right. \tag{0.6}$$

(Ω est un ouvert de $\mathbb{R}^N$ et $(A_i)_{i\in[1,N]}$ sont des matrices réelles $l \times l$).

Il est bien connu que, selon la manière dont on veut résoudre ces systèmes, il faut ou non imposer certaines conditions sur les matrices $(A_i)_{i\in[1,N]}$: si l'on cherche à résoudre localement (0.6) avec des conditions initiales analytiques, aucune condition n'est requise (théorème de Cauchy-Kowalewska, voir [33]); si l'on veut résoudre, lorsque $\Omega = \mathbb{R}^N$, ce système pour toute condition initiale $\mathcal{C}^\infty$, alors il faut supposer que, pour tout $\xi \in \mathbb{R}^N$, $A(\xi) = \sum_{i=1}^N \xi_i A_i$ a des valeurs propres réelles (théorème de Lax-Mizohata, voir [18]); si l'on veut résoudre (0.6) pour $\Omega = \mathbb{R}^N$ et toute condition initiale dans $(L^2(\mathbb{R}^N))^l$, un raisonnement passant par la transformée de Fourier nous montre qu'il faut supposer que $\sup_{\xi\in\mathbb{R}^N} ||iA(\xi)|| < \infty$ (voir [67]). Toutes ces conditions sont nécessaires et suffisantes dans le cadre considéré, et ont été énoncées de la plus faible à la plus forte.

Cependant, aucune de ces conditions ne recouvre exactement le cas qui intéresse les mathématiciens qui étudient les discrétisations des systèmes du premier ordre. En effet, de nombreux schémas numériques dans ce domaine (schéma de Roe, de VFRoe; voir [37] par exemple) demandent, pour discrétiser un système de la forme

$$\left\{\begin{array}{ll} u_t + \operatorname{div}(f(u)) = 0 & \text{dans }]0,T[\times\Omega, \\ u(0) = u_0 & \text{dans } \Omega, \end{array}\right. \tag{0.7}$$

(où $f = (f_1, \ldots, f_N)$ avec $f_i : \mathbb{R}^l \to \mathbb{R}^l$), de résoudre des problèmes de la forme (0.6) avec une condition initiale de type "Riemann" (i.e. constante de part et d'autre d'un certain hyperplan de $\mathbb{R}^N$). Ce genre de condition initiale ne rentre ni dans le cadre du théorème de Lax-Mizohata (elle n'est pas dans $\mathcal{C}^\infty(\mathbb{R}^N)$), ni dans le cadre "Fourier" (i.e. avec une condition initiale de type $L^2(\mathbb{R}^N)$). On pourrait éventuellement, à l'aide d'un théorème de propagation à vitesse finie ou en utilisant des espaces construits sur $L^2(\mathbb{R}^N)$ (de la forme $H^s(\mathbb{R}^N)$ pour $s < 0$), considérer que les conditions initiales de type Riemann "rentrent" dans le cadre Fourier, mais ce serait certainement assez inadapté: en effet, on demanderait alors à résoudre (0.6) pour beaucoup trop de condition initiales (pour tous les u_0 dans $L^2(\mathbb{R}^N)$ ou dans $H^s(\mathbb{R}^N)$) et non pas uniquement pour les conditions initiales qui nous intéressent dans ce cas particulier (conditions initiales de type Riemann).

Chapitre 6

Nous établissons, dans le chapitre 6, une condition nécessaire et suffisante pour que (0.6) ait une solution pour toute condition initiale de type Riemann; il faut noter que, grâce à un résultat de propagation à vitesse finie que nous prouvons dans ce chapitre, la solution en question est alors unique, à condition que $\Omega = \mathbb{R}^N$ (i.e. que l'on ne considère pas de condition au bord).
Cette condition nécessaire et suffisante (que nous appelons "hyperbolicité" de (0.6), comme dans [37], même si nous sommes conscients que la terminologie dans ce domaine n'est pas évidente) est simplement la suivante: pour tout $\xi \in \mathbb{R}^N$, $A(\xi) = \sum_{i=1}^N \xi_i A_i$ est diagonalisable sur $\mathbb{R}$. Elle est évidemment strictement plus forte que la condition obtenue dans le cadre du théorème de Lax-Mizohata et on prouve, au début du chapitre 7, qu'elle est strictement plus faible que la condition obtenue dans le cadre "Fourier".

Il faut aussi noter que, au cours de notre preuve, nous retrouvons deux phénomènes connus: en considérant le système unidimensionnel

$$\begin{cases} u_t + Au_x = 0 & \text{dans }]0,T[\times\mathbb{R}, \\ u(0) = u_0 & \text{dans } \mathbb{R}, \end{cases} \tag{0.8}$$

on voit que si A a des valeurs propres complexes (non-réelles), alors on ne pourra en général résoudre ce système que pour des conditions initiales analytiques (théorème de Lax-Mizohata) et, si A a des blocs de Jordan non triviaux, alors la solution de (0.8) est en général moins dérivable en espace que ne l'est u_0 (on peut perdre autant de dérivées que la taille du plus gros bloc de Jordan de A moins 1).

Chapitre 7

Dans la remarque 6.2, nous donnons un système particulier qui, affirmons-nous, montre la différence entre notre condition d'"hyperbolicité" et la condition obtenue par analyse de Fourier. Au début du chapitre 7, nous prouvons ces dires. Ce système est ensuite, dans le reste du chapitre, étudié un peu plus à fond, car il se révèle finalement assez riche.
Il permet par exemple de montrer que la condition d'hyperbolicité n'est en général pas suffisante pour résoudre (0.6) pour toute condition initiale dans L^∞ ou dans BV_{loc} (si l'on veut éviter de perdre des dérivées). Et il permet aussi de prouver que, contrairement à ce qui se passe dans le cas scalaire (cf [54]), la solution d'un système d'équations du genre (0.8) peut ne pas dépendre continuement de la matrice A définissant ce système: si $A_n \to A_\infty$ et même si, pour tout $n \in \mathbb{N} \cup \{\infty\}$, le problème (0.8) avec $A = A_n$ est bien posé au sens de Hadamard dans $(L^2(\mathbb{R}))^l$, la solution de (0.8) avec $A = A_n$ peut ne pas converger vers la solution de ce problème avec $A = A_\infty$.

Partie IV

Cette partie rassemble divers autres travaux de recherche.

Chapitre 8

Le chapitre 8 est un résultat de densité dans les espaces de Sobolev $W^{1,p}(\Omega)$. Nous prouvons que, sous certaines hypothèses sur l'ouvert considéré (Ω est soit un ouvert régulier de $\mathbb{R}^N$, soit un ouvert polygonal de $\mathbb{R}^N$), les fonctions régulières satisfaisant une condition de Neumann sur $\partial\Omega$ sont denses dans $W^{1,p}(\Omega)$; nous établissons aussi une généralisation au cas des conditions au bord mixtes et un contre-exemple dans le cas d'un ouvert insuffisament régulier (mais quand même à frontière lipschitzienne). Quelques applications de ces résultats sont données, comme l'écriture d'une formulation du problème de Neumann en faisant passer (comme c'est possible pour le problème de Dirichlet) toutes les dérivées sur les fonctions tests, ou encore la possibilité de simplifier certaines démonstrations et de généraliser certains théorèmes de convergence de schémas volumes finis (voir [45] et [55]).

Chapitre 9

Dans le chapitre 9, nous étudions une discrétisation éléments finis mixtes — volumes finis pour un système modélisant, de manière simplifiée, un écoulement diphasique (formé de deux phases immiscibles) au travers d'un milieu poreux. Le système se réduit à une équation elliptique donnant une des deux inconnues, la seconde étant obtenue par résolution d'une équation de transport dont le champ de vitesses est donné par le gradient de la première inconnue. En l'absence de termes de diffusion, ce cas de découplage est le seul sur lequel des résultats mathématiques sont connus quant à l'existence d'une solution dans un sens faible. Il existait déjà plusieurs résultats de convergence de schémas numériques pour ce système, avec différentes méthodes pour discrétiser l'équation elliptique (éléments finis conformes, volumes finis). Nous introduisons dans ce chapitre des maillages non usuels dans le cadre des éléments finis (les mailles ne sont pas forcément polygonales) et une méthode d'éléments finis mixtes pour l'équation elliptique, dont l'intérêt est de traiter de façon précise des cas hétérogènes et anisotropes sur des maillages très généraux (les méthodes citées précédemment posent des difficultés de fabrication de maillages compatibles avec les hétérogénéités et les anisotropies), ce qui correspond à un besoin industriel bien identifié. Le raisonnement est assez classique, mais les difficultés rencontrées pour traiter les preuves mathématiques dans le cas des mailles considérées demandent des traitements nouveaux.

Chapitre 10

Le chapitre 10 comporte essentiellement deux parties. Dans la première, nous introduisons la capacité parabolique à l'aide d'une approche fonctionnelle dans le même esprit que [62]. Nous établissons ensuite un théorème de décomposition des mesures (en espace-temps) ne chargeant pas les ensembles de capacité parabolique nulle, similaire au résultat obtenu dans [12] pour le cas elliptique. Dans la deuxième partie, nous prouvons, grâce à cette décomposition, l'existence et l'unicité d'une solution renormalisée pour un problème parabolique non-linéaire dont le second membre est une mesure ne chargeant pas les ensembles de capacité parabolique nulle.

Enfin, pour clore ce panorama, nous souhaiterions signaler quelques autres travaux de recherche qui ne figurent pas dans ce document mais ont, pour certains, des liens avec quelques-un des chapitres qui suivent.
Le premier de ces travaux s'est effectué hors du cadre de cette thèse. Il s'agit d'un article paru dans Nonlinear Analysis, TMA: "Optimal Pointwise Control of Semilinear Parabolic Equations" (vol. **39** (2000), pp 135-156), co-écrit avec Jean-Pierre Raymond suite à un stage de magistère effectué sous sa direction.
Le chapitre 9 (en particulier l'appendice) s'appuie beaucoup sur le polycopié [31]. Ce travail est en grande partie non original mais comporte certaines études sur les ouverts "faiblement lipschitziens" (appelés "sous-variété lipschitzienne N-dimensionnelle de $\mathbb{R}^N$" dans [42] et plus généraux que les ouverts à frontière lipschitzienne de [58]) — en particulier la preuve de l'existence d'une normale extérieure et du théorème de Stokes pour ces ouverts — qui sont, à notre connaissance, nouvelles.
Beaucoup de résultats de densité et d'intégration par parties (dissimulées) du chapitre 10 trouvent leur justification au travers des résultats de [32]. On retrouve dans ce polycopié des résultats classiques (tels que les théorèmes d'injections compactes d'Aubin et de Simon), mais certains résultats (comme la différence entre $L^\infty(0,T;L^\infty(\Omega))$ et $L^\infty(]0,T[\times\Omega)$, des résultats de densité ou la justification de certaines identifications) sont, semble-t-il, nouveaux.

- Le chapitre 1 est un article accepté pour publication dans Potential Analysis.
- Les sections 2.2 et 2.3, ainsi que la section A.4 de l'annexe (concernant le traitement des conditions au bord mixtes) sont en grande partie tirées, avec quelques légères modifications, de l'article [29] publié dans Advances in Differential Equations.

- Le chapitre 4 est un article co-écrit avec Thierry Gallouët et accepté pour publication dans Rendiconti di Matematica.
- Le chapitre 6 est un article actuellement soumis pour publication.
- Le chapitre 8 est un article actuellement soumis pour publication.
- Le chaptre 9 est un article en préparation, co-écrit avec Robert Eymard, Danielle Hilhorst et Xiao Dong Zhou.
- Le chapitre 10 est un article en préparation, co-écrit avec Alessio Porretta et Alain Prignet.

And I'm on the edge
Of an endless fall
Sure enough
He's come to call
Got to go now
Get on that bus
Me and the wanderlust.

Partie I

Problèmes Elliptiques Non Coercitifs

Chapitre 1

Non Coercive Linear Elliptic Problems

J. Droniou.

Abstract We study here some linear elliptic partial differential equations (with Dirichlet, Fourier or mixed boundary conditions), to which are added convection terms (first order perturbations) that entail the loss of the classical coercivity property. We prove existence, uniqueness and regularity results for the solutions to these problems.

1.1 Introduction

1.1.1 Notations

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz continuous boundary. We denote by $\mathbf{n}$ the unit normal to $\partial\Omega$ outward to Ω and by σ the measure on $\partial\Omega$.
$x \cdot y$ denotes the usual euclidean scalar product of two vectors $(x, y) \in \mathbb{R}^N$; $|\cdot|$ is the associated euclidean norm.
When E is a measurable subset of $\mathbb{R}^N$, $|E|$ denotes the Lebesgue measure of E.
For $q \in [1, \infty]$, q' denotes the conjugate exponent of q (that is to say $1/q + 1/q' = 1$). The space $(L^q(\Omega))^N$ is endowed with the norm $||F||_{(L^q(\Omega))^N} = ||\,|F|\,||_{L^q(\Omega)}$; $B(q, R)$ denotes the closed ball in $(L^q(\Omega))^N$ of center 0 and radius R.
If Γ is a measurable subset of $\partial\Omega$, $W^{1,q}_\Gamma(\Omega)$ is the space of all functions in $W^{1,q}(\Omega)$ (the usual Sobolev space) the trace of which is null on Γ; it is endowed with the same norm as $W^{1,q}(\Omega)$, that is to say $||v||_{W^{1,q}(\Omega)} = ||v||_{L^q(\Omega)} + ||\,|\nabla v|\,||_{L^q(\Omega)}$. When $q = 2$, we denote as usual $W^{1,2} = H^1$.
We take, when $N \geq 3$, $N_* = N$ and, when $N = 2$, $N_* \in]2, \infty[$.

1.1.2 The Equations

The kinds of equations we will study are

$$\left\{\begin{array}{lll} -\mathrm{div}(A\nabla\mathcal{U}) - \mathrm{div}(\mathbf{v}\mathcal{U}) + b\mathcal{U} = \mathcal{L} & \text{in} & \Omega, \\ \mathcal{U} = \mathcal{U}_d & \text{on} & \Gamma_d, \\ A\nabla\mathcal{U}\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n})\mathcal{U} = \mathcal{U}_f & \text{on} & \Gamma_f, \end{array}\right. \tag{1.1}$$

and

$$\left\{\begin{array}{lll} -\mathrm{div}(A^T\nabla\mathcal{V}) + \mathbf{v}\cdot\nabla\mathcal{V} + b\mathcal{V} = \mathcal{L} & \text{in} & \Omega, \\ \mathcal{V} = \mathcal{V}_d & \text{on} & \Gamma_d, \\ A^T\nabla\mathcal{V}\cdot\mathbf{n} + \lambda\mathcal{V} = \mathcal{V}_f & \text{on} & \Gamma_f \end{array}\right. \tag{1.2}$$

(where Γ_d and Γ_f are measurable subsets of $\partial\Omega$, the union of which is $\partial\Omega$ and such that $\sigma(\Gamma_d \cap \Gamma_f) = 0$).

In fact, we will only study the variational (or weak) formulations of these equations; using functions $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ the trace on $\partial\Omega$ of which are $\mathcal{U}_d$ and $\mathcal{V}_d$, searching weak solutions of (1.1) or (1.2) comes down to searching solutions of

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + \int_\Omega u\mathbf{v}\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,, \\ \qquad\qquad\qquad\qquad \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{array}\right. \tag{1.3}$$

or

$$\left\{\begin{array}{l} v \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla v\cdot\nabla\varphi + \int_\Omega \varphi\mathbf{v}\cdot\nabla v + \int_\Omega bv\varphi + \int_{\Gamma_f}\lambda v\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,, \\ \qquad\qquad\qquad\qquad \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{array}\right. \tag{1.4}$$

(with $u=\mathcal{U}-\widetilde{\mathcal{U}}$, $v=\mathcal{V}-\widetilde{\mathcal{V}}$ and L which takes into account $\mathcal{L}$ and $(\widetilde{\mathcal{U}},\mathcal{U}_f)$ or $(\widetilde{\mathcal{V}},\mathcal{V}_f)$).

In order that all the terms in (1.3) and (1.4) be defined, the minimal hypotheses on the datas are, thanks to the Sobolev injections: $A:\Omega\to M_N(\mathbb{R})$ is a matrix-valued essentially bounded measurable function, $b\in L^{\frac{N_*}{2}}(\Omega)$, $\lambda\in L^{N_*-1}(\partial\Omega)$ and $\mathbf{v}\in(L^{N_*}(\Omega))^N$.

The classical framework of study for linear elliptic problems is the Lax-Milgram Theorem, which demands the coercivity of the bilinear form appearing in (1.3) or (1.4), i.e. additional hypotheses on the datas.
The main coercivity hypothesis is on A, to ensure that the principal part of the operator is elliptic (see Hypothesis (1.9)).
In order that the lower order terms do not to cause the loss of this coercivity, it is usual to add then hypotheses on $\mathbf{v}$, b, and λ. For the pure Dirichlet condition ($\Gamma_f=\emptyset$), this can be

$$-\frac{1}{2}\mathrm{div}(\mathbf{v})+b\geq c \quad \text{in} \quad \mathcal{D}'(\Omega),$$

with c "small enough" in $L^{\frac{N_*}{2}}(\Omega)$ (in general, c is taken equal to 0) — notice that this condition adds an hypothesis on the *regularity* of $\mathbf{v}$ (when this inequality is satisfied, $\mathrm{div}(\mathbf{v})$ must be a Radon measure on Ω).
In the case of Fourier or mixed boundary conditions, to cleanly express these additional hypotheses, we need more regularity on $\mathbf{v}$ (to give a sense to $\mathbf{v}\cdot\mathbf{n}$). Moreover, in these cases, when we want to obtain regularity results, the minimal regularity on $\mathbf{v}$ seemed to be the Lipschitz continuity (because of the many integrates by parts we have then to do; see [29]).

Asking for the principal part ($-\mathrm{div}(A\nabla u)$ or $-\mathrm{div}(A^T\nabla v)$) to be coercive is quite natural when we search for solutions in $H^1(\Omega)$. One could wonder if the additional hypotheses on the lower order terms $\mathrm{div}(\mathbf{v}u)$ (or $\mathbf{v}\cdot\nabla u$), bu and λu are really necessary; we will see below that we cannot avoid some hypotheses on the zero-order terms bu and λu. Concerning the first order terms, works have already been done to get rid of the coercivity hypothesis on the convection term when it is in conservative form.

In [8], the author proves an existence result, and studies some qualitative properties, for entropy solutions of

$$\left\{\begin{array}{lll} -\mathrm{div}(a(x,u,\nabla u)) = f-\mathrm{div}(F+\Phi(u)) & \text{in} & \Omega, \\ u=0 & \text{on} & \partial\Omega, \end{array}\right. \tag{1.5}$$

where $\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator in divergence form acting on $W^{1,p}_0(\Omega)$ $(1-2/N<p<N)$, $f\in L^1(\Omega)$, $F\in(L^{p'}(\Omega))^N$ and Φ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$; due to the lack of growth properties on Φ, it is crucial in (1.5) to consider pure homogeneous Dirichlet boundary conditions and a Φ not depending on $x\in\Omega$.

In [43], the authors study the existence and uniqueness of renormalized solutions to

$$\left\{\begin{array}{lll}\lambda u - \mathrm{div}(a(x,\nabla u) + \Phi(x,u)) = f & \text{in} & \Omega, \\ (a(x,\nabla u) + \Phi(x,u))\cdot \mathbf{n} = 0 & \text{on} & \Gamma_f, \\ u = 0 & \text{on} & \Gamma_d,\end{array}\right. \tag{1.6}$$

where λ is a non-negative real number, $\mathrm{div}(a(x,\nabla u))$ is a Leray-Lions operator in divergence form — notice the independance of a with respect to u — acting on $W^{1,p}(\Omega)$, $f \in L^1(\Omega)$ and Φ is a Caratheodory function with growth properties; the problem is either pure Dirichlet ($\Gamma_f = \emptyset$) or mixed ($\Gamma_f \neq \emptyset$ but $\sigma(\Gamma_d) > 0$).

We prove, in Section 2, existence and uniqueness results for (1.3) and (1.4), with no coercivity hypothesis on the convection term. These results are not consequences of [8] or [43], because the natural space of entropy or renormalized solutions is not the usual Sobolev space $H^1(\Omega)$.
In Section 3, we will see that the regularity results we already have in the coercive case, when the right-hand side L is more regular (see [70] and [29]), are still true with general convections terms. Under stronger hypotheses ($\mathbf{v} \in (L^\infty(\Omega))^N$ and $L \in L^\infty(\Omega)$), the existence and regularity results appear in [30].
We then shortly describe in Section 4 how the regularity results of Section 3 can be transformed in existence and uniqueness results with measures as datas (as in [70] or [29]).

1.1.3 The Zero-Order Terms

We cannot, in general, solve Problems (1.3) and (1.4) for any $b \in L^{\frac{N_*}{2}}(\Omega)$ and $\lambda \in L^{N_*-1}(\partial\Omega)$. This is due to the existence of an eigenvalue for the Laplace operator.
Consider pure Dirichlet boundary conditions and take e an eigenvector of $-\Delta$ on $H^1_0(\Omega)$, that is to say $e \in H^1_0(\Omega)\backslash\{0\}$ such that $-\Delta e = le$ for a $l \in \mathbb{R}$ (in fact, we have then $l > 0$).
Take now $b \in \mathbb{R}$ and suppose there exists a solution $u \in H^1_0(\Omega)$ of $-\Delta u + bu = e$; that is to say, for all $\varphi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla\varphi + \int_\Omega bu\varphi = \int_\Omega e\varphi.$$

With $\varphi = e$, we get

$$(l+b)\int_\Omega ue = \int_\Omega e^2.$$

Since $e \neq 0$, this last equation can not be satisfied for $b = -l$; thus, there is no solution $u \in H^1_0(\Omega)$ of $-\Delta u - lu = e$.
The same kind of reasoning can be done in the mixed case, and this shows that we cannot avoid additional hypotheses on b and λ (i.e. we cannot only suppose integrability hypotheses on these datas).

In (1.3) we have considered convection terms only in conservative form; in (1.4), we have considered convection terms only in non-conservative form. A natural question is the following: can we consider, in the same equation, convection terms both in conservative and non-conservative form? That is to say, can we solve

$$\left\{\begin{array}{lll}-\mathrm{div}(A\nabla u) - \mathrm{div}(\mathbf{v}u) + \mathbf{w}\cdot\nabla u + bu = L & \text{in} & \Omega, \\ u = 0 & \text{on} & \Gamma_d, \\ A\nabla u\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n})u = 0 & \text{on} & \Gamma_f,\end{array}\right. \tag{1.7}$$

the same way we will solve (1.3) and (1.4) (i.e. without additional hypothesis on the convection terms)? The answer is no, and is due to the same objection as before. Indeed, take $\mathbf{v}$ a regular vector-valued function; since, for $u \in H^1_0(\Omega)$, we have $\mathrm{div}(\mathbf{v}u) - \mathbf{v}\cdot\nabla u = u\mathrm{div}(\mathbf{v})$, a solution in $H^1_0(\Omega)$ of $-\Delta u - \mathrm{div}(\mathbf{v}u) + \mathbf{v}\cdot\nabla u = L$ (that is to say Problem (1.7) in the case of pure Dirichlet boundary conditions, with $A = Id$, $b = 0$ and $\mathbf{w} = \mathbf{v}$) would be a solution to $-\Delta u - (\mathrm{div}(\mathbf{v}))u = L$; by taking a regular vector-valued function $\mathbf{v}$ such that $\mathrm{div}(\mathbf{v}) = l$, the preceding reasoning proves that, in general, this last problem has no solution.

Thus, (1.7) is not solvable without additional hypotheses on the first-order terms.
Problems (1.3) and (1.4) seems thus to be the most general problems we can solve, when we add no structural hypothesis on the first order terms.

1.1.4 Hypotheses

We make the following hypotheses on the datas.

$$\Gamma_d \text{ and } \Gamma_f \text{ are measurable subsets of } \partial\Omega \text{ such that } \sigma(\Gamma_d \cap \Gamma_f) = 0 \text{ and } \partial\Omega = \Gamma_d \cup \Gamma_f, \tag{1.8}$$

$$\begin{array}{c} A : \Omega \to M_N(\mathbb{R}) \text{ is a measurable matrix-valued function which satisfies:} \\ \exists \alpha_A > 0 \text{ such that } A(x)\xi \cdot \xi \geq \alpha_A |\xi|^2 \text{ for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^N, \\ \exists \Lambda_A > 0 \text{ such that } ||A(x)|| := \sup\{|A(x)\xi|\,,\ \xi \in \mathbb{R}^N\,,\ |\xi| = 1\} \leq \Lambda_A \text{ for a.e. } x \in \Omega, \end{array} \tag{1.9}$$

$$b \in L^{\frac{N_*}{2}}(\Omega)\,,\ b \geq 0 \text{ a.e. on } \Omega, \tag{1.10}$$

$$\lambda \in L^{N_*-1}(\partial\Omega)\,,\ \lambda \geq 0\ \sigma\text{-a.e. on } \partial\Omega, \tag{1.11}$$

$$\mathbf{v} \in (L^{N_*}(\Omega))^N, \tag{1.12}$$

$$L \in (H^1_{\Gamma_d}(\Omega))' \tag{1.13}$$

(recall that $N_* = N$ when $N \geq 3$ and that $N_* \in]2, \infty[$ when $N = 2$).

The non-convection parts of equations (1.3) and (1.4) are supposed to be coercive, that is to say:

$$\begin{array}{c} \exists b_0 > 0\,,\ \exists E \subset \Omega \text{ such that } b \geq b_0 \text{ on } E, \\ \exists \lambda_0 > 0\,,\ \exists S \subset \Gamma_f \text{ such that } \lambda \geq \lambda_0 \text{ on } S \text{ and either} \\ \sigma(\Gamma_d) > 0 \text{ or } |E| > 0 \text{ or } \sigma(S) > 0. \end{array} \tag{1.14}$$

The set of variables that give the coercivity of the principal part of the operators in (1.3) or (1.4) is denoted by $\mathcal{B} = (\Omega, \alpha_A, \Gamma_d, b_0, E, \lambda_0, S)$.

Remark 1.1 *It is well-known that, under Hypotheses (1.8)—(1.11) and (1.14), for all $q \in [1, 2]$, there exists $\mathcal{K}(q, \mathcal{B}) > 0$ such that, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,*

$$\mathcal{K}(q, \mathcal{B})||\varphi||^2_{H^1(\Omega)} \leq \alpha_A \int_\Omega |\nabla\varphi|^2 + \left(b_0 \int_E |\varphi|^q + \lambda_0 \int_S |\varphi|^q \, d\sigma\right)^{\frac{2}{q}}.$$

By denoting $C_S(\Omega, N_*)$ the norm of the Sobolev injection $H^1(\Omega) \hookrightarrow L^{\frac{2N_*}{N_*-2}}(\Omega)$ (see [1]), we also take

$$\chi \in [0, \tfrac{\mathcal{K}(2,\mathcal{B})}{C_S(\Omega, N_*)}[. \tag{1.15}$$

Remark 1.2 *When χ satisfies (1.15), we have, for all $\mathbf{w} \in B(N_*, \chi)$ and all $\varphi \in H^1_{\Gamma_d}(\Omega)$,*

$$\begin{aligned} &\int_\Omega A\nabla\varphi \cdot \nabla\varphi + \int_\Omega \varphi\mathbf{w} \cdot \nabla\varphi + \int_\Omega b\varphi^2 + \int_{\Gamma_f} \lambda\varphi^2 \, d\sigma \\ &\quad\geq\ \mathcal{K}(2, \mathcal{B})||\varphi||^2_{H^1(\Omega)} - ||\,|\mathbf{w}|\,||_{L^{N_*}(\Omega)}||\varphi||_{L^{\frac{2N_*}{N_*-2}}(\Omega)}||\,|\nabla\varphi|\,||_{L^2(\Omega)} \\ &\quad\geq\ (\mathcal{K}(2, \mathcal{B}) - \chi C_S(\Omega, N_*))||\varphi||^2_{H^1(\Omega)}, \end{aligned}$$

with $\mathcal{K}(2, \mathcal{B}) - \chi C_S(\Omega, N_*) > 0$ (this exactly means that the bilinear form $(\varphi, \psi) \to \int_\Omega A\nabla\varphi \cdot \nabla\psi + \int_\Omega \varphi\mathbf{w} \cdot \nabla\psi + \int_\Omega b\varphi\psi + \int_{\Gamma_f} \lambda\varphi\psi \, d\sigma$ is coercive on $H^1_{\Gamma_d}(\Omega)$).
When $\mathbf{v} \in B(N_*, \chi)$ with χ satisfying (1.15), by the Lax-Milgram Theorem, (1.3) and (1.4) have thus unique solutions; our aim is to prove that we do not need such an hypothesis on $\mathbf{v}$ to have existence and uniqueness results for these problems.

Remark 1.3 *When* $\mathbf{v}$ *only satisfies (1.12), Problems (1.3) and (1.4) are in general non-coercive not only in the sense of the Lax-Milgram Theorem (the classical tool for linear elliptic problems) but also in the sense of the Leray-Lions Theorem (the classical tool for nonlinear elliptic problems). Indeed, consider the pure Dirichlet boundary conditions with* $b = \lambda = 0$ *(for the sake of simplicity) and take* $\mathbf{w}$ *a regular function such that* $\mathrm{div}(\mathbf{w}) \neq 0$*; we can find* $u \in H_0^1(\Omega)$ *such that*

$$\int_\Omega u\mathbf{w} \cdot \nabla u = \frac{1}{2}\int_\Omega \mathbf{w} \cdot \nabla(u^2) \neq 0$$

(take $u \in \mathcal{C}_c^\infty(\Omega)\backslash\{0\}$*, the support of which is contained in* $\{x \in \Omega \mid \mathrm{div}(\mathbf{w})(x) < 0\}$ *or in* $\{x \in \Omega \mid \mathrm{div}(\mathbf{w})(x) > 0\}$*); let then*

$$s = -\frac{\int_\Omega A\nabla u \cdot \nabla u}{\int_\Omega u\mathbf{w} \cdot \nabla u} \quad \text{and} \quad \mathbf{v} = s\mathbf{w}.$$

The sequence $(u_n)_{n\geq 1} = (nu)_{n\geq 1} \in H_0^1(\Omega)$ *satisfies* $||u_n||_{H_0^1(\Omega)} \to \infty$ *as* $n \to \infty$ *and*

$$\int_\Omega A\nabla u_n \cdot \nabla u_n + \int_\Omega u_n\mathbf{v} \cdot \nabla u_n = 0 \quad \text{for all } n \geq 1,$$

which means that the operator in (1.3) or (1.4) is not coercive in the sense of Leray-Lions (see [53]).

Remark 1.4 *When* A *satisfies (1.9),* A^T *also satisfies (1.9); thus, in (1.4), we could replace* A^T *by* A*. We have written (1.4) with* A^T *so that the duality between (1.3) and (1.4) clearly appears.*

1.2 Existence and Uniqueness Results

1.2.1 The main result

Theorem 1.1 *Under Hypotheses (1.8)—(1.14), there exists a unique solution* u *to (1.3) and a unique solution* v *to (1.4). Moreover, if* $r > N$ *and* $\Lambda \geq 0$ *are such that* $\mathbf{v} \in B(N_*, \chi) + B(r, \Lambda)$*, with* χ *satisfying (1.15), and if* Λ_L *is an upper bound of* $||L||_{(H^1_{\Gamma_d}(\Omega))'}$*, there exists* C *only depending on* $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$ *such that* $||u||_{H^1(\Omega)} \leq C$ *and* $||v||_{H^1(\Omega)} \leq C$*.*

Remark 1.5 *For all* $\mathbf{v} \in (L^{N_*}(\Omega))^N$ *and all* $\eta > 0$*, there exists* $\Lambda > 0$ *such that* $\mathbf{v} \in B(N_*, \eta) + B(\infty, \Lambda)$*; however, this* Λ *does not only depend on the norm of* $\mathbf{v}$ *in* $(L^{N_*}(\Omega))^N$*. On the other hand, if* $\mathbf{v}$ *is in a compact subset* K *of* $(L^{N_*}(\Omega))^N$*, for example, we can choose* Λ *only depending on* K *and* η*.*

Remark 1.6 *In the pure Dirichlet case* $(\Gamma_f = \emptyset)$*, we do not need, in this theorem, the Lipschitz continuity hypothesis on the boundary of* Ω*.*

Proof of Theorem 1.1
The proof is made in several steps. The main tool to obtain existence and estimates on the solutions of (1.3) and (1.4) is the Leray-Schauder Topological Degree (see [26]).
The first three steps are devoted to prove an existence result for (1.3). This existence result is then used in the fourth and fifth steps to prove an *a priori* estimate on the solution of (1.4) that lead to an existence result for (1.4). Using the linearity of these equations and a duality argument, we prove, in the last step, the uniqueness results.
We will simultaneously obtain the existence of solutions to (1.3) and (1.4) and the estimates given in the theorem; thus, we take from now on $r > N$, $\Lambda \geq 0$ and χ satisfying (1.15), and we suppose that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $(\mathbf{v}_0, \mathbf{v}_1) \in (L^{N_*}(\Omega))^N \times (L^r(\Omega))^N$, $||\,|\mathbf{v}_0|\,||_{L^{N_*}(\Omega)} \leq \chi$ and $||\,|\mathbf{v}_1|\,||_{L^r(\Omega)} \leq \Lambda$. We will see that the bound in $H^1(\Omega)$ on the solutions we obtain only depends on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Step 1: a compact application for (1.3).

For all $\overline{u} \in H^1_{\Gamma_d}(\Omega)$, since $\overline{u}\mathbf{v} \in (L^2(\Omega))^N$ (because of the Sobolev injection $H^1(\Omega) \hookrightarrow L^{\frac{2N_*}{N_*-2}}(\Omega)$), there exists a unique $u = \mathcal{F}(\overline{u})$ solution to

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} - \int_\Omega \overline{u}\mathbf{v}\cdot\nabla\varphi\,, \\ \hfill \forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.16}$$

This defines an application $\mathcal{F} : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$.
It is quite easy to see that $\mathcal{F}$ is continuous; indeed, if $\overline{u}_n \to \overline{u}$ in $H^1_{\Gamma_d}(\Omega)$ as $n \to \infty$, then $\overline{u}_n\mathbf{v} \to \overline{u}\mathbf{v}$ in $(L^2(\Omega))^N$ so that $\mathcal{F}(\overline{u}_n) \to \mathcal{F}(\overline{u})$ in $H^1(\Omega)$.
Suppose that $(\overline{u}_n)_{n\geq 1}$ is a bounded sequence of $H^1_{\Gamma_d}(\Omega)$. There exists then $\overline{u} \in H^1_{\Gamma_d}(\Omega)$ such that, up to a subsequence, $\overline{u}_n \to \overline{u}$ a.e. on Ω and is bounded in $L^{\frac{2N_*}{N_*-2}}(\Omega)$; applying Lemma 1.1, we get $\overline{u}_n\mathbf{v} \to \overline{u}\mathbf{v}$ in $(L^2(\Omega))^N$, which implies $\mathcal{F}(\overline{u}_n) \to \mathcal{F}(\overline{u})$ in $H^1(\Omega)$. $\mathcal{F}$ is thus a compact operator.

A fixed point of $\mathcal{F}$ is a solution to (1.3). To prove, using the Leray-Schauder Topological Degree, that $\mathcal{F}$ has a fixed point, we have to find $R > 0$ such that, for all $t \in [0,1]$, there exists no solution of $u - t\mathcal{F}(u) = 0$ satisfying $||u||_{H^1(\Omega)} = R$. This is the aim of steps two and three.

Take $t \in [0,1]$ and suppose that u satisfies $u = t\mathcal{F}(u)$; we have then

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = t\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} - t\int_\Omega u\mathbf{v}\cdot\nabla\varphi\,, \\ \hfill \forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.17}$$

Notice that the equation in (1.17) can also be written as

$$\int_\Omega A\nabla u\cdot\nabla\varphi + t\int_\Omega u\mathbf{v}_0\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = t\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} - t\int_\Omega u\mathbf{v}_1\cdot\nabla\varphi. \tag{1.18}$$

Step 2: using the ideas of [10], we prove an estimate on $\ln(1+|u|)$.
Define, for $k \geq 0$, $T_k(s) = \min(k, \max(-s,k))$ and $r_k(s) = T_1(s - T_k(s))$. Since $b \geq 0$ a.e. on Ω, $\lambda \geq 0$ σ-a.e. on $\partial\Omega$ and $sr_k(s) \geq 0$ for all $s \in \mathbb{R}$, and since $\nabla(r_k(u)) = \mathbf{1}_{B_k}\nabla u$, with $\mathbf{1}_{B_k}$ the characteristic function of the set $B_k = \{x \in \Omega \mid k \leq |u| < k+1\}$, we find, by putting $\varphi = r_k(u)$ in (1.17),

$$\begin{aligned} &\alpha_A\int_\Omega|\nabla(r_k(u))|^2 + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u\,d\sigma \\ \leq\ & \int_\Omega A\nabla u\cdot\nabla(r_k(u)) + \int_\Omega bur_k(u) + \int_{\Gamma_f}\lambda ur_k(u)\,d\sigma \\ \leq\ & |\langle L, r_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u||\mathbf{v}||\nabla(r_k(u))| \\ \leq\ & ||L||_{(H^1_{\Gamma_d}(\Omega))'}||r_k(u)||_{H^1(\Omega)} + (k+1)||\,|\mathbf{v}|\,||_{L^2(B_k)}||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)}. \end{aligned}$$

But $|r_k(s)| \leq 1$ so that

$$||r_k(u)||_{H^1(\Omega)} = ||r_k(u)||_{L^2(\Omega)} + ||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)} \leq |\Omega|^{1/2} + ||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)}.$$

We obtain thus

$$\begin{aligned} &\alpha_A\int_\Omega|\nabla(r_k(u))|^2 + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u\,d\sigma \\ \leq\ & \Lambda_L|\Omega|^{1/2} + \Lambda_L||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)} + (k+1)||\,|\mathbf{v}|\,||_{L^2(B_k)}||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)} \\ \leq\ & \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{\alpha_A}{4}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + \frac{\alpha_A}{4}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + (k+1)^2\frac{||\,|\mathbf{v}|\,||^2_{L^2(B_k)}}{\alpha_A}, \end{aligned}$$

that is to say

$$\frac{\alpha_A}{2}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u\,d\sigma \le \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + (k+1)^2\frac{||\,|\mathbf{v}|\,||^2_{L^2(B_k)}}{\alpha_A}. \quad (1.19)$$

With $k=0$, since $sr_0(s)=|s|$ as soon as $|s|\ge 1$, (1.19) gives

$$\begin{aligned}
& b_0\int_E \ln(1+|u|) + \lambda_0\int_S \ln(1+|u|)\,d\sigma \\
\le\ & b_0\int_E |u| + \lambda_0\int_S |u|\,d\sigma \\
\le\ & b_0\int_{E\cap\{|u|\ge 1\}} r_0(u)u + \lambda_0\int_{S\cap\{|u|\ge 1\}} r_0(u)u\,d\sigma + b_0\int_{E\cap\{|u|\le 1\}}|u| + \lambda_0\int_{S\cap\{|u|\le 1\}}|u|\,d\sigma \\
\le\ & \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{||\,|\mathbf{v}|\,||^2_{L^2(\Omega)}}{\alpha_A} + b_0|E| + \lambda_0\sigma(S) \\
\le\ & \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{2}{\alpha_A}(||\,|\mathbf{v}_0|\,||^2_{L^2(\Omega)} + ||\,|\mathbf{v}_1|\,||^2_{L^2(\Omega)}) + b_0|E| + \lambda_0\sigma(S) \\
\le\ & \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{2}{\alpha_A}\left(|\Omega|^{1-\frac{2}{N_*}}\chi^2 + |\Omega|^{1-\frac{2}{r}}\Lambda^2\right) + b_0|E| + \lambda_0\sigma(S) = C_1 \qquad (1.20)
\end{aligned}$$

(recall that $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$ with $||\,|\mathbf{v}_0|\,||_{L^{N_*}(\Omega)}\le\chi$ and $||\,|\mathbf{v}_1|\,||_{L^r(\Omega)}\le\Lambda$), with C_1 only depending on $(N_*,\mathcal{B},\chi,r,\Lambda,\Lambda_L)$.
Since $(B_k)_{k\in\mathbb{N}}$ is a partition of Ω, and since $|u|\ge k$ on B_k, we find, using once again (1.19),

$$\begin{aligned}
||\,|\nabla(\ln(1+|u|))|\,||^2_{L^2(\Omega)} &= \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} \\
&= \sum_{k=0}^\infty \int_{B_k}\frac{|\nabla u|^2}{(1+|u|)^2} \\
&\le \sum_{k=0}^\infty \int_\Omega \frac{|\nabla(r_k(u))|^2}{(1+k)^2} \\
&\le \frac{2}{\alpha_A}\left(\Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A}\right)\sum_{k=0}^\infty\frac{1}{(1+k)^2} + \frac{2}{\alpha_A^2}\sum_{k=0}^\infty\int_{B_k}|\mathbf{v}|^2 \\
&\le \frac{2}{\alpha_A}\frac{\pi^2}{6}\left(\Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A}\right) + \frac{2||\,|\mathbf{v}_0| + |\mathbf{v}_1|\,||^2_{L^2(\Omega)}}{\alpha_A^2} \\
&\le \frac{2}{\alpha_A}\frac{\pi^2}{6}\left(\Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A}\right) + \frac{4|\Omega|^{1-\frac{2}{N_*}}\chi^2 + 4|\Omega|^{1-\frac{2}{r}}\Lambda^2}{\alpha_A^2} = C_2 \qquad (1.21)
\end{aligned}$$

where C_2 only depends on $(N_*,\mathcal{B},\chi,r,\Lambda,\Lambda_L)$.
Taking together (1.20) and (1.21) we get, thanks to Remark 1.1,

$$||\ln(1+|u|)||^2_{L^2(\Omega)} \le ||\ln(1+|u|)||^2_{H^1(\Omega)} \le \frac{1}{\mathcal{K}(1,\mathcal{B})}(\alpha_A C_2 + C_1^2) = C_3 \qquad (1.22)$$

with C_3 only depending on $(N_*,\mathcal{B},\chi,r,\Lambda,\Lambda_L)$.

Step 3: conclusion for (1.3).
We prove now a $H^1(\Omega)$ estimate on the solution of (1.17).

Take $\varphi = S_k(u) = u - T_k(u)$ in (1.18). Since $S_k(u)u \geq (S_k(u))^2$, we have, thanks to Remark 1.2 (notice that, for all $t \in [0,1]$, $t\mathbf{v}_0 \in B(N_*,\chi)$),

$$\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega,N_*))||S_k(u)||^2_{H^1(\Omega)}\\
\leq\ & \int_\Omega A\nabla(S_k(u))\cdot\nabla(S_k(u)) + t\int_\Omega S_k(u)\mathbf{v}_0\cdot\nabla S_k(u) + \int_\Omega b(S_k(u))^2 + \int_{\Gamma_f}\lambda(S_k(u))^2\,d\sigma\\
\leq\ & \int_\Omega A\nabla u\cdot\nabla(S_k(u)) + t\int_\Omega u\mathbf{v}_0\cdot\nabla S_k(u) + \int_\Omega buS_k(u) + \int_{\Gamma_f}\lambda uS_k(u)\,d\sigma\\
&+t\int_\Omega(S_k(u)-u)\mathbf{v}_0\cdot\nabla(S_k(u))\\
\leq\ & |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u||\mathbf{v}_1||\nabla(S_k(u))| + \int_\Omega |u-S_k(u)||\mathbf{v}_0||\nabla(S_k(u))|\\
\leq\ & |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u-S_k(u)|(|\mathbf{v}_0|+|\mathbf{v}_1|)|\nabla(S_k(u))| + \int_\Omega |S_k(u)||\mathbf{v}_1||\nabla(S_k(u))|.
\end{aligned}$$

But $|u - S_k(u)| \leq k$ and $\nabla(S_k(u)) = 0$ outside $E_k = \{|u| \geq k\}$, so that

$$\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega,N_*))||S_k(u)||^2_{H^1(\Omega)}\\
\leq\ & |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + k||\,|\mathbf{v}_0|+|\mathbf{v}_1|\,||_{L^2(E_k)}||S_k(u)||_{H^1(\Omega)}\\
&+||\,|\mathbf{v}_1|\,||_{L^r(\Omega)}||S_k(u)||_{L^{\frac{2r}{r-2}}(\Omega)}||S_k(u)||_{H^1(\Omega)}\\
\leq\ & |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + k\left(|E_k|^{\frac12-\frac1{N_*}}\chi + |E_k|^{\frac12-\frac1r}\Lambda\right)||S_k(u)||_{H^1(\Omega)}\\
&+\Lambda||S_k(u)||_{L^{\frac{2r}{r-2}}(\Omega)}||S_k(u)||_{H^1(\Omega)}.
\end{aligned}\tag{1.23}$$

Since $\frac{2r}{r-2} < \frac{2N}{N-2}$, there exists $q_r > \frac{2r}{r-2}$ only depending on r and N such that $H^1(\Omega) \hookrightarrow L^{q_r}(\Omega)$; we have thus, by denoting C_4 the norm of this injection (C_4 only depends on (Ω, r) — a dependence on Ω takes into account a dependence on N) and by noticing that $S_k(u) = 0$ outside E_k,

$$||S_k(u)||_{L^{\frac{2r}{r-2}}(\Omega)} \leq |E_k|^{\frac{r-2}{2r}-\frac1{q_r}}||S_k(u)||_{L^{q_r}(\Omega)} \leq C_4|E_k|^{\frac{r-2}{2r}-\frac1{q_r}}||S_k(u)||_{H^1(\Omega)},$$

which gives, in (1.23),

$$\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega,N_*))||S_k(u)||^2_{H^1(\Omega)}\\
\leq\ & |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}| + k\left(|E_k|^{\frac12-\frac1{N_*}}\chi + |E_k|^{\frac12-\frac1r}\Lambda\right)||S_k(u)||_{H^1(\Omega)}\\
&+C_4\Lambda|E_k|^{\frac{r-2}{2r}-\frac1{q_r}}||S_k(u)||^2_{H^1(\Omega)}
\end{aligned}\tag{1.24}$$

By the Tchebycheff inequality and (1.22), we have

$$|E_k| = |\{\ln(1+|u|)^2 \geq \ln(1+k)^2\}| \leq \frac{1}{(\ln(1+k))^2}||\ln(1+|u|)||^2_{L^2(\Omega)} \leq \frac{C_3}{(\ln(1+k))^2}.$$

Since $\frac{r-2}{2r} - \frac1{q_r} > 0$, there exists thus k_0 only depending on $(C_3, C_4, \Lambda, r, q_r, \mathcal{K}(2,\mathcal{B}), \chi, C_S(\Omega,N_*))$, i.e. only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$, such that, for all $k \geq k_0$, $C_4\Lambda|E_k|^{\frac{r-2}{2r}-\frac1{q_r}} \leq \frac{\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*)}{2}$. We deduce then form (1.24) that, for all $k \geq k_0$,

$$\begin{aligned}
&||S_k(u)||_{H^1(\Omega)}\\
\leq\ & \frac{2}{\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*)}\left(\frac{|\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}} + k\left(|E_k|^{\frac12-\frac1{N_*}}\chi + |E_k|^{\frac12-\frac1r}\Lambda\right)\right)
\end{aligned}\tag{1.25}$$

(we have not simplified so far, because this inequality will be useful in the proof of Proposition 1.2).
Taking $k = k_0$, and since $E_k \subset \Omega$, we get

$$||S_{k_0}(u)||_{H^1(\Omega)} \leq \frac{2}{\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*)} \left(\Lambda_L + k_0 \left(|\Omega|^{\frac{1}{2}-\frac{1}{N_*}}\chi + |\Omega|^{\frac{1}{2}-\frac{1}{r}}\Lambda\right)\right) = C_5 \tag{1.26}$$

with C_5 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.
Take now $\varphi = T_{k_0}(u)$ in (1.17). Since $uT_{k_0}(u) \geq (T_{k_0}(u))^2$ and $\nabla(T_{k_0}(u)) = \mathbf{1}_{\{|u|\leq k_0\}}\nabla u$, we have, by Remark 1.1,

$$\begin{aligned}
\mathcal{K}(2,\mathcal{B})||T_{k_0}(u)||^2_{H^1(\Omega)} &\leq \Lambda_L||T_{k_0}(u)||_{H^1(\Omega)} + \int_\Omega |u||\mathbf{v}||\nabla(T_{k_0}(u))| \\
&\leq \Lambda_L||T_{k_0}(u)||_{H^1(\Omega)} + k_0||\,|\mathbf{v}_0| + |\mathbf{v}_1|\,||_{L^2(\Omega)}||T_{k_0}(u))||_{H^1(\Omega)},
\end{aligned}$$

that is to say

$$||T_{k_0}(u)||_{H^1(\Omega)} \leq \frac{\Lambda_L + k_0\left(|\Omega|^{\frac{1}{2}-\frac{1}{N_*}}\chi + |\Omega|^{\frac{1}{2}-\frac{1}{r}}\Lambda\right)}{\mathcal{K}(2,\mathcal{B})} = C_6$$

with C_6 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$ (recall that k_0 only depends on these datas).
Since $u = T_{k_0}(u) + S_{k_0}(u)$, we deduce from this last inegality and (1.26) that

$$||u||_{H^1(\Omega)} \leq C_5 + C_6 = C_7,$$

with C_7 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Notice that we have just proven the estimate on the solution of (1.3) given in the theorem: if u is a solution of (1.3), then it is a solution of (1.17) with $t = 1$ and we have thus $||u||_{H^1(\Omega)} \leq C_7$.

Take now $R = C_7 + 1$. For all $t \in [0, 1]$ and all $u \in H^1_{\Gamma_d}(\Omega)$ solution of $u - t\mathcal{F}(u) = 0$, we have $||u||_{H^1(\Omega)} \neq R$; since $\mathcal{F}$ is a compact operator, the Leray Schauder Topological Degree allows us to see that $\mathcal{F}$ has a fixed point, that is to say a solution u of (1.3).

Step 4: a compact application for (1.4).
Let $\overline{v} \in H^1_{\Gamma_d}(\Omega)$; we have $\mathbf{v} \cdot \nabla\overline{v} \in L^{\frac{2N_*}{N_*+2}}(\Omega) \subset (H^1(\Omega))'$; there exists thus a unique solution $v = \mathcal{G}(\overline{v})$ to

$$\left\{\begin{array}{l}
v \in H^1_{\Gamma_d}(\Omega)\,, \\
\displaystyle\int A^T\nabla v \cdot \nabla\varphi + \int_\Omega bv\varphi + \int_{\Gamma_f} \lambda v\varphi\, d\sigma = \langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} - \int_\Omega \varphi\mathbf{v}\cdot\nabla\overline{v}\,, \\
\qquad\qquad\qquad \forall\varphi \in H^1_{\Gamma_d}(\Omega).
\end{array}\right. \tag{1.27}$$

This defines an application $\mathcal{G} : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$. It is quite easy to see that $\mathcal{G}$ is continuous; indeed, if $\overline{v}_n \to \overline{v}$ in $H^1_{\Gamma_d}(\Omega)$, then $\mathbf{v} \cdot \nabla\overline{v}_n \to \mathbf{v} \cdot \nabla\overline{v}$ in $(H^1(\Omega))'$, which implies $\mathcal{G}(\overline{v}_n) \to \mathcal{G}(\overline{v})$ in $H^1(\Omega)$.
We will now prove that $\mathcal{G}$ is a compact operator. Suppose that $(\overline{v}_n)_{n\geq1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$; then $(\mathbf{v} \cdot \nabla\overline{v}_n)_{n\geq1}$ is bounded in $(H^1(\Omega))'$ so that, using $\varphi = \mathcal{G}(\overline{v}_n) = v_n$ in the equation satisfied by v_n, we get

$$\mathcal{K}(2,\mathcal{B})||v_n||^2_{H^1(\Omega)} \leq (\Lambda_L + ||\mathbf{v} \cdot \nabla\overline{v}_n||_{(H^1(\Omega))'})||v_n||_{H^1(\Omega)},$$

which implies that $(v_n)_{n\geq1}$ is bounded in $H^1(\Omega)$.
Up to a subsequence, we can thus suppose that $(v_n)_{n\geq1}$ converges a.e. on Ω and is bounded in $L^{\frac{2N_*}{N_*-2}}(\Omega)$.
Let $n \geq 1$, $m \geq 1$; substract the equation satisfied by v_m to the equation satisfied by v_n and use $\varphi = v_n - v_m$ as a test function; by denoting M a bound on $(||\overline{v}_n||_{H^1(\Omega)})_{n\geq1}$, this gives

$$\begin{aligned}
\mathcal{K}(2,\mathcal{B})||v_n - v_m||^2_{H^1(\Omega)} &\leq \left|\int_\Omega (v_n - v_m)\mathbf{v} \cdot (\nabla\overline{v}_m - \nabla\overline{v}_n)\right| \\
&\leq 2M||\,|v_n\mathbf{v} - v_m\mathbf{v}|\,||_{L^2(\Omega)}.
\end{aligned} \tag{1.28}$$

But, since $\mathbf{v} \in (L^{N_*}(\Omega))^N$ and $(v_n)_{n\geq 1}$ is a bounded sequence of $L^{\frac{2N_*}{N_*-2}}(\Omega)$ which converges a.e. on Ω, Lemma 1.1 tells us that $(v_n\mathbf{v})_{n\geq 1}$ converges in $(L^2(\Omega))^N$, and is thus a Cauchy sequence in this space. We deduce from (1.28) that $(v_n)_{n\geq 1}$ is a Cauchy sequence in $H^1_{\Gamma_d}(\Omega)$ and converges in this space.
Since $\mathcal{G}$ is a compact operator, to prove that it has a fixed point, we just have to find $R>0$ such that, for all $t\in[0,1]$, there exists no solution of $v - t\mathcal{G}(v) = 0$ satisfying $||v||_{H^1(\Omega)} = R$.

Step 5: estimate on the solutions of $v - t\mathcal{G}(v) = 0$.
Let $t\in[0,1]$ and suppose that $v\in H^1_{\Gamma_d}(\Omega)$ satifies $v = t\mathcal{G}(v)$. We have then

$$\left\{\begin{array}{l} v\in H^1_{\Gamma_d}(\Omega)\,,\\ \displaystyle\int A^T\nabla v\cdot\nabla\varphi + t\int_\Omega \varphi\mathbf{v}\cdot\nabla v + \int_\Omega bv\varphi + \int_{\Gamma_f}\lambda v\varphi\,d\sigma = \langle tL,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\\ \qquad\qquad\qquad\qquad\qquad\forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.29}$$

Since, for all $t\in[0,1]$, $t\mathbf{v}\in B(N_*,\chi)+B(r,\Lambda)$, there exists, by the result of step 3, C_8 only depending on $(N_*,\mathcal{B},\chi,r,\Lambda)$ such that, for all $\theta\in(H^1_{\Gamma_d}(\Omega))'$ satisfying $||\theta||_{(H^1_{\Gamma_d}(\Omega))'}\leq 1$, we can find a solution u to

$$\left\{\begin{array}{l} u\in H^1_{\Gamma_d}(\Omega)\,,\ ||u||_{H^1(\Omega)}\leq C_8\,,\\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + t\int_\Omega u\mathbf{v}\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = \langle\theta,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\\ \qquad\qquad\qquad\qquad\qquad\forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.30}$$

By taking $\varphi = v$ in the equation satisfied by u and $\varphi = u$ in the equation satisfied by v, we get

$$\langle\theta,v\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} = \langle tL,u\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} \leq \Lambda_L C_8.$$

Since this inequality is satisfied for all $\theta\in(H^1_{\Gamma_d}(\Omega))'$ such that $||\theta||_{(H^1_{\Gamma_d}(\Omega))'}\leq 1$, we deduce that $||v||_{H^1(\Omega)}\leq\Lambda_L C_8$.

Notice that this gives the estimate of the theorem; indeed, if v is a solution of (1.4), then it is a solution of (1.29) with $t=1$ so that $||v||_{H^1(\Omega)}\leq\Lambda_L C_8$.

Take now $R = \Lambda_L C_8 + 1$. We have juste proven that, for any $t\in[0,1]$, any solution v to $v - t\mathcal{G}(v) = 0$ satisfies $||v||_{H^1(\Omega)} < R$; thus, by the Leray-Schauder Topological Degree, $\mathcal{G}$ has a fixed point, that is to say a solution of (1.4).

Step 6: uniqueness.
Since (1.3) is a linear problem, it is sufficient to prove that the only solution to (1.3) with $L=0$ is the null function. Let u be a solution to (1.3) with $L=0$; let v be a solution of (1.4) with $L=\mathrm{sgn}(u)\in(H^1_{\Gamma_d}(\Omega))'$ (the existence of a solution to this problem is ensured by step 5); by putting $\varphi = v$ in the equation satisfied by u and $\varphi = u$ in the equation satisfied by v, we get $\int_\Omega |u| = 0$, that is to say $u = 0$.
A similar reasoning gives the uniqueness of the solution to (1.4). ■

1.2.2 Existence and uniqueness in a nonlinear case

To prove the existence of a solution to (1.3), we have not really used the linearity with respect to u of the divergence part $\mathrm{div}(u\mathbf{v})$ (indeed, the tool used in the preceding proof — the Leray-Schauder Topological Degree — is a nonlinear tool). With exactly the same reasoning as in the first three steps of the proof of Theorem 1.1, we can prove the following result.

Theorem 1.2 *Under Hypotheses (1.8)—(1.11), (1.13), (1.14), if $\Phi:\Omega\times\mathbb{R}\to\mathbb{R}^N$ is a Caratheodory function satisfying*

$$\exists g\in L^{N_*}(\Omega) \text{ such that } |\Phi(x,s)|\leq g(x)(1+|s|) \text{ for a.e. } x\in\Omega, \text{ for all } s\in\mathbb{R}, \tag{1.31}$$

and if Λ_L is an upper bound of $||L||_{(H^1_{\Gamma_d}(\Omega))'}$, there exists a solution to

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + \int_\Omega \Phi(\cdot,u)\cdot\nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f}\lambda u\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,, \\ \qquad\qquad\qquad\qquad \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{array}\right. \tag{1.32}$$

such that $||u||_{H^1(\Omega)} \le C$, with C only depending([1]) on $(N_*,\mathcal{B},g,\Lambda_L)$.

Remark 1.7 *Since the proof of the existence of a solution to (1.4) strongly used the linearity of the equation (the* a priori *estimate on the solution to (1.4) comes from a duality argument), we cannot state, with this reasoning, an existence result for a nonlinear problem coming from Equation (1.4) (conversely to what we have done in Theorem 1.2 for Equation (1.3)).*

Adding a Lipschitz continuity hypothesis on Φ, it is also quite easy to obtain an uniqueness result for (1.32).

Proposition 1.1 *Under the hypotheses of Theorem 1.2, if Φ satisfies moreover*

$$\exists C>0\,,\ \exists h\in L^{N_*}(\Omega) \text{ such that } |\Phi(x,s)-\Phi(x,t)| \le C\left(h(x)+|s|^{\frac{2}{N_*-2}}+|t|^{\frac{2}{N_*-2}}\right)|s-t|$$
$$\text{for a.e. } x\in\Omega,\text{ for all } (s,t)\in\mathbb{R}^2,$$

then the solution to (1.32) is unique.

Proof of Proposition 1.1
Take two solutions u and $\overline{u}$ to (1.32) and define

$$\mathbf{v}(x) = \left\{\begin{array}{ll} \frac{\Phi(x,u(x))-\Phi(x,\overline{u}(x))}{u(x)-\overline{u}(x)} & \text{when } u(x)\neq\overline{u}(x), \\ 0 & \text{when } u(x)=\overline{u}(x). \end{array}\right.$$

Thanks to the Lipschitz continuity hypothesis on Φ, and since $(u,\overline{u}) \in H^1(\Omega) \subset L^{\frac{2N_*}{N_*-2}}(\Omega)$, we have $\mathbf{v} \in (L^{N_*}(\Omega))^N$; substracting the equation satisfied by $\overline{u}$ to the equation satisfied by u, we see that $w = u - \overline{u}$ satisfies (1.3) with $L = 0$. Since the solution to (1.3) is unique, this gives $w = 0$, that is to say $u = \overline{u}$. ■

Thanks to the existence, uniqueness and estimates results of Theorem 1.1, we could also, as it is classical in the coercive case, prove existence results for some other non-linear equations built from (1.3) and (1.4).

1.3 Regularity Results

In the coercive case, when the right-hand side satisfies([2])

$$\exists p > N \text{ such that } L \in (W^{1,p'}_{\Gamma_d}(\Omega))', \tag{1.33}$$

(and under additional properties on $\mathbf{v}$, b, λ and Γ_d) we already know that the solutions to (1.3) and (1.4) are Hölder continuous (see [70] in the pure Dirichlet case, and [29] for other boundary conditions and a convection term in conservative form). We will see that this property is still true in the non-coercive case.

[1] As in Theorem 1.1, C does not depend on g only through $||g||_{L^{N_*}(\Omega)}$, but this dependance could be precised by cutting g into two parts — one small in $L^{N_*}(\Omega)$, the other in $L^r(\Omega)$ for a $r > N$.

[2] There is a little abuse of notation here. By writing "the right-hand side satisfies (1.33)", we mean that we solve (1.3) or (1.4) with $L = \widetilde{L}_{|H^1_{\Gamma_d}(\Omega)}$ for a $\widetilde{L} \in (W^{1,p'}_{\Gamma_d}(\Omega))'$; in the following, we make this abuse of notation by confusing L with $\widetilde{L}$. Under Hypothesis (1.42), this is not an abuse since we can then prove that $H^1_{\Gamma_d}(\Omega)$ is densely imbedded in $W^{1,p'}_{\Gamma_d}(\Omega)$.

1.3.1 L^∞ bound

In the proof of Theorem 1.1, the role played by the convection term in conservative form $\mathrm{div}(\mathbf{v}u)$ is quite different from the role played by the convection term in non conservative form $\mathbf{v}\cdot\nabla u$ (the technique used to obtain estimates on the solution to (1.3) does not work to obtain estimates on the solution to (1.4)). As it is shown in [70], when considering regularity results, the difference between (1.3) and (1.4) is even more stronger; when the convection term is in non conservative form, Hypothesis (1.12) is enough, but when it is in conservative form, $\mathbf{v}$ must be (at least for technical reasons) slightly more integrable than what is strictly necessary to obtain the existence result.
This is why we will have to consider, when dealing with (1.3), the following hypothesis:

$$\exists r > N \text{ such that } \mathbf{v}\in (L^r(\Omega))^N. \tag{1.34}$$

When $\mathbf{v}$ satisfies this hypothesis, we denote by $\Lambda_{\mathbf{v}}$ an upper bound of $||\,|\mathbf{v}|\,||_{L^r(\Omega)}$.

The first regularity results deal with essential bounds on the solutions to (1.3) and (1.4)

Proposition 1.2 *Under Hypotheses (1.8)—(1.11), (1.14), (1.33) and (1.34), the solution u to (1.3) is in $L^\infty(\Omega)$. Moreover, if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, there exists C only depending on $(N_*,\mathcal{B},r,\Lambda_{\mathbf{v}},p,\Lambda_L)$ such that $||u||_{L^\infty(\Omega)}\le C$.*

Remark 1.8 *In dimension $N=2$, (1.12) implies (1.34) (i.e. there is no additional hypothesis on $\mathbf{v}$ with respect to the hypotheses of Theorem 1.1).*

Proposition 1.3 *Under Hypotheses (1.8)—(1.12), (1.14) and (1.33), the solution v to (1.4) is in $L^\infty(\Omega)$. Moreover, if $r>N$ and $\Lambda\ge 0$ are such that $\mathbf{v}\in B(N_*,\chi)+B(r,\Lambda)$, with χ satisfying (1.15), and if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, then there exists C only depending on $(N_*,\mathcal{B},\chi,r,\Lambda,p,\Lambda_L)$ such that $||v||_{L^\infty(\Omega)}\le C$.*

Proof of Proposition 1.2
The solution u of (1.3) is also a solution of (1.17) with $t=1$. Since $\mathbf{v}\in B(N_*,0)+B(r,\Lambda_{\mathbf{v}})$, the reasoning in the proof of Theorem 1.1 that has lead to (1.25) can be applied to u with $\chi=0$; there exists thus $k_0>0$ only depending on $(N_*,\mathcal{B},r,\Lambda_{\mathbf{v}},p,\Lambda_L)$ (notice that $|\Omega|^{\frac12-\frac1p}\Lambda_L$ is an upper bound of $||L||_{(H^1_{\Gamma_d}(\Omega))'}$) such that, for all $k\ge k_0$,

$$||S_k(u)||_{H^1(\Omega)}\le \frac{2}{\mathcal{K}(2,\mathcal{B})}\left(\frac{|\langle L,S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}}+\Lambda_{\mathbf{v}}k|E_k|^{\frac12-\frac1r}\right), \tag{1.35}$$

with $S_k(u)=u-T_k(u)=u-\min(k,\max(u,-k))$ and $E_k=\{x\in\Omega\mid |u(x)|\ge k\}$.
Since $S_k(u)=0$ outside E_k and $p'<2$, we have $||S_k(u)||_{W^{1,p'}(\Omega)}\le |E_k|^{\frac{1}{p'}-\frac12}||S_k(u)||_{H^1(\Omega)}$, so that

$$\frac{|\langle L,S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}}\le \Lambda_L\frac{||S_k(u)||_{W^{1,p'}(\Omega)}}{||S_k(u)||_{H^1(\Omega)}}\le \Lambda_L|E_k|^{\frac12-\frac1p}. \tag{1.36}$$

Let $h>k\ge k_0$. Since $|S_k(u)|\ge (h-k)$ on E_h, and thanks to the Sobolev injection $W^{1,1}(\Omega)\hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, there exists C_1 only depending on Ω such that

$$(h-k)|E_h|^{\frac{N-1}{N}}\le ||S_k(u)||_{L^{\frac{N}{N-1}}(\Omega)}\le C_1||S_k(u)||_{W^{1,1}(\Omega)}\le C_1|E_k|^{\frac12}||S_k(u)||_{H^1(\Omega)}. \tag{1.37}$$

(1.36) and (1.37) used in (1.35) give then, for all $h>k\ge k_0$,

$$|E_h|^{\frac{N-1}{N}}\le \frac{2C_1|E_k|^{\frac12}}{\mathcal{K}(2,\mathcal{B})(h-k)}\left(\Lambda_L|E_k|^{\frac12-\frac1p}+\Lambda_{\mathbf{v}}k|E_k|^{\frac12-\frac1r}\right)\le \frac{C_2}{h-k}(|E_k|^{1-\frac1p}+k|E_k|^{1-\frac1r}),$$

with C_2 only depending on $(\mathcal{B},\Lambda_{\mathbf{v}},\Lambda_L)$. Since, for $q\in\{r,p\}$, $|E_k|^{1-\frac{1}{q}}\leq|\Omega|^{\frac{1}{\inf(r,p)}-\frac{1}{q}}|E_k|^{1-\frac{1}{\inf(r,p)}}$, there exists thus C_3 only depending on $(\mathcal{B},r,\Lambda_{\mathbf{v}},p,\Lambda_L)$ such that, for all $h>k\geq k_0$,

$$|E_h|\leq\frac{C_3^\beta(1+k)^\beta}{(h-k)^\beta}|E_k|^\gamma$$

with $\beta=\frac{N}{N-1}>0$ and $\gamma=\beta(1-\frac{1}{\inf(r,p)})>1$ (recall that $r>N$ and $p>N$).
For all $h>k\geq 0$, we have then

$$|E_{h+k_0}|\leq\frac{C_3^\beta(1+k_0)^\beta(1+k)^\beta}{(h-k)^\beta}|E_{k+k_0}|^\gamma$$

(because $(1+k+k_0)\leq(1+k_0)(1+k)$), and Lemma 1.2 (a generalization of a classical lemma by Stampacchia) applied to $F(k)=|E_{k+k_0}|$ gives thus H only depending on $(C_3,k_0,\beta,\gamma,\Omega)$ (notice that $F(0)=|E_{k_0}|\leq|\Omega|$), i.e. on $(N_*,\mathcal{B},r,\Lambda_{\mathbf{v}},p,\Lambda_L)$, such that $|E_{H+k_0}|=0$, that is to say $|u|\leq H+k_0$ a.e. on Ω. ■

Proof of Proposition 1.3
The idea is identical to that of the preceding proof. We write $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$ with $\mathbf{v}_0\in B(N_*,\chi)$ and $\mathbf{v}_1\in B(r,\Lambda)$.
Since $vS_k(v)\geq(S_k(v))^2$ and $\nabla v=\nabla(S_k(v))$ a.e. on the set $\{S_k(v)\neq 0\}$, using $S_k(v)$ as a test function in (1.4), we get

$$\begin{aligned}&(\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*))||S_k(v)||^2_{H^1(\Omega)}\\ &\leq\ \int_\Omega A^T\nabla v\cdot\nabla(S_k(v))+\int_\Omega S_k(v)\mathbf{v}_0\cdot\nabla v+\int_\Omega bvS_k(v)+\int_{\Gamma_f}\lambda vS_k(v)\,d\sigma\\ &\leq\ |\langle L,S_k(v)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}|+\Lambda||S_k(v)||_{L^{\frac{2r}{r-2}}(\Omega)}||S_k(v)||_{H^1(\Omega)}.\end{aligned}\tag{1.38}$$

Since $\frac{2r}{r-2}<\frac{2N}{N-2}$, there exists $q_r>\frac{2r}{r-2}$ only depending on r and N such that $H^1(\Omega)\hookrightarrow L^{q_r}(\Omega)$; by denoting C_1 the norm of this injection (which only depends on r and Ω) and $E_k=\{x\in\Omega\mid|v(x)|\geq k\}$, we have then

$$||S_k(v)||_{L^{\frac{2r}{r-2}}(\Omega)}\leq C_1|E_k|^{\frac{r-2}{2r}-\frac{1}{q_r}}||S_k(v)||_{H^1(\Omega)}.$$

But, since $|\Omega|^{\frac{1}{2}-\frac{1}{p}}\Lambda_L$ is an upper bound of $||L||_{(H^1_{\Gamma_d}(\Omega))'}$, there exists, by Theorem 1.1, C_2 only depending on $(N_*,\mathcal{B},\chi,r,\Lambda,p,\Lambda_L)$ such that $||v||_{H^1(\Omega)}\leq C_2$, which implies $|E_k|\leq C_2^2/k^2$ for all $k\geq 0$. We can thus find $k_0>0$ only depending on $(N_*,\mathcal{B},\chi,r,\Lambda,p,\Lambda_L)$ such that, for all $k\geq k_0$, $C_1\Lambda|E_k|^{\frac{r-2}{2r}-\frac{1}{q_r}}\leq(\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*))/2$.
We have then, thanks to (1.38) and when $k\geq k_0$,

$$||S_k(v)||_{H^1(\Omega)}\leq\frac{2}{\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*)}\times\frac{|\langle L,S_k(v)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}|}{||S_k(v)||_{H^1(\Omega)}}.$$

This inequality is similar to (1.35) (it is even simpler), and we can then conclude as in the proof of Proposition 1.2. ■

1.3.2 Hölder continuity

To state the Hölder continuity results, we need (at least for technical reasons) stronger integrability hypotheses on b and λ; we replace thus (1.10) and (1.11) by

$$\begin{array}{c}\exists\overline{r}>N\text{ such that }b\in L^{\frac{\overline{r}}{2}}(\Omega)\,,\ \lambda\in L^{\overline{r}-1}(\partial\Omega)\\ \text{and }b\geq 0\text{ a.e. on }\Omega,\ \lambda\geq 0\ \sigma\text{-a.e. on }\partial\Omega.\end{array}\tag{1.39}$$

We denote by Λ_b an upper bound of $||b||_{L^{\frac{\overline{r}}{2}}(\Omega)}$ and by Λ_λ an upper bound of $||\lambda||_{L^{\overline{r}-1}(\partial\Omega)}$.

We also need an hypothesis on Γ_d and Γ_f; these sets must be "well-distributed" on $\partial\Omega$. We introduce thus two kinds of mappings of $\partial\Omega$:

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B := \{x \in \mathbb{R}^N \mid |x| < 1\} \text{ is a Lipschitz continuous} \\ \text{homeomorphism with a Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_+ := \{x \in B \mid x_N > 0\}, \\ h(O \cap \partial\Omega) = \{x \in \partial B_+ \mid x_N = 0\}, \end{array} \tag{1.40}$$

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B \text{ is a Lipschitz continuous homeomorphism} \\ \text{with Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_{++} := \{x \in B \mid x_N > 0\,,\ x_{N-1} > 0\}, \\ h(O \cap \Gamma_f) = \{x \in \partial B_{++} \mid x_{N-1} = 0\}, \\ h(O \cap \Gamma_d) = \{x \in \partial B_{++} \mid x_N = 0\}, \end{array} \tag{1.41}$$

and we suppose that

$$\begin{array}{c} \text{There exists a finite number of } (O_i, h_i)_{i\in[1,m]} \text{ such that} \\ \partial\Omega \subset \cup_{i=1}^m O_i \text{ and, for all } i \in [1,m],\ (O_i,h_i) \text{ is of one of the following types:} \\ \left| \begin{array}{ll} (D) & O_i \cap \partial\Omega = O_i \cap \Gamma_d \text{ and } (O_i,h_i) \text{ satisfies (1.40)} \\ (F) & O_i \cap \partial\Omega = O_i \cap \Gamma_f \text{ and } (O_i,h_i) \text{ satisfies (1.40)} \\ (DF) & (O_i,h_i) \text{ satisfies (1.41).} \end{array} \right. \end{array} \tag{1.42}$$

Corollary 1.1 *Under Hypotheses (1.8), (1.9), (1.14), (1.33), (1.34), (1.39) and (1.42), the solution u to (1.3) is Hölder continuous on Ω. More precisely, if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, there exists $\kappa > 0$ only depending on $(\Omega, \alpha_A, \Lambda_A, \overline{r}, r, p)$ and C only depending on $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, \Lambda_b, \Lambda_\lambda, r, \Lambda_{\mathbf{v}}, p, \Lambda_L)$ such that u satisfies $||u||_{\mathcal{C}^{0,\kappa}(\Omega)} \le C$.*

Remark 1.9 *$\mathcal{C}^{0,\kappa}(\Omega)$ denotes the space of κ-Hölder continuous functions, endowed with its usual norm.*

Remark 1.10 *Provided that the function g in (1.31) is in $L^r(\Omega)$ for a $r > N$, the results of Proposition 1.2 and Corollary 1.1 are also true for any solution of (1.32).*

Corollary 1.2 *Under Hypotheses (1.8), (1.9), (1.12), (1.14), (1.33), (1.39) and (1.42), the solution v to (1.4) is Hölder continuous on Ω. More precisely, if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, $r > N$ and $\Lambda \ge 0$, there exists $\eta > 0$ only depending on (N_*, Ω, α_A), $\kappa > 0$ only depending on $(N_*, \Omega, \alpha_A, \Lambda_A, \overline{r}, r, \Lambda, p)$ and C only depending on $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, \Lambda_b, \Lambda_\lambda, r, \Lambda, p, \Lambda_L)$ such that, when $\mathbf{v} \in B(N_*, \eta) + B(r, \Lambda)$, v satisfies $||v||_{\mathcal{C}^{0,\kappa}(\Omega)} \le C$.*

Proof of Corollaries 1.1 and 1.2
Thanks to Proposition 1.2 (respectively 1.3), the solution u to (1.3) (respectively v to (1.4)) is essentially bounded on Ω, and we have an estimate on its L^∞ norm. Thus, thanks to (1.34) and (1.39) (respectively (1.39)), the terms $\varphi \to \int_\Omega u\varphi$, $\varphi \to \int u\mathbf{v} \cdot \nabla\varphi$, $\varphi \to \int_\Omega bu\varphi$ and $\varphi \to \int_{\Gamma_f} \lambda u\varphi \, d\sigma$ (respectively $\varphi \to \int_\Omega bv\varphi$ and $\varphi \to \int_{\Gamma_f} \lambda v\varphi \, d\sigma$) are in $(W^{1,\inf(r,\overline{r})'}_{\Gamma_d}(\Omega))'$ (respectively $(W^{1,\overline{r}'}_{\Gamma_d}(\Omega))'$), and we have a bound on their norms in this space.
By putting these terms in the right-hand side, we notice then that u satisfies

$$\left\{ \begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla\varphi + \int_\Omega u\varphi = \langle \widetilde{L}, \varphi \rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{1.43}$$

with $\widetilde{L} \in (W^{1,l'}_{\Gamma_d}(\Omega))'$ for $l = \inf(\overline{r}, r, p) > N$. The results of [70] (in the pure Dirichlet case) or of [29] (for other boundary conditions) give then the Hölder continuity of u, as well as the estimates on the Hölder space to which u belongs and on its norm in this space.
For v, we get an equation of the kind

$$\left\{ \begin{array}{l} v \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T \nabla v \cdot \nabla \varphi + \int_\Omega \varphi \mathbf{v} \cdot \nabla v = \langle \widetilde{L}, \varphi \rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{1.44}$$

with $\widetilde{L} \in (W^{1,l'}_{\Gamma_d}(\Omega))'$ for $l = \inf(\overline{r}, p) > N$. In the pure Dirichlet case, the results of [70] give then the Hölder continuity of v; for other boundary conditions, a slight modification of the methods in [29]([3]) gives the Hölder continuity (as well as the estimates) of v. ■

1.4 The Duality Method for Non Regular Right Hand Sides

As it is shown in [70], the regularity results of Corollaries 1.1 and 1.2 can be transformed into existence and uniqueness results for weaker right-hand sides.

We suppose here Hypotheses (1.8), (1.9), (1.12), (1.14), (1.39) and (1.42).
Define $\mathcal{T} : (H^1_{\Gamma_d}(\Omega))' \to H^1_{\Gamma_d}(\Omega)$ such that, for all $L \in (H^1_{\Gamma_d}(\Omega))'$, $\mathcal{T}L$ is the unique solution to (1.4). Thanks to Theorem 1.1, $\mathcal{T}$ is well defined, linear and continuous.
Let $p \in]N, \infty[$. Thanks to Corollary 1.2,

$$\mathcal{T}_p = \mathcal{T}_{|(W^{1,p'}_{\Gamma_d}(\Omega))'} : (W^{1,p'}_{\Gamma_d}(\Omega))' \to H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$$

is well defined, linear and continuous([4]). The adjoint operator of $\mathcal{T}_p$ is a linear continuous application $\mathcal{T}_p^* : (H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))' \to W^{1,p'}_{\Gamma_d}(\Omega)$ (since $1 < p < \infty$, $W^{1,p'}_{\Gamma_d}(\Omega)$ is a reflexive space).

Let $\mathcal{M}(\overline{\Omega}) = (\mathcal{C}(\overline{\Omega}))'$ (identified, through the Riesz representation theorem, to the space of bounded measures on $\overline{\Omega}$). Since $H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is continuously and densely embedded in $\mathcal{C}(\overline{\Omega})$ and in $H^1(\Omega)$, $\mathcal{M}(\overline{\Omega})$ and $(H^1(\Omega))'$ are continuously embedded in $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$.
Thus, we can talk of $\mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$ as a subspace([5]) of $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$.

Let $\zeta \in \mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$. By definition, $f_p = \mathcal{T}_p^* \zeta$ is the unique solution to

$$\left\{ \begin{array}{l} f_p \in W^{1,p'}_{\Gamma_d}(\Omega)\,, \\ \forall L \in (W^{1,p'}_{\Gamma_d}(\Omega))'\,,\ \langle L, f_p \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \langle \zeta, \mathcal{T}_p L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \langle \zeta, \mathcal{T} L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array} \right. \tag{1.45}$$

Take now $q \in]N, p[$ and f_q the solution of (1.45) when p is replaced by q. Let $L \in (W^{1,p'}_{\Gamma_d}(\Omega))'$; since $W^{1,q'}_{\Gamma_d}(\Omega) \hookrightarrow W^{1,p'}_{\Gamma_d}(\Omega)$, $f_q \in W^{1,p'}_{\Gamma_d}(\Omega)$ and $L_{|W^{1,q'}_{\Gamma_d}(\Omega)} \in (W^{1,q'}_{\Gamma_d}(\Omega))'$, so that, by definition of f_q,

$$\langle L, f_q \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \langle L_{|W^{1,q'}_{\Gamma_d}(\Omega)}, f_q \rangle_{(W^{1,q'}_{\Gamma_d}(\Omega))', W^{1,q'}_{\Gamma_d}(\Omega)} = \langle \zeta, \mathcal{T} L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}.$$

Thus, f_q is also a solution to (1.45) and we have then $f_q = f_p$ for all $q \in]N, p[$.

[3]Voir Annexe A.
[4]$H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is endowed with the norm $||\cdot||_{H^1(\Omega)} + ||\cdot||_{\mathcal{C}(\overline{\Omega})}$.
[5]Endowed with the norm $||\zeta|| = \inf\{||\mu||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'}\,,\ (\mu, L) \in \mathcal{M}(\overline{\Omega}) \times (H^1(\Omega))'\,,\ \mu + L = \zeta\}$, this is a Banach space, and it is continuously embedded in $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$. In fact, one can show that $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))' = \mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$.

The solution to (1.45) belongs thus to $\cap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)$ and is in fact the unique solution to

$$\left\{\begin{array}{l} f \in \bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)\,, \\ \forall q < \frac{N}{N-1}\,,\ \forall L \in (W^{1,q}_{\Gamma_d}(\Omega))'\,,\ \langle L, f\rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)} = \langle \zeta, \mathcal{T}L\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{1.46}$$

The unique solution to (1.46) is called the duality solution of

$$\left\{\begin{array}{ll} -\mathrm{div}(A\nabla f) - \mathrm{div}(\mathbf{v}f) + bf = \zeta & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_d, \\ A\nabla f\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n})f = 0 & \text{on } \Gamma_f. \end{array}\right. \tag{1.47}$$

This gives a notion of solution to (1.1) when the right-hand side L is in $\mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$, for which we have thus existence and uniqueness (as well as estimates, since $\mathcal{T}_p^*$ is linear continuous — its norm is that of $\mathcal{T}_p$, which can be bounded thanks to the results of Theorem 1.1 and Corollary 1.2).
To understand why, by solving (1.46), we can say that, in a way, we have solved (1.47), we refer the reader to [29]. In particular, it is quite easy to see that, when $\zeta \in (H^1(\Omega))'$, the solution to (1.3) with $L = \zeta_{|H^1_{\Gamma_d}(\Omega)}$ is the solution to (1.46); we can also state integral formulations (one equivalent to (1.46), the other weaker than (1.46)) satisfied by the solution of (1.46) that makes it easier to see why this solution is a solution to (1.47).

Under Hypothesis (1.34), one can do the same reasoning using the regularity results on the solution to (1.3). In this case, we obtain a duality solution to

$$\left\{\begin{array}{ll} -\mathrm{div}(A^T\nabla f) + \mathbf{v}\cdot\nabla f + bf = \zeta & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_d, \\ A^T\nabla f\cdot\mathbf{n} + \lambda f = 0 & \text{on } \Gamma_f. \end{array}\right. \tag{1.48}$$

All the results on the duality solutions obtained in [29] do also apply here; in particular, we could state a stability result similar to the one of Theorem 4.1 in [29].

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1.5 Appendix: Technical Lemmas

Lemma 1.1 *Let $(p,q,r) \in [1,\infty]$ such that $q < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $g \in L^q(\Omega)$ and $(f_n)_{n\geq 1}$ is a bounded sequence of $L^p(\Omega)$ which converges a.e. on Ω to f, then $f_n g \to fg$ in $L^r(\Omega)$.*

Remark 1.11 *This results is also true when Ω is replaced by any measured space $(X, \mathcal{A}, \mu)$.*

Proof of Lemma 1.1
We have $f_n g \to fg$ a.e. on Ω. Since $r < \infty$ (because $q < \infty$) and Ω is of finite measure, thanks to the Vitali Theorem, we just have to prove the r-equi-integrability of $(f_n g)_{n\geq 1}$ to get the convergence in $L^r(\Omega)$ of this sequence.
Denote by M an upper bound of $(||f_n||_{L^p(\Omega)})_{n\geq 1}$. Let E be a measurable subset of Ω; by the Hölder inequality, we have

$$||f_n g||_{L^r(E)} \leq ||f_n||_{L^p(E)} ||g||_{L^q(E)} \leq M ||g||_{L^q(E)}.$$

Since $q < \infty$, we have $||g||_{L^q(E)} \to 0$ as $|E| \to 0$; this gives the r-equi-integrability of $(f_n g)_{n\geq 1}$ and concludes the proof of this lemma. ■

Lemma 1.2 *Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function. If there exist $\beta > 0$, $\gamma > 1$ and $C > 0$ such that*

$$\forall h > k \geq 0\,,\; F(h) \leq \frac{C^\beta (1+k)^\beta}{(h-k)^\beta} F(k)^\gamma$$

and if

$$H = \exp\left(\sum_{n \geq 0} \frac{2^{\frac{1}{\beta}} C F(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n} \right) < +\infty,$$

then $F(H) = 0$.

For the proof of this variant of Lemma 4.1, i) in [70], we refer the reader to [29].

Chapitre 2

Quelques résultats supplémentaires

2.1 Cas non-linéaire

L'outil que nous avons employé pour prouver l'existence d'une solution variationnelle à (1.3), le degré topologique, est non-linéaire et nous permet donc de traiter, comme nous l'avons vu dans la sous-section 1.2.2, des termes convectifs non-linéaires. Cependant, nous avons fortement utilisé le fait que la partie non-convective de (1.3) était linéaire; c'était nécessaire pour pouvoir définir l'application $\mathcal{F}$ dans l'étape 1 de la preuve du théorème 1.1 (il fallait que la solution de (1.16) soit unique).
Il existe des résultats (cf [11]) qui donnent l'unicité de la solution à certains problèmes de la forme

$$\left\{ \begin{array}{ll} -\mathrm{div}(a(x,u,\nabla u)) + b|u|^{p-2}u = L & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{array} \right. \tag{2.1}$$

où $-\mathrm{div}(a(x,u,\nabla u))$ est un opérateur de Leray-Lions agissant sur $W^{1,p}(\Omega)$. On pourrait donc adapter la méthode précédente à ces cas-là et prouver ainsi des résultats d'existence, dans un cadre "variationnel", pour des problèmes elliptiques non-linéaires de la forme

$$\left\{ \begin{array}{ll} -\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}(\Phi(x,u)) + b|u|^{p-2}u = L & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{array} \right. \tag{2.2}$$

Il faudrait cependant, pour faire ceci, ajouter des hypothèses sur a qui entrainent l'unicité de la solution variationnelle de (2.1); la méthode employée précédemment dans le cas linéaire n'est donc pas très adaptée, malgrè l'emploi du degré topologique, au cas non-linéaire.

Nous nous proposons ici de voir que des estimations similaires à celles du cas linéaire, associées à une technique d'approximation (déjà employée dans [8]), permettent néanmoins d'obtenir un résultat d'existence de solution variationnelle dans un cas non-linéaire non-coercitif.
La technique d'approximation que nous allons utiliser a aussi été employée dans [43] pour prouver l'existence, lorsque $L \in L^1(\Omega)$, d'une solution renormalisée à (2.2); cependant, que ce soit dans le cadre des solutions renormalisées de [43] ou dans le cadre des solutions entropiques de [8], on ne cherche des estimations dans l'espace d'énergie associé à l'opérateur que sur les tronquées des solutions (en particulier, on n'estime pas $u - T_k(u)$ dans cet espace). Les estimations que nous présentons ici dans le cas variationnel sont donc originales vis-a-vis des travaux précédents.

2.1.1 Hypothèses

Ω est un ouvert borné connexe de $\mathbb{R}^N$ à frontière lipschitzienne; Γ_d est une partie mesurable de $\partial\Omega$ et $\Gamma_f = \partial\Omega \backslash \Gamma_d$. σ représente la mesure sur $\partial\Omega$.

Soit $p \in]1, \infty[$. Si $p \leq N$, on prend $\overline{p} \in]\frac{Np}{N-1}, \frac{Np}{N-p}[$; si $p > N$, on pose $\overline{p} = +\infty$. Avec un tel choix de $\overline{p}$, $W^{1,p}(\Omega)$ s'injecte compactement dans $L^{\overline{p}}(\Omega)$ et $W^{1-1/p,p}(\partial\Omega)$ (l'espace des traces sur $\partial\Omega$ de fonctions dans $W^{1,p}(\Omega)$) s'injecte compactement dans $L^{\frac{N-1}{N}\overline{p}}(\partial\Omega)$.
On note $W^{1,p}_{\Gamma_d}(\Omega)$ l'espace des fonctions de $W^{1,p}(\Omega)$ dont la trace est nulle sur Γ_d.

Nous nous intéressons au problème

$$\left\{\begin{array}{ll} -\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}(\Phi(x,u)) + b|u|^{p-2}u = \mathcal{L} & \text{dans } \Omega, \\ u = 0 & \text{sur } \Gamma_d, \\ a(x,u,\nabla u)\cdot \mathbf{n} + \Phi(x,u)\cdot \mathbf{n} + \lambda |u|^{p-2}u = \mathcal{U}_f & \text{sur } \Gamma_f. \end{array}\right. \tag{2.3}$$

Les hypothèses sur la partie dominante de (2.3) sont:

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ est une fonction de Carathéodory,} \tag{2.4}$$

$$\begin{array}{c} \exists \alpha > 0 \text{ et } \Theta \in L^1(\Omega;\mathbb{R}^+) \text{ tels que } a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p - \Theta(x) \\ \text{pour presque-tout } x \in \Omega, \text{ pour tout } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N, \end{array} \tag{2.5}$$

$$\begin{array}{c} \exists \beta > 0\,,\ h \in L^{p'}(\Omega) \text{ et } \theta \in [0,\overline{p}] \text{ si } \overline{p} < +\infty \text{ ou } \theta \in [0,+\infty[\text{ si } \overline{p} = +\infty \text{ tels que} \\ |a(x,s,\xi)| \leq h(x) + \beta|s|^{\theta/p'} + \beta|\xi|^{p/p'} \text{ pour presque-tout } x \in \Omega, \text{ pour tout } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N, \end{array} \tag{2.6}$$

$$\begin{array}{c} (a(x,s,\xi) - a(x,s,\eta))\cdot(\xi - \eta) > 0 \\ \text{pour presque-tout } x \in \Omega, \text{ pour tout } (s,\xi,\eta) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \text{ tels que } \xi \neq \eta. \end{array} \tag{2.7}$$

Les termes de plus bas degré satisfont:

$$b \in L^{(\frac{\overline{p}}{p})'}(\Omega)\,,\ \lambda \in L^{(\frac{(N-1)\overline{p}}{Np})'}(\partial\Omega). \tag{2.8}$$

Afin d'obtenir, comme dans le cas linéaire, la coercitivité de la partie non-convective de (2.3), on fait l'hypothèse suivante:

$$b \geq 0 \text{ presque partout sur } \Omega\,,\ \lambda \geq 0\ \sigma\text{-presque partout sur } \partial\Omega \text{ et } \left\{\begin{array}{l} \sigma(\Gamma_d) > 0\,, \\ \text{ ou} \\ b \neq 0\,, \\ \text{ ou} \\ \lambda \neq 0. \end{array}\right. \tag{2.9}$$

Sous les hypothèses (2.8) et (2.9), un raisonnement par l'absurde montre que, pour tout $q \in [1,p]$, il existe $K > 0$ tel que, pour tout $u \in W^{1,p}_{\Gamma_d}(\Omega)$, on a

$$\alpha\int_\Omega |\nabla u|^p + \left(\int_\Omega b|u|^q\right)^{p/q} + \left(\int_{\Gamma_f} \lambda|u|^q\,d\sigma\right)^{p/q} \geq K||u||^p_{W^{1,p}(\Omega)}. \tag{2.10}$$

Remarque 2.1 *L'opérateur que nous considérons ici est un peu différent des opérateurs de Leray-Lions tels qu'on les étudie classiquement. Cependant, il n'est pas dur de voir (par les méthodes usuellement employées) que, sous les hypothèses (2.4)—(2.9), il existe une solution variationnelle à (2.3) lorsque* $\Phi \equiv 0$.

Enfin, la partie convective de (2.3) satisfait:

$$\begin{array}{c} \Phi : \Omega\times\mathbb{R} \to \mathbb{R}^N \text{ est une fonction de Carathéodory telle qu'il existe } g \in L^r(\Omega), \\ \text{avec } r = \frac{N}{p-1} \text{ si } p < N \text{ et } r \in]\frac{p}{p-1},\infty[\text{ si } p \geq N, \text{ satisfaisant} \\ |\Phi(x,s)| \leq g(x)(1+|s|^{p-1}) \text{ pour presque-tout } x \in \Omega \text{ et pour tout } s \in \mathbb{R}. \end{array} \tag{2.11}$$

2.1.2 Théorème d'existence

Théorème 2.1 *Sous les hypothèses (2.4)—(2.9) et (2.11), si* $L \in (W^{1,p}(\Omega))'$*, il existe une solution à*

$$\left\{\begin{array}{l} u \in W^{1,p}_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi + \int_\Omega \Phi(x,u)\cdot\nabla\varphi + \int_\Omega b|u|^{p-2}u\varphi + \int_{\Gamma_f}\lambda|u|^{p-2}u\varphi\,d\sigma \\ \qquad = \langle L,\varphi\rangle_{(W^{1,p}(\Omega))',W^{1,p}(\Omega)}\,,\ \forall\varphi\in W^{1,p}_{\Gamma_d}(\Omega). \end{array}\right. \tag{2.12}$$

Remarque 2.2 *Le problème (2.12) est une formulation faible de (2.3) (avec L qui prend en compte* $\mathcal{L}$ *et* $\mathcal{U}_f$*). On remarque aussi que, grâce aux injections de Sobolev et aux hypothèses d'intégrabilité des diverses données, tous les termes de l'équation de (2.12) ont un sens.*

Preuve du théorème 2.1
Comme signalé précédemment, nous allons prouver l'existence d'une solution à (2.12) au moyen d'une technique d'approximation.
En notant, comme d'habitude, $T_n(s) = \max(-n, \min(s,n))$ la troncature de niveau n, on pose $\Phi_n(x,s) = \Phi(x, T_n(s))$ et $a_n(x,s,\xi) = a(x,s,\xi) + \Phi_n(x,s)$. a_n est clairement une fonction de Carathéodory qui vérifie (2.7) (car Φ_n ne dépend pas de ξ); on a, pour presque tout $x \in \Omega$ et tout $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, par l'inégalité de Young,

$$\begin{array}{rcl} a_n(x,s,\xi)\cdot\xi & \geq & \alpha|\xi|^p - \Theta(x) - g(x)(1+n^{p-1})|\xi| \\ & \geq & \alpha|\xi|^p - \Theta(x) - Cg(x)^{p'}(1+n^{p-1})^{p'} - \dfrac{\alpha}{2}|\xi|^p \\ & = & \dfrac{\alpha}{2}|\xi|^p - \widetilde{\Theta}(x) \end{array}$$

où C ne dépend que de (p,α) et $\widetilde{\Theta} = \Theta + Cg^{p'}(1+n^{p-1})^{p'} \in L^1(\Omega)$ (par hypothèse sur g on a $g \in L^{p'}(\Omega)$).
Enfin, pour presque-tout $x \in \Omega$ et tout $(s,\xi) \in \mathbb{R}^N$, on a

$$|a_n(x,s,\xi)| \leq h(x) + g(x)(1+n^{p-1}) + \beta|s|^{\theta/p'} + \beta|\xi|^{p/p'}$$

avec $h + g(1+n^{p-1}) \in L^{p'}(\Omega)$.
a_n vérifie donc les hypothèses (2.4)—(2.7) et il existe (cf remarque 2.1) une solution([1]) à

$$\left\{\begin{array}{l} u_n \in W^{1,p}_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi + \int_\Omega \Phi(x,T_n(u_n))\cdot\nabla\varphi + \int_\Omega b|u_n|^{p-2}u_n\varphi + \int_{\Gamma_f}\lambda|u_n|^{p-2}u_n\varphi\,d\sigma \\ \qquad = \langle L,\varphi\rangle_{(W^{1,p}(\Omega))',W^{1,p}(\Omega)}\,,\ \forall\varphi\in W^{1,p}_{\Gamma_d}(\Omega). \end{array}\right. \tag{2.13}$$

Nous allons maintenant obtenir, de la même manière que dans le cas linéaire, des estimations sur $(u_n)_{n\geq 1}$. Dans toute la preuve qui suit, les différentes constantes C_i intervenant ne dépendent pas de n.

Etape 1: estimation sur $\ln(1+|u_n|)$.
Soit $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^p}$. En utilisant $\varphi(u_n)$ comme fonction test dans (2.13), on a

$$\begin{array}{l} \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\cdot\frac{\nabla u_n}{(1+|u_n|)^p} + \int_\Omega b|u_n|^{p-2}u_n\varphi(u_n) + \int_{\Gamma_f}\lambda|u_n|^{p-2}u_n\varphi(u_n)\,d\sigma \\ \displaystyle\quad\leq \int_\Omega g(x)(1+|u_n|^{p-1})\frac{|\nabla u_n|}{(1+|u_n|)^p} + ||L||_{(W^{1,p}(\Omega))'}||\varphi(u_n)||_{W^{1,p}(\Omega)}. \end{array} \tag{2.14}$$

[1] En fait, contrairement à ce que nous avons pu laissé penser au début de cette section, le degré topologique (ou le théorème de point fixe de Schauder) intervient aussi dans cette preuve: son utilisation est cachée dans l'application du théorème d'existence de Leray-Lions.

Or

$$a(x,u_n,\nabla u_n)\cdot\frac{\nabla u_n}{(1+|u_n|)^p}\geq\alpha\frac{|\nabla u_n|^p}{(1+|u_n|)^p}-\frac{\Theta(x)}{(1+|u_n|)^p}\geq\alpha|\nabla(\ln(1+|u_n|))|^p-\Theta(x). \quad (2.15)$$

φ étant impaire, croissante et ayant le même signe que s, on a, lorsque $|s|\geq 1$, $|s|^{p-2}s\varphi(s)\geq\varphi(1)|s|^{p-1}$. Ainsi, puisque $p>1$, la fonction continue $s\to|s|^{p-2}s\varphi(s)-\ln(1+|s|)$ tend vers l'infini lorsque $|s|\to\infty$. Il existe donc C_1 tel que, pour tout $s\in\mathbb{R}$, $|s|^{p-2}s\varphi(s)\geq\ln(1+|s|)-C_1$. b et λ étant positives, on en déduit

$$\begin{aligned}&\int_\Omega b|u_n|^{p-2}u_n\varphi(u_n)+\int_{\Gamma_f}\lambda|u_n|^{p-2}u_n\varphi(u_n)\,d\sigma\\ &\quad\geq\int_\Omega b\ln(1+|u_n|)+\int_{\Gamma_f}\lambda\ln(1+|u_n|)\,d\sigma-C_1||b||_{L^1(\Omega)}-C_1||\lambda||_{L^1(\partial\Omega)}. \quad (2.16)\end{aligned}$$

On a $1+|s|^{p-1}\leq 2(1+|s|)^{p-1}$ pour tout $s\in\mathbb{R}$, donc

$$\int_\Omega g(x)(1+|u_n|^{p-1})\frac{|\nabla u_n|}{(1+|u_n|)^p}\leq 2\int_\Omega g(x)\frac{|\nabla u_n|}{(1+|u_n|)}\leq 2||g||_{L^{p'}(\Omega)}||\,|\nabla(\ln(1+|u_n|)|\,||_{L^p(\Omega)}. \quad (2.17)$$

Enfin, puisque φ est bornée et $(1+|u_n|)^p\geq 1+|u_n|$, on a

$$\begin{aligned}||\varphi(u_n)||_{W^{1,p}(\Omega)}&\leq C_2+\left(\int_\Omega\left(\frac{|\nabla u_n|}{(1+|u_n|)^p}\right)^p\right)^{1/p}\\ &\leq C_2+\left(\int_\Omega\frac{|\nabla u_n|^p}{(1+|u_n|)^p}\right)^{1/p}\\ &\leq C_2+||\,|\nabla(\ln(1+|u_n|)|\,||_{L^p(\Omega)}. \quad (2.18)\end{aligned}$$

En injectant (2.15)—(2.18) dans (2.14) et par l'inégalité de Young, on en déduit

$$\begin{aligned}\alpha||\,|\nabla(\ln(1+|u_n|)|\,||^p_{L^p(\Omega)}+\int_\Omega b\ln(1+|u_n|)+\int_{\Gamma_f}\lambda\ln(1+|u_n|)\,d\sigma&\\ \leq C_3+\frac{\alpha}{2}||\,|\nabla(\ln(1+|u_n|)|\,||^p_{L^p(\Omega)}.&\end{aligned}$$

Ainsi, $(||\,|\nabla(\ln(1+|u_n|)|\,||_{L^p(\Omega)})_{n\geq 1}$, $(\int_\Omega b\ln(1+|u_n|))_{n\geq 1}$ et $(\int_{\Gamma_f}\lambda\ln(1+|u_n|)\,d\sigma)_{n\geq 1}$ sont des suites bornées; par (2.10) (appliqué à $q=1$), on en déduit que $(\ln(1+|u_n|))_{n\geq 1}$ est bornée dans $W^{1,p}(\Omega)$.

Etape 2: estimation sur $S_k(u_n)=u_n-T_k(u_n)$.
Soit $k>0$. En utilisant $S_k(u_n)$ comme fonction test dans (2.13), on a, puisque $\nabla S_k(u_n)=\mathbf{1}_{\{|u_n|\geq k\}}\nabla u_n$, $|u_n|^{p-2}u_nS_k(u_n)\geq|S_k(u_n)|^p$ et $|u_n|\leq k+|S_k(u_n)|$,

$$\begin{aligned}&\alpha\int_\Omega|\nabla S_k(u_n)|^p+\int_\Omega b|S_k(u_n)|^p+\int_{\Gamma_f}\lambda|S_k(u_n)|^p\,d\sigma\\ &\leq\int_\Omega g(x)(1+|u_n|^{p-1})|\nabla(S_k(u_n))|+||L||_{(W^{1,p}(\Omega))'}||S_k(u_n)||_{W^{1,p}(\Omega)}+||\Theta||_{L^1(\Omega)}\\ &\leq\int_\Omega g(x)(1+(2k)^{p-1}+2^{p-1}|S_k(u_n)|^{p-1})|\nabla(S_k(u_n))|+||L||_{(W^{1,p}(\Omega))'}||S_k(u_n)||_{W^{1,p}(\Omega)}+||\Theta||_{L^1(\Omega)}\\ &\leq 2^{p-1}\int_\Omega g(x)|S_k(u_n)|^{p-1}|\nabla(S_k(u_n))|+\Big((1+(2k)^{p-1})||g||_{L^{p'}(\Omega)}+||L||_{(W^{1,p}(\Omega))'}\Big)||S_k(u_n)||_{W^{1,p}(\Omega)}\\ &\quad+||\Theta||_{L^1(\Omega)}.\end{aligned}$$

Soit $\varepsilon > 0$ (que l'on fixera plus tard). On écrit $g = g_1 + g_2$ avec $g_1 \in L^r(\Omega)$ tel que $||g_1||_{L^r(\Omega)} \leq \varepsilon$ et $g_2 \in L^\infty(\Omega)$ (une telle écriture est toujours possible). Par les inégalités de Hölder et (2.10), on a donc

$$\begin{aligned} K||S_k(u_n)||^p_{W^{1,p}(\Omega)} &\leq 2^{p-1}||g_1||_{L^r(\Omega)}||\,|S_k(u_n)|^{p-1}||_{L^q(\Omega)}||S_k(u_n)||_{W^{1,p}(\Omega)} \\ &\quad +2^{p-1}||g_2||_{L^\infty(\Omega)}||\,|S_k(u_n)|^{p-1}||_{L^{p'}(\Omega)}||S_k(u_n)||_{W^{1,p}(\Omega)} \\ &\quad +C_4(1+k^{p-1})||S_k(u_n)||_{W^{1,p}(\Omega)} + ||\Theta||_{L^1(\Omega)} \\ &\leq 2^{p-1}\varepsilon||S_k(u_n)||^{p-1}_{L^{q(p-1)}(\Omega)}||S_k(u_n)||_{W^{1,p}(\Omega)} + C_5||S_k(u_n)||^{p-1}_{L^p(\Omega)}||S_k(u_n)||_{W^{1,p}(\Omega)} \\ &\quad +C_4(1+k^{p-1})||S_k(u_n)||_{W^{1,p}(\Omega)} + ||\Theta||_{L^1(\Omega)} \end{aligned} \tag{2.19}$$

où $q \in [p', \infty[$ est tel que $\frac{1}{r} + \frac{1}{q} = \frac{1}{p'}$ (comme $r > p'$, un tel q existe bien).
Si $p < N$, on a $r = N/(p-1)$ donc $q(p-1) = \frac{Np}{N-p}$; si $p \geq N$, on a $r > p/(p-1)$, donc $p \leq q(p-1) < \infty$. Dans les deux cas, il existe donc, par les injections de Sobolev, C_6 tel que $||S_k(u_n)||_{L^{q(p-1)}(\Omega)} \leq C_6||S_k(u_n)||_{W^{1,p}(\Omega)}$. De plus, par Hölder et l'injection de Sobolev $W^{1,p}(\Omega) \hookrightarrow L^{\overline{p}}(\Omega)$, et puisque $S_k(u_n) = 0$ hors de $E_k^n = \{|u_n| \geq k\}$, il existe C_7 tel que

$$||S_k(u_n)||_{L^p(\Omega)} \leq |E_k^n|^{\frac{1}{p}-\frac{1}{\overline{p}}}||S_k(u_n)||_{L^{\overline{p}}(\Omega)} \leq C_7|E_k^n|^{\frac{1}{p}-\frac{1}{\overline{p}}}||S_k(u_n)||_{W^{1,p}(\Omega)}.$$

En revenant dans (2.19), on obtient donc

$$\begin{aligned} K||S_k(u_n)||^p_{W^{1,p}(\Omega)} &\leq C_8\varepsilon||S_k(u_n)||^p_{W^{1,p}(\Omega)} + C_8\left(|E_k^n|^{\frac{1}{p}-\frac{1}{\overline{p}}}\right)^{p-1}||S_k(u_n)||^p_{W^{1,p}(\Omega)} \\ &\quad +C_8(1+k^{p-1})||S_k(u_n)||_{W^{1,p}(\Omega)} + ||\Theta||_{L^1(\Omega)}. \end{aligned}$$

Nous fixons maintenant $\varepsilon = K/(4C_8)$. Comme, par Tchebycheff, $|E_k^n| \leq \frac{||\ln(1+|u_n|)||^p_{L^p(\Omega)}}{(\ln(1+k))^p}$ et, par l'étape 1, $(\ln(1+|u_n|))_{n\geq 1}$ est bornée dans $L^p(\Omega)$, on peut fixer $k > 0$ tel que, pour tout $n \geq 1$, $C_8\left(|E_k^n|^{\frac{1}{p}-\frac{1}{\overline{p}}}\right)^{p-1} \leq K/4$. On en déduit alors, pour tout $n \geq 1$,

$$\frac{K}{2}||S_k(u_n)||^p_{W^{1,p}(\Omega)} \leq C_8(1+k^{p-1})||S_k(u_n)||_{W^{1,p}(\Omega)} + ||\Theta||_{L^1(\Omega)}.$$

Ainsi, pour ce k, $(S_k(u_n))_{n\geq 1}$ est bornée dans $W^{1,p}(\Omega)$.

Etape 3: estimation sur u_n.
En mettant, pour le k fixé dans l'étape 2, $T_k(u_n)$ comme fonction test dans (2.13), on a, puisque $|u_n|^{p-2}u_nT_k(u_n) \geq |T_k(u_n)|^p$ et $\nabla(T_k(u_n)) = 0$ là où $|u_n| \geq k$,

$$\begin{aligned} &\alpha\int_\Omega |\nabla(T_k(u_n))|^p + \int_\Omega b|T_k(u_n)|^p + \int_{\Gamma_f} \lambda|T_k(u_n)|^p \\ &\leq ||L||_{(W^{1,p}(\Omega))'}||T_k(u_n)||_{W^{1,p}(\Omega)} + \int_\Omega |g|(1+k^{p-1})|\nabla(T_k(u_n))| + ||\Theta||_{L^1(\Omega)} \\ &\leq (||L||_{(W^{1,p}(\Omega))'} + (1+k^{p-1})||g||_{L^{p'}(\Omega)})||T_k(u_n)||_{W^{1,p}(\Omega)} + ||\Theta||_{L^1(\Omega)}. \end{aligned}$$

Par (2.10), $(T_k(u_n))_{n\geq 1}$ est donc bornée dans $W^{1,p}(\Omega)$ et on en déduit, grâce à l'étape 2, que $(u_n)_{n\geq 1}$ est bornée dans $W^{1,p}(\Omega)$.

Etape 4: passage à la limite.
Quitte à extraire une suite, on peut donc supposer qu'il existe $u \in W^{1,p}_{\Gamma_d}(\Omega)$ tel que $u_n \to u$ faiblement dans $W^{1,p}_{\Gamma_d}(\Omega)$, fortement dans $L^{\overline{p}}(\Omega)$ et presque partout sur Ω. On peut aussi supposer (la trace $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ étant compacte — car $p > 1$) que les traces de $(u_n)_{n\geq 1}$ convergent σ-presque partout sur $\partial\Omega$ vers la trace de u.

Nous allons maintenant montrer que, à une sous-suite près, $\nabla u_n \to \nabla u$ presque partout sur Ω, en utilisant la même méthode que dans le cas variationnel coercitif (Leray-Lions).

On considère donc la fonction intégrable positive $f_n(x) = (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)$. Grâce à l'équation satisfaite par u_n, on a

$$\begin{aligned}\int_\Omega f_n &= \langle L, u_n - u\rangle_{(W^{1,p}(\Omega))',W^{1,p}(\Omega)} - \int_\Omega \Phi(x,T_n(u_n)) \cdot (\nabla u_n - \nabla u) - \int_\Omega b|u_n|^{p-2}u_n(u_n-u) \\ &\quad - \int_\Omega \lambda |u_n|^{p-2}u_n(u_n-u)\,d\sigma - \int_\Omega a(x,u_n,\nabla u)\cdot(\nabla u_n - \nabla u). \qquad (2.20)\end{aligned}$$

Comme $u_n \to u$ dans $W^{1,p}(\Omega)$ faible, on a $\langle L, u_n - u\rangle_{(W^{1,p}(\Omega))',W^{1,p}(\Omega)} \to 0$.
Puisque $0 \le \theta \le \overline{p}$ on a, par Hölder (avec l'exposant $\overline{p}/\theta \in [1,\infty]$) et pour tout $E \subset \Omega$ mesurable,

$$\int_E |u_n|^\theta \le ||u_n||^\theta_{L^{\overline{p}}(E)} |E|^{1-\frac{\theta}{\overline{p}}}.$$

Si $\overline{p} < \infty$, $(u_n)_{n\ge 1}$, qui converge dans $L^{\overline{p}}(\Omega)$, est $\overline{p}$-équi-intégrable et l'inégalité précédente nous montre alors que $(|u_n|^\theta)_{n\ge1}$ est 1-équi-intégrable; si $\overline{p} = \infty$, $1 - \frac{\theta}{\overline{p}} = 1$ et $(u_n)_{n\ge1}$ est bornée dans $L^{\overline{p}}(\Omega)$, donc la même inégalité nous montre encore que $(|u_n|^\theta)_{n\ge1}$ est 1-équi-intégrable. Ainsi, dans les deux cas, $(|u_n|^{\theta/p'})_{n\ge1}$ est p'-équi-intégrable et, par (2.6), $(a(x,u_n,\nabla u))_{n\ge1}$ est p'-équi-intégrable. Cette suite de fonctions convergeant presque partout vers $a(x,u,\nabla u)$, on en déduit que $a(x,u_n,\nabla u) \to a(x,u,\nabla u)$ dans $(L^{p'}(\Omega))^N$ et, puisque $u_n \to u$ faiblement dans $W^{1,p}(\Omega)$, que $\int_\Omega a(x,u_n,\nabla u)\cdot(\nabla u_n - \nabla u) \to 0$.
$(u_n)_{n\ge1}$ étant bornée dans $W^{1,p}(\Omega)$, elle est aussi bornée dans $L^{\overline{p}}(\Omega)$; $(|u_n|^{p-2}u_n(u_n-u))_{n\ge1}$ est donc une suite bornée dans $L^{\overline{p}/p}(\Omega)$ qui converge presque partout sur Ω vers 0. Puisque $b \in L^{(\overline{p}/p)'}(\Omega)$, avec $(\overline{p}/p)' < \infty$, le lemme 1.1 nous dit que $b|u_n|^{p-2}u_n(u_n-u) \to 0$ dans $L^1(\Omega)$.
De même, les traces de $(u_n)_{n\ge1}$ sont bornées dans $L^{(N-1)\overline{p}/N}(\partial\Omega)$ donc $((|u_n|^{p-2}u_n(u_n-u))_{|\partial\Omega})_{n\ge1}$ est une suite bornée dans $L^{(N-1)\overline{p}/Np}(\partial\Omega)$ qui converge σ-presque partout sur $\partial\Omega$ vers 0. Puisque $\lambda \in L^{((N-1)\overline{p}/Np)'}(\partial\Omega)$ avec $((N-1)\overline{p}/Np)' < \infty$, le lemme 1.1 (associé à la remarque 1.11) nous dit que $\lambda|u_n|^{p-2}u_n(u_n-u) \to 0$ dans $L^1(\partial\Omega)$.
Il reste à voir la convergence du terme $\int_\Omega \Phi(x,T_n(u_n))\cdot(\nabla u_n - \nabla u)$. La suite $(1+|u_n|^{p-1})_{n\ge1}$ converge presque partout sur Ω et est bornée dans $L^q(\Omega)$ avec $q = \frac{Np}{(N-p)(p-1)}$ si $p < N$ et $q < \infty$ si $p \ge N$([2]).
g étant dans $L^r(\Omega)$, le lemme 1.1 nous dit que $(g(1+|u_n|^{p-1}))_{n\ge1}$ converge dans $L^l(\Omega)$ avec $l \in [1,\infty[$ tel que $\frac{1}{r} + \frac{1}{q} = \frac{1}{l}$. Si $p < N$, on a $l = p'$; si $p \ge N$, puisque $r > p'$, on peut trouver $q < \infty$ tel que $l = p'$. Dans les deux cas, on constate donc que $(g(1+|u_n|^{p-1}))_{n\ge1}$ converge dans $L^{p'}(\Omega)$ et est donc p'-équi-intégrable sur Ω. Comme $|\Phi(x,T_n(u_n))| \le g(1+|u_n|^{p-1})$, on en déduit que $(\Phi(x,T_n(u_n)))_{n\ge1}$ est aussi p'-équi-intégrable sur Ω; cette suite de fonction convergeant presque partout sur Ω vers $\Phi(x,u)$, on a donc $\Phi(x,T_n(u_n)) \to \Phi(x,u)$ dans $(L^{p'}(\Omega))^N$. Puisque $u_n \to u$ faiblement dans $W^{1,p}(\Omega)$, cela nous donne donc $\int_\Omega \Phi(x,T_n(u_n))\cdot(\nabla u_n - \nabla u) \to 0$.
En rassemblant toutes ces convergences dans (2.20), on en déduit que $\int_\Omega f_n \to 0$, soit, puisque $f_n \ge 0$, que $f_n \to 0$ dans $L^1(\Omega)$ et, quitte à extraire une suite, presque partout sur Ω.
L'argument classique de [53] nous permet alors de voir que $\nabla u_n \to \nabla u$ presque partout sur Ω. En effet, en prenant $x \in \Omega$ tel que $a(x,\cdot,\cdot)$ est continue sur $\mathbb{R}\times\mathbb{R}^N$, $\Theta(x) < \infty$, $h(x) < \infty$, $\nabla u(x) \in \mathbb{R}^N$, $f_n(x) \to 0$, $u_n(x) \to u(x)$ et (2.5)-(2.7) soient valides, soit $(\nabla u_n(x))_{n\ge1}$ est non bornée dans $\mathbb{R}^N$, soit $(\nabla u_n(x))_{n\ge1}$ est bornée dans $\mathbb{R}^N$. Dans le premier cas, cela signifie qu'on peut extraire une suite, encore notée $(\nabla u_n(x))_{n\ge1}$, dont la norme tend vers l'infini; mais alors, par (2.5) et (2.6), on a

$$\begin{aligned} f_n(x) &\ge \alpha|\nabla u_n(x)|^p - \Theta(x) - h(x)|\nabla u(x)| - \beta|u_n(x)|^{\theta/p'}|\nabla u(x)| - \beta|\nabla u_n(x)|^{p/p'}|\nabla u(x)| \\ &\quad -|a(x,u_n(x),\nabla u(x))|(|\nabla u_n(x)| + |\nabla u(x)|) \to \infty\end{aligned}$$

[2]En fait, elle est aussi bornée dans $L^\infty(\Omega)$ si $p > N$.

(puisque $p > p/p'$ et $p > 1$), ce qui est une contradiction et montre que $(\nabla u_n(x))_{n\geq 1}$ est forcément bornée dans $\mathbb{R}^N$. Quitte à extraire une suite, encore notée $(\nabla u_n(x))_{n\geq 1}$, on peut donc supposer que $\nabla u_n(x) \to \xi$ dans $\mathbb{R}^N$; on a alors

$$f_n(x) \to 0 = (a(x,u(x),\xi) - a(x,u(x),\nabla u(x))) \cdot (\xi - \nabla u(x))$$

ce qui, par (2.7), implique $\xi = \nabla u(x)$; la seule limite des suites extraites de $(\nabla u_n(x))_{n\geq 1}$ étant $\nabla u(x)$, cela prouve bien que la suite bornée $(\nabla u_n(x))_{n\geq 1}$ converge vers $\nabla u(x)$.

On a donc trouvé une fonction $u \in W^{1,p}_{\Gamma_d}(\Omega)$ telle que $u_n \to u$ faiblement dans $W^{1,p}(\Omega)$, presque partout sur Ω, $\nabla u_n \to \nabla u$ presque partout sur Ω et $u_{n|\partial\Omega} \to u_{|\partial\Omega}$ σ-presque partout sur $\partial\Omega$.
Comme $(u_n)_{n\geq 1}$ est bornée dans $W^{1,p}(\Omega)$, $(a(x,u_n,\nabla u_n))_{n\geq 1}$ est bornée dans $(L^{p'}(\Omega))^N$ et converge donc, à une sous-suite près, faiblement dans cet espace; mais $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ presque partout sur Ω donc, par le théorème de compacité $L^p - L^q$, $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ fortement dans $(L^q(\Omega))^N$ dès que $q < p'$. La limite faible dans $(L^{p'}(\Omega))^N$ de $(a(x,u_n,\nabla u_n))_{n\geq 1}$ est donc $a(x,u,\nabla u)$. Ainsi, pour tout $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla\varphi \to \int_\Omega a(x,u,\nabla u) \cdot \nabla\varphi. \tag{2.21}$$

$|u_n|^{p-2}u_n \to |u|^{p-2}u$ presque partout sur Ω en étant bornée dans $L^{\overline{p}/(p-1)}(\Omega)$ donc, par le lemme 1.1, $b|u_n|^{p-2}u_n \to b|u|^{p-2}u$ dans $L^{\overline{p}'}(\Omega)$; ainsi, pour tout $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega b|u_n|^{p-2}u_n\varphi \to \int_\Omega b|u|^{p-2}u\varphi. \tag{2.22}$$

$|u_n|^{p-2}u_n \to |u|^{p-2}u$ σ-presque partout sur $\partial\Omega$ en étant bornée dans $L^{(N-1)\overline{p}/(N(p-1))}(\partial\Omega)$ donc, par le lemme 1.1 et la remarque 1.11, $\lambda|u_n|^{p-2}u_n \to \lambda|u|^{p-2}u$ dans $L^{((N-1)\overline{p}/N)'}(\partial\Omega)$; ainsi, pour tout $\varphi \in W^{1,p}(\Omega)$,

$$\int_{\Gamma_f} \lambda|u_n|^{p-2}u_n\varphi\, d\sigma \to \int_{\Gamma_f} \lambda|u|^{p-2}u\varphi\, d\sigma. \tag{2.23}$$

On a déjà vu que $\Phi(x,T_n(u_n)) \to \Phi(x,u)$ dans $(L^{p'}(\Omega))^N$, donc, pour tout $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega \Phi(x,T_n(u_n)) \cdot \nabla\varphi \to \int_\Omega \Phi(x,u) \cdot \nabla\varphi. \tag{2.24}$$

(2.21)—(2.24) permettent donc de passer à la limite dans l'équation satisfaite par u_n et de constater que u est bien une solution de (2.12). ∎

2.1.3 Théorème d'unicité

Le résultat d'unicité de [11], qui n'utilise pas l'hypothèse de coercitivité de l'opérateur, peut s'appliquer directement dans le cas non-coercitif étudié ici, lorsque l'on considère des conditions au bord de type Dirichlet; l'adaptation aux conditions au bord mixtes est assez triviale, mais traiter les conditions au bord de type Fourier demande une petite astuce ([3]).
Dans la preuve qui suit, la nouveauté par rapport à [11] réside donc essentiellement dans l'étape 1.

Théorème 2.2 *Sous les hypothèses (2.4)—(2.9), (2.11) et les deux suivantes:*

$$\begin{gathered}\exists\delta > 0\,,\ H \in L^{p'}(\Omega) \textit{ et } \varepsilon_0 > 0 \textit{ tels que}\\ |a(x,s,\xi) + \Phi(x,s) - a(x,t,\xi) - \Phi(x,t)| \leq \delta|s-t|(H(x) + |\xi|^{p-1} + |s|^{p^*/p'} + |t|^{p^*/p'})\\ \textit{pour presque-tout } x \in \Omega,\ \textit{pour tout } (s,t,\xi) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \textit{ satisfaisant } |s-t| < \varepsilon_0\end{gathered} \tag{2.25}$$

[3] A savoir: prouver que, si u et v sont des solutions de (2.12), alors $u = v$ presque partout sur $\{x \in \Omega \mid b(x) > 0\}$ et σ-presque partout sur $\{x \in \partial\Omega \mid \lambda(x) > 0\}$.

(où $p^* \in [1,\infty[$ est un exposant tel que $W^{1,p}(\Omega)$ s'injecte dans $L^{p^*}(\Omega)$ — i.e. $p^* \in [1, \frac{Np}{N-p}]$ si $p < N$ et $p^* \in [1,\infty[$ si $p \geq N$),

$$\begin{array}{c} 1 < p \leq 2 \text{ et } \exists \gamma > 0 \text{ tel que } (a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) \geq \gamma(|\xi|^{p-2} + |\eta|^{p-2})|\xi - \eta|^2 \\ \text{pour presque-tout } x \in \Omega, \text{ pour tout } (s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \\ \text{ou} \\ b > 0 \text{ presque partout sur } \Omega, \end{array} \tag{2.26}$$

la solution de (2.12) est unique.

Preuve du théorème 2.2
Supposons qu'il existe deux solutions u et v à (2.12). En faisant la différence des équations satisfaites par u et v et en utilisant la fonction test $\varphi = T_\varepsilon(w)$, où $w = u - v$ et $0 < \varepsilon < \varepsilon_0$, on a, par (2.25),

$$\int_\Omega (a(x,u,\nabla u) - a(x,u,\nabla v)) \cdot \nabla(T_\varepsilon(w)) \tag{2.27}$$

$$+ \int_\Omega b(|u|^{p-2}u - |v|^{p-2}v)T_\varepsilon(w) + \int_{\Gamma_f} \lambda(|u|^{p-2}u - |v|^{p-2}v)T_\varepsilon(w)\, d\sigma \tag{2.28}$$

$$= \int_\Omega (a(x,v,\nabla v) + \Phi(x,v) - a(x,u,\nabla v) - \Phi(x,u)) \cdot \nabla(T_\varepsilon(w))$$

$$\leq \delta \int_{A_\varepsilon} |w|(H + |\nabla v|^{p-1} + |u|^{p^*/p'} + |v|^{p^*/p'})|\nabla w| \tag{2.29}$$

où $A_\varepsilon = \{0 < |w| < \varepsilon\}$. On commence par traiter les termes de plus bas degré (2.28) puis on sépare les cas $p > 2$ et $p \leq 2$.

Etape 1: termes (2.28).
On remarque que le terme (2.27) est positif, que $T_\varepsilon(w)$ a le même signe que $|u|^{p-2}u - |v|^{p-2}v$ et que

$$|T_\varepsilon(w)| \geq \varepsilon \mathbf{1}_{B_\varepsilon}, \quad |T_\varepsilon(w)_{|\partial\Omega}| \geq \varepsilon \mathbf{1}_{C_\varepsilon},$$

où $B_\varepsilon = \{x \in \Omega \mid |w(x)| \geq \varepsilon\}$ et $C_\varepsilon = \{x \in \partial\Omega \mid |w(x)| \geq \varepsilon\}$.
Ainsi, par positivité de b et λ, (2.29) donne

$$\varepsilon \int_\Omega b \left||u|^{p-2}u - |v|^{p-2}v\right| \mathbf{1}_{B_\varepsilon} + \varepsilon \int_{\Gamma_f} \lambda \left||u|^{p-2}u - |v|^{p-2}v\right| \mathbf{1}_{C_\varepsilon}\, d\sigma \leq \varepsilon \int_{A_\varepsilon} \phi \tag{2.30}$$

où $\phi = \delta(H + |\nabla v|^{p-1} + |u|^{p^*/p'} + |v|^{p^*/p'})|\nabla w| \in L^1(\Omega)$ (car $(H, |\nabla v|^{p-1}, |u|^{p^*/p'}, |v|^{p^*/p'}) \in L^{p'}(\Omega)$ et $|\nabla w| \in L^p(\Omega)$). Par convergence dominée, puisque $\mathbf{1}_{A_\varepsilon} \to 0$ sur Ω, on a $\int_{A_\varepsilon} \phi \to 0$ lorsque $\varepsilon \to 0$. En divisant par ε dans (2.30), le lemme de Fatou donne, puisque $\mathbf{1}_{B_\varepsilon} \to \mathbf{1}_{\{x\in\Omega \mid w(x)\neq 0\}}$ sur Ω et $\mathbf{1}_{C_\varepsilon} \to \mathbf{1}_{\{x\in\partial\Omega \mid w(x)\neq 0\}}$ sur $\partial\Omega$ lorsque $\varepsilon \to 0$,

$$\int_{\{x\in\Omega \mid w(x)\neq 0\}} b \left||u|^{p-2}u - |v|^{p-2}v\right| = 0 \quad \text{et} \quad \int_{\{x\in\partial\Omega \mid w(x)\neq 0\}} \lambda \left||u|^{p-2}u - |v|^{p-2}v\right|\, d\sigma = 0.$$

On en déduit donc que $u = v$ presque partout sur $\{x \in \Omega \mid b(x) > 0\}$ et que $u = v$ σ-presque partout sur $\{x \in \partial\Omega \mid \lambda(x) > 0\}$.

Etape 2: si $p > 2$.
On a alors, à un ensemble de mesure nulle près, par (2.26), $\{x \in \Omega \mid b(x) > 0\} = \Omega$, donc $u = v$ presque partout sur Ω par l'étape précédente.

Etape 3: si $1 < p \leq 2$.

Par (2.26) et (2.29), puisque les termes (2.28) sont positifs, on a

$$\gamma\int_\Omega(|\nabla u|^{p-2}+|\nabla v|^{p-2})|\nabla(T_\varepsilon(w))|^2\le\delta\int_{A_\varepsilon}|w|(H+|\nabla v|^{p-1}+|u|^{p^*/p'}+|v|^{p^*/p'})|\nabla w|.$$

Mais, par Young, si $\varphi=u$ ou v,

$$\begin{aligned}\delta|w|H|\nabla w| &\le \frac{\gamma}{5}|\nabla u|^{p-2}|\nabla w|^2+C|w|^2H^2|\nabla u|^{2-p}\,,\\ \delta|w|\,|\nabla v|^{p-1}|\nabla w|=\delta|w|\,|\nabla v|^{\frac p2}|\nabla v|^{\frac p2-1}|\nabla w| &\le \frac{\gamma}{5}|\nabla v|^{p-2}|\nabla w|^2+C|w|^2|\nabla v|^p\,,\\ \delta|w|\,|\varphi|^{p^*/p'}|\nabla w| &\le \frac{\gamma}{5}|\nabla u|^{p-2}|\nabla w|^2+C|w|^2|\varphi|^{2p^*/p'}|\nabla u|^{2-p}\,,\end{aligned}$$

de sorte que

$$\begin{aligned}&\frac{\gamma}{5}\int_\Omega(|\nabla u|^{p-2}+|\nabla v|^{p-2})|\nabla(T_\varepsilon(w))|^2\\ &\le\ C\int_{A_\varepsilon}|w|^2(H^2|\nabla u|^{2-p}+|\nabla v|^p+(|u|^{2p^*/p'}+|v|^{2p^*/p'})|\nabla u|^{2-p})\le\varepsilon^2\int_{A_\varepsilon}\psi_1\end{aligned}\tag{2.31}$$

où $\psi_1=C((H^2+|u|^{2p^*/p'}+|v|^{2p^*/p'})|\nabla u|^{2-p}+|\nabla v|^p)$. Mais $H^2+|u|^{2p^*/p'}+|v|^{2p^*/p'}\in L^{p'/2}(\Omega)$ (on a bien $p'\ge 2$ puisque $p\le 2$), $|\nabla u|^{2-p}\in L^{p/(2-p)}(\Omega)$ ($p/(2-p)\in[1,\infty]$ car $2p\ge 2$) et $\frac{2}{p'}+\frac{2-p}{p}=1$, donc $\psi_1\in L^1(\Omega)$.
Par Cauchy-Schwarz, on déduit de (2.31) que

$$\int_\Omega|\nabla(T_\varepsilon(w))|\le C_0\left(\varepsilon^2\int_{A_\varepsilon}\psi_1\right)^{1/2}\left(\int_{A_\varepsilon}\psi_2\right)^{1/2}$$

où $\psi_2=(|\nabla u|^{p-2}+|\nabla v|^{p-2})^{-1}$. Or, pour tous réels (x,y) positifs, on a $(x+y)^{-1}\le x^{-1}+y^{-1}$, donc $\psi_2\le|\nabla u|^{2-p}+|\nabla v|^{2-p}\in L^1(\Omega)$ (car $0\le 2-p\le p$). Ainsi, puisque $\mathbf{1}_{A_\varepsilon}\to 0$ sur Ω, on a, pour $i\in\{1,2\}$, $\int_{A_\varepsilon}\psi_i\to 0$ lorsque $\varepsilon\to 0$.
On obtient finalement

$$\int_\Omega|\nabla(T_\varepsilon(w))|\le\varepsilon\omega(\varepsilon)\tag{2.32}$$

où $\omega(\varepsilon)\to 0$ lorsque $\varepsilon\to 0$.

Comme u et v sont dans $W^{1,p}_{\Gamma_d}(\Omega)$ et $u-v=0$ σ-presque partout sur $\partial\Omega$ là où $\lambda>0$ (cf étape 1), en notant $\Gamma_0=\Gamma_d\cup\{x\in\partial\Omega\mid\lambda(x)>0\}$, on a $w=u-v\in W^{1,p}_{\Gamma_0}(\Omega)$, donc $T_\varepsilon(w)\in W^{1,1}_{\Gamma_0}(\Omega)$. On sait aussi, toujours par l'étape 1, que $w=u-v=0$ presque partout sur Ω là où $b>0$.
Par (2.9), on a soit $\sigma(\Gamma_d)>0$, soit $\sigma(\{x\in\partial\Omega\mid\lambda(x)>0\})>0$, soit $|\{x\in\Omega\mid b(x)>0\}|>0$. Dans n'importe lequel des deux premiers cas, $\sigma(\Gamma_0)>0$ et l'inégalité de Poincaré nous donne donc C' tel que, pour tout $\varphi\in W^{1,1}_{\Gamma_0}(\Omega)$,

$$||\varphi||_{L^1(\Omega)}\le C'||\,|\nabla\varphi|\,||_{L^1(\Omega)}.\tag{2.33}$$

Dans le deuxième cas, puisque $E=\{x\in\Omega\mid b(x)>0\}$ est de mesure non-nulle, il est assez aisé de voir par l'absurde qu'il existe C' tel que, pour tout $\varphi\in W^{1,1}(\Omega)$ nulle sur E, on a encore (2.33).
Puisque $T_\varepsilon(w)\in W^{1,1}_{\Gamma_0}(\Omega)$ est nulle presque partout sur E, on peut donc lui appliquer (2.33) qui donne, associé à (2.32),

$$\int_\Omega|T_\varepsilon(w)|\le C'\varepsilon\omega(\varepsilon).$$

Soit $\delta\in]0,\varepsilon_0[$; si $0<\varepsilon<\delta$, on a $|T_\varepsilon(w)|\ge\varepsilon$ sur $B_\delta=\{x\in\Omega\mid|w(x)|\ge\delta\}$, donc par l'inégalité précédente, $|B_\delta|\le C'\omega(\varepsilon)$. En faisant $\varepsilon\to 0$, on en déduit que B_δ est de mesure nulle pour tout $\delta\in]0,\varepsilon_0[$, c'est à dire que $w=0$ presque partout sur Ω. ■

2.2 Contre-exemple à la régularité höldérienne

Considérons le problème mixte élémentaire suivant:

$$\begin{cases} -\Delta u = L & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \Gamma_d, \\ \nabla u \cdot \mathbf{n} = 0 & \text{sur} \quad \Gamma_f. \end{cases} \tag{2.34}$$

Lorsque L est régulier (par exemple dans $\mathcal{C}_c^\infty(\Omega)$, mais on a vu que l'on a besoin de beaucoup moins que cela), on a pu prouver la continuité sur $\overline{\Omega}$ de la solution variationnelle u de (2.34) *à condition* de supposer une hypothèse de “bonne répartition” de Γ_d sur $\partial\Omega$ (hypothèse (1.42)); cette hypothèse est bien sûr inutile lorsque l'on veut prouver la continuité de u à l'intérieur de Ω mais elle est cruciale lorsque l'on cherche à obtenir la continuité de u jusqu'au bord de Ω.
C'est ce que nous nous proposons de voir ici.

2.2.1 Construction d'un Γ_d singulier

Nous allons exhiber un borélien Γ_d de $\partial\Omega$ pour lequel la solution de (2.34) n'est pas continue sur $\overline{\Omega}$ (pour une très large classe de seconds membres réguliers L). L'idée est de trouver Γ_d suffisament “gros” pour que toute fonction continue nulle sur Γ_d soit nulle sur $\partial\Omega$, mais de sorte qu'il existe des fonctions de $H^1_{\Gamma_d}(\Omega)$ qui ne sont pas nulles sur $\partial\Omega$ tout entier.

On suppose $N = 3$ ([4]) et on prend donc Ω ouvert borné de $\mathbb{R}^3$ à frontière lipschitzienne.

Pour $x \in \mathbb{R}^3$ et $\alpha \in]0,1[$, on note

$$f_{x,\alpha}(y) = \left(1 - \frac{|y-x|}{\alpha}\right)^+.$$

$f_{x,\alpha}$ est une fonction de $H^1(\mathbb{R}^3)$ qui prend ses valeurs dans $[0,1]$ et dont le gradient $-\frac{1}{\alpha}\mathbf{1}_{B(x,\alpha)}\frac{\cdot - x}{|\cdot - x|}$ vérifie $|\nabla f_{x,\alpha}| = \frac{1}{\alpha}\mathbf{1}_{B(x,\alpha)}$ ($B(x,\alpha)$ désigne la boule euclidienne de centre x et de rayon α). On a donc $||\,|\nabla f_{x,\alpha}|\,||^2_{L^2(\mathbb{R}^3)} = \frac{1}{\alpha^2}|B(x,\alpha)| = C_0\alpha$, où $C_0 = |B(0,1)|$; comme $||f_{x,\alpha}||^2_{L^2(\mathbb{R}^3)} \le |B(x,\alpha)| = C_0\alpha^3$, on en déduit

$$||f_{x,\alpha}||_{H^1(\mathbb{R}^3)} \le C_1\sqrt{\alpha}, \tag{2.35}$$

où C_1 ne dépend ni de x ni de α (rappelons que l'on a pris $\alpha \le 1$). Enfin, on a

$$\int_{\partial\Omega} |f_{x,\alpha}|\,d\sigma \le \sigma(\partial\Omega \cap B(x,\alpha)) \le C_2\alpha^2, \tag{2.36}$$

où C_2 ne dépend que de Ω ([5]).

Prenons $\alpha_n = \frac{\varepsilon}{2^n}$ ($\varepsilon \in]0,1[$ sera fixé plus tard) et $(x_n)_{n\ge1}$ une suite dense dans $\partial\Omega$. On pose

$$\Gamma_d = \bigcup_{n\ge1} \left(\partial\Omega \cap B\left(x_n, \frac{\alpha_n}{2}\right)\right).$$

Notons que Γ_d est un ouvert dense de $\partial\Omega$.
Vu le choix de $(\alpha_n)_{n\ge1}$ et la propriété (2.35), la série des $(f_{x_n,\alpha_n})_{n\ge1}$ est absolument convergente dans $H^1(\mathbb{R}^3)$ et

$$f = \sum_{n\ge1} f_{x_n,\alpha_n}$$

[4] Le raisonnement qui suit pourrait être effectué avec n'importe quel $N \ge 3$. Il n'est cependant pas clair que le résultat que nous prouvons dans cette section 2.2 soit vrai en dimension $N = 2$.

[5] C'est une propriété générale: la mesure $(N-1)$-dimensionnelle de l'intersection d'un bord d'ouvert lipschitzien de $\mathbb{R}^N$ avec un ensemble de diamètre de l'ordre de α est un $\mathcal{O}(\alpha^{N-1})$.

est donc bien défini dans $H^1(\mathbb{R}^3)$. On a aussi, au sens des traces, $f = \sum_{n\geq 1} f_{x_n,\alpha_n}$ dans $L^2(\partial\Omega)$.
Pour tout $n \geq 1$, comme $f_{x_n,\alpha_n} \geq 1/2$ sur $\partial\Omega \cap B(x_n, \alpha_n/2)$ (cette fonction étant continue, sa trace sur $\partial\Omega$ est égale à sa restriction) et $f \geq f_{x_n,\alpha_n}$ sur $\partial\Omega$ (toutes les fonctions $(f_{x_m,\alpha_m})_{m\geq 1}$ sont positives), on a $f \geq 1/2$ sur $\partial\Omega \cap B(x_n, \alpha_n/2)$. Ainsi, $f \geq 1/2$ sur Γ_d.
La fonction $1 - 2T_{1/2}(f) \in H^1(\mathbb{R}^3)$ est nulle sur Γ_d ($T_{1/2}(f) = 1/2$ sur Γ_d, puisque $f \geq 1/2$ sur Γ_d) et sa restriction φ_0 à Ω vérifie donc $\varphi_0 \in H^1_{\Gamma_d}(\Omega)$. Nous allons maintenant choisir $\varepsilon \in]0,1[$ pour nous assurer que $\int_{\partial\Omega} \varphi_0 \, d\sigma > 0$ (et que donc $\varphi_0 \not\equiv 0$ sur $\partial\Omega$).
Puisque la série définissant f converge au sens des traces dans $L^2(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$, on a, par (2.36),

$$\int_{\partial\Omega} |2T_{1/2}(f)| \, d\sigma \leq 2\int_{\partial\Omega} f \, d\sigma = 2\sum_{n\geq 1} \int_{\partial\Omega} f_{x_n,\alpha_n} \, d\sigma \leq 2C_2\varepsilon^2 \sum_{n\geq 1} \frac{1}{(2^n)^2} = C_3\varepsilon^2$$

où C_3 ne dépend pas de ε. On prend alors $\varepsilon \in]0,1[$ tel que $C_3\varepsilon^2 \leq \sigma(\partial\Omega)/2$; avec ce choix, on a

$$\int_{\partial\Omega} \varphi_0 \, d\sigma = \sigma(\partial\Omega) - \int_{\partial\Omega} 2T_{1/2}(f) \, d\sigma \geq \frac{\sigma(\partial\Omega)}{2} > 0,$$

ce que l'on souhaitait.

Pour résumer, on a donc construit un ouvert dense Γ_d de $\partial\Omega$ tel qu'il existe $\varphi_0 \in H^1_{\Gamma_d}(\Omega)$ vérifiant $\int_{\partial\Omega} \varphi_0 \, d\sigma > 0$.

2.2.2 Non-continuité sur $\partial\Omega$ de la solution de (2.34)

Nous allons maintenant prouver le résultat suivant.

Théorème 2.3 *Avec le Γ_d construit dans la sous-section 2.2.1, pour tout $L \in \mathcal{C}_c^\infty(\Omega)$ positive non-nulle, la solution de (2.34) n'est pas continue sur $\overline{\Omega}$.*

En fait, nous allons prouver que la trace de la solution u de (2.34) n'est pas continue sur $\partial\Omega$, ce qui implique en particulier que u n'est pas continue sur $\overline{\Omega}$ (mais u est continue sur Ω).

Preuve du théorème 2.3
Prenons $L \in \mathcal{C}_c^\infty(\Omega)$ positive non nulle et raisonnons par l'absurde: on suppose que la trace de la solution $u \in H^1_{\Gamma_d}(\Omega)$ de (2.34) est continue sur $\partial\Omega$; étant nulle σ-presque partout sur Γ_d (car $u \in H^1_{\Gamma_d}(\Omega)$) et Γ_d étant un ouvert de $\partial\Omega$, cette trace est donc nulle partout sur Γ_d; puisque Γ_d est dense dans $\partial\Omega$, cette trace est finalement nulle sur $\partial\Omega$ en entier.
Soit ψ la solution variationnelle de

$$\left\{ \begin{array}{lll} -\Delta\psi = 0 & \text{dans} & \Omega, \\ \psi = 0 & \text{sur} & \Gamma_d, \\ \nabla\psi \cdot \mathbf{n} = 1 & \text{sur} & \Gamma_f, \end{array} \right. \tag{2.37}$$

c'est à dire $\psi \in H^1_{\Gamma_d}(\Omega)$ vérifiant

$$\int_\Omega \nabla\psi \cdot \nabla\varphi = \int_{\Gamma_f} \varphi \, d\sigma \,, \ \forall \varphi \in H^1_{\Gamma_d}(\Omega).$$

En mettant u comme fonction test dans l'équation satisfaite par ψ et ψ comme fonction test dans l'équation satisfaite par u, on trouve, puisque u est nulle sur $\partial\Omega$,

$$\int_\Omega L\psi = \int_{\Gamma_f} u \, d\sigma = 0. \tag{2.38}$$

Nous allons voir, en étudiant ψ, que c'est impossible.

On commence par constater que ψ n'est pas constante sur Ω. En effet, si ψ était constante, en prenant $\varphi_0 \in H^1_{\Gamma_d}(\Omega)$ (construite dans la sous-section 2.2.1) comme fonction test dans l'équation satisfaite par ψ, on aurait

$$0 = \int_\Omega \nabla\psi \cdot \nabla\varphi_0 = \int_{\Gamma_f} \varphi_0 \, d\sigma \neq 0$$

(car $\int_{\Gamma_f} \varphi_0 \, d\sigma = \int_{\partial\Omega} \varphi_0 \, d\sigma$ puisque φ_0 est nulle sur Γ_d), ce qui est une contradiction.
En utilisant ψ^- comme fonction test dans l'équation satisfaite par ψ, on a

$$-\int_\Omega |\nabla(\psi^-)|^2 = \int_{\Gamma_f} \psi^- \, d\sigma \geq 0,$$

donc $\nabla(\psi^-) = 0$ sur Ω, c'est à dire ψ^- constante sur Ω (on prend, pour simplifier, Ω connexe; dans le cas contraire, il faut raisonner sur chaque composante connexe de Ω). Mais ψ^- est nulle σ-presque partout sur Γ_d, qui n'est pas de σ-mesure nulle, donc ψ^- est en fait nulle presque partout sur Ω: $\psi \geq 0$ presque partout sur Ω.
Comme $\Delta\psi = 0$ dans Ω, ψ appartient à $\mathcal{C}^\infty(\Omega)$ et, n'étant pas constante, elle ne peut avoir de minimum dans Ω (cf [56]). Puisque ψ est positive sur Ω, on en déduit que $\psi > 0$ sur Ω. Mais on avait choisi $L \in \mathcal{C}^\infty_c(\Omega)$ positive, donc $L\psi \geq 0$ sur Ω et (2.38) nous donne donc $L\psi = 0$ sur Ω; ψ étant strictement positive sur Ω, on en déduit que $L = 0$ sur Ω, ce qui est une contradiction avec notre choix de L. ■

En conclusion, une hypothèse de "bonne répartition" de Γ_d (du genre (1.42)) est essentielle à l'extension du résultat de régularité höldérienne de Stampacchia au cas des conditions aux limites mixtes.

2.3 Différentes formulations pour les solutions par dualité

Nous étudions ici un peu plus précisément les solutions par dualité de (1.47) introduites brièvement dans la section 1.4.
Dans toute cette section, nous supposons donc les hypothèses (1.8), (1.9), (1.12), (1.14), (1.39) et (1.42).

2.3.1 Lien avec le cadre variationnel

Lorsque $\zeta \in (H^1(\Omega))'$, il existe une unique solution à (1.47) au sens variationnel: c'est la solution de (1.3) lorsque $L = \zeta_{|H^1_{\Gamma_d}(\Omega)}$, dont l'existence et l'unicité est assurée par le théorème 1.1.
Pour que la solution de (1.46) soit effectivement une solution, en un certain sens, de (1.47), il est naturel d'attendre que (1.46) redonne la solution variationnelle de (1.47) lorsque $\zeta \in (H^1(\Omega))'$. C'est effectivement le cas.

Pour voir cela, nous allons prouver que la solution variationnelle $u \in H^1_{\Gamma_d}(\Omega)$ de (1.47) est aussi la solution de (1.46).
On constate tout d'abord que l'on a effectivement $u \in \bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)$. Prenons ensuite $q < N/(N-1)$ et $L \in (W^{1,q}_{\Gamma_d}(\Omega))'$; par définition de u solution de (1.3) avec $\zeta_{|H^1_{\Gamma_d}(\Omega)}$ comme second membre, de $\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)}) \in H^1_{\Gamma_d}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, et puisque $\zeta \in (H^1(\Omega))'$, on a

$$\begin{aligned}
&\langle \zeta, \mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))',H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \\
&= \langle \zeta, \mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})\rangle_{(H^1(\Omega))',H^1(\Omega)} \\
&= \int_\Omega A\nabla u \cdot \nabla(\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})) + \int_\Omega u\mathbf{v} \cdot \nabla\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)}) + \int_\Omega bu\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)}) + \int_{\Gamma_f} \lambda u\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)}) \, d\sigma \\
&= \int_\Omega A^T\nabla(\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})) \cdot \nabla u + \int_\Omega u\mathbf{v} \cdot \nabla\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)}) + \int_\Omega b\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})u + \int_{\Gamma_f} \lambda\mathcal{T}(L_{|H^1_{\Gamma_d}(\Omega)})u \, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \langle L_{|H^1_{\Gamma_d}(\Omega)}, u\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} \\
&= \langle L, u\rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)}
\end{aligned}
$$

et u satisfait donc bien l'équation de (1.46).

2.3.2 Formulation intégrale forte de (1.46)

Il peut être plus facile de comprendre pourquoi (1.46) résout (1.47) une fois que l'on a transformé (1.46) en une formulation équivalente faisant intervenir, comme les formulations variationnelles usuelles, des intégrales. Nous avons cependant besoin, pour faire cela, de quelques préliminaires.

Soit $s \in]1, \infty[$. L'application

$$
\left\{ \begin{array}{r} \mathcal{C}^\infty_c(\Omega) \longrightarrow (W^{1,s}_{\Gamma_d}(\Omega))' \\ \varphi \longrightarrow (\phi \to \int_\Omega \varphi\phi) \end{array} \right.
$$

est une injection dense de $\mathcal{C}^\infty_c(\Omega)$ dans $(W^{1,s}_{\Gamma_d}(\Omega))'$ (la densité vient de la représentation de tout élément de $(W^{1,s}_{\Gamma_d}(\Omega))'$ comme somme d'un élément de $L^{s'}(\Omega)$ et de la divergence — en un certain sens — d'un élément de $(L^{s'}(\Omega))^N$). Cette injection nous permet de considérer $\mathcal{C}^\infty_c(\Omega)$ comme un sous-espace de $(W^{1,s}_{\Gamma_d}(\Omega))'$ (et cette identification est cohérente avec l'injection naturelle de ces espaces dans $\mathcal{D}'(\Omega)$ lorsque $\Gamma_d = \partial\Omega$).
Définissons $\Theta : H^1_{\Gamma_d}(\Omega) \to (H^1_{\Gamma_d}(\Omega))'$ par: pour tout $(\varphi, \psi) \in H^1_{\Gamma_d}(\Omega)$,

$$
\langle \Theta(\varphi), \psi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} = \int_\Omega A^T\nabla\varphi \cdot \nabla\psi + \int_\Omega \psi\mathbf{v}\cdot\nabla\varphi + \int_\Omega b\varphi\psi + \int_{\Gamma_f} \lambda\varphi\psi\, d\sigma.
$$

Notons deux propriétés de Θ, qui nous seront utiles par la suite.

i) Pour tout $\varphi \in H^1_{\Gamma_d}(\Omega)$, $\Theta(\varphi)_{|\mathcal{D}(\Omega)} = -\mathrm{div}(A^T\nabla\varphi) + \mathbf{v}\cdot\nabla\varphi + b\varphi$ dans $\mathcal{D}'(\Omega)$,

ii) $\Theta \circ \mathcal{T} = Id_{(H^1_{\Gamma_d}(\Omega))'}$ et $\mathcal{T} \circ \Theta = Id_{H^1_{\Gamma_d}(\Omega)}$.

Remarque 2.3 *Dans le cas Dirichlet pur, $\Gamma_d = \partial\Omega$ et on sait alors que $\Theta(\varphi) \in (H^1_0(\Omega))'$ est entièrement déterminé par ses valeurs sur $\mathcal{D}(\Omega)$; la propriété i) n'est alors plus uniquement valable en restriction à $\mathcal{D}(\Omega)$, mais dans $(H^1_0(\Omega))'$ en entier.*

Preuve des propriétés i) et ii)
La propriété i) étant complètement triviale, nous passons directement à la preuve de ii). Pour tout $L \in (H^1_{\Gamma_d}(\Omega))'$, par définition de $\mathcal{T}$ on a, lorsque $\psi \in H^1_{\Gamma_d}(\Omega)$,

$$
\begin{aligned}
\langle L, \psi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} &= \int_\Omega A^T\nabla(\mathcal{T}L)\cdot\nabla\psi + \int_\Omega \psi\mathbf{v}\cdot\nabla(\mathcal{T}L) + \int_\Omega b(\mathcal{T}L)\psi + \int_{\Gamma_f} \lambda(\mathcal{T}L)\psi\, d\sigma \\
&= \langle \Theta(\mathcal{T}L), \psi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)},
\end{aligned}
$$

ce qui signifie que $L = \Theta \circ \mathcal{T}(L)$ et prouve la première partie de ii).
Soit maintenant $\varphi \in H^1_{\Gamma_d}(\Omega)$. On a, par définition de Θ,

$$
\left\{ \begin{array}{l} \varphi \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla\varphi\cdot\nabla\psi + \int_\Omega \psi\mathbf{v}\cdot\nabla\varphi + \int_\Omega b\varphi\psi + \int_{\Gamma_f} \lambda\varphi\psi\, d\sigma = \langle\Theta(\varphi), \psi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}\,, \\ \qquad \forall\psi \in H^1_{\Gamma_d}(\Omega). \end{array} \right.
$$

Mais l'unique solution à ce problème (cf théorème 1.1) est $\mathcal{T}(\Theta(\varphi))$, donc $\varphi = \mathcal{T}(\Theta(\varphi))$ et la deuxième partie de la propriété ii) est prouvée. ■

On peut maintenant établir une formulation intégrale équivalente à (1.46).
Soit $p \in]N,\infty[$. Puisque $\mathcal{C}_c^\infty(\Omega)$ est densément injecté dans $(W^{1,p'}_{\Gamma_d}(\Omega))'$, (1.45) est équivalent à

$$\left\{\begin{array}{l} f_p \in W^{1,p'}_{\Gamma_d}(\Omega)\,,\\ \forall L \in \mathcal{C}_c^\infty(\Omega)\,,\ \langle L, f_p\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))',W^{1,p'}_{\Gamma_d}(\Omega)} = \int_\Omega L f_p = \langle \zeta, \mathcal{T}L\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))',H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{2.39}$$

Mais par le raisonnement ci-dessus, il y a une bijection

$$\mathcal{C}_c^\infty(\Omega) \longleftrightarrow \{\varphi \in H^1_{\Gamma_d}(\Omega)\,,\ \Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)\},$$

qui à $L \in \mathcal{C}_c^\infty(\Omega)$ associe $\varphi = \mathcal{T}L \in H^1_{\Gamma_d}(\Omega)$ (on a bien, alors, $\Theta(\mathcal{T}L) = L \in \mathcal{C}_c^\infty(\Omega)$) et qui, à $\varphi \in H^1_{\Gamma_d}(\Omega)$ telle que $\Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)$, associe $L = \Theta(\varphi)$. Notons aussi que, si $\varphi \in H^1_{\Gamma_d}(\Omega)$ vérifie $\Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega) \subset (W^{1,p'}_{\Gamma_d}(\Omega))'$, alors par le corollaire 1.2, $\varphi \in H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})$.
(1.45) est donc équivalent à

$$\left\{\begin{array}{l} f_p \in W^{1,p'}_{\Gamma_d}(\Omega)\,,\\ \forall\varphi \in H^1_{\Gamma_d}(\Omega) \text{ tel que } \Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)\,,\ \int_\Omega \Theta(\varphi) f_p = \langle \zeta, \varphi\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))',H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{2.40}$$

Mais, par le lemme fondamental des distributions, lorsque l'élément $\Theta(\varphi)$ de $(H^1_{\Gamma_d}(\Omega))'$ est dans $\mathcal{C}_c^\infty(\Omega)$, il est entièrement connu par ses valeurs sur $\mathcal{D}(\Omega)$; par la propriété i) de Θ, lorsque $\Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)$, on a donc $\Theta(\varphi) = -\mathrm{div}(A^T\nabla\varphi) + \mathbf{v}\cdot\nabla\varphi + b\varphi$.
Ainsi, (1.45) est équivalent à

$$\left\{\begin{array}{l} f_p \in W^{1,p'}_{\Gamma_d}(\Omega)\,,\\ \forall\varphi \in H^1_{\Gamma_d}(\Omega) \text{ tel que } \Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)\,,\\ \qquad \int_\Omega f_p(-\mathrm{div}(A^T\nabla\varphi) + \mathbf{v}\cdot\nabla\varphi + b\varphi) = \langle \zeta, \varphi\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))',H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{2.41}$$

Nous avons vu que la solution de (1.45) est aussi la solution de (1.46) (et, en particulier, appartient à $W^{1,q}_{\Gamma_d}(\Omega)$ pour tout $q < N/(N-1)$). On en déduit donc une forme équivalente de (1.46):

$$\left\{\begin{array}{l} f \in \bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)\,,\\ \forall\varphi \in H^1_{\Gamma_d}(\Omega) \text{ tel que } \Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)\,,\\ \qquad \int_\Omega f(-\mathrm{div}(A^T\nabla\varphi) + \mathbf{v}\cdot\nabla\varphi + b\varphi) = \langle \zeta, \varphi\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))',H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{2.42}$$

Cette formulation est en fait une formulation faible usuelle de (1.47) dans laquelle on a fait passer toutes les dérivées sur les fonctions tests (les termes de bord sont absorbés par la condition "$\Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)$").

Remarque 2.4 *Dans le cas purement Dirichlet, grâce à la remarque 2.3, la condition "$\Theta(\varphi) \in \mathcal{C}_c^\infty(\Omega)$" est équivalente à "$-\mathrm{div}(A^T\nabla\varphi) + \mathbf{v}\cdot\nabla\varphi + b\varphi \in \mathcal{C}_c^\infty(\Omega)$".*

2.3.3 Formulation intégrale faible de (1.47)

Nous avons vu, dans la partie précédente, qu'on peut obtenir une formulation intégrale équivalente à (1.46) en faisant passer toutes les dérivées sur les fonctions tests. Mais les formulations variationnelles usuelles ne font passer qu'une dérivée sur les fonctions tests; on peut donc chercher à obtenir une formulation faible de (1.47) en ne faisant passer qu'une dérivée sur les fonctions tests et tenter de voir les liens entre cette formulation et (1.46).

Soit $p \in]N,\infty[$ et $\varphi \in W^{1,p}_{\Gamma_d}(\Omega) \subset \mathcal{C}(\overline{\Omega})$. On a, lorsque $q < N/(N-1)$, $W^{1,q}(\Omega) \hookrightarrow L^{\frac{Nq}{N-q}}(\Omega)$ et $W^{1-1/q,q}(\partial\Omega) \hookrightarrow L^{\frac{(N-1)q}{N-q}}(\partial\Omega)$; comme $\frac{Nq}{N-q} \to \frac{N}{N-2}$ et $\frac{(N-1)q}{N-q} \to \frac{N-1}{N-2}$ lorsque $q \to \frac{N}{N-1}$, et puisque $\mathbf{v} \in (L^{N_*}(\Omega))^N$, $b \in L^{\frac{\overline{r}}{2}}(\Omega)$ et $\lambda \in L^{\overline{r}-1}(\partial\Omega)$ pour un $\overline{r} > N$, il existe $q < \frac{N}{N-1}$ tel que l'application

$$L_\varphi : \psi \in W^{1,q}(\Omega) \to \int_\Omega A^T\nabla\varphi\cdot\nabla\psi + \int_\Omega \psi\mathbf{v}\cdot\nabla\varphi + \int_\Omega b\varphi\psi + \int_{\Gamma_f} \lambda\varphi\psi\, d\sigma$$

est bien définie et dans $(W^{1,q}(\Omega))'$. On constate aussi que $\mathcal{T}L_\varphi = \varphi$, puisque $\mathcal{T}L_\varphi$ et φ sont deux solutions du problème

$$\left\{\begin{array}{l} w \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla w\cdot\nabla\psi + \int_\Omega \psi\mathbf{v}\cdot\nabla w + \int_\Omega bw\psi + \int_{\Gamma_f} \lambda w\psi\, d\sigma \\ \qquad = \langle (L_\varphi)_{|H^1_{\Gamma_d}(\Omega)}, \psi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}\,,\ \forall \psi \in H^1_{\Gamma_d}(\Omega) \end{array}\right. \tag{2.43}$$

qui admet une unique solution.
On a donc, si f est la solution de (1.46), $\langle L_\varphi, f\rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)} = \langle \zeta, \varphi\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}$. Ainsi, la solution de (1.46) vérifie

$$\left\{\begin{array}{l} f \in \displaystyle\bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi + \int_\Omega f\mathbf{v}\cdot\nabla\varphi + \int_\Omega bf\varphi + \int_{\Gamma_f} \lambda f\varphi\, d\sigma = \langle \zeta, \varphi\rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}\,, \\ \qquad \forall \varphi \in \displaystyle\bigcup_{p>N} W^{1,p}_{\Gamma_d}(\Omega). \end{array}\right. \tag{2.44}$$

Cette formulation est la formulation variationnelle naturelle de (1.47) (celle, en particulier, obtenue dans [10]). Elle n'est cependant pas équivalente à (1.46), car la solution de (2.44) n'est en général pas unique (voir [65]).

2.4 Théorèmes de stabilité pour les solutions par dualité

Nous prouvons ici des théorèmes de stabilité pour les solutions par dualité de (1.47) et (1.48).
Pour établir ces résultats, nous utilisons de manière cruciale le fait que les estimations de continuité que l'on a sur les solutions de (1.3) et (1.4) sont dans des espaces de Hölder, et donnent donc des résultats de compacité dans $\mathcal{C}(\overline{\Omega})$ sur ces solutions; ceci permet de prouver les convergences faibles dans les espaces $W^{1,q}_{\Gamma_d}(\Omega)$ des solutions par dualité de (1.47) et (1.48). La convergence forte des gradients de ces solutions est prouvée au travers de leur convergence en mesure, pour laquelle nous utilisons une idée introduite dans [38].

2.4.1 Résultat de stabilité pour la solution de (1.46)

On fait les hypothèses suivantes:

$$\begin{array}{c} \forall m \geq 0\,,\ A_m : \Omega \to M_N(\mathbb{R}) \text{ est une fonction mesurable,} \\ \exists \alpha_A > 0 \text{ tel que } A_m(x)\xi\cdot\xi \geq \alpha_A|\xi|^2 \text{ pour tout } m \geq 0, \text{ pour presque tout } x \in \Omega \\ \text{et pour tout } \xi \in \mathbb{R}^N, \\ \exists \Lambda_A \geq 0 \text{ tel que } ||A_m(x)|| \leq \Lambda_A \text{ pour tout } m \geq 0 \text{ et pour presque tout } x \in \Omega, \\ A_m \longrightarrow A_\infty \text{ presque partout sur } \Omega. \end{array} \tag{2.45}$$

$$\begin{array}{c} \forall m \geq 0\,,\ \mathbf{v}_m \in (L^{N_*}(\Omega))^N\,, \\ \mathbf{v}_m \longrightarrow \mathbf{v}_\infty \text{ dans } (L^{N_*}(\Omega))^N. \end{array} \tag{2.46}$$

Il existe $\overline{r} > N$ tel que

$$\begin{gathered}\forall m \geq 0\,,\ b_m \in L^{\frac{\overline{r}}{2}}(\Omega) \text{ et } b_m \geq 0 \text{ presque partout sur } \Omega\,,\\ b_m \to b_\infty \text{ faiblement dans } L^{\frac{\overline{r}}{2}}(\Omega)\end{gathered} \tag{2.47}$$

et

$$\begin{gathered}\forall m \geq 0\,, \lambda_m \in L^{\overline{r}-1}(\partial\Omega) \text{ et } \lambda_m \geq 0 \ \sigma\text{-presque partout sur } \partial\Omega\,,\\ \lambda_m \to \lambda_\infty \text{ faiblement dans } L^{\overline{r}-1}(\partial\Omega).\end{gathered} \tag{2.48}$$

$$\begin{gathered}\forall m \geq 0\,,\ \zeta_m = \mu_m + L_m \text{ avec } \mu_m \in \mathcal{M}(\overline{\Omega})\,,\ L_m \in (H^1(\Omega))' \text{ et}\\ \mu_m \to \mu_\infty \text{ dans } \mathcal{M}(\overline{\Omega}) \text{ faible-}*\,,\ L_m \to L_\infty \text{ fortement dans } (H^1(\Omega))'.\end{gathered} \tag{2.49}$$

On notera $\zeta_\infty = \mu_\infty + L_\infty \in \mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$.

On suppose aussi que les problèmes sont uniformément bien posés, c'est à dire

$$\begin{gathered}\exists b_0 > 0\,,\ \exists E \subset \Omega \text{ tel que, pour tout } m \geq 0,\ b_m \geq b_0 \text{ sur } E,\\ \exists \lambda_0 > 0\,,\ \exists S \subset \Gamma_f \text{ tel que, pour tout } m \geq 0,\ \lambda_m \geq \lambda_0 \text{ sur } S\\ \text{et soit } \sigma(\Gamma_d) > 0\,,\ \text{soit } |E| > 0\,,\ \text{soit } \sigma(S) > 0.\end{gathered} \tag{2.50}$$

Théorème 2.4 *Sous les hypothèses (1.8), (2.45)—(2.50) et (1.42), si f_m est la solution de (1.46) avec $(A_m, \mathbf{v}_m, b_m, \lambda_m, \zeta_m)$ à la place de $(A, \mathbf{v}, b, \lambda, \zeta)$ et f_∞ est la solution de (1.46) avec $(A_\infty, \mathbf{v}_\infty, b_\infty, \lambda_\infty, \zeta_\infty)$ à la place de $(A, \mathbf{v}, b, \lambda, \zeta)$, alors*

$$f_m \overset{m\to\infty}{\longrightarrow} f_\infty \textit{ fortement dans } W^{1,q}_{\Gamma_d}(\Omega) \textit{ pour tout } q < \tfrac{N}{N-1}.$$

Remarque 2.5 *Nous verrons aussi (lemme 2.4) que si dans l'hypothèse (2.49) on suppose seulement "$L_m \to L_\infty$ dans $(H^1(\Omega))'$ faible-$*$", alors on a $f_m \to f$ faiblement dans $W^{1,q}_{\Gamma_d}(\Omega)$ pour tout $q < N/(N-1)$. Cependant, sous la seule convergence faible-$*$ de $(L_m)_{m\geq 0}$, alors il est faux en général que $f_m \to f$ fortement dans $W^{1,1}(\Omega)$ (c'est déjà le cas pour les problèmes variationnels).*

2.4.2 Lemmes techniques

Les lemmes de cette partie sont relativement simples et classiques.

Lemme 2.1 *Si $l \in (H^1(\Omega))' \cap \mathcal{M}(\overline{\Omega})$, alors pour tout $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, on a $|\langle l, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)}| \leq ||l||_{\mathcal{M}(\overline{\Omega})} ||\varphi||_{L^\infty(\Omega)}$.*

Preuve du lemme 2.1
Par définition, dire que $l \in (H^1(\Omega))' \cap \mathcal{M}(\overline{\Omega})$, c'est dire que l est un élément de $(H^1(\Omega))'$ qui vérifie: il existe $C > 0$ tel que, pour tout $\varphi \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $|\langle l, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)}| \leq C||\varphi||_{\mathcal{C}(\overline{\Omega})}$; comme $H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ est dense dans $\mathcal{C}(\overline{\Omega})$ ($\mathcal{C}^1(\overline{\Omega})$ est dense dans $\mathcal{C}(\overline{\Omega})$), on peut alors étendre $l_{|H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}$ en une unique application de $(\mathcal{C}(\overline{\Omega}))' = \mathcal{M}(\overline{\Omega})$; on identifie alors l et cette unique extension.
Soit $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$; on note $M = ||\varphi||_{L^\infty(\Omega)}$. Soit $\varphi_n \in \mathcal{C}^\infty(\overline{\Omega})$ telles que $\varphi_n \to \varphi$ dans $H^1(\Omega)$; on sait alors que $T_M(\varphi_n) \to T_M(\varphi) = \varphi$ dans $H^1(\Omega)$.
Comme $T_M(\varphi_n) \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$, on a $|\langle l, T_M(\varphi_n)\rangle_{(H^1(\Omega))', H^1(\Omega)}| \leq ||l||_{\mathcal{M}(\overline{\Omega})} ||T_M(\varphi_n)||_{\mathcal{C}(\overline{\Omega})} \leq M||l||_{\mathcal{M}(\overline{\Omega})}$.
En passant à la limite $n \to \infty$, on en déduit le résultat souhaité. ■

Lemme 2.2 *Soit $A : \Omega \to M_N(\mathbb{R})$ mesurable essentiellement bornée et vérifiant, pour presque tout $x \in \Omega$ et pour tout $\xi \in \mathbb{R}^N$, $A(x)\xi \cdot \xi \geq 0$. Si $(a_m)_{m\geq 0} \in H^1(\Omega)$ converge faiblement vers a dans $H^1(\Omega)$, alors*

$$\int_\Omega A\nabla a \cdot \nabla a \leq \liminf_{m\to\infty} \int_\Omega A\nabla a_m \cdot \nabla a_m.$$

Preuve du lemme 2.2
Soit B la forme bilinéaire symétrique définie sur $H^1(\Omega)$ par $(A^T + A)/2$, c'est à dire

$$\forall (w, \widetilde{w}) \in H^1(\Omega)\,,\ B(w, \widetilde{w}) = \int_\Omega \frac{A^T + A}{2} \nabla w \cdot \nabla \widetilde{w}$$

L'hypothèse sur A nous permet de voir que, pour tout $w \in H^1(\Omega)$,

$$B(w, w) = \int_\Omega A\nabla w \cdot \nabla w \geq 0.$$

Ainsi, B étant une forme bilinéaire symétrique positive, on peut appliquer l'inégalité de Cauchy-Schwarz et obtenir, pour tout $m \geq 0$,

$$B(a, a_m)^2 \leq B(a_m, a_m)B(a, a). \tag{2.51}$$

Puisque $a_m \to a$ faiblement dans $H^1(\Omega)$, on a

$$B(a, a_m) = \int_\Omega \frac{1}{2}(A^T + A)\nabla a \cdot \nabla a_m \to \int_\Omega \frac{1}{2}(A^T + A)\nabla a \cdot \nabla a = \int_\Omega A\nabla a \cdot \nabla a.$$

En prenant la lim inf lorsque $m \to \infty$ dans (2.51), on obtient donc

$$\left(\int_\Omega A\nabla a \cdot \nabla a\right)^2 \leq \left(\liminf_{m\to\infty} \int_\Omega A\nabla a_m \cdot \nabla a_m\right) \int_\Omega A\nabla a \cdot \nabla a,$$

ce qui conclut la preuve de ce lemme. ■

Lemme 2.3 *Si $(\mathbf{v}_m)_{m\geq 0}$ est une suite convergeante dans $(L^{N_*}(\Omega))^N$, alors pour tout $\eta > 0$, il existe $\Lambda \geq 0$ tel que, pour tout $m \geq 0$, $\mathbf{v}_m \in B(N_*, \eta) + B(\infty, \Lambda)$.*

Remarque 2.6 *Prouver ce lemme consiste en fait à établir ce que nous avons dit en fin de remarque 1.5.*

Preuve du lemme 2.3
On va montrer en fait que, pour tout $\eta > 0$, il existe $n \geq 1$ tel que, pour tout $m \geq 0$, $||\,|T_n(\mathbf{v}_m) - \mathbf{v}_m|\,||_{L^{N_*}(\Omega)} < \eta$ (où, lorsque $X \in \mathbb{R}^N$, $T_n(X)$ représente le vecteur $(T_n(X_1), \ldots, T_n(X_N))$); on aura alors, pour tout $m \geq 0$, $\mathbf{v}_m = \mathbf{v}_m - T_n(\mathbf{v}_m) + T_n(\mathbf{v}_m)$ avec $\mathbf{v}_m - T_n(\mathbf{v}_m) \in B(N_*, \eta)$ et $T_n(\mathbf{v}_m) \in B(\infty, n\sqrt{N})$, ce qui conclura la preuve.
La démonstration se fait par l'absurde. Supposons donc qu'il existe $\eta > 0$ tel que, pour tout $n \geq 1$, il existe $m_n \geq 0$ vérifiant $||\,|T_n(\mathbf{v}_{m_n}) - \mathbf{v}_{m_n}|\,||_{L^{N_*}(\Omega)} > \eta$. $(\mathbf{v}_m)_{m\geq 0}$ étant convergeante dans $(L^{N_*}(\Omega))^N$, quitte à extraire une suite, on peut supposer que $\mathbf{v}_{m_n} \to \mathbf{v}$ dans $(L^{N_*}(\Omega))^N$ lorsque $n \to \infty$ (soit $(m_n)_{n\geq 1}$ est bornée, auquel cas il existe une suite extraite de $(m_n)_{n\geq 1}$ qui soit constante, soit $(m_n)_{n\geq 1}$ n'est pas bornée et on peut en extraire une suite qui tend vers l'infini). Puisque T_n est 1-lipschitzienne pour tout $n \geq 1$, on a alors $||\,|T_n(\mathbf{v}_{m_n}) - T_n(\mathbf{v})|\,||_{L^{N_*}(\Omega)} \leq ||\,|\mathbf{v}_{m_n} - \mathbf{v}|\,||_{L^{N_*}(\Omega)}$. Ainsi,

$$\begin{aligned}
&||\,|T_n(\mathbf{v}) - \mathbf{v}|\,||_{L^{N_*}(\Omega)} \\
&\quad \geq \quad ||\,|T_n(\mathbf{v}_{m_n}) - \mathbf{v}_{m_n}|\,||_{L^{N_*}(\Omega)} - (||\,|T_n(\mathbf{v}) - T_n(\mathbf{v}_{m_n})|\,||_{L^{N_*}(\Omega)} + ||\,|\mathbf{v}_{m_n} - \mathbf{v}|\,||_{L^{N_*}(\Omega)}) \\
&\quad \geq \quad \eta - 2||\,|\mathbf{v}_{m_n} - \mathbf{v}|\,||_{L^{N_*}(\Omega)} \to \eta > 0 \quad \text{lorsque } n \to \infty.
\end{aligned}$$

C'est une contradiction, car $T_n(\mathbf{v}) \to \mathbf{v}$ sur Ω en étant dominée par $|\mathbf{v}| \in L^{N_*}(\Omega)$, donc $T_n(\mathbf{v}) \to \mathbf{v}$ dans $(L^{N_*}(\Omega))^N$. ■

2.4.3 Convergence des solutions de (1.4)

Proposition 2.1 *Soit* $q < N/(N-1)$ *et* $L \in (W^{1,q}_{\Gamma_d}(\Omega))'$. *Sous les hypothèses (1.8), (2.45), (2.46), (2.47), (2.48), (2.50) et (1.42), en notant* v_m *la solution de (1.4) avec* $(A_m, \mathbf{v}_m, b_m, \lambda_m)$ *à la place de* $(A, \mathbf{v}, b, \lambda)$ *et* v_∞ *la solution de (1.4) avec* $(A_\infty, \mathbf{v}_\infty, b_\infty, \lambda_\infty)$ *à la place de* $(A, \mathbf{v}, b, \lambda)$, *on a* $v_m \to v_\infty$ *fortement dans* $H^1(\Omega)$ *et dans* $\mathcal{C}(\overline{\Omega})$.

Remarque 2.7 *Pour montrer la convergence forte de* $(v_m)_{m\geq 1}$ *vers* v *dans* $H^1(\Omega)$, *on a juste besoin de* $L \in (H^1_{\Gamma_d}(\Omega))'$ *et l'hypothèse (1.42) est inutile.*

Preuve de la proposition 2.1
Comme $\mathbf{v}_m \to \mathbf{v}_\infty$ fortement dans $(L^{N_*}(\Omega))^N$, pour tout $\eta > 0$, il existe Λ tel que, pour tout $m \geq 0$, $\mathbf{v}_m \in B(N_*, \eta) + B(\infty, \Lambda)$ (lemme 2.3); ainsi, le théorème 1.1 et le corollaire 1.2 donnent $\kappa > 0$ et $C > 0$ tels que, pour tout $m \geq 0$, $v_m \in H^1_{\Gamma_d}(\Omega) \cap C^{0,\kappa}(\Omega)$ et $||v_m||_{H^1(\Omega)} + ||v_m||_{\mathcal{C}^{0,\kappa}(\Omega)} \leq C$.

Etape 1: on montre que $v_m \to v_\infty$ faiblement dans $H^1(\Omega)$.
$(v_m)_{m\geq 0}$ est une suite bornée de $H^1_{\Gamma_d}(\Omega)$ et converge donc, quitte à extraire une suite, vers w faiblement dans $H^1_{\Gamma_d}(\Omega)$ et fortement dans $L^s(\Omega)$ pour tout $s < 2N/(N-2)$ (pour tout $s < \infty$ si $N = 2$); de plus, la trace de v_m converge fortement dans $L^s(\partial\Omega)$ vers la trace de w, pour tout $s < 2(N-1)/(N-2)$ (pour tout $s < \infty$ si $N = 2$). Pour tout $\varphi \in H^1_{\Gamma_d}(\Omega)$, on a alors

- $A_m\nabla\varphi \to A_\infty\nabla\varphi$ fortement dans $(L^2(\Omega))^N$ et $\nabla v_m \to \nabla w$ faiblement dans $(L^2(\Omega))^N$, donc
$$\int_\Omega A_m^T \nabla v_m \cdot \nabla\varphi \to \int_\Omega A_\infty^T \nabla w \cdot \nabla\varphi,$$
- $\mathbf{v}_m \to \mathbf{v}_\infty$ fortement dans $(L^{N_*}(\Omega))^N$, $\varphi \in L^{\frac{2N_*}{N_*-2}}(\Omega)$ et $\nabla v_m \to \nabla w$ faiblement dans $L^2(\Omega)$, donc puisque $\frac{1}{N_*} + \frac{N_*-2}{2N_*} + \frac{1}{2} = 1$,
$$\int_\Omega \varphi\mathbf{v}_m \cdot \nabla v_m \to \int_\Omega \varphi\mathbf{v}_\infty \cdot \nabla w,$$
- $b_m \to b_\infty$ faiblement dans $L^{\frac{\overline{r}}{2}}(\Omega)$, $\varphi \in L^{\frac{2\overline{r}}{\overline{r}-2}}(\Omega)$ et $v_m \to w$ fortement dans $L^{\frac{2\overline{r}}{\overline{r}-2}}(\Omega)$ (car $2\overline{r}/(\overline{r}-2) < 2N/(N-2)$), donc puisque $\frac{2}{\overline{r}} + \frac{\overline{r}-2}{2\overline{r}} + \frac{\overline{r}-2}{2\overline{r}} = 1$,
$$\int_\Omega b_m v_m \varphi \to \int_\Omega b_\infty w\varphi,$$
- $\lambda_m \to \lambda_\infty$ faiblement dans $L^{\overline{r}-1}(\partial\Omega)$, $\varphi \in L^{\frac{2(\overline{r}-1)}{\overline{r}-2}}(\partial\Omega)$ et $v_m \to w$ fortement dans $L^{\frac{2(\overline{r}-1)}{\overline{r}-2}}(\partial\Omega)$ (car $2(\overline{r}-1)/(\overline{r}-2) < 2(N-1)/(N-2)$), donc puisque $\frac{1}{\overline{r}-1} + \frac{\overline{r}-2}{2(\overline{r}-1)} + \frac{\overline{r}-2}{2(\overline{r}-1)} = 1$,
$$\int_{\Gamma_f} \lambda_m v_m \varphi \, d\sigma \to \int_{\Gamma_f} \lambda_\infty w\varphi \, d\sigma.$$

En passant donc à la limite $m \to \infty$ dans l'équation vérifiée par v_m, on constate que w est la solution de (1.4) avec $(A_\infty, \mathbf{v}_\infty, b_\infty, \lambda_\infty)$ à la place de $(A, \mathbf{v}, b, \lambda)$, c'est à dire que $w = v_\infty$; la seule limite faible dans $H^1(\Omega)$ d'une sous-suite de $(v_m)_{m\geq 0}$ étant (par le raisonnement précédent) v_∞, on en déduit que la suite tout entière $(v_m)_{m\geq 0}$ converge vers v_∞ faiblement dans $H^1(\Omega)$, et donc fortement dans $L^s(\Omega)$ pour tout $s < 2N/(N-2)$; on obtient aussi la convergence forte dans $L^s(\partial\Omega)$ pour tout $s < 2(N-1)/(N-2)$ des traces de $(v_m)_{m\geq 0}$ vers la trace de v_∞.

Etape 2: on montre que $v_m \to v_\infty$ fortement dans $H^1(\Omega)$.

En soustrayant l'équation satisfaite par v_∞ de l'équation satisfaite par v_m, on a, pour tout $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\int_\Omega A_m^T \nabla(v_m - v_\infty)\cdot\nabla\varphi + \int_\Omega (A_m^T - A_\infty^T)\nabla v_\infty\cdot\nabla\varphi + \int_\Omega \varphi\mathbf{v}_m\cdot\nabla(v_m - v_\infty)$$
$$+\int_\Omega \varphi(\mathbf{v}_m - \mathbf{v}_\infty)\cdot\nabla v_\infty + \int_\Omega b_m\varphi(v_m - v_\infty) + \int_\Omega (b_m - b_\infty)\varphi v_\infty + \int_{\Gamma_f}\lambda_m\varphi(v_m - v_\infty)\,d\sigma$$
$$+\int_{\Gamma_f}(\lambda_m - \lambda_\infty)\varphi v_\infty\,d\sigma = 0.$$

Soit $\chi \in]0, \frac{\mathcal{K}(2,\mathcal{B})}{C_S(\Omega,N_*)}[$. Comme il a été signalé au début de la preuve, il existe $\Lambda > 0$ tel que, pour tout $m \geq 0$, $\mathbf{v}_m \in B(N_*,\chi) + B(\infty,\Lambda)$; notons donc $\mathbf{v}_m = \mathbf{v}_m^{(1)} + \mathbf{v}_m^{(2)}$ avec $\mathbf{v}_m^{(1)} \in B(N_*,\chi)$ et $\mathbf{v}_m^{(2)} \in B(\infty,\Lambda)$. L'équation précédente peut s'écrire

$$\int_\Omega A_m^T\nabla(v_m - v_\infty)\cdot\nabla\varphi + \int_\Omega \varphi\mathbf{v}_m^{(1)}\cdot\nabla(v_m - v_\infty) + \int_\Omega b_m\varphi(v_m - v_\infty) + \int_{\Gamma_f}\lambda_m\varphi(v_m - v_\infty)\,d\sigma$$
$$= \int_\Omega (A_\infty^T - A_m^T)\nabla v_\infty\cdot\nabla\varphi - \int_\Omega \varphi\mathbf{v}_m^{(2)}\cdot\nabla(v_m - v_\infty) + \int_\Omega \varphi(\mathbf{v}_\infty - \mathbf{v}_m)\cdot\nabla v_\infty + \int_\Omega (b_\infty - b_m)\varphi v_\infty$$
$$+\int_{\Gamma_f}(\lambda_\infty - \lambda_m)\varphi v_\infty\,d\sigma.$$

En prenant $\varphi = v_m - v_\infty$ et en utilisant les propriétés de A_m, b_m, λ_m et $\mathbf{v}_m^{(1)}$, cela donne donc, par la remarque 1.2,

$$\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega,N_*))||v_m - v_\infty||^2_{H^1(\Omega)}\\
&\leq \quad ||\,|(A_m - A_\infty)\nabla v_\infty|\,||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)} + \Lambda||v_m - v_\infty||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)}\\
&\quad +||v_m - v_\infty||_{L^{\frac{2N_*}{N_*-2}}(\Omega)}||\,|\mathbf{v}_m - \mathbf{v}_\infty|\,||_{L^{N_*}(\Omega)}||\,|\nabla v_\infty|\,||_{L^2(\Omega)}\\
&\quad +\left|\int_\Omega (b_m - b_\infty)(v_m - v_\infty)v_\infty\right| + \left|\int_{\Gamma_f}(\lambda_m - \lambda_\infty)(v_m - v_\infty)v_\infty\,d\sigma\right|.
\end{aligned} \tag{2.52}$$

Or

- $||\,|(A_m - A_\infty)\nabla v_\infty|\,||_{L^2(\Omega)} \to 0$ ($A_m - A_\infty \to 0$ presque partout en étant bornée dans $L^\infty(\Omega; M_N(\mathbb{R}))$) et $(v_m)_{m\geq 0}$ est bornée dans $H^1(\Omega)$ donc $||\,|(A_m - A_\infty)\nabla v_\infty|\,||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)} \to 0$ lorsque $m \to \infty$,

- $||v_m - v_\infty||_{L^2(\Omega)} \to 0$ lorsque $n \to \infty$ (cf étape 1: on a $2 < 2N/(N-2)$) et $(v_m)_{m\geq 0}$ est bornée dans $H^1(\Omega)$ donc $\Lambda||v_m - v_\infty||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)} \to 0$ lorsque $m \to \infty$,

- $(v_m)_{m\geq 0}$ est bornée dans $L^{\frac{2N_*}{N_*-2}}(\Omega)$ (elle est bornée dans $H^1(\Omega)$) et $||\,|\mathbf{v}_m - \mathbf{v}_\infty|\,||_{L^{N_*}(\Omega)} \to 0$, donc $||v_m - v_\infty||_{L^{\frac{2N_*}{N_*-2}}(\Omega)}||\,|\mathbf{v}_m - \mathbf{v}_\infty|\,||_{L^{N_*}(\Omega)}||\,|\nabla v_\infty|\,||_{L^2(\Omega)} \to 0$ lorsque $m \to \infty$,

- $b_m - b_\infty \to 0$ faiblement dans $L^{\frac{\overline{r}}{2}}(\Omega)$, $v_m - v_\infty \to 0$ fortement dans $L^{\frac{2\overline{r}}{\overline{r}-2}}(\Omega)$ (cf étape 1: $2\overline{r}/(\overline{r}-2) < 2N/(N-2)$) et $v_\infty \in L^{\frac{2\overline{r}}{\overline{r}-2}}(\Omega)$, donc $\int_\Omega (b_m - b_\infty)(v_m - v_\infty)v_\infty \to 0$ lorsque $m \to \infty$,

- $\lambda_m - \lambda_\infty \to 0$ faiblement dans $L^{\overline{r}-1}(\partial\Omega)$, $v_m - v_\infty \to 0$ fortement dans $L^{\frac{2(\overline{r}-1)}{\overline{r}-2}}(\partial\Omega)$ (cf étape 1: on a $2(\overline{r}-1)/(\overline{r}-2) < 2(N-1)/(N-2)$) et $v_\infty \in L^{\frac{2(\overline{r}-1)}{\overline{r}-2}}(\partial\Omega)$, donc $\int_{\Gamma_f}(\lambda_m - \lambda_\infty)(v_m - v_\infty)v_\infty\,d\sigma \to 0$ lorsque $m \to \infty$.

Le terme de droite de (2.52) tend donc vers 0 lorsque $m \to \infty$, ce qui prouve que $v_m \to v_\infty$ dans $H^1(\Omega)$.

Etape 3: on montre que $v_m \to v_\infty$ dans $\mathcal{C}(\overline{\Omega})$.
$(v_m)_{m\geq 0}$ est bornée dans $\mathcal{C}^{0,\kappa}(\Omega)$, et donc relativement compacte dans $\mathcal{C}(\overline{\Omega})$; mais, la convergence dans $\mathcal{C}(\overline{\Omega})$ entrainant la convergence dans $L^2(\Omega)$, et puisque $v_m \to v_\infty$ dans $L^2(\Omega)$, la seule limite possible dans $\mathcal{C}(\overline{\Omega})$ des suites extraites de $(v_m)_{m\geq 0}$ est v_∞. On a donc bien $v_m \to v_\infty$ dans $\mathcal{C}(\overline{\Omega})$. ■

2.4.4 Preuve du théorème de stabilité

Lemme 2.4 *Sous les hypothèses (1.8), (2.45), (2.46), (2.47), (2.48), (2.50) et (1.42), si* $(\mu_m)_{m\geq 0} \in \mathcal{M}(\overline{\Omega})$ *converge vers* μ_∞ *dans* $\mathcal{M}(\overline{\Omega})$ *faible-∗ et* $(L_m)_{m\geq 0} \in (H^1(\Omega))'$ *converge vers* L_∞ *dans* $(H^1(\Omega))'$ *faible-∗, en notant* f_m *la solution de (1.46) avec* $(A_m, \mathbf{v}_m, b_m, \lambda_m, \mu_m + L_m)$ *à la place de* $(A, \mathbf{v}, b, \lambda, \zeta)$ *et* f_∞ *la solution de (1.46) avec* $(A_\infty, \mathbf{v}_\infty, b_\infty, \lambda_\infty, \mu_\infty + L_\infty)$ *à la place de* $(A, \mathbf{v}, b, \lambda, \zeta)$*, on a* $f_m \to f_\infty$ *faiblement dans* $W^{1,q}_{\Gamma_d}(\Omega)$ *pour tout* $q < N/(N-1)$ *et fortement dans* $L^q(\Omega)$ *pour tout* $q < N/(N-2)$.

Preuve du lemme 2.4
Soit, pour $m \in \mathbb{N} \cup \{\infty\}$, $\mathcal{T}_q^{(m)}$ l'application qui, à $L \in (W^{1,q}_{\Gamma_d}(\Omega))'$, associe la solution $\mathcal{T}_q^{(m)}(L) \in H^1_{\Gamma_d}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ de (1.4) avec $(A_m, \mathbf{v}_m, b_m, \lambda_m)$ à la place de $(A, \mathbf{v}, b, \lambda)$.
Soit $q < N/(N-1)$ et $L \in (W^{1,q}_{\Gamma_d}(\Omega))'$; par définition, pour $m \in \mathbb{N} \cup \{\infty\}$, $f_m = (\mathcal{T}_q^{(m)})^*(\zeta_m)$ avec $\zeta_m = \mu_m + L_m$, donc

$$\begin{aligned}
&\langle L, f_m - f_\infty \rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)} \\
&= \langle \zeta_m, \mathcal{T}_q^{(m)}(L) \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} - \langle \zeta_\infty, \mathcal{T}_q^{(\infty)}(L) \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \\
&= \langle \mu_m, \mathcal{T}_q^{(m)}(L) \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} - \langle \mu_\infty, \mathcal{T}_q^{(\infty)}(L) \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \qquad (2.53) \\
&\quad + \langle L_m, \mathcal{T}_q^{(m)}(L) \rangle_{(H^1(\Omega))', H^1(\Omega)} - \langle L_\infty, \mathcal{T}_q^{(\infty)}(L) \rangle_{(H^1(\Omega))', H^1(\Omega)}. \qquad (2.54)
\end{aligned}$$

Or, par la proposition 2.1, $\mathcal{T}_q^{(m)}(L) \to \mathcal{T}_q^{(\infty)}(L)$ dans $H^1(\Omega)$ et dans $\mathcal{C}(\overline{\Omega})$; donc, puisque $\mu_m \to \mu$ dans $\mathcal{M}(\overline{\Omega})$ faible-∗ et $L_m \to L$ dans $(H^1(\Omega))'$ faible-∗, les termes (2.53) et (2.54) tendent vers 0 lorsque $m \to \infty$.
On a donc bien $f_m \to f_\infty$ faiblement dans $W^{1,q}_{\Gamma_d}(\Omega)$ et, la convergence faible dans $W^{1,q}(\Omega)$ impliquant la convergence forte dans $L^r(\Omega)$ pour tout $r < Nq/(N-q)$, cela conclut la preuve de ce lemme. ■

Lemme 2.5 *Sous les hypothèses (1.8), (1.9), (1.12), (1.14), (1.39) et (1.42), si* $\zeta = \mu + L$ *avec* $\mu \in \mathcal{M}(\overline{\Omega})$ *et* $L \in (H^1(\Omega))'$*,* f *est la solution de (1.46) et* M *vérifie* $||\mathbf{v}||_{(L^{N_*}(\Omega))^N} + ||\mu||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'} \leq M$*, alors il existe* C *ne dépendant que de* $(N_*, \mathcal{B}, M)$ *tel que, pour tout* $k \in \mathbb{R}^+$*,* $T_k(f) \in H^1_{\Gamma_d}(\Omega)$ *et* $||T_k(f)||_{H^1(\Omega)} \leq C(k+1)$.

Preuve du lemme 2.5
Soit $(\mu_j)_{j\geq 1} \in (H^1(\Omega))' \cap \mathcal{M}(\overline{\Omega})$ tel que $\mu_j \to \mu$ dans $\mathcal{M}(\overline{\Omega})$ faible-∗ et, pour tout $j \geq 1$, $||\mu_j||_{\mathcal{M}(\overline{\Omega})} \leq ||\mu||_{\mathcal{M}(\overline{\Omega})}$. Notons $f^{(j)}$ la solution de (1.46) avec $\mu_j + L \in (H^1(\Omega))'$ à la place de ζ; on sait que $f^{(j)}$ est en fait la solution de (1.3) avec $\mu_j + L$ à la place de L (cf 2.3.1).
Ainsi, $T_k(f^{(j)}) \in H^1_{\Gamma_d}(\Omega)$ et, en utilisant cette fonction dans l'équation satisfaite par $f^{(j)}$, on a

$$\begin{aligned}
&\int_\Omega A\nabla(T_k(f^{(j)})) \cdot \nabla(T_k(f^{(j)})) + \int_\Omega f^{(j)}\mathbf{v} \cdot \nabla(T_k(f^{(j)})) + \int_\Omega b f^{(j)} T_k(f^{(j)}) + \int_{\Gamma_f} \lambda f^{(j)} T_k(f^{(j)})\, d\sigma \\
&\quad = \langle \mu_j + L, T_k(f^{(j)}) \rangle_{(H^1(\Omega))', H^1(\Omega)}.
\end{aligned}$$

Mais $f^{(j)}T_k(f^{(j)}) \geq (T_k(f^{(j)}))^2$ et, b et λ étant positives et supérieures, respectivement, à b_0 sur E et à λ_0 sur S, on a $\int_\Omega b f^{(j)} T_k(f^{(j)}) \geq b_0 \int_E (T_k(f^{(j)}))^2$ et $\int_{\Gamma_f} \lambda f^{(j)} T_k(f^{(j)})\, d\sigma \geq \lambda_0 \int_S (T_k(f^{(j)}))^2\, d\sigma$; de plus,

$|f^{(j)}| \leq k$ là où $\nabla(T_k(f^{(j)})) \neq 0$ donc on obtient, au vu du lemme 2.1,

$$\begin{aligned}
&\alpha_A|| \, |\nabla(T_k(f^{(j)}))| \, ||^2_{L^2(\Omega)} + b_0 \int_E (T_k(f^{(j)}))^2 + \lambda_0 \int_S (T_k(f^{(j)}))^2 \\
\leq \quad & k|| \, |\mathbf{v}| \, ||_{L^2(\Omega)} || \, |\nabla(T_k(f^{(j)}))| \, ||_{L^2(\Omega)} + k||\mu||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'} ||T_k(f^{(j)})||_{H^1(\Omega)} \\
\leq \quad & \frac{k^2}{\mathcal{K}(2,\mathcal{B})} |\Omega|^{1-\frac{2}{N_*}} ||\mathbf{v}||^2_{(L^{N_*}(\Omega))^N} + \frac{\mathcal{K}(2,\mathcal{B})}{4} ||T_k(f^{(j)})||^2_{H^1(\Omega)} + k||\mu||_{\mathcal{M}(\overline{\Omega})} \\
& + \frac{1}{\mathcal{K}(2,\mathcal{B})} ||L||^2_{(H^1(\Omega))'} + \frac{\mathcal{K}(2,\mathcal{B})}{4} ||T_k(f^{(j)})||^2_{H^1(\Omega)}.
\end{aligned}$$

Par la remarque 1.1, on a finalement

$$\frac{\mathcal{K}(2,\mathcal{B})}{2} ||T_k(f^{(j)})||^2_{H^1(\Omega)} \leq \frac{k^2}{\mathcal{K}(2,\mathcal{B})} |\Omega|^{1-\frac{2}{N_*}} M^2 + kM + \frac{1}{\mathcal{K}(2,\mathcal{B})} M^2 \leq \frac{\mathcal{K}(2,\mathcal{B})}{2} C^2 (k+1)^2$$

où C ne dépend que de $(N_*, \mathcal{B}, M)$. On a donc prouvé que $(T_k(f^{(j)}))_{j\geq 1}$ est bornée dans $H^1_{\Gamma_d}(\Omega)$ par $C(k+1)$. Mais, par le lemme 2.4 (en fait une version simplifiée de ce lemme) $f^{(j)} \to f$ fortement dans $L^1(\Omega)$, donc la suite $(T_k(f^{(j)}))_{j\geq 1}$ bornée par $C(k+1)$ dans $H^1_{\Gamma_d}(\Omega)$ converge faiblement dans cet espace vers $T_k(f)$ et on a bien $T_k(f) \in H^1_{\Gamma_d}(\Omega)$ avec $||T_k(f)||_{H^1(\Omega)} \leq C(k+1)$. ■

Lemme 2.6 *Sous les hypothèses (1.8), (1.9), (1.12), (1.14), (1.39) et (1.42), si* $\zeta = \mu + L$ *avec* $\mu \in \mathcal{M}(\overline{\Omega})$ *et* $L \in (H^1(\Omega))'$, f *est la solution de (1.46) et* M *vérifie*

$$||\mu||_{\mathcal{M}(\overline{\Omega})} + ||b||_{L^{\frac{\overline{r}}{2}}(\Omega)} + ||\lambda||_{L^{\overline{r}-1}(\partial\Omega)} + ||f||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)} + ||f||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)} \leq M,$$

alors, pour tout $k \in \mathbb{R}^+$, *tout* $\delta \in]0,1[$ *et tout* $\psi \in H^1_{\Gamma_d}(\Omega)$,

$$\begin{aligned}
&\int_\Omega A\nabla(T_\delta(T_{k+1}(f) - T_k(\psi))) \cdot \nabla(T_\delta(T_{k+1}(f) - T_k(\psi))) \\
\leq \quad & (2M^2 + M)\delta + \langle L, T_\delta(T_{k+1}(f) - T_k(\psi))\rangle_{(H^1(\Omega))', H^1(\Omega)} \\
& - \int_\Omega A\nabla(T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(f) - T_k(\psi))) - \int_\Omega T_{k+1}(f)\mathbf{v} \cdot \nabla(T_\delta(T_{k+1}(f) - T_k(\psi))).
\end{aligned}$$

Preuve du lemme 2.6

Soit $(\mu_j)_{j\geq 1} \in (H^1(\Omega))' \cap \mathcal{M}(\overline{\Omega})$ telle que $\mu_j \to \mu$ dans $\mathcal{M}(\overline{\Omega})$ faible-$*$ et, pour tout $j \geq 1$, $||\mu_j||_{\mathcal{M}(\overline{\Omega})} \leq ||\mu||_{\mathcal{M}(\overline{\Omega})}$.

Soit $f^{(j)}$ la solution de (1.46) avec $\mu_j + L$ à la place de ζ ($f^{(j)}$ est aussi la solution de (1.3) avec $\mu_j + L$ à la place de L); $f^{(j)}$ converge fortement vers f dans $L^1(\Omega)$ (lemme 2.4) et les suites $(T_{k+1}(f^{(j)}))_{j\geq 1}$ et $(T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi)))_{j\geq 1}$ sont bornées dans $H^1(\Omega)$ (lemme 2.5), donc $T_{k+1}(f^{(j)}) \to T_{k+1}(f)$ et $T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi)) \to T_\delta(T_{k+1}(f) - T_k(\psi))$ faiblement dans $H^1(\Omega)$. Quitte à extraire une suite, on peut aussi supposer que $f^{(j)} \to f$ presque partout sur Ω.

En utilisant $T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi)) \in H^1_{\Gamma_d}(\Omega)$ comme fonction test dans le problème variationnel satisfait par $f^{(j)}$, on a

$$\begin{aligned}
&\int_\Omega A\nabla f^{(j)} \cdot \nabla(T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))) \\
= \quad & -\int_\Omega f^{(j)}\mathbf{v} \cdot \nabla(T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))) - \int_\Omega b f^{(j)} T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi)) \\
& - \int_{\Gamma_f} \lambda f^{(j)} T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))\, d\sigma + \langle \mu_j, T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))\rangle_{(H^1(\Omega))', H^1(\Omega)} \\
& + \langle L, T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))\rangle_{(H^1(\Omega))', H^1(\Omega)}.
\end{aligned}$$

Lorsque $\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\neq 0$, on a $|T_{k+1}(f^{(j)})-T_k(\psi)|\leq\delta<1$, donc $|T_{k+1}(f^{(j)})|<k+1$, c'est à dire $f^{(j)}=T_{k+1}(f^{(j)})$; on en déduit que

$$\begin{aligned}
&\int_\Omega A\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\\
&= \int_\Omega A\nabla(T_{k+1}(f^{(j)})-T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\\
&\leq -\int_\Omega A\nabla(T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))-\int_\Omega T_{k+1}(f^{(j)})\mathbf{v}\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\\
&\quad+\delta||b||_{L^{\frac{\overline{r}}{2}}(\Omega)}||f^{(j)}||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)}+\delta||\lambda||_{L^{\overline{r}-1}(\partial\Omega)}||f^{(j)}||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)}\\
&\quad+\delta||\mu||_{\mathcal{M}(\overline{\Omega})}+\langle L,T_\delta(T_{k+1}(f^{(j)})-T_k(\psi))\rangle_{(H^1(\Omega))',H^1(\Omega)}. \qquad (2.55)
\end{aligned}$$

Mais

- $T_\delta(T_{k+1}(f^{(j)})-T_k(\psi))\to T_\delta(T_{k+1}(f)-T_k(\psi))$ faiblement dans $H^1(\Omega)$, donc, par le lemme 2.2,

$$\begin{aligned}
&\int_\Omega A\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))\\
&\leq \liminf_{j\to\infty}\int_\Omega A\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))
\end{aligned}$$

- $T_\delta(T_{k+1}(f^{(j)})-T_k(\psi))\to T_\delta(T_{k+1}(f)-T_k(\psi))$ faiblement dans $H^1(\Omega)$ et $A\nabla(T_k(\psi))\in(L^2(\Omega))^N$ donc

$$\int_\Omega A\nabla(T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\to\int_\Omega A\nabla(T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi))),$$

- $T_{k+1}(f^{(j)})\mathbf{v}\to T_{k+1}(f)\mathbf{v}$ presque partout sur Ω en étant majorée par $(k+1)|\mathbf{v}|\in L^2(\Omega)$, donc la convergence a aussi lieu dans $(L^2(\Omega))^N$ et, puisque $\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\to\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))$ faiblement dans $(L^2(\Omega))^N$, on a

$$\int_\Omega T_{k+1}(f^{(j)})\mathbf{v}\cdot\nabla(T_\delta(T_{k+1}(f^{(j)})-T_k(\psi)))\to\int_\Omega T_{k+1}(f)\mathbf{v}\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi))),$$

- $f^{(j)}\to f$ dans $L^q(\Omega)$ pour tout $q<N/(N-2)$ (lemme 2.4), donc $||f^{(j)}||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)}\to||f||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)}$ (car $\overline{r}/(\overline{r}-2)<N/(N-2)$),
- $f^{(j)}$ tendant vers f faiblement dans $W^{1,q}(\Omega)$ pour tout $q<N/(N-1)$, la trace de $f^{(j)}$ converge vers la trace de f fortement dans $L^s(\partial\Omega)$ pour tout $s<(N-1)/(N-2)$; ainsi, $||f^{(j)}||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)}\to||f||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)}$ (car $(\overline{r}-1)/(\overline{r}-2)<(N-1)/(N-2)$),
- $T_\delta(T_{k+1}(f^{(j)})-T_k(\psi))\to T_\delta(T_{k+1}(f)-T_k(\psi))$ faiblement dans $H^1(\Omega)$ donc $\langle L,T_\delta(T_{k+1}(f^{(j)})-T_k(\psi))\rangle_{(H^1(\Omega))',H^1(\Omega)}\to\langle L,T_\delta(T_{k+1}(f)-T_k(\psi))\rangle_{(H^1(\Omega))',H^1(\Omega)}$.

En prenant la $\liminf$ de (2.55), on déduit de ces convergences que

$$\begin{aligned}
&\int_\Omega A\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))\\
&\leq -\int_\Omega A\nabla(T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi))-\int_\Omega T_{k+1}(f)\mathbf{v}\cdot\nabla(T_\delta(T_{k+1}(f)-T_k(\psi)))\\
&\quad+\delta||b||_{L^{\frac{\overline{r}}{2}}(\Omega)}||f||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)}+\delta||\lambda||_{L^{\overline{r}-1}(\partial\Omega)}||f||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)}\\
&\quad+\delta||\mu||_{\mathcal{M}(\overline{\Omega})}+\langle L,T_\delta(T_{k+1}(f)-T_k(\psi))\rangle_{(H^1(\Omega))',H^1(\Omega)},
\end{aligned}$$

c'est à dire le résultat du lemme par hypothèse sur M. ∎

Preuve du théorème 2.4
On sait déjà que $f_m \to f_\infty$ faiblement dans $W^{1,q}_{\Gamma_d}(\Omega)$ et fortement dans $L^q(\Omega)$ pour tout $q < N/(N-1)$. On en déduit que $(f_m)_{m\geq 0}$ est bornée dans $W^{1,q}_{\Gamma_d}(\Omega)$ pour tout $q < N/(N-1)$. Nous allons prouver que, à une sous-suite près, $\nabla f_m \to \nabla f_\infty$ en mesure sur Ω; une fois ce résultat établi, nous aurons alors, à une sous-suite près, $\nabla f_m \to \nabla f_\infty$ presque partout sur Ω en étant bornée dans $(L^q(\Omega))^N$ pour tout $q < N/(N-1)$ et, par le lemme de compacité $L^p - L^q$, $\nabla f_m \to \nabla f_\infty$ dans $(L^q(\Omega))^N$ pour tout $q < N/(N-1)$; puisque la seule limite possible des sous-suites de $(\nabla f_m)_{m\geq 0}$ dans $L^q(\Omega)$ est ∇f_∞, ce raisonnement nous donne finalement la convergence de toute la suite $(\nabla f_m)_{m\geq 0}$ vers ∇f_∞ dans $(L^q(\Omega))^N$ (pour tout $q < N/(N-1)$).
Il faut donc prouver que, à une sous-suite près, pour tout $\eta > 0$, $|\{|\nabla f_m - \nabla f_\infty| > \eta\}| \to 0$ lorsque $m \to \infty$.

Quitte à extraire une suite, on peut supposer que $f_m \to f_\infty$ presque partout sur Ω (c'est la seule extraction de suite que nous effectuerons).
On a

$$\{|\nabla f_m - \nabla f_\infty| > \eta\} \subset \{|f_\infty| > k\} \cup \{|f_m - f_\infty| > \delta\} \cup E_{k,m,\delta}, \tag{2.56}$$

avec $\delta \in]0,1[$ et $E_{k,m,\delta} = \{|\nabla f_m - \nabla f_\infty| > \eta\} \cap \{|f_\infty| \leq k\} \cap \{|f_m - f_\infty| \leq \delta\}$.
Soit $\varepsilon > 0$ et fixons $k \in \mathbb{R}^+$ tel que $|\{|f_\infty| > k\}| \leq \varepsilon$.
Nous allons montrer que l'on peut choisir $\delta > 0$ et $m_1 > 0$ tel que, pour tout $m \geq m_1$, $|E_{k,m,\delta}| < \varepsilon$; une fois ceci effectué, pour δ ainsi fixé, on pourra prendre $m_2 \geq m_1$ tel que, pour tout $m \geq m_2$, $|\{|f_m - f_\infty| > \delta\}| < \varepsilon$ ($f_m \to f_\infty$ dans $L^1(\Omega)$, donc en mesure). On aura alors trouvé m_2 tel que, pour tout $m \geq m_2$, $|\{|\nabla f_m - \nabla f_\infty| > \eta\}| < 3\varepsilon$, ce qui concluera cette démonstration.

Puisque $\nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) = \nabla f_m - \nabla f_\infty$ sur $E_{k,m,\delta}$, on a

$$\alpha_A \eta^2 |E_{k,m,\delta}| \leq \int_\Omega A_m \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))).$$

$(\mu_m)_{m\geq 0}$ étant bornée dans $\mathcal{M}(\overline{\Omega})$, $(b_m)_{m\geq 0}$ étant bornée dans $L^{\frac{\overline{r}}{2}}(\Omega)$, $(\lambda_m)_{m\geq 0}$ étant bornée dans $L^{\overline{r}-1}(\Omega)$ et $(f_m)_{m\geq 0}$ étant bornée dans $W^{1,q}(\Omega)$ pour tout $q < N/(N-1)$ (ce qui implique que $(f_m)_{m\geq 0}$ est bornée dans $L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)$ et que la suite des traces de f_m est bornée dans $L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)$), on peut trouver $M > 0$ tel que, pour tout $m \geq 0$,

$$||\mu_m||_{\mathcal{M}(\overline{\Omega})} + ||b_m||_{L^{\frac{\overline{r}}{2}}(\Omega)} + ||\lambda_m||_{L^{\overline{r}-1}(\partial\Omega)} + ||f_m||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)} + ||f_m||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)} \leq M.$$

Par le lemme 2.6 appliqué à $(A_m, \mathbf{v}_m, b_m, \lambda_m, \zeta_m)$ au lieu de $(A, \mathbf{v}, b, \lambda, \zeta)$ et à $\psi = T_k(f_\infty) \in H^1_{\Gamma_d}(\Omega)$, on a alors, pour tout $m \geq 0$,

$$\begin{aligned}
\alpha_A \eta^2 |E_{k,m,\delta}| \;\leq\; & (2M^2 + M)\delta + \langle L_m, T_\delta(T_{k+1}(f_m) - T_k(f_\infty))\rangle_{(H^1(\Omega))', H^1(\Omega)} \\
& - \int_\Omega A_m \nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) \\
& - \int_\Omega T_{k+1}(f_m)\mathbf{v}_m \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))).
\end{aligned}$$

Prenons $\delta_0 = \alpha_A \eta^2 \varepsilon/(8M^2 + 4M)$; pour tout $\delta \leq \delta_0$, on a donc, $\forall m \geq 0$,

$$\begin{aligned}
|E_{k,m,\delta}| \;\leq\; & \frac{\varepsilon}{4} + \frac{1}{\alpha_A \eta^2} \left|\langle L_m, T_\delta(T_{k+1}(f_m) - T_k(f_\infty))\rangle_{(H^1(\Omega))', H^1(\Omega)}\right| \\
& + \frac{1}{\alpha_A \eta^2} \left| \int_\Omega A_m \nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))\right| \\
& + \frac{1}{\alpha_A \eta^2} \left| \int_\Omega T_{k+1}(f_m)\mathbf{v}_m \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))\right|.
\end{aligned} \tag{2.57}$$

Comme $L_m \to L_\infty$ fortement dans $(H^1(\Omega))'$ et $T_\delta(T_{k+1}(f_m) - T_k(f_\infty)) \to T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))$ faiblement dans $H^1(\Omega)$ ($(T_{k+1}(f_m))_{m\geq 0}$ est bornée dans $H^1(\Omega)$ par le lemme 2.5 et converge presque partout vers $T_{k+1}(f)$ sur Ω), on a

$$\begin{aligned}&\langle L_m, T_\delta(T_{k+1}(f_m) - T_k(f_\infty))\rangle_{(H^1(\Omega))',H^1(\Omega)}\\ &\quad\overset{m\to\infty}{\longrightarrow}\quad \langle L_\infty, T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))\rangle_{(H^1(\Omega))',H^1(\Omega)} = g(\delta).\end{aligned} \tag{2.58}$$

Comme $T_{k+1}(f_m) \to T_{k+1}(f)$ presque partout sur Ω en étant majoré par $k+1$ et $\mathbf{v}_m \to \mathbf{v}_\infty$ fortement dans $(L^2(\Omega))^N$, on a $T_{k+1}(f_m)\mathbf{v}_m \to T_{k+1}(f_\infty)\mathbf{v}_\infty$ dans $(L^2(\Omega))^N$; par convergence faible dans $H^1(\Omega)$ de $(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))_{m\geq 0}$ vers $T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))$, on en déduit que

$$\begin{aligned}&\int_\Omega T_{k+1}(f_m)\mathbf{v}_m \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))\\ &\quad\overset{m\to\infty}{\longrightarrow}\quad \int_\Omega T_{k+1}(f_\infty)\mathbf{v}_\infty \cdot \nabla(T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))) = h(\delta).\end{aligned} \tag{2.59}$$

Mais, pour tout $\psi \in H^1(\Omega)$, $T_\delta(\psi) \to 0$ dans $H^1(\Omega)$ lorsque $\delta \to 0$ (la convergence vers 0 dans $L^2(\Omega)$ est évidente puisque $|T_\delta(\psi)| \leq \delta$ sur Ω, et la convergence du gradient vient du fait que $\nabla(T_\delta(\psi)) = \mathbf{1}_{\{0<|\psi|<\delta\}}\nabla\psi$ et que $\mathbf{1}_{\{0<|\psi|<\delta\}} \to 0$ sur Ω en étant majorée par 1), donc $T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty)) \to 0$ dans $H^1(\Omega)$ lorsque $\delta \to 0$. Ainsi, $g(\delta) \to 0$ et $h(\delta) \to 0$ lorsque $\delta \to 0$. On fixe $\delta \in]0, \delta_0]$ tel que $|g(\delta)| \leq \alpha_A \eta^2 \varepsilon/8$ et $|h(\delta)| \leq \alpha_A\eta^2\varepsilon/8$.
Par (2.58), (2.59) et ce choix de δ, il existe donc M tel que, pour tout $m \geq M$,

$$\begin{gathered}\left|\langle L_m, T_\delta(T_{k+1}(f_m) - T_k(f_\infty))\rangle_{(H^1(\Omega))',H^1(\Omega)}\right| \leq \frac{\alpha_A\eta^2\varepsilon}{4}\\ \text{et}\\ \left|\int_\Omega T_{k+1}(f_m)\mathbf{v}_m \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))\right| \leq \frac{\alpha_A\eta^2\varepsilon}{4}.\end{gathered} \tag{2.60}$$

Comme $A_m\nabla(T_k(f_\infty)) \to A\nabla(T_k(f_\infty))$ dans $(L^2(\Omega))^N$ et $(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))_{m\geq 0}$ converge faiblement dans $H^1(\Omega)$ vers $T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))$, on a

$$\int_\Omega A_m\nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) \to \int_\Omega A\nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty))).$$

Mais

$$\begin{aligned}&A\nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_\infty) - T_k(f_\infty)))\\ &= A\nabla f_\infty \cdot \nabla(T_{k+1}(f_\infty) - T_k(f_\infty))\mathbf{1}_{\{|f_\infty|<k\}}\mathbf{1}_{\{|T_{k+1}(f_\infty)-T_k(f_\infty)|<\delta\}}\\ &= A\nabla f_\infty \cdot (\mathbf{1}_{\{|f_\infty|<k+1\}}\mathbf{1}_{\{|f_\infty|<k|\}}\nabla f_\infty - \mathbf{1}_{\{|f_\infty|<k\}}\mathbf{1}_{\{|f_\infty|<k\}}\nabla f_\infty)\\ &= A\nabla f_\infty \cdot (\mathbf{1}_{\{|f_\infty|<k\}}(\nabla f_\infty - \nabla f_\infty)) = 0\end{aligned}$$

et on a en fait

$$\int_\Omega A_m\nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) \to 0.$$

On peut donc trouver $M' > 0$ tel que, pour tout $m \geq M'$,

$$\left|\int_\Omega A_m\nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty)))\right| < \frac{\alpha_A\eta^2\varepsilon}{4}. \tag{2.61}$$

(2.57), (2.60) et (2.61) donnent donc, avec le choix précédement effectué de δ et pour tout $m \geq m_1 = \sup(M, M')$, $|E_{k,m,\delta}| \leq \varepsilon$, ce qui conclut la démonstration de ce théorème. ■

2.4.5 Résultat de stabilité pour la solution par dualité de (1.48)

La technique utilisée pour prouver le théorème 2.4 peut être employée pour obtenir le résultat suivant concernant la stabilité des solutions par dualité de (1.48).

Théorème 2.5 *Sous les hypothèses (1.8), (2.45), (2.47), (2.48), (2.49), (2.50), (1.42) et*

$$\begin{array}{c} \exists r > N \text{ tel que, } \forall m \geq 0\,,\ \mathbf{v}_m \in (L^r(\Omega))^N\,, \\ \mathbf{v}_m \to \mathbf{v}_\infty \text{ dans } (L^r(\Omega))^N\,, \end{array} \tag{2.62}$$

en notant f_m la solution par dualité de (1.48) avec $(A_m, \mathbf{v}_m, b_m, \lambda_m, \zeta_m)$ à la place de $(A, \mathbf{v}, b, \lambda, \zeta)$ et f_∞ la solution par dualité de (1.48) avec $(A_\infty, \mathbf{v}_\infty, b_\infty, \lambda_\infty, \zeta_\infty)$ à la place de $(A, \mathbf{v}, b, \lambda, \zeta)$, on a

$$f_m \overset{m\to\infty}{\longrightarrow} f_\infty \text{ fortement dans } W^{1,q}_{\Gamma_d}(\Omega) \text{ pour tout } q < \tfrac{N}{N-1}.$$

Voyons les adaptations à faire (issues du fait qu'il faut changer les termes $\int_\Omega \varphi \mathbf{v}_m \cdot \nabla v_m$ et $\int_\Omega f^{(j)} \mathbf{v} \cdot \nabla \varphi$ qui apparaissaient dans les preuves précédentes en $\int_\Omega v_m \mathbf{v}_m \cdot \nabla \varphi$ et $\int_\Omega \varphi \mathbf{v} \cdot \nabla f^{(j)}$) pour prouver le théorème 2.5.

Dans toutes les preuves, le changement de $(A_m)_{m\in\mathbb{N}\cup\{\infty\}}$ en $(A_m^T)_{m\in\mathbb{N}\cup\{\infty\}}$ n'apporte pas de difficulté.

Adaptation de la proposition 2.1

Enoncé: il faut remplacer l'hypothèse (2.46) par l'hypothèse (2.62), et "(1.4)" par "(1.3)".

Preuve: Dans l'étape 1, il faut prouver la convergence de $\int_\Omega v_m \mathbf{v}_m \cdot \nabla \varphi$ vers $\int_\Omega w \mathbf{v}_\infty \cdot \nabla \varphi$; mais $v_m \to w$ fortement dans $L^{\frac{2r}{r-2}}(\Omega)$ (car $2r/(r-2) < 2N/(N-2)$), $\mathbf{v}_m \to \mathbf{v}_\infty$ dans $(L^r(\Omega))^N$ et $\nabla \varphi \in (L^2(\Omega))^N$, donc puisque $\frac{r-2}{2r} + \frac{1}{r} + \frac{1}{2} = 1$, on a

$$\int_\Omega v_m \mathbf{v}_m \cdot \nabla \varphi \to \int_\Omega w \mathbf{v}_\infty \cdot \nabla \varphi.$$

Dans l'étape 2, après avoir soustrait l'équation satisfaite par v_∞ de l'équation satisfaite par v_m et pris $\varphi = v_m - v_\infty$ comme fonction test, on trouve (sans séparer $\mathbf{v}_m$ en deux parties),

$$\begin{aligned} &\mathcal{K}(2,\mathcal{B})||v_m - v_\infty||^2_{H^1(\Omega)} \\ &\quad\leq \ ||\,|(A_m^T - A_\infty^T)\nabla v_\infty|\,||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)} + \left|\int_\Omega (v_m \mathbf{v}_m - v_\infty \mathbf{v}_\infty)\cdot \nabla(v_m - v_\infty)\right| \\ &\qquad + \left|\int_\Omega (b_m - b_\infty)(v_m - v_\infty)v_\infty\right| + \left|\int_{\Gamma_f} (\lambda_m - \lambda_\infty)(v_m - v_\infty)v_\infty \, d\sigma\right|. \end{aligned}$$

Les termes $||\,|(A_m^T - A_\infty^T)\nabla v_\infty|\,||_{L^2(\Omega)}||\,|\nabla(v_m - v_\infty)|\,||_{L^2(\Omega)}$, $\int_\Omega (b_m - b_\infty)(v_m - v_\infty)v_\infty$ et $\int_{\Gamma_f}(\lambda_m - \lambda_\infty)(v_m - v_\infty)v_\infty \, d\sigma$ tendent vers 0 sans adaptation. Comme $v_m \to v_\infty$ fortement dans $L^{\frac{2r}{r-2}}(\Omega)$ (car $2r/(r-2) < 2N/(N-2)$), $\mathbf{v}_m \to \mathbf{v}$ fortement dans $(L^r(\Omega))^N$ et $\nabla(v_m - v) \to 0$ faiblement dans $(L^2(\Omega))^N$, le terme $\int_\Omega (v_m \mathbf{v}_m - v_\infty \mathbf{v}_\infty) \cdot \nabla(v_m - v_\infty)$ tend aussi vers 0.
Il n'y a rien à adapter dans l'étape 3.

Adaptation du lemme 2.4

Enoncé: il faut remplacer l'hypothèse (2.46) par l'hypothèse (2.62), et "(1.4)" par "(1.3)".

Preuve: pas d'adaptation à faire, le résultat découle directement de la proposition 2.1 adaptée.

Adaptation du lemme 2.5

Enoncé: il faut remplacer l'hypothèse (1.12) par l'hypothèse (1.34), f désigne la solution par dualité de (1.48), M doit vérifier

$$||\mathbf{v}||_{(L^r(\Omega))^N} + ||b||_{L^{\frac{\overline{r}}{2}}(\Omega)} + ||\lambda||_{L^{\overline{r}-1}(\partial\Omega)} + ||\mu||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'} \le M \tag{2.63}$$

et C dépend de $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, r, M)$.

Preuve: pour majorer le terme $\int_\Omega T_k(f^{(j)})\mathbf{v}\cdot\nabla f^{(j)}$, on écrit simplement

$$\left|\int_\Omega T_k(f^{(j)})\mathbf{v}\cdot\nabla f^{(j)}\right| \le k||\mathbf{v}||_{(L^r(\Omega))^N}||f^{(j)}||_{W^{1,r'}(\Omega)} \le kM||f^{(j)}||_{W^{1,r'}(\Omega)}. \tag{2.64}$$

Or $r' < N/(N-1)$ donc $f^{(j)} = (\mathcal{T}_{r'})^*(\mu_j + L)$, où $\mathcal{T}_{r'} : (W^{1,r'}_{\Gamma_d}(\Omega))' \to H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}(\overline{\Omega})$ désigne l'application qui, à $l \in (W^{1,r'}_{\Gamma_d}(\Omega))'$, associe la solution $\mathcal{T}_{r'}(l)$ de (1.3) avec l à la place de L. Le théorème 1.1 et le corollaire 1.1 donnent $C_0 > 0$ ne dépendant que de $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, r, M)$ tel que

$$||\mathcal{T}_{r'}||_{\mathcal{L}((W^{1,r'}_{\Gamma_d}(\Omega))';H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}(\overline{\Omega}))} \le C_0.$$

On a alors

$$||(\mathcal{T}_{r'})^*||_{\mathcal{L}((H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}(\overline{\Omega}))';W^{1,r'}_{\Gamma_d}(\Omega))} \le C_0.$$

Ainsi,

$$||f^{(j)}||_{W^{1,r'}(\Omega)} \le C_0||\mu_j + L||_{(H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}(\overline{\Omega}))'} \le C_0(||\mu_j||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'}) \le C_0 M,$$

et on a donc, dans (2.64),

$$\left|\int_\Omega T_k(f^{(j)})\mathbf{v}\cdot\nabla f^{(j)}\right| \le C_0 M^2 k,$$

ce qui suffit pour conclure comme dans la preuve du lemme 2.5 (et la borne obtenue sur $||T_k(f)||_{H^1(\Omega)}$ est en fait en $\sqrt{k+1}$).

Adaptation du lemme 2.6

Enoncé: il faut remplacer l'hypothèse (1.12) par l'hypothèse (1.34), f désigne la solution par dualité de (1.48) et M doit vérifier (2.63). La conclusion est modifiée en: il existe $C > 0$ ne dépendant que de $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, r, M)$ tel que, pour tout $k \in \mathbb{R}^+$, pour tout $\delta \in]0,1[$ et pour tout $\psi \in H^1_{\Gamma_d}(\Omega)$,

$$\begin{aligned}&\int_\Omega A^T\nabla(T_\delta(T_{k+1}(f) - T_k(\psi)))\cdot\nabla(T_\delta(T_{k+1}(f) - T_k(\psi)))\\ &\quad\le\ C\delta + \langle L, T_\delta(T_{k+1}(f) - T_k(\psi))\rangle_{(H^1(\Omega))',H^1(\Omega)} - \int_\Omega A^T\nabla(T_k(\psi))\cdot\nabla(T_\delta(T_{k+1}(f) - T_k(\psi))).\end{aligned}$$

Preuve: Puisque $r_0 = \inf(\overline{r}, r) > N$, donc $r_0' < N/(N-1)$, comme dans l'adaptation de la preuve du lemme 2.5, on va trouver C_0 ne dépendant que de $(N_*, \mathcal{B}, \Lambda_A, \overline{r}, r, M)$ tel que

$$||f^{(j)}||_{W^{1,r_0'}(\Omega)} \le C_0 M. \tag{2.65}$$

Dans l'équivalent de (2.55), on a, puisque $r_0' \ge r'$ et $r_0' \ge \overline{r}'$,

$$\begin{aligned}\left|\int_\Omega T_\delta(T_{k+1}(f^{(j)}) - T_k(\psi))\mathbf{v}\cdot\nabla f^{(j)}\right| &\le \delta||\mathbf{v}||_{(L^r(\Omega))^N}||f^{(j)}||_{W^{1,r'}(\Omega)} \le \delta M C_1||f||_{W^{1,r_0'}(\Omega)},\\ ||f||_{L^{\frac{\overline{r}}{\overline{r}-2}}(\Omega)} &\le C_2||f||_{W^{1,\overline{r}'}(\Omega)} \le C_3||f||_{W^{1,r_0'}(\Omega)},\\ ||f||_{L^{\frac{\overline{r}-1}{\overline{r}-2}}(\partial\Omega)} &\le C_4||f||_{W^{1,\overline{r}'}(\Omega)} \le C_5||f||_{W^{1,r_0'}(\Omega)},\end{aligned}$$

où $(C_1, C_2, C_3, C_4, C_5)$ ne dépendent que de $(\Omega, \overline{r}, r)$ (les injections de Sobolev sont correctes car, étant donné que $\overline{r} > N$, on a $\overline{r}/(\overline{r}-2) \leq N\overline{r}'/(N-\overline{r}')$ et $(\overline{r}-1)/(\overline{r}-2) \leq (N-1)\overline{r}'/(N-\overline{r}')$). Ces majorations associées à (2.65) donnent le résultat souhaité.

Adaptation de la preuve du théorème 2.4

Comme $(\mu_m)_{m\geq 0}$ est bornée dans $\mathcal{M}(\overline{\Omega})$, $(L_m)_{m\geq 0}$ est bornée dans $(H^1(\Omega))'$, $(\mathbf{v}_m)_{m\geq 0}$ est bornée dans $(L^r(\Omega))'$, $(b_m)_{m\geq 0}$ est bornée dans $L^{\frac{\overline{r}}{2}}(\Omega)$ et $(\lambda_m)_{m\geq 0}$ est bornée dans $L^{\overline{r}-1}(\Omega)$, on peut trouver M tel que, pour tout $m \geq 0$,

$$||\mathbf{v}_m||_{(L^r(\Omega))^N} + ||b_m||_{L^{\frac{\overline{r}}{2}}(\Omega)} + ||\lambda_m||_{L^{\overline{r}-1}(\partial\Omega)} + ||\mu_m||_{\mathcal{M}(\overline{\Omega})} + ||L_m||_{(H^1(\Omega))'} \leq M.$$

En appliquant alors l'adaptation du lemme 2.6, on obtient $C > 0$ tel que, pour tout $m \geq 0$,

$$\begin{aligned} \alpha_A \eta^2 |E_{k,m,\delta}| \quad &\leq \quad C\delta + |\langle L_m, T_\delta(T_{k+1}(f_m) - T_k(f_\infty))\rangle_{(H^1(\Omega))',H^1(\Omega)}| \\ &\quad + \left| \int_\Omega A^T \nabla(T_k(f_\infty)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f_\infty))) \right|, \end{aligned}$$

et on conclut comme dans la preuve du théorème 2.4 (en plus simple, même, puisque les termes (2.59) n'apparaissent pas ici).

Chapitre 3

Etude d'un Schéma Volumes Finis pour une Equation Elliptique non Coercitive

3.1 L'équation

Nous nous intéressons, dans cette partie, à un schéma de type volumes finis pour une équation de la forme (1.1).

Ω est ici un ouvert polygonal de $\mathbb{R}^N$, $N=2$ ou 3. Afin de simplifier l'étude, nous considérons uniquement le cas du Laplacien avec des conditions au bord de type Dirichlet homogènes (i.e. $A=Id$, $\Gamma_d=\partial\Omega$ et $\mathcal{U}_d=0$ dans (1.1)).
Le problème étudié est donc le suivant

$$\left\{\begin{array}{lll} -\Delta u+\operatorname{div}(\mathbf{v}u)+bu=f+\operatorname{div}(G) & \text{dans} & \Omega, \\ u=0 & \text{sur} & \partial\Omega, \end{array}\right. \tag{3.1}$$

et on effectue les hypothèses suivantes sur les données:

$$\mathbf{v}\in(\mathcal{C}(\overline{\Omega}))^N\,,\quad b\in L^\infty(\Omega) \text{ avec } b\geq 0 \text{ sur } \Omega,\quad f\in L^2(\Omega)\quad\text{et}\quad G\in(\mathcal{C}(\overline{\Omega}))^N.$$

La principale nouveauté du travail que nous présentons ici vient du fait que, contrairement au cas classique étudié dans [37], nous n'imposons aucune condition de signe sur la divergence de $\mathbf{v}$. Il faut aussi constater que $\mathbf{v}$ est seulement supposée continue, et non pas $\mathcal{C}^1$ comme c'est demandé usuellement.
Le terme "div(G)" n'apparait pas dans [37]; son introduction n'entraine pas de problème supplémentaire, mais elle est agréable car elle permet de déduire très simplement les estimations d'erreurs à partir des estimations a priori.

On pourrait affaiblir les hypothèses d'intégrabilité sur b et f, mais afin de ne pas compliquer le raisonnement, nous nous contenterons de celles-ci.

3.2 La discrétisation

La discrétisation de (3.1) par volumes finis demande l'introduction d'une notion dite de "maillage admissible" de Ω.

Définition 3.1 *Un maillage admissible de Ω, noté $\mathcal{M}$, est la donnée d'une famille finie $\mathcal{T}$ d'ouverts polygonaux convexes disjoints inclus dans Ω (les "mailles", ou "volumes de contrôle"), d'une famille finie $\mathcal{E}$ de sous-ensembles disjoints de $\overline{\Omega}$ contenus dans des hyperplans de $\mathbb{R}^N$ (les "interfaces") et d'une famille $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ de points de Ω tels que*

i) $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$,

ii) Tout $\sigma \in \mathcal{E}$ est un ouvert non vide du bord d'un élément K de $\mathcal{T}$,

iii) En notant $\mathcal{E}_K = \{\sigma \in \mathcal{E} \mid \sigma \subset \partial K\}$, $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$ pour tout $K \in \mathcal{T}$,

iv) Pour tous K et L distincts dans $\mathcal{T}$, soit la mesure $(N-1)$-dimensionnelle de $\overline{K}\cap\overline{L}$ est nulle, soit $\overline{K}\cap\overline{L} = \overline{\sigma}$ pour un $\sigma \in \mathcal{E}$, que l'on notera alors $\sigma = K|L$,

v) Pour tout $K \in \mathcal{T}$, $x_K \in K$,

vi) Pour tout $\sigma = K|L \in \mathcal{E}$, la droite (x_K, x_L) intersecte orthogonalement σ,

vii) Pour tout $\sigma \in \mathcal{E}$, $\sigma \subset \partial\Omega \cap \partial K$, la droite orthogonale à σ et passant par x_K rencontre σ.

La taille du maillage est alors définie par $h_{\mathcal{T}} = \sup_{K\in\mathcal{T}} \mathrm{diam}(K)$. On note $m(K)$ la mesure de Lebesgue de $K \in \mathcal{T}$. La normale à $\sigma \in \mathcal{E}_K$ extérieure à K sera notée $\mathbf{n}_{K,\sigma}$.
On pose $\mathcal{E}_{\rm int} = \{\sigma \in \mathcal{E} \mid \sigma \not\subset \partial\Omega\}$ l'ensemble des interfaces intérieures à Ω et $\mathcal{E}_{\rm ext} = \mathcal{E}\backslash\mathcal{E}_{\rm int}$. Lorsque $\sigma \in \mathcal{E}$, on note $m(\sigma)$ la mesure $(N-1)$-dimensionnelle de σ; si $\sigma = K|L \in \mathcal{E}_{\rm int}$, d_σ est la distance euclidienne entre les points (x_K, x_L) et $d_{K,\sigma}$ désigne la distance entre x_K et σ; lorsque $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$, $d_\sigma = d_{K,\sigma}$ est la distance entre x_K et σ. La transmissivité au travers d'une interface σ est $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$. γ désigne la mesure $(N-1)$-dimensionnelle sur les interfaces du maillage.

On peut maintenant définir la discrétisation par volumes finis de (3.1) sur un maillage admissible $\mathcal{M}$. En notant, pour $K \in \mathcal{T}$ et $\sigma \in \mathcal{E}_K$,

$$\begin{aligned} f_K &= \frac{1}{m(K)}\int_K f\,, \quad G_{K,\sigma} = \frac{1}{m(\sigma)}\int_\sigma G\cdot\mathbf{n}_{K,\sigma}\,d\gamma \\ b_K &= \frac{1}{m(K)}\int_K b\,, \quad \text{et} \quad v_{K,\sigma} = \int_\sigma \mathbf{v}\cdot\mathbf{n}_{K,\sigma}\,d\gamma\,, \end{aligned} \tag{3.2}$$

le schéma est défini par

$$\forall K \in \mathcal{T}\,,\ \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}u_{\sigma,+} + m(K)b_K u_K = m(K)f_K + \sum_{\sigma\in\mathcal{E}_K} m(\sigma)G_{K,\sigma}\,, \tag{3.3}$$

$$\begin{aligned} &\forall\sigma = K|L \in \mathcal{E}_{\rm int}\,, \quad F_{K,\sigma} = -\tau_\sigma(u_L - u_K)\,, \\ &\forall\sigma \in \mathcal{E}_{\rm ext}\cap\mathcal{E}_K\,, \quad F_{K,\sigma} = \tau_\sigma u_K\,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\forall\sigma = K|L \in \mathcal{E}_{\rm int}\,, \quad u_{\sigma,+} = u_K \text{ si } v_{K,\sigma} \geq 0\,, \quad u_{\sigma,+} = u_L \text{ dans le cas contraire,} \\ &\forall\sigma \in \mathcal{E}_{\rm ext}\cap\mathcal{E}_K\,, \quad u_{\sigma,+} = u_K \text{ si } v_{K,\sigma} \geq 0\,, \quad u_{\sigma,+} = 0 \text{ dans le cas contraire.} \end{aligned} \tag{3.5}$$

Les équations (3.3)—(3.5) forment un système linéaire de taille Card($\mathcal{T}$) en les inconnues $(u_K)_{K\in\mathcal{T}}$. On remarquera au passage que ce schéma, comme tout schéma de type volumes finis, conserve les flux: si $\sigma = K|L$, alors $F_{K,\sigma} = -F_{L,\sigma}$, $G_{K,\sigma} = -G_{L,\sigma}$ et $v_{K,\sigma} = -v_{L,\sigma}$.

Remarque 3.1 *Le choix du décentrement (3.5) (décentrement amont) est classique lorsque l'on impose la condition* $\mathrm{div}(\mathbf{v}) \geq 0$ *et permet d'éliminer les termes de convection dans les estimations; un tel décentrement est dans tous les cas nécessaire pour obtenir une stabilité inconditionnelle du schéma. Nous étudierons, dans la partie 3.6, d'autres choix pour le terme de convection, mais nous constaterons que, dans ce cas, la stabilité obtenue est en général conditionnelle (n'est valable que pour $h_{\mathcal{T}}$ assez petit).*

Remarque 3.2 *Il peut y avoir une petite indétermination dans le choix de $u_{\sigma,+}$. En effet, si $\sigma = K|L \in \mathcal{E}_{\rm int}$ et $v_{K,\sigma} = 0$, alors on a aussi $v_{L,\sigma} = 0$, de sorte qu'une lecture de (3.5) nous dit de prendre $u_{\sigma,+} = u_K$ et l'autre lecture nous dit de prendre $u_{\sigma,+} = u_L$. Cette indétermination n'est cependant pas grave car, $v_{K,\sigma}$ étant nul, le terme impliquant $u_{\sigma,+}$ n'apparait pas dans (3.3). Pour lever l'ambiguïté sur ces interfaces σ particulières (les interfaces pour lesquelles il y a indétermination ne dépendent que de $\mathbf{v}$ et du maillage, et non pas des inconnues $(u_K)_{K\in\mathcal{T}}$), on peut par exemple, dans ce cas-là, fixer arbitrairement un K tel que $\sigma \in \mathcal{E}_K$ et poser $u_{\sigma,+} = u_K$.*

3.3 Estimations a priori sur la solution approchée

Nous prouvons ici des estimations a priori, dans une norme adaptée au problème, sur les $(u_K)_{K\in\mathcal{T}}$ vérifiant (3.3)—(3.5).
Dans cette partie, il nous importe peu que les $(f_K)_{K\in\mathcal{T}}$ et $(G_{K,\sigma})_{K\in\mathcal{T},\,\sigma\in\mathcal{E}_K}$ soient issus de fonctions f et G au travers des définitions (3.2). C'est pourquoi nous prenons ici des seconds membres plus généraux, vérifiant seulement:

$$\begin{array}{l} \forall K\in\mathcal{T}\,,\ f_K\in\mathbb{R}\,,\\ \forall K\in\mathcal{T}\,,\ \forall\sigma\in\mathcal{E}_K\,,\ G_{K,\sigma}\in\mathbb{R}\text{ et, pour tout }\sigma=K|L\in\mathcal{E}_{\rm int},\ G_{K,\sigma}=-G_{L,\sigma}. \end{array} \quad (3.6)$$

On posera alors

$$\mathcal{N}(f_\mathcal{T},G_\mathcal{M}) = \left(\sum_{K\in\mathcal{T}} m(K)f_K^2\right)^{1/2} + \left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma G_\sigma^2\right)^{1/2}$$

où $G_\sigma = |G_{K,\sigma}|$ pour un $K\in\mathcal{T}$ tel que $\sigma\in\mathcal{E}_K$ (par hypothèse sur les $(G_{K,\sigma})_{K\in\mathcal{T},\,\sigma\in\mathcal{E}_K}$, la définition de G_σ ne dépend pas du $K\in\mathcal{T}$ choisi tel que $\sigma\in\mathcal{E}_K$).

La norme adaptée au problème, avec laquelle toutes les estimations sur les solutions de (3.3)—(3.5) seront obtenues, est la norme H_0^1 discrète suivante.

Définition 3.2 *Lorsque $\mathcal{M}$ est un maillage admissible sur Ω et $v_\mathcal{T} = (v_K)_{K\in\mathcal{T}}$, on pose*

$$||v_\mathcal{T}||_{1,\mathcal{M}} = \left(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma v_\mathcal{T})^2\right)^{1/2},$$

où $D_\sigma v_\mathcal{T} = |v_K - v_L|$ lorsque $\sigma = K|L \in \mathcal{E}_{\rm int}$ et $D_\sigma v_\mathcal{T} = |v_K|$ lorsque $\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}_K$.

En identifiant naturellement l'espace $X(\mathcal{M})$ des fonctions définies presque partout sur Ω et constantes sur chaque maille de $\mathcal{M}$ à l'espace $\prod_{K\in\mathcal{T}}\mathbb{R}$, on constate aisément que $||\cdot||_{1,\mathcal{M}}$ est une norme sur $X(\mathcal{M})$.

3.3.1 Lemmes généraux

La norme $||\cdot||_{1,\mathcal{M}}$ est la norme naturelle pour obtenir des estimations a priori sur les solutions de (3.3)—(3.5). Elle a aussi le double avantage d'être plus forte que la norme de $L^2(\Omega)$ et de donner en fait un résultat de compacité dans $L^2(\Omega)$, comme le montrent les lemmes suivants.

Lemme 3.1 *(inégalité de Poincaré discrète) Soit $\mathcal{M}$ un maillage admissible de Ω. Si $v_\mathcal{T}\in X(\mathcal{M})$, alors on a*

$$||v_\mathcal{T}||_{L^2(\Omega)} \le \text{diam}(\Omega)||v_\mathcal{T}||_{1,\mathcal{M}}.$$

(Voir le lemme 9.1 dans [37]).

Lemme 3.2 *(théorème de Rellich discret) Soit* $C > 0$. *Si* $\mathcal{M}_n$ *est une suite de maillages admissibles de* Ω *et* $v_n \in X(\mathcal{M}_n)$ *vérifie* $||v_n||_{1,\mathcal{M}_n} \le C$, *alors* $(v_n)_{n\ge1}$ *est relativement compacte dans* $L^2(\Omega)$; *de plus, si* $h_{\mathcal{T}_n} \to 0$ *lorsque* $n \to \infty$, *les valeurs d'adhérence de* $(v_n)_{n\ge1}$ *sont dans* $H_0^1(\Omega)$.

(Conséquence immédiate du lemme 9.3 et du théorème 14.2 de [37]).

L'obtention d'estimations a priori sur les solutions de (3.3)—(3.5) (i.e. (3.1) discrétisée) suit les mêmes idées que la preuve des estimations a priori sur les solutions de (3.1) (la seule différence étant que l'estimation sur $\ln(1+|u|)$ n'est pas obtenue en prenant des fonctions tests sous une forme de "rondelle" mais sous une forme globale, comme nous l'avons fait dans le cas non-linéaire de la section 2.1). Nous aurons donc besoin d'inégalités de Sobolev discrètes (lemme 9.5 dans [37]).

Lemme 3.3 *Soit* $\mathcal{M}$ *un maillage admissible de* Ω *et* $\zeta > 0$ *tel que*

$$\forall K \in \mathcal{T}\ ,\ \forall \sigma \in \mathcal{E}_K\ ,\ d_{K,\sigma} \ge \zeta d_\sigma .$$

Si $v_\mathcal{T} \in X(\mathcal{M})$, *alors il existe* C *ne dépendant que de* Ω *et* ζ *tel que, pour tout* $q \in [2,+\infty[$ *lorsque* $N = 2$ *ou tout* $q \in [2,6]$ *lorsque* $N = 3$,

$$||v_\mathcal{T}||_{L^q(\Omega)} \le Cq||v_\mathcal{T}||_{1,\mathcal{M}}.$$

3.3.2 Estimations a priori sur $\ln(1+|u_\mathcal{T}|)$

Proposition 3.1 *Soit* $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^2}$ *et* $\mathcal{M}$ *un maillage admissible. Si* $(u_K)_{K\in\mathcal{T}}$ *est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6), alors*

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) \le 2|\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + 2\mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 + 2N|\Omega|\,||\mathbf{v}||^2_{(L^\infty(\Omega))^N}\ , \tag{3.7}$$

où l'on a posé, lorsque $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$ *et, lorsque* $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$, $u_L = 0$.

Avant d'aborder la preuve de cette proposition, nous établissons un petit lemme technique.

Lemme 3.4 *Soit* $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^2}$. *Si* $(x,y) \in \mathbb{R}^2$ *ont même signe et* $|x| \le |y|$, *alors*

$$x^2(\varphi(x) - \varphi(y))^2 \le (y-x)(\varphi(y) - \varphi(x)). \tag{3.8}$$

Preuve du lemme 3.4
On commence par supposer que x et y sont positifs, de sorte que l'on a $0 \le x \le y$. Dans ce cas, $\varphi(y) - \varphi(x) = \int_x^y \frac{dt}{(1+t)^2} = \frac{1}{1+x} - \frac{1}{1+y} = \frac{y-x}{(1+x)(1+y)}$ et

$$\begin{aligned} x^2(\varphi(x) - \varphi(y))^2 &\le x^2\frac{|y-x|}{(1+x)(1+y)}|\varphi(x) - \varphi(y)| \\ &\le \frac{x^2}{(1+x)(1+y)}|y-x||\varphi(y) - \varphi(x)|. \end{aligned}$$

Or $(1+x)(1+y) \ge (1+x)^2 \ge x^2$ puisque $0 \le x \le y$; de plus, φ est croissante, donc $|y-x||\varphi(y)-\varphi(x)| = (y-x)(\varphi(y) - \varphi(x))$. On déduit alors, de l'inégalité précédente, (3.8) lorsque x et y sont positifs.
Si x et y sont négatifs, puisque φ est impaire, on a

$$x^2(\varphi(x) - \varphi(y))^2 = (-x)^2(-\varphi(-x) - (-\varphi(-y)))^2 = (-x)^2(\varphi(-x) - \varphi(-y))^2,$$

avec $|-x| = |x| \leq |y| = |-y|$ et $(-x, -y)$ positifs; en appliquant alors (3.8) prouvée précédemment pour des réels positifs, on en déduit

$$x^2(\varphi(x) - \varphi(y))^2 \leq (-y - (-x))(\varphi(-y) - \varphi(-x)) = (x - y)(\varphi(x) - \varphi(y)) = (y - x)(\varphi(y) - \varphi(x)),$$

ce qui conclut la preuve. ■

Preuve de la proposition 3.1
En multipliant chaque égalité de (3.3) par $\varphi(u_K)$ et en sommant sur $K \in \mathcal{T}$, on obtient

$$\begin{aligned} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\varphi(u_K) + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} u_{\sigma,+}\varphi(u_K) + \sum_{K\in\mathcal{T}} m(K) b_K u_K \varphi(u_K) \\ = \sum_{K\in\mathcal{T}} m(K) f_K \varphi(u_K) + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m(\sigma) G_{K,\sigma}\varphi(u_K). \end{aligned} \quad (3.9)$$

En rassemblant la sommation par interfaces et en utilisant (3.4), on a

$$\begin{aligned} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\varphi(u_K) &= \sum_{\sigma\in\mathcal{E}_{\rm int},\, \sigma=K|L} (F_{K,\sigma}\varphi(u_K) + F_{L,\sigma}\varphi(u_L)) + \sum_{\sigma\in\mathcal{E}_{\rm ext}\,, \sigma\in\mathcal{E}_K} \tau_\sigma u_K \varphi(u_K) \\ &= \sum_{\sigma\in\mathcal{E}_{\rm int},\, \sigma=K|L} \tau_\sigma (u_K - u_L)(\varphi(u_K) - \varphi(u_L)) + \sum_{\sigma\in\mathcal{E}_{\rm ext}\,, \sigma\in\mathcal{E}_K} \tau_\sigma u_K \varphi(u_K) \\ &= \sum_{\sigma\in\mathcal{E}} \tau_\sigma (u_K - u_L)(\varphi(u_K) - \varphi(u_L)) \end{aligned} \quad (3.10)$$

où l'on a posé, comme annoncé, $\sigma = K|L$ lorsque $\sigma \in \mathcal{E}_{\rm int}$ et $u_L = 0$ lorsque $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$.
Toujours en sommant par interfaces, on a, par conservativité des flux,

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} u_{\sigma,+}\varphi(u_K) \\ &= \sum_{\sigma\in\mathcal{E}_{\rm int},\, \sigma=K|L} u_{\sigma,+}(v_{K,\sigma}\varphi(u_K) + v_{L,\sigma}\varphi(u_L)) + \sum_{\sigma\in\mathcal{E}_{\rm ext},\, \sigma\in\mathcal{E}_K} u_{\sigma,+} v_{K,\sigma}\varphi(u_K) \\ &= \sum_{\sigma\in\mathcal{E}_{\rm int},\, \sigma=K|L} u_{\sigma,+} v_{K,\sigma}(\varphi(u_K) - \varphi(u_L)) + \sum_{\sigma\in\mathcal{E}_{\rm ext},\, \sigma\in\mathcal{E}_K} u_{\sigma,+} v_{K,\sigma}\varphi(u_K) \\ &= \sum_{\sigma\in\mathcal{E}} u_{\sigma,+} v_{K,\sigma}(\varphi(u_K) - \varphi(u_L)) \end{aligned}$$

(rappelons que $u_L = 0$ — donc $\varphi(u_L) = 0$ — lorsque $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$). Notons, lorsque $\sigma \in \mathcal{E}$, $v_\sigma = |v_{K,\sigma}|$ pour un $K \in \mathcal{T}$ tel que $\sigma \in \mathcal{E}_K$ (la définition de v_σ ne dépend pas du choix d'un tel K). Nous noterons aussi $u_{\sigma,-}$ le décentrement aval, i.e. $u_{\sigma,-}$ est tel que $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$, toujours en posant $\sigma = K|L$ si $\sigma \in \mathcal{E}_{\rm int}$ et $u_L = 0$ si $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$ (en passant sous silence le problème d'indétermination dont nous avons parlé en remarque 3.2, cela revient à poser $u_{\sigma,-} = u_L$ lorsque $v_{K,\sigma} \geq 0$ et $u_{\sigma,-} = u_K$ dans le cas contraire).
Prenons $\sigma \in \mathcal{E}$; si $v_{K,\sigma} \geq 0$, alors $u_{\sigma,+} = u_K$ et $u_{\sigma,-} = u_L$ donc $v_{K,\sigma}(\varphi(u_K) - \varphi(u_L)) = v_\sigma(\varphi(u_{\sigma,+}) - \varphi(u_{\sigma,-}))$; si $v_{K,\sigma} < 0$, alors $u_{\sigma,+} = u_L$ et $u_{\sigma,-} = u_K$ donc $v_{K,\sigma}(\varphi(u_K) - \varphi(u_L)) = -v_\sigma(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) = v_\sigma(\varphi(u_{\sigma,+}) - \varphi(u_{\sigma,-}))$ (le problème d'indétermination qui peut surgir lorsque $v_{K,\sigma} = 0$ ne se pose pas ici, puisqu'alors $v_{K,\sigma} = v_\sigma = 0$). Ainsi,

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} u_{\sigma,+}\varphi(u_K) = \sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,+}) - \varphi(u_{\sigma,-})). \quad (3.11)$$

b étant positive et $\varphi(s)$ étant du même signe que s,

$$\sum_{K\in\mathcal{T}} m(K) b_K u_K \varphi(u_K) \geq 0. \quad (3.12)$$

Comme φ est bornée par 1, on a, par Cauchy-Schwarz,

$$\begin{aligned}\left|\sum_{K\in\mathcal{T}} m(K)f_K\varphi(u_K)\right| &\leq \sum_{K\in\mathcal{T}} m(K)|f_K| \\ &\leq \left(\sum_{K\in\mathcal{T}} m(K)\right)^{1/2}\left(\sum_{K\in\mathcal{T}} m(K)f_K^2\right)^{1/2} \\ &\leq |\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}). \end{aligned} \tag{3.13}$$

Par hypothèse de conservativité sur les $(G_{K,\sigma})_{K\in\mathcal{T},\,\sigma\in\mathcal{E}_K}$,

$$\begin{aligned}\left|\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)G_{K,\sigma}\varphi(u_K)\right| &= \left|\sum_{\sigma\in\mathcal{E}} m(\sigma)G_{K,\sigma}(\varphi(u_K)-\varphi(u_L))\right| \\ &\leq \left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma G_\sigma^2\right)^{1/2}\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(\varphi(u_K)-\varphi(u_L))^2\right)^{1/2} \\ &\leq \mathcal{N}(f_\mathcal{T}, G_\mathcal{M})\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(\varphi(u_K)-\varphi(u_L))^2\right)^{1/2}.\end{aligned}$$

Or φ est croissante et 1-lipschitzienne (φ' est majorée par 1) donc, pour tous $(x,y)\in\mathbb{R}^2$, $(\varphi(x)-\varphi(y))^2 \leq |x-y||\varphi(x)-\varphi(y)| = (x-y)(\varphi(x)-\varphi(y))$; ainsi,

$$\sum_{\sigma\in\mathcal{E}}\tau_\sigma(\varphi(u_K)-\varphi(u_L))^2 \leq \sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_K-u_L)(\varphi(u_K)-\varphi(u_L)).$$

Par l'inégalité de Young, on en déduit donc que

$$\left|\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)G_{K,\sigma}\varphi(u_K)\right| \leq \mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 + \frac{1}{4}\sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_K-u_L)(\varphi(u_K)-\varphi(u_L)). \tag{3.14}$$

En injectant (3.10), (3.11), (3.12), (3.13) et (3.14) dans (3.9), on obtient

$$\begin{aligned}&\frac{3}{4}\sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_K-u_L)(\varphi(u_K)-\varphi(u_L)) \\ &\quad\leq |\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + \mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 - \sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,+})-\varphi(u_{\sigma,-})) \\ &\quad= |\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + \mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 + \sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})).\end{aligned} \tag{3.15}$$

Nous allons maintenant étudier un peu plus précisément chaque terme de la dernière somme, cas par cas. On utilise pour cela le fait que φ est croissante.

- Si $u_{\sigma,+}\geq u_{\sigma,-}$ et $u_{\sigma,+}\geq 0$, alors $\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})\leq 0$ et $u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+}))\leq 0$.
- Si $u_{\sigma,+}\geq u_{\sigma,-}$ et $u_{\sigma,+}<0$, alors $0>u_{\sigma,+}\geq u_{\sigma,-}$, donc $(u_{\sigma,+},u_{\sigma,-})$ ont le même signe et $|u_{\sigma,+}|\leq|u_{\sigma,-}|$.
- Si $u_{\sigma,+}<u_{\sigma,-}$ et $u_{\sigma,+}\geq 0$, alors $0\leq u_{\sigma,+}<u_{\sigma,-}$, donc $(u_{\sigma,+},u_{\sigma,-})$ ont le même signe et $|u_{\sigma,+}|\leq|u_{\sigma,-}|$.
- Si $u_{\sigma,+}<u_{\sigma,-}$ et $u_{\sigma,+}<0$, alors $\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})\geq 0$ et $u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+}))\leq 0$.

En notant alors $\mathcal{A}_1 = \{\sigma \in \mathcal{E} \mid u_{\sigma,+} \geq u_{\sigma,-}\,,\ u_{\sigma,+} < 0\}$ et $\mathcal{A}_2 = \{\sigma \in \mathcal{E} \mid u_{\sigma,+} < u_{\sigma,-}\,,\ u_{\sigma,+} \geq 0\}$, on constate que, pour tout $\sigma \in \mathcal{E}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2)$, $v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) \leq 0$. Ainsi,

$$\sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) \leq \sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})).$$

Comme $v_\sigma \leq m(\sigma)||\mathbf{v}||_{(L^\infty(\Omega))^N}$, on en déduit, par Cauchy-Schwarz,

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) \\
&\leq \ ||\mathbf{v}||_{(L^\infty(\Omega))^N} \sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} m(\sigma)|u_{\sigma,+}||\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})| \\
&\leq \ ||\mathbf{v}||_{(L^\infty(\Omega))^N} \left(\sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} m(\sigma)d_\sigma\right)^{1/2} \left(\sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} \tau_\sigma u_{\sigma,+}^2(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+}))^2\right)^{1/2}.
\end{aligned}$$

Or $\sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} m(\sigma)d_\sigma \leq \sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma = N|\Omega|$ et, dès que $\sigma \in \mathcal{A}_1 \cup \mathcal{A}_2$, $(u_{\sigma,+}, u_{\sigma,-})$ ont le même signe et $|u_{\sigma,+}| \leq |u_{\sigma,-}|$, donc par le lemme 3.4 et l'inégalité de Young,

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) \\
&\leq \ (N|\Omega|)^{1/2}||\mathbf{v}||_{(L^\infty(\Omega))^N} \left(\sum_{\sigma\in\mathcal{A}_1\cup\mathcal{A}_2} \tau_\sigma(u_{\sigma,-} - u_{\sigma,+})(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+}))\right)^{1/2} \\
&\leq \ N|\Omega|\,||\mathbf{v}||^2_{(L^\infty(\Omega))^N} + \frac{1}{4}\sum_{\sigma\in\mathcal{E}} \tau_\sigma(u_{\sigma,-} - u_{\sigma,+})(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})).
\end{aligned}$$

Pour tout $\sigma \in \mathcal{E}$, on a $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$, donc $(u_{\sigma,-} - u_{\sigma,+})(\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) = (u_K - u_L)(\varphi(u_K) - \varphi(u_L))$. En revenant à (3.15), on obtient donc

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) \leq 2|\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + 2\mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 + 2N|\Omega|\,||\mathbf{v}||^2_{(L^\infty(\Omega))^N},$$

ce qui conclut la preuve de cette proposition. ∎

Corollaire 3.1 *Si $\mathcal{M}$ est un maillage admissible et $u_\mathcal{T} = (u_K)_{K\in\mathcal{T}}$ est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6), alors*

$$||\ln(1 + |u_\mathcal{T}|)||_{1,\mathcal{M}} \leq \left(2|\Omega|^{1/2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + 2\mathcal{N}(f_\mathcal{T}, G_\mathcal{M})^2 + 2N|\Omega|\,||\mathbf{v}||^2_{(L^\infty(\Omega))^N}\right)^{1/2}$$

Preuve du corollaire 3.1
On constate que, pour tout $s \in \mathbb{R}$, $\ln(1 + |s|) = \int_0^s \frac{\mathrm{sgn}(t)\,dt}{1+|t|}$. Ainsi, pour tous $(x, y) \in \mathbb{R}^2$, par Cauchy-Schwarz et par croissance de φ,

$$\begin{aligned}
(\ln(1 + |x|) - \ln(1 + |y|))^2 \ &= \ \left(\int_y^x \frac{\mathrm{sgn}(t)\,dt}{1 + |t|}\right)^2 \\
&\leq \ |x - y|\left|\int_y^x \frac{dt}{(1 + |t|)^2}\right| = |x - y||\varphi(x) - \varphi(y)| = (x - y)(\varphi(x) - \varphi(y)).
\end{aligned}$$

En utilisant cette majoration et l'estimation de la proposition 3.1, on en déduit le résultat du corollaire.
∎

3.3.3 Estimations a priori sur $u_{\mathcal{T}}$

Munis de l'estimation sur $\ln(1+|u_{\mathcal{T}}|)$, nous pouvons maintenant, comme dans le cas continu, en déduire une estimation sur $u_{\mathcal{T}}$ solution de (3.3)—(3.5).

Proposition 3.2 *Soit $\mathcal{M}$ un maillage admissible et $\zeta > 0$ tel que*

$$\forall K \in \mathcal{T}\,,\ \forall \sigma \in \mathcal{E}_K\,,\ d_{K,\sigma} \geq \zeta d_\sigma.$$

Il existe C ne dépendant que de $(||\mathbf{v}||_{(L^\infty(\Omega))^N}, \Omega, \zeta)$ tel que, si $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6), on a $||u_{\mathcal{T}}||_{1,\mathcal{M}} \leq C\mathcal{N}(f_{\mathcal{T}}, G_{\mathcal{M}})$.

Preuve de la proposition (3.2)
Dans toute cette preuve, les constantes C_i ne dépendent que de $(||\mathbf{v}||_{(L^\infty(\Omega))^N}, \Omega, \zeta)$.

Etape 1: estimation sur $S_k(u_{\mathcal{T}})$.
Nous supposons ici que $u_{\mathcal{T}}$ est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6) et $\mathcal{N}(f_{\mathcal{T}}, G_{\mathcal{M}}) \leq 1$. En notant, comme dans le cas continu, pour $k \in \mathbb{R}$, $S_k(s) = s - T_k(s) = s - \min(k, \max(s, -k))$, nous cherchons des estimations sur $S_k(u_{\mathcal{T}})$ pour un k bien choisi.
En multipliant chaque égalité de (3.3) par $S_k(u_K)$, en sommant sur $K \in \mathcal{T}$ et en rassemblant les sommes par interfaces, on a

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} \tau_\sigma(u_K - u_L)(S_k(u_K) - S_k(u_L)) &= \sum_{K\in\mathcal{T}} m(K) f_K S_k(u_K) + \sum_{\sigma\in\mathcal{E}} m(\sigma) G_{K,\sigma}(S_k(u_K) - S_k(u_L)) \\ &\quad - \sum_{K\in\mathcal{T}} m(K) b_K u_K S_k(u_K) - \sum_{\sigma\in\mathcal{E}} u_{\sigma,+} v_{K,\sigma}(S_k(u_K) - S_k(u_L)),\end{aligned}$$

où, comme précédemment, on a posé, lorsque $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$ et, lorsque $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$, $u_L = 0$.
Comme S_k est croissante et 1-lipschitzienne, on a $(S_k(u_K) - S_k(u_L))^2 \leq (u_K - u_L)(S_k(u_K) - S_k(u_L))$; b étant positive et $S_k(s)$ étant du même signe que s, $b_K u_K S_k(u_K) \geq 0$. Par Cauchy-Schwarz, et puisque $\mathcal{N}(f_{\mathcal{T}}, G_{\mathcal{M}}) \leq 1$,

$$\left|\sum_{K\in\mathcal{T}} m(K) f_K S_k(u_K)\right| \leq \left(\sum_{K\in\mathcal{T}} m(K) f_K^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}} m(K) S_k(u_K)^2\right)^{1/2} \leq ||S_k(u_{\mathcal{T}})||_{L^2(\Omega)}$$

et

$$\begin{aligned}\left|\sum_{\sigma\in\mathcal{E}} m(\sigma) G_{K,\sigma}(S_k(u_K) - S_k(u_L))\right| &\leq \left(\sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma G_\sigma^2\right)^{1/2} \left(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (S_k(u_K) - S_k(u_L))^2\right)^{1/2} \\ &\leq ||S_k(u_{\mathcal{T}})||_{1,\mathcal{M}}.\end{aligned}$$

On a donc, par le lemme 3.1 (inégalité de Poincaré discrète),

$$||S_k(u_{\mathcal{T}})||_{1,\mathcal{M}}^2 \leq C_1 ||S_k(u_{\mathcal{T}})||_{1,\mathcal{M}} + C_2 \sum_{\sigma\in\mathcal{E}} m(\sigma)|u_{\sigma,+}|\,|S_k(u_K) - S_k(u_L)|. \tag{3.16}$$

Or

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} m(\sigma)|u_{\sigma,+}||S_k(u_K) - S_k(u_L)| &\leq \left(\sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma u_{\sigma,+}^2\right)^{1/2} \left(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (S_k(u_K) - S_k(u_L))^2\right)^{1/2} \\ &\leq \left(\sum_{K\in\mathcal{T}} u_K^2 \left(\sum_{\sigma\in\mathcal{E}_K \,|\, v_{K,\sigma}\geq 0} m(\sigma) d_\sigma\right)\right)^{1/2} ||S_k(u_{\mathcal{T}})||_{1,\mathcal{M}}.\end{aligned}$$

Puisque $d_\sigma \le \frac{1}{\zeta} d_{K,\sigma}$ pour tout $\sigma \in \mathcal{E}_K$, on a

$$\sum_{\sigma\in\mathcal{E}_K \mid v_{K,\sigma}\ge 0} m(\sigma) d_\sigma \le \frac{1}{\zeta}\sum_{\sigma\in\mathcal{E}_K} m(\sigma) d_{K,\sigma} = \frac{N}{\zeta} m(K),$$

donc

$$\sum_{\sigma\in\mathcal{E}} m(\sigma)|u_{\sigma,+}||S_k(u_K) - S_k(u_L)| \le C_3 ||u_\mathcal{T}||_{L^2(\Omega)} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}}.$$

Mais $u_\mathcal{T} = T_k(u_\mathcal{T}) + S_k(u_\mathcal{T})$, donc $||u_\mathcal{T}||_{L^2(\Omega)} \le k|\Omega|^{1/2} + ||S_k(u_\mathcal{T})||_{L^2(\Omega)}$, ce qui donne

$$\sum_{\sigma\in\mathcal{E}} m(\sigma)|u_{\sigma,+}||S_k(u_K) - S_k(u_L)| \le C_3 k|\Omega|^{1/2} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}} + C_3 ||S_k(u_\mathcal{T})||_{L^2(\Omega)} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}}. \quad (3.17)$$

Soit $p > 2$; puisque $S_k(u_\mathcal{T}) = 0$ hors de $E_k = \{|u_\mathcal{T}| \ge k\}$, on a, par Hölder,

$$||S_k(u_\mathcal{T})||_{L^2(\Omega)} \le |E_k|^{\frac{1}{2}-\frac{1}{p}} ||S_k(u_\mathcal{T})||_{L^p(\Omega)}.$$

Comme, par le lemme 3.3 (inégalité de Sobolev discrète), il existe $p > 2$ (ne dépendant que de N) et C_4 tels que

$$||S_k(u_\mathcal{T})||_{L^p(\Omega)} \le C_4 ||S_k(u_\mathcal{T})||_{1,\mathcal{M}},$$

on en déduit que

$$||S_k(u_\mathcal{T})||_{L^2(\Omega)} \le C_4 |E_k|^{\frac{1}{2}-\frac{1}{p}} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}},$$

ce qui implique, dans (3.17),

$$\sum_{\sigma\in\mathcal{E}} m(\sigma)|u_{\sigma,+}||S_k(u_K) - S_k(u_L)| \le C_3 k|\Omega|^{1/2} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}} + C_3 C_4 |E_k|^{\frac{1}{2}-\frac{1}{p}} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}}^2.$$

En injectant cette inégalité dans (3.16), on a donc

$$||S_k(u_\mathcal{T})||_{1,\mathcal{M}}^2 \le C_1 ||S_k(u_\mathcal{T})||_{1,\mathcal{M}} + C_5 k ||S_k(u_\mathcal{T})||_{1,\mathcal{M}} + C_5 |E_k|^{\frac{1}{2}-\frac{1}{p}} ||S_k(u_\mathcal{T})||_{1,\mathcal{M}}^2.$$

On a, grâce au corollaire 3.1 et au lemme 3.1, $||\ln(1+|u_\mathcal{T}|)||_{L^2(\Omega)}^2 \le C_6$. Donc, par Tchebycheff,

$$|E_k| = |\{\ln(1+|u_\mathcal{T}|) \ge \ln(1+k)\}| \le \frac{C_6}{(\ln(1+k))^2}.$$

Puisque $\frac{1}{2} - \frac{1}{p} > 0$, il existe donc k_0 ne dépendant que de (C_5, C_6, p), i.e. ne dépendant que de $(||\mathbf{v}||_{(L^\infty(\Omega))^N}, \Omega, \zeta)$, tel que $C_5 |E_{k_0}|^{\frac{1}{2}-\frac{1}{p}} \le 1/2$. On a alors

$$||S_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}}^2 \le 2C_1 ||S_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}} + 2C_5 k_0 ||S_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}},$$

ce qui donne

$$||S_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}} \le C_7. \quad (3.18)$$

Etape 2: estimation sur $u_\mathcal{T}$.
Toujours en supposant que $u_\mathcal{T}$ est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6) et $\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) \le 1$, nous obtenons une estimation sur $||u_\mathcal{T}||_{1,\mathcal{M}}$.
Pour le k_0 trouvé dans la partie précédente, on cherche d'abord une estimation sur $T_{k_0}(u_\mathcal{T})$. En multipliant chaque terme de (3.3) par $T_{k_0}(u_K)$, en sommant sur $K \in \mathcal{T}$ et en rassemblant les sommes par

interfaces, on trouve, puisque T_{k_0} est croissante et 1-lipschitzienne,

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}}\tau_\sigma(T_{k_0}(u_K)-T_{k_0}(u_L))^2 &\leq \sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_K-u_L)(T_{k_0}(u_K)-T_{k_0}(u_L))\\ &= \sum_{K\in\mathcal{T}} m(K)f_K T_{k_0}(u_K)+\sum_{\sigma\in\mathcal{E}} m(\sigma)G_{K,\sigma}(T_{k_0}(u_K)-T_{k_0}(u_L))\\ &\quad -\sum_{K\in\mathcal{T}} m(K)b_K u_K T_{k_0}(u_K)-\sum_{\sigma\in\mathcal{E}} u_{\sigma,+}v_{K,\sigma}(T_{k_0}(u_K)-T_{k_0}(u_L)).\end{aligned}$$

Or $u_K T_{k_0}(u_K)\geq 0$ et $|T_{k_0}(u_K)|\leq k_0$, donc, par Cauchy-Schwarz, on obtient

$$\begin{aligned}||T_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}}^2 &\leq k_0\left(\sum_{K\in\mathcal{T}} m(K)\right)^{1/2}\left(\sum_{K\in\mathcal{T}} m(K)f_K^2\right)^{1/2}\\ &\quad +\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma G_\sigma^2\right)^{1/2}\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(T_{k_0}(u_K)-T_{k_0}(u_L))^2\right)^{1/2}\\ &\quad +C_8\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma u_{\sigma,+}^2\right)^{1/2}\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(T_{k_0}(u_K)-T_{k_0}(u_L))^2\right)^{1/2}\\ &\leq k_0|\Omega|^{1/2}+||T_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}}+C_8\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma u_{\sigma,+}^2\right)^{1/2}||T_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}}. \quad (3.19)\end{aligned}$$

On a déjà vu que

$$\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma u_{\sigma,+}^2\right)^{1/2}\leq C_9||u_\mathcal{T}||_{L^2(\Omega)}.$$

De plus, puisque $|u_\mathcal{T}|\leq k_0+|S_{k_0}(u_\mathcal{T})|$, par (3.18) et le lemme 3.1, on a $||u_\mathcal{T}||_{L^2(\Omega)}\leq C_{10}$.
Ainsi, (3.19) donne $||T_{k_0}(u_\mathcal{T})||_{1,\mathcal{M}}\leq C_{11}$. Puisque $u_\mathcal{T}=T_{k_0}(u_\mathcal{T})+S_{k_0}(u_\mathcal{T})$, cette inégalité et (3.18) donnent finalement

$$||u_\mathcal{T}||_{1,\mathcal{M}}\leq C_{12}.$$

Etape 3: conclusion.
Supposons maintenant que $u_\mathcal{T}$ est une solution de (3.3)—(3.5) avec un second membre vérifiant (3.6) et prenons $\lambda>\mathcal{N}(f_\mathcal{T},G_\mathcal{M})$ (ceci pour éviter de diviser par 0...). Comme le système (3.3)—(3.5) est linéaire, $u_\mathcal{T}/\lambda$ est solution de ce système avec $(f_K/\lambda)_{K\in\mathcal{T}}$, $(G_{K,\sigma}/\lambda)_{K\in\mathcal{T},\sigma\in\mathcal{E}_K}$ comme second membre; ce second membre vérifiant (3.6) et $\mathcal{N}(f_\mathcal{T}/\lambda,G_\mathcal{M}/\lambda)\leq 1$, on en déduit, par ce qui précède, que $||u_\mathcal{T}/\lambda||_{1,\mathcal{M}}\leq C_{12}$, c'est à dire que $||u_\mathcal{T}||_{1,\mathcal{M}}\leq C_{12}\lambda$. En faisant tendre λ vers $\mathcal{N}(f_\mathcal{T},G_\mathcal{M})$, on obtient finalement le résultat voulu. ■

3.4 Existence, unicité et convergence de la solution approchée

Lemme 3.5 *Soit $\mathcal{M}$ un maillage admissible de Ω et $\zeta>0$ tel que, pour tout $K\in\mathcal{T}$ et tout $\sigma\in\mathcal{E}_K$, $d_{K,\sigma}\geq\zeta d_\sigma$. Soit $(f_K)_{K\in\mathcal{T}}$ et $(G_{K,\sigma})_{K\in\mathcal{T},\sigma\in\mathcal{E}_K}$ définis par (3.2). Il existe alors une unique solution $u_\mathcal{T}=(u_K)_{K\in\mathcal{T}}$ à (3.3)—(3.5) et elle vérifie $||u_\mathcal{T}||_{1,\mathcal{M}}\leq C(||f||_{L^2(\Omega)}+||G||_{(L^\infty(\Omega))^N})$, avec C ne dépendant que de $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, ζ et Ω.*

Preuve du lemme 3.5
Le système (3.3)—(3.5) est linéaire carré en $(u_K)_{K\in\mathcal{T}}$. De plus, l'estimation a priori de la proposition 3.2 prouve que, si le second membre de ce système est nul (ce second membre vérifie (3.6)), alors toute

solution correspondante est nulle. Ce système est donc injectif et, étant de dimension finie, il est bijectif; il existe donc, pour tout second membre, une unique solution à (3.3)—(3.5).
Supposons maintenant que le second membre soit donné par (3.2). Alors il vérifie clairement (3.6) et, par la proposition (3.2), la solution correspondante vérifie $||u_\mathcal{T}||_{1,\mathcal{M}} \le C\mathcal{N}(f_\mathcal{T}, G_\mathcal{M})$ où C ne dépend que de $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, ζ et Ω. Or, pour tout $K \in \mathcal{T}$ et $\sigma \in \mathcal{E}_K$, $f_K^2 \le \frac{1}{m(K)} \int_K |f|^2$ et $G_\sigma \le ||G||_{(L^\infty(\Omega))^N}$, donc

$$\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) \le ||f||_{L^2(\Omega)} + ||G||_{(L^\infty(\Omega))^N} \left(\sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma \right)^{1/2} = ||f||_{L^2(\Omega)} + ||G||_{(L^\infty(\Omega))^N} (N|\Omega|)^{1/2},$$

ce qui donne bien l'estimation cherchée dans le lemme. ■

Maintenant que nous savons qu'il existe une solution au problème (3.1) discrétisé, et que nous avons des estimations sur cette solution, nous pouvons montrer un théorème de convergence.

Théorème 3.1 *Soit $\zeta > 0$. On considère des maillages $\mathcal{M}$ de Ω tels que, pour tout $K \in \mathcal{T}$ et pour tout $\sigma \in \mathcal{E}_K$, on a $d_{K,\sigma} \ge \zeta d_\sigma$. Alors la solution $u_\mathcal{T}$ de (3.3)—(3.5) converge, lorsque $h_\mathcal{T} \to 0$, dans $L^2(\Omega)$ vers la solution variationnelle u de (3.1).*

Remarque 3.3 *L'hypothèse de minoration de $d_{K,\sigma}/d_\sigma$ n'est utile que pour obtenir une estimation sur les solutions discrétisées, i.e. pour pouvoir appliquer le lemme 3.5; nous ne l'utiliserons pas pour prouver que les limites des solutions discrétisées sont des solutions de (3.1). Ainsi, notre preuve de convergence permet d'étendre les résultats de convergence de [37] au cas $\mathbf{v} \in (\mathcal{C}(\overline{\Omega}))^N$ et $G \in (\mathcal{C}(\overline{\Omega}))^N$ non nul, sans hypothèse supplémentaire sur les maillages.*

Preuve du théorème 3.1
Par l'estimation du lemme 3.5, on sait que $||u_\mathcal{T}||_{1,\mathcal{M}}$ est borné indépendamment du maillage $\mathcal{M}$. Le lemme 3.2 nous dit alors que l'ensemble Y des solutions discrètes (i.e. solutions de (3.3)—(3.5) pour un maillage $\mathcal{M}$ satisfaisant $d_{K,\sigma} \ge \zeta d_\sigma$ pour tout $K \in \mathcal{T}$ et tout $\sigma \in \mathcal{E}_K$) est relativement compact dans $L^2(\Omega)$ et que les valeurs d'adhérence de suites $(u_n)_{n\ge1}$ de cet ensemble, correspondant à des maillages dont la taille $h_{\mathcal{T}_n}$ tend vers 0, sont dans $H_0^1(\Omega)$. Nous allons montrer que toute valeur d'adhérence de ce genre est une solution variationnelle de (3.1); cette solution étant unique (théorème 1.1), cela nous donnera bien la convergence, lorsque $h_\mathcal{T} \to 0$ et sans extraire de suite, de $u_\mathcal{T}$ vers la solution de (3.1).

Soit donc $u_\mathcal{T} \in Y$ qui converge dans $L^2(\Omega)$, lorsque $h_\mathcal{T} \to 0$, vers un certain $u \in H_0^1(\Omega)$. On prend $\psi \in \mathcal{C}_c^\infty(\Omega)$ et on veut donc montrer que

$$\int_\Omega \nabla u \cdot \nabla\psi - \int_\Omega u\mathbf{v} \cdot \nabla\psi + \int_\Omega bu\psi = \int_\Omega f\psi - \int_\Omega G \cdot \nabla\psi. \quad (3.20)$$

On s'inspire pour cela en tout point de la preuve de convergence dans [37], en faisant la modification nécessaire au fait que $\mathbf{v}$ et G ne sont que continues ici.
Dans la preuve qui suit, les constantes C_i ne dépendent que de ψ, Ω et d'un majorant (que l'on a) de $||u_\mathcal{T}||_{1,\mathcal{M}}$.

On prend $h_\mathcal{T}$ assez petit tel que $\psi = 0$ sur K pour tout $K \in \mathcal{T}$ vérifiant $\overline{K} \cap \partial\Omega \ne \emptyset$. En multipliant par $\psi(x_K)$ l'équation satisfaite par $u_\mathcal{T}$ et en sommant sur $K \in \mathcal{T}$, on a

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma (u_K - u_L)(\psi(x_K) - \psi(x_L)) + \sum_{\sigma\in\mathcal{E}} v_{K,\sigma} u_{\sigma,+} (\psi(x_K) - \psi(x_L)) + \sum_{K\in\mathcal{T}} m(K) b_K u_K \psi(x_K)$$
$$= \sum_{K\in\mathcal{T}} m(K) f_K \psi(x_K) + \sum_{\sigma\in\mathcal{E}} m(\sigma) G_{K,\sigma} (\psi(x_K) - \psi(x_L)). \quad (3.21)$$

Notons T_1, T_2, T_3, T_4 et T_5 les termes de cette égalité.

Etape 1: convergence de T_3 et T_4.
Soit $\psi_{\mathcal{T}} \in X(\mathcal{M})$ définie par $\psi_K = \psi(x_K)$; on a $\psi_{\mathcal{T}} \to \psi$ dans $L^p(\Omega)$ pour tout $p < \infty$. Ainsi, puisque $u_{\mathcal{T}} \to u$ et $f_{\mathcal{T}} \to f$ dans $L^2(\Omega)$ et $b_{\mathcal{T}} \to b$ dans $L^p(\Omega)$ pour tout $p < \infty$, on a

$$T_3 \to \int_\Omega bu\psi \quad \text{et} \quad T_4 \to \int_\Omega f\psi.$$

Etape 2: convergence de T_1.
On commence par remarquer que

$$T_1' = -\sum_{\sigma\in\mathcal{E}}(u_K - u_L)\int_\sigma \nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma = -\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} u_K\int_\sigma \nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma = -\int_\Omega u_{\mathcal{T}}\Delta\psi$$

converge vers $-\int_\Omega u\Delta\psi = \int_\Omega \nabla u\cdot\nabla\psi$. De plus, en notant

$$R_{K,\sigma} = \frac{\psi(x_L)-\psi(x_K)}{d_\sigma} - \frac{1}{m(\sigma)}\int_\sigma \nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma,$$

on a, par régularité de ψ, $|R_{K,\sigma}| \le C_1 h_{\mathcal{T}}$. On peut alors comparer T_1 et T_1':

$$\begin{aligned}
|T_1 - T_1'| &\le \sum_{\sigma\in\mathcal{E}} m(\sigma)|u_K-u_L|\left|\frac{\psi(x_K)-\psi(x_L)}{d_\sigma} + \frac{1}{m(\sigma)}\int_\sigma \nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma\right| \\
&\le C_1 h_{\mathcal{T}}\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_K-u_L)^2\right)^{1/2}\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma\right)^{1/2} \\
&\le C_1(N|\Omega|)^{1/2}||u_{\mathcal{T}}||_{1,\mathcal{M}} h_{\mathcal{T}}.
\end{aligned}$$

Puisque $||u_{\mathcal{T}}||_{1,\mathcal{M}}$ est borné, on a donc $T_1 - T_1' \to 0$ et donc $T_1 \to \int_\Omega \nabla u\cdot\nabla\psi$.

Etape 3: convergence de T_2.
Nous désirons montrer que T_2 tend vers $-\int_\Omega u\mathbf{v}\cdot\nabla\psi$. Pour cela, on se donne $\varepsilon > 0$ et $\mathbf{w} \in (\mathcal{C}^1(\overline{\Omega}))^N$ telle que $||\mathbf{v}-\mathbf{w}||_{(L^\infty(\Omega))^N} \le \varepsilon$. En notant $w_{K,\sigma} = \int_\sigma \mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma$ et

$$T_2' = \sum_{\sigma\in\mathcal{E}} w_{K,\sigma}u_{\sigma,+}(\psi(x_K)-\psi(x_L)),$$

on a

$$\begin{aligned}
|T_2 - T_2'| &= \left|\sum_{\sigma\in\mathcal{E}}(v_{K,\sigma}-w_{K,\sigma})u_{\sigma,+}(\psi(x_K)-\psi(x_L))\right| \\
&\le C_2\varepsilon\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma|u_{\sigma,+}| \\
&= C_2\varepsilon\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,+}|u_{\sigma,+}| + \sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,-}|u_{\sigma,+}|\right)
\end{aligned}$$

où, comme on s'y attend,

pour $\sigma \in \mathcal{E}_{\text{int}}$, $\quad d_{\sigma,+} = d_{K,\sigma}$ avec le $K \in \mathcal{T}$ tel que $\sigma \in \mathcal{E}_K$ choisi pour définir $u_{\sigma,+} = u_K$,
pour $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$, $\quad d_{\sigma,+} = d_{K,\sigma}$ si $v_{K,\sigma} \ge 0$, $d_{\sigma,+} = 0$ dans le cas contraire

et $d_{\sigma,-} = d_\sigma - d_{\sigma,+}$. On notera $\mathcal{E}_{K,+} = \{\sigma \in \mathcal{E}_K \mid K \text{ a été choisi pour définir } u_{\sigma,+} = u_K\}$.

Par l'estimation que l'on a sur $||u_{\mathcal{T}}||_{1,\mathcal{M}}$ et le lemme de Poincaré discret,

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,+}|u_{\sigma,+}| &= \sum_{K\in\mathcal{T}}|u_K|\sum_{\sigma\in\mathcal{E}_{K,+}} m(\sigma)d_{K,\sigma}\\ &\leq \sum_{K\in\mathcal{T}}|u_K|\sum_{\sigma\in\mathcal{E}_K} m(\sigma)d_{K,\sigma}\\ &\leq N||u_{\mathcal{T}}||_{L^1(\Omega)}\\ &\leq C_3.\end{aligned}$$

et

$$\begin{aligned}&\sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,-}|u_{\sigma,+}|\\ &\leq \sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,-}|u_{\sigma,-}| + \sum_{\sigma\in\mathcal{E}} m(\sigma)d_{\sigma,-}|u_{\sigma,+}-u_{\sigma,-}|\\ &\leq \sum_{K\in\mathcal{T}}|u_K|\sum_{\sigma\in\mathcal{E}_K\setminus\mathcal{E}_{K,+}} m(\sigma)d_{K,\sigma} + h_{\mathcal{T}}\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma\right)^{1/2}\left(\sum_{\sigma\in\mathcal{E}}\tau_\sigma(u_{\sigma,+}-u_{\sigma,-})^2\right)^{1/2}\\ &\leq N||u_{\mathcal{T}}||_{L^1(\Omega)} + h_{\mathcal{T}}(N|\Omega|)^{1/2}||u_{\mathcal{T}}||_{1,\mathcal{M}}\\ &\leq C_4.\end{aligned}$$

Ainsi,

$$|T_2 - T_2'| \leq C_5\varepsilon. \tag{3.22}$$

Etudions maintenant T_2'.

$$\begin{aligned}T_2' &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} w_{K,\sigma}u_{\sigma,+}\psi(x_K)\\ &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} w_{K,\sigma}(u_{\sigma,+}-u_K)\psi(x_K) + \sum_{K\in\mathcal{T}} u_K\psi(x_K)\sum_{\sigma\in\mathcal{E}_K} w_{K,\sigma}\\ &= T_2'' + T_2'''\end{aligned}$$

Par régularité de $\mathbf{w}$, on a

$$T_2''' = \sum_{K\in\mathcal{T}} u_K\psi(x_K)\int_{\partial K}\mathbf{w}\cdot\mathbf{n}_K\,d\gamma = \sum_{K\in\mathcal{T}} u_K\psi(x_K)\int_K \mathrm{div}(\mathbf{w}) \to \int_\Omega u\psi\mathrm{div}(\mathbf{w}).$$

De plus,

$$T_2'' = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}(u_{\sigma,+}-u_K)\int_\sigma \psi\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}(u_{\sigma,+}-u_K)\int_\sigma(\psi(x_K)-\psi)\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma.$$

Mais, comme le support de ψ ne touche pas les mailles au bord de Ω,

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} u_{\sigma,+}\int_\sigma\psi\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} u_{\sigma,+}\left(\int_\sigma\psi\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma + \int_\sigma\psi\mathbf{w}\cdot\mathbf{n}_{L,\sigma}\,d\gamma\right) = 0$$

car $\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}$ lorsque $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$. Comme

$$-\sum_{K\in\mathcal{T}} u_K\sum_{\sigma\in\mathcal{E}_K}\int_\sigma\psi\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma = -\sum_{K\in\mathcal{T}} u_K\int_K\mathrm{div}(\psi\mathbf{w}) \to -\int_\Omega u\mathrm{div}(\psi\mathbf{w}),$$

et

$$\begin{aligned}
\left|\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}(u_{\sigma,+}-u_K)\int_\sigma(\psi(x_K)-\psi)\mathbf{w}\cdot\mathbf{n}_{K,\sigma}\,d\gamma\right| &\leq C_6h_{\mathcal{T}}||\mathbf{w}||_{(L^\infty(\Omega))^N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)|u_{\sigma,+}-u_K| \\
&\leq C_6h_{\mathcal{T}}||\mathbf{w}||_{(L^\infty(\Omega))^N}\sum_{\sigma\in\mathcal{E}}m(\sigma)|u_L-u_K| \\
&\leq C_6h_{\mathcal{T}}||\mathbf{w}||_{(L^\infty(\Omega))^N}\left(\sum_{\sigma\in\mathcal{E}}m(\sigma)d_\sigma\right)^{1/2}||u_{\mathcal{T}}||_{1,\mathcal{M}} \\
&\leq C_7h_{\mathcal{T}},
\end{aligned}$$

on en déduit que $T_2'' \to -\int_\Omega u\mathrm{div}(\psi\mathbf{w})$. Ainsi, $T_2' \to -\int_\Omega u\mathrm{div}(\psi\mathbf{w}) + \int_\Omega u\psi\mathrm{div}(\mathbf{w}) = -\int_\Omega u\mathbf{w}\cdot\nabla\psi$; il existe donc $\eta > 0$ tel que, pour tout $h_{\mathcal{T}} < \eta$,

$$\left|T_2' - \left(-\int_\Omega u\mathbf{w}\cdot\nabla\psi\right)\right| \leq \varepsilon. \tag{3.23}$$

Mais

$$\left|\int_\Omega u\mathbf{v}\cdot\nabla\psi - \int_\Omega u\mathbf{w}\cdot\nabla\psi\right| \leq \varepsilon||u||_{L^2(\Omega)}||\,|\nabla\psi|\,||_{L^2(\Omega)} = C_8\varepsilon. \tag{3.24}$$

(3.22), (3.23) et (3.24) donnent donc, dès que $h_{\mathcal{T}} < \eta$,

$$\left|T_2 - \left(-\int_\Omega u\mathbf{v}\cdot\nabla\psi\right)\right| \leq C_9\varepsilon,$$

ce qui signifie bien que $T_2' \to -\int_\Omega u\mathbf{v}\cdot\nabla\psi$.

Etape 4: convergence de T_5.
La méthode employée pour prouver la convergence de T_5 est très semblable, en beaucoup plus simple (il s'agit de prendre $u_{\sigma,+} \equiv 1$ dans la démonstration précédente!), à la méthode déjà employée pour prouver la convergence de T_2.
Soit $\varepsilon > 0$ et $H \in (\mathcal{C}^1(\overline{\Omega}))^N$ telle que $||G-H||_{(L^\infty(\Omega))^N} \leq \varepsilon$. On introduit

$$T_5' = \sum_{\sigma\in\mathcal{E}} m(\sigma)H_{K,\sigma}(\psi(x_K)-\psi(x_L)),$$

où $H_{K,\sigma} = m(\sigma)^{-1}\int_\sigma H\cdot\mathbf{n}_{K,\sigma}\,d\gamma$. On a alors

$$|T_5 - T_5'| \leq C_{10}\varepsilon\sum_{\sigma\in\mathcal{E}}m(\sigma)d_\sigma = C_{11}\varepsilon.$$

Mais, par régularité de H et de ψ,

$$\begin{aligned}
T_5' &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)H_{K,\sigma}\psi(x_K) \\
&= \sum_{K\in\mathcal{T}}\psi(x_K)\int_K\mathrm{div}(H) \to \int_\Omega\mathrm{div}(H)\psi = -\int_\Omega H\cdot\nabla\psi.
\end{aligned}$$

Il existe donc $\eta > 0$ tel que, pour $h_{\mathcal{T}} < \eta$, $|T_5' + \int_\Omega H\cdot\nabla\psi| \leq \varepsilon$. Comme

$$\left|\int_\Omega H\cdot\nabla\psi - \int_\Omega G\cdot\nabla\psi\right| \leq C_{12}\varepsilon,$$

on déduit de tout ceci que, dès que $h_{\mathcal{T}} < \eta$, on a

$$\left| T_5 - \left(- \int_\Omega G \cdot \nabla \psi \right) \right| \le (1 + C_{11} + C_{12})\varepsilon,$$

ce qui prouve bien que $T_5 \to - \int_\Omega G \cdot \nabla \psi$.
Grâce aux convergences de T_1, T_2, T_3, T_4 et T_5, on constate que $u \in H_0^1(\Omega)$ vérifie (3.20), ce qui conclut la preuve de ce théorème. ■

3.5 Estimation d'erreur

Nous prouvons ici, lorsque $G = 0$ et $\mathbf{v} \in (\mathcal{C}^1(\overline{\Omega}))^N$, une estimation de l'erreur entre la solution approchée $u_{\mathcal{T}}$ et la solution exacte u de (3.1), lorsque cette dernière est dans $H^2(\Omega)$. Pour simplifier, nous prendrons aussi b constant.

Théorème 3.2 *On suppose que $G = 0$, que $\mathbf{v} \in (\mathcal{C}^1(\overline{\Omega}))^N$ et que b est constant. Soit $\zeta > 0$. On considère un maillage $\mathcal{M}$ de Ω tel que, pour tout $K \in \mathcal{T}$ et tout $\sigma \in \mathcal{E}_K$, on a $d_{K,\sigma} \ge \zeta d_\sigma$. On note $u_{\mathcal{T}}$ la solution de (3.3)—(3.5) et u la solution de (3.1). On suppose que $u \in H^2(\Omega)$ (ce qui est par exemple le cas si Ω est convexe). Alors, en définissant $e_{\mathcal{T}} \in X(\mathcal{M})$ par $e_K = u_K - u(x_K)$, il existe C ne dépendant que de $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, Ω et ζ tel que*

$$||e_{\mathcal{T}}||_{1,\mathcal{M}} \le C h_{\mathcal{T}} ||u||_{H^2(\Omega)}.$$

Preuve du théorème 3.2
Ici encore, on suit fidèlement [37].
Dans cette preuve, les constantes C_i ne dépendent que de b, $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, Ω et ζ.

u étant dans $H^2(\Omega)$, elle vérifie (3.1) au sens fort. On peut donc intégrer l'équation de (3.1) sur chaque maille K pour trouver

$$\sum_{\sigma \in \mathcal{E}_K} \overline{F}_{K,\sigma} + \sum_{\sigma \in \mathcal{E}_K} \overline{V}_{K,\sigma} + b \int_K u = \int_K f \tag{3.25}$$

où $\overline{F}_{K,\sigma} = - \int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma}$ et $\overline{V}_{K,\sigma} = \int_\sigma u\mathbf{v} \cdot \mathbf{n}_{K,\sigma}$. En posant les flux approchés

$$\begin{aligned} F^*_{K,\sigma} &= -\tau_\sigma(u(x_L) - u(x_K)) && \text{lorsque } \sigma = K|L \in \mathcal{E}_{\text{int}}\,, \\ F^*_{K,\sigma} &= \tau_\sigma u(x_K) && \text{lorsque } \sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K\,, \end{aligned}$$

et

$$V^*_{K,\sigma} = v_{K,\sigma} u(x_{\sigma,+})$$

où, pour $\sigma = K|L \in \mathcal{E}_{\text{int}}$ (respectivement $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$), $x_{\sigma,+} = x_K$ lorsque $v_{K,\sigma} \ge 0$ et $x_{\sigma,+} = x_L$ (respectivement $x_{\sigma,+} \in \sigma$ — ce qui implique $u(x_{\sigma,+}) = 0$) dans le cas contraire, il a été prouvé dans [41] que les erreurs de consistance

$$R_{K,\sigma} = \frac{1}{m(\sigma)}(F^*_{K,\sigma} - \overline{F}_{K,\sigma})\,, \quad r_{K,\sigma} = \frac{1}{m(\sigma)}(V^*_{K,\sigma} - \overline{V}_{K,\sigma}) \quad \text{et} \quad \rho_K = u(x_K) - \frac{1}{m(K)} \int_K u$$

vérifient

$$\begin{aligned} &|R_{K,\sigma}| \le C_1 h_{\mathcal{T}} (m(\sigma) d_\sigma)^{-1/2} ||u||_{H^2(\mathcal{V}_\sigma)}\,, \quad |r_{K,\sigma}| \le C_1 h_{\mathcal{T}} (m(\sigma) d_\sigma)^{-1/4} ||u||_{W^{1,4}(\mathcal{V}_\sigma)} \\ &\text{et} \quad |\rho_K| \le h_{\mathcal{T}} m(K)^{-1/4} ||u||_{W^{1,4}(K)} \end{aligned} \tag{3.26}$$

où $\mathcal{V}_\sigma = \{t x_K + (1-t)x\,,\ x \in \sigma\,,\ t \in]0,1[\} \cup \{t x_L + (1-t)x\,,\ x \in \sigma\,,\ t \in]0,1[\}$ lorsque $\sigma = K|L \in \mathcal{E}_{\text{int}}$ et $\mathcal{V}_\sigma = \{t x_K + (1-t)x\,,\ x \in \sigma\,,\ t \in]0,1[\}$ lorsque $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$ (les estimations de [41] sont plus générales, mais nous n'utiliserons que cette forme). Remarquez que, puisque $u \in H^2(\Omega)$ et $N = 2$ ou 3, on a bien $u \in W^{1,4}(\Omega)$.

Par (3.25) et la définition de $R_{K,\sigma}$, $r_{K,\sigma}$ et ρ_K, on a donc, pour tout $K \in \mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}_K} F^*_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} V^*_{K,\sigma} + m(K)bu(x_K) = m(K)f_K + m(K)b\rho_K + \sum_{\sigma\in\mathcal{E}_K} m(\sigma)(R_{K,\sigma} + r_{K,\sigma}).$$

En soustrayant cette équation à celle vérifiée par $u_{\mathcal{T}}$, on constate donc que $e_{\mathcal{T}}$ satisfait, pour tout $K \in \mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}_K} \widetilde{F}_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}e_{\sigma,+} + m(K)be_K = -m(K)b\rho_K - \sum_{\sigma\in\mathcal{E}_K} m(\sigma)(R_{K,\sigma} + r_{K,\sigma}),$$

où $\widetilde{F}_{K,\sigma} = -\tau_\sigma(e_L - e_K)$ si $\sigma = K|L \in \mathcal{E}_{\rm int}$, $\widetilde{F}_{K,\sigma} = \tau_\sigma e_K$ si $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$, et $e_{\sigma,+}$ est le décentrement amont de $e_{\mathcal{T}}$.
$e_{\mathcal{T}}$ est donc la solution de (3.3)—(3.5) avec, pour second membre,

$$((-b\rho_K)_{K\in\mathcal{T}}, (-R_{K,\sigma} - r_{K,\sigma})_{K\in\mathcal{T},\sigma\in\mathcal{E}_K}).$$

Ce second membre vérifiant (3.6) (car les flux $\overline{F}$, $\overline{V}$, F^* et V^* sont conservatifs), on peut alors appliquer la proposition 3.2:

$$||e_{\mathcal{T}}||_{1,\mathcal{M}} \le C_2\mathcal{N}((b\rho_K)_{K\in\mathcal{T}}, (R_{K,\sigma} + r_{K,\sigma})_{K\in\mathcal{T},\sigma\in\mathcal{E}_K}). \tag{3.27}$$

Mais, par (3.26), Cauchy-Schwarz et les inégalité de Sobolev, on a

$$\begin{aligned}
\sum_{K\in\mathcal{T}} m(K)b^2\rho_K^2 &\le b^2h_{\mathcal{T}}^2 \sum_{K\in\mathcal{T}} m(K)^{1/2}||u||^2_{W^{1,4}(K)} \\
&\le b^2h_{\mathcal{T}}^2 \left(\sum_{K\in\mathcal{T}} m(K)\right)^{1/2} \left(\sum_{K\in\mathcal{T}} ||u||^4_{W^{1,4}(K)}\right)^{1/2} \\
&\le b^2h_{\mathcal{T}}^2|\Omega|^{1/2}||u||^2_{W^{1,4}(\Omega)} \\
&\le C_3h_{\mathcal{T}}^2||u||^2_{H^2(\Omega)}
\end{aligned}$$

et, en notant $R_\sigma = |R_{K,\sigma}|$ et $r_\sigma = |r_{K,\sigma}|$ lorsque $\sigma \in \mathcal{E}_K$ (ces définitions ne dépendent que de σ, non du K tel que $\sigma \in \mathcal{E}_K$),

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma(R_\sigma + r_\sigma)^2 &\le 2C_1^2h_{\mathcal{T}}^2 \left(\sum_{\sigma\in\mathcal{E}} ||u||^2_{H^2(\mathcal{V}_\sigma)} + \sum_{\sigma\in\mathcal{E}} (m(\sigma)d_\sigma)^{1/2}||u||^2_{W^{1,4}(\mathcal{V}_\sigma)}\right) \\
&\le 2C_1^2h_{\mathcal{T}}^2 \left(||u||^2_{H^2(\Omega)} + \left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma\right)^{1/2} \left(\sum_{\sigma\in\mathcal{E}} ||u||^4_{W^{1,4}(\mathcal{V}_\sigma)}\right)^{1/2}\right) \\
&\le C_4h_{\mathcal{T}}^2||u||^2_{H^2(\Omega)}
\end{aligned}$$

(en effet, $\cup_\sigma \mathcal{V}_\sigma = \Omega$ à un ensemble de mesure nulle près, et l'union est disjointe). En rassemblant ces inégalités dans (3.27), on en déduit le résultat du théorème. ■

3.6 Concernant le décentrement amont

Comme nous l'avons signalé dans la remarque 3.1, on peut aussi étudier des schémas non décentrés amont, i.e. pour lesquels $u_{\sigma,+}$ n'est pas donné par (3.5).
Nous allons voir ici que, quel que soit le choix linéaire que l'on effectue entre u_K et u_L pour $u_{\sigma,+}$, on a des estimations sur la solution du schéma correspondant, à condition que la taille $h_{\mathcal{T}}$ du maillage soit assez petite. Munis de ces estimations, la convergence se fait alors comme dans le cas décentré amont.

Plus précisément: si $\mathcal{M}$ est un maillage de Ω, on considère la discrétisation suivante de (3.1)

$$\forall K \in \mathcal{T}\,,\ \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}u_\sigma + m(K)b_K u_K = m(K)f_K + \sum_{\sigma\in\mathcal{E}_K} m(\sigma)G_{K,\sigma}\,, \tag{3.28}$$

$$\begin{array}{ll} \forall\sigma = K|L \in \mathcal{E}_{\rm int}\,, & F_{K,\sigma} = -\tau_\sigma(u_L - u_K)\,,\\ \forall\sigma \in \mathcal{E}_{\rm ext}\cap\mathcal{E}_K\,, & F_{K,\sigma} = \tau_\sigma u_K\,, \end{array} \tag{3.29}$$

$$\begin{array}{ll} \forall\sigma = K|L \in \mathcal{E}_{\rm int}\,, & u_\sigma = \alpha_{K,\sigma}u_K + \alpha_{L,\sigma}u_L\,,\\ \forall\sigma \in \mathcal{E}_{\rm ext}\cap\mathcal{E}_K\,, & u_{\sigma,+} = \alpha_{K,\sigma}u_K\,, \end{array} \tag{3.30}$$

où $(\alpha_{K,\sigma})_{K\in\mathcal{T},\,\sigma\in\mathcal{E}_K}$ sont fixés (ne dépendent que des données $(\mathbf{v}, f, G)$ du problème) dans $[0,1]$ de sorte que $\alpha_{K,\sigma} + \alpha_{L,\sigma} = 1$ dès que $\sigma = K|L \in \mathcal{E}_{\rm int}$.
Le choix du décentrement amont consiste à prendre, modulo l'indétermination non gênante dont nous avons parlé en remarque 3.2, $\alpha_{K,\sigma} = 1$ si $v_{K,\sigma} \geq 0$ et $\alpha_{K,\sigma} = 0$ dans le cas contraire.
Le choix centré consiste à prendre $\alpha_{K,\sigma} = \frac{1}{2}$ pour tout $K \in \mathcal{T}$ et tout $\sigma \in \mathcal{E}_K$.

L'obtention d'estimations a priori sur les solutions du système linéaire (3.28)—(3.30) se fait en le comparant au système (3.3)—(3.5).

Proposition 3.3 *Soit $\zeta > 0$ tel que le maillage $\mathcal{M}$ vérifie: pour tout $K \in \mathcal{T}$, pour tout $\sigma \in \mathcal{E}_K$, $d_{K,\sigma} \geq \zeta d_\sigma$. Il existe $h_0 > 0$ et C ne dépendant que de $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, Ω et ζ tel que, si $h_\mathcal{T} \leq h_0$, toute solution $u_\mathcal{T}$ de (3.28)—(3.30) avec un second membre vérifiant (3.6) satisfait $||u_\mathcal{T}||_{1,\mathcal{M}} \leq C\mathcal{N}(f_\mathcal{T}, G_\mathcal{M})$.*

Preuve de la proposition 3.3
On introduit le décentrement amont $u_{\sigma,+}$ de $u_\mathcal{T}$ comme défini dans (3.5). $u_\mathcal{T}$ vérifie alors, pour tout $K \in \mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}u_{\sigma,+} + m(K)b_K u_K = m(K)f_K + \sum_{\sigma\in\mathcal{E}_K} m(\sigma)\left(G_{K,\sigma} + \frac{v_{K,\sigma}}{m(\sigma)}(u_{\sigma,+} - u_\sigma)\right),$$

c'est à dire une équation de la forme (3.3)—(3.5) où l'on a remplace $G_{K,\sigma}$ par $\widetilde{G}_{K,\sigma} = G_{K,\sigma} + \frac{v_{K,\sigma}}{m(\sigma)}(u_{\sigma,+} - u_\sigma)$.
Les flux $(v_{K,\sigma})_{K\in\mathcal{T},\,\sigma\in\mathcal{E}_K}$ étant conservatifs, ce second membre vérifie (3.6) et on a donc, par la proposition 3.2,

$$||u_\mathcal{T}||_{1,\mathcal{M}} \leq C_1\mathcal{N}(f_\mathcal{T}, \widetilde{G}_\mathcal{M}),$$

où C_1 ne dépend que de $||\mathbf{v}||_{(L^\infty(\Omega))^N}$, Ω et ζ.
Pour estimer $\mathcal{N}(f_\mathcal{T}, \widetilde{G}_\mathcal{M})$, on écrit

$$\begin{aligned} &\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma \left|G_{K,\sigma} + \frac{v_{K,\sigma}}{m(\sigma)}(u_{\sigma,+} - u_\sigma)\right|^2\right)^{1/2}\\ &\leq\ \sqrt{2}\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma G_\sigma^2\right)^{1/2} + \sqrt{2}\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma\left(\frac{v_\sigma}{m(\sigma)}(u_{\sigma,+} - u_\sigma)\right)^2\right)^{1/2}. \end{aligned}$$

Comme $v_\sigma \leq ||\mathbf{v}||_{(L^\infty(\Omega))^N} m(\sigma)$ et $|u_{\sigma,+} - u_\sigma| \leq D_\sigma u$ (car $(u_{\sigma,+}, u_\sigma) \in [u_K, u_L]$ si $\sigma = K|L \in \mathcal{E}_{\rm int}$ et $(u_{\sigma,+}, u_\sigma) \in [0, u_K]$ si $\sigma \in \mathcal{E}_{\rm ext}\cap\mathcal{E}_K$), on peut écrire

$$\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma\left(\frac{v_\sigma}{m(\sigma)}(u_{\sigma,+} - u_\sigma)\right)^2\right)^{1/2} \quad\leq\quad ||\mathbf{v}||_{(L^\infty(\Omega))^N}\left(\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma(D_\sigma u)^2\right)^{1/2}$$

$$\begin{aligned} &\leq \quad ||\mathbf{v}||_{(L^\infty(\Omega))^N} \left(\sum_{\sigma \in \mathcal{E}} d_\sigma^2 \tau_\sigma (D_\sigma u)^2 \right)^{1/2} \\ &\leq \quad 2h_\mathcal{T} ||\mathbf{v}||_{(L^\infty(\Omega))^N} ||u_\mathcal{T}||_{1,\mathcal{M}}. \end{aligned}$$

Ainsi,

$$||u_\mathcal{T}||_{1,\mathcal{M}} \leq C_1\sqrt{2}\mathcal{N}(f_\mathcal{T}, G_\mathcal{M}) + 2C_1\sqrt{2}h_\mathcal{T}||\mathbf{v}||_{(L^\infty(\Omega))^N}||u_\mathcal{T}||_{1,\mathcal{M}}.$$

Cette inégalité nous montre que le résultat de la proposition est valide avec $h_0 = 1/(4\sqrt{2}C_1||\mathbf{v}||_{(L^\infty(\Omega))^N})$ et $C = 2\sqrt{2}C_1$. ■

Muni de cette estimation conditionnelle (valable uniquement pour $h_\mathcal{T}$ petit), on obtient, exactement comme dans l'étude du système décentré amont, des résultats de convergence et d'estimation d'erreur pour les solutions de (3.28)—(3.30), à condition de supposer à chaque fois $h_\mathcal{T}$ assez petit.

Partie II

Unicité des Solutions Obtenues comme Limites d'Approximations

Chapitre 4

A uniqueness result for quasilinear elliptic equations with measures as data

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Abstract We prove here a uniqueness result for Solutions Obtained as the Limit of Approximations of quasilinear elliptic equations with different kinds of boundary conditions and measures as data.

4.1 Introduction

4.1.1 Notations

In this paper, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with a Lipschitz continuous boundary. The unit normal to $\partial\Omega$ outward to Ω is denoted by $\mathbf{n}$. We denote by $x \cdot y$ the usual Euclidean product of two vectors $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$; the associated Euclidean norm is written $|.|$. The Lebesgue measure of a measurable subset E in $\mathbb{R}^N$ is denoted by $|E|$; σ is the Lebesgue measure on $\partial\Omega$ (i.e. the (N–1)-dimensional Hausdorff measure). Γ_d and Γ_f are measurable subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_d \cup \Gamma_f$ and $\sigma(\Gamma_d \cap \Gamma_f) = 0$.
For $q \in [1, +\infty]$, we denote by q' the conjugate exponent of q (i.e. $q' = q/(q-1)$). $W^{1,q}(\Omega)$ is the usual Sobolev space, endowed with the norm $||u||_{W^{1,q}(\Omega)} = ||u||_{L^q(\Omega)} + ||\,|\nabla u|\,||_{L^q(\Omega)}$. $W^{1,q}_{\Gamma_d}(\Omega)$ is the space of functions of $W^{1,q}(\Omega)$ which have a null trace on Γ_d.
When $q = 2$, we write $H^1_{\Gamma_d}(\Omega)$ instead of $W^{1,q}_{\Gamma_d}(\Omega)$. The space of the traces of functions in $H^1_{\Gamma_d}(\Omega)$ is denoted by $H^{1/2}_{\Gamma_d}(\Omega)$ and it is endowed with the norm

$$||u||_{H^{1/2}_{\Gamma_d}(\Omega)} = \inf\{||f||_{H^1(\Omega)} \mid f \in H^1_{\Gamma_d}(\Omega)\,,\ f_{|\partial\Omega} = u\}.$$

The hypotheses on the function a that will define our quasilinear elliptic equation are the following:

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Caratheodory function,} \tag{4.1}$$

$$\begin{array}{c}\exists \gamma > 0\,,\ \Theta \in L^1(\Omega) \text{ such that } a(x, s, \xi) \cdot \xi \geq \gamma|\xi|^2 - \Theta(x)\\ \text{for a.e. } x \in \Omega, \text{ for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,\end{array} \tag{4.2}$$

$$\begin{array}{c}\exists \beta > 0 \text{ and } h \in L^2(\Omega) \text{ such that } |a(x, s, \xi)| \leq h(x) + \beta|s| + \beta|\xi|\\ \text{for a.e. } x \in \Omega, \text{ for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,\end{array} \tag{4.3}$$

$$\begin{array}{c}\exists \alpha > 0 \text{ such that } (a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) \geq \alpha|\xi - \eta|^2\\ \text{for a.e. } x \in \Omega, \text{ for all } (s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N,\end{array} \tag{4.4}$$

$$\begin{array}{c}\exists \Lambda > 0 \text{ such that } |a(x, s, \xi) - a(x, s, \eta)| \leq \Lambda|\xi - \eta|\\ \text{for a.e. } x \in \Omega, \text{ for all } (s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N,\end{array} \tag{4.5}$$

$$\begin{array}{c}\exists\delta>0 \text{ such that}\\ |a(x,s,\xi)-a(x,t,\xi)|\le\delta|s-t| \text{ for a.e. } x\in\Omega,\\ \text{for all } (s,t,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N.\end{array}\tag{4.6}$$

Remark 4.1 *Hypotheses (4.1)—(4.3) are classical for the Leray-Lions operators in divergence form acting on $H^1(\Omega)$; Hypothesis (4.4) is a stronger form of the classical monotonicity hypothesis*

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)>0 \textit{ for a.e. } x\in\Omega, \textit{ for all } (s,\xi,\eta)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \textit{ with } \xi\neq\eta.\tag{4.7}$$

of the Leray-Lions operators, but is nevertheless classical when we want to obtain a uniqueness result, even in the variational case (see [11]). Hypothesis (4.5) is not really demanding, since, for example, $a(x,s,\xi)=\widetilde{a}(s)\xi$ (with $\widetilde{a}\in L^\infty(\Omega)$) satisfies this hypothesis, but Hypothesis (4.6) is really strong and we would rather like to impose a weaker hypothesis, of the kind

$$\begin{array}{c}\exists\delta>0 \textit{ such that } |a(x,s,\xi)-a(x,t,\xi)|\le\delta|s-t|(1+|s|+|t|+|\xi|)\\ \textit{for a.e. } x\in\Omega, \textit{ for all } (s,t,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\end{array}$$

to handle the case $a(x,s,\xi)=\widetilde{a}(s)\xi$ with $\widetilde{a}$ Lipschitz continuous.

Remark 4.2 *There are however many functions which satisfy Hypotheses (4.1)—(4.6). For example, for $M\ge0$, $a(x,s,\xi)=(1+\inf(M,\ln(1+|s|+|\xi|)))\xi+\phi(x,s)$, with $\phi:\Omega\times\mathbb{R}\to\mathbb{R}^N$ a Caratheodory function, Lipschitz continuous with respect to $s\in\mathbb{R}$ (with a Lipschitz constant not depending on $x\in\Omega$) and such that $\sup_{s\in\mathbb{R}}|\phi(.,s)|\in L^2(\Omega)$.*

Consider the problem

$$\left\{\begin{array}{l}-\mathrm{div}(a(x,u,\nabla u))=f \quad \text{in} \quad \Omega,\\ u=0 \quad \text{on} \quad \partial\Omega.\end{array}\right.\tag{4.8}$$

It is well known (see [10]) that, when f is a bounded measure on Ω and a satisfies (4.1)—(4.3) and (4.7), we can find a solution to this problem (even when we consider an operator acting on $W_0^{1,p}(\Omega)$, $1<p<\infty$ — see also [4] when $p<1-\frac{2}{N}$ —, not only on $H_0^1(\Omega)$). The main idea of [10] is to approximate f by regular functions, find estimates on the corresponding solutions and pass to the limit.
Moreover, when a does not depend on s and f is a function in $L^1(\Omega)$, we can find (see [4]) a formulation (so-called "entropy formulation") for (4.8) which ensures the uniqueness of the solution (the existence is still obtained by approximation).

In [57], the author defines another sense of solution, the "solution by transposition", which gives an existence and uniqueness result when a still does not depend on s but f is a bounded measure. This definition requires the introduction of a particular matrix-valued function $M(.,.):\mathbb{R}^N\times\mathbb{R}^N\longrightarrow M_N(\mathbb{R})$ satisfying a few properties (general algebraic properties, completely independent of a); the formulation by transposition uses then the matrix $M(\nabla u-\nabla v, a(.,\nabla u)-a(.,\nabla v))$, where u is the solution by transposition and v is any function in $H_0^1(\Omega)$. There can be many different possible choices of the matrix $M(.,.)$ (the matrix chosen by the author depends on a parameter λ, which is any real number in $]0,\alpha[$, where α is given by (4.4)). The solution by transposition seems thus to depend on the particular choice of M; however, an additional work allows to see that, with the methods of [57], we can prove the uniqueness of the solution obtained as the limit of approximations (when a is independent of s).

When f is a bounded measure, a satisfies (4.1)—(4.5) but does not depend on s and is of class $\mathcal{C}^1$ with respect to ξ, the uniqueness of the solution obtained as the limit of approximations of Problem (4.8) is proven in [6].
We will prove here that the ideas of [6] can lead to a uniqueness result when f is a bounded measure, a depends on s (but satifies (4.6)) and is only Lipschitz continuous with respect to ξ. The main difficulty brought by the dependence of a on s is in the resolution of the "dual equation" (4.19) in which the

operator is not coercive (because of the convection term). We will also consider more general boundary conditions; they bring a few more difficulties (in particular the regularity result we need on the solution of (4.19)) which are solved by the results of [29].

The boundary conditions we consider are of the mixed or Fourier kind (that is to say a condition on u on Γ_d and a condition on $a(x,u,\nabla u)\cdot\mathbf{n}+\lambda u$ on Γ_f).
To get the coercivity that will ensure the existence of a solution, we add the assumption

$$\begin{array}{c}\sigma(\Gamma_d)>0 \text{ and } \lambda\in L^\infty(\partial\Omega)\,,\ \lambda\ge 0\ \sigma\text{-a.e. on } \partial\Omega\\ \text{or}\\ \Gamma_d=\emptyset \text{ and } \lambda\in L^\infty(\partial\Omega)\,,\ \lambda\ge 0\ \sigma\text{-a.e. on } \partial\Omega,\ \sigma(\{x\in\partial\Omega\mid\lambda(x)>0\})\neq 0.\end{array}\tag{4.9}$$

Remark 4.3 *Under Hypothesis (4.9), a classical reasoning shows that, for all $q\in[1,+\infty[$, $\overline{q}\in[1,q]$ and $\rho>0$, there exists $\mathcal{K}_{q,\overline{q}}(\rho,\Omega,\Gamma_d,\lambda)>0$ such that, for all $v\in W^{1,q}_{\Gamma_d}(\Omega)$, we have*

$$\rho\int_\Omega|\nabla v|^q+\left(\int_{\Gamma_f}\lambda|v|^{\overline{q}}\,d\sigma\right)^{q/\overline{q}}\ge\mathcal{K}_{q,\overline{q}}(\rho,\Omega,\Gamma_d,\lambda)||v||^q_{W^{1,q}(\Omega)}.\tag{4.10}$$

The proof of uniqueness we present here uses an existence and regularity result of a solution to a dual problem. To obtain the required regularity result, we need some hypotheses on the way Γ_d and Γ_f are distributed along $\partial\Omega$.
Let us introduce two kinds of mapping of $\partial\Omega$:

$$\begin{array}{c}O \text{ is an open subset of } \mathbb{R}^N,\\ h:O\to B:=\{x\in\mathbb{R}^N\mid |x|<1\} \text{ is a Lipschitz continuous}\\ \text{homeomorphism with a Lipschitz continuous inverse mapping,}\\ h(O\cap\Omega)=B_+:=\{x\in B\mid x_N>0\},\\ h(O\cap\partial\Omega)=B^{N-1}:=\{x\in\partial B_+\mid x_N=0\}\end{array}\tag{4.11}$$

(since Ω has a Lipschitz continuous boundary, there exists a finite number of $(O_i,h_i)_{i\in[1,m]}$, such that, for all $i\in[1,m]$, (O_i,h_i) satisfies (4.11) and $\partial\Omega\subset\cup_{i=1}^m O_i$) and

$$\begin{array}{c}O \text{ is an open subset of } \mathbb{R}^N,\\ h:O\to B \text{ is a Lipschitz continuous homeomorphism}\\ \text{with a Lipschitz continuous inverse mapping,}\\ h(O\cap\Omega)=B_{++}:=\{x\in B\mid x_N>0\,,\ x_{N-1}>0\},\\ h(O\cap\Gamma_f)=\Gamma_1:=\{x\in\partial B_{++}\mid x_{N-1}=0\},\\ h(O\cap\Gamma_d)=\Gamma_2:=\{x\in\partial B_{++}\mid x_N=0\}.\end{array}\tag{4.12}$$

The additional assumption we make on Γ_d and Γ_f is the following:

$$\begin{array}{c}\text{There exists a finite number of } (O_i,h_i)_{i\in[1,m]} \text{ such that}\\ \partial\Omega\subset\cup_{i=1}^m O_i \text{ and, for all } i\in[1,m],\ (O_i,h_i) \text{ is of one of the following types:}\\ \left|\begin{array}{ll}(D) & O_i\cap\partial\Omega=O_i\cap\Gamma_d \text{ and } (O_i,h_i) \text{ satisfies (4.11)}\\ (F) & O_i\cap\partial\Omega=O_i\cap\Gamma_f \text{ and } (O_i,h_i) \text{ satisfies (4.11)}\\ (DF) & (O_i,h_i) \text{ satisfies (4.12).}\end{array}\right.\end{array}\tag{4.13}$$

4.1.2 The SOLA and the main result

We recall here some facts about the solutions obtained as the limit of approximations for quasilinear elliptic equations with measures as data.

We denote by $\mathcal{M}(\Omega)$ the space of bounded measures on Ω and $\mathcal{M}(\partial\Omega)$ the space of bounded measures on $\partial\Omega$.
If $\mu \in \mathcal{M}(\Omega)$ and $\mu^\partial \in \mathcal{M}(\partial\Omega)$, we consider the problem

$$\left\{\begin{array}{l} -\mathrm{div}(a(x,u,\nabla u)) = \mu \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_d, \\ a(x,u,\nabla u)\cdot \mathbf{n} + \lambda u = \mu^\partial \quad \text{on} \quad \Gamma_f. \end{array}\right. \tag{4.14}$$

The technique of approximation introduced in [10] is the following: let $(\mu_n)_{n\geq 1} \in \mathcal{M}(\Omega)\cap (H^1(\Omega))'$([1]) such that $\mu_n \to \mu$ for the weak-$*$ topology of $(\mathcal{C}(\overline{\Omega}))'$, $(\mu_n^\partial)_{n\geq 1} \in \mathcal{M}(\partial\Omega)\cap(H^{1/2}(\partial\Omega))'$ such that $\mu_n^\partial \to \mu^\partial$ for the weak-$*$ topology of $\mathcal{M}(\partial\Omega)$ and take u_n a solution to

$$\left\{\begin{array}{l} u_n \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi + \int_{\Gamma_f} \lambda u_n \varphi\, d\sigma = \langle \mu_n, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} \\ \qquad\qquad + \langle \mu_n^\partial, \varphi\rangle_{(H^{1/2}_{\Gamma_d}(\partial\Omega))', H^{1/2}_{\Gamma_d}(\partial\Omega)}\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{4.15}$$

We can prove that the sequence $(u_n)_{n\geq 1}$ is bounded in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q < N/(N-1)$; thus, up to a subsequence, $u_n \to u$ strongly in $L^q(\Omega)$ and weakly in $W^{1,q}_{\Gamma_d}(\Omega)$; it is then possible to prove that, up to a subsequence, $\nabla u_n \to \nabla u$ a.e. on Ω, which allows us to pass to the limit in the equation of (4.15) to see that u satisfies

$$\left\{\begin{array}{l} u \in \displaystyle\bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi + \int_{\Gamma_f}\lambda u\varphi = \int_\Omega \varphi\, d\mu + \int_{\partial\Omega}\varphi\, d\mu^\partial\,,\ \forall\varphi \in \bigcup_{r>N} W^{1,r}_{\Gamma_d}(\Omega). \end{array}\right. \tag{4.16}$$

A Solution Obtained as the Limit of Approximations (a SOLA) for (4.14) is any u obtained by the method detailed above.

Remark 4.4 *In [10], where the SOLA (without this name, used for the first time in [22]) have been introduced, the authors study the pure homogeneous Dirichlet case (with $\Theta = 0$). But the adaptation of their methods to the non-homogeneous mixed or Fourier case is quite straightforward (see [66] for the Fourier case with $\Theta \equiv 0$), even with a non-null $\Theta \in L^1(\Omega)$.*

When $N \geq 3$, the solution of (4.16) is not always unique; indeed, a counter-example by J. Serrin [68] modified by A. Prignet [65] gives a non-null solution of (4.16) in the linear ($a(x,s,\xi) = A(x)\xi$) Dirichlet case when $\mu = \mu^\partial = 0$ (see also [29] for the adaptation of this counter-example to the mixed case).
However, there is uniqueness of the SOLA for this problem, and this is the main result of this paper:

Theorem 4.1 *Under Hypotheses (4.1)—(4.6), (4.9) and (4.13), Problem (4.14) has one and only one SOLA.*

[1] This means that μ_n is a measure on Ω such that there exists $C > 0$ satisfying, for all $\varphi \in \mathcal{C}(\overline{\Omega})\cap H^1(\Omega)$, $|\int \varphi\, d\mu_n| \leq C||\varphi||_{H^1(\Omega)}$; by density of $\mathcal{C}(\overline{\Omega})\cap H^1(\Omega)$ in $H^1(\Omega)$, there exists then a unique extension of μ_n as an element of $(H^1(\Omega))'$. The same kind of definition and consideration apply to $\mu_n^\partial \in \mathcal{M}(\partial\Omega)\cap(H^{1/2}(\partial\Omega))'$.

Remark 4.5 *In fact, the proof of the existence of a SOLA to (4.14) does not use all our hypotheses on a (it only uses (4.1)—(4.3), (4.7) and (4.9)). Our proof of the uniqueness of the SOLA does not use all the Hypotheses we put on a too; indeed, we will see that we do not use (4.2) and (4.3) in this paper, we only use the fact that a SOLA exists. Thus, this result of uniqueness can be extented to other equations for which we know a SOLA exists. For example, in [8], L. Boccardo proves a wide existence result (for a pure Dirichlet problem — this is quite important — with a right-hand side in L^1) that entails the existence of a SOLA for an operator defined by a function of the kind*

$$a(x,s,\xi) = a_0(x,s,\xi) + \phi(s),$$

where a_0 satisfies (4.1)—(4.6) and $\phi : \mathbb{R} \to \mathbb{R}^N$ is a Lipschitz continuous function; the hypotheses on ϕ in [8] are in fact much weaker and require thus $f \in L^1(\Omega)$, but our stronger hypotheses allow us to take a right-hand side in $\mathcal{M}(\Omega)$. Thus, a satisfies (4.1), (4.4)—(4.6) and the existence and uniqueness result of Theorem 4.1 is still valid for such an operator in the pure Dirichlet case.

We will also see that this uniqueness result implies the following (very simple) stability result.

Theorem 4.2 *Let $(\mu_n)_{n\geq 1} \in \mathcal{M}(\Omega)$ converges to μ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and $(\mu_n^\partial)_{n\geq 1} \in \mathcal{M}(\partial\Omega)$ converges to μ^∂ in $\mathcal{M}(\partial\Omega)$ weak-$*$. Under Hypotheses (4.1)—(4.6), (4.9) and (4.13), if u_n is the SOLA of (4.14) with (μ_n, μ_n^∂) instead of (μ, μ^∂) and u is the SOLA of (4.14), then $u_n \to u$ strongly in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q < \frac{N}{N-1}$.*

Remark 4.6 *In fact, we will prove the following more general result: under Hypotheses (4.1)—(4.3), (4.7) and (4.9), if u_n is* a *SOLA — of a slightly particular kind, see in the proof of Theorem 4.2 — of (4.14) with (μ_n, μ_n^∂) instead of (μ, μ^∂), there exists a subsequence $(u_{n_k})_{k\geq 1}$ and* a *SOLA u of (4.14) such that $u_{n_k} \stackrel{k\to\infty}{\longrightarrow} u$ strongly in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q < N/(N-1)$. The fact that we can, with stronger hypotheses, get rid of the subsequence is of course due to the uniqueness of the SOLA in this case.*

Remark 4.7 *Once again, the proof of this stability result only uses the existence and uniqueness of the SOLA, not all the hypotheses on a (especially, we do not use (4.2) and (4.3)); thus Theorem 4.2 is also valid for other kinds of quasilinear equations for which we know a SOLA exists, such as the example given in Remark 4.5.*

A uniqueness result for a linear equation is very often linked to an existence result for a dual equation. It is also the case here, although (4.14) is not a linear problem; so, before the proof of Theorem 4.1, we study in Section 2 an equation which will appear as the dual equation of a problem coming from (4.14).

4.2 The "dual" equation

We make the following hypotheses:

$$\begin{array}{c} A : \Omega \to M_N(\mathbb{R}) \text{ is a measurable matrix valued function which satisfies:} \\ \exists \alpha > 0 \text{ such that } A(x)\xi \cdot \xi \geq \alpha|\xi|^2 \text{ for a.e. } x \in \Omega\,, \text{ for all } \xi \in \mathbb{R}^N, \\ \exists M > 0 \text{ such that } ||A(x)|| := \sup\left\{|A(x)\xi|\,,\ \xi \in \mathbb{R}^N\,,\ |\xi| = 1\right\} \leq M \text{ for a.e. } x \in \Omega, \end{array} \quad (4.17)$$

$$\mathbf{v} \in (L^\infty(\Omega))^N\,, \quad (4.18)$$

and we take α_A a coercivity constant for A, Λ_A an essential upper bound of $||A(.)||$ on Ω and $\Lambda_{\mathbf{v}}$ an upper bound of $||\,|\mathbf{v}|\,||_{L^\infty(\Omega)}$.

We will prove the following existence result:

Theorem 4.3 *Under Hypotheses (4.17), (4.18), (4.9) and (4.13), if $\theta \in L^\infty(\Omega)$ then, by denoting by Λ_θ an upper bound of $||\theta||_{L^\infty(\Omega)}$, there exists $\kappa \in]0,1[$ depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda)$, C_0 depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$ and C_1 depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \Lambda_\theta)$ such that there exists a solution to*

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega) \cap \mathcal{C}^{0,\kappa}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_\Omega \mathbf{v}\cdot\nabla f\varphi + \int_{\Gamma_f} \lambda f\varphi\, d\sigma = \int_\Omega \theta\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{4.19}$$

satisfying $||f||_{\mathcal{C}^{0,\kappa}(\Omega)} \le C_0$ and $||f||_{H^1(\Omega)} \le C_1$.

Remark 4.8 *We have denoted by $\mathcal{C}^{0,\kappa}(\Omega)$ the space of κ-Hölder continuous functions on Ω, endowed with the norm*

$$||f||_{\mathcal{C}^{0,\kappa}(\Omega)} = ||f||_{L^\infty(\Omega)} + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\kappa}.$$

Remark 4.9 *Without Hypothesis (4.13), we obtain a solution of the equation in Problem (4.19) in the space $H^1_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$, with the same kind of estimates (we will notice it in the course of the proof); Hypothesis (4.13) is only useful to apply the results of [29] in order to obtain the Hölder continuity of the solution.*

To prove the existence result of Theorem 4.3, we need an *a priori* estimate on the solutions of (4.19) (an L^1 estimate is enough). This is the aim of Lemma 4.1 for the proof of which the authors wish to thank Lucio Boccardo (for having given them the key estimate of Step 2).

Lemma 4.1 *Let A satisfy (4.17), $\mathbf{w} \in (L^\infty(\Omega))^N$ and $\tau \in L^\infty(\Omega)$; we denote by $\Lambda_{\mathbf{w}}$ an upper bound of $||\,|\mathbf{w}|\,||_{L^\infty(\Omega)}$ and Λ_τ an upper bound of $||\tau||_{L^\infty(\Omega)}$. Under Hypothesis (4.9), there exists C_0 depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{w}}, \lambda, \Lambda_\tau)$ and a solution to*

$$\left\{ \begin{array}{l} g \in H^1_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla g \cdot \nabla\varphi + \int_\Omega g\mathbf{w}\cdot\nabla\varphi + \int_{\Gamma_f} \lambda g\varphi\, d\sigma = \int_\Omega \tau\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{4.20}$$

such that $||g||_{H^1(\Omega)} + ||g||_{L^\infty(\Omega)} \le C_0$.

Remark 4.10 *Once we know that g satisfies (4.20), since $\varphi \to \int_\Omega g\mathbf{v}\cdot\nabla\varphi$ is in $(W^{1,1}_{\Gamma_d}(\Omega))'$ (because g is essentially bounded), the results of [29] show that, under Hypothesis (4.13), g is in fact Hölder continuous on $\overline{\Omega}$.*

Remark 4.11 *The conclusions of Theorem 4.3 and Lemma 4.1 also hold when θ or τ only belong to $\bigcup_{p>N}(W^{1,p'}_{\Gamma_d}(\Omega))'$ (the proof of this uses the same ideas we present here; see [70] or [29] for the details concerning the treatment of right-hand sides of this kind).*

Remark 4.12 *(Lucio Boccardo [7]) A close examination of the second step of the proof of Lemma 4.1 shows that the bound we obtain on $||\ln(1+|g_n|)||_{H^1_{\Gamma_d}(\Omega)}$ depends on the L^1-norm of the right-hand side τ. Thus, we can easily prove (by approximation) an existence result for*

$$\left\{ \begin{array}{l} -\mathrm{div}(A^T\nabla g) - \mathrm{div}(g\mathbf{v}) = \tau \quad \text{in} \quad \Omega\,, \\ g = 0 \quad \text{on} \quad \Gamma_d\,, \\ A^T\nabla g\cdot\mathbf{n} + \lambda g = 0 \quad \text{on} \quad \Gamma_f\,, \end{array} \right. \tag{4.21}$$

(this problem has, when τ is regular, (4.20) as variational formulation) when τ is a bounded measure on Ω; we must however be careful with the formulation of (4.21) since we only obtain a "solution" g such that, for all $k \ge 0$, $T_k(g) \in H^1_{\Gamma_d}(\Omega)$ (where $T_k(s) = \min(k, \max(s,-k))$).

Remark 4.13 *Using the results of Theorem 4.3 and Lemma 4.1 and the ideas of their proofs, we can prove, when $L \in (H^1(\Omega))'$, the existence and uniqueness of solutions to*

$$\left\{\begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi + \int_\Omega \mathbf{v}\cdot\nabla f\varphi + \int_{\Gamma_f}\lambda f\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{4.22}$$

and

$$\left\{\begin{array}{l} g \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla g\cdot\nabla\varphi + \int_\Omega g\mathbf{v}\cdot\nabla\varphi + \int_{\Gamma_f}\lambda g\varphi\,d\sigma = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega).\end{array}\right. \tag{4.23}$$

Remark 4.14 *In fact, to prove Lemma 4.1 and Theorem 4.3 (as well as the results of Remark 4.13), we only need $\mathbf{v} \in (L^r(\Omega))^N$ with a $r > N$. But since such an hypothesis on $\mathbf{v}$ would not allow us to consider really better conditions in Theorem 4.1 (using the result of Theorem 4.3 with $\mathbf{v} \in (L^r(\Omega))^N$ for a $r > N$ would allow us to weaken Hypothesis (4.6), but not enough to handle the case of functions of the form $a(s,\xi) = \widetilde{a}(s)\xi$), we prefer to consider the stronger Hypothesis (4.18), which is sufficient to our purpose here.*

Proof of Lemma 4.1
We will approximate Problem (4.20) by problems for which we have, thanks to the Schauder fixed point theorem, a solution; then, by proving estimates on the solutions of these approximate problems, we will obtain a solution to (4.20) (without passing to the limit!).

Step 1: the approximate problems.
For $t \geq 0$, define $T_t(s) = \min(t, \max(-t, s))$. Let n be an integer and, if $\overline{g} \in L^2(\Omega)$, define $F(\overline{g}) = g$ as the unique solution to

$$\left\{\begin{array}{l} g \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla g\cdot\nabla\varphi + \int_{\Gamma_f}\lambda g\varphi\,d\sigma = \int_\Omega \tau\varphi - \int_\Omega T_n(\overline{g})\mathbf{w}\cdot\nabla\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{4.24}$$

(the bilinear form is coercive on $H^1_{\Gamma_d}(\Omega)$ thanks to (4.10) applied to $q = \overline{q} = 2$ and $\rho = \alpha_A$).
We notice that $F : L^2(\Omega) \to L^2(\Omega)$ is continuous; indeed, if $\overline{g}_m \to \overline{g}_\infty$ in $L^2(\Omega)$, and if (for $m \in \mathbb{N}$ or $m = \infty$) L_m is the linear form

$$\langle L_m,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} = \int_\Omega \tau\varphi - \int_\Omega T_n(\overline{g}_m)\mathbf{w}\cdot\nabla\varphi,$$

then $L_m \to L_\infty$ in $(H^1_{\Gamma_d}(\Omega))'$, so that $g_m = F(\overline{g}_m) \to g_\infty = F(\overline{g}_\infty)$ in $H^1_{\Gamma_d}(\Omega)$, thus in $L^2(\Omega)$.
Moreover, there exists $R > 0$ such that, for all $\overline{g} \in L^2(\Omega)$, $||F(\overline{g})||_{H^1(\Omega)} \leq R$; indeed, by taking g as a test function in (4.24), we get

$$\alpha_A||\,|\nabla g|\,||^2_{L^2(\Omega)} + \int_{\Gamma_f}\lambda|g|^2\,d\sigma \leq ||\tau||_{(H^1_{\Gamma_d}(\Omega))'}||g||_{H^1(\Omega)} + n||\,|\mathbf{w}|\,||_{L^2(\Omega)}||g||_{H^1(\Omega)},$$

which gives, thanks to (4.10),

$$\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)||g||_{H^1(\Omega)} \leq ||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n||\,|\mathbf{w}|\,||_{L^2(\Omega)};$$

thus, $R = \mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)^{-1}(||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n||\,|\mathbf{w}|\,||_{L^2(\Omega)})$ satisfies the property.
$F : L^2(\Omega) \to L^2(\Omega)$ is thus a compact application (thanks to the Rellich theorem) which sends the whole space $L^2(\Omega)$ in the ball of center 0 and radius R in $L^2(\Omega)$.

By the Schauder fixed point theorem, F has a fixed point in the ball of center 0 and radius R; we have thus proven that there exists g_n solution to

$$\left\{\begin{array}{l} g_n \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T\nabla g_n\cdot\nabla\varphi + \int_\Omega T_n(g_n)\mathbf{w}\cdot\nabla\varphi + \int_{\Gamma_f}\lambda g_n\varphi\,d\sigma = \int_\Omega \tau\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{4.25}$$

satisfying

$$\begin{array}{rcl} ||g_n||_{H^1(\Omega)} & \leq & \mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)^{-1}(||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n||\,|\mathbf{w}|\,||_{L^2(\Omega)}) \\ & \leq & \mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)^{-1}(\Lambda_\tau|\Omega|^{\frac12} + n\Lambda_{\mathbf{w}}|\Omega|^{\frac12}). \end{array}$$

Step 2: we prove that $(\ln(1+|g_n|))_{n\geq1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$, using the technique introduced in [10]. Let us first prove an estimate on $\int_{\Gamma_f}\lambda|g_n|\,d\sigma$. Take $\varphi = T_k(g_n)/k\in H^1_{\Gamma_d}(\Omega)$ as a test function in (4.25). We obtain, since $|T_k(s)/k|\leq 1$ for all $s\in\mathbb{R}$ and $\nabla(T_k(g_n)) = \mathbf{1}_{\{0<|g_n|<k\}}\nabla g_n$ a.e. on Ω (where $\mathbf{1}_E$ is the characteristic function of a set E),

$$\begin{array}{rcl} \displaystyle\int_{\Gamma_f}\lambda\frac{T_k(g_n)}{k}g_n\,d\sigma & \leq & \displaystyle\frac1k\int_\Omega A^T\nabla g_n\nabla(T_k(g_n)) + \int_{\Gamma_f}\lambda\frac{T_k(g_n)}{k}g_n\,d\sigma \\ & \leq & \displaystyle\int_\Omega|\tau| + \int_{\{0<|g_n|<k\}}|\mathbf{w}||g_n|\frac{|\nabla g_n|}{k} \\ & \leq & \displaystyle\int_\Omega|\tau| + ||\,|\mathbf{w}|\,||_{L^2(\Omega)}\left(\int_{\{0<|g_n|<k\}}|\nabla g_n|^2\right)^{\frac12}. \end{array} \tag{4.26}$$

But $g_nT_k(g_n)/k\to|g_n|$ on $\partial\Omega$ as $k\to0$ (if $g_n(x)=0$, $g_n(x)T_k(g_n(x))/k=0$ and, if $g_n(x)\neq0$, $T_k(g_n(x))/k\to\mathrm{sgn}(g_n(x))$) and $|g_nT_k(g_n)/k|\leq|g_n|\in L^1(\partial\Omega)$; thus, by the dominated convergence theorem, $\int_{\Gamma_f}\lambda g_n(T_k(g_n)/k)\,d\sigma\to\int_{\Gamma_f}\lambda|g_n|$. Moreover, since $\nabla g_n\in L^2(\Omega)$ and $|\{0<|g_n|<k\}|\to0$ as $k\to0$ (this is the non-increasing continuity of the measure, associated to the fact that $\cap_{k>0}\{0<|g_n|<k\}=\emptyset$), we obtain $\int_{\{0<|g_n|<k\}}|\nabla g_n|^2\to0$ as $k\to0$. Thus, passing to the limit $k\to0$ in (4.26), we obtain

$$\int_{\Gamma_f}\lambda\ln(1+|g_n|)\,d\sigma \leq \int_{\Gamma_f}\lambda|g_n|\,d\sigma \leq \int_\Omega|\tau| \leq |\Omega|\Lambda_\tau. \tag{4.27}$$

Let us now prove an estimate on the derivatives of g_n. Let $k\in\mathbb{N}$ and denote $r_k(s)=T_1(s-T_k(s))$, that is to say

$$\left\{\begin{array}{ll} r_k(s)=-1 & \text{if } s<-k-1 \\ r_k(s)=s+k & \text{if } -k-1\leq s\leq -k \\ r_k(s)=0 & \text{if } -k<s<k \\ r_k(s)=s-k & \text{if } k\leq s\leq k+1 \\ r_k(s)=1 & \text{if } k+1<s. \end{array}\right.$$

We know that $r_k(g_n)\in H^1_{\Gamma_d}(\Omega)$ with $\nabla(r_k(g_n))=\mathbf{1}_{B^n_k}\nabla g_n$, where $B^n_k=\{x\in\Omega\mid k\leq|g_n|<k+1\}$. Using $r_k(g_n)$ as a test function in (4.25), we get thus, since $|g_n|\leq k+1$ on B^n_k and $g_nr_k(g_n)\geq0$ on $\partial\Omega$,

$$\begin{array}{rcl} \alpha_A||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} & \leq & \displaystyle\int_\Omega A^T\nabla(r_k(g_n))\cdot\nabla(r_k(g_n)) + \int_{\Gamma_f}\lambda g_nr_k(g_n)\,d\sigma \\ & = & \displaystyle\int_\Omega A^T\nabla g_n\cdot\nabla(r_k(g_n)) + \int_{\Gamma_f}\lambda g_nr_k(g_n)\,d\sigma \\ & = & \displaystyle\int_\Omega\tau r_k(g_n) - \int_\Omega T_n(g_n)\mathbf{w}\cdot\nabla(r_k(g_n)) \end{array}$$

$$\begin{aligned}
&\leq ||\tau||_{L^1(\Omega)} + \int_{B_k^n} |\mathbf{w}||g_n||\nabla(r_k(g_n))| \\
&\leq \Lambda_\tau|\Omega| + (k+1)||\,|\mathbf{w}|\,||_{L^2(B_k^n)}||\,|\nabla(r_k(g_n))|\,||_{L^2(\Omega)} \\
&\leq \Lambda_\tau|\Omega| + \frac{\alpha_A}{2}||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} + \frac{||\,|\mathbf{w}|\,||^2_{L^2(B_k^n)}}{2\alpha_A}(k+1)^2.
\end{aligned}$$

Thus, we obtain

$$||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} \leq \frac{2\Lambda_\tau|\Omega|}{\alpha_A} + \frac{||\,|\mathbf{w}|\,||^2_{L^2(B_k^n)}}{\alpha_A^2}(k+1)^2. \tag{4.28}$$

We will use this to show that $(\nabla(\ln(1+|g_n|)))_{n\geq 1}$ is bounded in $L^2(\Omega)$.
We have, since Ω is the disjoint union of $(B_k^n)_{k\geq 0}$, and $|g_n|\geq k$ on B_k^n,

$$\begin{aligned}
\int_\Omega |\nabla(\ln(1+|g_n|))|^2 &= \int_\Omega \frac{|\nabla(|g_n|)|^2}{(1+|g_n|)^2} \\
&= \sum_{k\geq 0}\int_{B_k^n} \frac{|\nabla g_n|^2}{(1+|g_n|)^2} \\
&\leq \sum_{k\geq 0}\int_\Omega \frac{|\nabla(r_k(g_n))|^2}{(1+k)^2}.
\end{aligned}$$

Using (4.28), this gives

$$\begin{aligned}
\int_\Omega |\nabla(\ln(1+|g_n|))|^2 &\leq \frac{2\Lambda_\tau|\Omega|}{\alpha_A}\sum_{k\geq 0}\frac{1}{(1+k)^2} + \frac{1}{\alpha_A^2}\sum_{k\geq 0}\int_{B_k^n}|\mathbf{w}|^2 \\
&\leq \frac{\pi^2\Lambda_\tau|\Omega|}{3\alpha_A} + \frac{||\,|\mathbf{w}|\,||^2_{L^2(\Omega)}}{\alpha_A^2}.
\end{aligned}$$

This last estimate, associated to (4.27) and to (4.10) (with $q=2$ and $\overline{q}=1$) gives

$$||\ln(1+|g_n|)||^2_{H^1(\Omega)} \leq \frac{1}{\mathcal{K}_{2,1}(1,\Omega,\Gamma_d,\lambda)}\left(\frac{\pi^2\Lambda_\tau|\Omega|}{3\alpha_A} + \frac{\Lambda_{\mathbf{w}}^2|\Omega|}{\alpha_A^2} + |\Omega|^2\Lambda_\tau^2\right) := C_1$$

(C_1 depends on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$).

Step 3: we conclude by proving that $(g_n)_{n\geq 1}$ is bounded in $L^\infty(\Omega)$.
Let $S_k(s) = s - T_k(s)$; we have $S_k(g_n)\in H^1_{\Gamma_d}(\Omega)$ with $\nabla(S_k(g_n)) = \mathbf{1}_{E_k^n}\nabla g_n$ (where $E_k^n = \{x\in\Omega \mid |g_n(x)|>k\}$). Since $S_k(g_n)=0$ outside E_k^n and since $g_nS_k(g_n) = |g_n||S_k(g_n)|\geq |S_k(g_n)|^2$, we have, using $S_k(g_n)$ as a test function in (4.25),

$$\begin{aligned}
&\alpha_A||\,|\nabla(S_k(g_n))|\,||^2_{L^2(\Omega)} + \int_{\Gamma_f}\lambda|S_k(g_n)|^2\,d\sigma \\
&\leq \int_\Omega A^T\nabla g_n\cdot\nabla(S_k(g_n)) + \int_{\Gamma_f}\lambda g_nS_k(g_n)\,d\sigma \\
&\leq \Lambda_\tau\int_\Omega |S_k(g_n)| + \int_\Omega |\mathbf{w}||g_n||\nabla(S_k(g_n))| \\
&\leq \Lambda_\tau||S_k(g_n)||_{L^2(\Omega)}|E_k^n|^{\frac{1}{2}} + \int_{E_k^n}|\mathbf{w}|(|S_k(g_n)|+k)|\nabla(S_k(g_n))| \\
&\leq \Lambda_\tau||S_k(g_n)||_{L^2(\Omega)}|E_k^n|^{\frac{1}{2}} + ||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}(k||\,|\mathbf{w}|\,||_{L^2(E_k^n)} + ||\,|\mathbf{w}|S_k(g_n)||_{L^2(E_k^n)}) \\
&\leq \Lambda_\tau||S_k(g_n)||_{H^1(\Omega)}|E_k^n|^{\frac{1}{2}} + k\Lambda_{\mathbf{w}}||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}|E_k^n|^{\frac{1}{2}} \\
&\quad + ||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}\Lambda_{\mathbf{w}}||S_k(g_n)||_{L^2(E_k^n)}.
\end{aligned} \tag{4.29}$$

Thanks to the Hölder inequality we have, when $p > 2$,

$$||S_k(g_n)||_{L^2(E_k^n)} \leq ||S_k(g_n)||_{L^p(\Omega)} |E_k^n|^{\frac{1}{2}-\frac{1}{p}}.$$

Since $2 < 2N/(N-2)$, there exists, by the Sobolev injection, $p > 2$ and C_2 only depending on Ω such that

$$||S_k(g_n)||_{L^p(\Omega)} \leq C_2 ||S_k(g_n)||_{H^1(\Omega)}.$$

Thus, with (4.29) and (4.10), we get

$$\begin{aligned}\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda) ||S_k(g_n)||^2_{H^1(\Omega)} \leq\ & \Lambda_\tau ||S_k(g_n)||_{H^1(\Omega)} |E_k^n|^{\frac{1}{2}} + \Lambda_{\mathbf{w}} k ||S_k(g_n)||_{H^1(\Omega)} |E_k^n|^{\frac{1}{2}} \\ & + C_2 \Lambda_{\mathbf{w}} |E_k^n|^{\frac{1}{2}-\frac{1}{p}} ||S_k(g_n)||^2_{H^1(\Omega)}. \end{aligned} \tag{4.30}$$

The Tchebycheff inequality reads

$$\begin{aligned} |E_k^n| = |\{\ln(1+|g_n|) > \ln(1+k)\}| &\leq \frac{1}{(\ln(1+k))^2} || \ln(1+|g_n|)||^2_{L^2(\Omega)} \\ &\leq \frac{1}{(\ln(1+k))^2} || \ln(1+|g_n|)||^2_{H^1(\Omega)} \\ &\leq \frac{C_1^2}{(\ln(1+k))^2} \end{aligned}$$

where C_1 is the constant given by Step 2. Since $1/2 > 1/p$, there exists thus k_0 depending on C_2, $\Lambda_{\mathbf{w}}$, p, C_1 and $\mathcal{K}_{2,2}(\alpha_A, \Omega, \Lambda_d, \lambda)$, i.e. depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{w}}, \lambda, \Lambda_\tau)$, such that, for all $k \geq k_0$ and all $n \geq 1$, $C_2 \Lambda_{\mathbf{w}} |E_k^n|^{\frac{1}{2}-\frac{1}{p}} \leq \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)/2$.
We obtain thus, for all $k \geq k_0$, thanks to (4.30),

$$||S_k(g_n)||_{H^1(\Omega)} \leq \left(\frac{2\Lambda_\tau}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} + \frac{2\Lambda_{\mathbf{w}} k}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} \right) |E_k^n|^{\frac{1}{2}} \leq C_3(1+k)|E_k^n|^{\frac{1}{2}},$$

where C_3 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{w}}, \lambda, \Lambda_\tau)$.
By noticing that, when $h > k$, $|S_k(g_n)| \geq (h-k)$ on E_h^n, we get, thanks to the Sobolev injection $W^{1,1}(\Omega) \hookrightarrow L^{N/(N-1)}(\Omega)$ (the norm of which, denoted by C_4, only depends on Ω),

$$\begin{aligned} (h-k)|E_h^n|^{(N-1)/N} &\leq ||S_k(g_n)||_{L^{N/(N-1)}(\Omega)} \\ &\leq C_4 ||S_k(g_n)||_{W^{1,1}(\Omega)} \\ &\leq C_4 |E_k^n|^{\frac{1}{2}} ||S_k(g_n)||_{H^1(\Omega)} \\ &\leq C_3 C_4 (1+k)|E_k^n|. \end{aligned}$$

Thus, as soon as $h > k \geq k_0$, we have, with $\beta = N/(N-1) > 1$,

$$|E_h^n| \leq \frac{(C_3C_4)^\beta (1+k)^\beta}{(h-k)^\beta} |E_k^n|^\beta \leq \frac{(C_3C_4(1+k_0))^\beta (1+k-k_0)^\beta}{(h-k)^\beta} |E_k^n|^\beta$$

(because, when $k \geq k_0$, $(1+k_0)(1+k-k_0) \geq 1+k$). Lemma 4.2 given just after the end of this proof, and applied to the non-increasing function $G_n(k) = |E_{k+k_0}^n|$, allows us to see that, if

$$H_0 = \exp\left(\sum_{m \geq 0} \frac{2^{\frac{1}{\beta}} C_3 C_4 (1+k_0) |\Omega|^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m} \right) \geq \exp\left(\sum_{m \geq 0} \frac{2^{\frac{1}{\beta}} C_3 C_4 (1+k_0) G_n(0)^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m} \right),$$

(notice that $H_0 < +\infty$ depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{w}}, \lambda, \Lambda_\tau)$), then $G_n(H_0) = 0$, that is to say $|g_n| \leq H_0 + k_0$ a.e. on Ω for all $n \geq 1$.

Thus, by taking n_0 an integer greater than $H_0 + k_0$ (n_0 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{w}}, \lambda, \Lambda_\tau)$) and letting $g = g_{n_0}$, we have a solution to (4.20) (because $T_{n_0}(g_{n_0}) = g_{n_0} = g$) which satisfies $||g||_{L^\infty(\Omega)} \le H_0 + k_0$ and $||g||_{H^1(\Omega)} = ||g_{n_0}||_{H^1(\Omega)} \le \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)^{-1}(\Lambda_\tau|\Omega|^{\frac{1}{2}} + n_0\Lambda_{\mathbf{w}}|\Omega|^{\frac{1}{2}})$. This completes the proof of Lemma 4.1.■

Lemma 4.2 *Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function. If there exist $\beta > 1$ and $C > 0$ such that*

$$\forall h > k \ge 0\,,\ G(h) \le \frac{C^\beta(1+k)^\beta}{(h-k)^\beta} G(k)^\beta$$

then, with

$$H = \exp\left(\sum_{m\ge 0} \frac{2^{\frac{1}{\beta}} C G(0)^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m}\right) < +\infty,$$

we have $G(H) = 0$.

For the proof of this lemma, which is a slight generalization of a lemma by G. Stampacchia ([70] Lemma 4.1, i)), we refer the reader to Lemma 2.2 in [29].

Proof of Theorem 4.3
The proof of the existence of a solution to

$$\left\{\begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi + \int_\Omega \mathbf{v}\cdot\nabla f\varphi + \int_{\Gamma_f} \lambda f\varphi\, d\sigma = \int_\Omega \theta\varphi\,,\ \forall\varphi \in H^1_{\Gamma_d}(\Omega), \end{array}\right. \tag{4.31}$$

(i.e. Problem (4.19) without the regularity $f \in \mathcal{C}^{0,\kappa}(\Omega)$) uses the topological degree (see [26]); the proof of the Hölder continuity of the solution, as well as the estimates in the Hölder space, uses a result of [29].

Step 1: on a cut-off problem.
Let n be an integer. Recall that $T_n(s) = \min(n, \max(-n, s))$. We know that, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$, $T_n(\varphi) \in H^1_{\Gamma_d}(\Omega)$ with $\nabla(T_n(\varphi)) = \mathbf{1}_{\{|\varphi|<n\}}\nabla\varphi$.
Let $\overline{f} \in H^1_{\Gamma_d}(\Omega)$; since $\mathbf{v}\cdot\nabla(T_n(\overline{f})) \in L^2(\Omega) \subset (H^1_{\Gamma_d}(\Omega))'$, there exists a unique solution $f = F(\overline{f})$ to

$$\left\{\begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi + \int_{\Gamma_f} \lambda f\varphi\, d\sigma = \int_\Omega \theta\varphi - \int_\Omega \mathbf{v}\cdot\nabla(T_n(\overline{f}))\varphi\,,\ \forall\varphi \in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{4.32}$$

This defines an application $F : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$.
We will prove, using the topological degree, that F has a fixed point (conversely to the proof of Lemma 4.1, the Schauder fixed point theorem seems not applicable here).

Notice first that F is continuous; indeed, if $\overline{f}_m \to \overline{f}$ in $H^1_{\Gamma_d}(\Omega)$, then $T_n(\overline{f}_m) \to T_n(\overline{f})$ in $H^1_{\Gamma_d}(\Omega)$, so that $\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)) \to \mathbf{v}\cdot\nabla(T_n(\overline{f}))$ in $L^2(\Omega)$, thus also in $(H^1_{\Gamma_d}(\Omega))'$ and the solution $F(\overline{f}_m)$ of (4.32) when $\overline{f}$ is replaced by $\overline{f}_m$ tends thus in $H^1_{\Gamma_d}(\Omega)$ to the solution $F(\overline{f})$ of (4.32).

We will now prove that, if $(\overline{f}_m)_{m\ge1}$ is a bounded sequence in $H^1_{\Gamma_d}(\Omega)$, then there exists a subsequence (still denoted $(\overline{f}_m)_{m\ge1}$) such that $(F(\overline{f}_m))_{m\ge1}$ converges in $H^1_{\Gamma_d}(\Omega)$. Since $(\overline{f}_m)_{m\ge1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$, $(\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)))_{m\ge1}$ is bounded in $L^2(\Omega)$ and there exists thus a subsequence, still denoted $(\overline{f}_m)_{m\ge1}$, such that $\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)) \to \Phi$ weakly in $L^2(\Omega)$.

Since $(F(\overline{f}_m))_{m\geq 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$ (because of the coercivity of the operator in (4.32) and of the fact that $(\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)))_{m\geq 1}$ is bounded in $L^2(\Omega)$), its trace is bounded in $L^2(\partial\Omega)$ and we can also suppose that, up to a subsequence, $(F(\overline{f}_m))_{m\geq 1}$ converges to F_0, weakly in $H^1_{\Gamma_d}(\Omega)$, strongly in $L^2(\Omega)$ and its trace weakly in $L^2(\partial\Omega)$; we see then that F_0 is the solution to

$$\left\{\begin{array}{l} F_0\in H^1_{\Gamma_d}(\Omega)\,,\\ \displaystyle\int_\Omega A\nabla F_0\cdot\nabla\varphi+\int_{\Gamma_f}\lambda F_0\varphi\,d\sigma=\int_\Omega\theta\varphi-\int_\Omega\Phi\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega).\end{array}\right. \tag{4.33}$$

We have now to prove that the convergence of $(F(\overline{f}_m))_{m\geq 1}$ to F_0 is strong in $H^1_{\Gamma_d}(\Omega)$; to see this, we subtract the equation satisfied by F_0 from the equation satisfied by $F(\overline{f}_m)$ and we use the test function $\varphi=F(\overline{f}_m)-F_0\in H^1_{\Gamma_d}(\Omega)$ to find

$$\begin{aligned}
&\alpha_A||\,|\nabla(F(\overline{f}_m)-F_0)|\,||^2_{L^2(\Omega)}+\int_{\Gamma_f}\lambda|F(\overline{f}_m)-F_0|^2\,d\sigma\\
&\leq\ \int_\Omega A\nabla(F(\overline{f}_m)-F_0)\cdot\nabla(F(\overline{f}_m)-F_0)+\int_{\Gamma_f}\lambda(F(\overline{f}_m)-F_0)(F(\overline{f}_m)-F_0)\,d\sigma\\
&=\ \int_\Omega(\Phi-\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)))(F(\overline{f}_m)-F_0)\\
&\leq\ ||\Phi-\mathbf{v}\cdot\nabla(T_n(\overline{f}_m))||_{L^2(\Omega)}||F(\overline{f}_m)-F_0||_{L^2(\Omega)}.
\end{aligned}$$

Since $(\mathbf{v}\cdot\nabla(T_n(\overline{f}_m)))_{m\geq 1}$ is bounded in $L^2(\Omega)$ and $F(\overline{f}_m)\to F_0$ in $L^2(\Omega)$, this inequality, associated to (4.10), gives

$$||F(\overline{f}_m)-F_0||_{H^1(\Omega)}\to 0.$$

Thus, $F:H^1_{\Gamma_d}(\Omega)\to H^1_{\Gamma_d}(\Omega)$ is a compact operator. To prove that F has a fixed point by an application of the Leray-Schauder topological degree, it remains to find $R>0$ such that, if $t\in[0,1]$ and $\overline{f}\in H^1_{\Gamma_d}(\Omega)$ satisfies $\overline{f}-tF(\overline{f})=0$, then $||\overline{f}||_{H^1_{\Gamma_d}(\Omega)}\neq R$.
Suppose we have such a $t\in[0,1]$ and such a $\overline{f}\in H^1_{\Gamma_d}(\Omega)$; then $\overline{f}$ satisfies

$$\int_\Omega A\nabla\overline{f}\cdot\nabla\varphi+\int_{\Gamma_f}\lambda\overline{f}\varphi\,d\sigma=t\int_\Omega\theta\varphi-t\int_\Omega\mathbf{v}\cdot\nabla(T_n(\overline{f}))\varphi\qquad\text{for all }\varphi\in H^1_{\Gamma_d}(\Omega).$$

Take $\varphi=\overline{f}$; since $\nabla(T_n(\overline{f}))=\mathbf{1}_{\{|\overline{f}|<n\}}\nabla\overline{f}$, we find, with (4.10),

$$\begin{aligned}
\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)||\overline{f}||^2_{H^1(\Omega)}\ &\leq\ \alpha_A\int_\Omega|\nabla\overline{f}|^2+\int_{\Gamma_f}\lambda|\overline{f}|^2\,d\sigma\\
&\leq\ ||\theta||_{(H^1_{\Gamma_d}(\Omega))'}||\overline{f}||_{H^1(\Omega)}+n||\,|\mathbf{v}|\,||_{L^2(\Omega)}||\,|\nabla\overline{f}|\,||_{L^2(\Omega)}\\
&\leq\ \Big(||\theta||_{(H^1_{\Gamma_d}(\Omega))'}+n||\,|\mathbf{v}|\,||_{L^2(\Omega)}\Big)||\overline{f}||_{H^1(\Omega)},
\end{aligned}$$

which gives

$$||\overline{f}||_{H^1(\Omega)}\leq\frac{||\theta||_{(H^1_{\Gamma_d}(\Omega))'}}{\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)}+\frac{n||\,|\mathbf{v}|\,||_{L^2(\Omega)}}{\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)}.$$

Thus, by taking $R=1+(||\theta||_{(H^1_{\Gamma_d}(\Omega))'}+n||\,|\mathbf{v}|\,||_{L^2(\Omega)})/\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)$, we deduce from the properties of the topological degree that F has a fixed point in the ball of center 0 and radius R in $H^1_{\Gamma_d}(\Omega)$.
We denote by f_n such a fixed point, which satisfies

$$\left\{\begin{array}{l} f_n\in H^1_{\Gamma_d}(\Omega)\,,\\ \displaystyle\int_\Omega A\nabla f_n\cdot\nabla\varphi+\int_\Omega\mathbf{v}\cdot\nabla(T_n(f_n))\varphi+\int_{\Gamma_f}\lambda f_n\varphi\,d\sigma=\int_\Omega\theta\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{4.34}$$

and $||f_n||_{H^1(\Omega)} \leq 1 + (||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n\Lambda_{\mathbf{v}}|\Omega|^{\frac{1}{2}})/\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)$.

Step 2: we prove an L^1 estimate for the sequence $(f_n)_{n\geq 1}$ constructed in Step 1.
Let $\mathbf{w}_n = \mathbf{1}_{\{|f_n|<n\}}\mathbf{v}$; we have, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\int_\Omega A\nabla f_n \cdot \nabla\varphi + \int_\Omega \mathbf{w}_n \cdot \nabla f_n \varphi + \int_{\Gamma_f} \lambda f_n \varphi \, d\sigma = \int_\Omega \theta\varphi. \tag{4.35}$$

Since $\Lambda_{\mathbf{v}}$ is an upper bound for $||\,|\mathbf{w}_n|\,||_{L^\infty(\Omega)}$, we can find, thanks to Lemma 4.1, a $g_n \in H^1_{\Gamma_d}(\Omega)$ satisfying, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\int_\Omega A^T\nabla g_n \cdot \nabla\varphi + \int_\Omega g_n \mathbf{w}_n \cdot \nabla\varphi + \int_{\Gamma_f} \lambda g_n \varphi \, d\sigma = \int_\Omega \mathrm{sgn}(f_n)\varphi, \tag{4.36}$$

and such that $||g_n||_{L^\infty(\Omega)} \leq K_0$, where K_0 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda)$ but not n (sgn denotes the sign function, and we have thus $||\mathrm{sgn}(f_n)||_{L^\infty(\Omega)} \leq 1$).
By putting $\varphi = f_n$ in (4.36) and $\varphi = g_n$ in (4.35), we get

$$||f_n||_{L^1(\Omega)} = \int_\Omega \mathrm{sgn}(f_n) f_n = \int_\Omega \theta g_n \leq K_0||\theta||_{L^1(\Omega)}. \tag{4.37}$$

Step 3: with the same methods as in Step 3 of the proof of Lemma 4.1, we prove an L^∞ estimate on $(f_n)_{n\geq 1}$.
Define S_k as in Step 3 of the proof of Lemma 4.1, and use $S_k(f_n)$ as a test function in (4.34): we get, by denoting $E^n_k = \{x \in \Omega \mid |f_n(x)| > k\}$, and since $f_n S_k(f_n) \geq |S_k(f_n)|^2$,

$$\begin{aligned}
&\alpha_A||\,|\nabla(S_k(f_n))|\,||^2_{L^2(\Omega)} + \int_{\Gamma_f} \lambda |S_k(f_n)|^2 \, d\sigma && (4.38)\\
&\leq \int_\Omega A\nabla f_n \cdot \nabla(S_k(f_n)) + \int_{\Gamma_f} \lambda f_n S_k(f_n)\, d\sigma \\
&\leq \Lambda_\theta ||S_k(f_n)||_{L^2(\Omega)}|E^n_k|^{\frac{1}{2}} + \int_{\{|f_n|<n\}} |\mathbf{v}||\nabla f_n||S_k(f_n)| \\
&\leq \Lambda_\theta ||S_k(f_n)||_{L^2(\Omega)}|E^n_k|^{\frac{1}{2}} + \int_\Omega |\mathbf{v}||\nabla(S_k(f_n))||S_k(f_n)| \\
&\leq \Lambda_\theta ||S_k(f_n)||_{H^1(\Omega)}|E^n_k|^{\frac{1}{2}} + ||\,|\nabla(S_k(f_n))|\,||_{L^2(\Omega)}||\,|\mathbf{v}|S_k(f_n)||_{L^2(\Omega)} \\
&\leq \Lambda_\theta ||S_k(f_n)||_{H^1(\Omega)}|E^n_k|^{\frac{1}{2}} + ||\,|\nabla(S_k(f_n))|\,||_{L^2(\Omega)}\Lambda_{\mathbf{v}}||S_k(f_n)||_{L^2(\Omega)}, && (4.39)
\end{aligned}$$

because $\nabla f_n = \nabla(S_k(f_n))$ where $S_k(f_n) \neq 0$.
As before, we notice that, thanks to the Sobolev injection of H^1, there exists $p > 2$ and K_1 only depending on Ω such that

$$\begin{aligned}
||S_k(f_n)||_{L^2(\Omega)} &\leq ||S_k(f_n)||_{L^p(\Omega)}|E^n_k|^{\frac{1}{2}-\frac{1}{p}} \\
&\leq K_1||S_k(f_n)||_{H^1(\Omega)}|E^n_k|^{\frac{1}{2}-\frac{1}{p}},
\end{aligned}$$

which gives, introduced in (4.39) and thanks to (4.10),

$$\begin{aligned}
&\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)||S_k(f_n)||^2_{H^1(\Omega)} \\
&\quad\leq \Lambda_\theta||S_k(f_n)||_{H^1(\Omega)}|E^n_k|^{\frac{1}{2}} + K_1\Lambda_{\mathbf{v}}|E^n_k|^{\frac{1}{2}-\frac{1}{p}}||S_k(f_n)||^2_{H^1(\Omega)}. && (4.40)
\end{aligned}$$

But, with (4.37) and the Tchebycheff inequality, we see that

$$|E^n_k| \leq \frac{1}{k}||f_n||_{L^1(\Omega)} \leq \frac{K_0|\Omega|\Lambda_\theta}{k};$$

there exists thus k_0 depending on $(K_1, \Lambda_{\mathbf{v}}, p, K_0, \Omega, \Lambda_\theta, \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda))$ (i.e. on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$), such that, for all $n \geq 1$ and all $k \geq k_0$, $K_1 \Lambda_{\mathbf{v}} |E_k^n|^{\frac{1}{2}-\frac{1}{p}} \leq \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)/2$.
Returning to (4.40), we have then, for all $k \geq k_0$,

$$||S_k(f_n)||_{H^1(\Omega)} \leq \frac{2\Lambda_\theta}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} |E_k^n|^{\frac{1}{2}} = K_2 |E_k^n|^{\frac{1}{2}}$$

where K_2 depends on $(\Omega, \Gamma_d, \alpha_A, \lambda, \Lambda_\theta)$.
Then, reasoning as in the end of Step 3 of the proof of Lemma 4.1, we get, for all $h > k \geq k_0$,

$$|E_h^n| \leq \frac{M K_2^\beta}{(h-k)^\beta} |E_k^n|^\beta,$$

with $\beta = N/(N-1) > 1$ and M depending on Ω.
Using Lemma 4.2 (or, more directly, Lemma 4.1 i) in [70]), we see thus that there exists H_0 depending on (Ω, β, M, K_2), i.e. depending on $(\Omega, \Gamma_d, \alpha_A, \lambda, \Lambda_\theta)$ [notice that a dependence on Ω takes into account a dependence on N] such that, for all $n \geq 1$, $|E_{H_0+k_0}^n| = 0$, that is to say $||f_n||_{L^\infty(\Omega)} \leq K_3 = H_0 + k_0$, where K_3 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$.

By taking any integer $n_0 \geq K_3$ (such a choice of n_0 depends on K_3, thus on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$) and letting $f = f_{n_0}$, we get a solution to

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega) \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_\Omega \mathbf{v} \cdot \nabla f \varphi + \int_{\Gamma_f} \lambda f \varphi \, d\sigma = \int_\Omega \theta\varphi \,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{4.41}$$

(because, since $n_0 \geq K_3 \geq ||f_{n_0}||_{L^\infty(\Omega)}$, $T_{n_0}(f_{n_0}) = f_{n_0} = f$) such that

$$||f||_{H^1(\Omega)} \leq 1 + \frac{||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n_0 \Lambda_{\mathbf{v}} |\Omega|^{\frac{1}{2}}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} \leq 1 + \frac{\Lambda_\theta |\Omega|^{\frac{1}{2}} + n_0 \Lambda_{\mathbf{v}} |\Omega|^{\frac{1}{2}}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} := C_1,$$

where C_1 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$; notice also that

$$||f||_{L^\infty(\Omega)} \leq K_3. \tag{4.42}$$

Since, up to now, we have not used Hypothesis (4.13), this proves what we have claimed in Remark 4.9.

Step 4: conclusion.
It remains to prove that the solution $f \in H^1_{\Gamma_d}(\Omega)$ of (4.41) we found in the preceding section is in fact in $\mathcal{C}^{0,\kappa}(\Omega)$ for a $\kappa > 0$. This is the only part of the proof where we need Hypothesis (4.13).

We have, for all $\varphi \in H^1(\Omega)$,

$$\left| \int_\Omega \varphi \mathbf{v} \cdot \nabla\varphi \right| \leq || \, |\nabla\varphi| \, ||_{L^2(\Omega)} \Lambda_{\mathbf{v}} ||\varphi||_{L^2(\Omega)} \leq \frac{\alpha_A}{2} || \, |\nabla\varphi| \, ||^2_{L^2(\Omega)} + \frac{\Lambda_{\mathbf{v}}^2}{2\alpha_A} ||\varphi||^2_{L^2(\Omega)}.$$

Thus, by taking $b = 1 + \frac{\Lambda_{\mathbf{v}}^2}{2\alpha_A}$, the bilinear continuous form

$$(\varphi, \psi) \in H^1(\Omega) \to \int_\Omega A\nabla\varphi \cdot \nabla\psi + \int_\Omega \mathbf{v} \cdot \nabla\varphi\psi + \int_\Omega b\varphi\psi$$

is coercive (notice that the choice of b depends on $(\Omega, \alpha_A, \Lambda_{\mathbf{v}})$).

f is the solution to

$$\left\{\begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi + \int_\Omega \mathbf{v}\cdot\nabla f\varphi + \int_{\Gamma_f}\lambda f\varphi\,d\sigma + \int_\Omega bf\varphi = \int_\Omega \widetilde{\theta}\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega), \end{array}\right. \tag{4.43}$$

where $\widetilde{\theta} = \theta + bf \in L^\infty(\Omega)$.
Thus, $\widetilde{\theta} \in (W^{1,1}_{\Gamma_d}(\Omega))'$ and, thanks to (4.42), the norm of $\widetilde{\theta}$ in $(W^{1,1}_{\Gamma_d}(\Omega))'$ is bounded by K_4 depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$. With our choice of b, a slight adaptation of the methods of [70] and [29] shows then that (thanks to Hypothesis (4.13)), there exists $\kappa \in]0,1[$ depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, b)$, i.e. depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda)$ and K_5 depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, b, K_4)$, i.e. depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$, such that the solution f of (4.43) is in $\mathcal{C}^{0,\kappa}(\Omega)$ with $||f||_{\mathcal{C}^{0,\kappa}(\Omega)} \le K_5$. ■

4.3 Proof of the uniqueness and stability theorems

We will use, in the course of this proof, the following result.

Lemma 4.3 *Let $f : \Omega \to \mathbb{R}$, $F : \Omega \to \mathbb{R}^N$ and $G : \Omega \to \mathbb{R}^N$ be measurable functions such that $|F - G| \in L^1(\Omega)$. Under Hypotheses (4.1), (4.4) and (4.5), there exists a measurable matrix-valued function $M : \Omega \to M_N(\mathbb{R})$ such that*

$$M(x)\tau\cdot\tau \ge \alpha|\tau|^2 \text{ for a.e. } x\in\Omega, \text{ for all } \tau\in\mathbb{R}^N, \tag{4.44}$$

$$||M(x)|| \le \Lambda \text{ for a.e. } x\in\Omega, \tag{4.45}$$

$$a(x,f(x),F(x)) - a(x,f(x),G(x)) = M(x)(F(x)-G(x)) \text{ for a.e. } x\in\Omega. \tag{4.46}$$

Remark 4.15 *Notice that α and Λ do not depend on f, F or G (only on a).*

Proof of Lemma 4.3
When a is of class $\mathcal{C}^1$ with respect to ξ, it is very simple: just take

$$M(x) = \int_0^1 \frac{\partial a}{\partial\xi}(x, f(x), F(x) + t(G(x) - F(x))\,dt$$

(where $\frac{\partial a}{\partial\xi}$, the partial derivative of a with respect to ξ, is identified to a $N\times N$ matrix; it is easy to see that this partial derivative satisfies (4.44) and (4.45)).
When a is only Lipschitz continuous with respect to ξ, it has a partial derivative for a.e. $\xi\in\mathbb{R}^N$, but we cannot take the preceding expression since $F(.) + t(G(.) - F(.))$ could take (on the whole of Ω and for any $t\in[0,1]$) its values where a is not derivable with respect to ξ.
We solve this problem by the following trick: by denoting $(\rho_n)_{n\ge1}$ a sequence of mollifiers in $\mathbb{R}^N$, we take $a_n(x,s,\xi) = (a(x,s,.)*\rho_n)(\xi)$; a_n is a Caratheodory function which is of class $\mathcal{C}^1$ with respect to ξ. We have thus

$$a_n(x,f(x),F(x)) - a_n(x,f(x),G(x)) = M_n(x)(F(x)-G(x)), \tag{4.47}$$

where $M_n(x) = \int_0^1 \frac{\partial a_n}{\partial\xi}(x,f(x),F(x)+t(G(x)-F(x))\,dt$; by noticing that $\frac{\partial a_n}{\partial\xi}(x,s,\xi) = (\frac{\partial a}{\partial\xi}(x,s,.)*\rho_n)(\xi)$, we see that $\frac{\partial a_n}{\partial\xi}$ — and thus M_n — satisfies (4.44) and (4.45) for all $n\ge1$.
Thus, $(M_n)_{n\ge1}$ being a bounded sequence in $(L^\infty(\Omega))^{N\times N}$, there exists a subsequence, still denoted $(M_n)_{n\ge1}$, which converges to M in $(L^\infty(\Omega))^{N\times N}$ weak-$*$; it is then quite clear that M satisfies (4.44) and (4.45). Moreover, since $|F-G|\in L^1(\Omega)$, $M_n(F-G)\to M(F-G)$ in the sense of distributions. Since $a_n(x,f(x),F(x)) - a_n(x,f(x),G(x)) \to a(x,f(x),F(x)) - a(x,f(x),G(x))$ for a.e. $x\in\Omega$ (for all $x\in\Omega$ such that $a(x,.,.)$ is continuous) and is dominated by $\Lambda|F-G|\in L^1(\Omega)$, the convergence is also true in

$(L^1(\Omega))^N$ (and thus in the sense of distributions). By passing to the limit (in the sense of distributions) in (4.47), and since the limits are functions, we get

$$a(x,f(x),F(x)) - a(x,f(x),G(x)) = M(x)(F(x)-G(x)) \quad \text{for a.e. } x\in\Omega,$$

and the measurable matrix valued function M is thus convenient.■

Proof of Theorem 4.1
Let $\mu \in \mathcal{M}(\Omega)$, $\mu^\partial \in \mathcal{M}(\partial\Omega)$ and u, v two SOLA of (4.14).
By definition, there exists $(\mu_n,\nu_n)_{n\geq 1} \in \mathcal{M}(\Omega)\cap (H^1(\Omega))'$ satisfying $\mu_n \to \mu$ and $\nu_n \to \mu$ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$, $(\mu_n^\partial,\nu_n^\partial)_{n\geq 1} \in \mathcal{M}(\partial\Omega)\cap (H^{1/2}(\Omega))'$ satisfying $\mu_n^\partial \to \mu^\partial$ and $\nu_n^\partial \to \mu^\partial$ in $\mathcal{M}(\partial\Omega)$ weak-$*$, u_n a solution of (4.15) and v_n a solution of (4.15) with (ν_n,ν_n^∂) instead of (μ_n,μ_n^∂) such that $u_n \to u$ and $v_n \to v$ in $L^1(\Omega)$ (in fact, the convergence is much stronger but we will not need it).
By substracting the equation satisfied by v_n from the equation satisfied by u_n, we have, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\begin{aligned}&\int_\Omega (a(x,u_n,\nabla u_n) - a(x,v_n,\nabla v_n))\cdot\nabla\varphi + \int_{\Gamma_f}\lambda(u_n-v_n)\varphi\, d\sigma\\ &= \langle \mu_n-\nu_n,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} + \langle \mu_n^\partial-\nu_n^\partial,\varphi\rangle_{(H^{1/2}_{\Gamma_d}(\Omega))',H^{1/2}_{\Gamma_d}(\Omega)}.\end{aligned} \tag{4.48}$$

Let $\mathcal{V}:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ defined, for all $(x,s,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$, by

$$\begin{cases}\mathcal{V}(x,s,t,\xi) = \frac{a(x,s,\xi)-a(x,t,\xi)}{s-t} & \text{if } s\neq t,\\ \mathcal{V}(x,s,t,\xi) = 0 & \text{if } s=t.\end{cases}$$

Thanks to Hypothesis (4.1), $\mathcal{V}$ is Borel-measurable (it is Borel-measurable on the Borel set $\{s\neq t\}$ and on the Borel set $\{s=t\}$) and, by (4.6), $|\mathcal{V}(x,s,t,\xi)|\leq\delta$ for a.e. $x\in\Omega$, for all $(s,t,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$; we also have, for all $(x,s,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$,

$$a(x,s,\xi)-a(x,t,\xi) = (s-t)\mathcal{V}(x,s,t,\xi).$$

$\mathcal{V}$ being Borel-measurable and u_n, v_n, ∇v_n being measurable, $\mathbf{v}_n(.) = \mathcal{V}(.,u_n(.),v_n(.),\nabla v_n(.))$ is measurable on Ω and, for a.e. $x\in\Omega$, we have $|\mathbf{v}_n(x)|\leq\delta$.
By denoting $M_n:\Omega\to M_N(\mathbb{R})$ the measurable matrix-valued function given by Lemma 4.3 applied to $f=u_n$, $F=\nabla u_n$ and $G=\nabla v_n$ (notice that $|F-G|\in L^2(\Omega)\subset L^1(\Omega)$), we obtain, for a.e. $x\in\Omega$,

$$\begin{aligned}&a(x,u_n(x),\nabla u_n(x)) - a(x,v_n(x),\nabla v_n(x))\\ &\quad= a(x,u_n(x),\nabla u_n(x)) - a(x,u_n(x),\nabla v_n(x)) + a(x,u_n(x),\nabla v_n(x)) - a(x,v_n(x),\nabla v_n(x))\\ &\quad= M_n(x)(\nabla u_n(x)-\nabla v_n(x)) + (u_n(x)-v_n(x))\mathbf{v}_n(x).\end{aligned}$$

By (4.48), $w_n = u_n - v_n$ is thus a solution to

$$\begin{cases} w_n\in H^1_{\Gamma_d}(\Omega),\\ \displaystyle\int_\Omega M_n\nabla w_n\cdot\nabla\varphi + \int_\Omega w_n\mathbf{v}_n\cdot\nabla\varphi + \int_{\Gamma_f}\lambda w_n\varphi\,d\sigma = \langle\mu_n-\nu_n,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\\ \qquad\qquad + \langle\mu_n^\partial-\nu_n^\partial,\varphi\rangle_{(H^{1/2}_{\Gamma_d}(\partial\Omega))',H^{1/2}_{\Gamma_d}(\partial\Omega)},\ \forall\varphi\in H^1_{\Gamma_d}(\Omega).\end{cases} \tag{4.49}$$

M_n^T is a measurable matrix-valued function which satisfies Properties (4.44) and (4.45) (notice that α and Λ do not depend on n) and we have $\mathbf{v}_n\in L^\infty(\Omega)$ with $\delta\geq ||\,|\mathbf{v}_n|\,||_{L^\infty(\Omega)}$ (notice that δ does not depend on n).

Thanks to Theorem 4.3, since $\mathrm{sgn}(u-v)\in L^\infty(\Omega)$, there exists $\kappa>0$ and $C>0$ depending on $(\Omega,\Gamma_d,\alpha,\Lambda,\delta,\lambda)$ (i.e. κ and C do not depend on n) and, for all $n\geq 1$, a solution to

$$\left\{\begin{array}{l} f_n\in H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}^{0,\kappa}(\Omega)\,,\\ \displaystyle\int_\Omega M_n^T\nabla f_n\cdot\nabla\varphi+\int_\Omega \mathbf{v}_n\cdot\nabla f_n\varphi+\int_{\Gamma_f}\lambda f_n\varphi\,d\sigma=\int_\Omega \mathrm{sgn}(u-v)\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{4.50}$$

such that $||f_n||_{\mathcal{C}^{0,\kappa}(\Omega)}\leq C$.
Using f_n as a test function in (4.49) and w_n as a test function in (4.50), we obtain

$$\begin{aligned}\int_\Omega w_n\mathrm{sgn}(u-v) &= \int_\Omega M_n\nabla w_n\cdot\nabla f_n+\int_\Omega w_n\mathbf{v}_n\cdot\nabla f_n+\int_{\Gamma_f}\lambda w_n f_n\,d\sigma\\ &= \int_\Omega f_n\,d(\mu_n-\nu_n)+\int_{\partial\Omega} f_n\,d(\mu_n^\partial-\nu_n^\partial).\end{aligned} \tag{4.51}$$

Since $(f_n)_{n\geq1}$ is bounded in $\mathcal{C}^{0,\kappa}(\Omega)$, it is relatively compact in $\mathcal{C}(\overline{\Omega})$ (thanks to the Ascoli-Arzelà theorem) and we can thus suppose that, up to a subsequence still denoted $(f_n)_{n\geq1}$, we have $f_n\to f$ in $\mathcal{C}(\overline{\Omega})$. Since $\mu_n-\nu_n\to 0$ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and $\mu_n^\partial-\nu_n^\partial\to 0$ in $\mathcal{M}(\partial\Omega)$ weak-$*$, we get

$$\int_\Omega f_n\,d(\mu_n-\nu_n)+\int_{\partial\Omega} f_n\,d(\mu_n^\partial-\nu_n^\partial)\to 0.$$

Using the fact that $w_n\to u-v$ in $L^1(\Omega)$, we deduce then from (4.51), by passing to the limit $n\to\infty$, that

$$0=\int_\Omega \mathrm{sgn}(u-v)(u-v)=\int_\Omega |u-v|,$$

which gives $u=v$ a.e. on Ω and concludes the proof.■

Proof of Theorem 4.2
We first prove the more general result stated in Remark 4.6. We suppose thus, to begin, only Hypotheses (4.1)—(4.3), (4.7) and (4.9) and we take $(u_n)_{n\geq1}$ satisfying: for all $n\geq1$, there exists three sequences $(\mu_{n,m})_{m\geq1}\in\mathcal{M}(\Omega)\cap(H^1(\Omega))'$, $(\mu^\partial_{n,m})_{m\geq1}\in\mathcal{M}(\partial\Omega)\cap(H^{1/2}(\partial\Omega))'$ and $(u_{n,m})_{m\geq1}\in H^1_{\Gamma_d}(\Omega)$ such that

$$\begin{array}{l}\mu_{n,m}\stackrel{m\to\infty}{\longrightarrow}\mu_n \text{ in } (\mathcal{C}(\overline{\Omega}))' \text{ weak-}*,\ \mu^\partial_{n,m}\stackrel{m\to\infty}{\longrightarrow}\mu^\partial_n \text{ in } \mathcal{M}(\partial\Omega) \text{ weak-}*,\\ \exists C>0 \text{ such that } ||\mu_{n,m}||_{\mathcal{M}(\Omega)}+||\mu^\partial_{n,m}||_{\mathcal{M}(\partial\Omega)}\leq C \text{ for all } n\geq1 \text{ and } m\geq1,\\ \forall m\geq1\,,\ u_{n,m} \text{ is a solution of (4.15) with } (\mu_{n,m},\mu^\partial_{n,m}) \text{ instead of } (\mu_n,\mu^\partial_n),\\ u_{n,m}\stackrel{m\to\infty}{\longrightarrow}u_n \text{ in } W^{1,q}_{\Gamma_d}(\Omega) \text{ for all } q\in[1,N/(N-1)[\end{array} \tag{4.52}$$

((4.52) is the additional hypothesis we must make — see below for the reason).

Let $\{\varphi_k\,,\ k\geq1\}$ (respectively $\{\psi_k\,,\ k\geq1\}$) be a countable dense subset of $\mathcal{C}(\overline{\Omega})$ (respectively $\mathcal{C}(\partial\Omega)$). For all $n\geq1$, there exists $m_n\geq1$ such that

- $\left|\int_\Omega\varphi_k\,d\mu_{n,m_n}-\int_\Omega\varphi_k\,d\mu_n\right|\leq 1/n$ for all $k\in[1,n]$,
- $\left|\int_{\partial\Omega}\psi_k\,d\mu^\partial_{n,m_n}-\int_{\partial\Omega}\psi_k\,d\mu^\partial_n\right|\leq 1/n$ for all $k\in[1,n]$,
- $||u_{n,m_n}-u_n||_{W^{1,N/(N-1)-1/n}_{\Gamma_d}(\Omega)}\leq 1/n$.

It is then quite clear that $(\nu_n)_{n\geq1}=(\mu_{n,m_n})_{n\geq1}\in\mathcal{M}(\Omega)\cap(H^1(\Omega))'$ and $(\nu^\partial_n)_{n\geq1}=(\mu^\partial_{n,m_n})_{n\geq1}\in\mathcal{M}(\partial\Omega)\cap(H^{1/2}(\partial\Omega))'$ converge respectively to μ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and to μ^∂ in $\mathcal{M}(\partial\Omega)$ weak-$*$. Indeed,

$(\nu_n)_{n\geq 1} = (\mu_{n,m_n})_{n\geq 1}$ is bounded in $\mathcal{M}(\Omega)$ by C (this is where we need (4.52)) and, for all $k \geq 1$, if $n \geq k$,

$$\left|\int_\Omega \varphi_k\,d\nu_n - \int_\Omega \varphi_k\,d\mu\right| \leq \frac{1}{n} + \left|\int_\Omega \varphi_k\,d\mu_n - \int_\Omega \varphi_k\,d\mu\right| \to 0 \text{ as } n \to \infty.$$

The bound of $(\nu_n)_{n\geq 1}$ and this convergence on a dense subset of $\mathcal{C}(\overline{\Omega})$ gives the weak-$*$ convergence. We can do the same for $(\nu_n^\partial)_{n\geq 1}$.
Thus, by definition of a SOLA, since $v_n = u_{n,m_n}$ is a solution of (4.15) with (ν_n, ν_n^∂) instead of (μ_n, μ_n^∂), there exists a subsequence $(v_{n_k})_{k\geq 1}$ and a SOLA u of (4.14) such that $v_{n_k} \to u$ in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q \in [1, N/(N-1)[$. Let $q \in [1, N/(N-1)[$; for all $k \geq (N/(N-1) - q)^{-1}$, since $n_k \geq k$, we have then (with $r_k = N/(N-1) - 1/n_k > q$),

$$\begin{aligned}
||u_{n_k} - u||_{W^{1,q}(\Omega)} &\leq ||u_{n_k} - v_{n_k}||_{W^{1,q}(\Omega)} + ||v_{n_k} - u||_{W^{1,q}(\Omega)} \\
&\leq |\Omega|^{1/q - 1/r_k} ||u_{n_k} - v_{n_k}||_{W^{1,r_k}(\Omega)} + ||v_{n_k} - u||_{W^{1,q}(\Omega)} \\
&\leq \frac{\sup(1, |\Omega|)}{n_k} + ||v_{n_k} - u||_{W^{1,q}(\Omega)} \to 0 \text{ as } k \to \infty,
\end{aligned}$$

which gives the convergence of $(u_{n_k})_{k\geq 1}$ to u in $W^{1,q}_{\Gamma_d}(\Omega)$, for all $q \in [1, N/(N-1)[$.

Suppose now that we add the hypotheses of Theorem 4.1, or that we are in the case of Remark 4.5. We have then the uniqueness of the SOLA.
The SOLA u_n thus does not depend on the way we approximate (μ_n, μ_n^∂), and we can always take $(\mu_{n,m}, \mu_{n,m}^\partial)_{m\geq 1}$ which approximate these measures and satisfy moreover $||\mu_{n,m}||_{\mathcal{M}(\Omega)} \leq ||\mu_n||_{\mathcal{M}(\Omega)}$ and $||\mu_{n,m}^\partial||_{\mathcal{M}(\partial\Omega)} \leq ||\mu_n^\partial||_{\mathcal{M}(\partial\Omega)}$ for all $m \geq 1$; in this case, since $(\mu_n)_{n\geq 1}$ is bounded in $\mathcal{M}(\Omega)$ and $(\mu_n^\partial)_{n\geq 1}$ is bounded in $\mathcal{M}(\partial\Omega)$ (they converge for the weak-$*$ topology), we see that $(\mu_{n,m}, \mu_{n,m}^\partial)_{n\geq 1, m\geq 1}$ satisfy (4.52).
By supposing that $(u_n)_{n\geq 1}$ does not converge to the SOLA u of (4.14), we would take $\varepsilon > 0$ and a subsequence, still denoted $(u_n)_{n\geq 1}$, such that, for a $q_0 \in [1, N/(N-1)[$, $||u_n - u||_{W^{1,q_0}(\Omega)} > \varepsilon$ for all n. Applying the preceding reasoning, we get a subsequence $(u_{n_k})_{k\geq 1}$ which converges in $W^{1,q_0}_{\Gamma_d}(\Omega)$ to a SOLA v of (4.14). The SOLA being unique, we have in fact $u = v$ and this leads to a contradiction, thus proving Theorem 4.2 and Remark 4.7. ■

Chapitre 5

Remarques sur l'unicité des SOLA

5.1 A propos de la définition des SOLA

Dans l'article originel [10], les mesures du second membre sont approximées par des fonctions de $L^1(\Omega)\cap H^{-1}(\Omega)$. Lorsque l'on cherche à obtenir uniquement l'existence de solutions à (4.14), il importe peu de savoir avec quel genre d'outil (fonctions, mesures...) on approche le second membre.
Cependant, lorsque l'on s'intéresse à l'unicité des solutions ainsi approchées, il faut préciser exactement de quelle manière ces solutions ont étés approchées: *a priori*, on pourrait obtenir plus de solutions si l'on se permet des approximations avec des mesures que si l'on se permet uniquement des approximations avec des fonctions; c'est pourquoi nous avons choisi de permettre, dans la définition de SOLA, des approximations du second membre les plus larges possibles, i.e. non seulement par des fonctions mais aussi par des mesures.

La question qui se pose cependant, maintenant, est de savoir si tout le raisonnement de [10] est valable avec le genre d'approximation que l'on a permis. C'est effectivement le cas.
Les seules propriétés utiles de l'approximation $(L_n)_{n\geq 1}$ de $\mu+\mu^\partial\in\mathcal{M}(\overline{\Omega})$, lorsque l'on veut pouvoir appliquer le raisonnement de [10], sont les suivantes:

$$L_n\in (H^1(\Omega))', \tag{5.1}$$

$$\forall r>N\,,\ \forall\varphi\in W^{1,r}_{\Gamma_d}(\Omega)\,,\ \langle L_n,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}\to\int_\Omega\varphi\,d\mu+\int_{\partial\Omega}\varphi\,d\mu^\partial \tag{5.2}$$

et

$$\exists C>0 \text{ tel que },\forall n\geq 1\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\cap L^\infty(\Omega)\,,\ \left|\langle L_n,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}\right|\leq C||\varphi||_{L^\infty(\Omega)}. \tag{5.3}$$

Ces propriétés sont effectivement vérifiées pour le genre d'approximation que nous avons choisi.
En effet, notre approximation consiste à prendre $L_n=\mu_n+\mu_n^\partial$ avec $\mu_n\in\mathcal{M}(\Omega)$ et $\mu_n^\partial\in\mathcal{M}(\partial\Omega)$ telles qu'il existe $M>0$ vérifiant, pour tout $\varphi\in\mathcal{C}(\overline{\Omega})\cap H^1(\Omega)$,

$$\left|\int_\Omega\varphi\,d\mu_n+\int_{\partial\Omega}\varphi\,d\mu_n^\partial\right|\leq M||\varphi||_{H^1(\Omega)}. \tag{5.4}$$

Cette inégalité et la densité de $\mathcal{C}(\overline{\Omega})\cap H^1(\Omega)$ dans $H^1(\Omega)$ impliquent que L_n peut s'étendre de manière unique en une application de $(H^1(\Omega))'$, de sorte que (5.1) est vérifée.
Par définition de $\mu_n\to\mu$ dans $\mathcal{M}(\overline{\Omega})$ faible-$*$ et de $\mu_n^\partial\to\mu^\partial$ dans $\mathcal{M}(\partial\Omega)$ faible-$*$, puisque $\varphi\in W^{1,r}_{\Gamma_d}(\Omega)$ est continue sur $\overline{\Omega}$ lorsque $r>N$, on a $\langle L_n,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}=\int_\Omega\varphi\,d\mu_n+\int_{\partial\Omega}\varphi\,d\mu_n^\partial\to\int_\Omega\varphi\,d\mu+\int_{\partial\Omega}\varphi\,d\mu^\partial$, ce qui prouve (5.2).
Comme $(L_n)_{n\geq 1}$ converge dans $\mathcal{M}(\overline{\Omega})$ faible-$*$, elle est borné dans cet espace; notons C un majorant de $||L_n||_{\mathcal{M}(\overline{\Omega})}$. Soit $\varphi\in H^1(\Omega)\cap L^\infty(\Omega)$ et prenons $\varphi_m\in\mathcal{C}^\infty(\overline{\Omega})$ qui converge vers φ dans $H^1(\Omega)$; les

fonctions $T_{||\varphi||_{L^\infty(\Omega)}}(\varphi_m)$ (où $T_M = \min(M, \max(-M, s))$) sont dans $\mathcal{C}(\overline{\Omega}) \cap H^1(\Omega)$ et convergent vers $T_{||\varphi||_{L^\infty(\Omega)}}(\varphi) = \varphi$ dans $H^1(\Omega)$. Par définition de C, pour tout $n \geq 1$,

$$\left|\langle L_n, T_{||\varphi||_{L^\infty(\Omega)}}(\varphi_m)\rangle_{(H^1(\Omega))', H^1(\Omega)}\right| \leq C||T_{||\varphi||_{L^\infty(\Omega)}}(\varphi_m)||_{L^\infty(\Omega)} \leq C||\varphi||_{L^\infty(\Omega)}.$$

En passant à la limite $m \to \infty$ dans cette inégalité, on en déduit (5.3).

Remarque 5.1 *Lorsque (comme c'est le cas quand on suppose l'hypothèse (4.13)) $\mathcal{C}(\overline{\Omega}) \cap H^1_{\Gamma_d}(\Omega)$ est dense dans $H^1_{\Gamma_d}(\Omega)$, alors au lieu de prendre une approximation $L_n \in \mathcal{M}(\overline{\Omega})$ vérifiant (5.4) pour tout $\varphi \in \mathcal{C}(\overline{\Omega}) \cap H^1(\Omega)$, on peut se contenter d'une approximation vérifiant cette inégalité pour $\varphi \in \mathcal{C}(\overline{\Omega}) \cap H^1_{\Gamma_d}(\Omega)$; L_n s'étend alors en un élément de $(H^1_{\Gamma_d}(\Omega))'$ (et ensuite en un élément de $(H^1(\Omega))'$ par Hahn-Banach).*

5.2 Contre-exemple à l'unicité des SOLA

Nous avons vu (théorème 2.3) que l'hypothèse (4.13) de "bonne répartition" de Γ_d est essentielle au résultat de régularité höldérienne jusqu'au bord des solutions de problèmes elliptiques, résultat que nous utilisons fortement pour prouver l'unicité de la SOLA.
Nous allons prouver ici que cette hypothèse (4.13) est tout aussi essentielle à l'unicité de la SOLA (ou tout du moins que, sans une hypothèse de ce genre sur la répartition de Γ_d le long de $\partial\Omega$, on n'a plus le résultat d'unicité de la SOLA).

Soit $\Omega =]0,1[^3$ (pour simplifier). On prend Γ_d construit, pour ce Ω, dans la sous-section 2.2.1.

Théorème 5.1 *Avec ce choix de Γ_d, pour tout $\mu \in \mathcal{M}(\Omega)$ et $\mu^\partial \in \mathcal{M}(\partial\Omega)$, il existe au moins deux SOLA à*

$$\left\{\begin{array}{lll} -\Delta v = \mu & \textit{dans} & \Omega, \\ v = 0 & \textit{sur} & \Gamma_d, \\ \nabla v \cdot \mathbf{n} = \mu^\partial & \textit{sur} & \Gamma_f. \end{array}\right. \quad (5.5)$$

Preuve du théorème 5.1
Nous commençons par construire une mesure $\mu_0^\partial \in \mathcal{M}(\partial\Omega)$ particulière, puis nous prouvons la non-unicité de la SOLA de (5.5) avec $\mu^\partial = \mu_0^\partial$ et $\mu = 0$; enfin, nous concluons lorsque μ et μ^∂ sont quelconques.

Etape 1: construction de μ_0^∂ particulière.
Prenons $L \in \mathcal{C}_c^\infty(\Omega)$ positive non-nulle; on sait alors que la trace de la solution variationnelle u de (2.34) n'est pas continue sur $\partial\Omega$ (cf sous-section 2.2.2).
En particulier, u n'est pas nulle σ-presque partout sur $\partial\Omega$. Prenons alors $x_0 \in \partial\Omega$ tel que

$$x_0 \in (]0,1[^2 \times \{0,1\}) \cup (]0,1[\times\{0,1\}\times]0,1[) \cup (\{0,1\}\times]0,1[^2)$$

(i.e. x_0 est sur une des parties ouvertes planes de $\partial\Omega$), x_0 est un point de Lebesgue de $u_{|\partial\Omega}$ et $u(x_0) \neq 0$ (comme on a pris x_0 sur une partie plane de $\partial\Omega$, la notion de point de Lebesgue est simplement la notion classique pour des fonctions définies sur des ouverts de $\mathbb{R}^2$). Pour simplifier les notations, nous supposons par exemple que $x_0 \in]0,1[^2 \times \{0\}$.
La mesure particulière que nous cherchons est $\mu_0^\partial = \delta_{x_0} \in \mathcal{M}(\partial\Omega)$, la masse de Dirac située en x_0.

Etape 2: Non-unicité de la SOLA de (5.5) lorsque $\mu = 0$ et $\mu^\partial = \mu_0^\partial$.
Une première SOLA de ce problème est la fonction nulle.
Pour voir cela, on prend $(y_n)_{n\geq 1} \in \Gamma_d$ qui converge vers x_0 (c'est possible car Γ_d est dense dans $\partial\Omega$). x_0 étant dans $]0,1[^2\times\{0\}$, on peut supposer que tous les $(y_n)_{n\geq 1}$ sont aussi dans $]0,1[^2\times\{0\}$; on notera alors, pour $r > 0$, $D(y_n, r) = \{z \in \mathbb{R}^2 \times \{0\} \mid |z - y_n| < r\}$ (i.e. le disque dans $\mathbb{R}^2 \times \{0\}$ de centre y_n et de rayon r).

Γ_d étant ouvert dans $\partial\Omega$, il existe, pour tout $n \geq 1$, $\beta_n \in]0, 1/n[$ tel que $D(y_n, \beta_n) \subset (]0,1[^2 \times \{0\}) \cap \Gamma_d$. Nous notons $\mu_n = \frac{1}{\sigma(D(y_n,\beta_n))}\mathbf{1}_{D(y_n,\beta_n)} \in L^\infty(\partial\Omega) \subset \mathcal{M}(\partial\Omega) \cap (H^{1/2}(\partial\Omega))'$. Puisque $y_n \to x_0$ et $\beta_n \to 0$, on a clairement $\mu_n \to \mu^\partial = \delta_{x_0}$ dans $\mathcal{M}(\partial\Omega)$ faible-$*$.
Soit v_n la solution variationnelle de

$$\begin{cases} -\Delta v_n = 0 & \text{dans} \quad \Omega, \\ v_n = 0 & \text{sur} \quad \Gamma_d, \\ \nabla v_n \cdot \mathbf{n} = \mu_n & \text{sur} \quad \Gamma_f. \end{cases} \tag{5.6}$$

On sait que, à une sous-suite près, $(v_n)_{n\geq 1}$ converge vers une SOLA de (5.5) avec $\mu = 0$ et $\mu^\partial = \delta_{x_0}$. Mais, pour tout $n \geq 1$, v_n est, par définition, la seule fonction de $H^1_{\Gamma_d}(\Omega)$ qui vérifie

$$\int_\Omega \nabla v_n \cdot \nabla\varphi = \int_{\Gamma_f} \varphi\mu_n \, d\sigma$$

pour tout $\varphi \in H^1_{\Gamma_d}(\Omega)$. Or $\mu_n = 0$ sur Γ_f (car $D(y_n, \beta_n) \subset \Gamma_d$), donc $\int_{\Gamma_f} \varphi\mu_n \, d\sigma = 0$; ainsi, v_n est la seule fonction de $H^1_{\Gamma_d}(\Omega)$ qui vérifie $\int_\Omega \nabla v_n \cdot \nabla\varphi = 0$ pour tout $\varphi \in H^1_{\Gamma_d}(\Omega)$, ce qui signifie que v_n est la fonction nulle.
La fonction nulle est donc bien une SOLA de (5.5) avec $\mu = 0$ et $\mu^\partial = \delta_{x_0}$.

Nous allons maintenant prouver qu'il existe une SOLA non-nulle de (5.5) avec $\mu = 0$ et $\mu^\partial = \delta_{x_0}$, ce qui concluera cette étape.
Soit, pour n assez grand (tel que $D(x_0, 1/n) \subset]0,1[^2 \times \{0\}$), $\nu_n = \frac{1}{\sigma(D(x_0,1/n))}\mathbf{1}_{D(x_0,1/n)} \in L^\infty(\partial\Omega)$. On a $\nu_n \to \mu_0^\partial = \delta_{x_0}$ dans $\mathcal{M}(\partial\Omega)$ faible-$*$ et la solution variationnelle de

$$\begin{cases} -\Delta w_n = 0 & \text{dans} \quad \Omega, \\ w_n = 0 & \text{sur} \quad \Gamma_d, \\ \nabla w_n \cdot \mathbf{n} = \nu_n & \text{sur} \quad \Gamma_f, \end{cases} \tag{5.7}$$

c'est à dire la fonction $w_n \in H^1_{\Gamma_d}(\Omega)$ qui vérifie

$$\int_\Omega \nabla w_n \cdot \nabla\varphi = \int_{\Gamma_f} \nu_n\varphi \, d\sigma \, , \; \forall\varphi \in H^1_{\Gamma_d}(\Omega),$$

converge à une sous-suite près, encore notée $(w_n)_{n\geq 1}$, dans $L^1(\Omega)$ (et même mieux) vers une SOLA de (5.5) avec $\mu = 0$ et $\mu^\partial = \delta_{x_0}$. Notons w cette SOLA.
En utilisant u (la solution variationnelle de (2.34) fixée dans l'étape 1) dans l'équation satisfaite par w_n, on obtient, puisque $u = 0$ sur Γ_d,

$$\int_{\partial\Omega} \nu_n u \, d\sigma = \int_{\Gamma_f} \nu_n u \, d\sigma = \int_\Omega L w_n.$$

Mais, lorsque $n \to \infty$, par définition de $\nu_n = \frac{1}{\sigma(D(x_0,1/n))}\mathbf{1}_{D(x_0,1/n)}$ et du fait que x_0 est un point de Lebesgue de $u_{|\partial\Omega}$, on a $\int_{\partial\Omega} \nu_n u \, d\sigma \to u(x_0) \neq 0$ (par choix de x_0). Ainsi, puisque $w_n \to w$ dans $L^1(\Omega)$, on a

$$\int_\Omega L w = u(x_0) \neq 0,$$

ce qui prouve que w n'est pas nulle.

Etape 3: linéarité du problème.
Les SOLA de (5.5) "dépendent linéairement" de μ et μ^∂, c'est à dire que si, pour $i = 1$ et 2, $\mu_i \in \mathcal{M}(\Omega)$, $\mu_i^\partial \in \mathcal{M}(\partial\Omega)$ et u_i est une SOLA de (5.5) avec $(\mu, \mu^\partial) = (\mu_i, \mu_i^\partial)$, alors $u_1 + u_2$ est une SOLA de (5.5) avec $(\mu, \mu^\partial) = (\mu_1 + \mu_2, \mu_1^\partial + \mu_2^\partial)$.

En effet, en prenant, pour $i = 1$ et 2, $\mu_i^{(n)} \in \mathcal{M}(\Omega) \cap (H^1(\Omega))'$ et $\mu_i^{\partial,(n)} \in \mathcal{M}(\partial\Omega) \cap (H^{1/2}(\partial\Omega))'$ qui convergent respectivement vers μ_i dans $\mathcal{M}(\overline{\Omega})$ faible-$*$ et vers μ_i^∂ dans $\mathcal{M}(\partial\Omega)$ faible-$*$, et telles que la solution variationnelle de

$$\left\{\begin{array}{lll} -\Delta u_i^{(n)} = \mu_i^{(n)} & \text{dans} & \Omega, \\ u_i^{(n)} = 0 & \text{sur} & \Gamma_d, \\ \nabla u_i^{(n)} \cdot \mathbf{n} = \mu_i^{\partial,(n)} & \text{sur} & \Gamma_f \end{array}\right. \tag{5.8}$$

converge vers u_i, alors $\mu_1^{(n)} + \mu_2^{(n)} \in \mathcal{M}(\Omega) \cap (H^1(\Omega))'$ converge vers $\mu_1 + \mu_2$ dans $\mathcal{M}(\overline{\Omega})$ faible-$*$, $\mu_1^{\partial,(n)} + \mu_2^{\partial,(n)} \in \mathcal{M}(\partial\Omega) \cap (H^{1/2}(\partial\Omega))'$ converge vers $\mu_1^\partial + \mu_2^\partial$ dans $\mathcal{M}(\partial\Omega)$ faible-$*$, donc la solution variationnelle de

$$\left\{\begin{array}{lll} -\Delta u^{(n)} = \mu_1^{(n)} + \mu_2^{(n)} & \text{dans} & \Omega, \\ u^{(n)} = 0 & \text{sur} & \Gamma_d, \\ \nabla u^{(n)} \cdot \mathbf{n} = \mu_1^{\partial,(n)} + \mu_2^{\partial,(n)} & \text{sur} & \Gamma_f, \end{array}\right. \tag{5.9}$$

qui n'est autre que $u^{(n)} = u_1^{(n)} + u_2^{(n)}$, converge à une sous-suite près vers une SOLA de (5.5) avec $(\mu, \mu^\partial) = (\mu_1 + \mu_2, \mu_1^\partial, \mu_2^\partial)$. Comme $u_1^{(n)} + u_2^{(n)}$ converge vers $u_1 + u_2$, cela prouve que $u_1 + u_2$ est, comme annoncé, une SOLA de ce problème.

Etape 4: conclusion.
Soit maintenant $\mu_* \in \mathcal{M}(\Omega)$ et $\mu_*^\partial \in \mathcal{M}(\partial\Omega)$ quelconques.
On prend u une SOLA de (5.5) pour ces données. Comme w et 0 sont des SOLA de (5.5) avec $\mu = 0$ et $\mu^\partial = \mu_0^\partial$ (cf étape 2), l'étape 3 nous permet de voir que $w - 0 = w$ est une SOLA de (5.5) avec $\mu = 0$ et $\mu^\partial = 0$; ainsi, toujours par l'étape 3, $u + w$ est une SOLA de (5.5) avec $\mu = \mu_*$ et $\mu^\partial = \mu_*^\partial$. w n'étant pas nulle, on a donc trouvé deux SOLA distinctes u et $u + w$ de (5.5) avec $\mu = \mu_*$ et $\mu^\partial = \mu_*^\partial$, ce qui conclut la preuve de ce théorème. ■

Partie III

La Condition d'Hyperbolicité pour les Systèmes Linéaires du Premier Ordre

Chapitre 6

A New Approach for the Hyperbolicity Condition of First Order Systems

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Abstract We study here first order linear systems of partial differential equations with constant coefficients. We prove a necessary and sufficient condition for such systems to have solutions when we consider initial conditions of Riemann type. We also prove that this condition is necessary when dealing with more regular initial conditions.

6.1 Introduction

We study systems of the form

$$\begin{cases} u_t(x,t) + \displaystyle\sum_{i=1}^{N} A_i u_{x_i}(x,t) = 0, & x \in \Omega\,,\ t \in]0,T[, \\ u(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{6.1}$$

where Ω is an open set of $\mathbb{R}^N$ ($N \geq 1$), $T > 0$, $u = (u_1, \ldots, u_l) : \Omega \times [0,T[\to \mathbb{R}^l$, $(A_1, \ldots, A_N)$ are $l \times l$ real matrices and $u^0 = (u_1^0, \ldots, u_l^0) : \Omega \to \mathbb{R}^l$ is an initial condition. We will not handle the boundary conditions, which are anyway unnecessary to obtain the hyperbolicity condition.

When $\Omega = \mathbb{R}^N$, a classical way to study this kind of problem is to use the Fourier Transform. In this case, the natural space of solutions is built on $L^2(\mathbb{R}^N)$ (for example $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^N))$); when we ask for Problem (6.1) to be well-posed in this space in the sense of Hadamard, we can then prove that the matrices $(A_j)_{j\in[1,N]}$ must satisfy: $\sup_{\xi\in\mathbb{R}^N} ||\exp(i\sum_{j=1}^N \xi_j A_j)|| < +\infty$, see e.g. [67] (this condition implies, but is not equivalent to, the diagonalizability of $\sum_{j=1}^N \xi_j A_j$ for all $\xi \in \mathbb{R}^N$; see [48] and item ii) in Remark 6.2). Notice that the well-known Friedrichs systems (that is to say systems of the kind (6.1) such that there exists a symmetric definite positive matrix S satisfying: for all $j \in [1,N]$, SA_j is symmetric) always satisfy this condition (see [67]).
Because of the use of the Fourier transform, this method is limited to the case $\Omega = \mathbb{R}^N$ and demands that (6.1) be well-posed in $L^2(\mathbb{R}^N)$.

When we search for classical $\mathcal{C}^\infty$ global ($\Omega = \mathbb{R}^N$) solutions, the Lax-Mizohata theorem (see [18]) gives a result of the same kind. This theorem however demands that (6.1) be well-posed (in $\mathcal{C}^\infty(\mathbb{R}^N \times [0,\infty[)$) not only for a null right-hand side, but for any $\mathcal{C}^\infty$ right-hand side; in this case, it states that, for any $\xi \in \mathbb{R}^N$, the matrix $\sum_{i=1}^N \xi_i A_i$ must have real eigenvalues (the Lax-Mizohata theorem is not limited to constant-coefficient problems of the first order); in fact, since our system has constant coefficients, it is even here a sufficient condition for the system to be well-posed in $\mathcal{C}^\infty$ (this is the Garding theorem). Notice however that the important part in the proof of the Lax-Mizohata theorem is not the initial condition, but the non-null right hand side (see [18]), which we do not need here; it is also very important that the datas be $\mathcal{C}^\infty$ regular. Indeed, we prove here that, when we take less regular datas, the matrix $\sum_{i=1}^N \xi_i A_i$ must not only have real eigenvalues, but must also be diagonalizable on $\mathbb{R}$.

Our aim here is to study (6.1) when the functions u^0 and u are neither in L^2, nor regular; in fact, we will consider very specific initial conditions (of Riemann type) and solutions in the largest possible space of functions (the largest space of functions which is endowed in the space of distributions), that is to say L^1_{loc}. The study of the Riemann problem coming from (6.1) is quite interesting (see item ii) in Remark 6.3).
We will prove, using methods that seem new, that, when we ask for (6.1) to have a local weak solution in $(L^1_{\text{loc}}(\Omega \times [0,T[))^l$ for any initial condition of Riemann type, then the matrices $(A_i)_{i\in[1,N]}$ must satisfy a so-called hyperbolicity condition; this condition will also appear to be a sufficient one. We will not need to suppose the well-posedness of the system in the sense of Hadamard, neither to consider boundary conditions: the mere existence of a local solution to the equation inside the open set of study is enough to obtain the hyperbolicity condition.

6.2 Definitions, remarks and results

Definition 6.1 *(local weak solution) Let $u^0 \in (L^1_{\text{loc}}(\Omega))^l$ and $T > 0$. A function $u \in (L^1_{\text{loc}}(\Omega \times [0,T[))^l$ is a local weak solution on $\Omega \times [0,T[$ to (6.1) if, for all $\varphi \in \mathcal{C}_c^\infty(\Omega \times [0,T[)$,*

$$\int_0^T \int_\Omega u(x,t)\varphi_t(x,t)\,dx\,dt + \int_0^T \int_\Omega \sum_{i=1}^N A_i u(x,t)\varphi_{x_i}(x,t)\,dx\,dt + \int_\Omega u^0(x)\varphi(x,0)\,dx = 0. \tag{6.2}$$

Remark 6.1 *i) Of course, a solution to (6.1) in the sense of Definition 6.1 is also a solution in the sense of the distributions on $\Omega\times]0,T[$.*

ii) Notice that (6.2) is an equality in $\mathbb{R}^l$. Moreover, to handle the initial condition, we take test functions φ which do not necessary vanish at $t = 0$ ($\mathcal{C}_c^\infty(\Omega \times [0,T[)$ is the space of the restrictions to $\Omega \times [0,T[$ of functions in $\mathcal{C}_c^\infty(\Omega\times]-\infty,T[)$); since we do not take into account boundary conditions, these test functions vanish on the boundary of Ω.

iii) The technique of Hölmgren associated to a simple form of the Cauchy-Kowalewska theorem gives a finite speed propagation result on the local weak solutions of (6.1) (see Corollary 6.2). In particular, when $\Omega = \mathbb{R}^N$, the local weak solution to (6.1) is unique.

Definition 6.2 *The system (6.1) is hyperbolic if, for all $\xi \in \mathbb{R}^N$, $\sum_{i=1}^N \xi_i A_i$ is diagonalizable on $\mathbb{R}$.*

Remark 6.2 *i) There are numerous definitions of "hyperbolicity" for a first order linear system of equations; some only ask for the matrix $\sum_{i=1}^N \xi_i A_i$ to have real eigenvalues ([18], [46]). The definition we have chosen here is the one that appears in [37]; it is stronger than in [18] but weaker than in [67] (see item ii) below).*

ii) The hyperbolicity condition in [67], namely $\sup_{\xi\in\mathbb{R}^N} ||\exp(i\sum_{j=1}^N \xi_j A_j)|| < +\infty$, is equivalent to the existence of $P : S^{N-1} \to \mathrm{GL}(l;\mathbb{R})$ such that, for all $\xi \in S^{N-1}$, $P(\xi)^{-1}\sum_{j=1}^N \xi_j A_j P(\xi)$ is diagonal and $\sup_{\xi\in S^{N-1}} ||P(\xi)||\,||P(\xi)^{-1}|| < +\infty$ (see [48]). When $N = 1$ or $N = l = 2$, this definition is equivalent to Definition 6.2 (in fact, in these cases, a system which is hyperbolic in the sense *of Definition 6.2 is a Friedrichs system); however, as soon as $N \geq 2$ and $l \geq 3$, the definition of hyperbolicity in [67] is strictly stronger than our definition. For example, when $N = 2$ and $l = 3$, the system defined by*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

is hyperbolic in the sense of Definition 6.2 but not in the sense given in [67] (when we take matrices $P(\xi)$ that diagonalize $\xi_1 A_1 + \xi_2 A_2$, we notice that $||P(\xi)|| \, ||P(\xi)^{-1}||$ must explode when $\xi \to (1, 0)$ with $\xi_2 \neq 0$).

When Ω is an open set of $\mathbb{R}^N$ and $a \in \Omega$, we define $E(\Omega, a)$ as the set of functions $f : \Omega \to \mathbb{R}^l$ for which there exists $\xi \in \mathbb{R}^N \backslash \{0\}$ and $f^* \in \mathbb{R}^l$ such that, for all $x \in \Omega$, $f(x) = f^* \mathbf{1}_{\mathbb{R}^-}((x - a) \cdot \xi)$, where $\mathbf{1}_E$ denotes the characteristic function of a set $E \subset \mathbb{R}$ and $X \cdot Y$ is the usual Euclidean product of two vectors $(X, Y) \in \mathbb{R}^N \times \mathbb{R}^N$ (that is to say, $f = 0$ on one of the half-space defined by the affine hyperplane passing by a and having ξ as normal vector, and $f = f^*$ on the other half-space).
The main result of our paper is the following:

Theorem 6.1 *Let Ω be an open set of $\mathbb{R}^N$ and $a \in \Omega$. There is equivalence between:*

1) For all $u_0 \in E(\Omega, a)$, there exists $T > 0$ and a local weak solution $u \in (L^1_{\mathrm{loc}}(\Omega \times [0, T[))^l$ on $\Omega \times [0, T[$ to (6.1).

2) The system (6.1) is hyperbolic.

Remark 6.3 *i) In fact, we will see that, when the system is hyperbolic, the local weak solutions we obtain, for initial conditions of Riemann type, exist for all $T > 0$.*

ii) Some important numerical schemes to compute the solutions of non-linear systems of the kind

$$\begin{cases} u_t + \sum_{i=1}^{N} (f_i(u))_{x_i} = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{6.3}$$

reduce to solving linear Riemann problems in one space dimension (for exemple, the Roe and VFRoe schemes, see [37]). In fact, the linear Riemann problems that occur in the Roe and VFRoes schemes are often given by matrices of the kind $\sum_{i=1}^N \xi_i f_i'(\overline{u})$, for some $\xi \in S^{N-1}$ and $\overline{u} \in D \subset \mathbb{R}^l$ (where D is the admissible values of the solution u to (6.3)); thus, thanks to Theorem 6.1, we see that, to use such schemes, a necessary and sufficient condition is that $\sum_{i=1}^N \xi_i f_i'(\overline{u})$ be diagonalizable on $\mathbb{R}$ for all $\xi \in S^{N-1}$ and all $\overline{u} \in D$: this is what [37] calls the hyperbolicity condition for non-linear systems (and we notice that this is consistant with Definition 6.2 in the case of linear systems).

iii) In [67], as we have seen in item ii) of Remark 6.2, the hyperbolicity condition obtained on the matrices $(A_i)_{i \in [1,N]}$ is in general stronger than our hyperbolicity condition; it is however not so surprising since, to obtain this condition, [67] asks for (6.1) to be well-posed in the sense of Hadamard in $L^2(\mathbb{R}^N)$, which is a stronger hypothesis (more initial conditions, more properties on the solutions) than the one of 1) $\Longrightarrow$ 2) in Theorem 6.1.

Definition 6.3 *The system (6.1) is solvable on an open set Ω of $\mathbb{R}^N$ if, for any initial condition $u^0 \in (L^\infty(\Omega))^l$, there exists $T > 0$ and at least one local weak solution $u \in (L^1_{\text{loc}}(\Omega \times [0, T[))^l$ on $\Omega \times [0, T[$ to (6.1).*

Remark 6.4 *i) Problem (6.1) can also be studied with more regular initial conditions, for example $u^0 \in (\mathcal{C}^k(\Omega))^l$ (with $k \in \mathbb{N}$). In this case, the definition of "solvability" of (6.1) would also demand that the corresponding solution be as regular in space as u^0 (since (6.1) is a first order linear system), that is to say in $(L^1([0, T]; \mathcal{C}^k(\Omega))^l$. We also study the "solvability" of (6.1) under this definition.*

ii) Notice that our definition of "solvable" is much weaker than the classical (Hadamard's) definition of "well-posed"; indeed, this last definition demands the uniqueness of the solution to (6.1) (which, in our case, would oblige us to handle some boundary conditions) as well as the continuous dependence of this solution with respect to the initial and boundary conditions. In fact, for systems of the kind of (6.1), there is some sort of uniqueness result: a finite speed propagation result; since we will need this result, we will talk about it later on.

An immediate consequence (since the functions of $E(\Omega, a)$ are in $(L^\infty(\Omega))^l$) of Theorem 6.1 is the following corollary.

Corollary 6.1 *If the system (6.1) is solvable on an open set of $\mathbb{R}^N$ in the sense of Definition 6.3, then it is hyperbolic.*

Remark 6.5 *i) It is however not sure that the hyperbolicity condition is a sufficient property for (6.1) to be solvable. This comes from the fact that, as we will see in the proof of Theorem 6.1, when we solve an hyperbolic system of the kind (6.1) with an initial condition of Riemann type, the bound we can obtain on the solution with respect to the initial condition depends on the norm of the matrices $(P(\xi), P(\xi)^{-1})$ which diagonalize $\sum_{i=1}^N \xi_i A_i$; and as we have seen in item ii) of Remark 6.2, there are examples of hyperbolic systems such that the norm of the matrices $P(\xi)$ or $P(\xi)^{-1}$ explodes when ξ tends to some ξ_0.*

ii) We will see in Section 6.3 that, in the 1-dimensional case ($N = 1$), the hyperbolicity of (6.1) is in fact equivalent to its solvability (in the sense of Definition 6.3 or in the sense of item i) in Remark 6.4).

As said in item i) of Remark 6.4, we also have a result concerning regular initial conditions.

Theorem 6.2 *Let Ω be an open set of $\mathbb{R}^N$. If, for any $u_0 \in (\mathcal{C}^k_c(\Omega))^l$, there exists $T > 0$ and a local weak solution $u \in (L^1([0, T]; \mathcal{C}^k(\Omega))^l$ on $\Omega \times [0, T[$ to (6.1), then this system is hyperbolic.*

Remark 6.6 *i) $\mathcal{C}^k_c(I)$ denotes the space of functions $I \to \mathbb{R}$ which have a compact support in I and are k times continuously derivable on I.*

ii) In the course of the proof of this theorem, we will find back the Lax-Mizohata result (without the need of a non-null right-hand side) in our particular case: when we ask for (6.1) to have a $\mathcal{C}^\infty$ solution for any $\mathcal{C}^\infty$ initial condition, then $\sum_{i=1}^N \xi_i A_i$ must have real eigenvalues.

6.2.1 A preliminary result

In all the sequel, $|\cdot|_\infty$ denotes the norm of the supremum on $\mathbb{R}^N$ and "dist" the associated distance. e is the vector $(1, \ldots, 1)$ of $\mathbb{R}^N$.

Lemma 6.1 *Let Ω be an open set of $\mathbb{R}^N$, $P \in \mathrm{GL}(l,\mathbb{R})$, $x^0 \in \mathbb{R}^N$, $(\lambda,\mu) \in \mathbb{R}\times\mathbb{R}^*$ and O be an open set relatively compact in $\Omega_{x^0,\mu} = x^0 + \mu\Omega$. If $u^0 \in (L^1_{\mathrm{loc}}(\Omega))^l$ and $u \in (L^1_{\mathrm{loc}}(\Omega\times[0,T[))^l$ is a local weak solution on $\Omega\times[0,T[$ to (6.1), then, by denoting $T_O = \inf\left(T, \frac{1}{|\lambda|}\mathrm{dist}(O,\mathbb{R}^N\backslash\Omega_{x^0,\mu})\right)$, the function $v \in (L^1_{\mathrm{loc}}(O\times[0,T_O[))^l$ defined by*

$$v(x,t) = Pu\left(\frac{1}{\mu}(x-x^0) + \frac{\lambda}{\mu}te, t\right), \qquad (x,t) \in O\times[0,T_O[, \tag{6.4}$$

is a local weak solution on $O\times[0,T_O[$ to the system (6.1) with $\mu PA_iP^{-1} - \lambda Id$ instead of A_i ($i \in [1,N]$) and $v^0(x) = Pu^0((x-x^0)/\mu) \in (L^1(O))^l$ instead of u_0.

The proof of this lemma is quite straightforward and we leave it to the reader.

Remark 6.7 *i) When $\lambda = 0$, we can take any open set $O \subset \Omega_{x^0,\mu}$ and $T_O = T$.*

ii) We thus deduce that, for $a \in \Omega$, if the system (6.1) has a local weak solution for any initial condition in $E(\Omega,a)$, then the system (6.1) with $\mu PA_iP^{-1} - \lambda Id$ instead of A_i ($i \in [1,N]$) has a local weak solution for any initial condition in $E(O, x^0+\mu a)$. Of course, one can notice that the hyperbolicity condition is invariant with respect to the transformations up above of the system.

6.3 The 1-dimensional case

We study here the 1-dimensional case, i.e. $N = 1$. The system (6.1) is thus reduced to

$$\begin{cases} u_t(x,t) + Au_x(x,t) = 0, & x \in \Omega\,,\ t \in]0,T[, \\ u(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{6.5}$$

and Ω is an open set in $\mathbb{R}$.

6.3.1 Particularities of the 1-dimensional case

The case $N = 1$ is very particular. Indeed, we first notice that an hyperbolic system in one dimension is always solvable in the sense of Definition 6.3; in fact, in this case, (6.5) has a local weak solution for any initial condition $u^0 \in (L^1(\Omega))^l$ (or $(L^1_{\mathrm{loc}}(\Omega))^l$ if $\Omega = \mathbb{R}$) and this solution is as regular as u^0.
To see this, choose a basis $(e_1,\ldots,e_l)$ of $\mathbb{R}^l$ which is made of eigenvectors of A; if $u^0(\cdot) = u^0_1(\cdot)e_1 + \ldots + u^0_l(\cdot)e_l \in (L^1_{\mathrm{loc}}(\mathbb{R}))^l$ and λ_i is the eigenvalue of A associated to e_i, then $u(x,t) = u^0_1(x-\lambda_1 t)e_1 + \ldots + u^0_l(x-\lambda_l t)e_l \in (L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+))^l$ is a (in fact *the*) local weak solution on $\mathbb{R}\times\mathbb{R}^+$ to (6.5); if $v^0 \in (L^1(\Omega))^l$, then define $u^0 \in (L^1(\mathbb{R}))^l$ as the extension of v^0 by 0 outside Ω, take $u \in (L^1(\mathbb{R}\times\mathbb{R}^+))^l$ as defined above: $u_{|\Omega\times\mathbb{R}^+} \in (L^1(\Omega\times\mathbb{R}^+))^l$ is a local weak solution on $\Omega\times\mathbb{R}^+$ to the system (6.5) with v^0 instead of u^0.
This also proves 2) $\Longrightarrow$ 1) in Theorem 6.1 when $N = 1$.

Remark 6.8 *Notice also that, in the one dimensional case and if $\Omega = \mathbb{R}$, our definition of solvability is equivalent to Hadamard's definition of well-posedness. Indeed, if (6.5) is solvable in the sense of Definition 6.3, Corollary 6.1 tells us that A is diagonalizable on $\mathbb{R}$; we know then (they have been computed up above) the expressions of the solutions to (6.5), and it is clear, with these expressions, that this system is well posed in the sense of Hadamard.*

We will see (Corollary 6.2) that, in any dimension N, we have a "local uniqueness" result for the solution of (hyperbolic or not) systems of the kind (6.1).
But an interesting consequence of the particularity of the 1-dimensional case is that the proof of this uniqueness result is, when the system is hyperbolic, far simplier than in the general case.

With Theorem 6.1 or 6.2, the following proposition tells us that the existence of solutions to (6.5) implies a local uniqueness of these solutions. The trick which allows us to prove Theorem 6.1 and 6.2 by induction on the size l of the system (6.5) is then the following: we split the system of size l (say S) in two systems, one of size $l-2$ (say S1) and the other of size 2 (say S2), in such a way that the existence of solutions to S implies the existence of solutions to S1; by induction (S1 is of size $l-2$) and Proposition 6.1, we get the uniqueness of the solution to S1 and we reduce thus the study of S to the study of the remaining system $S2$ (of size 2).
We state this very particular and simple case ($N = 1$, A is diagonalizable) of Corollary 6.2 to see that, in dimension $N = 1$ (in converse to the case $N \geq 2$), thanks to the trick described up above, we need no general result on the uniqueness of the solutions to (6.5) to obtain the hyperbolicity condition.

Proposition 6.1 *If A is diagonalizable on $\mathbb{R}$ then, for any open set O relatively compact in Ω, there exists $T_O > 0$ such that, when $u^0 = 0$ a.e. on Ω, any local weak solution on $\Omega \times [0, T[$ to (6.5) is null a.e. on $O\times]0, \inf(T_O, T)[$.*

To prove this classical finite speed propagation result, one juste need to take a basis of $\mathbb{R}^l$ in which A is diagonal; in this basis, each component of the solution satisfies a scalar transport equation and the result is then quite obvious; we leave the details to the reader.

6.3.2 Necessity of the Hyperbolicity Condition

Proof of Theorems 6.1 and 6.2 when $N = 1$
We have already seen, at the beginning of Subsection 6.3.1, that (when $N = 1$) 2) $\Longrightarrow$ 1) in Theorem 6.1 holds. It remains to prove Theorem 6.2 and 1) $\Longrightarrow$ 2) in Theorem 6.1.

This proof is made by contradiction. We will in fact prove by induction on the size l of A that, if A is not diagonalizable on $\mathbb{R}$, then

i) there exists $u^* \in \mathbb{R}^l$ such that, for any open interval I, for any $a \in I$ and for any $T > 0$, (6.5) has no local weak solution in $(L^1_{\text{loc}}(I \times [0, T]))^l$ for the initial condition $u^0 = u^* \mathbf{1}_{I \cap]-\infty, a[}$.

ii) for any open interval I and for any $k \in \mathbb{N}$, there exists $u^0 \in (\mathcal{C}^k_c(I))^l$ such that, for any $T > 0$, (6.5) has no local weak solution in $(L^1([0, T]; \mathcal{C}^k(I)))^l$ for the initial condition u^0.

The case $l = 1$ being trivial (every matrix of size 1 is diagonalizable on $\mathbb{R}$), we begin with the case $l = 2$.

Step 1: $l = 2$.
Let I be an open interval of $\mathbb{R}$ and $A \in M_2(\mathbb{R})$. Suppose that A is not diagonalizable on $\mathbb{R}$; two cases arise then: A has two different complex conjugate eigenvalues or A has only one real eigenvalue and is not in the space $\mathbb{R}Id$.

Step 1.1.
Let us handle the case when the eigenvalues of A are different complex conjugate; by denoting $\lambda + \frac{i}{\mu}$ and $\lambda - \frac{i}{\mu}$ ($(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^*$) these eigenvalues, there exists $P \in \text{GL}(2, \mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda & \frac{1}{\mu} \\ -\frac{1}{\mu} & \lambda \end{pmatrix} = \lambda Id + \frac{1}{\mu} A_0, \tag{6.6}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us take J an open interval, $T'>0$ and $v^0\in L^1(J)$ such that there exists $v=(v_1,v_2)\in(L^1_{\rm loc}(J\times[0,T'[))^2$ a local weak solution on $J\times[0,T'[$ to the system (6.5) with A_0 instead of A and $(v^0,0)$ as an initial condition. We have thus, for all $\psi\in\mathcal{C}_c^\infty(J\times[0,T'[)$,

$$\int_0^{T'}\int_J v_1(x,t)\psi_t(x,t)\,dx\,dt+\int_0^{T'}\int_J v_2(x,t)\psi_x(x,t)\,dx\,dt+\int_J v^0(x)\psi(x,0)\,dx=0, \tag{6.7}$$

$$\int_0^{T'}\int_J v_2(x,t)\psi_t(x,t)\,dx\,dt-\int_0^{T'}\int_J v_1(x,t)\psi_x(x,t)\,dx\,dt=0. \tag{6.8}$$

Let $\varphi\in\mathcal{C}_c^\infty(J\times[0,T'[)$. By applying (6.7) with $\psi=\varphi_t$ and (6.8) with $\psi=\varphi_x$, we find

$$\int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt+\int_J v^0(x)\varphi_t(x,0)\,dx=0. \tag{6.9}$$

Define $\widetilde{v_1}\in L^1_{\rm loc}(J\times]-T',T'[)$ by $\widetilde{v_1}(x,t)=v_1(x,t)$ if $t\geq 0$ and $\widetilde{v_1}(x,t)=v_1(x,-t)$ if $t<0$. If $\varphi\in\mathcal{D}(J\times]-T',T'[)$, a change of variable gives

$$\begin{aligned}\int_{-T'}^0\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt &= \int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\varphi(x,-t)\,dx\,dt\\ &= \int_0^{T'}\int_J v_1(x,t)\Delta_{(x,t)}\widetilde{\varphi}(x,t)\,dx\,dt,\end{aligned}$$

where $\widetilde{\varphi}\in\mathcal{C}_c^\infty(J\times[0,T'[)$ is defined by $\widetilde{\varphi}(x,t)=\varphi(x,-t)$. Using (6.9) with $\widetilde{\varphi}$ instead of φ, we get thus

$$\int_{-T'}^0\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt=-\int_J v^0(x)\widetilde{\varphi}_t(x,0)\,dx=\int_J v^0(x)\varphi_t(x,0)\,dx, \tag{6.10}$$

since $\widetilde{\varphi}_t(x,t)=-\varphi_t(x,-t)$.
The definition of $\widetilde{v_1}$, associated to (6.9) and (6.10), gives then, for all $\varphi\in\mathcal{D}(J\times]-T',T'[)$,

$$\int_{-T'}^{T'}\int_J \widetilde{v_1}(x,t)\Delta_{(x,t)}\varphi(x,t)\,dx\,dt=0,$$

that is to say $\Delta\widetilde{v_1}=0$ in $\mathcal{D}'(J\times]-T',T'[)$; by Lemma 6.2, $\widetilde{v_1}$ is then in $\mathcal{C}^\infty(J\times]-T',T'[)$ (and even real analytic). We have thus $v_1\in\mathcal{C}^\infty(J\times[0,T'[)$ and $(\widetilde{v_1})_t(x,0)=0$. Thanks to some classical integrates by part in (6.9), we see then that v^0 is the restriction of $\widetilde{v_1}$ to $J\times\{0\}$ and must thus be in $\mathcal{C}^\infty(J)$ (and even real analytic).

To prove items i) and ii) in this case, we take

$u^*=P^{-1}\binom{1}{0}$, I an open interval of $\mathbb{R}$, $a\in I$, $u^0=u^*\mathbf{1}_{I\cap]-\infty,a[}$ and $T>0$	in the case of item i),
I an open interval of $\mathbb{R}$, $w\in\mathcal{C}_c^k(I)\backslash\mathcal{C}^\infty(I)$, $a\in I$ such that w is not infinitely differentiable at a, $u^0=P^{-1}\binom{w}{0}$ and $T>0$	in the case of item ii),

(notice that u^* does not depend on I, a or T) and we suppose that there exists a local weak solution on $I\times[0,T[$ to (6.5). By taking J an open interval, relatively compact in μI, which contains μa, Lemma 6.1 tells us that there exists a local weak solution on $J\times[0,T'[$ (for a $T'\in]0,T]$) to the system defined by A_0 with the initial condition $Pu^0(x/\mu)=(v^0(x),0)$; thanks to the preceding reasoning, this means that v^0 is $\mathcal{C}^\infty$ on J, which is a contradiction since $\mu a\in J$.
Since v^0 must even be real analytic on J, we also obtain a stronger form of the Lax-Mizohata result in our particular case: there exists $u^0\in(\mathcal{C}^\infty(I))^2$ such that (6.1) has no local weak solution in $(L^1_{\rm loc}(I\times[0,T[))^2$

(take u^0 which is not real analytic). Using the finite speed propagation result of Corollary 6.2, this result can be extented to the cases $l \geq 3$ and $N \geq 2$.

Step 1.2
Let us now treat the case when A has a unique real eigenvalue without being diagonalizable on $\mathbb{R}$. By denoting λ this eigenvalue, there exists $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda Id + A_1, \tag{6.11}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let J be an open interval, $T' > 0$ and $v^0 \in L^1(J)$ such that there exists a local weak solution $v = (v_1, v_2) \in (L^1_{\text{loc}}(J \times [0, T'[))^2$ to the system (6.5) with A_1 instead of A and $(0, v^0)$ as an initial condition. We thus have, for all $\varphi \in \mathcal{C}_c^\infty(J \times [0, T'[)$,

$$\int_0^{T'} \int_J v_1(x,t)\varphi_t(x,t)\,dx\,dt + \int_0^{T'} \int_J v_2(x,t)\varphi_x(x,t)\,dx\,dt = 0, \tag{6.12}$$

$$\int_0^{T'} \int_J v_2(x,t)\varphi_t(x,t) + \int_J v^0(x)\varphi(x,0)\,dx = 0. \tag{6.13}$$

By taking $\varphi \in \mathcal{D}(J\times]0, T'[)$, (6.13) tells us that $(v_2)_t = 0$ in $\mathcal{D}'(J\times]0, T'[)$, so that v_2 only depends on x: $v_2(x,t) = \widetilde{v_2}(x)$ a.e. on $J\times]0, T'[$. Take then $\varphi(x,t) = \zeta(x)\theta(t)$, where $\zeta \in \mathcal{D}(J)$ and $\theta \in \mathcal{C}_c^\infty([0, T'[)$, $\theta(0) = 1$. The same equation (6.13) for this φ tells us that

$$\int_J \widetilde{v_2}(x)\zeta(x)\,dx = \int_J v^0(x)\zeta(x)\,dx,$$

that is to say $\widetilde{v_2} = v^0$ a.e. on J.
Then, with $\varphi(x,t) = \zeta(x)\theta(t)$, where $\zeta \in \mathcal{D}(J)$ and $\theta \in \mathcal{D}(]0, T'[)$ is such that $\int_0^{T'} \theta(t)\,dt = 1$, (6.12) gives us

$$\int_J \zeta(x) \left(\int_0^{T'} v_1(x,t)\theta'(t)\,dt \right) = -\int_J \zeta'(x)v^0(x)\,dx,$$

which means that

$$(v^0)' = \int_0^{T'} v_1(\cdot,t)\theta'(t)\,dt \quad \text{ in } \quad \mathcal{D}'(J). \tag{6.14}$$

We can now conclude this step: take

- $u^* = P^{-1}\binom{0}{1}$, I open interval of $\mathbb{R}$, $a \in I$, $u^0 = u^* \mathbf{1}_{I \cap]-\infty, a[}$ and $T > 0$ — in the case of item i),
- I open interval of $\mathbb{R}$, $w \in \mathcal{C}_c^k(I) \backslash \mathcal{C}^{k+1}(I)$, $a \in I$ such that w is not $\mathcal{C}^{k+1}$ at a, $u^0 = P^{-1}\binom{0}{w}$ and $T > 0$ — in the case of item ii),

(notice that u^* does not depend on I, a or T) and suppose that there exists a local weak solution

$u \in (L^1_{\text{loc}}(I \times [0, T[))^2$ in the case of item i),
$u \in (L^1([0, T]; \mathcal{C}^k(I)))^2$ in the case of item ii)

on $I \times [0, T[$ to (6.5). Then by taking J an open interval, relatively compact in I, which contains a, Lemma 6.1 tells us that $v(x,t) = Pu(x + \lambda t, t)$ (defined on $J \times [0, T'[$ for a $T' \in]0, T]$) is a local weak solution on $J \times [0,T'[$ to the system (6.5) with A_1 instead of A and $Pu^0_{|J} = (0, v^0) \in (L^1(J))^2$ instead of u^0. In the case of item i), $v \in (L^1_{\rm loc}(J \times [0,T'[))^2$ and the right-hand side of (6.14) defines thus a function in $L^1_{\rm loc}(J)$; this means that $v^0 = \mathbf{1}_{J\cap]-\infty,a[} \in W^{1,1}_{\rm loc}(J)$, which is impossible since $a \in J$. In the case of item ii), $v \in (L^1([0,T'];\mathcal{C}^k(J)))^2$ and (6.14) shows us that $(v^0)' \in \mathcal{C}^k(J)$, i.e. that $v^0 = w \in \mathcal{C}^{k+1}(J)$, which is impossible since $a \in J$.

i) and ii) are thus fully proved in the cases $l = 1$ and $l = 2$. We prove the general case by induction.

Step 2: $l \geq 3$.
We suppose that i) and ii) are true for systems of size $l - 1$ or less, and we prove, by contradiction, that they are true for systems of size l. Since the ideas of the proofs of both items are the same, we only detail the proof of item i).

Suppose that $A \in M_l(\mathbb{R})$ is not diagonalizable on $\mathbb{R}$ and that, nevertheless, we have: for every $u^* \in \mathbb{R}^l$, there exists an open interval I, $a \in I$ and $T > 0$ such that (6.5) has a local weak solution $u \in (L^1_{\rm loc}(I \times [0,T[))^l$ for the initial condition $u^0 = u^*\mathbf{1}_{I\cap]-\infty,a[}$.
Since A is not diagonalizable on $\mathbb{R}$, there exists $P \in \mathrm{GL}(l,\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \widetilde{A} & C \\ 0 & B \end{pmatrix},$$

where $C \in M_{2,l-2}(\mathbb{R})$, $B \in M_{l-2}(\mathbb{R})$, 0 is the zero of $M_{l-2,2}(\mathbb{R})$ and $\widetilde{A} \in M_2(\mathbb{R})$ is of type (6.6) or (6.11) (depending whether all the eigenvalue of A are real or not).

Let us first see that B is diagonalizable on $\mathbb{R}$.
Take $v^* = (v^*_1, \ldots, v^*_{l-2}) \in \mathbb{R}^{l-2}$ and denote $u^* = P^{-1}(0, 0, v^*_1, \ldots, v^*_{l-2}) \in \mathbb{R}^l$; by hypothesis on A, there exists an open interval I, $a \in I$, $T > 0$ and a local weak solution $u \in (L^1_{\rm loc}(I \times [0,T[))^l$ on $I \times [0,T[$ to the system (6.5) with $u^*\mathbf{1}_{I\cap]-\infty,a[}$ instead of u^0. Thus, Pu is a local weak solution on $I \times [0,T[$ to the system (6.5) with PAP^{-1} instead of A and $(0, 0, v^*_1, \ldots, v^*_{l-2})\mathbf{1}_{I\cap]-\infty,a[}$ instead of u^0.
The $l - 2$ last equations of the system satisfied by $Pu = (u_1, u_2, v_1, \ldots, v_{l-2}) \in (L^1_{\rm loc}(I \times [0,T[))^l$ let us see that $v = (v_1, \ldots, v_{l-2}) \in (L^1_{\rm loc}(I \times [0,T[))^{l-2}$ is a local weak solution to the system defined by B with an initial condition $v^*\mathbf{1}_{I\cap]-\infty,a[}$.
By induction, the result is true for systems of size $l-2$ (the size of B) and B is thus diagonalizable on $\mathbb{R}$.

Since the result is true for systems of size 2 (Step 1), and since $\widetilde{A}$ is not diagonalizable on $\mathbb{R}$, there exists $(u^*_1, u^*_2) \in \mathbb{R}^2$ such that, for any open interval K, any $b \in K$ and any $T' > 0$, the system defined by $\widetilde{A}$ has no local weak solution in $(L^1_{\rm loc}(K \times [0,T'[))^2$ for the initial condition $(u^*_1, u^*_2)\mathbf{1}_{K\cap]-\infty,b[}$.
Take $u^* = (u^*_1, u^*_2, 0, \ldots, 0) \in \mathbb{R}^l$; by hypothesis on A, there exists an open interval I, $a \in I$ and $T > 0$ such that the system defined by A has a local weak solution $\widetilde{u} \in (L^1_{\rm loc}(I \times [0,T[))^l$ for the initial condition $u^0 = P^{-1}u^*\mathbf{1}_{I\cap]-\infty,a[}$.
The $l - 2$ last equations of the system satisfied by $P\widetilde{u} = u = (u_1, \ldots, u_l)$ let us see that $(u_3, \ldots, u_l)$ is a local weak solution on $I \times [0,T[$ of the system defined by B for a null initial condition; thus, since B is diagonalizable on $\mathbb{R}$, Proposition 6.1 let us see that, if J is a relatively compact open interval of I, there exists $T_J \in]0,T]$ such that $(u_3, \ldots, u_l) = 0$ a.e. on $J \times]0, T_J[$.
Taking J which contains a and returning to the first two equations of the system satisfied by u, we see that $(u_1, u_2) \in (L^1_{\rm loc}(J \times [0,T_J[))^2$ is a local weak solution on $J \times [0,T_J[$ of the system defined by $\widetilde{A}$ for the initial conditon $(u^*_1, u^*_2)\mathbf{1}_{J\cap]-\infty,a[}$: this is a contradiction with the choice of (u^*_1, u^*_2). ■

Remark 6.9 *i) Denote by $\mathcal{C}^{-k}(\Omega)$ the space of the distributions on Ω having order k; it is endowed with the weak-$*$ topology of the distributions on Ω. With very few changes (some integrals changed into duality products), the preceding proof also shows that, if Ω is an open set of $\mathbb{R}^N$ and if we ask* *for (6.5) to have a local weak solution in* $(\mathcal{C}([0,T];\mathcal{C}^{-k}(\Omega)))^l$ *for any initial condition in* $(\mathcal{C}^{-k}(\Omega))^l$*, then (6.5) must be hyperbolic.*

ii) In fact, we could also prove, with the same method, that if A *has a non-trivial Jordan block of size* $m \geq 2$*, there exists a vector* $u^* \in \mathbb{R}^l$ *such that, if (6.5) has a local solution* $u \in (L^1([0,T];\mathcal{C}^k(\Omega)))^l$ *with* $u_0 = wu^*$ *for some* $w \in L^1(\Omega)$*, then* w *must be in* $\mathcal{C}^{k+m-1}(\Omega)$*.*

6.4 The multi-dimensional case

We now study the case $N \geq 2$. The proof of Theorems 6.1 (1) $\Longrightarrow$ 2)) and 6.2 in this case uses the result for $N = 1$, by taking particular initial conditions that only depends on x_1; but to use the result for $N = 1$, we need to know that the corresponding solutions also only depend on x_1. This is why we need the following uniqueness results.

6.4.1 On the uniqueness of solutions to (6.1)

Here, $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^N$. $||\cdot||$ is any norm on $\mathbb{R}^l$ and, when $B \in M_l(\mathbb{R})$, $||B||$ denotes the induced norm. The following proposition is a form of the Hölmgren theorem, but we state and prove it to give some estimates on the space-time range of uniqueness of the solution to (6.1).

Proposition 6.2 *(Finite Speed Propagation) Let* Ω *be an open set in* $\mathbb{R}^N$*,* $u_0 \in (L^1_{\rm loc}(\Omega))^l$ *and* $u \in (L^1_{\rm loc}(\Omega \times [0,T[))^l$ *a local weak solution on* $\Omega \times [0,T[$ *to (6.1). Let* $M = \sum_{i=1}^N ||A_i^T||$*,* $x^0 \in \Omega$ *and* $B(x^0,r)$ *a ball (for the Euclidean norm) of center* x^0 *and radius* r *relatively compact in* Ω*. If* $\varepsilon < \inf(1/4Mr(N+1), T/r^2)$ *and* $u^0 = 0$ *a.e. on* $B(x^0,r)$ *then, by denoting* $\mathcal{U} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid x \in B(x^0,r),\ 0 < t < \varepsilon(r^2 - |x-x^0|^2)\}$*, we have* $u = 0$ *a.e. on* $\mathcal{U}$*.*

Remark 6.10 *The condition* $\varepsilon r^2 < T$ *ensures that* $\mathcal{U} \subset \Omega \times [0,T[$*.*

Proof of Proposition 6.2
Thanks to Lemma 6.1, we see that $v(x,t) = u(x^0 + x, t)$ defines a function $v \in (L^1_{\rm loc}(B(0,r) \times [0,T[))^l$ which is a local weak solution to

$$\left\{\begin{array}{ll} v_t(x,t) + \sum_{i=1}^N A_i v_{x_i}(x,t) = 0, & (x,t) \in B(0,r) \times]0,T[, \\ v(x,0) = 0, & x \in B(0,r). \end{array}\right. \tag{6.15}$$

Proving that $u = 0$ a.e. on $\mathcal{U}$ is equivalent to proving that $v = 0$ a.e. on $\mathcal{V} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r,\ 0 < t < \varepsilon(r^2 - |x|^2)\} \subset B(0,r) \times]0,T[$.

Let $\varphi = (\varphi_1, \ldots, \varphi_l) \in (\mathcal{C}_c^\infty(B(0,r) \times [0,T[))^l$; by adding, for $j = 1$ to l, the $j^{\rm th}$ component of the equation satisfied by v when we use φ_j as a test function, and by denoting $X \cdot Y$ the scalar product of two vectors $(X,Y) \in \mathbb{R}^l$, we find

$$\int_{B(0,r) \times]0,T[} v(x,t) \cdot \varphi_t(x,t) + \sum_{i=1}^N A_i v(x,t) \cdot \varphi_{x_i}(x,t)\, dx\, dt = 0,$$

that is to say, with $B_i = A_i^T$,

$$\int_{B(0,r) \times]0,T[} v(x,t) \cdot \left(\varphi_t(x,t) + \sum_{i=1}^N B_i \varphi_{x_i}(x,t)\right) dx\, dt = 0, \quad \forall \varphi \in (\mathcal{C}_c^\infty(B(0,r) \times [0,T[))^l. \tag{6.16}$$

We will see that, thanks to Lemma 6.3, (6.16) implies $v = 0$ a.e. on $\mathcal{V}$.

Since $4Mr(N+1)\varepsilon < 1$ (with $M = \sum_{i=1}^N ||B_i||$), thanks to Lemma 6.3, there exists, for all $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^l$ polynomial function, $f \in (\mathcal{C}^\infty(\overline{V}))^l$ (with $V = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r\,,\ |t| < \varepsilon(r^2 - |x|^2)\}$) solution to (6.27).
Suppose that we can use $\varphi = f\mathbf{1}_{\mathcal{V}}$ as a test function in (6.16); we would then obtain

$$\int_{\mathcal{V}} v(x,t) \cdot F(x,t)\, dx\, dt = 0 \tag{6.17}$$

for all polynomial function F; since these functions are dense in $(\mathcal{C}(\overline{\mathcal{V}}))^l$ and since $v \in (L^1(\mathcal{V}))^l$, this equation would also be true for all $F \in (\mathcal{C}(\overline{\mathcal{V}}))^l$, which is enough to see that $v = 0$ a.e. on $\mathcal{V}$.

Our aim now is to prove that, by approximating f in a precise way, (6.17) is true, which will conclude the proof of this proposition.
It is, in fact, fairly simple thanks to Lemma 6.4: with the sequence $(\gamma_n)_{n\geq 1}$ given in this lemma, we notice that, for all $n \geq 1$, $f_n = (\gamma_n f)_{|B(0,r)\times[0,T[} \in (\mathcal{C}_c^\infty(B(0,r) \times [0,T[))^l$ (because $f \in (\mathcal{C}^\infty(\overline{V}))^l$ and $\mathrm{supp}(\gamma_n)$ is a compact set of $V \subset B(0,r)\times]-\infty, T[$); thus, $\varphi = f_n$ is valid in (6.16) and we get

$$\int_{\mathcal{V}} v(x,t) \cdot \left((f_n)_t(x,t) + \sum_{i=1}^N B_i (f_n)_{x_i}(x,t) \right) dx\, dt = 0, \quad \forall n \geq 1. \tag{6.18}$$

Moreover, for $X \in \{x_1, \ldots, x_N, t\}$, we have

$$\frac{\partial f_n}{\partial X} = \frac{\partial \gamma_n}{\partial X} f_n + \gamma_n \frac{\partial f}{\partial X}.$$

The properties of γ_n gives us $\partial_X \gamma_n(x,t) = 0$ when $|x| < r - 1/n$ and $0 \leq t < \varepsilon(r^2 - |x|^2) - K/n$ (with K given in Lemma 6.4), and $\gamma_n(x,t) \to 1$ for all $(x,t) \in \mathcal{V}$. Thus, $\partial_X f_n(x,t) \to \partial_X f(x,t)$ for all $(x,t) \in \mathcal{V}$. We want now to prove that $\partial_X f_n$ is bounded in $(L^\infty(\mathcal{V}))^l$. Let $n > 1/r$.

- If $|x| < r - 1/n$ and $0 \leq t < \varepsilon(r^2 - |x|^2) - K/n$, then

$$\left|\left| \frac{\partial f_n}{\partial X}(x,t) \right|\right| \leq |\gamma_n(x,t)| \left|\left| \frac{\partial f}{\partial X}(x,t) \right|\right| \leq \left|\left| \frac{\partial f}{\partial X} \right|\right|_{(L^\infty(\mathcal{V}))^l}. \tag{6.19}$$

- If $(x,t) \in \overline{\mathcal{V}}$ are such that $r - 1/n \leq |x|$ or $\varepsilon(r^2 - |x|^2) - K/n \leq t$, then there exists $x' \in \overline{B}(0,r)$ such that $|x - x'| \leq 1/n$ and $|t - \varepsilon(r^2 - |x'|^2)| \leq \sup(2\varepsilon r, K)/n$ (take $x' = rx/|x|$ if $t < \varepsilon(r^2 - |x|^2) - K/n$ and $x' = x$ else).

  Since f is a Lipschitz continuous function on $\overline{V}$ (it is in $(\mathcal{C}^\infty(\overline{V}))^l$), and since $f(x', \varepsilon(r^2 - |x'|^2) = 0$ for all $|x'| \leq r$, there exists thus $C_0 > 0$ such that, for all $n \geq 1$, $||f(x,t)|| \leq C_0/n$ as soon as $r - 1/n \leq |x| \leq r$ or $\varepsilon(r^2 - |x|^2) - K/n \leq t \leq \varepsilon(r^2 - |x|^2)$. This gives then, with the properties of the derivatives of γ_n, for all such (x,t),

$$\left|\left| \frac{\partial f_n}{\partial X}(x,t) \right|\right| \leq \left| \frac{\partial \gamma_n}{\partial X}(x,t) \right| ||f(x,t)|| + |\gamma_n(x,t)| \left|\left| \frac{\partial f}{\partial X}(x,t) \right|\right| \leq CC_0 + \left|\left| \frac{\partial f}{\partial X} \right|\right|_{(L^\infty(\mathcal{V}))^l}. \tag{6.20}$$

The derivatives of f_n converge thus on $\mathcal{V}$ to the derivatives of f, and, thanks to (6.19) and (6.20), are bounded in $(L^\infty(\mathcal{V}))^l$; using the dominated convergence theorem, we can pass to the limit $n \to \infty$ in (6.18) to obtain

$$\int_{\mathcal{V}} v(x,t) \cdot \left(f_t(x,t) + \sum_{i=1}^N B_i f_{x_i}(x,t) \right) dx\, dt = \int_{\mathcal{V}} v(x,t) \cdot F(x,t)\, dx\, dt = 0,$$

i.e. what we wanted. ■

Corollary 6.2 *Let* $\Omega =]-3\nu\sqrt{N}, 3\nu\sqrt{N}[^N$ *(*$\nu \in \mathbb{R}_*^+$*),* $(A_1,\dots,A_N) \in M_l(\mathbb{R})$ *and* $T > 0$*. There exists* $T_0 \in]0,T]$ *such that, if* $u \in (L^1_{\mathrm{loc}}(\Omega \times [0,T[))^l$ *is a local weak solution on* $\Omega \times [0,T[$ *to (6.1) for an initial condition which is null a.e. on* $]-2\nu\sqrt{N}, 2\nu\sqrt{N}[^N$*, then* $u = 0$ *a.e. on* $]-\nu,\nu[^N \times]0,T_0[$.

Proof of Corollary 6.2
Let $r = 2\nu\sqrt{N}$ and choose ε satisfying the hypothesis of Proposition 6.2 (notice that ε only depends on N, r, $\sum_{i=1}^N ||A_i^T||$ and T). Let $T_0 = \varepsilon(r^2 - (r/2)^2) = 3\varepsilon r^2/4$ (which only depends on N, r, $(A_1,\dots,A_N)$, a and T).
By applying Proposition 6.2 to u and $x^0 = 0$ (in which case $B(0,r) \subset]-2\nu\sqrt{N}, 2\nu\sqrt{N}[^N$ and the initial condition corresponding to u is null a.e. on $B(0,r)$), we get $u = 0$ a.e. on $\mathcal{U} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < 2\nu\sqrt{N},\ 0 < t < \varepsilon(r^2 - |x|^2)\}$.
For all $x \in]-\nu,\nu[^N$, one has $|x| < \nu\sqrt{N} = r/2$ so that $\varepsilon(r^2 - |x|^2) > T_0$; thus, $]-\nu,\nu[^N \times]0,T_0[\subset \mathcal{U}$ and the corollary is proved. ■

6.4.2 Proof of the main result

We can now prove Theorems 6.1 and 6.2 when $N \geq 2$. The proof of Theorem 6.2 being very similar to the proof of 1) $\Longrightarrow$ 2) in Theorem 6.1, we only detail the proof of Theorem 6.1.

Proof of Theorem 6.1 when $N \geq 2$
We prove 1) $\Longrightarrow$ 2) in the first two steps, and 2) $\Longrightarrow$ 1) in the third step.

Step 1: the case $\xi = (1,0,\dots,0)$.
We suppose that Ω is an open set of $\mathbb{R}^N$, that a belongs to Ω and that, for any initial condition in $E(\Omega,a)$, (6.1) has a local weak solution on $\Omega \times [0,T[$ (for some $T > 0$ depending on the initial condition), and we will prove that, under these hypotheses, A_1 is diagonalizable on $\mathbb{R}$.
Thanks to item ii) of Remark 6.7, we can suppose (by taking an hypercube $O =]-6\nu\sqrt{N}, 6\nu\sqrt{N}[$ in $-a + \Omega$, $P = Id$, $\mu = 1$ and $x^0 = -a$) that (6.1) has a local weak solution for any initial condition in $E(O,0)$.
To show that A_1 is diagonalizable on $\mathbb{R}$, we will prove that the system (6.5) with A_1 instead of A has, for any initial condition in $E(]-\nu,\nu[,0)$, a local weak solution on $]-\nu,\nu[\times[0,T[$ (for a $T > 0$ depending on the initial condition); once this result is proved, and since we have already proven Theorem 6.1 when $N = 1$, this will allow us to conclude that A_1 is indeed diagonalizable on $\mathbb{R}$.

Let $v^0 \in E(]\nu,\nu[,0)$; we have, for some $\xi_1 \in \mathbb{R}^*$ and some $v^* \in \mathbb{R}^l$, $v^0(s) = v^* \mathbf{1}_{\mathbb{R}^-}(s\xi_1)$. Define $u^0 \in E(O,0)$ by $u^0(x) = v^* \mathbf{1}_{\mathbb{R}^-}(x \cdot \xi)$, where $\xi = (\xi_1, 0, \dots, 0)$, that is to say $u^0(x) = v^* \mathbf{1}_{\mathbb{R}^-}(x_1\xi_1)$.
By hypothesis, there exists $T > 0$ and a local weak solution $u \in (L^1_{\mathrm{loc}}(O \times [0,T[))^l$ on $O \times [0,T[$ to (6.1).

Let $h = (0,h_2,\dots,h_N) \in]-3\nu\sqrt{N}, 3\nu\sqrt{N}[^N$ and define, on $]-3\nu\sqrt{N}, 3\nu\sqrt{N}[$, $u_h(x) = u(x+h)$. Thanks to Lemma 6.1 and the first item of Remark 6.7 (with $P = Id$, $\lambda = 0$, $\mu = 1$, $x^0 = -h$ and $]-3\nu\sqrt{N}, 3\nu\sqrt{N}[^N$ as the open set included in $x^0 + O$), we see that u_h is a local weak solution on $]-3\nu\sqrt{N}, 3\nu\sqrt{N}[\times[0,T[$ to the system (6.1) with an initial condition ${u^0}_h(x) = u^0(x+h) = u^0(x)$ (by definition of u^0 and h). Thus, $u_h - u$ is a local weak solution on $]-3\nu\sqrt{N}, 3\nu\sqrt{N}[\times[0,T[$ to the system (6.1) with a null initial condition.
Corollary 6.2 tells us that there exists $T_0 \in]0,T]$ *not depending on* h such that $u_h - u = 0$ a.e. on $]-\nu,\nu[^N \times]0,T_0[$.
Thus, for a.e. $(x,t) \in]-\nu,\nu[^N \times]0,T_0[$ and any $h = (0,h_2,\dots,h_N) \in]-2\nu,2\nu[^N \subset]-3\nu\sqrt{N}, 3\nu\sqrt{N}[^N$, we have $u(x+h,t) = u(x,t)$; this means that $u_{|]-\nu,\nu[^N\times]0,T_0[}$ only depends on the first space variable x_1 and on t, i.e. that there exists $v \in (L^1_{\mathrm{loc}}(]-\nu,\nu[\times[0,T_0[))^l$ such that, for a.e. $(x,t) \in]-\nu,\nu[^N \times [0,T_0]$, $u(x,t) = v(x_1,t)$.

Since u is a local weak solution to (6.1) on $O \times [0,T[$, it is also a local weak solution of the same system on $]-\nu,\nu[^N\times[0,T_0[$, that is to say: for all $\varphi \in \mathcal{C}_c^\infty(]-\nu,\nu[^N\times[0,T_0[)$,

$$\int_{]-\nu,\nu[^N\times]0,T_0[} u(x,t)\varphi_t(x,t) + \sum_{i=1}^N A_i u(x,t)\varphi_{x_i}(x,t)\,dx\,dt + \int_{]-\nu,\nu[^N} v^0(x_1)\varphi(x,0)\,dx = 0. \tag{6.21}$$

But, for all $\varphi \in \mathcal{C}_c^\infty(]-\nu,\nu[^N\times[0,T_0[)$, the Fubini theorem gives, if $i \geq 2$,

$$\begin{aligned}\int_0^{T_0}\int_{]-\nu,\nu[^N} & A_i u(x,t)\varphi_{x_i}(x,t)\,dx\,dt \\ &= \int_0^{T_0}\int_{]-\nu,\nu[^{N-1}} A_i v(x_1,t)\left(\int_{-\nu}^{\nu}\varphi_{x_i}(x,t)\,dx_i\right)dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_N\,dt = 0. \end{aligned}\tag{6.22}$$

Take now $\psi \in \mathcal{C}_c^\infty(]-\nu,\nu[\times[0,T_0[)$ and $\varphi(x,t) = \psi(x_1,t)\gamma(x_2,\ldots,x_N)$, with $\gamma \in \mathcal{C}_c^\infty(]-\nu,\nu[^{N-1})$ the integral of which is equal to 1. (6.21) and (6.22) gives us

$$\int_{]-\nu,\nu[\times]0,T_0[} v(x_1,t)\psi_t(x_1,t) + A_1 v(x_1,t)\psi_{x_1}(x_1,t)\,dx_1\,dt + \int_{]-\nu,\nu[} v^0(x_1)\psi(x_1,0)\,dx_1 = 0,$$

which exactly means that $v \in (L^1_{\text{loc}}(]-\nu,\nu[^N\times[0,T_0[))^l$ is a local weak solution on $]-\nu,\nu[\times[0,T_0[$ to (6.5) with A_1 instead of A and v^0 instead of u^0. Since v^0 was an arbitrary function in $E(]-\nu,\nu[,0)$, this concludes this step.

Step 2: the general case.
We suppose that Ω is an open set of $\mathbb{R}^N$, that a belongs to Ω and that, for any initial condition in $E(\Omega,a)$, (6.1) has a local weak solution on $\Omega\times[0,T[$ (for some $T>0$ depending on the initial condition).
Take $\xi \in \mathbb{R}^N\backslash\{0\}$ (the case $\xi = 0$ being trivial). We want to show that $\sum_{i=1}^N A_i\xi_i$ is diagonalizable on $\mathbb{R}$; obviously, by dividing this matrix by $|\xi|$, we can consider that $|\xi| = 1$.

We notice first an easy property of (6.1) with respect to orthogonal changes of variables.
Let $Q = ((q_{i,j}))_{(i,j)\in[1,N]^2}$ be an orthogonal $N\times N$ matrix, and denote $\Omega' = Q^{-1}(\Omega)$.
Take $u^0 \in (L^1_{\text{loc}}(\Omega'))^l$ and $u \in (L^1_{\text{loc}}(\Omega'\times[0,T[))^l$ (for some $T>0$) and define $v^0 \in (L^1_{\text{loc}}(\Omega))^l$ and $v \in (L^1_{\text{loc}}(\Omega\times[0,T[))^l$ by $v^0(x) = u^0(Q^{-1}x)$ and $v(x,t) = u(Q^{-1}x,t)$.
Then one has, for any $\varphi \in \mathcal{C}_c^\infty(\Omega'\times[0,T[)$, by denoting $\psi(x,t) = \varphi(Q^{-1}x,t) \in \mathcal{C}_c^\infty(\Omega\times[0,T[)$ (which satisfies $\psi_t(x,t) = \varphi_t(Q^{-1}x,t)$ and $\nabla_x\psi(x,t) = (Q^{-1})^T\nabla_x\varphi(Q^{-1}x,t) = Q\nabla_x\varphi(Q^{-1}x,t)$),

$$\begin{aligned}\int_0^T\int_\Omega & v(x,t)\psi_t(x,t) + \sum_{i=1}^N A_i v(x,t)\psi_{x_i}(x,t)\,dx\,dt + \int_\Omega v^0(x)\psi(x,0)\,dx \\ &= \int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t) + \sum_{i=1}^N A_i u(x,t)\left(Q\nabla_x\varphi\right)_i(x,t)\,dx\,dt + \int_{\Omega'} u^0(x)\varphi(x,0)\,dx \\ &= \int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t) + \sum_{i=1}^N A_i u(x,t)\sum_{j=1}^N q_{i,j}\varphi_{x_j}(x,t)\,dx\,dt + \int_{\Omega'} u^0(x)\varphi(x,0)\,dx \\ &= \int_0^T\int_{\Omega'} u(x,t)\varphi_t(x,t) + \sum_{j=1}^N\left(\sum_{i=1}^N q_{i,j}A_i\right)u(x,t)\varphi_{x_j}(x,t)\,dx\,dt + \int_{\Omega'} u^0(x)\varphi(x,0)\,dx.\end{aligned}$$

Thus, if v is a local weak solution on $\Omega\times[0,T[$ to (6.1) with v^0 as initial condition, then u is a local weak solution on $\Omega'\times[0,T[$ to (6.1) with $\mathcal{A}_j = \sum_{i=1}^N q_{i,j}A_i$ instead of A_j ($j\in[1,N]$) and u^0 as initial condition.

Take now an orthogonal matrix Q such that $q_{i,1} = \xi_i$ for all $i \in [1, N]$ (this is possible since $|\xi| = 1$). We still denote $\Omega' = Q^{-1}(\Omega)$.
Let $u^0 \in E(\Omega', Q^{-1}a)$; u^0 can be written as $u^* \mathbf{1}_{\mathbb{R}^-}((\cdot - Q^{-1}a) \cdot \eta)$ for some $u^* \in \mathbb{R}^l$ and $\eta \in \mathbb{R}^N \backslash \{0\}$. $v^0 = u^0 \circ Q^{-1}$ is then $v^0(x) = u^* \mathbf{1}_{\mathbb{R}^-}((Q^{-1}x - Q^{-1}a) \cdot \eta) = u^* \mathbf{1}_{\mathbb{R}^-}((x-a) \cdot Q\eta)$ (because $Q^{-1} = Q^T$ since Q is an orthogonal matrix), and we have then $v^0 \in E(\Omega, a)$; by hypothesis, there exists thus $T > 0$ and a local weak solution $v \in (L^1_{\mathrm{loc}}(\Omega \times [0, T[))^l$ on $\Omega \times [0, T[$ to (6.1) for the initial condition v^0.
Thus, by the preceding calculus, $u(x, t) = v(Qx, t)$ is a local weak solution on $\Omega' \times [0, T[$ to (6.1) with $\mathcal{A}_j = \sum_{i=1}^N q_{i,j} A_i$ instead of A_j ($j \in [1, N]$) and u^0 as initial condition. Since u^0 is an arbitrary function of $E(\Omega', Q^{-1}a)$, the reasoning of Step 1 shows that $\mathcal{A}_1 = \sum_{i=1}^N \xi_i A_i$ is diagonalizable on $\mathbb{R}$.

Step 3: proof of 2) $\Longrightarrow$ 1).
Let $u^0 = u^* \mathbf{1}_{\mathbb{R}^-}((\cdot - a) \cdot \xi) \in E(\Omega, a)$. We can always suppose that $|\xi| = 1$ (because $\mathbf{1}_{\mathbb{R}^-}((\cdot - a) \cdot \xi) = \mathbf{1}_{\mathbb{R}^-}((\cdot - a) \cdot \frac{\xi}{|\xi|})$).

The idea is of course (as indicated by the first step of this proof) to search for a solution u only depending on the coordinate along ξ, that is to say $u(x, t) = v((x - a) \cdot \xi, t)$. We see then that v must satisfy

$$\left\{ \begin{array}{ll} v_t(z,t) + \left(\sum_{i=1}^N \xi_i A_i\right) v_z(z,t) = 0 & z \in \mathbb{R}\,,\ t > 0 \\ v(z,0) = u^* \mathbf{1}_{\mathbb{R}^-}(z) & z \in \mathbb{R}. \end{array} \right. \tag{6.23}$$

This problem is a 1-dimensional one and, since the matrix $\sum_{i=1}^N \xi_i A_i$ is diagonalizable on $\mathbb{R}$, we can solve it.

Take $(e_1, \ldots, e_l)$ a basis of $\mathbb{R}^l$ made of eigenvectors of $\sum_{i=1}^N \xi_i A_i$ and denote by λ_α the eigenvalue of $\sum_{i=1}^N \xi_i A_i$ associated to e_α ($\alpha \in [1, l]$). We write $u^* = \sum_{\alpha=1}^l u^*_\alpha e_\alpha$.
Define $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^N \times [0, \infty[))^l$ by $u(x, t) = \sum_{\alpha=1}^l u^*_\alpha \mathbf{1}_{\mathbb{R}^-}((x - a) \cdot \xi - \lambda_\alpha t) e_\alpha$. Then, using the calculus made in Step 2 (to reduce to the case $\xi = (1, 0, \ldots, 0)$) and some simple changes of variables, one can see that u is a weak solution on $\mathbb{R}^N \times [0, \infty[$ to (6.1) with u^0 as initial condition. ■

6.5 About the linearization of a particular non-hyperbolic problem

Recall that a non-linear problem

$$\left\{ \begin{array}{l} u_t + (f(u))_x = 0, \\ u(x,0) = u^0(x) \end{array} \right. \tag{6.24}$$

is (unconditionaly) hyperbolic if, for every $\overline{u} \in \mathbb{R}^l$, the matrix $f'(\overline{u})$ is diagonalizable on $\mathbb{R}$ (i.e. the linearized problem around any state $\overline{u}$ is hyperbolic).
We have already noticed (see item ii) of Remark 6.3) that the hyperbolicity of a non-linear system of the kind (6.24) is interesting, for example when we want to apply classical Finite Volumes Schemes to the problem.

Consider the 2×2 following classical system (this is the pressureless gas system):

$$\left\{ \begin{array}{ll} \begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in \mathbb{R}\,,\ t \in \mathbb{R}^+_*, \\ \begin{pmatrix} \rho \\ u \end{pmatrix}(x,0) = \begin{pmatrix} \rho^0 \\ u^0 \end{pmatrix}(x), & x \in \mathbb{R}, \end{array} \right.$$

This system is equivalent, for regular solutions such that $\rho \neq 0$, to

$$\left\{ \begin{array}{ll} \left(\begin{array}{c} \rho \\ u \end{array}\right)_t + \left(\begin{array}{c} \rho u \\ u^2/2 \end{array}\right)_x = \left(\begin{array}{c} 0 \\ 0 \end{array}\right), & x \in \mathbb{R}\,,\ t \in \mathbb{R}_*^+, \\ \left(\begin{array}{c} \rho \\ u \end{array}\right)(x,0) = \left(\begin{array}{c} \rho^0 \\ u^0 \end{array}\right)(x), & x \in \mathbb{R}, \end{array} \right. \tag{6.25}$$

Take now $u^0(x) = -1$ if $x < 0$, $u^0(x) = +1$ if $x > 0$ and $\rho^0 \in L^\infty(\mathbb{R})$.
A straightforward computation lets us see that the functions

$$\rho(x,t) = \left\{ \begin{array}{ll} \rho^0(x-t) & \text{if } x < -t, \\ 0 & \text{if } -t \leq x \leq t, \\ \rho^0(x+t) & \text{if } t < x \end{array} \right.$$

and

$$u(x,t) = \left\{ \begin{array}{ll} -1 & \text{if } x < -t, \\ \dfrac{x}{t} & \text{if } -t \leq x \leq t, \\ +1 & \text{if } t < x, \end{array} \right. \quad \text{(that is to say the entropy solution of the Burgers problem)}$$

define a (space and time global) weak solution to (6.25) in the following natural sense: $(\rho, u) \in (L^\infty(\mathbb{R} \times [0,+\infty[))^2$ and, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times [0,+\infty[)$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(\begin{array}{c} \rho \\ u \end{array}\right)(x,t)\varphi_t(x,t)\,dx\,dt + \int_0^{+\infty}\int_{\mathbb{R}} \left(\begin{array}{c} \rho u \\ u^2/2 \end{array}\right)(x,t)\varphi_x(x,t)\,dx\,dt + \int_{\mathbb{R}} \left(\begin{array}{c} \rho^0 \\ u^0 \end{array}\right)(x)\varphi(x,0)\,dx = 0.$$

However, the linearization of (6.25) around a state $(\overline{\rho}, \overline{u}) \in \mathbb{R}^2$ is the linear system given by the matrix

$$A(\overline{\rho}, \overline{u}) = \left(\begin{array}{cc} \overline{u} & \overline{\rho} \\ 0 & \overline{u} \end{array}\right).$$

This matrix being diagonalizable only if $\overline{\rho} = 0$, the system (6.25) is not unconditionaly hyperbolic (it is conditionaly hyperbolic in the sense that the hyperbolicity of the linearized system depends on the state around which the linearization has been made).
Moreover, a close examination of the step 1.2 of the proof of Theorem 6.1 when $N = 1$ shows that the initial condition (ρ^0, u^0) we have chosen here is precisely the kind of initial condition for which the system defined by $A(\overline{\rho}, \overline{u})$ has no local weak solution (when $\overline{\rho} \neq 0$, which is for example the case when we choose $\rho^0 \equiv 1$ and we linearize around a state in the image of the initial condition).
On this example, we have shown that a non-linear system can have a weak solution for a particular initial datum, although the corresponding linearized system has no solution for this initial datum.

Notice however that, in our example, the solution we have found partly pass, for any $t > 0$, in the hyperbolicity zone of (6.25) (i.e. $\{(\overline{\rho}, \overline{u}) \in \mathbb{R}^2 \,|\, \overline{\rho} = 0\}$).

6.6 Appendix

6.6.1 Harmonic functions

The following lemma is a very classical result, but we give here a simple and self-contained proof.

Lemma 6.2 *Let Ω be an open set of $\mathbb{R}^N$, with $N \geq 1$. If $u \in L^1_{\text{loc}}(\Omega)$ satisfies $\Delta u = 0$ in $\mathcal{D}'(\Omega)$, then $u \in \mathcal{C}^\infty(\Omega)$.*

Proof of Lemma 6.2
Let us first introduce (or recall) some notations. $|.|$ designates the Euclidean norm in $\mathbb{R}^N$; when $r \in \mathbb{R}^+$, B_r is the Euclidean ball in $\mathbb{R}^N$, of center 0 and radius r and, for $x \in \mathbb{R}^N$, $B(x,r) = x + B_r$. When E is a borelian set of $\mathbb{R}^N$, $|E|$ stands for the Lebesgue measure of E.

Let us take $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ whose support is contained in B_1 and which satisfies

$$\rho \geq 0\,,\ \int_{\mathbb{R}^N} \rho(x)\,dx = 1 \text{ and } \rho(x) = \rho(y) \text{ whenever } |x| = |y|.$$

Define $\rho_n(x) = n^N \rho(nx)$. The function

$$u_n(x) = \int_\Omega u(t)\rho_n(x-t)\,dt = \int_{B_{1/n}} u(x+t)\rho_n(t)\,dt \qquad (\text{since } \rho_n(-\cdot) = \rho_n(\cdot))$$

is well defined and $\mathcal{C}^\infty$ on the open set $\Omega_n = \{x \in \Omega \mid d(x,\Omega^c) > 1/n\}$; moreover, one can verify that $\Delta u = 0$ in $\mathcal{D}'(\Omega)$ implies $\Delta u_n = 0$ in $\mathcal{D}'(\Omega_n)$ (thus in a classical way since u_n is regular).
u_n being regular and harmonic on Ω_n, it is well known that, for all $x \in \Omega_n$ and all $r < d(x,\Omega_n^c)$,

$$u_n(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} u_n(y)\,dy. \tag{6.26}$$

Since $u \in L^1_{\mathrm{loc}}(\Omega)$, we know that u_n (extented by 0 outside Ω_n) converges, as $n \to \infty$, to u in $L^1_{\mathrm{loc}}(\Omega)$, thus a.e. up to a subsequence. For a point $x \in \Omega$ where $u_n(x) \to u(x)$, i.e. for a.e. $x \in \Omega$, and for $r < d(x,\Omega^c)$, passing to the limit $n \to \infty$ in (6.26) (which is satisfied as soon as $n > 1/(\mathrm{dist}(x,\Omega^c) - r)$) gives us

$$u(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy = \frac{1}{|B_r|}\int_\Omega u(y)\mathbf{1}_{B(x,r)}(y)\,dy.$$

By the dominated convergence theorem, we see from this formula that u is (a.e. equal to) a continuous function on Ω; moreover, taking $s < r < d(x,\Omega^c)$, we obtain

$$\begin{aligned}\frac{1}{|B(x,r)\backslash B(x,s)|}\int_{B(x,r)\backslash B(x,s)} u(y)\,dy &= \frac{1}{|B(x,r)| - |B(x,s)|}\int_{B(x,r)} u(y)\,dy \\ &\quad - \frac{1}{|B(x,r)| - |B(x,s)|}\int_{B(x,s)} u(y)\,dy \\ &= \frac{|B(x,r)|}{|B(x,r)| - |B(x,s)|}u(x) - \frac{|B(x,s)|}{|B(x,r)| - |B(x,s)|}u(x) = u(x).\end{aligned}$$

Using the polar coordinates and passing to the limit $s \to r$ gives then, since u is continuous,

$$u(x) = \frac{1}{\sigma(S_1)}\int_{S_1} u(x+ry)\,d\sigma(y) \qquad \text{for all } x \in \Omega \text{ and } r < d(x,\Omega^c),$$

where $S_1 = \{y \in \mathbb{R}^N \mid |y| = 1\} = \partial B_1$ and σ is the $(N-1)$-dimensional measure on S_1.
Using the polar coordinates once again, and the fact that ρ_n is spherically invariant (that is to say, $\rho_n(x) = \theta_n(|x|)$ for a $\theta_n : \mathbb{R}^+ \to \mathbb{R}^+$), we get, for all $n \geq 1$ and all $x \in \Omega_n$,

$$\begin{aligned}u_n(x) &= \int_0^{1/n}\left(\int_{S_1} u(x+ry)\,d\sigma(y)\right) r^{N-1}\theta_n(r)\,dr \\ &= u(x)\int_0^{1/n} \sigma(S_1) r^{N-1}\theta_n(r)\,dr \\ &= u(x)\int_0^{1/n}\int_{S_1} \rho_n(ry) r^{N-1}\,d\sigma(y)\,dr = u(x)\int_{B_{1/n}} \rho_n(t)\,dt = u(x),\end{aligned}$$

which implies that $u \in \mathcal{C}^\infty(\Omega_n)$ for all $n \geq 1$, that is to say $u \in \mathcal{C}^\infty(\Omega)$. ∎

6.6.2 Lemmas for Proposition 6.2

Recall that $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$, $||\cdot||$ is any norm on $\mathbb{R}^l$ and, when $B \in M_l(\mathbb{R})$, $||B||$ denotes the induced norm.
The following result is a (very simple) particular case of the Cauchy-Kowalewska Theorem (see [33]). We however state and prove it, because we need the precise estimates on the time-space range of existence of the "local" solution given by the Cauchy-Kowalewska Theorem.

Lemma 6.3 *Let $(B_1, \ldots, B_N)$ be $l \times l$ real matrices; we denote $M = \sum_{i=1}^N ||B_i||$. Let $r \in \mathbb{R}_*^+$, and $\varepsilon > 0$ such that $4Mr(N+1)\varepsilon < 1$ and denote $V = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < r\,,\ |t| < \varepsilon(r^2 - |x|^2)\}$. Then, for any polynomial function $F : \mathbb{R}^{N+1} \to \mathbb{R}^l$, there exists $f \in (\mathcal{C}^\infty(\overline{V}))^l$ such that*

$$\left\{\begin{array}{ll} f_t(x,t) + \displaystyle\sum_{i=1}^N B_i f_{x_i}(x,t) = F(x,t), & (x,t) \in V, \\ f(x, \varepsilon(r^2 - |x|^2)) = 0, & |x| \le r. \end{array}\right. \tag{6.27}$$

Proof of Lemma 6.3
If $M = 0$, then $B_i = 0$ for all $i \in [1,N]$ and the result is quite simple: the function $f(x,t) = \int_{\varepsilon(r^2-|x|^2)}^t F(x,s)\,ds$ defined and $\mathcal{C}^\infty$ on $\mathbb{R}^N \times \mathbb{R}$ is a solution to (6.27).
We suppose thus from now on that $M > 0$.

Let $\theta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be the $\mathcal{C}^\infty$-diffeomorphism $\theta(y,s) = (2My, s + \varepsilon(r^2 - |2My|^2))$. Denote $G = F \circ \theta$ (G is also a polynomial function), $U = \{(y,s) \in \mathbb{R}^{N+1} \mid |y| < r/2M\,,\ |s| < r/2M\}$ and suppose we have found $g \in (\mathcal{C}^\infty(\overline{U}))^l$ solution of

$$\left\{\begin{array}{ll} g_s(y,s) + 4M\varepsilon \left(\sum\limits_{i=1}^N y_i B_i\right) g_s(y,s) + \frac{1}{2M} \sum\limits_{i=1}^N B_i g_{y_i}(y,s) = G(y,s), & (y,s) \in U, \\ g(y,0) = 0, & |y| \le r/2M. \end{array}\right. \tag{6.28}$$

Then, $f = g \circ \theta^{-1} \in (\mathcal{C}^\infty(\overline{\theta(U)}))^l$ and, since $\theta(U) \supset V$ (because $4Mr\varepsilon < 1/(N+1) < 1$), $f \in (\mathcal{C}^\infty(\overline{V}))^l$; moreover, $f(x,t) = g(x/2M, t - \varepsilon(r^2 - |x|^2))$ and we have thus, for all $(x,t) \in V$,

$$\begin{aligned} f_t(x,t) + \sum_{i=1}^N B_i f_{x_i}(x,t) &= g_s(x/2M, t - \varepsilon(r^2 - |x|^2))) + 2\varepsilon \sum_{i=1}^N 2M \frac{x_i}{2M} B_i g_s(x/2M, t - \varepsilon(r^2 - |x|^2)) \\ &\quad + \frac{1}{2M} \sum_{i=1}^N B_i g_{y_i}(x/2M, t - \varepsilon(r^2 - |x|^2)) \\ &= G(x/2M, t - \varepsilon(r^2 - |x|^2)) = F(x,t), \end{aligned}$$

with $f(x, \varepsilon(r^2 - |x|^2)) = g(x/2M, 0) = 0$ as soon as $|x| \le r$. Thus, solving (6.28) is enough to prove the lemma.

We search a solution to (6.28) under the form of a power series:

$$g(y,s) = \sum_{\alpha \in \mathbb{N}^N\,,\ k \in \mathbb{N}} g_{\alpha,k} y^\alpha s^k$$

(where $y^\alpha = y_1^{\alpha_1} \cdots y_N^{\alpha_N}$) with $g_{\alpha,k} \in \mathbb{R}^l$. It is then well known that, by denoting $D_y^\alpha = D_{y_1}^{\alpha_1} \cdots D_{y_N}^{\alpha_N}$ and $\alpha! = \alpha_1! \ldots \alpha_N!$, we must have $g_{\alpha,k} = D_y^\alpha D_s^k g(0,0)/\alpha! k!$. We will now compute these coefficients by supposing that g satisfies (6.28) and, then, show that the obtained coefficients define a $\mathcal{C}^\infty$ function on $\overline{U}$.

Using the second equation of (6.28), we see that $D_y^\alpha g(0,0) = 0$ for all $\alpha \in \mathbb{N}^N$. By applying, for $(\alpha, k) = (\alpha_1, \dots, \alpha_N, k) \in \mathbb{N}^N \times \mathbb{N}$, $D_y^\alpha D_s^k$ to the first equation of (6.28) and using Leibniz' formula, we find

$$\begin{aligned} D_y^\alpha D_s^{k+1} g_s(y,s) \quad &+ \quad 4M\varepsilon \sum_{i=1}^N (y_i B_i D_y^\alpha D_s^{k+1} g(y,s) + \alpha_i B_i D_y^{\alpha - e_i} D_s^{k+1} g(y,s)) \\ &+ \quad \frac{1}{2M} \sum_{i=1}^N B_i D_y^{\alpha+e_i} D_s^k g(y,s) = D_y^\alpha D_s^k G(y,s), \end{aligned}$$

where e_i is the element of $\mathbb{N}^N$ which has 1 on the i^{th} position and 0 on the other positions ($D_y^{\alpha-e_i}$ is the null operator if $\alpha_i = 0$). Thus,

$$D_y^\alpha D_s^{k+1} g(0,0) = D_y^\alpha D_s^k G(0,0) - 4M\varepsilon \sum_{i=1}^N \alpha_i B_i D_y^{\alpha-e_i} D_s^{k+1} g(0,0) - \frac{1}{2M} \sum_{i=1}^N B_i D_y^{\alpha+e_i} D_s^k g(0,0).$$

Define $a_{\alpha,0} = 0$ (for all $\alpha \in \mathbb{N}^N$) and, by induction on k and $|\alpha| = \sum_{i=1}^N \alpha_i$, for $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$,

$$a_{\alpha,k+1} = D_y^\alpha D_s^k G(0,0) - 4M\varepsilon \sum_{i=1}^N \alpha_i B_i a_{\alpha-e_i,k+1} - \frac{1}{2M} \sum_{i=1}^N B_i a_{\alpha+e_i,k} \tag{6.29}$$

(no matter the definition of $a_{\alpha-e_i,k+1}$ when $\alpha_i = 0$: since these coefficients are always multiplied by α_i, they can be omited in the sums up above).

We will now obtain an estimate on these $a_{\alpha,k}$. By denoting $K_n = \frac{1}{n!} \sup\{||a_{\alpha,k}||\,,\ |\alpha| + k = n\}$, (6.29) gives us, for all $n \geq 0$ and with $(\alpha,k) \in \mathbb{N}^{N+1}$ such that $|\alpha| + k = n$,

$$\begin{aligned} ||a_{\alpha,k+1}|| \quad &\leq \quad ||D_y^\alpha D_s^k G(0,0)|| + 4M\varepsilon \sum_{i=1}^N ||B_i|| \alpha_i n! K_n + \frac{1}{2M} \sum_{i=1}^N ||B_i|| (n+1)! K_{n+1} \\ &\leq \quad ||D_y^\alpha D_s^k G(0,0)|| + 4M^2 \varepsilon (n+1)! K_n + \frac{1}{2}(n+1)! K_{n+1}. \end{aligned}$$

Taking the suppremum of this on all $(\alpha,k) \in \mathbb{N}^{N+1}$ with $|\alpha| + k = n$ (recall that $a_{\alpha,0} = 0$ for all $\alpha \in \mathbb{N}^N$) and diving by $(n+1)!$, we get

$$K_{n+1} \leq \frac{\sup_{|\alpha|+k=n} ||D_y^\alpha D_s^k G(0,0)||}{(n+1)!} + 4M^2 \varepsilon K_n + \frac{1}{2} K_{n+1},$$

that is to say, with $C_n = \frac{2}{(n+1)!} \sup_{|\alpha|+k=n} ||D_y^\alpha D_s^k G(0,0)||$,

$$K_{n+1} \leq C_n + 8M^2 \varepsilon K_n \qquad \text{for all } n \geq 0.$$

We deduce from this the estimate that, for all $n \geq 1$ (recall that $K_0 = 0$),

$$K_n \leq C_{n-1} + (8M^2\varepsilon) C_{n-2} + \cdots + (8M^2\varepsilon)^l C_{n-1-l} + \cdots + (8M^2\varepsilon)^{n-1} C_0.$$

But, by denoting n_0 the degree of the polynomial G, we have $C_{n-1-l} = 0$ as soon as $n - 1 - l > n_0$; by taking $C = \sup(C_0, \dots, C_{n_0})$ and $n \geq n_0 + 1$, the preceding estimate is reduced to

$$K_n \quad \leq \quad C(8M^2\varepsilon)^{n-1-n_0}(1 + 8M^2\varepsilon + \ldots + (8M^2\varepsilon)^{n_0}).$$

Thanks to the hypothesis on ε, there exists thus $R > r/2M$ such that $8M^2\varepsilon < 1/(N+1)R$; we can thus find $\overline{K} \in \mathbb{R}^+$ such that, for all $n \geq 0$, $K_n \leq \overline{K}/((N+1)R)^n$, that is to say:

$$\text{for all } (\alpha,k) \in \mathbb{N}^N,\ ||a_{\alpha,k}|| \leq \frac{\overline{K}(|\alpha|+k)!}{((N+1)R)^{|\alpha|+k}}. \tag{6.30}$$

Using the fact that, as soon as $\sup_{i\in[1,N]}|y_i|<R$ and $|s|<R$, the series

$$\sum_{\alpha\in\mathbb{N}^N\,,\,k\in\mathbb{N}}\frac{(|\alpha|+k)!}{\alpha!k!}\frac{|y_1|^{\alpha_1}\cdots|y_N|^{\alpha_N}|s|^k}{((N+1)R)^{|\alpha|+k}}$$

$$=\sum_{n\geq 1}\sum_{|\alpha|+k=n}\frac{(|\alpha|+k)!}{\alpha!k!}\left(\frac{|y_1|}{(N+1)R}\right)^{\alpha_1}\cdots\left(\frac{|y_N|}{(N+1)R}\right)^{\alpha_N}\left(\frac{|s|}{(N+1)R}\right)^k$$

$$=\sum_{n\geq 0}\left(\frac{|y_1|}{(N+1)R}+\ldots+\frac{|y_N|}{(N+1)R}+\frac{|s|}{(N+1)R}\right)^n$$

$$=\frac{1}{1-(|y_1|/(N+1)R+\ldots+|y_N|/(N+1)R+|s|/(N+1)R)}$$

converges (because $(|y_1|+\ldots+|y_N|+|s|)/((N+1)R)<1$), we can prove, thanks to (6.30), that

$$g(y,s)=\sum_{\alpha\in\mathbb{N}^N\,,\,k\in\mathbb{N}}\frac{a_{\alpha,k}}{\alpha!k!}y^\alpha s^k$$

converges on $W=\{(y,s)\in\mathbb{R}^N\times\mathbb{R}\mid\sup_{i\in[1,N]}|y_i|<R\,,\;|s|<R\}$ and defines a function which is $\mathcal{C}^\infty$ on W (the derivative of g being obtained as the series of the derivatives of $\frac{a_{\alpha,k}}{\alpha!k!}y^\alpha s^k$).
Since $a_{\alpha,0}=0$ for all $\alpha\in\mathbb{N}^N$, we have $g(y,0)=0$ as soon as $(y,0)\in W$. Moreover, $S(y,s)=g_s(y,s)+4M\varepsilon(\sum_{i=1}^N y_iB_i)g_s(y,s)+\frac{1}{2M}\sum_{i=1}^N B_ig_{y_i}(y,s)-G(y,s)$ is a power series on W, the derivatives of which all vanish at $(y,s)=(0,0)$ (the $a_{\alpha,k}$ have been constructed to satisfy this property): thus, $S\equiv 0$ on W.
But $W\supset\{(y,s)\in\mathbb{R}^N\times\mathbb{R}\mid|y|<R\,,\;|s|<R\}\supset\overline{U}$ (recall that $R>r/2M$), so that $g\in(C^\infty(\overline{U}))^l$ satisfies (6.28). ■

Lemma 6.4 *Let $(r,\varepsilon)\in(\mathbb{R}_*^+)^2$ and $K>1+2\varepsilon r+\varepsilon$. There exists $C>0$ and $(\gamma_n)_{n\geq 1}\in\mathcal{C}_c^\infty(\mathbb{R}^N\times\mathbb{R})$ such that, by denoting $V_n=\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<r-1/n\,,\;|t|<\varepsilon(r^2-|x|^2)-K/n\}$, we have, for all $n\geq 1$:*

$$\begin{aligned}&\mathrm{supp}(\gamma_n)\subset V=\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<r\,,\;|t|<\varepsilon(r^2-|x|^2)\},\\&||\gamma_n||_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq 1,\\&\gamma_n\equiv 1 \text{ on } V_n,\\&||\nabla_{(x,t)}\gamma_n||_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq Cn.\end{aligned}$$

Proof of Lemma 6.4
Take $\rho\in\mathcal{C}_c^\infty(\mathbb{R}^{N+1})$, the support of which is included in $\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<1\,,\;|t|<1\}$, $\rho\geq 0$ and $\int_{\mathbb{R}^{N+1}}\rho(x,t)\,dx\,dt=1$. Let $\rho_n(x,t)=n^{N+1}\rho(nx,nt)$; the support of ρ_n is included in $\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid|x|<1/n\,,\;|t|<1/n\}$
Define $\gamma_n=\mathbf{1}_{V_{2n}}*\rho_{2n}$; γ_n is in $\mathcal{C}_c^\infty(\mathbb{R}^N\times\mathbb{R})$ and is bounded by 1 in $L^\infty(\mathbb{R}^N\times\mathbb{R})$ (Young inequality).
For $X\in\{x_1,\ldots,x_N,t\}$, one has

$$\frac{\partial\gamma_n}{\partial X}=\mathbf{1}_{V_{2n}}*\frac{\partial\rho_{2n}}{\partial X},$$

and, since $\partial_X\rho_{2n}(x,t)=2n\times(2n)^{N+1}\partial_X\rho(2nx,2nt)$, we get, thanks to Young inequality,

$$\left|\left|\frac{\partial\gamma_n}{\partial X}\right|\right|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}\leq\left|\left|\frac{\partial\rho_{2n}}{\partial X}\right|\right|_{L^1(\mathbb{R}^N\times\mathbb{R})}\leq 2n\left|\left|\frac{\partial\rho}{\partial X}\right|\right|_{L^1(\mathbb{R}^N\times\mathbb{R})}.$$

To conclude the proof of this lemma, we will need the following computations. Let $(p,q) \in \mathbb{N}^*$ and $(x,t) \in V_p$, $(x',t') \in \mathbb{R}^N \times \mathbb{R}$ such that $|x'| < 1/q$ and $|t'| < 1/q$; we notice first that

$$|x+x'| \leq |x| + |x'| < r - \frac{1}{p} + \frac{1}{q}. \tag{6.31}$$

Since $|x+x'|^2 = |x|^2 + 2x\cdot x' + |x'|^2 \leq |x|^2 + 2|x|/q + 1/q^2 \leq |x|^2 + 2r/q + 1/q$, we have

$$\begin{aligned} \varepsilon(r^2 - |x+x'|^2) &> |t| + \frac{K}{p} - \frac{2\varepsilon r}{q} - \frac{\varepsilon}{q} + |t'| - \frac{1}{q} \\ &\geq |t+t'| + \frac{K}{2p}\left(2 - \frac{4\varepsilon r p}{Kq} - \frac{2p\varepsilon}{Kq} - \frac{2p}{Kq}\right). \end{aligned} \tag{6.32}$$

Let us now check that $\mathrm{supp}(\gamma_n) \subset V$. We know that $\mathrm{supp}(\gamma_n) \subset V_{2n} + \mathrm{supp}(\rho_{2n})$; by taking $p = q = 2n$ in (6.31) and (6.32) we see, thanks to the hypothesis on K, that $V_{2n} + \mathrm{supp}(\rho_{2n}) \subset V$.
The last property on γ_n is also easy to check. If $(x,t) \in V_n$ then, with $(p,q) = (n,2n)$ in (6.31) and (6.32), one sees, thanks to the hypothesis on K, that $(x,t) - \mathrm{supp}(\rho_n) \subset V_{2n}$ so that

$$\gamma_n(x,t) = \int_{V_{2n}\cap(x,t)-\mathrm{supp}(\rho_{2n})} \rho_{2n}((x,t)-(y,s))\,dy\,ds = \int_{\mathrm{supp}(\rho_{2n})} \rho_{2n}(y,s)\,dy\,ds = 1.$$

The sequence $(\gamma_n)_{n\geq 1}$ satisfies thus the conclusions of the lemma. ■

Chapitre 7

Un contre-exemple intéressant

7.1 Hyperbolique "Fourier" et Hyperbolique au sens de [37]

Lorsque l'on demande que le problème soit bien posé au sens de Hadamard dans $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^N))$, une petite étude de Fourier (voir [67]) donne la condition nécessaire suivante: $\sup_{\xi\in\mathbb{R}^N} ||e^{i\sum_{j=1}^N \xi_j A_j}|| < +\infty$, ce qui revient à dire (cf [48]) que les matrices $\sum_{j=1}^N \xi_j A_j$ doivent être uniformément diagonalisables lorsque $\xi \in \mathbb{R}^N$. Etant donné l'homogénéité de ces matrices, c'est équivalent à dire qu'il existe $P : S^{N-1} \to \mathrm{GL}(l;\mathbb{R})$ tel que $\sup_{\xi\in S^{N-1}} ||P(\xi)||\,||P(\xi)^{-1}|| < +\infty$ et, pour tout $\xi \in S^{N-1}$, $P(\xi)^{-1}\sum_{j=1}^N \xi_j A_j P(\xi)$ est diagonale. Cette condition semble plus forte que notre condition d'hyperbolicité (où l'on ne demande pas de borne sur $||P(\xi)||\,||P(\xi)^{-1}||$). En fait, en dimension $N = 1$, ces conditions sont trivialement équivalentes (car S^{N-1} est alors réduit à deux points, et il n'y a qu'une seule matrice à diagonaliser); lorsque $N = l = 2$, on peut prouver que tout système hyperbolique (en notre sens) est symétrisable, et donc hyperbolique au sens de [67]. Cependant, lorsque $N \geq 2$ et $l \geq 3$, les deux notions d'hyperbolicité diffèrent et celle de [67] est strictement plus forte, en général, que la notre (comme cela a déjà été signalé en remarque 6.2).

7.1.1 Cas $N = l = 2$ (d'après l'exercice 3.9 de [67])

On suppose donc que $N = l = 2$ et que le système est hyperbolique au sens de la définition 6.2; nous allons montrer que ce système est alors symétrisable (i.e. qu'il existe une matrice S symétrique définie positive telle que SA_1 et SA_2 soient symétriques), ce qui impliquera en particulier qu'il est hyperbolique au sens de [67].

Commençons par remarquer que l'on peut se ramener au cas où A_1 est diagonale; en effet, par hypothèse A_1 est diagonalisable, i.e. il existe P inversible telle que PA_1P^{-1} soit diagonale.
Supposons que nous ayons trouvé S symétrique définie positive telle que SPA_1P^{-1} et SPA_2P^{-1} soient diagonales; alors $S' = P^TSP$ est symétrique définie positive et, pour $i \in \{1, 2\}$,

$$S'A_i = P^TSPA_i = P^T(SPA_iP^{-1})P$$

est symétrique.
Quitte à remplacer $(A_i)_{i\in[1,2]}$ par $(PA_iP^{-1})_{i\in[1,2]}$, ce qui ne change pas l'hyperbolicité du système, on peut donc supposer que A_1 est diagonale.

Si A_1 est proportionnelle à l'identité, disons aId, alors le résultat est évident; en prenant S symétrique définie positive telle que SA_2 est symétrique (un tel S existe toujours: il suffit de prendre une matrice P inversible telle que PA_2P^{-1} soit diagonale, puis de poser $S = P^TP$), on constate que $SA_1 = aS$ est aussi symétrique.
On peut donc maintenant supposer que $A_1 = \mathrm{diag}(\alpha, \beta)$ avec $\alpha \neq \beta$.

Ecrivons $A_2 = ((a_{i,j}))_{(i,j)\in[1,2]^2}$; pour tout $\lambda \in \mathbb{R}$, le polynôme caractéristique de $\lambda A_1 + A_2$, c'est à dire $X^2 - (a_{1,1} + \lambda\alpha + a_{2,2} + \lambda\beta)X + (a_{1,1} + \lambda\alpha)(a_{2,2} + \lambda\beta) - a_{1,2}a_{2,1}$ ne doit avoir que des racines réelles. Son discriminant

$$\begin{aligned}\Delta &= (a_{1,1} + \lambda\alpha + a_{2,2} + \lambda\beta)^2 + 4a_{1,2}a_{2,1} - 4(a_{1,1} + \lambda\alpha)(a_{2,2} + \lambda\beta) \\ &= (a_{1,1} + \lambda\alpha - (a_{2,2} + \lambda\beta))^2 + 4a_{1,2}a_{2,1} \\ &= (\alpha - \beta)^2\lambda^2 - 2(\alpha - \beta)(a_{1,1} - a_{2,2})\lambda + 4a_{1,2}a_{2,1} + (a_{1,1} - a_{2,2})^2\end{aligned}$$

doit donc toujours être positif ou nul. Ce discriminant étant lui-même un polynôme en λ, ses racines doivent donc être complexes non réelles ou réelles doubles; cela signifie que son propre discriminant en λ doit être négatif ou nul:

$$(\alpha - \beta)^2(a_{1,1} - a_{2,2})^2 - (4a_{1,2}a_{2,1} + (a_{1,1} - a_{2,2})^2)(\alpha - \beta)^2 \le 0.$$

Comme $\alpha \neq \beta$, cela implique donc $a_{1,2}a_{2,1} \ge 0$.
Si $a_{1,2} = 0$, alors $\lambda A_1 + A_2$ s'écrit

$$\begin{pmatrix} a_{1,1} + \lambda\alpha & a_{2,1} \\ 0 & a_{2,2} + \lambda\beta \end{pmatrix}.$$

Cette matrice devant être diagonalisable pour tout $\lambda \in \mathbb{R}$, en particulier pour $\lambda = (a_{2,2} - a_{1,1})/(\alpha - \beta)$ (λ pour lequel les éléments diagonaux sont égaux), on en déduit que $a_{2,1} = 0$. De même, si $a_{2,1} = 0$, on prouve que $a_{1,2} = 0$.
Ainsi, on a soit $a_{1,2} = a_{2,1} = 0$, soit $a_{1,2}a_{2,1} > 0$; dans le premier cas, A_2 est diagonale et A_1 et A_2 sont donc déjà symétriques. Dans le deuxième cas, $a_{1,2}$ et $a_{2,1}$ ayant le même signe,

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{a_{2,1}}{a_{1,2}} \end{pmatrix}$$

est symétrique définie positive et on constate par le calcul que SA_1 et SA_2 sont symétriques.

Le fait qu'un système symétrisable soit hyperbolique au sens de [67] est assez simple à voir.
En notant S une matrice symétrique définie positive telle que, pour tout $i \in [1, N]$, SA_i soit symétrique, on prend H la racine carrée symétrique définie positive de S et on constate que $HA_iH^{-1} = H^{-1}SA_iH^{-1}$ est symétrique; ainsi, pour tout $\xi \in \mathbb{R}^N$, $H\sum_{i=1}^N \xi_i A_i H^{-1}$ est symétrique, donc diagonalisable dans une base orthonormée: on peut trouver $\overline{P}(\xi)$ orthogonale telle que $\overline{P}(\xi)H\sum_{i=1}^N \xi_i A_i H^{-1}\overline{P}(\xi)^{-1}$ soit diagonale.
Ainsi, $P(\xi) = \overline{P}(\xi)H$ diagonalise $\sum_{i=1}^N \xi_i A_i$ et, puisque $\overline{P}(\xi)$ est orthogonale, $\sup_{\xi\in\mathbb{R}^N} ||P(\xi)|| < +\infty$ et $\sup_{\xi\in\mathbb{R}^N} ||P(\xi)^{-1}|| < +\infty$, ce qui suffit à voir que le système est hyperbolique au sens de [67].

7.1.2 Cas général

Comme nous l'avons dit, lorsque $N \ge 2$ et $l \ge 3$, les notions d'hyperbolicité de [67] et de la définition 6.2 diffèrent. Voici un exemple. Prenons $N = 2$, $l = 3$ et

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{et} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}. \tag{7.1}$$

A_1 est clairement diagonalisable (elle est diagonale!) et, pour $\lambda \in \mathbb{R}$, une étude du polynôme caractéristique de $\lambda A_1 + A_2$ donne pour valeurs propres 0, $(\lambda + \sqrt{\lambda^2 + 16})/2$ et $(\lambda - \sqrt{\lambda^2 + 16})/2$; ces valeurs propres étant deux à deux distinctes, $\lambda A_1 + A_2$ est diagonalisable.

Ainsi, pour tout $\xi \in \mathbb{R}^2$, $\xi_1 A_1 + \xi_2 A_2$ est diagonalisable: le système considéré est donc hyperbolique au sens de la définition 6.2 (mais non strictement hyperbolique, car A_1 a 0 comme valeur propre double). Nous allons cependant prouver que ce système n'est pas hyperbolique au sens de [67].

On voit assez simplement que, pour $\lambda \in \mathbb{R}$, les sous-espaces propres de $\lambda A_1 + A_2$ sont respectivement

$$\ker(\lambda A_1 + A_2 - 0Id) = \mathbb{R}(-4, 2-\lambda, 2)^T$$
$$\ker\left(\lambda A_1 + A_2 - \frac{\lambda + \sqrt{\lambda^2+16}}{2} Id\right) = \mathbb{R}(0, 4, \lambda + \sqrt{\lambda^2+16})^T$$
$$\ker\left(\lambda A_1 + A_2 - \frac{\lambda - \sqrt{\lambda^2+16}}{2} Id\right) = \mathbb{R}(0, 4, \lambda - \sqrt{\lambda^2+16})^T.$$

Lorsque $Q^{-1}(\lambda A_1 + A_2)Q$ est diagonale, les colonnes de la matrice Q sont des vecteurs propres de $\lambda A_1 + A_2$, donc des multiples des vecteurs générateurs des sous-espaces propres décrits ci-dessus (car tous ces sous-espaces propres sont de dimension 1); il existe donc une matrice de permutation $\mathcal{P}$ et trois réels non-nuls α_1, α_2 et α_3 tels que

$$Q = \begin{pmatrix} -4\alpha_1 & 0 & 0 \\ (2-\lambda)\alpha_1 & 4\alpha_2 & 4\alpha_3 \\ 2\alpha_1 & (\lambda + \sqrt{\lambda^2+16})\alpha_2 & (\lambda - \sqrt{\lambda^2+16})\alpha_3 \end{pmatrix} \mathcal{P}.$$

Notons

$$Q_\alpha = \begin{pmatrix} -4\alpha_1 & 0 & 0 \\ (2-\lambda)\alpha_1 & 4\alpha_2 & 4\alpha_3 \\ 2\alpha_1 & (\lambda + \sqrt{\lambda^2+16})\alpha_2 & (\lambda - \sqrt{\lambda^2+16})\alpha_3 \end{pmatrix}.$$

Un simple calcul par co-matrice donne

$$Q_\alpha^{-1} = \begin{pmatrix} \frac{-1}{4\alpha_1} & 0 & 0 \\ \frac{8-(2-\lambda)(\lambda - \sqrt{\lambda^2+16})}{32\alpha_2\sqrt{\lambda^2+16}} & \frac{\sqrt{\lambda^2+16}-\lambda}{8\alpha_2\sqrt{\lambda^2+16}} & \frac{1}{2\alpha_2\sqrt{\lambda^2+16}} \\ \frac{(2-\lambda)(\lambda+\sqrt{\lambda^2+16})-8}{32\alpha_3\sqrt{\lambda^2+16}} & \frac{\lambda+\sqrt{\lambda^2+16}}{8\alpha_3\sqrt{\lambda^2+16}} & \frac{-1}{2\alpha_3\sqrt{\lambda^2+16}} \end{pmatrix}.$$

Soit N la norme sur $M_l(\mathbb{R})$ du suppremum des coefficients; cette norme étant équivalente à $||\cdot||$, il existe $C > 0$ tel que $N(\cdot) \le C||\cdot||$.
Puisque $Q_\alpha = Q\mathcal{P}^{-1}$ et $Q_\alpha^{-1} = \mathcal{P}Q^{-1}$, avec $\mathcal{P}$ et $\mathcal{P}^{-1}$ des matrices de permutation, on a $N(Q) = N(Q_\alpha)$ et $N(Q^{-1}) = N(Q_\alpha^{-1})$, donc $N(Q_\alpha)N(Q_\alpha^{-1}) \le C^2||Q||\,||Q^{-1}||$.
Le coefficient $(2,1)$ de Q_α nous dit que $N(Q_\alpha) \ge |2-\lambda||\alpha_1|$, et le coefficient $(1,1)$ de Q_α^{-1} donne $N(Q_\alpha^{-1}) \ge 1/4|\alpha_1|$; ainsi, $N(Q_\alpha)N(Q_\alpha^{-1}) \ge |2-\lambda|/4$.
On a donc prouvé que, pour toute matrice Q diagonalisant $\lambda A_1 + A_2$, on a $||Q||\,||Q||^{-1} \ge |2-\lambda|/4C^2$.

Soit $P : S^1 \to \mathrm{GL}(l;\mathbb{R})$ telle que, pour tout $\xi \in S^1$, $P(\xi)^{-1}(\xi_1 A_1 + \xi_2 A_2)P(\xi)$ soit diagonale. Lorsque $\xi_2 \neq 0$, la matrice $P(\xi)$ diagonalise $\frac{\xi_1}{\xi_2}A_1 + A_2$ et on a donc $||P(\xi)||\,||P(\xi)^{-1}|| \ge |2 - \xi_1/\xi_2|/4C^2$, ce qui prouve que $||P(\xi)||\,||P(\xi)^{-1}||$ ne peut être borné sur S^1 (cette quantité tend, lorsque $\xi \to (1,0)$ avec $\xi_2 \neq 0$, vers l'infini).
Le système considéré n'est donc pas hyperbolique au sens de [67].

7.2 Non-résolubilités du système défini par (A_1, A_2)

Lorsque la condition initiale u^0 d'un système hyperbolique de la forme (6.1) ne dépend que d'une direction $\xi \in S^{N-1}$ (i.e. $u_0(x) = f(x\cdot\xi)$), on peut exprimer explicitement la solution correspondante en fonction de u^0.

Faisons-le, dans un cas particulier, pour le système défini par les matrices (7.1).

Soit $f \in L^1_{\text{loc}}(\mathbb{R})$ et $\lambda \in \mathbb{R}$. Considérons, en notant $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1) \in S^1$, la condition initiale

$$u^{0,(\lambda)}(x) = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} f(x \cdot \xi). \tag{7.2}$$

Puisque la décomposition du vecteur $(4,0,0)^T$ sur la base propre de $\xi_1 A_1 + \xi_2 A_2 = \frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ donnée dans la partie 7.1.2 est

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ \lambda - 2 \\ -2 \end{pmatrix} + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix}$$

avec

$$\alpha(\lambda) = \frac{2-\lambda}{8} + \frac{1}{\sqrt{\lambda^2+16}} + \frac{\lambda(\lambda-2)}{8\sqrt{\lambda^2+16}} = \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \text{lorsque } \lambda \to \infty$$

et

$$\beta(\lambda) = \frac{2-\lambda}{8} - \frac{1}{\sqrt{\lambda^2+16}} - \frac{\lambda(\lambda-2)}{8\sqrt{\lambda^2+16}} = -\frac{\lambda}{4}(1+\gamma(\lambda)) \quad \text{où } \gamma(\lambda) = \mathcal{O}(1/\lambda) \text{ lorsque } \lambda \to \infty,$$

la solution u du système définit par (A_1, A_2) (lorsque $\Omega = \mathbb{R}^2$) pour la condition initiale $u^{0,(\lambda)}$ est

$$\begin{aligned} u^{(\lambda)}(x,t) &= \begin{pmatrix} 4 \\ \lambda - 2 \\ 2 \end{pmatrix} f(x \cdot \xi - c_1(\lambda)t) + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} f(x \cdot \xi - c_2(\lambda)t) \\ &\quad + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix} f(x \cdot \xi - c_3(\lambda)t) \end{aligned} \tag{7.3}$$

où $c_1(\lambda) = 0$ est la valeur propre associée au premier vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$, $c_2(\lambda) = \frac{\lambda+\sqrt{\lambda^2+16}}{2\sqrt{\lambda^2+1}} = 1 + \varepsilon(\lambda)$ est la valeur propre associée au deuxième vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ (on a $\varepsilon(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to \infty$) et $c_3(\lambda) = \frac{\lambda-\sqrt{\lambda^2+16}}{2\sqrt{\lambda^2+1}} = -\frac{4}{\lambda^2}(1+\eta(\lambda))$ est la valeur propre associée au troisième vecteur propre de $\frac{1}{\sqrt{\lambda^2+1}}(\lambda A_1 + A_2)$ (on a $\eta(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to \infty$).
En notant $\tau_h f(z) = f(z+h)$, on peut donc écrire $u^{(\lambda)}(x,t) = v^{(\lambda)}(x \cdot \xi, t)$ avec

$$\begin{aligned} v^{(\lambda)}(z,t) &= \begin{pmatrix} 4 \\ \lambda - 2 \\ 2 \end{pmatrix} f(z) + \alpha(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix} \tau_{-c_2(\lambda)t} f(z) \\ &\quad + \beta(\lambda) \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix} \tau_{-c_3(\lambda)t} f(z). \end{aligned} \tag{7.4}$$

Un changement de variable donné par Q^T, où Q est une matrice orthognale telle que $Q\xi = (1,0)$ (ce qui est possible puisque $|\xi| = 1$), nous permet donc de voir que (puisque $]-1,1[^2 \subset B_2$ — rappelons que B_2 est la boule euclidienne de $\mathbb{R}^2$ de centre 0 et de rayon 2)

$$\int_0^1 \int_{B_2} |u^{(\lambda)}(x,t)| \, dx dt$$

$$
\begin{aligned}
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(x \cdot \xi, t)| \, dx dt \\
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(Q^T x \cdot \xi, t)| \, dx dt \\
&= \int_0^1 \int_{B_2} |v^{(\lambda)}(x_1, t)| \, dx dt \\
&\geq \int_0^1 \int_{-1}^1 \int_{-1}^1 |v^{(\lambda)}(x_1, t)| \, dx_1 \, dx_2 \, dt \\
&\geq \int_0^1 \int_{-1}^1 |v^{(\lambda)}(z, t)| \, dz \, dt \\
&\geq \int_0^1 \int_{-1}^1 |v_2^{(\lambda)}(z, t)| \, dz \, dt \\
&\geq \int_0^1 \int_{-1}^1 |(\lambda - 2)f + 4\alpha(\lambda)\tau_{-c_2(\lambda)t} f + 4\beta(\lambda)\tau_{-c_3(\lambda)t} f| \, dz \, dt \\
&\geq \int_0^1 \int_{-1}^1 |(\lambda - 2)f - \lambda(1 + \gamma(\lambda))\tau_{-c_3(\lambda)t} f| \, dz \, dt - ||4\alpha(\lambda)\tau_{-c_2(\lambda)t} f||_{L^1(]-1,1[\times]0,1[)} \\
&\geq |\lambda| \int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)| \, dz \, dt - 4|\alpha(\lambda)| \, ||f||_{L^1(]-1-|c_2(\lambda)|,1+|c_2(\lambda)|[)} - 2||f||_{L^1(]-1,1[)} \\
&\quad -|\lambda\gamma(\lambda)| \, ||f||_{L^1(]-1-|c_3(\lambda)|,1+|c_3(\lambda)|[)}.
\end{aligned}
$$

En notant

$$
M_1 = \sup_{\lambda \in \mathbb{R}} |\alpha(\lambda)| < \infty \,, \quad M_2 = \sup_{\lambda \in \mathbb{R}} |\lambda\gamma(\lambda)| < \infty \,, \quad C_2 = \sup_{\lambda \in \mathbb{R}} |c_2(\lambda)| < \infty \quad \text{et } C_3 = \sup_{\lambda \in \mathbb{R}} |c_3(\lambda)| < \infty,
$$

on en déduit

$$
\begin{aligned}
& |\lambda| \int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)| \, dz \, dt \\
& \leq 4M_1||f||_{L^1(]-1-C_2,1+C_2[)} + 2||f||_{L^1(]-1,1[)} + M_2||f||_{L^1(]-1-C_3,1+C_3[)} \\
& \quad +||v^{(\lambda)}||_{(L^1(]-1,1[\times[0,1[))^3} \qquad (7.5) \\
& \leq 4M_1||f||_{L^1(]-1-C_2,1+C_2[)} + 2||f||_{L^1(]-1,1[)} + M_2||f||_{L^1(]-1-C_3,1+C_3[)} \\
& \quad +||u^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3}. \qquad (7.6)
\end{aligned}
$$

7.2.1 Une fonction f particulière

L'équation (7.6) nous dit que, si $u^{(\lambda)}$ est bornée dans $L^1(B_2 \times [0, 1[)^3$, alors

$$
\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)| \, dz \, dt = \mathcal{O}\left(\frac{1}{|\lambda|}\right).
$$

Nous exhibons ici une fonction f, continue bornée sur $\mathbb{R}$, qui ne vérifie pas cette propriété.

Lemme 7.1 *Si $\alpha \in]0, 1[$, il existe $f \in \mathcal{C}(\mathbb{R})$, $C > 0$ et $h_0 \in]0, 1[$ tels que, pour tout $h \in]0, h_0[$, en notant $f_h(x) = \frac{1}{h} \int_x^{x+h} f(t) \, dt$, on a*

$$
\int_0^1 |f(z) - f_h(z)| \, dz \geq Ch^\alpha.
$$

Preuve du lemme 7.1
On cherche f sous la forme $f(z) = z^p \sin(z^{-n})$, où p et n sont des réels strictement positifs à déterminer.

Grâce à une intégration par parties, on a

$$\begin{aligned}\int f = \int z^p \sin(z^{-n})\, dz &= \frac{1}{n}\int z^{p+n+1}\left(nz^{-n-1}\sin(z^{-n})\right)\, dx\\ &= \frac{1}{n}\int z^{p+n+1}\frac{d}{dz}(\cos(z^{-n}))\, dz\\ &= \frac{1}{n}z^{p+n+1}\cos(z^{-n}) - \frac{n+p+1}{n}\int z^{p+n}\cos(z^{-n})\, dz.\end{aligned}$$

Fixons $\beta \in]0, \alpha[$; lorsque $h \leq 1$ et $0 \leq z \leq h^\beta - h$, on a donc

$$\begin{aligned}|f_h(z)| &= \left|\frac{1}{hn}\left((z+h)^{p+n+1}\cos((z+h)^{-n}) - z^{p+n+1}\cos(z^{-n})\right)\right.\\ &\quad \left. -\frac{n+p+1}{hn}\int_z^{z+h} s^{p+n}\cos(s^{-n})\, ds\right|\\ &\leq \frac{2h^{(p+n+1)\beta-1}}{n} + \frac{(n+p+1)h^{(p+n)\beta}}{n}\\ &\leq C_1 h^{(p+n+1)\beta-1}\end{aligned}$$

où $C_1 < +\infty$ ne dépend que de n et p (rappelons que $\beta \in]0, 1[$ et que $h \leq 1$, de sorte que $h^{1-\beta} \leq 1$). Ainsi,

$$\int_0^{h^\beta - h} |f_h(z)|\, dz \leq C_1 h^{(p+n+2)\beta-1}. \tag{7.7}$$

On veut maintenant minorer $\int_0^{h^\beta-h} |f(z)|\, dz = \int_0^{h^\beta-h} z^p|\sin(z^{-n})|\, dz$. Avec le changement de variable $y = z^{-n}$ et en notant $\mu_h = h^\beta - h$, on a

$$\begin{aligned}\int_0^{h^\beta-h} |f(z)|\, dz &= \frac{1}{n}\int_{\mu_h^{-n}}^{\infty} y^{-\frac{1}{n}-1}y^{-\frac{p}{n}}|\sin(y)|\, dy\\ &\geq \frac{1}{n}\sum_{k\geq k_h}\int_{2k\pi}^{2k\pi+\pi} y^{-\frac{p+n+1}{n}}\sin(y)\, dy,\end{aligned}$$

où $k_h = [\mu_h^{-n}/2\pi] + 1$ ($[\cdot]$ désigne la partie entière d'un réel). Comme $\sin \geq \sqrt{2}/2$ sur $[2k\pi + \pi/4, 2k\pi + 3\pi/4]$, on en déduit

$$\begin{aligned}&\int_0^{h^\beta-h} |f(z)|\, dz\\ &\geq \frac{\sqrt{2}}{2n}\sum_{k\geq k_h}\int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} y^{-\frac{p+n+1}{n}}\, dy\\ &\geq \frac{\sqrt{2}}{2n}\sum_{k\geq k_h}\frac{1}{\frac{p+n+1}{n}-1}\left(\left(2k\pi+\frac{\pi}{4}\right)^{1-\frac{p+n+1}{n}} - \left(2k\pi+\frac{3\pi}{4}\right)^{1-\frac{p+n+1}{n}}\right).\end{aligned}$$

Mais, par le théorème des accroissements finis, pour tout $k \in \mathbb{N}$, il existe $\theta \in [0, 1]$ tel que

$$\left(2k\pi+\frac{\pi}{4}\right)^{1-\frac{p+n+1}{n}} - \left(2k\pi+\frac{3\pi}{4}\right)^{1-\frac{p+n+1}{n}} = \frac{\pi}{2}\times\left(\frac{p+n+1}{n}-1\right)\left(2k\pi+\frac{\pi}{4}+\theta\frac{\pi}{2}\right)^{-\frac{p+n+1}{n}}$$

$$\begin{aligned} &\geq C_2\,(2(k+1)\pi)^{-\frac{p+n+1}{n}} \\ &\geq C_3 k^{-\frac{p+n+1}{n}} \end{aligned}$$

où $C_2 > 0$ et $C_3 > 0$ ne dépendent que de n et p. Ainsi,

$$\int_0^{h^\beta - h} |f(z)|\, dz \quad \geq \quad C_4 \sum_{k \geq k_h} k^{-\frac{p+n+1}{n}}$$

avec $C_4 > 0$ ne dépendant que de n et p. La fonction $s \to s^{-(p+n+1)/n}$ étant décroissante, on a

$$\sum_{k \geq k_h} k^{-\frac{p+n+1}{n}} \geq \int_{k_h}^{\infty} s^{-\frac{p+n+1}{n}}\, ds = \frac{1}{\frac{n+p+1}{n} - 1} k_h^{1-\frac{n+p+1}{n}} = \frac{1}{\frac{n+p+1}{n} - 1} k_h^{-\frac{p+1}{n}}.$$

En constatant que $k_h = [\mu_h^{-n}/2\pi] + 1 \leq (\mu_h^{-n}/2\pi) + 2$, on en déduit

$$\int_0^{h^\beta - h} |f(z)|\, dz \quad \geq \quad C_5 \left(\frac{\mu_h^{-n}}{2\pi} + 2 \right)^{-\frac{p+1}{n}} = C_5 \left(\frac{1}{2\pi} + 2\mu_h^n \right)^{-\frac{p+1}{n}} \mu_h^{p+1}$$

où $C_5 > 0$ ne dépend que de n et p. Puisque $\mu_h = h^\beta - h \in [h^\beta/2, 1]$ lorsque $h \leq (1/2)^{1/(1-\beta)}$, on a finalement, pour $h \leq (1/2)^{1/(1-\beta)}$,

$$\int_0^{h^\beta - h} |f(z)|\, dz \quad \geq \quad C_5 \left(\frac{1}{2\pi} + 2 \right)^{-\frac{p+1}{n}} \frac{h^{(p+1)\beta}}{2^{p+1}} = C_6 h^{(p+1)\beta} \tag{7.8}$$

où $C_6 > 0$ ne dépend que de n et p.

De (7.7) et (7.8), on tire, lorsque $h \leq (1/2)^{1/(1-\beta)}$,

$$\begin{aligned} ||f - f_h||_{L^1(]0,1[)} &\geq ||f - f_h||_{L^1(]0,h^\beta - h[)} \\ &\geq ||f||_{L^1(]0,h^\beta - h[)} - ||f_h||_{L^1(]0,h^\beta - h[)} \\ &\geq C_6 h^{(p+1)\beta} - C_1 h^{(p+n+2)\beta - 1} \\ &\geq (C_6 - C_1 h^{(n+1)\beta - 1}) h^{(p+1)\beta}. \end{aligned}$$

On choisit maintenant $p > 0$ tel que $(p+1)\beta = \alpha$ (ce choix est possible car $\beta < \alpha$) et $n > 0$ tel que $(n+1)\beta - 1 > 0$; il existe alors $h_0 \in]0, (1/2)^{1/(1-\beta)}[$ ne dépendant que de n et β tel que, pour tout $h \leq h_0$, $C_6 - C_1 h^{(n+1)\beta - 1} \geq C_6/2$; on a alors, pour $h \leq h_0$,

$$||f - f_h||_{L^1(]0,1[)} \geq \frac{C_6}{2} h^\alpha,$$

ce qui conclut la preuve de ce lemme. ∎

Prenons $\alpha = 1/4$ et $f \in \mathcal{C}(\mathbb{R})$, $C > 0$ et $h_0 \in]0, 1[$ donnés par le lemme 7.1 pour cet α.
On a, pour $r > 0$,

$$f_r(z) = \frac{1}{r} \int_z^{z+r} f(t)\, dt = \int_0^1 f(z + rt)\, dt = \int_0^1 \tau_{rt} f(z)\, dt.$$

Comme $c_3(\lambda) = -\frac{4}{\lambda^2}(1 + \eta(\lambda))$ avec $\eta(\lambda) = \mathcal{O}(1/\lambda^2)$ lorsque $\lambda \to +\infty$, il existe $\lambda_1 > 0$ tel que, pour tout $\lambda \geq \lambda_1$, $\lambda^{-2} \leq -c_3(\lambda) \leq h_0$; ainsi, pour tout $\lambda \geq \lambda_1$,

$$\int_{-1}^1 \int_0^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\, dt dz \quad \geq \quad \int_0^1 \left| \int_0^1 (f(z) - \tau_{-c_3(\lambda)t} f(z))\, dt \right| dz$$

$$
\begin{aligned}
&\geq \int_0^1 \left| f(z) - \int_0^1 \tau_{-c_3(\lambda)t} f(z)\,dt \right| dz \\
&\geq \int_0^1 |f(z) - f_{-c_3(\lambda)}(z)|\,dz \\
&\geq C|c_3(\lambda)|^\alpha \geq C\lambda^{-\frac{1}{2}}. \qquad (7.9)
\end{aligned}
$$

Cette relation ne faisant intervenir les valeurs de f que sur un compact de $\mathbb{R}$, on peut aussi supposer, quitte à changer f hors de ce compact, que $f \in \mathcal{C}_c(\mathbb{R})$.

7.2.2 Non-résolubilité $L^\infty - L^1_{\text{loc}}$

Lorsque la condition initiale est de type Riemann, la forme très particulière des solutions permet de voir que l'on a une estimation sur la solution dans les espaces $(L^q_{\text{loc}}(\mathbb{R}^2\times]0,\infty[))^3$ en fonction de la norme de la condition initiale dans $(L^p_{\text{loc}}(\mathbb{R}^2))^3$ (pour $q > 1$ assez petit et $p < \infty$ assez grand), estimation qui est indépendante de la direction de la condition initiale de type Riemann.
Cependant, cette estimation ne permet pas de résoudre le système considéré pour toute condition initiale dans $(L^\infty(\mathbb{R}^2))^3$ (ou même dans un espace plus petit).

Supposons en effet que, pour toute condition initiale $u^0 \in E = (\mathcal{C}_b(\mathbb{R}^2))^3$, on puisse trouver une solution $u \in F = (L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ au système défini par (A_1, A_2) pour la condition initiale u^0; par le corollaire 6.2, on a unicité de cette solution et l'application $L : u^0 \in E \to u \in F$ est alors bien définie; on voit immédiatement que L est linéaire.
Les espaces E et F étant métriques complets, L est continue si et seulement si son graphe est fermé. Or, si $(u^0_n)_{n\geq 1} \in E$ converge vers u^0 dans E et $(L(u^0_n))_{n\geq 1} = (u_n)_{n\geq 1} \in F$ converge vers u dans F, alors on constate en passant à la limite dans l'équation faible satisfaite par $(u^n)_{n\geq 1}$ que u est la solution du système défini par (A_1, A_2) pour la condition initiale u^0, c'est à dire que $u = L(u^0)$. Le graphe de L étant fermé, L est donc bien continue.

En particulier, en prenant la fonction $f \in \mathcal{C}_b(\mathbb{R})$ construite en 7.2.1, $\lambda \in \mathbb{R}$ et $u^{0,(\lambda)} \in E$ définie par (7.2) (rappelons que $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1)$), on a $||u^{(\lambda)}||_{(L^1(B_2\times[0,1]))^3} \leq C||u^{0,(\lambda)}||_E \leq 4C||f||_{\mathcal{C}_b(\mathbb{R})}$, où C ne dépend pas de λ (on a traduit la continuité de L et le fait que $u^{(\lambda)} = L(u^{0,(\lambda)})$). En injectant cette estimation dans (7.6), on en déduit qu'il existe $C' > 0$ tel que, pour tout $\lambda \in \mathbb{R}$,

$$
\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\,dz\,dt \leq \frac{C'}{|\lambda|}.
$$

f vérifiant (7.9), on en déduit que $C'\lambda^{-1} \geq C\lambda^{-1/2}$ pour tout λ assez grand, avec C et C' indépendants de λ, ce qui est une contradiction.

7.2.3 Non-résolubilité $BV_{\text{loc}} - BV_{\text{loc}}$

Supposons maintenant que, pour tout $u_0 \in (BV_{\text{loc}}(\mathbb{R}^2))^3$, il existe une solution $u \in (L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ au système défini par (A_1, A_2), pour la condition initiale u^0, qui soit dans $(L^1_{\text{loc}}([0,\infty[; BV(B_2)))^3$ ([1]).
Comme au début de la sous-section 7.2.2, par le théorème du graphe fermé, l'application $L : u^0 \in (BV_{\text{loc}}(\mathbb{R}^2))^3 \to u \in (L^1_{\text{loc}}(\mathbb{R}^2 \times [0,\infty[))^3 \cap (L^1_{\text{loc}}([0,\infty[; BV(B_2))^3$ serait bien définie et linéaire continue. En particulier, il existerait $R > 0$ et $C > 0$ tel que, pour tout $u^0 \in (BV_{\text{loc}}(\mathbb{R}^2))^3$,

$$
\begin{aligned}
&||D_1 L(u^0)||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} + ||D_2 L(u^0)||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} \\
&\quad \leq C||u^0||_{(L^1(B_R))^3} + C||D_1 u^0||_{(\mathcal{M}_b(B_R))^3} + C||D_2 u^0||_{(\mathcal{M}_b(B_R))^3}. \qquad (7.10)
\end{aligned}
$$

[1] Rappelons que, pour $r > 0$, $BV(B_r)$ est le Banach formé des fonctions $f \in L^1(B_r)$ telles que $D_1 f$ et $D_2 f$ soient des mesures finies sur B_r, muni de la norme $||f||_{L^1(B_r)} + ||D_1 f||_{\mathcal{M}_b(B_r)} + ||D_2 f||_{\mathcal{M}_b(B_r)}$. $BV_{\text{loc}}(\mathbb{R}^2)$ est l'espace métrique complet formé des fonctions $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ qui sont dans $BV(B_r)$ pour tout $r > 0$; il est muni de la famille dénombrable de semi-normes $||f||_{L^1(B_n)} + ||D_1 f||_{\mathcal{M}_b(B_n)} + ||D_2 f||_{\mathcal{M}_b(B_n)}$ $(n \geq 1)$.

Soit $f \in \mathcal{C}_b(\mathbb{R})$ et $F(z) = \int_0^z f(s)\,ds$; F est dans $\mathcal{C}^1(\mathbb{R})$. On se donne $\lambda \in \mathbb{R}$ et $\xi = \frac{1}{\sqrt{\lambda^2+1}}(\lambda, 1)$.
Lorsque $U^{0,(\lambda)}(x) = (4,0,0)^T F(x\cdot\xi)$, on voit sur l'expression de la solution $U^{(\lambda)}$ du système défini par (A_1, A_2) pour la condition initiale $U^{0,(\lambda)}$ ($U^{(\lambda)}$ est donnée par (7.3) où l'on a changé f en F) que $U^{(\lambda)} \in (\mathcal{C}^1(\mathbb{R}^2\times[0,\infty[))^3$; grâce à cette régularité, et puisque $|\xi| = 1$, on constate que $u^{(\lambda)}(x,t) = \xi_1\partial_{x_1}U^{(\lambda)}(x,t) + \xi_2\partial_{x_2}U^{(\lambda)}(x,t) \in (\mathcal{C}(\mathbb{R}^2\times[0,\infty[))^3$ est solution du système défini par (A_1, A_2) avec la condition initiale $u^{0,(\lambda)}(x) = (4,0,0)^T f(x\cdot\xi)$.
f et $u^{(\lambda)}$ sont donc liés par (7.6). Mais, puisque $U^{(\lambda)}$ est de classe $\mathcal{C}^1$,

$$\begin{aligned}||u^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3} &\leq |\xi_1|\,||\partial_{x_1}U^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3} + |\xi_2|\,||\partial_{x_2}U^{(\lambda)}||_{(L^1(B_2\times[0,1[))^3}\\ &\leq ||D_1U^{(\lambda)}||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3} + ||D_2U^{(\lambda)}||_{(L^1([0,1[;\mathcal{M}_b(B_2)))^3}. \end{aligned} \tag{7.11}$$

De plus, par un changement de variable défini par Q^T (où Q est une matrice orthogonale telle que $Q\xi = (1,0)$),

$$\begin{aligned}&||U^{0,(\lambda)}||_{(L^1(B_R))^3} + ||D_1U^{0,(\lambda)}||_{(\mathcal{M}_b(B_R))^3} + ||D_2U^{0,(\lambda)}||_{(\mathcal{M}_b(B_R))^3}\\ &\leq 4\int_{B_R}|F(x_1)|\,dx_1\,dx_2 + 4|\xi_1|\int_{B_R}|f(x_1)|\,dx_1\,dx_2 + 4|\xi_2|\int_{B_R}|f(x_1)|\,dx_1\,dx_2\\ &\leq 4|B_R|R\,||f||_{\mathcal{C}_b(\mathbb{R})} + 8|B_R|\,||f||_{\mathcal{C}_b(\mathbb{R})}, \end{aligned} \tag{7.12}$$

car, lorsque $x \in B_R$, $|F(x_1)| \leq |x_1|\,||f||_{\mathcal{C}_b(\mathbb{R})} \leq R||f||_{\mathcal{C}_b(\mathbb{R})}$.
On déduit donc, de (7.6), (7.10) appliqué à U^0 au lieu de u^0 et de (7.11), (7.12) que

$$\begin{aligned}&|\lambda|\int_0^1\int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t}f(z)|\,dz\,dt\\ &\leq (4M_1(2+2C_2) + 4 + M_2(2+2C_3) + 4C|B_R|R + 8C|B_R|)||f||_{\mathcal{C}_b(\mathbb{R})},\end{aligned}$$

ce qui nous conduit, grâce à (7.9) et comme en 7.2.2, à une contradiction.
On ne peut donc, pour tout $u^0 \in (BV_{\mathrm{loc}}(\mathbb{R}^2))^3$, trouver une solution $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^2\times[0,\infty[))^3$ qui soit dans $(L^1_{\mathrm{loc}}([0,\infty[;BV(B_2)))^3$.

7.3 Instabilité d'un système hyperbolique par rapport au flux

Considérons le problème

$$\begin{cases} (w_\varepsilon)_t(z,t) + (g_\varepsilon(w_\varepsilon))_z(z,t) = 0 & t>0\,,\ z\in\mathbb{R},\\ w_\varepsilon(z,0) = w^0(z) & z\in\mathbb{R},\end{cases} \tag{7.13}$$

où $g_\varepsilon : \mathbb{R}^l \to \mathbb{R}^l$ et $w^0 : \mathbb{R}\to\mathbb{R}^l$. On suppose que g_ε converge (en un sens à préciser) vers une certaine fonction g lorsque $\varepsilon \to 0$ (w^0 reste fixé). La question naturelle est de savoir si "la" solution w_ε de (7.13) tend vers "la" solution w du système (7.13) avec g à la place de g_ε.

Lucier [54] a déjà apporté une réponse à cette question dans le cas scalaire (i.e. $l = 1$): si $g_\varepsilon \to g$ dans l'espace des fonctions lipschitziennes sur $\mathbb{R}$ et si $w^0 \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, alors la solution entropique de (7.13) converge, lorsque $\varepsilon \to 0$, vers la solution entropique de (7.13) où g_ε est remplacé par g. En fait, grâce au principe de comparaison L^1 des solutions entropiques par rapport à leur condition initiale (cf. [67]), ce résultat reste vrai lorsque l'on suppose seulement $w^0 \in L^1_{\mathrm{loc}}(\mathbb{R})$.

La question suivante serait de savoir si ce résultat se généralise au cas des systèmes. La première difficulté est bien sûr de définir "la" solution de (7.13): en général, lorsque g_ε est non linéaire, on n'a pas de théorème d'unicité pour (7.13).

On peut donc commencer par considérer le cas linéaire, pour lequel la solution faible est unique (corollaire 6.2). Prenons $l = 3$ et $g_\varepsilon(w) = B(\varepsilon)w$, avec $B(\varepsilon) \in M_3(\mathbb{R})$ diagonalisable sur $\mathbb{R}$ ([2]). Le système alors considéré est

$$\left\{ \begin{array}{ll} (w_\varepsilon)_t(z,t) + B(\varepsilon)(w_\varepsilon)_z(z,t) = 0 & t > 0\,,\ z \in \mathbb{R}, \\ w_\varepsilon(z,0) = w^0(z) & z \in \mathbb{R}, \end{array} \right. \tag{7.14}$$

et on suppose que $B(\varepsilon) \to B$ dans $M_3(\mathbb{R})$ lorsque $\varepsilon \to 0$, avec B diagonalisable sur $\mathbb{R}$. On voudrait savoir si $w_\varepsilon \to w$, au moins dans $(\mathcal{M}_b(B_R \times [0,R[))^3$ faible-$*$ pour tout $R > 0$ (i.e. contre toute fonction de $\mathcal{C}_c(\mathbb{R}^2 \times [0,\infty[))$, avec w solution de (7.14) lorsque $B(\varepsilon)$ est remplacé par B (remarquons que, avec nos hypothèses, le problème (7.14) est, pour tout $\varepsilon \geq 0$, bien posé au sens de Hadamard dans $(L^2(\mathbb{R}))^3$). Mais, contrairement au cas scalaire, même dans ce cas simple, nous allons voir que la réponse est négative en général. Et le contre-exemple que nous construisons se base encore une fois sur les matrices (A_1, A_2) données par (7.1).

En notant $\lambda = \varepsilon^{-1}$ on prend $B(\varepsilon) = \frac{\lambda}{\sqrt{\lambda^2+1}}A_1 + \frac{1}{\sqrt{\lambda^2+1}}A_2$ et $w^0(z) = (4,0,0)^T f(z)$ avec f construite en 7.2.1. $B(\varepsilon)$ converge, lorsque $\varepsilon \to 0$, vers A_1 (et toutes ces matrices sont bien diagonalisables sur $\mathbb{R}$).
La solution w_ε de (7.14) est alors $v^{(\lambda)}$ donnée par (7.4). Supposons que, pour un $\widetilde{\lambda} > 0$, $(v^{(\lambda)})_{\lambda \geq \widetilde{\lambda}}$ soit bornée dans $(L^1_{\mathrm{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ (ce qui est le cas si w_ε converge, lorsque $\varepsilon \to 0$, dans $(\mathcal{M}_b(B_R \times [0,R[))^3$ faible-$*$ pour tout $R > 0$); grâce à (7.5), on aurait, pour $\lambda \geq \widetilde{\lambda}$,

$$\int_0^1 \int_{-1}^1 |f(z) - \tau_{-c_3(\lambda)t} f(z)|\, dz\, dt \leq \frac{C'}{|\lambda|}.$$

où C' ne dépend pas de λ. f vérifiant (7.9) pour λ assez grand, cela nous mène à une contradiction.
w_ε ne peut donc être bornée, pour ε assez petit, dans $(L^1_{\mathrm{loc}}(\mathbb{R}^2 \times [0,\infty[))^3$ et ne converge donc pas (au sens donné précédemment) vers w solution de (7.14) lorsque $B(\varepsilon)$ est remplacé par A_1.

La cause de cette non-convergence peut s'expliquer de la manière suivante.
Prenons $B : \mathbb{R} \to M_3(\mathbb{R})$ ([3]) continue telle que $B(s)$ est diagonalisable sur $\mathbb{R}$ pour tout $s \in \mathbb{R}$. Si $B(\cdot)$ a des valeurs propres de multiplicité constante (par rapport à s), alors on peut suivre continuement ces valeurs propres et, au moins au voisinage de $s = 0$, une base de $\mathbb{R}^3$ formée de vecteurs propres de $B(\cdot)$. On peut alors voir, sur la formule explicite des solutions décomposées sur ces bases de vecteurs propres, que la solution de (7.14) converge, lorsque $\varepsilon \to 0$, vers la solution de (7.14) avec $B(0)$ à la place de $B(\varepsilon)$. Dans l'exemple précédent, la multiplicité des valeurs propres de $B(s)$ dépend de s (lorsque $s = 0$, 0 est valeur propre double de $B(0) = A_1$, tandis que pour $s \neq 0$, toutes les valeurs propres de $B(s)$ sont distinctes). De plus, lorsque l'on regarde le comportement des vecteurs propres normalisés

$$\frac{1}{\sqrt{4^2 + (\lambda-2)^2 + 2^2}} \begin{pmatrix} 4 \\ \lambda - 2 \\ -2 \end{pmatrix}$$

$$\frac{1}{\sqrt{4^2 + (\lambda - \sqrt{\lambda^2+16})^2}} \begin{pmatrix} 0 \\ 4 \\ \lambda - \sqrt{\lambda^2+16} \end{pmatrix}$$

$$\frac{1}{\sqrt{4^2 + (\lambda + \sqrt{\lambda^2+16})^2}} \begin{pmatrix} 0 \\ 4 \\ \lambda + \sqrt{\lambda^2+16} \end{pmatrix}$$

de $B(\varepsilon)$, on constate que deux de ces vecteurs propres tendent vers $(0,1,0)^T$, que le troisième tend vers $(0,0,1)^T$ et que la direction propre $(1,0,0)^T$ de $B(0) = A_1$ n'est jamais atteinte en passant à la limite; la condition initiale que nous avions choisie était justement portée par cette direction propre particulière.

[2] Dans le cas monodimensionnel ($N = 1$) considéré ici, cette condition est notre condition d'hyperbolicité, qui est équivalente à celle de [67].

[3] Ce que nous disons ici n'est pas limité au cas $l = 3$.

On peut aussi comprendre cette non-convergence au travers de la condition exprimant que (7.14) est bien posé dans $(L^2(\mathbb{R}))^3$: cette condition est $\sup_{\xi\in\mathbb{R}} ||e^{i\xi B(\varepsilon)}|| < \infty$. Pour $\varepsilon \geq 0$ fixé, elle est satisfaite (puisqu'elle est alors équivalente à la diagonalisabilité sur $\mathbb{R}$ de $B(\varepsilon)$). Mais elle n'est pas satisfaite uniformément lorsque $\varepsilon \to 0$, i.e. pour tout $\varepsilon_0 > 0$ on a $\sup_{0\leq\varepsilon\leq\varepsilon_0}(\sup_{\xi\in\mathbb{R}} ||e^{i\xi B(\varepsilon)}||) = +\infty$; lorsque cette condition est satisfaite uniformément pour $\varepsilon \to 0$ et $w^0 \in (L^2(\mathbb{R}))^3$, il n'est pas très dur de voir que la solution de (7.14) pour $\varepsilon > 0$ converge (au moins dans $(L^2(]0,T[\times\mathbb{R}))^3$) vers la solution de ce problème pour $\varepsilon = 0$.

Partie IV

Autres Travaux

Chapitre 8

A Density Result in Sobolev Spaces

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Abstract We prove, when $1 \leq p < \infty$ and Ω is a polygonal or regular open set in $\mathbb{R}^N$, the density in $W^{1,p}(\Omega)$ of a space of regular functions satisfying a Neumann condition on $\partial\Omega$. We also give some applications of this result and a generalization concerning mixed Dirichlet-Neumann boundary conditions.

8.1 Introduction

8.1.1 Definitions

N is an integer greater than or equal to 2. The usual Euclidean scalar product of two vectors (x, y) of $\mathbb{R}^N$ is denoted by $x \cdot y$; $|\cdot|$ is the induced norm and "dist" the associated distance. For $\delta > 0$, $B_N(\delta)$ is the Euclidean ball in $\mathbb{R}^N$ of center 0 and radius δ. When E is a measurable subset of $\mathbb{R}^N$, $|E|$ is the Lebesgue measure of E. For $x \in \mathbb{R}^N$, we denote $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.
If Ω is an open set of $\mathbb{R}^N$ and $p \in [1, \infty[$, $W^{1,p}(\Omega)$ is the usual Sobolev space, endowed with the norm $||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\,|\nabla u|\,||_{L^p(\Omega)}$.

Definition 8.1 *Let $k \in \mathbb{N}$ and U be an open set of $\mathbb{R}^l$ ($l \geq 1$) or of a submanifold of $\mathbb{R}^l$. A function $\varphi : U \to \mathbb{R}$ is $\mathcal{C}^{k,1}$-continuous on U if φ is k times continuously differentiable on U, if the k first derivatives of φ are bounded on U and if the $k^{\rm th}$ derivative of φ is Lipschitz continuous on U. A function is $\mathcal{C}^{\infty,1}$-continuous on U if it is $\mathcal{C}^{k,1}$-continuous on U for all $k \in \mathbb{N}$. When a function takes its values into $\mathbb{R}^m$ for a $m \geq 1$, it is $\mathcal{C}^{k,1}$-continuous if each of its component is $\mathcal{C}^{k,1}$-continuous.*

We denote by $\mathcal{C}^{k,1}(U)$ the set of $\mathcal{C}^{k,1}$-continuous functions on U and by $\mathcal{C}^{k,1}_c(U)$ the set of functions in $\mathcal{C}^{k,1}(U)$ which have a compact support in U. Notice that, for all $k \in \mathbb{N} \cup \{\infty\}$, if $\varphi \in (\mathcal{C}^{k,1}(U))^m$ and $f \in \mathcal{C}^{k,1}(\mathbb{R}^m)$, then $f \circ \varphi \in \mathcal{C}^{k,1}(U)$.

Definition 8.2 *Let Ω be an open bounded set of $\mathbb{R}^N$ ($N \geq 2$) and $k \in \mathbb{N} \cup \{\infty\}$.*

i) *Ω has a $\mathcal{C}^{k,1}$-continuous boundary if, for all $a \in \partial\Omega$, there exists an orthonormal coordinate system $\mathcal{R}$ centered at a, an open set V of $\mathbb{R}^N$ containing a, such that $V = V' \times]-\alpha, \alpha[$ in $\mathcal{R}$, and a $\mathcal{C}^{k,1}$-continuous function $\eta : V' \to]-\alpha, \alpha[$ such that, in $\mathcal{R}$, $\partial\Omega \cap V = \{(y', \eta(y'))\,,\ y' \in V'\}$ and $\Omega \cap V = \{(y', y_N) \in V \mid y_N > \eta(y')\}$.*

ii) *Ω has a Lipschitz continuous boundary if it has a $\mathcal{C}^{0,1}$-continuous boundary.*

iii) *Ω is polygonal if it has a Lipschitz continuous boundary and if its boundary is included in a finite union of affine hyperplanes.*

In the sequel, the open sets Ω we consider have at least a Lipschitz continuous boundary. We can then define a $(N-1)$-dimensional measure σ on $\partial\Omega$ and a unit normal $\mathbf{n} \in (L^\infty(\partial\Omega))^N$ to $\partial\Omega$ outward to Ω (when Ω has a $\mathcal{C}^{k,1}$-continuous boundary, $\mathbf{n} \in (\mathcal{C}^{k-1,1}(\partial\Omega))^N$).
For $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $a \in \partial\Omega$, we denote, when such a quantity exists,

$$\frac{\partial\varphi}{\partial\mathbf{n}}(a) = \lim_{t\to 0} \frac{\varphi(a + t\mathbf{n}(a)) - \varphi(a)}{t}.$$

If φ is $\mathcal{C}^1$-continuous, this limit exists and is equal to $\nabla\varphi(a) \cdot \mathbf{n}(a)$.

Definition 8.3 *Let $k \in \mathbb{N} \cup \{\infty\}$. We define $E^k(\Omega)$ as the space of the restrictions to Ω of functions $\varphi \in \mathcal{C}_c^{k,1}(\mathbb{R}^N)$ satisfying, for σ-a.e. $a \in \partial\Omega$, $\frac{\partial\varphi}{\partial\mathbf{n}}(a) = 0$.*

8.1.2 Main Results

Theorem 8.1 *Let $p \in [1, \infty[$. If $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary, then $E^{k-1}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Remark 8.1 *i) There is, in [55], an alternate proof of this result to the one we present here. However, this proof (which relies on the idea of transporting the problem with well-chosen diffeomorphisms) can only be applied to open sets with at least $\mathcal{C}^{2,1}$-continuous boundaries.*

ii) We will in fact prove a more general result than Theorem 8.1, not asking for Ω to have a $\mathcal{C}^{k,1}$-continuous boundary "everywhere" (see Theorem 8.3).

Theorem 8.2 *Let $p \in [1, \infty[$. If Ω is a polygonal open set of $\mathbb{R}^N$, then $E^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Remark 8.2 *i) We will see in Section 8.4 that there exists open sets Ω with a Lipschitz continuous boundary such that the space of the restrictions to Ω of functions in $\mathcal{C}^1(\mathbb{R}^N)$ satisfying a Neumann boundary condition on $\partial\Omega$ is not dense in $W^{1,p}(\Omega)$.*

ii) (Thierry Gallouët [40]) There is an alternate result to Theorem 8.1 which avoid the loss of a derivative (with respect to the regularity of the open set): if Ω has a $C^{1,1}$-continuous boundary or is polygonal convex, then for all $u \in H^1(\Omega)$, there exists $(u_n)_{n\geq 1} \in H^2(\Omega)$ satisfying, for all $n \geq 1$, $\nabla u_n \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$ and such that $u_n \to u$ in $H^1(\Omega)$.

The idea is to solve the following Neumann problem

$$\begin{cases} v_\varepsilon - \varepsilon\Delta v_\varepsilon = u & \textit{dans } \Omega, \\ \nabla v_\varepsilon \cdot \mathbf{n} = 0 & \textit{sur } \partial\Omega. \end{cases} \tag{8.1}$$

Ω having a $\mathcal{C}^{1,1}$-continuous boundary or being polygonal convex, the variational solution to this problem is in $H^2(\Omega)$; by multiplying the equation by Δv_ε, we notice that $(v_\varepsilon)_{\varepsilon>0}$ is bounded in $H^1(\Omega)$ and that it converges weakly in this space to u; by Mazur's lemma, a convex combination of the $(v_\varepsilon)_{\varepsilon>0}$ converges strongly to u.

If this technique avoids the loss of a derivative (we get the density of H^2 functions when Ω has a $\mathcal{C}^{1,1}$-continuous boundary), in contrary to Theorem 8.1 (density of $\mathcal{C}^{0,1}$-continuous functions under the same hypothesis), the derivatives are however far less regular than in Theorem 8.1 (in L^2 instead of L^∞). Moreover, in the case of a polygonal open set, Theorem 8.2 gives a far better result than the method up above.

8.2 Theorem 8.1 and a generalization

As said before, we will prove a more general result than Theorem 8.1. To state this result, we need a generalization of Definition 8.2: when $\mathcal{K}$ is a compact subset of $\mathbb{R}^N$ and $k \in \mathbb{N} \cup \{\infty\}$, we say that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$ if it satisfies item i) of Definition 8.2 not for all $a \in \partial\Omega$, but for all $a \in \partial\Omega \cap \mathcal{K}$.

With this definition, the generalization of Theorem 8.1 is the following.

Theorem 8.3 *Let $p \in [1, \infty[$ and Ω be an open set of $\mathbb{R}^N$ with a Lipschitz continuous boundary. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$, $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$ and suppose that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$. If K is a compact subset of $\mathbb{R}^N$ and $u \in W^{1,p}(\Omega)$ has a compact support in the interior of K ([1]), there exists a sequence of functions $(u_n)_{n\geq 1} \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with supports in the interior of K such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and, for all $n \geq 1$ and all $a \in \partial\Omega \cap \mathcal{K}$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$.*

Remark 8.3 *i) The global hypothesis on the boundary of Ω (i.e. the Lipschitz continuity of $\partial\Omega$) is only used, in the proof, to ensure that the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ are dense in $W^{1,p}(\Omega)$; thus, we could replace this hypothesis by a weaker one (for example asking that Ω satisfies the segment property, see [1]).*

ii) Notice that the Neumann boundary condition is satisfied for all *$a \in \partial\Omega \cap \mathcal{K}$, even when $k = 1$ (in which case it is even not obvious that $\frac{\partial u_n}{\partial \mathbf{n}}(a)$ is defined for σ-a.e. $a \in \partial\Omega \cap \mathcal{K}$, let alone for all $a \in \partial\Omega \cap \mathcal{K}$).*

This Theorem is an easy consequence of the following proposition (which states in fact the result of Theorem 8.3 when u is regular and $K = \mathcal{K}$).

Proposition 8.1 *Let $p \in [1, \infty[$, $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$ and Ω be an open set of $\mathbb{R}^N$. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ has its support in the interior of $\mathcal{K}$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, there exists a sequence of functions $(u_n)_{n\geq 1} \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with supports in the interior of $\mathcal{K}$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and, for all $n \geq 1$ and all $a \in \partial\Omega$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$.*

Proof of Theorem 8.3
Thanks to the definition of "Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$", we see that there exists a compact set $\mathcal{K}'$ of $\mathbb{R}^N$ containing $\mathcal{K}$ in its interior such that Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}'$; let $\theta \in \mathcal{C}_c^\infty(\mathrm{int}(\mathcal{K}'))$ such that $\theta \equiv 1$ on the neighborhood of $\mathcal{K}$.

Ω having a Lipschitz continuous boundary, there exists $(\varphi_n)_{n\geq 1} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\Omega)$.
Let $\Theta \in \mathcal{C}_c^\infty(\mathrm{int}(K))$ such that $\Theta \equiv 1$ on the neighborhood of $\mathrm{supp}(u)$ and define $v_n = \Theta\theta\varphi_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. $v_n \to \Theta\theta u = \theta u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and the support of v_n is included in the interior of $\mathcal{K}' \cap K$; by Proposition 8.1, there exists thus $w_n \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}' \cap K$ such that $||v_n - w_n||_{W^{1,p}(\Omega)} \leq 1/n$ and, for all $a \in \partial\Omega$, $\frac{\partial w_n}{\partial \mathbf{n}}(a) = 0$.
Let $u_n = w_n + (1-\theta)\Theta\varphi_n \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$; the support of u_n is a compact subset of the interior of K (since $\mathrm{supp}(w_n) \cup \mathrm{supp}(\Theta) \subset \mathrm{int}(K)$). Since $1 - \theta \equiv 0$ on the neighborhood of $\mathcal{K}$, one has $\frac{\partial((1-\theta)\Theta\varphi_n)}{\partial \mathbf{n}} \equiv 0$ on $\mathcal{K} \cap \partial\Omega$, so that, for all $a \in \mathcal{K} \cap \partial\Omega$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$. Since $w_n \to \theta u$ and $(1-\theta)\Theta\varphi_n \to (1-\theta)\Theta u = (1-\theta)u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, we have $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, and this concludes the proof of the theorem.
■

It remains now to prove Proposition 8.1.

[1] That is to say, the extension of u to $\mathbb{R}^N$ by 0 outside Ω has a compact support in $\mathrm{int}(K)$

The idea is the following: the function $g = \nabla u \cdot \mathbf{n}$ is $\mathcal{C}^{k-1,1}$-continuous on the neighborhood of $\partial\Omega \cap \mathcal{K}$; we construct a sequence $(\gamma_n)_{n\geq 1} \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$ which converges to 0 in $W^{1,p}(\Omega)$ and such that, for all $n \geq 1$, $\frac{\partial \gamma_n}{\partial \mathbf{n}} = g$ on $\partial\Omega \cap \mathcal{K}$; the sequence $u_n = u - \gamma_n$ converges then to u in $W^{1,p}(\Omega)$ and satisfies the Neumann boundary condition on $\partial\Omega \cap \mathcal{K}$.
The main difficulty of this proof is to construct the sequence $(\gamma_n)_{n\geq 1}$.

The first lemma is quite classical when $\partial\Omega$ is a regular submanifold of $\mathbb{R}^N$. We however prove it completely because, when $\partial\Omega$ is only $\mathcal{C}^{1,1}$-continuous, the main tool of the proof is not so common.

Lemma 8.1 *Let Ω be an open set of $\mathbb{R}^N$. If $\mathcal{K}$ is a compact subset of $\mathbb{R}^N$, $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, there exists an open set U of $\mathbb{R}^N$ containing $\partial\Omega \cap \mathcal{K}$ and a $\mathcal{C}^{k-1,1}$-continuous application $P : U \to \partial\Omega$ such that, for all $y \in U$, $P(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$. Moreover, for all $a \in \mathcal{K} \cap \partial\Omega$, there exists $t_a > 0$ such that, for all $|t| < t_a$, $P(a + t\mathbf{n}(a)) = a$.*

Remark 8.4 *We will also see, in the course of the proof, that U can be chosen so that*

$$\forall y \in U\backslash\partial\Omega\,,\ \mathbf{n}(P(y)) \cdot (y - P(y)) \neq 0. \tag{8.2}$$

Proof of Lemma 8.1
Step 1: local construction.
We prove in this step that, for all $a \in \partial\Omega \cap \mathcal{K}$, there exists an open set U_a of $\mathbb{R}^N$ containing a and a $\mathcal{C}^{k-1,1}$-continuous application $P_a : U_a \to \partial\Omega$ such that, for all $y \in U_a$, $P_a(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$.

Let $a \in \partial\Omega \cap \mathcal{K}$ and $\mathcal{R}$, $V = V' \times]-\alpha, \alpha[$ and $\eta : V' \to]-\alpha, \alpha[$ given for a by the definition of "Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$". From now on, all the coordinates are taken in $\mathcal{R}$ (notice that the norm and the distance are not modified by this change of coordinates).

Let us first study, for a given $y = (y', y_N)$, the solutions x' to $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
$F(x', y) = x' - y' + (\eta(x') - y_N)\nabla\eta(x')$ is $\mathcal{C}^{k-1,1}$-continuous on $V' \times \mathbb{R}^N$ and is null at $(x', y) = (0, 0)$. Moreover, when it exists, $\frac{\partial F}{\partial x'}(x', y) = Id + \nabla\eta(x')\nabla\eta(x')^T + (\eta(x') - y_N)\eta''(x')$ (where $\eta''(x')$ is confused with the Hessian matrix of η).
If $k \geq 2$, then F being $\mathcal{C}^{k-1}$-continuous and $\frac{\partial F}{\partial x'}(0, 0) = Id + \nabla\eta(0)\nabla\eta(0)^T$ being definite positive, thus invertible, the classical implicit function theorem gives an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a $\mathcal{C}^{k-1}$-continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$. Moreover, since $f'(y) = -\left(\frac{\partial F}{\partial x'}(f(y), y)\right)^{-1} \circ \frac{\partial F}{\partial y}(f(y), y)$ and F is $\mathcal{C}^{k-1,1}$-continuous, f is in fact $\mathcal{C}^{k-1,1}$-continuous (even if it means to reduce U).
If $k = 1$, then $\nabla\eta$ is Lipschitz continuous on V'; there exists thus $C > 0$ such that, for every $x' \in V'$, if $\eta''(x')$ exists, then $||\eta''(x')|| \leq C$ ($||\cdot||$ denotes a norm on the space of $(N-1)\times(N-1)$ matrices). Thus, for all $\xi \in \mathbb{R}^{N-1}$, if (x', y) is such that $\frac{\partial F}{\partial x'}(x', y)$ exists, we have

$$\frac{\partial F}{\partial x'}(x', y)\xi \cdot \xi \geq |\xi|^2 + (\nabla\eta(x')^T\xi)^2 - C|\eta(x') - y_N||\xi|^2.$$

Supposing that $(x', y) \to (0, 0)$ and that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ exists, passing to the limit in this inequality lets us see that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ is a 1-coercive matrix (that is to say a $(N-1)\times(N-1)$ matrix A such that, for all $\xi \in \mathbb{R}^{N-1}$, $A\xi \cdot \xi \geq |\xi|^2$). Thus, any convex combination that can be made with such limits is also 1-coercive; this implies that, by denoting S the set of $(x', y) \in V' \times \mathbb{R}^N$ such that F is differentiable with respect to x' at (x', y), the set

$$\mathrm{co}\left\{\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)\,,\ (x', y) \in S\right\}$$

is made of invertible matrices. The Lipschitz implicit function theorem of [19] gives then an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a Lipschitz continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$.

Let $\beta > 0$ such that $B_{N-1}(\beta) \subset W$ and $B_N(\beta/2) \subset U$; let $y \in B_N(0, \beta/2)$. By compacity of $\partial\Omega$, there exists some points in $\partial\Omega$ that are at distance $\mathrm{dist}(y, \partial\Omega)$ of y. Moreover, since $0 \in \partial\Omega$, when x is such a point, we have $|x| \le |y| + |x - y| \le |y| + |y - 0| < \beta$, that is to say $x \in B_N(\beta)$.
x can thus be written as $(x', \eta(x'))$ for a $x' \in B_{N-1}(\beta) \subset W$; x' is then a minimum of the $\mathcal{C}^1$-continuous function $|\cdot - y'|^2 + |\eta(\cdot) - y_N|^2$ on V' and we deduce that $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
Since $(x', y) \in W \times U$, x' is unique and $x' = f(y)$ (f has been constructed up above).
There can thus be only one projection of y on $\partial\Omega$; it is given by a function of y which is $\mathcal{C}^{k-1,1}$-continuous on $B_N(\delta/2)$. This concludes this step (with $U_a = B_N(\delta/2)$ and $P_a(y) = (f(y), \eta(f(y)))$ in $\mathcal{R}$).

Step 2: we cover the compact set $\mathcal{K} \cap \partial\Omega$ by a finite number of U_{a_i}, $i = 1, \ldots, l$, constructed in step 1. $\cup_{i=1}^l U_{a_i}$ being an open set of $\mathbb{R}^N$ containing $\mathcal{K} \cap \partial\Omega$, there exists an open set U of $\mathbb{R}^N$ containing $\mathcal{K} \cap \partial\Omega$ and relatively compact in $\cup_{i=1}^l U_{a_i}$. Define $P : U \to \partial\Omega$ by: $\forall y \in U$, $P(y)$ is the unique point of $\partial\Omega$ at distance $\mathrm{dist}(y, \partial\Omega)$ of y (since $y \in U_{a_i}$ for a certain $i \in [1, l]$, we know that this point exists and is unique).
By construction of P and of the $(P_{a_i})_{i \in [1,l]}$, and by uniqueness of the point at distance $\mathrm{dist}(y, \partial\Omega)$ of y when $y \in U$, we have $P = P_{a_i}$ on U_{a_i}. P is thus $\mathcal{C}^{k-1,1}$-continuous on U.

Let us now check that, for all $a \in \mathcal{K} \cap \partial\Omega$ and t small enough, we have $P(a + t\mathbf{n}(a)) = a$. Since the projection of a point of U on $\partial\Omega$ is unique, we have, on the neighborhood of a, $P = P_a$. By the study made in step 1, and using the notations introduced on the neighborhood of a (in which case the expression of $\mathbf{n}(a)$ is $(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}(\nabla\eta(0), -1)^T$), we see that, for t small enough, $P(a + t\mathbf{n}(a)) = (x', \eta(x'))$ where x' is the unique solution on the neighborhood of 0 to $x' - t(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}\nabla\eta(0) + (\eta(x') + t(\sqrt{1 + |\nabla\eta(0)|^2})^{-1})\nabla\eta(x') = 0$; but $x' = 0$ is a solution to this equation. This means that $P(a + t\mathbf{n}(a)) = (0, \eta(0)) = 0$ in $\mathcal{R}$, that is to say $P(a + t\mathbf{n}(a)) = a$.

To conclude this proof, we see that the open set U given above satisfies (8.2).
Let $y \in U$; there exists $i \in [1, l]$ such that $y \in U_{a_i}$; by the study made in step 1, and with the notations of this step, we have $P(y) = (x', \eta(x'))$ where $x' \in V'$ satisfies $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$. If $\mathbf{n}(P(y)) \cdot (y - P(y)) = 0$, then $(\nabla\eta(x'), -1)^T \cdot (y' - x', y_N - \eta(x'))^T = 0$ (because $\mathbf{n}(P(y)) = (\sqrt{1 + |\nabla\eta(x')|^2})^{-1}(\nabla\eta(x'), -1)$), so that $(y' - x') \cdot \nabla\eta(x') - (y_N - \eta(x')) = 0$. By using the equation satisfied by x', we deduce that $(\eta(x') - y_N)(|\nabla\eta(x')|^2 + 1) = 0$, that is to say $y_N = \eta(x')$ and, once again thanks to the equation satisfied by x', $x' = y'$. This gives $y = P(y) \in \partial\Omega$.
Thus, if $y \in U \backslash \partial\Omega$, we have $\mathbf{n}(P(y)) \cdot (y - P(y)) \neq 0$.■

The following lemma gives the existence of the $(\gamma_n)_{n \ge 1}$ needed in the proof of Proposition 8.1.

Lemma 8.2 *Let $p \in [1, +\infty[$, Ω be an open set of $\mathbb{R}^N$ and $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$. If $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$, $g \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ has its support in the interior of $\mathcal{K}$ and Ω has a $\mathcal{C}^{k,1}$-continuous boundary on the neighborhood of $\mathcal{K}$, then for all $\varepsilon > 0$, there exists $\gamma \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$ such that $||\gamma||_{W^{1,p}(\Omega)} < \varepsilon$ and, for all $a \in \partial\Omega$, $\frac{\partial\gamma}{\partial\mathbf{n}}(a) = g(a)$.*

Proof of Lemma 8.2
Let U and P given for $\mathcal{K}$ by Lemma 8.1; we can suppose that U is bounded and satisfies (8.2). Let $\theta \in \mathcal{C}_c^\infty(\mathrm{int}(\mathcal{K}) \cap U)$ such that $\theta \equiv 1$ on the neighborhood of $\mathrm{supp}(g) \cap \partial\Omega$.
Let $h \in \mathcal{C}_c^\infty(]-1, 1[)$ such that $h(0) = 0$ and $h'(0) = 1$; when $\delta > 0$, we take $h_\delta(x) = \delta h(x/\delta)$.
Define $\gamma_\delta(y) = \theta(y) g(P(y)) h_\delta(\mathbf{n}(P(y)) \cdot (y - P(y)))$; this function is well defined and $\mathcal{C}^{k-1,1}$-continuous on U; since its support is a compact subset of $\mathrm{int}(\mathcal{K}) \cap U$, its extension to $\mathbb{R}^N$ by 0 outside U is in $\mathcal{C}^{k-1,1}(\mathbb{R}^N)$ and has a compact support in the interior of $\mathcal{K}$.

Let us first check that, for all $a \in \partial\Omega$, $\frac{\partial\gamma_\delta}{\partial\mathbf{n}}(a)$ exists and is equal to $g(a)$. We study different cases, depending on the position of a on $\partial\Omega$.
When $a \in \partial\Omega\backslash\mathcal{K}$, it is quite clear because, for t small enough, $(a, a+t\mathbf{n}(a)) \not\in \text{supp}(\theta)$ so that $\gamma_\delta(a+t\mathbf{n}(a)) = \gamma_\delta(a) = 0 = g(a)$.
When $a \in \partial\Omega \cap (\mathcal{K}\backslash\text{supp}(g))$, we have, by Lemma 8.1, $P(a+t\mathbf{n}(a)) = a$ for t small enough, so that $g(P(a+t\mathbf{n}(a))) = g(a) = 0$; this implies $\gamma_\delta(a+t\mathbf{n}(a)) = \gamma_\delta(a) = 0 = g(a)$.
When $a \in \partial\Omega \cap \text{supp}(g)$, then, for t small enough, $\theta(a+t\mathbf{n}(a)) = \theta(a) = 1$ ($\theta \equiv 1$ on the neighborhood of $\text{supp}(g) \cap \partial\Omega$) and $P(a+t\mathbf{n}(a)) = a$, so that $\gamma_\delta(a+t\mathbf{n}(a)) - \gamma_\delta(a) = g(a)h_\delta(\mathbf{n}(a)\cdot(a+t\mathbf{n}(a)-a)) - g(a)h_\delta(\mathbf{n}(a)\cdot(a-a)) = g(a)h_\delta(t)$; since $h_\delta(0) = 0$ and $h'_\delta(0) = 1$, we deduce that $\frac{\partial\gamma_\delta}{\partial\mathbf{n}}(a)$ exists and is equal to $g(a)$.

Let us now prove that $\gamma_\delta \to 0$ in $W^{1,p}(\Omega)$ as $\delta \to 0$; this will conclude the proof of the lemma (by taking $\gamma = \gamma_\delta$ for δ small enough).
Notice first that, for all $x \in \Omega$, $|\gamma_\delta(x)| \leq \delta||h||_{L^\infty(\mathbb{R})}||\theta||_{L^\infty(\mathbb{R}^N)}||g||_{L^\infty(\mathbb{R}^N)}$; thus, when $\delta \to 0$, $\gamma_\delta \to 0$ in $L^\infty(\Omega)$, and so in $L^p(\Omega)$.
Since h_δ is regular and $\theta g \circ P$, $n \circ P \cdot (Id - P)$ are Lipschitz continuous, we have, on U,

$$\begin{aligned}\nabla\gamma_\delta &= h_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\theta g\circ P)\\ &\quad +\theta g\circ P h'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P)).\end{aligned}$$

But $||h_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\theta g\circ P)||_{L^\infty(\Omega)} \leq \delta||h||_{L^\infty(\mathbb{R})}||\nabla(\theta g\circ P)||_{L^\infty(\Omega)} \to 0$ as $\delta \to 0$. Moreover, by (8.2), for all $y \in U \cap \Omega$, $\mathbf{n}(P(y))\cdot(y-P(y)) \neq 0$, so that $h'_\delta(n(P(y))\cdot(y-P(y))) \to 0$ as $\delta \to 0$ (the support of h'_δ is included in $]-\delta,\delta[$); thus, $\theta g\circ P h'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P)) \to 0$ on Ω; since $||h'_\delta||_{L^\infty(\mathbb{R})} \leq ||h'||_{L^\infty(\mathbb{R})}$, we deduce, by the dominated convergence theorem, that $\theta g\circ P h'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P)) \to 0$ in $L^p(\Omega)$ ($n\circ P\cdot(Id-P)$ is Lipschitz continuous on U, thus its gradient is essentially bounded on U). ■

Proof of Proposition 8.1
Step 1: we prove that $\nabla u\cdot\mathbf{n} : \partial\Omega \to \mathbb{R}$ has an extension $g \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$.

Cover $\mathcal{K}\cap\partial\Omega$ by a finite number of open sets $(V_i)_{i\in[1,l]}$ of $\mathbb{R}^N$ such that, for all $i \in [1,l]$, there exists an orthonormal coordinate system $\mathcal{R}_i$ in which $V_i = V'_i\times]-\alpha_i,\alpha_i[$ and a $\mathcal{C}^{k,1}$-continuous application $\eta_i : V'_i \to]-\alpha_i,\alpha_i[$ satisfying, in $\mathcal{R}_i$, $\Omega\cap V_i = \{(y', y_N) \in V_i \mid y_N > \eta_i(y')\}$ and $\partial\Omega\cap V_i = \{(y',\eta_i(y'))\,,\ y' \in V'_i\}$.
Take $(\theta_i)_{i\in[1,l]}$ such that, for all $i \in [1,l]$, $\theta_i \in \mathcal{C}_c^\infty(V_i)$ and $\sum_{i=1}^l \theta_i \equiv 1$ on $\mathcal{K}\cap\partial\Omega$.
Let $i \in [1,l]$. Using the coordinates $\mathcal{R}_i$, we have, at a point $(y',\eta_i(y')) \in \partial\Omega\cap V_i$, $\mathbf{n}(y',\eta_i(y')) = (\sqrt{1+|\nabla\eta_i(y')|^2})^{-1}(\nabla\eta_i(y'),-1)^T$; define then, for $(y',y_N) \in V_i$,

$$g_i(y',y_N) = \theta_i(y',\eta_i(y'))\nabla u(y',\eta_i(y'))\cdot((\sqrt{1+|\nabla\eta_i(y')|^2})^{-1}(\nabla\eta_i(y'),-1)^T)$$

(i.e. g_i does not depend on y_N). g_i is $\mathcal{C}^{k-1,1}$-continuous on V_i and has a compact support in V_i; its extension, still denoted g_i, to $\mathbb{R}^N$ by 0 outside V_i is thus in $\mathcal{C}^{k-1,1}(\mathbb{R}^N)$.

Let $\Theta \in \mathcal{C}_c^\infty(\text{int}(\mathcal{K}))$ such that $\Theta \equiv 1$ on the neighborhood of $\text{supp}(u)$ and $g = \Theta\sum_{i=1}^l g_i$. $g \in \mathcal{C}^{k-1,1}(\mathbb{R}^N)$ has a compact support in the interior of $\mathcal{K}$. On $\partial\Omega$, one has $g_i = \theta_i\nabla u\cdot\mathbf{n}$, so that $g = \Theta\sum_{i=1}^l \theta_i\nabla u\cdot\mathbf{n} = (\sum_{i=1}^l\theta_i)\Theta\nabla u\cdot\mathbf{n}$; but $\Theta = \sum_{i=1}^l\theta_i \equiv 1$ on $\text{supp}(\nabla u\cdot\mathbf{n})$ (because $\text{supp}(\nabla u\cdot\mathbf{n}) \subset \mathcal{K}\cap\partial\Omega$ and $\sum_{i=1}^l\theta_i \equiv 1$ on $\mathcal{K}\cap\partial\Omega$), and we have thus $g = \nabla u\cdot\mathbf{n}$ on $\partial\Omega$.

Step 2: conclusion.
By Lemma 8.2, we can find, for all $n \geq 1$, $\gamma_n \in \mathcal{C}_c^{k-1,1}(\mathbb{R}^N)$ with support in the interior of $\mathcal{K}$ such that $||\gamma_n||_{W^{1,p}(\Omega)} \leq 1/n$ and $\frac{\partial\gamma_n}{\partial\mathbf{n}} = g = \nabla u\cdot\mathbf{n}$ on $\partial\Omega$. The sequence $(u-\gamma_n)_{n\geq 1}$ satisfies thus the conclusion of the proposition. ■

8.3 Polygonal open set

The idea of the proof of Theorem 8.2 is to approximate any regular function by functions that, on the neighborhood of each affine part of $\partial\Omega$, only depend on the coordinates along this affine part (for example, on the neighborhood of a vertex of $\partial\Omega$, the approximating functions will be constant; on the neighborhood of an edge of $\partial\Omega$, it will only depend on the 1-dimensional coordinate along this edge; etc...).
We first introduce some notations and then prove a lemma that entails Theorem 8.2.

Let Ω be a polygonal open set of $\mathbb{R}^N$ and $H_1,\dots,H_{r_1}$ some affine hyperplanes, the union of which contains $\partial\Omega$ (we also choose these hyperplanes pairwise distincts and such that, for all $i\in[1,r_1]$, $H_i\cap\partial\Omega\neq\emptyset$).
For $i\in[1,r_1]$, let $F_i^1=H_i\cap\partial\Omega$. Define $q=\sup\{s\geq 1\mid \exists J\subset[1,r_1]$ of cardinal s such that $\cap_{i\in J}F_i^1\neq\emptyset\}$; for $d\in[1,q]$, we denote $J_1^d,\dots,J_{r_d}^d$ the subsets of cardinal d of $[1,r_1]$ such that, for all $i\in[1,r_d]$, $F_i^d=\cap_{l\in J_i^d}F_l^1\neq\emptyset$.
When $d\in[1,q]$ and $i\in[1,r_d]$, $A(F_i^d)$ denotes the affine space of minimal dimension containing F_i^d ([2]) and $V(F_i^d)$ the vector space parallel to $A(F_i^d)$. The orthogonal projection on $V(F_i^d)$ is denoted by $\widetilde{P}_i^d$; we also take $f_i^d\in A(F_i^d)$, so that the orthogonal projection of a point x on $A(F_i^d)$ is $P_i^d(x)=f_i^d+\widetilde{P}_i^d(x-f_i^d)$.

When $d\in[1,q+1]$, we say that a function $u\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_d$ if, for all $m\in[d,q]$ and for all $i\in[1,r_m]$, $u=u\circ P_i^m$ on the neighborhood of F_i^m (i.e., on the neighborhood of F_i^m, u only depends on the coordinates along F_i^m). Notice that any function in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_{q+1}$ (there exists no $m\in[q+1,q]$).

Lemma 8.3 *Let $p\in[1,\infty[$, $d\in[2,q+1]$ and K be a compact subset of $\mathbb{R}^N$. If $u\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ has its support in the interior of K and satisfies $\mathcal{B}_d$, there exists a sequence of fuctions $u_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ with supports in the interior of K such that $u_n\to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$ and, for all $n\geq 1$, u_n satisfies $\mathcal{B}_{d-1}$.*

Proof of Lemma 8.3
Before beginning the proof itself, let us make some remarks:

$$\text{If } i\neq j,\ F_i^{d-1}\cap F_j^{d-1} \text{ is either empty or equal to a certain } F_k^m, \text{ for a } m\in[d,q] \text{ and a } k\in[1,r_m]. \tag{8.3}$$

$$\text{If } F_j^m\subset F_i^d \text{ and } x\in F_j^m+B_N(\delta), \text{ then } P_i^d(x)\in F_j^m+B_N(\delta). \tag{8.4}$$

$$\text{If } F_j^m\subset F_i^d, \text{ then } P_j^m\circ P_i^d=P_i^d\circ P_j^m=P_j^m. \tag{8.5}$$

• *Proof of (8.3)*: by definition, $F_i^{d-1}=\cap_{l\in J_i^{d-1}}F_l^1$ and $F_j^{d-1}=\cap_{l\in J_j^{d-1}}F_l^1$, with J_i^{d-1} and J_j^{d-1} distinct (since $i\neq j$) subsets of cardinal $d-1$ of $[1,r_1]$. Thus, $J_i^{d-1}\cup J_j^{d-1}$ has a cardinal $m\geq d$; if $F_i^{d-1}\cap F_j^{d-1}\neq\emptyset$, then $\cap_{l\in(J_i^{d-1}\cup J_j^{d-1})}F_l^1=F_i^{d-1}\cap F_j^{d-1}$ is not empty, so that, by definition, $m\leq q$ and $J_i^{d-1}\cup J_j^{d-1}$ is a certain J_k^m, for a $k\in[1,r_m]$; we get then $F_k^m=\cap_{l\in J_k^m}F_l^1=F_i^{d-1}\cap F_j^{d-1}$.
• *Proof of (8.4)*: we have $x=z+h$, where $z\in F_j^m\subset A(F_i^d)$ and $|h|<\delta$; since z belongs to $A(F_i^d)$, it is equal to its projection on this affine space, so that $P_i^d(x)=f_i^d+\widetilde{P}_i^d(z-f_i^d)+\widetilde{P}_i^d(h)=z+\widetilde{P}_i^d(h)$; $\widetilde{P}_i^d$ being an orthogonal projection, we have $|\widetilde{P}_i^d(h)|\leq|h|<\delta$, and this proves the result.
• *Proof of (8.5)*: we first notice that the range of P_j^m is included in $A(F_j^m)$, thus in $A(F_i^d)$; since P_i^d is equal to the identity mapping on $A(F_i^d)$, we deduce that $P_i^d\circ P_j^m=P_j^m$; it remains to prove the second equality. Let $x\in\mathbb{R}^N$; $P_j^m(x)$ is the unique $z\in A(F_j^m)$ such that $(x-z)\perp V(F_j^m)$; but $x-P_j^m(P_i^d(x))=x-P_i^d(x)+P_i^d(x)-P_j^m(P_i^d(x))$ with $x-P_i^d(x)$ orthogonal to $V(F_i^d)$ (by definition of P_i^d), thus also to $V(F_j^m)\subset V(F_i^d)$, and $P_i^d(x)-P_j^m(P_i^d(x))$ orthogonal to $V(F_j^m)$ (by definition of P_j^m); thus, $P_j^m(P_i^d(x))$ is in $A(F_j^m)$ and satisfies $(x-P_j^m(P_i^d(x)))\perp V(F_j^m)$, which implies $P_j^m(P_i^d(x))=P_j^m(x)$.

[2] We could also have taken $A(F_i^d)=\cap_{l\in J_i^d}H_l$.

Step 1: we define a sequence of functions $(v_n)_{n\geq 1}$.
Let $\delta > 0$ such that, for all $m \in [d,q]$ and all $i \in [1,r_m]$, $u = u \circ P_i^m$ on $F_i^m + B_N(\delta)$.

When $i \in [1,r_{d-1}]$, we define $L_i = \{(m,k) \in \mathbb{N}^2 \mid d \leq m \leq q\,,\ 1 \leq k \leq r_m\,,\ F_k^m \subset F_i^{d-1}\}$. $G_i^{d-1} = F_i^{d-1} \backslash (\cup_{(m,k)\in L_i}(F_k^m + B_N(\delta/2)))$ is a compact set which does not intersect the compact set $\cup_{j\neq i} F_j^{d-1}$; indeed, suppose that the intersection of these compact sets contains a point: this point is then in $F_i^{d-1} \cap F_j^{d-1}$ for a $j \neq i$, thus, by (8.3), in some F_k^m for a $m \in [d,q]$ and a $k \in [1,r_m]$ such that $F_k^m \subset F_i^{d-1}$; this point can then not belong to G_i^{d-1}.
Thus, $\delta_0 = \inf(\delta, \inf\{\mathrm{dist}(G_i^{d-1}, \cup_{j\neq i} F_j^{d-1})\,,\ i \in [1,r_{d-1}]\})$ is positive. Take then, for all $i \in [1,r_{d-1}]$, $\theta_i \in \mathcal{C}_c^\infty\left(F_i^{d-1} + B_N(\delta_0/2)\right)$ such that $\theta_i \equiv 1$ on $F_i^{d-1} + B_N(\delta_0/4)$.
Let $\gamma \in \mathcal{C}_c^\infty(]-2,2[)$ such that $\gamma \equiv 1$ on $]-1,1[$; we denote $\gamma_n(t) = \gamma(nt)$ (notice that $\gamma_n(|\cdot|)$ is $\mathcal{C}^\infty$-continuous, since γ_n is constant on a neighborhood of 0); by denoting Id the identity mapping of $\mathbb{R}^N$, we let

$$v_n = \sum_{i=1}^{r_{d-1}} \theta_i \gamma_n\left(\left|Id - P_i^{d-1}\right|\right)\left(u - u \circ P_i^{d-1}\right) \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

Step 2: we prove that $v_n \to 0$ in $W^{1,p}(\Omega)$ as $n \to \infty$.
We have, as $n \to \infty$, for all $i \in [1,r_{d-1}]$, $\gamma_n(|x - P_i^{d-1}(x)|) \to 0$ when $x \neq P_i^{d-1}(x)$, that is to say when $x \notin A(F_i^{d-1})$. The sets $(A(F_i^{d-1}))_{i\in[1,r_{d-1}]}$ being of null measure (they are included in some hyperplanes), this means that $v_n \to 0$ a.e. on Ω. By the dominated convergence theorem, since Ω is bounded and

$$||v_n||_{L^\infty(\Omega)} \leq 2\sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\mathbb{R}^N)} ||\gamma||_{L^\infty(\mathbb{R})} ||u||_{L^\infty(\mathbb{R}^N)},$$

v_n tends thus to 0 in $L^p(\Omega)$ as $n \to \infty$.
Let us now study the gradient of v_n. It is the sum of

$$\sum_{i=1}^{r_{d-1}} \gamma_n\left(\left|Id - P_i^{d-1}\right|\right) \nabla\left(\theta_i\left(u - u \circ P_i^{d-1}\right)\right) \tag{8.6}$$

and

$$\sum_{i=1}^{r_{d-1}} \theta_i\left(u - u \circ P_i^{d-1}\right)\gamma_n'\left(\left|Id - P_i^{d-1}\right|\right)\zeta_i \tag{8.7}$$

where $\zeta_i = \nabla(|Id - P_i^{d-1}|) = (Id - \widetilde{P}_i^{d-1})\frac{Id - P_i^{d-1}}{|Id - P_i^{d-1}|}$ (because $I - \widetilde{P}_i^{d-1}$ is symmetric).
By the same argument than before, the term (8.6) tends to 0 in $L^p(\Omega)$ as $n \to \infty$.
$\widetilde{P}_i^{d-1}$ being an orthogonal projection, we have, for a.e. $x \in \mathbb{R}^N$ (for all $x \notin A(F_i^{d-1})$),

$$|\zeta_i(x)| \leq \left|\frac{x - P_i^{d-1}(x)}{|x - P_i^{d-1}(x)|}\right| \leq 1.$$

Thus, the norm in $L^p(\Omega)$ of (8.7) is bounded by

$$\sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\Omega)} \left|\left|(u - u \circ P_i^{d-1})\gamma_n'\left(\left|Id - P_i^{d-1}\right|\right)\right|\right|_{L^p(\Omega)}.$$

But, $\gamma_n'(|x - P_i^{d-1}(x)|) = 0$ when $|x - P_i^{d-1}(x)| \geq 2/n$, that is to say when $x \notin A(F_i^{d-1}) + B_N(2/n)$ (recall that $|x - P_i^{d-1}(x)|$ is the distance between x and $A(F_i^{d-1})$); thus, using the Lipschitz continuity of u and the estimate $||\gamma_n'||_{L^\infty(\mathbb{R})} \leq n||\gamma'||_{L^\infty(\mathbb{R})}$, we can bound the norm in $L^p(\Omega)$ of (8.7) by

$$2||\gamma'||_{L^\infty(\mathbb{R})}\mathrm{lip}(u) \sum_{i=1}^{r_{d-1}} ||\theta_i||_{L^\infty(\Omega)} \left|\Omega \cap \left(A(F_i^{d-1}) + B_N(2/n)\right)\right|^{1/p}. \tag{8.8}$$

Since Ω is of finite measure and $\cap_{n\geq 1}(A(F_i^{d-1})+B_N(2/n))=A(F_i^{d-1})$ is a non-increasing intersection of null measure, (8.8) tends to 0 as $n\to\infty$.
Both terms (8.6) and (8.7) going to 0 in $L^p(\Omega)$ as $n\to\infty$, we deduce that $v_n\to 0$ in $W^{1,p}(\Omega)$ as $n\to\infty$.

Step 3: study of v_n on the neighborhood of a F_i^{d-1}.
Let $i\in[1,r_{d-1}]$ and $U_{i,n}$ be the open set $F_i^{d-1}+B_N(\inf(\delta_0/4,1/n))$. If $x\in U_{i,n}$, then $|x-P_i^{d-1}(x)|=\mathrm{dist}(x,A(F_i^{d-1}))\leq \mathrm{dist}(x,F_i^{d-1})<1/n$, so that $\gamma_n(|x-P_i^{d-1}(x)|)=1$. Thus, on $U_{i,n}$,

$$v_n=u-u\circ P_i^{d-1}+\sum_{j\neq i}\theta_j\gamma_n(|Id-P_j^{d-1}|)(u-u\circ P_j^{d-1}).$$

Let $j\neq i$ and $x\in U_{i,n}$ such that $\theta_j(x)\neq 0$. We have then $x\in(F_i^{d-1}+B_N(\delta_0/4))\cap(F_j^{d-1}+B_N(\delta_0/2))$; by writing $x=z+h$ with $z\in F_j^{d-1}$ and $|h|<\delta_0/2$, we have $z\in F_j^{d-1}\cap(F_i^{d-1}+B_N(3\delta_0/4))$, thus $z\in\cup_{(m,k)\in L_j}(F_k^m+B_N(\delta/2))$ by definition of δ_0 (z cannot belong to G_j^{d-1} since the distance between G_j^{d-1} and F_i^{d-1} is greater than or equal to δ_0); since $|x-z|<\delta_0/2\leq\delta/2$, we deduce that $x\in F_k^m+B_N(\delta)$ for a $m\in[d,q]$ and a $k\in[1,r_m]$ such that $F_k^m\subset F_j^{d-1}$. By (8.4), we get then $(x,P_j^{d-1}(x))\in(F_k^m+B_N(\delta))^2$, which gives, by definition of δ and by (8.5), $u(x)=u(P_k^m(x))$ and $u(P_j^{d-1}(x))=u(P_k^m(P_j^{d-1}(x)))=u(P_k^m(x))$, which implies $u(x)-u(P_j^{d-1}(x))=0$.
We have thus, on $U_{i,n}$, $v_n=u-u\circ P_i^{d-1}$.

Step 4: conclusion.
Let $\Theta\in\mathcal{C}_c^\infty(\mathrm{int}(K))$ and $\varepsilon>0$ such that $\Theta\equiv 1$ on $\mathrm{supp}(u)+B_N(\varepsilon)$.
Define $u_n=u-\Theta v_n\in\mathcal{C}_c^\infty(\mathbb{R}^N)$; $u_n\to u$ in $W^{1,p}(\Omega)$ as $n\to\infty$, the support of u_n is included in the interior of K and, for all $i\in[1,r_{d-1}]$, we have, on $U_{i,n}$, $u_n=u-\Theta u+\Theta(u\circ P_i^{d-1})=(1-\Theta)u+\Theta(u\circ P_i^{d-1})=\Theta(u\circ P_i^{d-1})$ (because $1-\Theta\equiv 0$ on the neighborhood of $\mathrm{supp}(u)$).
Let $\mathcal{U}_{i,n}=F_i^{d-1}+B_N(\inf(\delta_0/4,1/n,\varepsilon/2))\subset U_{i,n}$. If $x\in\mathcal{U}_{i,n}\cap(\mathrm{supp}(u)+B_N(\varepsilon))$, then $u_n(x)=\Theta(x)u(P_i^{d-1}(x))=u(P_i^{d-1}(x))$ because $\Theta\equiv 1$ on $\mathrm{supp}(u)+B_N(\varepsilon)$. If $x\in\mathcal{U}_{i,n}\backslash(\mathrm{supp}(u)+B_N(\varepsilon))$, then $x=z+h$ with $z\in F_i^{d-1}$ and $|h|<\varepsilon/2$, so that ($z\in A(F_i^{d-1})$ is equal to its projection on this space) $P_i^{d-1}(x)=z+\widetilde{P}_i^{d-1}(h)$, thus $|P_i^{d-1}(x)-x|\leq|h|+|\widetilde{P}_i^{d-1}(h)|<\varepsilon/2+\varepsilon/2$ (because $\widetilde{P}_i^{d-1}$ is an orthogonal projection on a vector space and satisfies thus $|\widetilde{P}_i^{d-1}(h)|\leq|h|$); we get then $P_i^{d-1}(x)\notin\mathrm{supp}(u)$ (because $\mathrm{dist}(x,\mathrm{supp}(u))\geq\varepsilon$), which gives $u_n(x)=\Theta(x)u(P_i^{d-1}(x))=0=u(P_i^{d-1}(x))$.
Thus, for all $i\in[1,r_{d-1}]$,

$$u_n=u\circ P_i^{d-1}\quad\text{on}\quad \mathcal{U}_{i,n}=F_i^{d-1}+B_N(\inf(\delta_0/4,1/n,\varepsilon/2)).\tag{8.9}$$

If $x\in\mathcal{U}_{i,n}$, then by (8.4), $P_i^{d-1}(x)\in\mathcal{U}_{i,n}$, so that, by (8.9) and (8.5),

$$u_n(P_i^{d-1}(x))=u(P_i^{d-1}(P_i^{d-1}(x)))=u(P_i^{d-1}(x))=u_n(x);$$

thus, $u_n=u_n\circ P_i^{d-1}$ on the neighborhood of F_i^{d-1}, for all $i\in[1,r_{d-1}]$.
It remains to prove that u_n satisfies $\mathcal{B}_d$. Let $m\in[d,q]$ and $i\in[1,r_m]$. There exists $j\in[1,r_{d-1}]$ such that $F_i^m\subset F_j^{d-1}$ ([3]); let $W=F_i^m+B_N(\inf(\delta_0/4,1/n,\varepsilon/2))\subset\mathcal{U}_{j,n}$.
When $x\in W$, by (8.9), $u_n(x)=u(P_j^{d-1}(x))$. But, by (8.4), $P_j^{d-1}(x)\in W\subset F_i^m+B_N(\delta)$; the definition of δ and (8.5) give thus

$$u_n(x)=u(P_j^{d-1}(x))=u(P_i^m(P_j^{d-1}(x)))=u(P_i^m(x)).\tag{8.10}$$

Moreover, by (8.4), $P_i^m(x)\in W\subset\mathcal{U}_{j,n}$, which gives, thanks to (8.9) and (8.5),

$$u_n(P_i^m(x))=u(P_j^{d-1}(P_i^m(x)))=u(P_i^m(x)).\tag{8.11}$$

[3]Indeed, one just need to take $J\subset J_i^m$ of cardinal $d-1$ and to notice that $\cap_{l\in J}F_l^1\supset\cap_{l\in J_i^m}F_l^1=F_i^m\neq\emptyset$, so that J is a J_j^{d-1} for a $j\in[1,r_{d-1}]$ and $F_j^{d-1}=\cap_{l\in J}F_l^1\supset F_i^m$.

(8.10) and (8.11) give $u_n = u_n \circ P_i^m$ on W, neighborhood of F_i^m.
u_n satisfies thus $\mathcal{B}_{d-1}$, and this concludes the proof of the lemma. ■

Proof of Theorem 8.2
Denote, for $d \in [1, q+1]$, $\mathcal{E}_d$ the space of the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfying $\mathcal{B}_d$.
Ω having a Lipschitz continuous boundary, $\mathcal{E}_{q+1}$ (the space of the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$) is dense in $W^{1,p}(\Omega)$. Lemma 8.3 allows us to see that $\mathcal{E}_q$ is then dense in $W^{1,p}(\Omega)$ and, by induction, that $\mathcal{E}_1$ is dense in $W^{1,p}(\Omega)$. We will prove that $\mathcal{E}_1 \subset E^\infty(\Omega)$, which will conclude the proof of this theorem.

Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ satisfying $\mathcal{B}_1$, $d \in [1, q]$ and $i \in [1, r_d]$. On the neighborhood of F_i^d, we have $u = u \circ P_i^d$, so that $\nabla u = (\widetilde{P}_i^d)^T \nabla u \circ P_i^d = \widetilde{P}_i^d \nabla u \circ P_i^d$ (an orthogonal projection is always symmetric); thus, by denoting, for $l \in [1, r_1]$, $\widetilde{H}_l$ the vector hyperspace parallel to H_l, we have

$$\nabla u \in V(F_i^d) \subset \bigcap_{l \in J_i^d} \widetilde{H}_l \text{ on the neighborhood of } F_i^d. \tag{8.12}$$

Let $x \in \partial\Omega$ and $J_x = \{l \in [1, r_1] \mid x \in H_l\}$; since $\cap_{l \in J_x} F_l^1$ is not empty (it contains x), J_x is a J_i^d for a $d \in [1, q]$ and a $i \in [1, r_d]$.
On the neighborhood of x, the only hyperplanes $(H_l)_l$ that intersect $\partial\Omega$ are $(H_l)_{l \in J_i^d}$ (because, when $l \notin J_i^d = J_x$, the compact sets $\{x\}$ and $F_l^1 = \partial\Omega \cap H_l$ are disjoint); thus, for σ-a.e. $y \in \partial\Omega$ on the neighborhood of x, there exists $l \in J_i^d$ such that $\mathbf{n}(y)$ is orthogonal to $\widetilde{H}_l$. Since $x \in F_i^d$, we deduce from (8.12) that $\nabla u \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$ on the neighborhood of x.
We have thus proven that $\nabla u \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$, that is to say $u_{|\Omega} \in E^\infty(\Omega)$. ■

Remark 8.5 *One can of course prove a similar result for open sets with singularities on the boundary that are of the same kind than the singularities on the boundary of polygonal open sets. For example, if we can transform locally, by a $\mathcal{C}^{r,1}$-diffeomorphism ($r \geq 1$) that preserves the outer normal ([4]), the singularities of an open set Ω into the singularities of a polygonal open set, we obtain the density of $E^r(\Omega)$ in $W^{1,p}(\Omega)$. This gives in fact another proof (the one in [55]) of Theorem 8.1, but only for $k \geq 2$.*
A crucial example of this is $\Omega = O \times]0, T[$ where O is an open set of $\mathbb{R}^{N-1}$ with a $\mathcal{C}^{r+1,1}$-continuous boundary. Though Ω has a boundary which is only Lipschitz continuous, the singularities of this boundary are, up to a $\mathcal{C}^{r,1}$-diffeomorphism, equivalent to the singularities of a polygonal open set.

8.4 Applications, Counter-example and Generalization

8.4.1 A new formulation for the Neumann problem

The classical variational formulation of the Neumann problem

$$\left\{ \begin{array}{ll} -\Delta u = L & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{8.13}$$

is the following:

$$\left\{ \begin{array}{l} u \in H^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi = \langle L, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}, \ \forall \varphi \in H^1(\Omega). \end{array} \right. \tag{8.14}$$

With Theorem 8.1 or 8.2 and an integrate by parts, we see that (8.14) is equivalent, when Ω has a $\mathcal{C}^{k+1,1}$-continuous boundary (with $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$) or is polygonal (in which case we take $k = \infty$

[4]Such diffeomorphisms can be constructed thanks to the flow of a vector field which is, on $\partial\Omega$, equal to the unit normal.

below),

$$\left\{\begin{array}{l} u \in H^1(\Omega), \\ -\displaystyle\int_\Omega u\Delta\varphi = \langle L, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)}\ ,\ \forall \varphi \in \mathcal{C}^{k,1}(\Omega) \text{ such that } \nabla\varphi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \end{array}\right. \tag{8.15}$$

This means that, exactly as for the Dirichlet problem, we have found a formulation of (8.13) — equivalent to the variational formulation, thus implying existence and uniqueness of a solution — that allows to put all the derivatives on the test functions.
This formulation can be useful, for example, to simplify the proof of the convergence of the finite volume discretization of the Neumann problem on polygonal open sets (see [37]): (8.15) allows to prove that the finite volume approximation converges to the variational solution without the need of a discrete trace theorem, with the same methods as in the Dirichlet case.

With Theorem 8.4 below, we can do the same for some mixed Dirichlet-Neumann problems.

8.4.2 Application to the convergence of a finite volume scheme

In [45], the authors prove the convergence of a finite volume scheme for a diffusion problem with mixed Dirichlet-Neumann-Signorini boundary conditions. It is classical, when studying finite volume schemes, to consider polygonal open sets of $\mathbb{R}^N$ (see [37]); in [45], the authors must however make an additional assumption on the open set: they must suppose that the open set is convex.
This restriction comes from the same restriction as in Remark 8.2: the authors need that an element of a Hodge decomposition be in H^2, which is ensured by the convexity of the open set (since this element comes from the resolution of a Neumann problem).

Theorem 8.2 allows us to see that the results of [45] are still true without the convexity hypothesis on the open set; moreover, it also simplifies quite a lot the proof of the result in [45] in which the Hodge decomposition was involved (with Theorem 8.2, the functions appearing in this proof are not only in H^2, but also $\mathcal{C}^\infty$-continuous, which makes the error estimates easier to obtain).

We will talk about another application of our results to finite volume scheme in item ii) of Remark 8.6.

8.4.3 Counter-example

Though polygonal open sets are not very regular, the singularities of their boundaries are of a kind that allows the density in $W^{1,p}$ of $\mathcal{C}^\infty$-continuous functions satisfying a Neumann boundary condition.
There is no similar result for general open sets with only Lipschiz continuous boundary; the loss of regularity noticed in Theorem 8.1 gives us the intuition of this (for open sets with $\mathcal{C}^{k,1}$-continuous boundary, we only get the density of $\mathcal{C}^{k-1,1}$-contiuous functions), and the following example gives us the proof of this intuition.

Let $(s_n)_{n\geq 1}$ be an enumeration of the rationals in $]-1,1[$ and $\eta(s) = \sum_{n\geq 1} 2^{-n}\sup(0, s-s_n) - c$ (where c is chosen so that $\eta(0)=0$); η is Lipschitz continuous on $]-1,1[$ and its derivative is $\eta'(s) = \sum_{n\geq 1} 2^{-n}\mathbf{1}_{]s_n,1[}(s) = \sum_{n\,|\,s>s_n} 2^{-n}$ ($\mathbf{1}_{]s_n,s[}$ is the characteristic function of the set $]s_n,s[$). Let Ω be an open set of $\mathbb{R}^2$ with a Lipschitz continuous boundary and such that $\Omega\cap]-1,1[\times]-1,1[=\{(s,t)\in\,]-1,1[\times]-1,1[\ |\ t>\eta(s)\}$; we will denote $\Lambda = \{(s,\eta(s))\ ,\ s\in]-1,1[\}\subset\partial\Omega$.
Let $\varphi\in\mathcal{C}^1(\mathbb{R}^N)$ such that $\nabla\varphi\cdot\mathbf{n}=0$ on $\partial\Omega$. We have then, for a.e. $s\in]-1,1[$,

$$0 = (\sqrt{1+|\eta'(s)|^2})\nabla\varphi(s,\eta(s))\cdot\mathbf{n}(s,\eta(s)) = \eta'(s)\frac{\partial\varphi}{\partial x_1}(s,\eta(s)) - \frac{\partial\varphi}{\partial x_2}(s,\eta(s)). \tag{8.16}$$

Let $n \geq 1$; there exists two sequences $(s_k^{n,+})_{k\geq 1}$ and $(s_k^{n,-})_{k\geq 1}$ converging to s_n and such that $s_k^{n,+} > s_n$, $s_k^{n,-} < s_n$ and (8.16) is satisfied for all $s \in \{s_k^{n,+}, s_k^{n,-}\ ,\ k \geq 1\}$. We have then $\eta'(s_k^{n,+}) \to \sum_{m\,|\,s_n \geq s_m} 2^{-m}$ and $\eta'(s_k^{n,-}) \to \sum_{m\,|\,s_n > s_m} 2^{-m}$ as $k \to \infty$.
By substracting (8.16) applied to $s_k^{n,-}$ to (8.16) applied to $s_k^{n,+}$, and then passing to the limit $k \to \infty$, we get $2^{-n}\frac{\partial \varphi}{\partial x_1}(s_n, \eta(s_n)) = 0$ for all $n \geq 1$; $(s_n)_{n\geq 1}$ being dense in $]-1,1[$, we deduce that the continuous function $\frac{\partial \varphi}{\partial x_1}(\cdot, \eta(\cdot))$ is null on $]-1,1[$ and, thanks to (8.16), that $\frac{\partial \varphi}{\partial x_2}(\cdot, \eta(\cdot))$ is also null on $]-1,1[$. Thus, $\varphi(\cdot, \eta(\cdot))$ is constant on $]-1,1[$: φ is constant on Λ.
Any limit in $W^{1,p}(\Omega)$ of functions $\varphi \in \mathcal{C}^1(\mathbb{R}^N)$ satisfying $\nabla\varphi \cdot \mathbf{n} = 0$ on $\partial\Omega$ is thus constant σ-a.e. on Λ; since there exists functions in $W^{1,p}(\Omega)$ that are not constant σ-a.e. on Λ (for example, $u(x) = x_1$), we deduce that $\{\varphi_{|\Omega}\ ,\ \varphi \in \mathcal{C}^1(\mathbb{R}^N)\ ,\ \nabla\varphi \cdot \mathbf{n} = 0 \text{ sur } \partial\Omega\}$ cannot be dense in $W^{1,p}(\Omega)$.

8.4.4 Mixed Dirichlet-Neumann boundary conditions

Let Γ be a measurable subset of $\partial\Omega$; we denote by $E^k_\Gamma(\Omega)$ the set of functions in $E^k(\Omega)$ the supports of which do not intersect Γ (such functions are, in particular, null on Γ).
$W^{1,p}_\Gamma(\Omega)$ is the space of functions in $W^{1,p}(\Omega)$ the trace of which is null σ-a.e. on Γ (if $u \in W^{1,p}(\Omega)$ is a limit in $W^{1,p}(\Omega)$ of a sequence in $E^k_\Gamma(\Omega)$, then we have $u \in W^{1,p}_\Gamma(\Omega)$). It is endowed with the same norm as $W^{1,p}(\Omega)$.

By denoting $B_+ = \{(y', y_N) \in B_N(1) \mid y_N > 0\}$, $D = \{(y', 0) \in B_N(1)\}$, $B_{++} = \{(y'', y_{N-1}, y_N) \in B_+ \mid y_{N-1} > 0\}$, $D_+ = \{(y'', y_{N-1}, 0) \in D \mid y_{N-1} \geq 0\}$, we make the additional assumption:

$$\begin{array}{c}\Gamma \text{ is closed and, for all } a \in \Gamma, \text{ there exists an open } U \text{ of } \mathbb{R}^N \text{ containing } a\\ \text{and a Lipschitz continuous homeomorphism } \phi : U \to B_N(1) \text{ with a}\\ \text{Lipschitz continuous inverse mapping such that one of the following cases occurs:}\\ \text{i) } U\cap\Gamma = U\cap\partial\Omega\ ,\ \phi(U\cap\Omega) = B_+ \text{ and } \phi(U\cap\partial\Omega) = D,\\ \text{ii) } \left\{\begin{array}{l}\phi(U\cap\Omega) = B_{++}\ ,\ \phi(U\cap\partial\Omega) = D_+ \cup \{(y'', 0, y_N) \in B_N(1) \mid y_N > 0\}\\ \text{and } \phi(U\cap\Gamma) = D_+\end{array}\right.\end{array} \quad (8.17)$$

An important example of a Γ satisfying this property is, when $\Omega = O\times]0,T[$ with O open set of $\mathbb{R}^{N-1}$ with a Lipschitz continuous boundary, $\Gamma = \overline{O} \times \{T\}$.

Theorem 8.4 *Let $p \in [1,+\infty[$. If $k \in \mathbb{N}\backslash\{0\}$ or $k = \infty$, Ω is an open set of $\mathbb{R}^N$ with a $\mathcal{C}^{k,1}$-continuous boundary and $\Gamma \subset \partial\Omega$ satisfies (8.17), then $E^{k-1}_\Gamma(\Omega)$ is dense in $W^{1,p}_\Gamma(\Omega)$. If Ω is a polygonal open set of $\mathbb{R}^N$ and $\Gamma \subset \partial\Omega$ satisfies (8.17), then $E^\infty_\Gamma(\Omega)$ is dense in $W^{1,p}_\Gamma(\Omega)$.*

Remark 8.6 *i) Of course, we have the same kind of results when we can locally transform the open set with a diffeomorphism that preserves the outer normal (see Remark 8.5); for example, if $\Omega = O\times]0,T[$ with O open set of $\mathbb{R}^{N-1}$ with a $\mathcal{C}^{k+1,1}$-continuous boundary ($k \geq 1$) and $\Gamma = \overline{O} \times \{T\}$, we can prove the density of $E^k_\Gamma(\Omega)$ in $W^{1,p}_\Gamma(\Omega)$.*

ii) In [55], the author uses a similar result to prove the convergence of a finite volume scheme for a diffusion and non-instantaneous dissolution problem in porous medium, when the medium is represented by an open set with regular boundary (at least $\mathcal{C}^3$-continuous, see item i) of Remark 8.1). Theorem 8.4 allows to extend the results of [55] to polygonal open sets, which are quite natural when dealing with finite volume schemes (see [37]).

Proof of Theorem 8.4
Step 1: we prove that any function $u \in W^{1,p}_\Gamma(\Omega)$ can be approximated in $W^{1,p}(\Omega)$ by functions in $\mathcal{C}^\infty_c(\mathbb{R}^N)$ the supports of which do not intersect Γ.

Cover first the compact set Γ by a finite number of mappings $(U_i, \phi_i)_{i\in[1,l]}$ given by (8.17). We take, for $i \in [1, l]$, $\theta_i \in \mathcal{C}_c^\infty(U_i)$ such that $\sum_{i=1}^l \theta_i \equiv 1$ on the neighborhood of Γ.
Define, for $i \in [1, l]$, $u_i = \theta_i u$. The function $v_i = u_i \circ \phi_i^{-1}$ is in $W^{1,p}_{\phi_i(U_i\cap\Gamma)}(\phi_i(U_i \cap \Omega))$ and its support is relatively compact in $B_N(1)$ (it is included in the support of $\theta_i \circ \phi_i^{-1}$). We will now handle separately the cases when (U_i, ϕ_i) satisfies i) or ii) in (8.17).

- Case i): we have $v_i \in W^{1,p}_D(B_+)$; the extension w_i of v_i to $B_N(1)$ by 0 outside B_+ is then in $W^{1,p}(B_N(1))$ and its support is a compact subset of $B_N(1)$ included in $\overline{B_+}$; the extension $\widetilde{w_i}$ of w_i to $\mathbb{R}^N$ by 0 outside $B_N(1)$ is thus in $W^{1,p}(\mathbb{R}^N)$. Let $f_{i,n}(y) = \widetilde{w_i}(y', y_N - 1/n)$; $f_{i,n}$ is in $W^{1,p}(\mathbb{R}^N)$ and the sequence $(f_{i,n})_{n\geq 1}$ converges to w_i in $W^{1,p}(B_N(1))$ (thus to v_i in $W^{1,p}(\phi_i(U_i \cap \Omega))$); moreover, for n large enough, $\mathrm{supp}(f_{i,n}) \subset \mathrm{supp}(w_i) + (0, \dots, 0, 1/n)$ is a compact subset of B_+ and does not intersect $\phi_i(U_i \cap \Gamma) = D$.
- Case ii): we have $v_i \in W^{1,p}_{D_+}(B_{++})$. The function $\widetilde{v_i} : B_+ \to \mathbb{R}$ defined a.e. by $\widetilde{v_i} = v_i$ on B_{++} and $\widetilde{v_i}(y) = v_i(y'', -y_{N-1}, y_N)$ if $y_{N-1} < 0$ is in $W^{1,p}_D(B_+)$ and its support is relatively compact in $B_N(1)$. By the reasoning made in Case i), there exists thus $(f_{i,n})_{n\geq 1} \in W^{1,p}(\mathbb{R}^N)$ which converges to $\widetilde{v_i}$ in $W^{1,p}(B_+)$ (thus to v_i in $W^{1,p}(\phi_i(U_i \cap \Omega))$) and such that, for n large enough, $\mathrm{supp}(f_{i,n})$ is a compact subset of $B_N(1)$ that does not intersect $D \supset \phi_i(U_i \cap \Gamma) = D_+$.

In both cases, taking n large enough, the support of $g_{i,n} = f_{i,n} \circ \phi_i \in W^{1,p}(U_i)$ is compact in U_i, does not intersect $U_i \cap \Gamma$ and the sequence $(g_{i,n})_{n\geq 1}$ converges to u_i in $W^{1,p}(U_i \cap \Omega)$.
For n large enough, the extension $G_{i,n}$ of $g_{i,n}$ by 0 outside U_i is thus in $W^{1,p}(\mathbb{R}^N)$, has a compact support which does not intersect Γ and, since $\mathrm{supp}(u_i)$ is relatively compact in U_i, we have $G_{i,n} \to u_i$ in $W^{1,p}(\Omega)$ as $n \to \infty$.
Let, for n large enough, $\mathcal{U}_n = \sum_{i=1}^l G_{i,n}$; this function of $W^{1,p}(\mathbb{R}^N)$ has a compact support which does not intersect Γ and $\mathcal{U}_n \to (\sum_{i=1}^l \theta_i)u$ in $W^{1,p}(\Omega)$.
Since $\mathrm{supp}(\mathcal{U}_n)$ is a compact set that does not intersect Γ, there exists $\mathcal{V}_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ the support of which does not intersect Γ and such that $||\mathcal{U}_n - \mathcal{V}_n||_{W^{1,p}(\mathbb{R}^N)} \leq 1/n$ (take $(\rho_m)_{m\geq 1}$ a sequence of mollifiers such that $\mathrm{supp}(\rho_m) \subset B_N(1/m)$; since $\mathrm{supp}(\mathcal{U}_n * \rho_m) \subset \mathrm{supp}(\mathcal{U}_n) + B_N(1/m)$ and $\mathrm{supp}(\mathcal{U}_n)$ and Γ are disjoint compact sets, for m large enough, one has $\mathrm{supp}(\mathcal{U}_n * \rho_m) \cap \Gamma = \emptyset$; since $\mathcal{U}_n * \rho_m \to \mathcal{U}_n$ in $W^{1,p}(\mathbb{R}^N)$ as $m \to \infty$, one sees that, for a m large enough, $\mathcal{V}_n = \mathcal{U}_n * \rho_m$ is convenient).
Let $\mathcal{W}_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\mathcal{W}_n \to u$ in $W^{1,p}(\Omega)$.
The sequence of functions $\mathcal{V}_n + (1 - \sum_{i=1}^l \theta_i)\mathcal{W}_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ converges to $(\sum_{i=1}^l \theta_i)u + (1 - \sum_{i=1}^l \theta_i)u = u$ in $W^{1,p}(\Omega)$ and $\mathrm{supp}(\mathcal{V}_n + (1 - \sum_{i=1}^l \theta_i)\mathcal{W}_n) \subset \mathrm{supp}(\mathcal{V}_n) \cup \mathrm{supp}(1 - \sum_{i=1}^l \theta_i)$, that is to say a compact set that does not intersect Γ. This concludes Step 1.

Step 2: To prove the theorem, it is thus sufficient to approximate, in $W^{1,p}(\Omega)$, any function $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ the support of which does not intersect Γ by functions in $E^{k-1}(\Omega)$ the supports of which do not intersect Γ.

Let K be a compact subset of $\mathbb{R}^N$ containing $\mathrm{supp}(u)$ in its interior and such that $K \cap \Gamma = \emptyset$.
In the case when Ω has a $\mathcal{C}^{k,1}$-continuous boundary (for a $k \geq 1$), Theorem 8.3 applied to these u and K concludes the proof.
In the case when Ω is a polygonal open set, Lemma 8.3 allows to see, by induction, that there exists a sequence of functions $u_n \in \mathcal{C}_c^\infty(\mathrm{int}(K))$ satisfying $\mathcal{B}_1$ and converging to u in $W^{1,p}(\Omega)$. Since the restrictions to Ω of functions satisfying $\mathcal{B}_1$ are in $E^\infty(\Omega)$ (as seen in the proof of Theorem 8.2), this concludes the proof. ∎

Chapitre 9

Convergence of a finite volume - mixed finite element method for a system of a hyperbolic and an elliptic equations

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Abstract : This paper gives a proof of convergence for the approximate solution of a system of an elliptic equation and of a hyperbolic equation, describing the conservation of two immiscible incompressible phases flowing in a porous medium. The approximate solution is obtained by a mixed finite element method on a large class of meshes for the elliptic equation and a finite volume method for the hyperbolic equation. Since the considered meshes are not necessarily structured, the proof uses a weak total variation inequality, which cannot yield a BV-estimate. We thus prove, under an L^∞ estimate, the weak convergence of the finite volume approximation. The strong convergence proof is then sketched under regularity assumptions which ensure that the fluxes are Lipschitz-continuous.

9.1 Introduction

The purpose of oil reservoir simulation implies to account for several phenomena such as chemical reactions, thermodynamical equilibrium and polyphasic flows. Since the full model is complex too much, a simplified model, describing the flow of two incompressible immiscible fluids through a porous medium, has been extensively studied. In this simplified model, two fluid phases, oil and water, flow through the pores of some possibly heterogeneous and anisotropic porous medium; water is injected through injection wells in order to displace the oil towards production wells. Here we neglect the gravity effects as well as the capillary pressure, which leads to the study of a first order conservation law for the saturation of one of the phases coupled with an elliptic equation for the pressure. Assuming the total mobility of the two phases to be constant and the mobility of water to be linear, the conservation equations of the two phases in a domain Ω yield the following system of equations.

$$u_t(x,t) - \mathrm{div}(u(x,t)\mathbf{\Lambda}(x)\nabla p(x)) = s(x,t)f^+(x) - u(x,t)f^-(x),$$

$$(1-u)_t(x,t) - \mathrm{div}((1-u(x,t))\mathbf{\Lambda}(x)\nabla p(x)) = (1-s(x,t))f^+(x) - (1-u(x,t))f^-(x),$$

for $(x,t) \in \Omega\times\mathbb{R}^+$. In the above equations, the saturation of the water phase is denoted by u, the common pressure of both phases is denoted by p. The absolute permeability $\mathbf{\Lambda}$ is a symmetric definite positive matrix (in anisotropic media, the eigenvalues of the matrix $\mathbf{\Lambda}$ are not all identical) which depends on the space variable in heterogeneous media (the symmetry hypothesis has no influence on the mathematical

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study of the problem). The function f represents the internal source terms, corresponding to the presence of wells drilled into the reservoir (f^+ and f^- denote the positive and negative parts of f). A positive source term corresponds to an injection well, a negative one corresponds to a production well. The function s represents the fraction of the water phase in the injected source term, and the saturation u of the water in place is the fraction of water in the produced source term. This problem, completed with initial and boundary conditions, is rewritten as follows.

$$u_t(x,t) + \text{div}(u\mathbf{q})(x,t) + u(x,t)f^-(x) = s(x,t)f^+(x) \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}^+, \tag{9.1}$$

$$\mathbf{\Lambda}^{-1}(x)\mathbf{q}(x) + \nabla p(x) = 0 \text{ for a.e. } x \in \Omega, \tag{9.2}$$

$$\text{div } \mathbf{q}(x) = f(x) \text{ for a.e. } x \in \Omega, \tag{9.3}$$

$$\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \text{ for a.e. } x \in \partial\Omega, \tag{9.4}$$

$$u(x,t) = \overline{u}(x,t) \text{ for a.e. } (x,t) \in \partial\Omega^- \times \mathbb{R}^+, \tag{9.5}$$

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \Omega, \tag{9.6}$$

Notice that the boundary condition for the saturation is only given on the part $\partial\Omega^-$ of the boundary where the flow enters into the domain, that means $\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \leq 0$.
In Eqs (9.1)-(9.6) (refered in the following as Problem (P)) the following hypotheses (refered in the following as Hypotheses (H)) are used.

Hypotheses (H):

1. *Ω is an open bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3 in pratical) such that, locally, Ω either has a $\mathcal{C}^{1,1}$ regular boundary or is convex.*
2. *$\mathbf{\Lambda}(x)$ is a measurable application from Ω to the set of symmetric real $d \times d$ matrices, such that there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfying $\lambda_1|z| \leq |\mathbf{\Lambda}(x)z| \leq \lambda_2|z|$ for almost every $x \in \Omega$ and all $z \in \mathbb{R}^d$.*
3. *$f \in L^2(\Omega)$.*
4. *$g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ for some $\mathbf{q}_0 \in (H^1(\Omega))^d$ and*

$$\int_\Omega f(x)\,dx - \int_{\partial\Omega} g(x)\,d\gamma(x) = 0.$$

5. *$\overline{u} \in L^\infty(\partial\Omega^- \times \mathbb{R}^+)$ where $\partial\Omega^- = \{x \in \partial\Omega, g(x) \leq 0\}$.*
6. *$u_0 \in L^\infty(\Omega)$.*
7. *$s \in L^\infty(\Omega \times \mathbb{R}^+)$.*

Here and in the following, when U is an open subset of $\mathbb{R}^d$ with a sufficient regular boundary (see Definition 9.2), we denote by $\mathbf{n}_{\partial U}$ the unit normal to ∂U outward to U and by γ the $(d-1)$-dimensional measure on ∂U. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ and $x \cdot y$ denotes the Euclidean scalar product of $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. When X is a subset of $\mathbb{R}^d$, $\delta(X)$ denotes the diameter of X, that is to say $\delta(X) = \sup_{(x,y)\in X^2} |x-y|$. $B(z,r)$ denotes the Euclidean ball of center $z \in \mathbb{R}^d$ and radius $r > 0$.

Remark 9.1 *Since we allow Ω to have a non-regular boundary, there is no convenient way to characterize the regularity condition on g. Indeed, if Ω has a $\mathcal{C}^{1,1}$-regular boundary, it is easy to see that $g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ if and only if $g \in H^{1/2}(\partial\Omega)$, but on the non-regular parts of $\partial\Omega$, this condition is not necessary and it is not even obvious that it is sufficient. For example, take $\Omega =]0,1[^2$, $g = 1$ on $(\{0\}\times]0,1[) \cup (\{1\}\times]0,1[)$ and $g = 0$ on $(]0,1[\times\{0\}) \cup (]0,1[\times\{1\})$; then g does not belong to $H^{1/2}(\partial\Omega)$, but g can be written as $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ with $\mathbf{q}_0(x,y) = (-1+2x, 0) \in (H^1(\Omega))^2$.*

A weak solution of Problem (P) is defined by the following sense.

Definition 9.1 *Under Hypotheses (H), a weak solution of (P) is given by $(u,p,\mathbf{q}) \in L^\infty(\Omega \times \mathbb{R}^+) \times L^2(\Omega) \times H_g(\mathrm{div},\Omega)$ such that*

$$\int_{\mathbb{R}^+}\int_\Omega u(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + \mathbf{q}(x)\cdot\nabla\phi(x,t) - \phi(x,t)f^-(x)\right)dx\,dt =$$
$$-\int_\Omega u_0(x)\phi(x,0)\,dx + \int_{\mathbb{R}^+}\int_{\partial\Omega^-} \overline{u}(x,t)\phi(x,t)g(x)\,d\gamma(x)\,dt - \int_{\mathbb{R}^+}\int_\Omega \phi(x,t)s(x,t)f^+(x)\,dx\,dt, \tag{9.7}$$

$\forall \phi \in C_c^1(\mathbb{R}^d \times \mathbb{R})$ *such that* $\phi = 0$ *on* $\partial\Omega^+ \times \mathbb{R}^+ = (\partial\Omega \setminus \partial\Omega^-) \times \mathbb{R}^+$,

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{q}(x)\,dx - \int_\Omega p(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = 0,\ \forall \mathbf{y} \in H_0(\mathrm{div},\Omega), \tag{9.8}$$

$$\int_\Omega v(x)\mathrm{div}\ \mathbf{q}(x)\,dx = \int_\Omega f(x)v(x)\,dx,\ \forall v \in L^2(\Omega), \tag{9.9}$$

and

$$\int_\Omega p(x)\,dx = 0. \tag{9.10}$$

where the function spaces $H(\mathrm{div},\Omega)$, $H_0(\mathrm{div},\Omega)$ and $H_g(\mathrm{div},\Omega)$ are defined by

$$H(\mathrm{div},\Omega) = \{\mathbf{q} \in (L^2(\Omega))^d\,,\ \mathrm{div}\ \mathbf{q} \in L^2(\Omega)\}\,,\quad H_0(\mathrm{div},\Omega) = \{\mathbf{q} \in H(\mathrm{div},\Omega)\,,\ \mathbf{q}\cdot\mathbf{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$$
$$\text{and}\quad H_g(\mathrm{div},\Omega) = \{\mathbf{q} \in H(\mathrm{div},\Omega)\,,\ \mathbf{q}\cdot\mathbf{n}_{\partial\Omega} = g \text{ on } \partial\Omega\}.$$

A number of numerical schemes for this problem in the case $\mathbf{\Lambda} = Id$ have already been discussed in the literature. Nevertheless, the numerical schemes used to approximate the solution of this simplified model, which is a system of an elliptic equation and a scalar hyperbolic equation, have only recently be studied from a convergence point of view. In particular Eymard and Gallouët [34] have proven the convergence of a numerical scheme involving a finite volume method for the computation of the saturation u and a standard finite element for the computation of the pressure p whereas Vignal [73] presents a convergence proof for a finite volume method for the discretization of both equations. Here we also discretize the conservation law for the saturation by means of a finite volume method but apply the mixed finite element method to discretize the elliptic equation. Error estimates have been derived by Jaffré and Roberts [47] for a semi-discretized problem in the simulation of miscible displacements involving an elliptic equation for the pressure coupled to a parabolic equation for saturation. For the numerical discretization they combine the mixed finite element method with an upstream weighting scheme. More recently Ölberger [60] has derived error estimates in the case that the finite volume method is applied for the discretization of a parabolic equation instead of the first order conservation law (9.1).

Here we deal with a mixed finite element method with an original basis for the elliptic equation. On a partition of the domain, the hypotheses on which are very large, we define the generalization of the

Raviart-Thomas space. The proof of the "inf-sup" condition and the proof that the interpolation error of regular functions tends to zero with the space step make use of Lipschitz-continuous homeomorphisms with Lipschitz-continuous inverse mappings and of some trace inequalities, for which the constants are given as functions of the size of the domain (the classical proofs of the trace inequalities with null average, by the way of contradiction, being unable to yield the relation of such constants with the domain). Then the hyperbolic equation is discretized by the classical upstream weighting scheme, and the convergence proof of the scheme is obtained, thanks to some "weak BV" inequalities. Such inequalities have only recently be introduced and proved for the proof of convergence of finite volume schemes on unstructured meshes for hyperbolic equations, since strong BV estimates have not been actually obtained on the discrete approximation. Thus this paper completes a lot of previous numerical works in which this scheme has been used on particular meshes (generally triangular meshes).
The organization of this paper is as follows. In Section 9.2, we present the numerical scheme that we use. In Section 9.3, we prove a convergence result for the mixed finite element method. In Section 9.4, we deal with the finite volume scheme, concluding to the weak convergence of a subsequence without additional regularity hypotheses on the data, and to the strong convergence otherwise.

9.2 The discretization

9.2.1 Admissible discretizations

In order to define the scheme, a notion of admissible discretization is given, which is used below in the definition of approximate discrete solutions.

Definition 9.2 (Admissible discretization of Ω) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary. An admissible discretization $\mathcal{D}$ of Ω is given by a finite set $\mathcal{M}$ of open subsets $K \subset \Omega$ with weakly Lipschitz-continuous boundaries and a finite set $\mathcal{A}$ of disjoint subsets $a \subset \overline{\Omega}$ such that:*

(i) $\cup_{K\in\mathcal{M}}\overline{K} = \overline{\Omega}$,

(ii) For all $K \in \mathcal{M}$, there exists a Lipschitz-continuous homeomorphism $\mathcal{L}_K$ from $\overline{K}$ to $\overline{B(0,\delta(K))}$ such that the inverse mapping is Lipschitz-continuous as well. One denotes by C_K the maximum value of both Lipschitz constants and by m_K the Lebesgue measure of K.

(iii) For all $(K,L) \in \mathcal{M}^2$ with $K \neq L$, one has $K \cap L = \emptyset$.

(iv) For all $a \in \mathcal{A}$, there exists $K \in \mathcal{M}$ such that a is a non-empty open subset of ∂K. By denoting $\mathcal{A}_K = \{a \in \mathcal{A} \mid a \subset \partial K\}$, we assume that $\partial K = \cup_{a\in\mathcal{A}_K}\overline{a}$. We denote by m_a the $(d-1)$-dimensional measure of a.

(v) The sets $\mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{A}_e \subset \mathcal{A}$ are defined by $A_i = \{a \in \mathcal{A}, \exists (K,L) \in \mathcal{M}^2,\ K \neq L,\ a \subset \partial K \cap \partial L\}$ and $A_e = \{a \in \mathcal{A}, \exists K \in \mathcal{M}, a \subset \partial K \cap \partial\Omega\}$ ([4]). One assumes that $(\mathcal{A}_i, \mathcal{A}_e)$ forms a partition of $\mathcal{A}$.

(vi) For all $a \in \mathcal{A}_i$, one between the two different $(K,L) \in \mathcal{M}^2$ such that $a \subset \partial K \cap \partial L$ is selected. Then we denote $K(a) = K$ and $L(a) = L$, and we set $\varepsilon_{K,a} = 1$ and $\varepsilon_{L,a} = -1$. The normal vector $\mathbf{n}_a(x)$ to a at $x \in a$ is defined by $\mathbf{n}_a(x) = \mathbf{n}_{\partial K}(x) = -\mathbf{n}_{\partial L}(x)$([5]). For all $a \in \mathcal{A}_e$, let K be the unique element of $\mathcal{M}$ such that $a \subset \partial K \cap \partial\Omega$. Then one denotes $K(a) = K$ and $\varepsilon_{K,a} = 1$. The normal vector $\mathbf{n}_a(x)$ to a at $x \in a$ is defined by $\mathbf{n}_a(x) = \mathbf{n}_{\partial\Omega}(x) = \mathbf{n}_{\partial K}(x)$ ([6]).

(vii) For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one assumes that there exists $x_{K,a} \in a$ and $\zeta_{K,a} > 0$ such that $a \supset \partial K \cap B(x_{K,a}, \zeta_{K,a}\delta(K))$.

[4] One can then show that, when $a \in \mathcal{A}_i$, the $\{K,L\} \subset \mathcal{M}$ such that $K \neq L$ and $a \subset \partial K \cap \partial L$ are unique; this is the same, when $a \in A_e$, for the $K \in \mathcal{M}$ such that $a \subset \partial K \cap \partial\Omega$.
[5] We can indeed show that, in such a situation, we have $\mathbf{n}_{\partial K} = -\mathbf{n}_{\partial L}$ on a.
[6] As for the preceding case, this equality between $\mathbf{n}_{\partial\Omega}$ and $\mathbf{n}_{\partial K}$ is not supposed, it can be proved.

By denoting $\overline{\mathbf{n}_a}$ the mean value of $\mathbf{n}_a$ on a, the thinness of the discretization $\mathcal{D}$ (controling the size of $\mathcal{D}$ and the behaviour of the edges of $\mathcal{D}$) is defined by

$$\mathrm{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}}\left(\delta(K), \max_{a\in\mathcal{A}_K}\left(\frac{1}{\sqrt{m_a}}||\mathbf{n}_a - \overline{\mathbf{n}_a}||_{L^2(a)}\right)\right) \tag{9.11}$$

and a geometrical factor, linked to the regularity of the discretization, is defined by

$$\mathrm{regul}(\mathcal{D}) = \max_{K\in\mathcal{M}}\left(C_K, \max_{a\in\mathcal{A}_K}\left(\frac{1}{\zeta_{K,a}}\right)\right). \tag{9.12}$$

Remark 9.2 *The definition of an open set with weakly Lipschitz-continuous boundary is given in [31] or in [42] under the name "d-dimensional Lipschitz-continuous submanifold of $\mathbb{R}^d$". It is far weaker than the definition of Lipschitz-continuous boundary given in [58].*

Remark 9.3 *The above definition is easily satisfied for a large variety of meshes. In the case $d = 2$, subsets K such that ∂K is defined in polar coordinates from an origin $M_K \in K$ by a 2π- periodic continuous piecewise C^1 function satisfy condition (ii). That is the case for convex polyhedra, such as triangles or parallelograms for example.*

Remark 9.4 *In the above definition, one cannot define edges by the sets $\partial K \cap \partial L$ or $\partial K \cap \partial\Omega$; indeed,* $\mathrm{thin}(\mathcal{D})$ *is destined to tend to* 0 *(in order to obtain the convergence results), which can lead to share the sets $\partial K \cap \partial L$ in different edges. In fact,* $\mathrm{thin}(\mathcal{D}) \to 0$ *means that the size of the discretization tends to* 0 *and that the edges become more and more planar.*
Notice that if Ω is polyhedral and the edges are planes, then $\mathrm{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}}\delta(K)$ *is simply the size of the discretization.*

Remark 9.5 *Hypothesis (vii) is only used for the study of the convergence of the finite volume scheme to the solution of the hyperbolic equation. It is not used in the proof of convergence of the mixed finite element method. Notice that this hypothesis, along with Hypothesis (ii) and Lemma 9.11, implies $m_a \geq C\delta(K)^{d-1}$, where C only depends on d, C_K and $\zeta_{K,a}$.*

9.2.2 Discrete function spaces

One now defines the set of basis functions for the mixed finite element method, which is a generalization of the Raviart-Thomas space $RT_0^0(\mathcal{M})$ (see [14], [72] or [59]).

Definition 9.3 (Discrete function spaces) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2. For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one denotes by $w_{K,a} \in H^1(K)$ the unique variational solution with $\int_K w_{K,a}(x)\,dx = 0$ of the Neumann problem*

$$\Delta w_{K,a}(x) = \frac{m_a}{m_K} \text{ for a.e. } x \in K,$$

and

$$\begin{aligned} \nabla w_{K,a}(x)\cdot\mathbf{n}_{\partial K}(x) &= 1 \quad \text{for a.e. } x \in a, \\ \nabla w_{K,a}(x)\cdot\mathbf{n}_{\partial K}(x) &= 0 \quad \text{for a.e. } x \in \partial K \setminus a. \end{aligned}$$

One then defines the function $\mathbf{w}_{K,a}$ from Ω to $\mathbb{R}^d$ by $\mathbf{w}_{K,a}(x) = \nabla w_{K,a}(x)$ for a.e. $x \in K$ and $\mathbf{w}_{K,a}(x) = 0$ for all $x \in \Omega \setminus K$.
One defines, for all $a \in \mathcal{A}_i$, $\mathbf{w}_a = \mathbf{w}_{K(a),a} - \mathbf{w}_{L(a),a}$ and, for all $a \in \mathcal{A}_e$, $\mathbf{w}_a = \mathbf{w}_{K(a),a}$. Then one gets $\mathbf{w}_a \in H(\mathrm{div},\Omega)$. The set $\mathbf{Q}_\mathcal{D} \subset H(\mathrm{div},\Omega)$ is the space generated by the functions $(\mathbf{w}_a)_{a\in\mathcal{A}}$, the set

$\mathbf{Q}_{\mathcal{D},0} \subset H_0(\mathrm{div},\Omega)$ *is the space generated by the functions* $(\mathbf{w}_a)_{a\in\mathcal{A}_i}$, *and for any* $b \in L^2(\partial\Omega)$, *the set* $\mathbf{Q}_{\mathcal{D},b} \subset \mathbf{Q}_{\mathcal{D}}$ *is the space* $\left\{\mathbf{q} + \sum_{a\in\mathcal{A}_e} \frac{1}{m_a}\int_a b(x)\,d\gamma(x)\ \mathbf{w}_a\ ,\ \mathbf{q}\in\mathbf{Q}_{\mathcal{D},0}\right\}$.

$V_{\mathcal{D}} \in L^2(\Omega)$ *is the space of functions* $f = \sum_{K\in\mathcal{M}} \alpha_K\ \chi_K$ *(where, for all* $K \in \mathcal{M}$, $\alpha_K \in \mathbb{R}$ *and* χ_K *is the characteristic function of* K*) such that* $\int_\Omega f(x)\,dx = \sum_{K\in\mathcal{M}} m_K\alpha_K = 0$.

9.2.3 The mixed finite element scheme

The mixed finite element approximate of (9.2)-(9.4) is a pair of functions

$$(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g},$$

solution of

$$\int_\Omega v(x)\mathrm{div}\ \mathbf{q}_{\mathcal{D}}(x)\,dx = \int_\Omega f(x)v(x)\,dx, \forall v \in V_{\mathcal{D}}, \tag{9.13}$$

and

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{q}_{\mathcal{D}}(x)\,dx - \int_\Omega p_{\mathcal{D}}(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = 0,\ \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}. \tag{9.14}$$

The unknown functions can be written as

$$\mathbf{q}_{\mathcal{D}} = \sum_{a\in\mathcal{A}} q_a\mathbf{w}_a$$

and

$$p_{\mathcal{D}} = \sum_{K\in\mathcal{M}} p_K\ \chi_K.$$

Then equations (9.13) and (9.14) lead to the following system of linear equations, with unknowns $(q_a)_{a\in\mathcal{A}}$ and $(p_K)_{K\in\mathcal{M}}$:

$$\sum_{a'\in\mathcal{A}} q_{a'} \int_\Omega \mathbf{w}_a(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{w}_{a'}(x)\,dx - m_a(p_{K(a)} - p_{L(a)}) = 0,\ \forall a\in\mathcal{A}_i,$$

$$q_a = g_a,\ \forall a \in \mathcal{A}_e,$$

$$\sum_{a\in\mathcal{A}_K} m_a q_a\ \varepsilon_{K,a} = f_K,\ \forall K\in\mathcal{M}, \tag{9.15}$$

$$\sum_{K\in\mathcal{M}} m_K p_K = 0,$$

where we denote

$$f_K = \int_K f(x)\,dx,\ \forall K\in\mathcal{M}, \tag{9.16}$$

and

$$g_a = \frac{1}{m_a}\int_a g(x)\,d\gamma(x),\ \forall a\in\mathcal{A}_e. \tag{9.17}$$

The existence and uniqueness of a solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}})$ to system (9.13)-(9.14) is stated in the following lemma.

Lemma 9.1 (Existence and uniqueness of the discrete approximation) *Let us assume hypotheses (H). Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2. Then system (9.13)-(9.14) defines one and only one approximate solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$.*

Proof. Since Lemma 9.4 (which is proved below) shows that the only solution of a linear system with the same matrix as (9.13)-(9.14) and a null right hand side is null, this matrix is invertible. This proves the lemma.

9.2.4 The finite volume scheme

One denotes, for all $K \in \mathcal{M}$ and $a \in \mathcal{A}_K$, $F_{K,a} = m_a q_a \varepsilon_{K,a}$ (then $F_{K(a),a} + F_{L(a),a} = 0$ holds for all $a \in \mathcal{A}_i$).

One then discretizes the hyperbolic problem. Let $\Delta t > 0$ be a constant time step. One defines a discrete source term

$$s_K^n = \frac{1}{\Delta t m_K} \int_{n\Delta t}^{(n+1)\Delta t} \int_K s(x,t)\, dx\, dt, \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}. \tag{9.18}$$

Prolonging by 0 the function $\overline{u}$ on $\partial\Omega^+ \times \mathbb{R}_+$, one defines

$$\overline{u}_a^n = \frac{1}{\Delta t m_a} \int_{n\Delta t}^{(n+1)\Delta t} \int_a \overline{u}(x)\, d\gamma(x)\, dt, \ \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}. \tag{9.19}$$

The discretization of the initial value (Eq. (9.6)) is given by

$$u_K^0 = \frac{1}{m_K} \int_K u_0(x) dx, \ \forall K \in \mathcal{M}. \tag{9.20}$$

The finite volume scheme discretization of equation (9.1) is written:

$$m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} + \sum_{a \in \mathcal{A}_K} u_a^n F_{K,a} = s_K^n f_K^+ - u_K^n f_K^-, \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}, \tag{9.21}$$

where u_a^n is defined by :

$$\begin{array}{l} u_a^n = u_{K(a)}^n \text{ if } q_a > 0, \text{ else } u_a^n = u_{L(a)}^n, \ \forall a \in \mathcal{A}_i, \ \forall n \in \mathbb{N} \\ u_a^n = u_{K(a)}^n \text{ if } q_a > 0, \text{ else } u_a^n = \overline{u}_a^n, \ \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}. \end{array} \tag{9.22}$$

For a given discretization $\mathcal{D}$ and a time step Δt, we can define the approximate solution by:

$$u_{\mathcal{D},\Delta t}(x,t) = u_K^n, \text{ for a.e. } (x,t) \in K \times [n\Delta t, (n+1)\Delta t), \ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}. \tag{9.23}$$

9.3 The convergence of the mixed method

One has the following result.

Theorem 9.1 (Convergence of the mixed finite element scheme) *Under Hypotheses (H), let ξ be a fixed positive real value and let $\mathcal{D}$ be a discretization of Ω in the sense of definition 9.2 such that* $\text{regul}(\mathcal{D}) \le \xi$. *Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10) and $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13)-(9.14).*
Then

$$\begin{array}{lll} \lim\limits_{\text{thin}(\mathcal{D}) \to 0} \|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|_{H(\text{div},\Omega)} & = & 0, \\ \lim\limits_{\text{thin}(\mathcal{D}) \to 0} \|p - p_{\mathcal{D}}\|_{L^2(\Omega)} & = & 0. \end{array} \tag{9.24}$$

In order to prove Theorem 9.1, some lemmata must be previously shown. The next lemma deals with an interpolation result for regular functions.

Lemma 9.2 (Interpolation of regular functions) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \text{regul}(\mathcal{D})$. Let $\mathbf{q} \in (H^1(\Omega))^d$. Let $\mathbf{y} \in H(\text{div}, \Omega)$ be defined by*

$$\mathbf{y} = \sum_{a \in \mathcal{A}} \frac{1}{m_a} \int_a \mathbf{q}(x) \cdot \mathbf{n}_a(x)\, d\gamma(x)\ \mathbf{w}_a.$$

Then we have $\text{div}\ \mathbf{y} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K \text{div}\ \mathbf{q}(x)\, dx\ \chi_K$ *and there exists $C_1 > 0$ which only depends on d and ξ such that*

$$\|\mathbf{q} - \mathbf{y}\|_{L^2(\Omega)} \leq C_1 \,\text{thin}(\mathcal{D}) \|\mathbf{q}\|_{(H^1(\Omega))^d}. \tag{9.25}$$

One can notice then that, when $\text{thin}(\mathcal{D}) \to 0$, the function y such defined tends to q in $H(\text{div}, \Omega)$.

Proof. In the following proof, C_i denotes different positive real values which only depend on ξ and d. The proof of $\text{div}\ \mathbf{y} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K \text{div}\ \mathbf{q}(x)\, dx\ \chi_K$ is straightforward, since $\text{div}\ \mathbf{w}_a = 0$ on K if $K \notin \{K(a), L(a)\}$ and $\text{div}\ \mathbf{w}_a = \varepsilon_{K,a} \frac{m_a}{m_K}$ on K if $K \in \{K(a), L(a)\}$.
Let $K \in \mathcal{M}$. Let us define the function $w \in H^1(K)$ by

$$w = \sum_{a \in \mathcal{A}_K} \left(\frac{1}{m_a} \int_a \mathbf{q}(x) \cdot \mathbf{n}_{\partial K}(x)\, d\gamma(x) \right)\ w_{K,a},$$

which is such that $\nabla w(x) = \mathbf{y}(x)$ for a.e. $x \in K$. Similarly, denoting $\tilde{\mathbf{q}} = \frac{1}{m_K} \int_K \mathbf{q}(x)\, dx$, we define $\tilde{w} \in H^1(K)$ by

$$\tilde{w} = \sum_{a \in \mathcal{A}_K} \left(\frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x)\, d\gamma(x) \right)\ w_{K,a}.$$

We get

$$\|\mathbf{q} - \mathbf{y}\|^2_{L^2(K)} \leq 3\|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)} + 3\|\tilde{\mathbf{q}} - \nabla\tilde{w}\|^2_{L^2(K)} + 3\|\nabla\tilde{w} - \nabla w\|^2_{L^2(K)}.$$

Let us first deal with $A = \|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)}$. Thanks to the Cauchy-Schwarz inequality, one has

$$A \leq \frac{1}{m_K} \int_K \int_K |\mathbf{q}(x) - \mathbf{q}(y)|^2\, dx\, dy,$$

which yields, using (9.63) proved in Lemma 9.13,

$$\|\mathbf{q} - \tilde{\mathbf{q}}\|^2_{L^2(K)} \leq C_2\, \delta(K)^2 \|\mathbf{q}\|^2_{(H^1(K))^d}. \tag{9.26}$$

We now turn to the study of $B = \|\tilde{\mathbf{q}} - \nabla\tilde{w}\|^2_{L^2(K)}$. One defines the function $h \in H^2(K)$ by $h(x) = \tilde{\mathbf{q}} \cdot x - \frac{1}{m_K} \int_K (\tilde{\mathbf{q}} \cdot y)\, dy$. This function thus satisfies $\nabla h = \tilde{\mathbf{q}}$ and $\int_K h(x)\, dx = 0$. Since $h - \tilde{w}$ is the variational solution of a Neumann problem on K with null average and $\Delta(h - \tilde{w})$ constant, one gets

$$B = \sum_{a \in \mathcal{A}_K} \int_a (h(x) - \tilde{w}(x)) \left(\tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x) - \frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(y)\, d\gamma(y) \right) d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, we deduce

$$B^2 \le B' \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - h(x))^2 \, d\gamma(x),$$

where

$$B' = \sum_{a\in\mathcal{A}_K} \int_a \left(\tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(x) - \frac{1}{m_a} \int_a \tilde{\mathbf{q}} \cdot \mathbf{n}_{\partial K}(y) \, d\gamma(y) \right)^2 d\gamma(x).$$

We use (9.56) proved in Lemma 9.12. It yields $\sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - h(x))^2 \, d\gamma(x) \le C_3 \, \delta(K) B$, thus we obtain

$$B \le C_3 \, \delta(K) B'. \tag{9.27}$$

We have, by definition of $\text{thin}(\mathcal{D})$,

$$\begin{aligned} \delta(K)B' &\le \delta(K)|\tilde{\mathbf{q}}|^2 \sum_{a\in\mathcal{A}_K} \int_a (\mathbf{n}_a(x) - \overline{\mathbf{n}_a})^2 \, d\gamma(x) \\ &\le \frac{\delta(K)}{m_K} \int_K |\mathbf{q}(x)|^2 \, dx \sum_{a\in\mathcal{A}_K} \text{thin}(\mathcal{D})^2 m_a \\ &\le C_4 \, \text{thin}(\mathcal{D})^2 \int_K |\mathbf{q}(x)|^2 \, dx \times \frac{\delta(K) m_{\partial K}}{m_K}. \end{aligned}$$

Using $m_K \ge C_5 \, \delta(K)^d$ and $m_{\partial K} \le C_6 \, \delta(K)^{d-1}$ (hypothesis (ii) of Definition 9.2 and Lemma 9.11), relation (9.27) gives

$$\|\tilde{\mathbf{q}} - \nabla \tilde{w}\|^2_{L^2(K)} \le C_7 \, \text{thin}(\mathcal{D})^2 \|\mathbf{q}\|^2_{L^2(K)}. \tag{9.28}$$

We finaly study the term $C = \|\nabla \tilde{w} - \nabla w\|^2_{L^2(K)}$. We have

$$C = \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x)) \left(\frac{1}{m_a} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y)) \cdot \mathbf{n}_{\partial K}(y) \, d\gamma(y) \right) d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, one has

$$C^2 \le C' \sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x))^2 \, d\gamma(x),$$

where

$$C' = \sum_{a\in\mathcal{A}_K} \int_a \left(\frac{1}{m_a} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y)) \cdot \mathbf{n}_{\partial K}(y) \, d\gamma(y) \right)^2 d\gamma(x)$$

Thanks again to (9.56) given by Lemma 9.12, we get $\sum_{a\in\mathcal{A}_K} \int_a (\tilde{w}(x) - w(x))^2 \, d\gamma(x) \le C_3 \, \delta(K) C$, which leads to

$$C \le C_3 \, \delta(K) C'. \tag{9.29}$$

Turning to the study of C', and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} C' &\leq \sum_{a\in\mathcal{A}_K} \int_a (\tilde{\mathbf{q}} - \mathbf{q}(y))^2 \, d\gamma(y) = \int_{\partial K} (\tilde{\mathbf{q}} - \mathbf{q}(y))^2 \, d\gamma(y) \\ &\leq \int_{\partial K} \frac{1}{m_K} \int_K (\mathbf{q}(z) - \mathbf{q}(y))^2 \, dz \, d\gamma(y). \end{aligned} \quad (9.30)$$

Thanks again to Lemma 9.12, we get

$$C' \leq C_2 \, \delta(K) \|\mathbf{q}\|^2_{(H^1(K))^d},$$

and therefore, thanks to (9.29) and (9.30), there exists $C_8 > 0$ such that

$$\|\nabla\tilde{w} - \nabla w\|^2_{L^2(K)} \leq C_8 \, \delta(K)^2 \|\mathbf{q}\|^2_{(H^1(K))^d}. \quad (9.31)$$

Summing relations (9.26), (9.28) and (9.31) on $K \in \mathcal{M}$ gives (9.25).

Lemma 9.3 *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $v \in V_{\mathcal{D}}$ and let $h \in H^2(\Omega)$ be the variational solution of $-\Delta h = v$ on Ω, with a homogeneous Neumann boundary condition and $\int_\Omega h(x)\,dx = 0$ (the existence of such a function resulting from the regularity hypotheses on Ω, see [42]). Let us define $\mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}$ by*

$$\mathbf{y} = \sum_{a\in\mathcal{A}} \left(\frac{1}{m_a} \int_a \nabla h(x) \cdot \mathbf{n}_a \, d\gamma(x) dx \right) \mathbf{w}_a. \quad (9.32)$$

Then there exists C_9, only depending on Ω, d and ξ such that $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq C_9 \, \|v\|_{L^2(\Omega)}$.

Proof.
Using $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq \|\mathbf{y} - \nabla h\|_{(L^2(\Omega))^d} + \|\nabla h\|_{(L^2(\Omega))^d}$, one applies Lemma 9.2 for $\mathbf{q} = \nabla h$, since $h \in H^2(\Omega)$ implies $\nabla h \in (H^1(\Omega))^d$. We thus get $\|\mathbf{y}\|_{(L^2(\Omega))^d} \leq (C_1 \,\mathrm{thin}(\mathcal{D}) + 1)\|h\|_{H^2(\Omega)}$. By hypothesis (H), one has $\|h\|_{H^2(\Omega)} \leq C_\Omega \|v\|_{L^2(\Omega)}$, which concludes the proof since $\mathrm{thin}(\mathcal{D}) \leq \max(\delta(\Omega), 2)$.

By noticing that the $\mathbf{y}$ defined by (9.32) satisfies div $\mathbf{y} = -v$, this lemma can also be stated in terms of an “inf-sup” condition.

Corollary 9.1 (Discrete “inf-sup” condition) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \mathrm{regul}(\mathcal{D})$. Then there exists $C_9 > 0$, only depending on Ω, d and ξ such that*

$$\inf_{v\in V_{\mathcal{D}}} \sup_{\mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}} \frac{\displaystyle\int_\Omega v(x) \ \mathrm{div}\ \mathbf{y}(x)\, dx}{\|v\|_{L^2(\Omega)} \ \|\mathbf{y}\|_{(L^2(\Omega))^d}} \geq \frac{1}{C_9}.$$

The following lemmata express the classical proof of the convergence of mixed finite element methods under an “inf-sup” condition and an interpolation result (detailed in [14] or [59] for example). We prove them for the sake of completeness, thus verifying that our hypotheses are sufficient to apply this convergence proof.

Lemma 9.4 (Estimate on the discrete approximations) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 9.2 and let $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $h \in L^2(\Omega)$ and $\mathbf{r} \in (L^2(\Omega))^d$ be given.*
Then, there exists one and only one $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},0}$ solution of

$$\int_\Omega \mathrm{div}\ \mathbf{q}_{\mathcal{D}}(x) \ v(x)\, dx = \int_\Omega h(x) \ v(x)\, dx, \ \forall v \in V_{\mathcal{D}}, \quad (9.33)$$

and

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)\mathbf{q}_\mathcal{D}(x)\,dx - \int_\Omega p_\mathcal{D}(x)\ \mathrm{div}\ \mathbf{y}(x)\,dx = \int_\Omega \mathbf{r}(x)\cdot\mathbf{y}(x)\,dx,\ \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}, \tag{9.34}$$

and there exists C_{10}, only depending on Ω, d, ξ, λ_1 and λ_2 such that

$$\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} + \|p_\mathcal{D}\|^2_{L^2(\Omega)} \le C_{10}\,(\|\mathbf{r}\|^2_{(L^2(\Omega))^d} + \|h\|^2_{L^2(\Omega)}). \tag{9.35}$$

Proof. We first remark that proving (9.35) for any solution $(p_\mathcal{D},\mathbf{q}_\mathcal{D})\in V_\mathcal{D}\times\mathbf{Q}_{\mathcal{D},0}$ to (9.33)-(9.34) is sufficient to prove that for a null right hand side, the discrete unknowns are null, and therefore that the linear system is invertible. For the proof of (9.35), one chooses, in (9.34), $\mathbf{y}=\mathbf{q}_\mathcal{D}$, and in (9.33), $v=p_\mathcal{D}$. It leads to

$$\frac{1}{\lambda_2}\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} \le \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d} + \|h\|_{L^2(\Omega)}\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{9.36}$$

One then applies Lemma 9.3, which gives the existence of $\mathbf{y}_0\in\mathbf{Q}_{\mathcal{D},0}$ such that div $\mathbf{y}_0=p_\mathcal{D}$ a.e. in Ω and

$$\|\mathbf{y}_0\|_{(L^2(\Omega))^d} \le C_9\,\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{9.37}$$

Introducing $\mathbf{y}_0$ in (9.34), one gets

$$\|p_\mathcal{D}\|^2_{L^2(\Omega)} \le \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d},$$

which gives, thanks to (9.37),

$$\|p_\mathcal{D}\|_{L^2(\Omega)} \le C_9\left(\|\mathbf{r}\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\right). \tag{9.38}$$

Thanks to (9.36) and (9.38), one gets (9.35).

Lemma 9.5 (Bound on the approximation error by the interpolation error)
Under Hypotheses (H), let $\xi>0$ and $\mathcal{D}$ be a discretization of Ω in the sense of definition 9.2 such that $\mathrm{regul}(\mathcal{D})\le\xi$. *Let $(p,\mathbf{q})\in L^2(\Omega)\times H_g(\mathrm{div},\Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10) and $(p_\mathcal{D},\mathbf{q}_\mathcal{D})\in V_\mathcal{D}\times\mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14). Let $\tilde{\mathbf{q}}_\mathcal{D}\in\mathbf{Q}_{\mathcal{D},g}$ be given and let $\tilde p_\mathcal{D}\in V_\mathcal{D}$ be defined by $\tilde p_\mathcal{D}=\sum_{K\in\mathcal{M}}\frac{1}{m_K}\int_K p(x)\,dx\ \chi_K$.*

Then there exists C_{11}, only depending on Ω, d, ξ, λ_1 and λ_2 such that

$$\|\mathbf{q}-\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} + \|p-p_\mathcal{D}\|^2_{L^2(\Omega)} \le C_{11}\,(\|\mathbf{q}-\tilde{\mathbf{q}}_\mathcal{D}\|^2_{H(\mathrm{div},\Omega)} + \|p-\tilde p_\mathcal{D}\|^2_{L^2(\Omega)}). \tag{9.39}$$

Proof. One gets, using the variational formulations (9.8)-(9.9) and (9.13)-(9.14):

$$\int_\Omega \mathrm{div}(\mathbf{q}_\mathcal{D}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))v(x)\,dx = \int_\Omega \mathrm{div}(\mathbf{q}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))v(x)\,dx,\ \forall v\in V_\mathcal{D},$$

and

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)(\mathbf{q}_\mathcal{D}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))\,dx - \int_\Omega (p_\mathcal{D}(x)-\tilde p_\mathcal{D}(x))\mathrm{div}\ \mathbf{y}(x)\,dx =$$
$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}^{-1}(x)(\mathbf{q}(x)-\tilde{\mathbf{q}}_\mathcal{D}(x))\,dx - \int_\Omega (p(x)-\tilde p_\mathcal{D}(x))\mathrm{div}\ \mathbf{y}(x)\,dx,\ \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}.$$

For all $\mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}$, thanks to the definition of $\tilde{p}_{\mathcal{D}}$, one gets $\int_\Omega (p(x) - \tilde{p}_{\mathcal{D}}(x)) \text{div } \mathbf{y}(x)\, dx = 0$. Thus $(p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}})$ is the solution of (9.33) and (9.34) with $\mathbf{r} = \mathbf{\Lambda}^{-1}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$ and $h = \text{div}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$. Applying Lemma 9.4 yields

$$\|\mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)} \leq C_{10} \left(\frac{1}{\lambda_1} \|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|\text{div } \mathbf{q} - \text{div } \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)} \right).$$

Using the Cauchy-Schwarz inequality, this leads to

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p - p_{\mathcal{D}}\|^2_{L^2(\Omega)} \quad \leq \quad & 2\left(\frac{C_{10}}{\lambda_1} + 1\right) \|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + 2C_{10}\, \|\text{div } \mathbf{q} - \text{div } \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)} \\ & + 2\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}, \end{aligned}$$

which gives (9.39).

Proof of Theorem 9.1. We apply Lemma 9.5. On the one hand, thanks again to (9.64) proved in Lemma 9.13, the following inequality holds:

$$\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)} \leq C_2 \,\text{thin}(\mathcal{D})^2 \|\nabla p\|^2_{L^2(\Omega)},$$

(notice that, when $p \in L^2(\Omega)$ satisfies (9.8), we have in fact $p \in H^1(\Omega)$) and therefore $\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}$ tends to 0 as thin($\mathcal{D}$) tends to 0. On the other hand, it suffices to prove that one can choose $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ such that $\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|_{H(\text{div},\Omega)}$ is as small as desired. Notice that, in general, the property $\mathbf{q} \in (H^1(\Omega))^d \cap H_g(\text{div}, \Omega)$ is wrong. Therefore, one takes $\mathbf{q}_0 \in (H^1(\Omega))^d$ such that $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega} = g$; then, $\mathbf{q} - \mathbf{q}_0 \in H_0(\text{div}, \Omega)$ and since Hypotheses (H) are sufficient to prove that Ω is locally star-shaped, we can approximate $\mathbf{q} - \mathbf{q}_0$ in $H_0(\text{div}, \Omega)$ by regular functions with compact support in Ω (see [71]); thus, $\mathbf{q}$ can be approximated in $H_g(\text{div}, \Omega)$ by $\tilde{\mathbf{q}} \in (H^1(\Omega))^d \cap H_g(\text{div}, \Omega)$. Then, applying Lemma 9.2, one can approximate $\tilde{\mathbf{q}}$ by $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ as close as demanded by letting thin($\mathcal{D}$) tend to zero.

9.4 The convergence of the finite volume method

We now show the following theorem.

Theorem 9.2 (Convergence of the finite volume scheme) *Under Hypotheses (H), let ξ and $\alpha \in (0, 1)$ be fixed positive real values. Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of definition 9.2 such that for all $m \in \mathbb{N}$, $\text{regul}(\mathcal{D}_m) \leq \xi$ and $\lim_{m\to+\infty} \text{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote $(p_m, \mathbf{q}_m)$ the solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ given by (9.13) and (9.14) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\Delta t_m > 0$, denoted Δt, such that the condition*

$$\Delta t \leq (1 - \alpha) \inf_{K\in\mathcal{M}} \frac{m_K}{\sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^+ + f_K^-}, \tag{9.40}$$

holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\Delta t}$ defined by (9.18)-(9.23).

*Then there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, still denoted $(u_m)_{m\in\mathbb{N}}$, which converges for the weak * topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+)$ solution of (9.7).*

If we add some hypotheses giving that $\mathbf{q}$ is Lipschitz continuous on $\overline{\Omega}$ (for example, $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2) then
- the function u is unique
- the whole sequence $(u_m)_{m\in\mathbb{N}}$ converges to u in $L^p(\Omega\times]0, T[)$ for all $p \in [1, \infty)$ and all $T > 0$.

The proof of Theorem (9.2) is classical, and has been developped for various choices of the discretization of the velocity field $\mathbf{q}$ (see [16], [34] and [73]). The originality of this proof is the use of the technical lemma 9.14, which is non standard.

9.4.1 L^∞ estimate

The purpose of this section is to prove the following result.

Lemma 9.6 (L^∞ stability of the finite volume scheme) *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ an admissible discretization in the sense of definition 9.2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\delta t > 0$ such that*

$$\delta t \leq \inf_{K\in\mathcal{M}} \frac{m_K}{\sum_{a\in\mathcal{A}_K} m_a(q_a\varepsilon_{K,a})^+ + f_K^-}. \tag{9.41}$$

Then the approximate solution $u_{\mathcal{D},\delta t}$ given by (9.18)-(9.23) is such that

$$\|u_{\mathcal{D},\delta t}\|_{L^\infty(\Omega\times\mathbb{R}^+)} \leq \max(\|u_0\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\partial\Omega^-\times\mathbb{R}^+)}, \|s\|_{L^\infty(\Omega\times\mathbb{R}^+)}). \tag{9.42}$$

Proof. According to the scheme (9.21), we have

$$u_K^{n+1} = u_K^n - \frac{\delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} u_a^n F_{K,a} + u_K^n f_K^- - s_K^n f_K^+\right),$$

which gives

$$u_K^{n+1} = u_K^n\left(1 - \frac{\delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^+ + f_K^-\right)\right) + \frac{\delta t}{m_K}\sum_{a\in\mathcal{A}_K} F_{K,a}^- u_a^n + \frac{\delta t}{m_K} f_K^+ s_K^n, \tag{9.43}$$

Thanks to the stability condition (9.41), equation (9.43) expresses u_K^{n+1} as a convex combination of the values u_K^n, $\overline{u}_a^n$, s_K^n. An easy proof by induction concludes the proof of the lemma.

Remark 9.6 *If the data are regular enough, the term $\sum_{a\in\mathcal{A}_K} m_a|q_a|$ behaves like size$(\mathcal{D})^{d-1}$ as size$(\mathcal{D})$ tends to 0, and the condition (9.41) takes the form $\delta t \leq C$size$(\mathcal{D})$ (where size$(\mathcal{D}) = \max_{K\in\mathcal{M}} \delta(K)$).*

9.4.2 A weak inequality on the spatial variations

Lemma 9.7 (Weak spatial variations inequality) *Under hypotheses (H), let $\xi > 0$, $\alpha \in (0,1)$, $T > 0$, and let $\mathcal{D}$ be an admissible discretization in the sense of definition 9.2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\delta t > 0$ such that the condition (9.40) holds. Let N_T be such that $N_T\delta t \leq T < (N_T+1)\delta t$ and let $(u_K^n)_{K\in\mathcal{M},\ n\in\mathbb{N}}$, $(u_a^n)_{a\in\mathcal{A},\ n\in\mathbb{N}}$ be defined by (9.18)-(9.22).*

Then there exists C_{12}, which only depends on d, Ω, T, ξ, α, f, s, g, $\overline{u}$ and u_0 (but not on $\mathcal{D}$ or δt), such that

$$\sum_{n=0}^{N_T}\delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} m_a(q_a\varepsilon_{K,a})^-(u_a^n - u_K^n)^2\right) \leq C_{12}\,. \tag{9.44}$$

Remark 9.7 *In references [16], [34] and [73], a weak BV-estimate is obtained from (9.44). We do not do so here, since in the convergence proof, the use of Lemma 9.14 takes advantage of a local bound of the diameter of each control volume. Otherwise, we should assume the existence of some $\beta > 0$ with*

$$\delta(K) \;\geq\; \beta \; size(\mathcal{D}), \;\; \forall K \in \mathcal{M}.$$

Proof. First, the discrete elliptic scheme (9.15) is used to get

$$\sum_{a \in \mathcal{A}_K} F^+_{K,a} + f^-_K = \sum_{a \in \mathcal{A}_K} F^-_{K,a} + f^+_K, \tag{9.45}$$

and therefore the scheme (9.21) also writes

$$m_K(u_K^{n+1} - u_K^n) + \Delta t \left(\sum_{a \in \mathcal{A}_K} F^-_{K,a}(u^n_K - u^n_a) + f^+_K(u^n_K - s^n_K) \right) = 0, \;\; \forall K \in \mathcal{M}, \;\; \forall n \in \mathbb{N}. \tag{9.46}$$

For all $n \in \mathbb{N}$ and $K \in \mathcal{M}$, let us multiply the equation (9.46) by u^n_K and sum the result on $K \in \mathcal{M}$ and $n = 0, \ldots, N_T$. It gives $T_1 + T_2 = 0$ with

$$T_1 = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{M}} m_K(u^{n+1}_K - u^n_K)u^n_K$$

and

$$T_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{M}} \left(\sum_{a \in \mathcal{A}_K} F^-_{K,a}(u^n_K - u^n_a)u^n_K + f^+_K(u^n_K - s^n_K)u^n_K \right).$$

Writing $u^{n+1}_K u^n_K = -\frac{1}{2}(u^{n+1}_K - u^n_K)^2 + \frac{1}{2}(u^{n+1}_K)^2 + \frac{1}{2}(u^n_K)^2$, one gets

$$T_1 = T_{11} + T_{12},$$

where

$$T_{11} = -\frac{1}{2} \sum_{n=0}^{N_T} \sum_{K \in \mathcal{M}} m_K(u^{n+1}_K - u^n_K)^2.$$

and

$$T_{12} = \frac{1}{2} \left(\sum_{K \in \mathcal{M}} m_K((u^{N_T+1}_K)^2 - (u^0_K)^2) \right).$$

Using (9.46) and the Cauchy-Schwarz inequality gives

$$m^2_K(u^{n+1}_K - u^n_K)^2 \leq \Delta t \left(\sum_{a \in \mathcal{A}_K} F^-_{K,a} + f^+_K \right) \left(\Delta t \sum_{a \in \mathcal{A}_K} F^-_{K,a}(u^n_a - u^n_K)^2 + f^+_K(s^n_K - u^n_K)^2 \right),$$
$$\forall K \in \mathcal{M}, \;\; \forall n \in \mathbb{N}.$$

Using condition (9.40) and equation (9.45), one gets

$$m_K(u_K^{n+1}-u_K^n)^2 \le (1-\alpha)\left(\Delta t\sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n-u_K^n)^2+f_K^+(s_K^n-u_K^n)^2\right),\quad \forall K\in\mathcal{M},\ \forall n\in\mathbb{N}. \tag{9.47}$$

One computes T_2. One gets $T_2 = T_{21}+T_{22}$ with

$$T_{21}=\frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K}F_{K,a}^-(u_a^n-u_K^n)^2+f_K^+(s_K^n-u_K^n)^2\right)$$

and

$$T_{22}=\frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K}F_{K,a}^-((u_K^n)^2-(u_a^n)^2)+f_K^+((u_K^n)^2-(s_K^n)^2)\right).$$

We get thus, thanks to (9.47),

$$T_{11}+T_{21}\ge \alpha T_{21}.$$

Thanks to (9.45), the term T_{22} can be rewritten as

$$T_{22}=\frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K}F_{K,a}(u_a^n)^2+f_K^-(u_K^n)^2-f_K^+(s_K^n)^2\right).$$

Thus, gathering by edges, one gets

$$T_{22}=\frac{1}{2}\sum_{n=0}^{N_T}\Delta t\left(\sum_{a\in\mathcal{A}_e}m_a g_a(u_a^n)^2+\sum_{K\in\mathcal{M}}(f_K^-(u_K^n)^2-f_K^+(s_K^n)^2)\right).$$

Since terms T_{12} and T_{22} can easily be bounded, using Lemma 9.6 (since condition (9.41) is weaker than (9.40)), we thus get (9.44).

9.4.3 The proof of the convergence theorem 9.2

We first notice that Lemma 9.6 gives the existence of a subsequence u_m and of a function $u\in L^\infty(\Omega\times\mathbb{R}^+)$ such that u_m converges to u for the weak * topology of $L^\infty(\Omega\times\mathbb{R}^+)$ as $m\to+\infty$. Recall that we have proved above (Theorem 9.1) that $\mathbf{q}_m$ tends to $\mathbf{q}$ in $H(\mathrm{div},\Omega)$ as $m\to+\infty$. This section is devoted to the proof that u satisfies (9.7) (the uniqueness part of the proof being studied in the next section).

Let $\phi\in C_c^1(\mathbb{R}^d\times\mathbb{R})$ be such that $\phi=0$ on $\partial\Omega\backslash\partial\Omega^-\times\mathbb{R}^+$. Let $T>0$ be such that

$$\phi=0 \quad \text{on} \quad \mathbb{R}^d\times[T,+\infty[. \tag{9.48}$$

In this proof, we denote by C_i various positive real values which only depend on d, Ω, ϕ, T, ξ, α, s, f, g, $\overline{u}$, u_0 and not on $\mathcal{D}$ or Δt.

In the following, we use the notations $\mathcal{D}=\mathcal{D}_m$ and $\Delta t=\Delta t_m$. Let us denote N_T the integer value such that $N_T\Delta t\le T<(N_T+1)\Delta t$. Setting

$$\phi_K^n=\frac{1}{\Delta t\ m_K}\int_K\int_{n\Delta t}^{(n+1)\Delta t}\phi(x,t)\,dx\,dt,\ \forall K\in\mathcal{M},\ \forall n\in\mathbb{N},$$

one multiplies the equality (9.46) by ϕ_K^n and sum on $K\in\mathcal{M}$ and $n\in\mathbb{N}$. One obtains $E_1+E_2=0$ with

$$E_1 = \sum_{n=0}^{N_T} \sum_{K\in\mathcal{M}} m_K (u_K^{n+1} - u_K^n)\phi_K^n,$$

and

$$E_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \left(\sum_{a\in\mathcal{A}_K} F_{K,a}^- (u_K^n - u_a^n)\phi_K^n + f_K^+ (u_K^n - s_K^n)\phi_K^n \right).$$

We also define

$$\phi_a^n = \frac{1}{\Delta t \ m_a} \int_a \int_{n\Delta t}^{(n+1)\Delta t} \phi(x,t)\, d\gamma(x)\, dt.$$

Let us study E_1. Thanks to (9.48), for all $K \in \mathcal{M}$, $\phi_K^{N_T+1} = 0$ holds and therefore

$$E_1 = \sum_{n=1}^{N_T+1} \sum_{K\in\mathcal{M}} m_K u_K^n (\phi_K^{n-1} - \phi_K^n) - \sum_{K\in\mathcal{M}} m_K u_K^0 \phi_K^0.$$

Using the weak * convergence of $(u_m)_{m\in\mathbb{N}}$ to u, we deduce the convergence of E_1 to

$$-\int_{\Omega\times\mathbb{R}^+} u(x,t) \frac{\partial \phi}{\partial t}(x,t)\, dx\, dt - \int_\Omega u_0(x)\phi(x,0)\, dx.$$

Next we consider the term E_2. It can be written, using (9.22) and gathering by edges, as

$$\begin{aligned} E_2 \quad = \quad & \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_i} m_a q_a \phi_{K_d(a)}^n (u_{L(a)}^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a q_a \phi_{K_d(a)}^n (u_a^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} f_K^+ \phi_K^n (u_K^n - s_K^n), \end{aligned} \tag{9.49}$$

where we define, for all $a \in \mathcal{A}_i$, $K_d(a)$ (the "downstream" control volume) by $K_d(a) = K(a)$ if $q_a \leq 0$, else $K_d(a) = L(a)$, and for all $a \in \mathcal{A}_e$, $K_d(a) = K(a)$. We set

$$\begin{aligned} f_\mathcal{D}(x) = \frac{1}{m_K} f_K, \quad & \text{for a.e.} \quad x \in K,\ \forall K \in \mathcal{M}, \\ s_{\mathcal{D},\Delta t}(x,t) = s_K^n, \quad & \text{for a.e.} \quad (x,t) \in K \times [n\Delta t, (n+1)\Delta t),\ \forall K \in \mathcal{M},\ \forall n \in \mathbb{N}, \\ \overline{u}_{\mathcal{D},\Delta t}(x,t) = \overline{u}_a^n, \quad & \text{for a.e.} \quad (x,t) \in a \times [n\Delta t, (n+1)\Delta t),\ \forall a \in \mathcal{A}_e,\ \forall n \in \mathbb{N}, \\ g_\mathcal{D}(x) = g_a, \quad & \text{for a.e.} \quad x \in a,\ \forall a \in \mathcal{A}_e, \end{aligned}$$

where f_K, g_a, s_K^n and $\overline{u}_a^n$ are respectively defined by (9.16), (9.17), (9.18) and (9.19). We define E_3 by

$$\begin{aligned} E_3 \quad = \quad & -\int_{\Omega\times\mathbb{R}^+} u_{\mathcal{D},\Delta t}(x,t) \mathbf{q}_\mathcal{D}(x) \cdot \nabla\phi(x,t)\, dx\, dt \\ & + \int_{\partial\Omega\times\mathbb{R}^+} \overline{u}_{\mathcal{D},\Delta t}(x,t) g_\mathcal{D}(x) \phi(x,t)\, d\gamma(x)\, dt \\ & + \int_{\Omega\times\mathbb{R}^+} (u_{\mathcal{D},\Delta t}(x,t) f_\mathcal{D}^-(x) - s_{\mathcal{D},\Delta t}(x,t) f_\mathcal{D}^+(x))\phi(x,t)\, dx\, dt \end{aligned}$$

Since $u_{\mathcal{D},\Delta t}$ converges to u for the weak * topology of $L^\infty(\Omega\times\mathbb{R}^+)$ and since $\mathbf{q}_{\mathcal{D}}$ converges strongly to $\mathbf{q}$ in $L^2(\Omega)$ as $m\to+\infty$, in view of the definitions of $\overline{u}_{\mathcal{D},\Delta t}$ and $g_{\mathcal{D}}$, we deduce the convergence of E_3 to $-\int_{\Omega\times\mathbb{R}^+} u(x,t)\mathbf{q}(x)\cdot\nabla\phi(x,t)\,dx\,dt+\int_{\partial\Omega^-\times\mathbb{R}^+}\overline{u}(x,t)g(x)\phi(x,t)\,d\gamma(x)\,dt+\int_{\Omega\times\mathbb{R}^+}(u(x,t)f^-(x)-s(x,t)f^+(x))\phi(x,t)\,dx\,dt$ as $m\to+\infty$.
Using (9.15) and the definition of $\mathbf{w}_{K,a}$, one can rewrite E_3 as

$$\begin{aligned}E_3 &= \sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_i} m_a q_a\phi_a^n(u_{L(a)}^n-u_{K(a)}^n)\\ &+\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a q_a\phi_a^n(\overline{u}_a^n-u_{K(a)}^n)\\ &+\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}(u_K^n-s_K^n)f_K^+\phi_K^n.\end{aligned}\tag{9.50}$$

From (9.49) and (9.50), we can deduce that

$$|E_3-E_2|\le E_4+E_5,$$

with

$$E_4=\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a|q_a||\phi_a^n||\overline{u}_a^n-u_a^n|$$

and

$$E_5=\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_i} m_a|q_a|\;|\phi_a^n-\phi_{K_d(a)}^n|\;|u_{K(a)}^n-u_{L(a)}^n|+\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a|q_a|\;|\phi_a^n-\phi_{K_d(a)}^n|\;|u_a^n-u_{K(a)}^n|.$$

Let us first study E_4. Since, for all $a\in\mathcal{A}_e$, relation (9.22) implies $u_a^n=\overline{u}_a^n$ when $q_a\le 0$, we can write

$$E_4=\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e\,,\,q_a>0} m_a|q_a||\phi_a^n||\overline{u}_a^n-u_a^n|.$$

For all $a\in\mathcal{A}_e$ such that $q_a=m_a^{-1}\int_a g(x)\,d\gamma(x)>0$, one has $\partial\Omega^+\cap a\neq\emptyset$ (recall that $\partial\Omega^+=\{x\in\partial\Omega\mid g(x)>0\}$); thus, since $\phi=0$ on $\partial\Omega^+\times\mathbb{R}^+$, there exists $x\in a$ such that $\phi(x,t)=0$ for all $t\ge 0$. By denoting C_{13} the Lipschitz constant of ϕ, one has then $|\phi(y,t)|\le C_{13}\,\delta(a)$ for all $y\in a$ and $t\ge 0$, which implies $|\phi_a^n|\le C_{13}\,\delta(a)$. Using (9.42), one gets then

$$\begin{aligned}E_4 &\le C_{14}\,\mathrm{thin}(\mathcal{D})\sum_{n=0}^{N_T}\Delta t\sum_{a\in\mathcal{A}_e} m_a|q_a|\\ &\le C_{14}\,\mathrm{thin}(\mathcal{D})(T+\Delta t)\sum_{a\in\mathcal{A}_e}\int_a|g(x)|\,d\gamma(x)\\ &= C_{14}\,\mathrm{thin}(\mathcal{D})(T+\Delta t)\int_{\partial\Omega}|g(x)|\,d\gamma(x),\end{aligned}$$

which shows that E_4 tends to 0 as $m\to+\infty$.
We turn now to the study of E_5. Thanks to the Cauchy-Schwarz inequality, we obtain

$$E_5^2 \le C_{13}{}^2 \left(\sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}} m_a |q_a| \delta(K_d(a))^2 \right) \left(\sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^- (u_a^n - u_K^n)^2 \right).$$

This gives, using Lemma 9.7 and the Cauchy-Schwarz inequality,

$$E_5^2 \le C_{15}\, \mathrm{thin}(\mathcal{D}) \left(\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a)) \right)^{1/2} \left(\sum_{a\in\mathcal{A}} m_a \delta(K_d(a)) \right)^{1/2}.$$

One can then apply Lemma 9.14, which yields

$$\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a)) \le C_{16} \sum_{a\in\mathcal{A}} \left(\int_{K_d(a)} \mathbf{q}_{\mathcal{D}}^2(x)\, dx + \delta(K_d(a))^2 \int_{K_d(a)} (\mathrm{div}\mathbf{q}_{\mathcal{D}}(x))^2\, dx \right).$$

Under Hypotheses (H) (and in particular item (vii)), one gets that $\mathrm{card}\mathcal{A}_K \le C_{17}$. Therefore, since $\mathbf{q}_{\mathcal{D}}$ converges to $\mathbf{q}$ in $H(\mathrm{div}, \Omega)$, it is bounded and

$$\sum_{a\in\mathcal{A}} m_a q_a^2 \delta(K_d(a)) \le C_{18}.$$

Using item (ii) of Hypotheses (H), one gets

$$\sum_{a\in\mathcal{A}} m_a \delta(K_d(a)) \le C_{19}.$$

Therefore, one can conclude

$$E_5 \le C_{20} \sqrt{\mathrm{thin}(\mathcal{D})},$$

which shows that E_2 tends to $-\int_{\Omega\times\mathbb{R}^+} u(x,t)\mathbf{q}(x)\cdot\nabla\phi(x,t)\, dx\, dt + \int_{\partial\Omega^-\times\mathbb{R}^+} \overline{u}(x,t) g(x) \phi(x,t)\, d\gamma(x)\, dt + \int_{\Omega\times\mathbb{R}^+} (u(x,t) f^-(x) - s(x,t) f^+(x))\phi(x,t)\, dx\, dt$ as $m \to +\infty$, and concludes the proof of Theorem 9.2.

9.5 Uniqueness of the weak solution under regularity on the data

We do not handle in details this part, since it does not involve the particular discrete frame we have developed in this paper. Some details can be found in [35], [15], [37] for example. We first state the following discrete result.

Lemma 9.8 *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ be an admissible discretization in the sense of definition 9.2 such that $\xi \ge \mathrm{regul}(\mathcal{D})$. Let $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (9.13) and (9.14) and let $\Delta t > 0$ such that the CFL condition (9.41) holds.*

Then the approximate solution $u_{\mathcal{D},\Delta t}$ given by (9.18)-(9.23) is such that

$$m_K(\eta(u_K^{n+1}) - \eta(u_K^n)) + \Delta t \left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-(\eta(u_K^n) - \eta(u_a^n)) + f_K^+ \eta'(u_K^n)(u_K^n - s_K^n) \right) \le 0,$$
$$\forall K \in \mathcal{M},\ \forall n \in \mathbb{N},\ \forall \eta \in C^1(\mathbb{R},\mathbb{R}) \text{ with } \eta'' \ge 0.$$

The proof of this lemma is easy, starting from the discrete relation (9.46) and multiplying it by $\eta'(u_K^n)$. From this lemma, one gets, letting $\text{thin}(\mathcal{D}) \to 0$, the following result, which proves the convergence of the scheme to a solution of the hyperbolic problem in a very weak sense ([36], [28]).

Lemma 9.9 (Convergence of the finite volume scheme to an entropy process solution) *Under Hypotheses (H), let $\xi > 0$ and $\alpha \in (0,1)$ be fixed real values. Let $(p,\mathbf{q}) \in L^2(\Omega) \times H_g(\text{div},\Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of definition 9.2 such that for all $m \in \mathbb{N}$, $\text{regul}(\mathcal{D}_m) \le \xi$ and $\lim_{m\to+\infty} \text{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote by $(p_m, \mathbf{q}_m)$ the solution $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ given by (9.13) and (9.14) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\Delta t_m > 0$, denoted Δt, such that the CFL condition (9.40) holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\Delta t}$ defined by (9.18)-(9.23).*

*Then there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted $(u_m)_{m\in\mathbb{N}}$, which converges for the nonlinear weak * topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$, solution of*

$$\int_{\mathbb{R}^+}\int_\Omega\int_0^1 \left(\begin{array}{l} \eta(u(x,t,\alpha))\dfrac{\partial\phi}{\partial t}(x,t) + \eta(u(x,t,\alpha))\text{div}(\phi(x,t)\mathbf{q}(x)) + \\ \eta'(u(x,t,\alpha))\phi(x,t)f^+(x)(s(x,t)-u(x,t,\alpha)) \end{array}\right) d\alpha\, dx\, dt$$
$$+\int_\Omega \eta(u_0(x))\ \phi(x,0)\, dx - \int_{\mathbb{R}^+}\int_{\partial\Omega^-} \eta(\overline{u}(x,t))\ \phi(x,t)g(x)\, d\gamma(x)\, dt \ge 0, \tag{9.51}$$

$\forall \phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^+)$ *such that* $\phi = 0$ *on* $\partial\Omega^+ \times \mathbb{R}^+ = (\partial\Omega \setminus \partial\Omega^-) \times \mathbb{R}^+$,
$\forall \eta \in C^1(\mathbb{R},\mathbb{R})$ *with* $\eta'' \ge 0$.

The proof of the above lemma is fully similar to the one which is given in section 9.4.3. Using the classical "doubling variable technique" and Krushkov entropies [49] lead to a result of uniqueness, under sufficient hypotheses on the data giving that $\mathbf{q}$ is Lipschitz-continuous (see [61] or [74] for the particular problem of handling the boundary conditions).

Lemma 9.10 (Uniqueness of the entropy process solution) *Under Hypotheses (H), and the additional hypotheses $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2 (for example), let $(p,\mathbf{q}) \in L^2(\Omega) \times H_g(\text{div},\Omega)$ be the unique weak solution of the problem (9.8) and (9.9) with the condition (9.10).*

Then $\mathbf{q}$ is Lipschitz-continuous in $\overline{\Omega}$, there exists one and only one function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$, solution of (9.51), and there exists one and only one $\tilde{u} \in L^\infty(\Omega \times \mathbb{R}^+)$, solution of (9.7), such that, for a.e. $(x,t,\alpha) \in \Omega \times \mathbb{R}^+ \times (0,1)$, $u(x,t,\alpha) = \tilde{u}(x,t)$.

This result of uniqueness yields the convergence in $L^p(\Omega\times]0,T[)$, for all $p \in [1,\infty)$ and $T > 0$, of $(u_m)_{m\in\mathbb{N}}$ to the unique solution $\tilde{u}$ of the problem.

9.6 Appendix : technical lemmata

Lemma 9.11 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism ϕ from $Q_{\delta(K)} =]-\delta(K), \delta(K)[^d$ to K with Lipschitz-continuous inverse mapping; we denote by ξ an upper bound of the Lipschitz constants of ϕ and ϕ^{-1}. Then there exists $C_{21} > 0$ only depending on ξ and d such that, for all $f \in L^1(\partial K)$, $f \ge 0$,*

$$C_{21}{}^{-1}\int_{\partial Q_{\delta(K)}} f\circ\phi(x)\, d\gamma(x) \le \int_{\partial K} f(x)\, d\gamma(x) \le C_{21}\int_{\partial Q_{\delta(K)}} f\circ\phi(x)\, d\gamma(x). \tag{9.52}$$

Notice that a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between two open sets has a unique extension as a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the closures of the open sets, and that this extension defines a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the boundaries of the open sets.

Remark 9.8 *The most useful inequality (and the easiest to obtain) in the following will be the second one of (9.52). We have also stated the first one in order that (9.52) allows to see that, when A is a measurable subset of ∂K, $\gamma(A)$ and $\gamma(\phi^{-1}(A))$ ([7]) are comparable, with constants only depending on an upper bound on the Lipschitz constants of ϕ and ϕ^{-1}.*

Proof.
We denote $\delta = \delta(K)$.
It is well known (see e.g. [31]) that the application

$$f \in L^1(\partial K) \to f \circ \phi \in L^1(\partial Q_\delta) \tag{9.53}$$

is an isomorphism; we want here to estimate the norm of this application (and of its inverse mapping) only in terms of ϕ and ϕ^{-1} (with bounds not depending on δ).
Let us first recall the definition of the integral on ∂K when K is an open set with weakly Lipschitz-continuous boundary: if V is an open set of $\mathbb{R}^d$ and $\tau :]-1,1[^{d-1} \to \partial K \cap V$ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping, then for $f \in L^1(\partial K)$, one has

$$\int_{\partial K \cap V} f(x)\, d\gamma(x) = \int_{]-1,1[^{d-1}} f \circ \tau(x) |\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau|(x)\, dx,$$

where $\partial_i \tau$ denotes the i-th partial derivative of τ (which is, by the Rademacher Theorem, a function in $(L^\infty(]-1,1[^{d-1}))^d$ and is essentially bounded by $\mathrm{lip}(\tau)$) and $\wedge$ is the inner product in $\mathbb{R}^d$.

With this definition, one can verify that the $(d-1)$-dimensional measure on ∂Q_δ is the $(d-1)$-Lebesgue measure on each piece of hyperplane the union of which is ∂Q_δ. One can also notice that $\partial Q_\delta = A \sqcup \sqcup_{i=1}^d (]-\delta,\delta[^{i-1} \times \{-\delta\} \times]-\delta,\delta[^{d-i} \sqcup]-\delta,\delta[^{i-1} \times \{\delta\} \times]-\delta,\delta[^{d-i})$ where $\gamma(A) = 0$ (A is made of sets of dimension $d-2$).
Since (9.53) is an isomorphism, the sets of null measure on ∂Q_δ are transported by ϕ on sets of null measure on ∂K. Thus, by denoting $H_{i,\pm} =]-\delta,\delta[^{i-1} \times \{\pm\delta\} \times]-\delta,\delta[^{d-i}$, one has, up to a set of null measure, $\partial K = \sqcup_{i=1}^d (\phi(H_{i,+}) \sqcup \phi(H_{i,-}))$. If $f \in L^1(\partial K)$, $f \geq 0$, the integral of f on ∂K can thus be estimated if we estimate all the integrals of f on $\phi(H_{i,\pm})$.
Let us do it for $H_{1,+}$, the other terms being studied the same way.

Define $\tau :]-1,1[^{d-1} \to \partial K \cap \phi(H_{1,+})$ by $\tau(x) = \phi(\delta, \delta x)$. τ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping; thus, by definition of the integral on ∂K,

$$\begin{aligned} \int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) &= \int_{]-1,1[^{d-1}} f \circ \tau(x) |\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau|(x)\, dx \\ &= \int_{]-1,1[^{d-1}} f \circ \phi(\delta, \delta x) \delta^{d-1} \left| \frac{\partial \phi}{\partial y_2}(\delta, \delta x) \wedge \cdots \wedge \frac{\partial \phi}{\partial y_d}(\delta, \delta x) \right| (x)\, dx. \end{aligned} \tag{9.54}$$

Thus, by a change of variable,

$$\int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) = \int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y) \left| \frac{\partial \phi}{\partial y_2}(\delta, y) \wedge \cdots \wedge \frac{\partial \phi}{\partial y_d}(\delta, y) \right| (y)\, dy.$$

[7] Recall that γ denotes the $(d-1)$-dimensional measure on the boundary of any open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary.

Since ϕ is Lipschitz-continuous, we have, for all $i \in [2,d]$, $||\frac{\partial \phi}{\partial y_i}||_{L^\infty(H_{1,+})} \le \mathrm{lip}(\phi)$ and there exists thus C_{22} only depending on ξ and d such that

$$\int_{\partial K \cap \phi(H_{1,+})} f(x)\, d\gamma(x) \le C_{22} \int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y)\, dy.$$

But, as we previously noticed, the $(d-1)$-dimensional measure on $H_{1,+}$ is the $(d-1)$-Lebesgue measure on this piece of hyperplane, and thus

$$\int_{]-\delta,\delta[^{d-1}} f \circ \phi(\delta, y)\, dy = \int_{H_{1,+}} f \circ \phi(x)\, d\gamma(x),$$

which proves the second inequality of (9.52).

The proof of the first inequality of (9.52) relies on a lemma (mainly algebraic) stating that there exists C_{23} only depending on d such that

$$|\partial_1 \tau \wedge \cdots \wedge \partial_{d-1}\tau| \ge C_{23}\, (\mathrm{lip}(\tau^{-1}))^{-(d-1)} \tag{9.55}$$

(see [31]). Since $\tau^{-1}(z) = \delta^{-1}((\phi^{-1}(z))_2, \ldots, (\phi^{-1}(z))_d)$, one has $\mathrm{lip}(\tau^{-1}) \le \xi \delta^{-1}$; using this in (9.55) and returning to (9.54) we get, thanks again to a change of variable, the first inequality of (9.52).

Lemma 9.12 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote by m_K the measure of K. One assumes that there exists a Lipschitz-continuous homeomorphism $\mathcal{L}$ from K to $B(0,\delta(K))$ with Lipschitz-continuous inverse mapping. Let ξ be a real value greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g \in H^1(K)$. The trace of g on ∂K is still denoted by g.*
Then there exists $C_3 > 0$, only depending on ξ and d, such that

$$\frac{1}{m_K} \int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) \le C_3\, \delta(K) \int_K (\nabla g(x))^2\, dx,$$

Thus, if $\int_K g(x)\, dx = 0$ holds, one has

$$\int_{\partial K} g(x)^2\, d\gamma(x) \le C_3\, \delta(K) \int_K (\nabla g(x))^2\, dx. \tag{9.56}$$

Proof. In the following proof, C_i denotes real values which only depend on d and ξ; δ denotes $\delta(K)$. The application $F : x \to (|x| / \sup_{i\in[1,d]} |x_i|)x$ is a Lipschitz-continuous homeomorphism with Lipschitz continuous inverse mapping between $B(0,\delta)$ and $Q =]-\delta,\delta[^d$; moreover, the Lipschitz constants of F and F^{-1} only depend on d. Thus, there exists a Lipschitz-continuous homeomorphism ϕ from Q to K, with Lipschitz continuous inverse mapping, such that the Lipschitz constants of ϕ and ϕ^{-1} are bounded by C_{24} only depending on d and ξ.
According to Lemma 9.11, there exists C_{25} only depending on d and ξ such that

$$\begin{aligned}\int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) &\le C_{25} \int_{\partial Q} \int_K (g(\phi(y')) - g(x))^2\, dx\, d\gamma(y') \\ &= C_{25} \int_{\partial Q} \int_Q (g(\phi(y')) - g(\phi(x')))^2 J_{\phi,d}(x')\, dx'\, d\gamma(y'),\end{aligned}$$

where $J_{\phi,d}(x')$ is the absolute value of the jacobian in the change of variable ϕ. Setting $h = g \circ \phi$, one has $h \in H^1(Q)$. Then one gets the existence of $C_{26} > 0$, only depending on d and ξ, such that

$$\int_{\partial K} \int_K (g(y) - g(x))^2\, dx\, d\gamma(y) \le C_{26} \int_{\partial Q} \int_Q (h(y) - h(x))^2\, dx\, d\gamma(y).$$

The change of variable $x = \phi^{-1}(x')$ proves the existence of $C_{27} > 0$, only depending on d and ξ such that

$$\int_Q (\nabla h(x))^2 \, dx \leq C_{27} \int_K (\nabla g(x'))^2 \, dx'. \tag{9.57}$$

Therefore, if one proves the existence of $C_{28} > 0$, only depending on d and ξ, such that

$$\int_{\partial Q} \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \leq C_{28} \, \delta^{d+1} \int_Q (\nabla h(x))^2 \, dx, \tag{9.58}$$

one gets (9.12) from (9.57) and (9.58) and the fact that the existence of $\mathcal{L}$ ensures that there exists $C_5 > 0$ with $m_K \geq C_5 \, \delta^d$.

In order to prove (9.58), one may assume by a classical argument of density that $h \in C^1(\overline{Q})$. Since Q is a cube with $2d$ edges, it suffices to prove the existence of $C_{29} > 0$, only depending on d and ξ, such that

$$\int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \leq C_{29} \, \delta^{d+1} \int_Q (\nabla h(x))^2 \, dx, \tag{9.59}$$

where $\sigma = \{-\delta\} \times [-\delta, \delta]^{d-1}$, to get (9.58) with $C_{28} = 2dC_{29}$. Let $H = [-\delta, \delta]^{d-1}$ and $Q^+ = [0, \delta] \times H$. We can now write, for all $z \in Q^+$,

$$\int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \leq 2 \int_\sigma \int_Q (h(y) - h(z))^2 \, dx \, d\gamma(y) + 2 \int_\sigma \int_Q (h(z) - h(x))^2 \, dx \, d\gamma(y).$$

An integration with respect to $z \in Q^+$ leads to

$$2^{d-1}\delta^d \int_\sigma \int_Q (h(y) - h(x))^2 \, dx \, d\gamma(y) \leq 2(2\delta)^d A + 2(2\delta)^{d-1} B, \tag{9.60}$$

with

$$A = \int_\sigma \int_{Q^+} (h(y) - h(z))^2 \, dz \, d\gamma(y),$$

and

$$B = \int_{Q^+} \int_Q (h(z) - h(x))^2 \, dx \, dz.$$

Let us first study A. By definition,

$$A = \int_H \int_H \int_0^\delta (h((-\delta, y)) - h((a, b)))^2 \, da \, db \, dy,$$

and therefore,

$$A = \int_H \int_H \int_0^\delta \left(\int_0^1 \nabla h((-\delta + \theta(a + \delta), y + \theta(b - y))) \cdot (a + \delta, b - y) \, d\theta \right)^2 \, da \, db \, dy.$$

Using the Cauchy-Schwarz inequality, one gets

$$A \leq (2\delta)^2 d \int_H \int_H \int_0^\delta \int_0^1 (\nabla h((-\delta + \theta(a + \delta), y + \theta(b - y))))^2 \, d\theta \, da \, db \, dy.$$

Using the Fubini Theorem and the two changes of variable $b \to b' = b - y \in H_2 = [-2\delta, 2\delta]^{d-1}$, $y \to y' = y + \theta b' \in H$, we obtain then

$$A \le (2\delta)^2 d \int_{H_2} \int_0^\delta \int_0^1 \int_H \left(\nabla h((-\delta + \theta(a+\delta), y'))\right)^2 \, dy' \, d\theta \, da \, db'.$$

We now change the variable θ into $t = -\delta + \theta(a+\delta)$. This yields:

$$A \le (2\delta)^2 (4\delta)^{d-1} d \int_0^\delta \int_{-\delta}^a \int_H \left(\nabla h((t, y'))\right)^2 \frac{1}{a+\delta} \, dy' \, dt \, da.$$

Since, for all $a \in [0, \delta]$, $\frac{1}{a+\delta} \le \frac{1}{\delta}$, one gets, setting $x = (t, y)$,

$$A \le 2^{2d} \delta^{d+1} d \int_Q \left(\nabla h(x)\right)^2 \, dx. \tag{9.61}$$

Let us now study B. We have

$$B \le (2\delta)^2 d \int_{Q^+} \int_Q \int_0^1 \left(\nabla h(x + \theta(z - x))\right)^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the the two changes of variable $z \to z' = z - x \in Q_2 = [-2\delta, 2\delta]^d$, $x \to x' = x + \theta z' \in Q$, we get

$$B \le (2\delta)^2 d \int_{Q_2} \int_Q \left(\nabla h(x')\right)^2 \, dx' \, dz',$$

which gives

$$B \le 2^{2d+2} \delta^{d+2} d \int_Q \left(\nabla h(x')\right)^2 \, dx'. \tag{9.62}$$

Thus, using (9.60), (9.61) and (9.62), one concludes the proof of (9.59).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (9.56) is then a direct consequence of

$$\int_{\partial K} g(x)^2 \, d\gamma(x) = \int_{\partial K} \left(g(x)^2 - \frac{1}{m_K} \int_K g(y) \, dy \right) d\gamma(x) \le \frac{1}{m_K} \int_{\partial K} \int_K \left(g(x) - g(y)\right)^2 \, dx \, d\gamma(y).$$

Lemma 9.13 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote the measure of K by m_K. One assumes that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from $B(0, \delta(K))$ to K. Let ξ be a real value greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g \in H^1(K)$.*
Then there exists $C_2 > 0$, only depending on ξ and d, such that

$$\frac{1}{m_K} \int_K \int_K (g(y) - g(x))^2 \, dx \, dy \le C_2 \, \delta(K)^2 \int_K \left(\nabla g(x)\right)^2 \, dx. \tag{9.63}$$

Thus, if $\int_K g(x) \, dx = 0$ holds, one has

$$\int_K g^2(x) \, dx \le C_2 \, \delta(K)^2 \int_K (\nabla g(x))^2 \, dx. \tag{9.64}$$

Proof. We denote $\delta = \delta(K)$. Using the change of variables $x' = \mathcal{L}(x)$ and $y' = \mathcal{L}(y)$, and writing for simplicity of notations $B = B(0, \delta)$, one gets the existence of C_{30}, only depending on d and ξ, such that

$$\int_K \int_K (g(y) - g(x))^2 \, dx \, dy \le C_{30} \int_B \int_B (g(\mathcal{L}(y')) - g(\mathcal{L}(x')))^2 \, dx' \, dy'.$$

Setting $h = g \circ \mathcal{L}$, one has $h \in H^1(B)$. Then one gets the existence of $C_{31} > 0$, only depending on d and ξ, such that

$$\int_B (\nabla h(x))^2 \, dx \leq C_{31} \int_K (\nabla g(x'))^2 \, dx'. \tag{9.65}$$

Thus, if one proves the existence of $C_{32} > 0$, only depending on d and ξ, such that

$$\int_B \int_B (h(y) - h(x))^2 \, dx \, dy \leq C_{32} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx, \tag{9.66}$$

one gets (9.13) from (9.65), (9.66) and the fact that the existence of $\mathcal{L}$ ensures that there exists C_5 with $m_K \geq C_5 \, \delta^d$. In order to prove (9.66), one may assume by a classical argument of density that $h \in C^1(\overline{B})$. One sets

$$A = \int_B \int_B (h(z) - h(x))^2 \, dx \, dz.$$

Using the Cauchy-Schwarz inequality, we get

$$A \leq (2\delta)^2 d \int_B \int_B \int_0^1 (\nabla h(x + \theta(z - x)))^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the changes of variable $z \to z' = z - x \in B_2 := B(0, 2\delta)$, $x \to x' = x + \theta z'$, we get

$$A \leq (2\delta)^2 d \int_{B_2} \int_B (\nabla h(x))^2 \, dx' \, dz',$$

which gives the existence of some C_{33}, only depending on d, such that

$$A \leq C_{33} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx.$$

This concludes the proof of (9.66).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (9.64) follows, remarking that in such a case

$$\int_K g^2(x) \, dx = \int_K \left(g(x) - \frac{1}{m_K} \int_K g(y) \, dy \right)^2 dx \leq \frac{1}{m_K} \int_K \int_K (g(x) - g(y))^2 \, dx \, dy.$$

Lemma 9.14 *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from K to $B(0, \delta(K))$. One denotes by ξ an upper bound of both Lipschitz constants. Let $a \subset \partial K$, such that there exists $x_0 \in a$ and $\zeta > 0$ with*

$$\partial K \cap B(x_0, \zeta\delta(K)) \subset a$$

Let m_a denote the $d-1$ Lebesque measure of a. Let $\mathbf{q} \in H(\mathrm{div}, K)$ such that $\mathbf{q} \cdot \mathbf{n}_{\partial K} \in L^2(\partial K)$ and there exists $q_a \in \mathbb{R}$ with $\mathbf{q}(x) \cdot \mathbf{n}_{\partial K}(x) = q_a$ for a.e. $x \in a$.
Then there exists C_{16}, only depending on d, ξ and ζ, such that

$$m_a q_a^2 \leq C_{16} \left(\frac{1}{\delta} \int_K \mathbf{q}^2(x) \, dx + \delta \int_K (\mathrm{div} \ \mathbf{q}(x))^2 \, dx \right) \tag{9.67}$$

Proof. Denoting $\delta = \delta(K)$, let $X \in \partial B(0,\delta)$ and $\eta \in (0,1]$. We have $\{Z \in \partial B(0,\delta) \mid Z\cdot X \geq (1-\eta)\delta^2\} = \partial B(0,\delta)\cap B(X,\sqrt{2\eta}\delta)$. Indeed, take $Z \in \partial B(0,\delta)$ and denote $h = Z - X$. One has, since $|Z|^2 = |X|^2 = \delta^2$, $|h|^2 = 2\delta^2 - 2Z\cdot X$; thus, $|h|^2 \leq 2\eta\delta^2$ if and only if $Z\cdot X \geq (1-\eta)\delta^2$.
Define

$$\mathcal{B}_\eta = \{y \in \partial K \mid \mathcal{L}(y)\cdot\mathcal{L}(x_0) \geq (1-\eta)\delta^2\} = \mathcal{L}^{-1}(\partial B(0,\delta)\cap B(\mathcal{L}(x_0),\sqrt{2\eta}\delta)).$$

Let $F(x) = (|x|/\sup_{i\in[1,d]}|x_i|)x$. $\mathcal{L}^{-1}\circ F^{-1}$ is a Lipschitz continuous homeomorphism with Lipschitz-continuous inverse mapping between K and $Q_\delta =]-\delta,\delta[^d$; moreover, the Lipschitz constants of $\mathcal{L}^{-1}\circ F^{-1}$ and its inverse mapping are bounded by a real number only depending on d and ξ. Thus, by Lemma 9.11 applied to $f = \chi_{\mathcal{B}_\eta}$,

$$\gamma(\mathcal{B}_\eta) \geq C_{34}\,\gamma(F\circ\mathcal{L}(\mathcal{B}_\eta)) = C_{34}\,\gamma(F(\partial B(0,\delta)\cap B(\mathcal{L}(x_0),\sqrt{2\eta}\delta))),$$

with C_{34} only depending on d and ξ. It is easy to see that $\gamma(F(\partial B(0,\delta)\cap B(\mathcal{L}(x_0),\sqrt{2\eta}\delta))) \geq C_{35}\,\delta^{d-1}$, where C_{35} only depends on d and η (the set $F(\partial B(0,\delta)\cap B(\mathcal{L}(x_0),\sqrt{2\eta}\delta))$ contains a significant part of a $(d-1)$-dimensional ball on ∂Q_δ with radius of order δ). Thus, one has

$$\gamma(B_\eta) \geq C_{36}\,\delta^{d-1}, \tag{9.68}$$

with C_{36} only depends on d, ξ and η.
Now, let $\eta_0 = \inf(1,(\zeta/\xi)^2/2) \in (0,1]$ (η_0 only depends on ζ and ξ); since $\mathcal{L}^{-1}$ is Lipschitz-continuous with constant ξ, one has

$$\mathcal{B}_{\eta_0} \subset \partial K \cap B(x_0,\zeta\delta) \subset a. \tag{9.69}$$

Let us define the function $v \in H^1(K)$ by

$$v(x) = \psi\left(\frac{\mathcal{L}(x)\cdot\mathcal{L}(x_0)}{\delta^2}\right),\ \forall x \in K,$$

where the function $\psi \in C([-1,1],[0,1])$ is defined by $\psi(s) = 0$ for all $s \in [-1,1-\eta_0]$, $\psi(s) = \frac{2(s+\eta_0-1)}{\eta_0}$ for all $s \in [1-\eta_0,1-\eta_0/2]$, $\psi(s) = 1$ for all $s \in [1-\eta_0/2,1]$. One has therefore $v(x) \in [0,1]$ for all $x \in \overline{K}$, $v = 1$ on $\mathcal{B}_{\eta_0/2}$ and $v = 0$ on $\partial K\setminus\mathcal{B}_{\eta_0} \supset \partial K \setminus a$ and

$$\nabla v(x) = \frac{\psi'\left(\frac{\mathcal{L}(x)\cdot\mathcal{L}(x_0)}{\delta^2}\right)}{\delta^2}(D\mathcal{L}(x))^T\mathcal{L}(x_0).$$

Thus, since $|\mathcal{L}(x_0)| \leq \delta$, we have $\|\nabla v\|_{L^\infty(K)} \leq \frac{C_{37}}{\delta}$ where C_{37} only depends on d, ξ and ζ. For all $x \in \partial K\setminus a$, $v(x) = 0$, and therefore the following relation holds

$$\int_K \nabla v(x)\cdot\mathbf{q}(x)\,dx = -\int_K v(x)\ \mathrm{div}\ \mathbf{q}(x)\,dx + q_a\int_a v(x)\,d\gamma(x).$$

We have $\int_a v(x)\,d\gamma(x) \geq \gamma(\mathcal{B}_{\eta_0/2})$ (because v is non-negative and has value 1 on $\mathcal{B}_{\eta_0/2}$) and thus, by (9.68), $\int_a v(x)\,d\gamma(x) \geq C_{38}\,\delta^{d-1}$ with C_{38} only depending on d, ξ and ζ. Since $||\nabla v(x)||_{L^\infty(K)} \leq \frac{C_{37}}{\delta}$ and $m_K \leq C_{39}\,\delta^d$, one therefore gets

$$q_a^2 \leq C_{40}\left(\delta^{d-2-2(d-1)}\int_K \mathbf{q}(x)^2\,dx \ +\ \delta^{d-2(d-1)}\int_K(\mathrm{div}\ \mathbf{q}(x))^2\,dx\right),$$

which leads to (9.67), since $m_a \leq C_{41}\,\delta^{d-1}$.

Chapitre 10

Parabolic Capacity and soft measures for nonlinear equations

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10.1 Introduction

Let Ω be a bounded, open subset of $\mathbf{R}^N$, T a positive number and $Q =]0,T[\times\Omega$. Let p be a real number, with $1 < p < \infty$, and let p' be its conjugate Hölder exponent (i.e. $1/p + 1/p' = 1$).
In this paper we deal with the parabolic initial boundary value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in }]0,T[\times\Omega, \\ u = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{10.1}$$

where A is a nonlinear monotone and coercive operator in divergence form which acts from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a model example, problem (10.1) includes the p–Laplace evolution equation:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu & \text{in }]0,T[\times\Omega, \\ u = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{10.2}$$

The main feature of our study is the presence of singular data μ and u_0, which are bounded measures (respectively on Q and on Ω). It is well known that, if $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, J. L. Lions [51] proved existence and uniqueness of a weak solution. Under the general assumption that μ and u_0 are bounded measures, the existence of a distributional solution was proved in [9], by approximating (10.1) with problems having regular data and using compactness arguments.
Unfortunately, due to the lack of regularity of the solutions, the distributional formulation is not strong enough to have uniqueness, as it can be proved by adapting the counterexample of J. Serrin for the stationary problem (see [68] and the refinement in [65]). In case of linear operators the difficulty can be overcome by defining the solution through the adjoint operator, this method is used in [70] for the stationary problem and yields a formulation having a unique solution. However, for nonlinear operators a new concept of solution needs to be defined to get a well–posed problem. In case of problem (10.1) with L^1 data, this was done independently in [5] and in [64] (see also [3]), where the notions of *renormalized* solution, and of *entropy* solution, respectively, were introduced. Both these approaches are able to obtain existence and uniqueness of solutions if $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$.
Our main goal here is to extend the result of existence and uniqueness to a larger class of measures which includes the L^1 case. Precisely, we prove (in the framework of renormalized solutions) that problem

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(10.1) has a unique solution for every measure μ which is zero on subsets of zero capacity, where the notion of capacity is suitably defined according to the operator $u_t + A(u)$. In fact, the importance of the measures not charging sets of null capacity was first observed in the stationary case in [12], where the authors prove existence and uniqueness of entropy solutions (as introduced in [4]) of the elliptic problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10.3}$$

if μ is a measure which does not charge the sets of zero p–capacity, i.e. the capacity defined from the Sobolev space $W_0^{1,p}(\Omega)$. Actually, this result relies on the fact that every such measure belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.
In order to use a similar approach in the evolution case, one needs to develop the theory of capacity related to the parabolic operator $u_t + A(u)$ and then investigate the relationships between time–space dependent measures and capacity. We introduce here the notion of capacity defined from the parabolic p–laplace equation in the same spirit of [62], where the standard notion of capacity constructed from the heat operator is presented in a useful functional approach (without any tool of potential theory or linear arguments). Indeed, letting

$$W = \left\{ u \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega)),\ u_t \in L^{p'}(0,T; (W_0^{1,p}(\Omega) \cap L^2(\Omega))') \right\},$$

we define the capacity of a set B as, roughly speaking, minimizing the norm of W for functions greater than 1 on B. This approach allows us to use the same arguments as in [20] and then to obtain a representation theorem for measures that are zero on subsets of Q that are of zero capacity (see Definition 10.5).
Thus our first main result extends the one in [12] for stationary measures and capacity.

Theorem 10.1 *Let μ be a bounded measure on Q which does not charge the sets of null capacity. Then there exist $g_1 \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$ and $h \in L^1(Q)$, such that*

$$\int_Q \varphi\, d\mu = \int_0^T \langle g_1, \varphi \rangle\, dt - \int_0^T \langle \varphi_t, g_2 \rangle\, dt + \int_Q h\, \varphi\, dxdt\,, \tag{10.4}$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $(W_0^{1,p}(\Omega)\cap L^2(\Omega))'$ and $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ ([3]).

Thanks to this decomposition result, for such class of measures (continuous with respect to capacity) we can still set our problem (10.1) in the framework of renormalized solutions. The idea is that, since μ can be splitted as in (10.4), problem (10.1) can be formally rewritten as $(u - g_2)_t + A(u) = g_1 + h$, and the renormalization argument can be applied to the difference $u - g_2$. We leave to Section 3 the precise definition of renormalized solution, let us state here our result of existence and uniqueness of renormalized solutions.

Theorem 10.2 *Let μ be a bounded measure on Q which is zero on subsets of Q that have zero capacity, and let $u_0 \in L^1(\Omega)$. Then there exists a unique renormalized solution u (see Definition 10.7) of (10.1). Moreover u satisfies the additional regularity: $u \in L^\infty(0,T; L^1(\Omega))$ and $T_k(u) = \max(-k, \min(k,u)) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$.*

Let us stress that, as far as the initial datum is concerned, the class of measure data which do not charge the parabolic capacity of the operator reduces to consider u_0 in $L^1(\Omega)$, so that no improvement can be obtained with respect to previous results. This is a consequence of the following lemma, which we prove in Section 2.

[3] Notice that, since $W^{-1,p'}(\Omega) \hookrightarrow (W_0^{1,p}(\Omega) \cap L^2(\Omega))'$, we have $g_1 \in L^{p'}(0,T; (W_0^{1,p}(\Omega) \cap L^2(\Omega))')$ so that the term involving g_1 in (10.4) is well defined.

Theorem 10.3 *Let B be a Borel set in Ω. Let $t_0 \in]0,T[$ fixed. One has*

$$\mathrm{cap}_p(\{t_0\} \times B) = 0 \quad \textit{if and only if} \quad \mathrm{meas}_{\Omega}(B) = 0.$$

A counterpart of Lemma 10.3 will also be proved (Theorem 10.6), stating that, for any interval $(t_0, t_1) \subset (0,T)$, $\mathrm{cap}_p((t_0,t_1) \times B) = 0$ if and only if the elliptic capacity (defined from $W_0^{1,p}(\Omega)$) of B is zero.
The plan of the paper is the following. In the next section, we give the definition and prove the basic properties of parabolic capacity, among which the existence of a unique cap–quasi continuous representative for functions in W. We also prove Theorem 10.3 as far as the restriction of capacity to sections $\{t\} \times \Omega$ is concerned. We investigate then the link between measures defined on the σ–algebra of borelians of Q and the previously defined capacity, and we prove the decomposition theorem stated above. In the third section we give first a result of existence and uniqueness for (10.1) if $\mu \in W'$, the dual space of W, which seems a natural extension of the classical result of J.L. Lions. Finally, we give the definition of renormalized solution and we prove existence and uniqueness.

In the sequel C will denote a constant that can change from line to line. For v a function of (t,x) and for k a real number, we will denote, for example, $\{v > k\}$ the set $\{(t,x) \in Q : v > k\}$, while χ_A denotes the characteristic function of a set A.

10.2 Parabolic capacity and measures

10.2.1 Capacity

The approach followed to define the capacity is in the same spirit of Pierre ([62]).

Definition 10.1 *Let us define $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$, and*

$$W = \left\{ u \in L^p(0,T;V),\ u_t \in L^{p'}(0,T;V') \right\},$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$. We will also use the non-homogeneous (when $p \neq 2$) quantity, linked to the energy estimates,

$$[u]_W = \|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;V')} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}. \tag{10.5}$$

Remark 10.1 *Since $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, we see that W is continuously embedded in $\mathcal{C}([0,T];L^2(\Omega))$ (cf [24]), which means that there exists $C > 0$ such that, for all $u \in W$, $\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C\,\|u\|_W$. Thus one has, for all $u \in W$,*

$$[u]_W \leq C\ \max\left\{ \|u\|^p_W, \|u\|^{p'}_W \right\}, \quad \|u\|_W \leq C\ \max\left\{ [u]_W^{\frac{1}{p}}, [u]_W^{\frac{1}{p'}} \right\}. \tag{10.6}$$

Remark 10.2 *When $\theta \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^N)$ and $u \in W$, then $\theta u \in W$ and there exists $C(\theta)$ not depending on u such that $\|\theta u\|_W \leq C(\theta)\|u\|_W$. Indeed, when $u \in L^p(0,T;V)$, it is quite obvious, by the regularity of θ, that $\theta u \in L^p(0,T;V)$ with $\|\theta u\|_{L^p(0,T;V)} \leq C(\theta)\|u\|_{L^p(0,T;V)}$. For the time derivative, it is a little bit tricky; we have, in the sense of distributions, $(\theta u)_t = \theta_t u + \theta u_t$. The second term is not a problem: since $u_t \in L^{p'}(0,T;V')$, one has $\theta u_t \in L^{p'}(0,T;V')$ and $\|\theta u_t\|_{L^{p'}(0,T;V')} \leq C(\theta)\|u_t\|_{L^{p'}(0,T;V')}$. For the first term, that is $\theta_t u$, we must use the injection of W in $\mathcal{C}([0,T];L^2(\Omega))$, thus also in $L^{p'}(0,T;L^2(\Omega))$; thanks to this injection, it is then easy to get $\theta_t u \in L^{p'}(0,T;L^2(\Omega))$ with $\|\theta_t u\|_{L^{p'}(0,T;L^2(\Omega))} \leq C(\theta)\|u\|_W$; $L^2(\Omega)$ being injected in V', we have $L^{p'}(0,T;L^2(\Omega)) \hookrightarrow L^{p'}(0,T;V')$ which gives $\theta_t u \in L^{p'}(0,T;V')$ and $\|\theta_t u\|_{L^{p'}(0,T;V')} \leq C(\theta)\|u\|_W$.*

Remark 10.3 *Since* $L^{p'}(0,T;V') = (L^p(0,T;V))'$ *(see [32], V is a separable reflexive space), and since* $L^p(0,T;V) = L^p(0,T;W_0^{1,p}(\Omega)) \cap L^p(0,T;L^2(\Omega)) = E \cap F$, *with* $E \cap F$ *being dense in* E *and* F, *we have* $L^{p'}(0,T;V') = E' + F' = L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega))$ *and the norms of these spaces are equivalent.*

In fact, the natural space that appears in the study of the p–laplacian parabolic operator is not W but $\widetilde{W} \subset W$, defined as follows.

Definition 10.2 *We define*

$$\widetilde{W} = \left\{ u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \right\}.$$

Remark 10.4 $\widetilde{W}$ *is continuously embedded in* W.

We will define the parabolic capacity using the space W, whereas a more natural definition would perhaps start from $\widetilde{W}$. However, using this space instead of W would entail some technical difficulties and since, as we will notice, the sets of null capacity with regards to W are the same than the sets of null capacity with regards to $\widetilde{W}$, there is no loss in working with W instead of $\widetilde{W}$ (see Remark 10.7).

Definition 10.3 *If* $U \subset Q$ *is an open set, we define*

$$\mathrm{cap}_p(U) = \inf\{\|u\|_W : u \in W,\ u \geq \chi_U \text{ almost everywhere in } \quad Q\} \tag{10.7}$$

(we will use the convention that $\inf \emptyset = +\infty$*), then for any borelian subset* $B \subset Q$ *the definition is extended by setting:*

$$\mathrm{cap}_p(B) = \inf\left\{\mathrm{cap}_p(U),\ U \textit{ open subset of } Q,\ B \subset U\right\}. \tag{10.8}$$

Proposition 10.1 *The capacity previously defined satisfies the subadditivity property, that is*

$$\mathrm{cap}_p\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathrm{cap}_p(E_i), \tag{10.9}$$

for every collection of borelian sets E_i.

Proof. Let, for all $i \geq 1$, U_i be an open set containing E_i such that $\mathrm{cap}_p(U_i) \leq \mathrm{cap}_p(E_i) + \frac{\varepsilon}{2^i}$, and let u_i be such that $u_i \geq \chi_{U_i}$ a. e. in Q and $\|u_i\|_W \leq \mathrm{cap}_p(U_i) + \frac{\varepsilon}{2^i}$. Without loss of generality we can assume that $\sum_{i=1}^{\infty} \mathrm{cap}_p(E_i) < \infty$ (otherwise (10.9) is trivial); this implies that $\sum_{i=1}^{\infty} u_i$ is strongly convergent in W. Let then $u = \sum_{i\geq 1} u_i$; clearly $u \geq \chi_U$ a.e. in Q where $U = \bigcup_{i=1}^{\infty} U_i$, so that, U being open,

$$\mathrm{cap}_p(U) \leq \|u\|_W \leq \sum_{i=1}^{\infty} \|u_i\|_W \leq \sum_{i=1}^{\infty} \mathrm{cap}_p(E_i) + 2\varepsilon.$$

Since $\bigcup_{i=1}^{\infty} E_i \subset U$ this implies (10.9). ∎

Remark 10.5 *As usual, the capacity defined above depends in fact of the open ambient set* Q *and we should have denoted* $\mathrm{cap}_p(B,Q)$ *to stress on this dependance. However, Proposition 10.1, along with Remark 10.2, allows to see that, when* B *is a borel set of* Q *and* $\mathrm{cap}_p(B,Q) = 0$, *then* $\mathrm{cap}_p(B,U) = 0$ *for all open sets* $U \subset Q$ *containing* B. *Indeed, take a sequence of compacts* $K_n \subset U$ *with* $U = \bigcup_{n\geq 1} K_n$, *then we have* $\mathrm{cap}_p(B,U) = \mathrm{cap}_p(\cup_{n\geq 1} B \cap K_n, U) \leq \sum_{n\geq 1} \mathrm{cap}_p(B \cap K_n, U)$. *Since* K_n *is a compact subset of* U *and since* $\mathrm{cap}_p(B \cap K_n, Q) = 0$, *we can prove, using a nonnegative function* $\zeta_n \in \mathcal{C}_c^\infty(U)$ *such that* $\zeta_n \equiv 1$ *on a neighborhood of* K_n, *that* $\mathrm{cap}_p(B \cap K_n, U) = 0$ *for any* n, *which proves our assertion.*

The definition of capacity can be alternatively done starting from the compact sets in Q, as follows. We denote $\mathcal{C}_c^\infty([0,T]\times\Omega)$ the space of restrictions to Q of smooth functions in $\mathbb{R}\times\mathbb{R}^N$ with compact support in $\mathbb{R}\times\Omega$.

Definition 10.4 *Let K be a compact subset of Q. The p-capacity of K with respect to Q is defined as:*

$$\mathrm{CAP}(K) = \inf\left\{\|u\|_W : u\in\mathcal{C}_c^\infty([0,T]\times\Omega),\ u\geq\chi_K\right\}.$$

The p-capacity of any open subset U of Q is then defined by:

$$\mathrm{CAP}(U) = \sup\left\{\mathrm{CAP}(K),\ K \text{ compact},\ K\subset U\right\},$$

and the p-capacity of any Borelian set $B\subset Q$ by

$$\mathrm{CAP}(B) = \inf\left\{\mathrm{CAP}(U),\ U \textit{ open subset of } Q,\ B\subset U\right\}.$$

This second definition of capacity given for compact subsets is motivated by the following theorem.

Theorem 10.4 *Let Ω be an open bounded set in $\mathbb{R}^N$ and $1<p<\infty$. Then $\mathcal{C}_c^\infty([0,T]\times\Omega)$ is dense in W.*

The proof of this theorem will be given in the appendix.

Remark 10.6 *Notice also that, when $u\in W$ has a compact support in Q and $(\rho_n)_{n\geq1}$ is a space-time regularizing kernel, then $u*\rho_n$ is well defined (at least for n large enough), is a function of $\mathcal{C}_c^\infty(Q)$ and $u*\rho_n\to u$ in W (see Lemma 10.7 in the appendix).*

Proposition 10.2 *The capacity* CAP *satisfies the subadditivity property.*

Proof. Let us first prove the subadditivity for finite unions of open sets, starting from compact sets. Indeed, let K_1, K_2 be compact subsets of Q, then there exist two functions u_1, $u_2\in\mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $u_i\geq\chi_{K_i}$ and $\|u_i\|_W\leq\mathrm{CAP}(K_i)+\varepsilon$, $i=1,2$. Since

$$u_1+u_2\in\mathcal{C}_c^\infty([0,T]\times\Omega)\,,\, u_1+u_2\geq\chi_{K_1\cup K_2}\,,\quad \|u_1+u_2\|_W\leq\|u_1\|_W+\|u_2\|_W\,,$$

it follows that $\mathrm{CAP}(K_1\cup K_2)\leq\mathrm{CAP}(K_1)+\mathrm{CAP}(K_2)$. Let now A, B be open subsets of Q, and let K be a compact subset of $A\cup B$. It is easy to find compact subsets K_A, K_B such that $K=K_A\cup K_B$, with $K_A\subset A$ and $K_B\subset B$ (for instance, define $F=\{z\in A : \mathrm{dist}(z,A^c)\geq\frac{m}{2}\}$ where $m=\min\limits_{z\in K}[\mathrm{dist}(z,A^c)+\mathrm{dist}(z,B^c)]$, then $K_A=K\cap F$ and $K_B=K\cap\overline{F^c}$ fit the requirement). Therefore we have $\mathrm{CAP}(K)\leq\mathrm{CAP}(K_A)+\mathrm{CAP}(K_B)\leq\mathrm{CAP}(A)+\mathrm{CAP}(B)$, and taking the supremum over $K\subset A\cup B$ we get

$$\mathrm{CAP}(A\cup B)\leq\mathrm{CAP}(A)+\mathrm{CAP}(B)\,,\quad \text{for all open sets}\quad A\,,B\subset Q\,. \tag{10.10}$$

Finally, let $\{E_i\}_{i\geq1}$ be borelian subsets of Q, and let $E=\bigcup_{i\geq1}E_i$. Assume that $\sum_{i\geq1}\mathrm{CAP}(E_i)<\infty$ and let A_i be open sets such that $E_i\subset A_i$ and $\mathrm{CAP}(A_i)\leq\mathrm{CAP}(E_i)+\frac{\varepsilon}{2^i}$, so that $\sum_{i\geq1}\mathrm{CAP}(A_i)\leq\sum_{i\geq1}\mathrm{CAP}(E_i)+\varepsilon$. Let $A=\bigcup_{i\geq1}A_i$, and take a compact subset $K\subset A$. Since the A_i are a covering of K, there exists a finite number l such that $K\subset\bigcup_{i=1}^{l}A_i$, hence using (10.10) we get

$$\mathrm{CAP}(K)\leq\mathrm{CAP}\left(\bigcup_{i=1}^{l}A_i\right)\leq\sum_{i=1}^{l}\mathrm{CAP}(A_i)\leq\sum_{i=1}^{\infty}\mathrm{CAP}(E_i)+\varepsilon\,.$$

Taking the supremum over $K\subset A$ and since $E\subset A$ we have

$$\mathrm{CAP}(E)\leq\mathrm{CAP}(A)\leq\sum_{i=1}^{\infty}\mathrm{CAP}(E_i)+\varepsilon\,,$$

which concludes the proof as ε tends to zero. ■

Note that, in the elliptic case, the two possible constructions of the capacity in the space $W_0^{1,p}(\Omega)$ (from the open sets of from the compacts) coincide. Here, we are not able to prove the same result (because of approximation difficulties), nevertheless we have that both capacities yield the same sets of zero capacity, which is in fact what matters.

Proposition 10.3 *Let B be a borelian subset of Q. Then one has* $\mathrm{CAP}(B) = 0$ *if and only if* $\mathrm{cap}_p(B) = 0$.

Proof. We first prove that $\mathrm{CAP}(B) \geq \mathrm{cap}_p(B)$ for every borelian set B, which will imply $\mathrm{cap}_p(B) = 0$ whenever $\mathrm{CAP}(B) = 0$.
Indeed, let A be open. Assume that $\mathrm{CAP}(A)$ is finite, and let $K_n = \{x \in A : \mathrm{dist}(x, \partial A) \geq \frac{1}{n}\}$. By definition there exists a sequence $\{\varphi_n\}$ of functions in $\mathcal{C}_c^\infty([0,T] \times \Omega)$ such that

$$\varphi_n \geq \chi_{K_n} \quad \text{in } Q, \qquad \|\varphi_n\|_W \leq \mathrm{CAP}(K_n) + \frac{1}{n} \leq \mathrm{CAP}(A) + \frac{1}{n}\,.$$

In particular we have that φ_n is a bounded sequence in W, which is a reflexive space, so that there exists a subsequence, not relabeled, and a function $\varphi \in W$ such that:

$$\begin{aligned} &\varphi_n \to \varphi && \text{weakly in } L^p(0,T;V), \\ &(\varphi_n)_t \to \varphi_t && \text{weakly in } L^{p'}(0,T;V'), \\ &\varphi_n \to \varphi && \text{almost everywhere in } Q. \end{aligned}$$

Last convergence is a consequence of standard compactness arguments (see [69]). Since $\varphi_n \geq \chi_{K_n}$ for every n, we deduce that $\varphi \in W$ and $\varphi \geq \chi_A$ almost everywhere in Q, so that φ can be used in Definition 10.3 above. By lower semicontinuity of the norm we get, as n tends to infinity:

$$\|\varphi\|_W \leq \mathrm{CAP}(A)\,,$$

which yields that $\mathrm{cap}_p(A) \leq \mathrm{CAP}(A)$. This inequality being satisfied for all open sets A, we deduce from the definition that it is also true for all borelians of Q.
Now, let us obtain the reverse implication. We take B a borelian such that $\mathrm{cap}_p(B) = 0$. Since CAP is sub-additive, it is enough to prove that, for any compact $K \subset Q$, one has $\mathrm{CAP}(B \cap K) = 0$. We take thus K a compact subset of Q and $\zeta \in \mathcal{C}_c^\infty(Q)$ such that $\zeta = 1$ on an open set O which contains K.
Since $\mathrm{cap}_p(B) = 0$, there exists, for all $\varepsilon > 0$, an open set A_ε containing B such that $\mathrm{cap}_p(A_\varepsilon) < \varepsilon$; we can then take $u \in W$ such that

$$u \geq \chi_{A_\varepsilon} \qquad \text{a.e. in } Q, \qquad \|u\|_W \leq 2\varepsilon\,.$$

We have $\zeta u \in W$ and $\|\zeta u\|_W \leq 2C(\zeta)\varepsilon$, with $C(\zeta)$ only depending on ζ (see Remark 10.2).
We will now estimate $\mathrm{CAP}(A_\varepsilon \cap O)$. Let L be a compact subset of $A_\varepsilon \cap O$ and $(\rho_n)_{n\geq 1}$ be a regularizing kernel in $\mathbb{R} \times \mathbb{R}^N$; since ζu has a compact support in Q, $(\zeta u) * \rho_n \in \mathcal{C}_c^\infty(Q)$ is well defined (at least for n large enough) and $(\zeta u) * \rho_n$ strongly converges to ζu in W (see Remark 10.6 and the appendix). We can thus fix $n(L,\varepsilon)$ such that $\|(\zeta u) * \rho_{n(L,\varepsilon)} - (\zeta u)\|_W \leq \varepsilon$ and $(\zeta u) * \rho_{n(L,\varepsilon)} \geq 1$ in L (recall that $\zeta u \geq 1$ on the open set $A_\varepsilon \cap O$ and that L is a compact subset of $A_\varepsilon \cap O$); with this choice of $n(L,\varepsilon)$, $v = (\zeta u) * \rho_{n(L,\varepsilon)} \in \mathcal{C}_c^\infty(Q) \subset \mathcal{C}_c^\infty([0,T] \times \Omega)$ and $v \geq \chi_L$. Thus, $\mathrm{CAP}(L) \leq \|v\|_W \leq \|v - \zeta u\|_W + \|\zeta u\|_W \leq (1 + 2C(\zeta))\varepsilon$. This being true for any compact subset L of the open set $A_\varepsilon \cap O$, we deduce that $\mathrm{CAP}(A_\varepsilon \cap O) \leq (1 + 2C(\zeta))\varepsilon$. But $B \cap K \subset A_\varepsilon \cap O$, so that $\mathrm{CAP}(B \cap K) \leq (1 + 2C(\zeta))\varepsilon$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we deduce that $\mathrm{CAP}(B \cap K) = 0$. ■

However, henceforth we will make use of Definition 10.3 of capacity, which can be handled more easily in many situations, as in the following result, where we give the characterization of sets of null capacity contained in the sections $\{t_0\} \times \Omega$ of the parabolic cylinder.

Theorem 10.5 *Let B be a borelian set in Ω. Let $t_0 \in]0,T[$ fixed. One has*

$$\mathrm{cap}_p(\{t_0\} \times B) = 0 \quad \textit{if and only if} \quad \mathrm{meas}_\Omega(B) = 0.$$

Proof. Assume first that $\mathrm{cap}_p(\{t_0\} \times B) = 0$ and let K be any compact set contained in B, so that $\mathrm{cap}_p(\{t_0\} \times K) = 0$. Since, by Proposition 10.3, we also have that $\mathrm{CAP}(\{t_0\} \times K) = 0$, then, for all $\delta > 0$ there exists a function $\psi_\delta \in \mathcal{C}_c^\infty([0,T] \times \Omega)$ such that $\|\psi_\delta\|_W \le \delta$ and $\psi_\delta(t_0) \ge 1$ on K. Since W is continuously imbedded in $\mathcal{C}([0,T], L^2(\Omega))$, we have

$$\mathrm{meas}_\Omega(K) \le \int_K |\psi_\delta(t_0)|^2\, dx \le \|\psi_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\|\psi_\delta\|^2_W \le C\delta^2,$$

so we deduce that $\mathrm{meas}_\Omega(K) \le C\delta^2$, and from the arbitrariness of δ then $\mathrm{meas}_\Omega(K) = 0$. Since this is true for any compact subset contained in B, by regularity of the Lebesgue measure we conclude that $\mathrm{meas}_\Omega(B) = 0$.

Conversely, if $\mathrm{meas}_\Omega(B) = 0$ then there exists, for all $\delta > 0$, an open set A_δ such that $B \subset A_\delta$ and $\mathrm{meas}_\Omega(A_\delta) < \delta$. Let us consider δ fixed in the following, and let K_n be a sequence of compact sets contained in A_δ such that $K_n \subset K_{n+1} \subset \ldots$, $\bigcup_{n=1}^\infty K_n = A_\delta$. Let $\varphi_n \in C_c(A_\delta)$ be such that $0 \le \varphi_n \le 1$, $\varphi_n \equiv 1$ on K_n and $\varphi_n \le \varphi_{n+1}$. Then we solve for $t \in [t_0, T]$,

$$\begin{cases} (\psi_n)_t - \mathrm{div}(|\nabla \psi_n|^{p-2}\nabla\psi_n) = 0 & \text{in }]t_0,T[\times\Omega, \\ \psi_n = 0 & \text{on }]t_0,T[\times\partial\Omega, \\ \psi_n(t_0) = \varphi_n & \text{in } \Omega. \end{cases} \tag{10.11}$$

Clearly we have that $\psi_n \in L^p(t_0,T;W_0^{1,p}(\Omega)) \cap L^\infty(t_0,T;L^2(\Omega))$ and $(\psi_n)_t \in L^{p'}(t_0,T;W^{-1,p'}(\Omega))$. Let us construct a function $\widetilde{\psi}_n$ defined on $[0,T]$, by setting

$$\begin{cases} \widetilde{\psi}_n = \psi_n & \text{in }]t_0,T] \times \Omega, \\ \widetilde{\psi}_\delta = \psi_n\left(T - \dfrac{t(T-t_0)}{t_0}\right) & \text{in } [0,t_0] \times \Omega. \end{cases}$$

It is not difficult to see that $\widetilde{\psi}_n$ belongs to W and by the energy estimates obtained from (10.11) by using ψ_n itself as test function we have (recall the notation in (10.5)) :

$$[\widetilde{\psi}_n]_W \le C\|\varphi_n\|^2_{L^2(\Omega)} \le C\mathrm{meas}(A_\delta) \le C\delta\,. \tag{10.12}$$

By regularity results on the p–laplacian evolution equation (see [27]) we have that ψ_n is continuous in $[t_0,T] \times \Omega$, hence $\widetilde{\psi}_n \in \mathcal{C}([0,T] \times \Omega)$. Thus we can define the open set $U_n := \{\widetilde{\psi}_n > \frac{1}{2}\}$. Since U_n is open and $2\widetilde{\psi}_n \ge \chi_{U_n}$ we have

$$\mathrm{cap}_p(U_n) \le 2\|\widetilde{\psi}_n\|_W \le C\max(\delta^{\frac{1}{p}}, \delta^{\frac{1}{p'}})\,. \tag{10.13}$$

Since $\{\varphi_n\}$ is nondecreasing we have that $\{\widetilde{\psi}_n\}$ is nondecreasing as well, hence $U_n \subset U_{n+1}$, and $\mathrm{cap}_p(U_n)$ is also a nondecreasing sequence, and bounded too. Setting $U_\infty = \bigcup_{n=1}^\infty U_n$, we have that

$$\mathrm{cap}_p(U_\infty) = \lim_{n\to\infty} \mathrm{cap}_p(U_n)\,. \tag{10.14}$$

Indeed, since $U_n \subset U_\infty$ we have $\lim_{n\to\infty} \mathrm{cap}_p(U_n) \le \mathrm{cap}_p(U_\infty)$. On the other hand, let $u_n \in W$ be such that

$$u_n \ge \chi_{U_n} \quad \text{a.e. in } Q \qquad \text{and} \qquad \|u_n\|_W \le \mathrm{cap}_p(U_n) + \frac{1}{n}\,,$$

(in fact, it can also be chosen u_n such that $\|u_n\|_W = \mathrm{cap}_p(U_n)$, but this is not essential). It follows from (10.13) that u_n is a bounded sequence in W, hence there exists a function $u \in W$ such that, up to a subsequence,

$$u_n \to u \qquad \text{weakly in } W \text{ and a.e. in } Q.$$

The almost everywhere convergence of this subsequence and the fact that $(U_n)_{n\geq 1}$ is nondecreasing imply that $u \geq \chi_{U_\infty}$ almost everywhere in Q; since U_∞ is open, we get

$$\mathrm{cap}_p(U_\infty) \leq \|u\|_W \leq \liminf_{n\to\infty} \|u_n\|_W \leq \lim_{n\to\infty} \mathrm{cap}_p(U_n)\,,$$

so that (10.14) is proved. Since $\varphi_n = 1$ on K_n for each n and $A_\delta = \bigcup_{n=1}^\infty K_n$, we have that U_∞ is an open set which contains $\{t_0\}\times A_\delta \supset \{t_0\}\times B$, so that we conclude from (10.14) and (10.13)

$$\mathrm{cap}_p(\{t_0\}\times B) \leq \mathrm{cap}_p(U_\infty) \leq C\max(\delta^{\frac{1}{p}}, \delta^{\frac{1}{p'}})\,,$$

which implies that $\mathrm{cap}_p(\{t_0\}\times B) = 0$. ■

The following result can be considered a counterpart of the previous result, since we consider subsets $(0,T)\times B$, $B\subset\Omega$.

Theorem 10.6 *Let $B\subset\Omega$ be a borelian set, and $0\leq t_0 < t_1 \leq T$. Then we have*

$$\mathrm{cap}_p((t_0,t_1)\times B) = 0 \quad \textit{if and only if} \quad \mathrm{cap}_p^e(B) = 0,$$

where cap_p^e denotes the elliptic capacity defined from $W_0^{1,p}(\Omega)$.

Proof.
If $\mathrm{cap}_p^e(B) = 0$, then there exists, for all $0<\delta<1$, an open set U_δ with $B\subset U_\delta$ such that $\mathrm{cap}_p^e(U_\delta) < \delta$. It is then a well-known result of the elliptic capacity that we can choose $v_\delta \in W_0^{1,p}(\Omega)$ with $1 \geq v_\delta \geq \chi_{U_\delta}$ a.e. in Ω and $||v_\delta||_{W_0^{1,p}(\Omega)} \leq \delta$. If $p\geq 2$, this also gives $||v_\delta||_{L^2(\Omega)} \leq C\delta$ (C not depending on δ); if $p<2$, since $|v_\delta|\leq 1$, we have $\int_\Omega |v_\delta|^2 \leq \int_\Omega |v_\delta|^p \leq C\delta^p$ (C still not depending on δ), that is to say $||v_\delta||_{L^2(\Omega)} \leq \sqrt{C}\delta^{p/2}$. In either case, we have thus $v_\delta \geq \chi_{U_\delta}$ such that $||v_\delta||_{W_0^{1,p}(\Omega)} + ||v_\delta||_{L^2(\Omega)} \leq C\max(\delta,\delta^{p/2})$. Using then $u(t,x) = v_\delta(x)$ in (10.7) for the definition of the parabolic capacity of $(t_0,t_1)\times U_\delta$ we deduce that $\mathrm{cap}_p((t_0,t_1)\times U_\delta) \leq C\max(\delta,\delta^{p/2})$, and then as δ goes to zero we get $\mathrm{cap}_p((t_0,t_1)\times B) = 0$.
Conversely, assume that $\mathrm{cap}_p((t_0,t_1)\times B) = 0$ and take $t_0 < t_0' < t_1' < t_1$. Since $\mathrm{cap}_p((t_0,t_1)\times B) = 0$, for all $\delta>0$, there exists an open set A_δ such that $((t_0,t_1)\times B)\subset A_\delta$ and $\mathrm{cap}_p(A_\delta) < \delta$. For every fixed $x\in B$, since the compact set $[t_0',t_1']\times\{x\}$ is contained in the open set A_δ, there exists an open set $U_x\subset\Omega$ such that $(t_0',t_1')\times\{x\}\subset(t_0',t_1')\times U_x\subset A_\delta$. Hence, setting $U = \bigcup_{x\in B} U_x$ we have that $B\subset U\subset\Omega$, U is an open set and $(t_0',t_1')\times U\subset A_\delta$, so that $\mathrm{cap}_p((t_0',t_1')\times U) \leq \mathrm{cap}_p(A_\delta) < \delta$. Let then $u_\delta \in W$ be such that $u_\delta \geq \chi_{(t_0',t_1')\times U}$ and $\|u_\delta\|_W \leq \delta$. Defining

$$v_\delta = \frac{1}{t_1'-t_0'}\int_{t_0'}^{t_1'} u_\delta\,dt\,,$$

we easily check that $v_\delta \in W_0^{1,p}(\Omega)$, $v_\delta \geq \chi_U$ almost everywhere in Ω and

$$\|v_\delta\|_{W_0^{1,p}(\Omega)} \leq \frac{1}{t_1'-t_0'}\int_{t_0'}^{t_1'} \|u_\delta\|_V\,dxdt \leq \frac{T^{\frac{p-1}{p}}}{t_1'-t_0'}\|u_\delta\|_W \leq \frac{T^{\frac{p-1}{p}}}{t_1'-t_0'}\,\delta\,.$$

Since U is open and contains B, the arbitrariness of δ implies $\mathrm{cap}_p^e(B) = 0$.

■

10.2.2 Quasicontinuous functions

Let us recall that a function u is called cap–quasi continuous if for every $\varepsilon>0$ there exists an open set F_ε, with $\mathrm{cap}_p(F_\varepsilon)\leq\varepsilon$, and such that $u_{|Q\setminus F_\varepsilon}$ (the restriction of u to $Q\setminus F_\varepsilon$) is continuous in $Q\setminus F_\varepsilon$. As usual, a property will be said to hold cap–quasi everywhere if it holds everywhere except on a set of zero capacity. The following lemma is essential to prove the existence of a cap–quasicontinuous representative in W. In fact, remark that if $u\in W$, one may have $|u|\notin W$, since the time derivative may lack of regularity. To overcome this obstacle we use some ideas contained in [62].

Lemma 10.1 *(i) Let u belong to W; then there exists a function z in $\widetilde{W}$ (see Definition 10.2) such that $|u| \le z$ and*

$$\|z\|_{\widetilde{W}} \le C \max\left\{ \|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}} \right\}. \tag{10.15}$$

(ii) If u belongs to $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ and u_t is in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ then there exists $z \in \widetilde{W}$ such that $|u| \le z$ and:

$$\begin{aligned} [z]_W \le C \Big(&\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\ &+ \|u\|_{L^\infty(Q)} \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} \Big). \end{aligned}$$

Remark 10.7 *In case (i), notice that, when $\|u\|_W$ is small, so is $\|z\|_{\widetilde{W}}$; this allows to prove that the sets of null capacity coming from W are the same than the sets of null capacity coming from $\widetilde{W}$.*
The case (ii) of Lemma 10.1 will not be useful to us, but we state and prove it because it allows to see that, if u is as in this case, then u has a unique cap–quasi continuous representative (see also Remark 10.12).

Proof. We divide the proof in two steps. We will denote $\Delta_p(u_\varepsilon) = \mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$.
Step 1. Let us consider the penalizing problem

$$\begin{cases} (u_\varepsilon)_t - \Delta_p(u_\varepsilon) = \frac{1}{\varepsilon}(u_\varepsilon - u)^- & \text{in }]0,T[\times\Omega, \\ u_\varepsilon = 0 & \text{on }]0,T[\times\partial\Omega, \\ u_\varepsilon(0) = u^+(0) & \text{in } \Omega, \end{cases} \tag{10.16}$$

which admits a nonnegative solution u_ε in $\mathcal{C}([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ by results in [51]. Choosing $u_\varepsilon - u$ as test function in (10.16) we get, for every t in $[0,T]$:

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon - u|^2(t)}{2}\,dx + \int_0^t\int_\Omega |\nabla u_\varepsilon|^p\,dxdt \le &\int_0^t\int_\Omega |\nabla u|\,|\nabla u_\varepsilon|^{p-1}\,dxdt \\ &+ \frac{1}{\varepsilon}\int_0^t\int_\Omega (u_\varepsilon - u)(u_\varepsilon - u)^- \\ &- \int_0^t \langle u_t, u_\varepsilon - u\rangle\,dt + \frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}, \end{aligned}$$

which yields, using also Young's inequality, and $(u_\varepsilon - u)(u_\varepsilon - u)^- \le 0$,

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon - u|^2(t)}{2}\,dx + \frac{1}{2}\int_0^t\int_\Omega |\nabla u_\varepsilon|^p\,dxdt \le C &\int_Q |\nabla u|^p\,dxdt \\ &- \int_0^t \langle u_t, u_\varepsilon - u\rangle\,dt + \frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{10.17}$$

If we are in case (i), u is in W and we have

$$\begin{aligned} &\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle dt\right| \\ &\le \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_V \\ &\le \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_{W_0^{1,p}(\Omega)} + \int_0^T \|u_t\|_{V'}\,\|u_\varepsilon - u\|_{L^2(\Omega)} \\ &\le \|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^1(0,T;V')}\,\|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le \|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u_t\|_{L^{p'}(0,T;V')}\,\|u_\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \end{aligned}$$

so that we easily deduce from (10.17), using Young's inequality:

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C \max\left\{\|u\|^p_W, \|u\|^{p'}_W\right\}. \tag{10.18}$$

If we are in case (ii), then the duality product $\int_0^t \langle u_t, u_\varepsilon - u\rangle$ in (10.17) is between the spaces

$$L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q) \quad \text{ and } \quad L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q),$$

and we need to prove an $L^\infty(Q)$ estimate on u_ε. This can be easily achieved by choosing $G_k(u_\varepsilon) = (u_\varepsilon - k)^+$ (let us recall that $u_\varepsilon \geq 0$) as test function in (10.16), with $k = \|u\|_{L^\infty(Q)}$: since $G'_k = \chi_{]k,\infty[} = (G'_k)^p$, we have

$$\int_Q |\nabla G_k(u_\varepsilon)|^p \, dxdt = \int_Q G'_k(u_\varepsilon)|\nabla u_\varepsilon|^p \, dxdt \leq \frac{1}{\varepsilon}\int_Q (u_\varepsilon - u)^- \, G_k(u_\varepsilon)\, dxdt\,,$$

and since $(u_\varepsilon - u)^- \, G_k(u_\varepsilon) = 0$ for $k = \|u\|_{L^\infty(Q)}$, we deduce that $\|u_\varepsilon\|_{L^\infty(Q)} \leq \|u\|_{L^\infty(Q)}$. Thus, writing $u_t = u_t^1 + u_t^2$ with $u_t^1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_t^2 \in L^1(Q)$ such that $\|u_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|u_t^2\|_{L^1(Q)} \leq 2\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$,

$$\begin{aligned}
\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle \, dt\right| &\leq \int_0^T \|u_t^1\|_{W^{-1,p'}(\Omega)}\|u_\varepsilon - u\|_{W_0^{1,p}(\Omega)}\, dt + \|u_t^2\|_{L^1(Q)}\|u_\varepsilon - u\|_{L^\infty(Q)} \\
&\leq C\|u_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \frac{1}{4}\|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + C\|u\|_{L^\infty(Q)}\|u_t^2\|_{L^1(Q)}\,.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle \, dt\right| \leq C\|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\
&+ \frac{1}{4}\|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega)))} + C\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega)))} + C\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}\,.
\end{aligned}$$

We deduce from (10.17) that, for all $t \in [0,T]$,

$$\begin{aligned}
\int_\Omega |u_\varepsilon - u|^2(t)\, dx + \int_0^t\int_\Omega |\nabla u_\varepsilon|^p \, dxdt \leq C\Bigg(\int_Q |\nabla u|^p\, dxdt + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\
+ \|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Bigg)\,,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \\
&\leq \; C\Bigg(\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\
&\qquad + \|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Bigg)\,.
\end{aligned} \tag{10.19}$$

From (10.18) or (10.19) we deduce that there exists a nonnegative function w in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ such that (up to subsequences)

$$u_\varepsilon \to w \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Note also that if $\varepsilon < \eta$ then $u_\varepsilon \geq u_\eta$; indeed, we have

$$\begin{aligned}
&-\int_0^t \langle (u_\varepsilon - u_\eta)_t, (u_\varepsilon - u_\eta)^-\rangle \, dt - \int_0^t\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u_\eta|^{p-2}\nabla u_\eta)\nabla(u_\varepsilon - u_\eta)^- \, dxdt \\
&\quad = -\int_0^t\int_\Omega \left(\frac{1}{\varepsilon}(u_\varepsilon - u)^- - \frac{1}{\eta}(u_\eta - u)^-\right)(u_\varepsilon - u_\eta)^- \, dxdt\,,
\end{aligned}$$

which yields, using the fact that the second term of the last equation is non negative and integrating by parts,

$$\begin{aligned}\frac{1}{2}\int_\Omega |(u_\varepsilon - u_\eta)^-(t)|^2\,dx &\le \int_0^t\!\!\int_\Omega (u_\varepsilon - u_\eta)^-\left(\frac{1}{\eta}(u_\eta - u)^- - \frac{1}{\varepsilon}(u_\varepsilon - u)^-\right)dxdt\\ &\le \int_0^t\!\!\int_\Omega (u_\varepsilon - u_\eta)^-(u_\eta - u)^-\left(\frac{1}{\eta} - \frac{1}{\varepsilon}\right)dxdt \le 0\,,\end{aligned}$$

for every t in $]0,T[$. Thus $(u_\varepsilon)_\varepsilon$ is a non negative sequence bounded in $L^1(Q)$, moreover it is increasing as ε tends to zero, hence thanks to the monotone convergence theorem, u_ε converges to w in $L^1(Q)$ and almost everywhere in Q. We have, choosing $(u_\varepsilon - u)^-$ as test function in (10.16),

$$\int_0^T \langle (u_\varepsilon)_t, (u_\varepsilon - u)^-\rangle\,dt + \int_0^T |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (u_\varepsilon - u)^-\,dxdt = \frac{1}{\varepsilon}\int_Q |(u_\varepsilon - u)^-|^2\,dxdt\,,$$

which implies

$$\begin{aligned}\frac{1}{\varepsilon}\int_Q |(u_\varepsilon - u)^-|^2\,dxdt + \int_\Omega \frac{|(u_\varepsilon - u)^-|^2(T)}{2}\,dx = &\int_0^T \langle u_t, (u_\varepsilon - u)^-\rangle\,dt\\ &+ \int_Q |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\,\nabla (u_\varepsilon - u)^-\,dxdt\,.\end{aligned}$$

Using either (10.18) in case (i) or (10.19) and the L^∞ estimate in case (ii) we deduce:

$$\frac{1}{\varepsilon}\int_Q |(u_\varepsilon - u)^-|^2\,dxdt \le C\,, \tag{10.20}$$

which implies, by Fatou's lemma, that $w \ge u$, and $w \ge u^+$ since $w \ge 0$.
Step 2: Let us now replace u_ε by a sequence converging in $\widetilde{W}$. Precisely, we define z_ε the solution of the following parabolic problem:

$$\begin{cases} -z^\varepsilon_t - \Delta_p z^\varepsilon = -2\Delta_p u_\varepsilon & \text{in }]0,T[\times\Omega,\\ z^\varepsilon = 0 & \text{on }]0,T[\times\partial\Omega,\\ z^\varepsilon(T) = u_\varepsilon(T) & \text{in } \Omega.\end{cases} \tag{10.21}$$

Since $-2\Delta_p u_\varepsilon \ge -(u_\varepsilon)_t - \Delta_p(u_\varepsilon)$ in distributional sense, we can easily deduce from (10.21) that $z^\varepsilon \ge u_\varepsilon$. Moreover using z^ε itself as test function and integrating between t and T, we have the following energy estimates:

$$\begin{aligned}&\|z^\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z^\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} \le C(\|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))})\\ &\|z^\varepsilon_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C(\|z^\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))})\,.\end{aligned} \tag{10.22}$$

In virtue of (10.22), we get that z^ε is bounded in $\widetilde{W}$, hence there exists a function $z \in L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and a function $w \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that (up to subsequences) $z^\varepsilon \to z$ weakly in $L^p(0,T;W^{1,p}_0(\Omega))$ and weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and $z^\varepsilon_t \to w$ weakly-$*$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$; it is then quite easy to see that $z_t = w$, so that z is in fact in $\widetilde{W}$. The classical compactness argument contained in [69] implies that z^ε is also compact in $L^1(Q)$. Thus we deduce, up to subsequences, that z^ε almost everywhere converges to z in Q, and since $z^\varepsilon \ge u_\varepsilon$ passing to the limit we obtain that:

$$z \ge w \ge u^+ \qquad \text{a.e. in } Q.$$

Moreover, using either (10.18) or (10.19) and (10.22), we deduce that, if u is in W, then

$$\|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \|z_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C\max\left\{\|u\|^p_W, \|u\|^{p'}_W\right\}\,,$$

which implies (10.15), and if u is in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and u_t belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, then

$$[z]_W \leq C(\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\ +\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\big) .$$

A similar construction can be made for the negative part u^-, so the conclusion of the lemma follows by writing $|u| = u^+ + u^-$. ∎

The previous lemma has the following important consequence.

Proposition 10.4 *If u is cap–quasi continuous and belongs to W, then, for all $t > 0$,*

$$\mathrm{cap}_p(\{|u| > t\}) \leq \frac{C}{t}\max\left\{\|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}}\right\} . \tag{10.23}$$

Proof. Let us first handle a simple case, that is to say $u \in \mathcal{C}_c^\infty([0,T]\times\Omega)$; then the set $\{|u| > t\}$ is open and its capacity can be computed according to (10.7). By Lemma 10.1 there exists a function $z \geq |u|$ satisfying (10.15). Since $\frac{z}{t} \geq 1$ on the set $\{|u| > t\}$ we have:

$$\mathrm{cap}_p(\{|u| > t\}) \leq \frac{\|z\|_W}{t} \leq \frac{C}{t}\max\left\{\|u\|_W^{\frac{p}{p'}}, \|u\|_W^{\frac{p'}{p}}\right\}$$

Let us now prove the general case: u is cap–quasi continuous and belongs to W. Let $\varepsilon > 0$ and A_ε be an open set such that $\mathrm{cap}_p(A_\varepsilon) \leq \varepsilon$ and $u_{|Q\setminus A_\varepsilon}$ is continuous in $Q\setminus A_\varepsilon$; by definition, this implies that $\{|u_{|Q\setminus A_\varepsilon}| > t\}\cap(Q\setminus A_\varepsilon)$ is an open set of $Q\setminus A_\varepsilon$, i.e. that there exists an open set U of $\mathbb{R}^N$ such that $\{|u_{|Q\setminus A_\varepsilon}| > t\}\cap(Q\setminus A_\varepsilon) = U\cap(Q\setminus A_\varepsilon)$. Thus,

$$\{|u| > t\}\cup A_\varepsilon = \big(\{|u_{|Q\setminus A_\varepsilon}| > t\}\cap(Q\setminus A_\varepsilon)\big)\cup A_\varepsilon = (U\cup A_\varepsilon)\cap Q$$

is an open set. Let then $z \in W$ be such that $z \geq |u|$ and (10.15) holds; let $w \in W$ be such that $\|w\|_W \leq \mathrm{cap}_p(A_\varepsilon)+\varepsilon \leq 2\varepsilon$ and $w \geq \chi_{A_\varepsilon}$; we have $w + \frac{z}{t} \geq 1$ almost everywhere on $\{\,|u| > t\}\cup A_\varepsilon$, hence

$$\mathrm{cap}_p(\{|u| > t\}\cup A_\varepsilon) \leq \|w\|_W + \frac{1}{t}\|z\|_W \leq 2\varepsilon + \frac{1}{t}\|z\|_W .$$

Thus we get

$$\mathrm{cap}_p(\{|u| > t\}) \leq 2\varepsilon + \frac{1}{t}\|z\|_W ,$$

which implies again (10.23). ∎

We can now prove the result on quasicontinuity, whose proof follows the standard approach with the help of Lemma 10.1.

Lemma 10.2 *Any element v of W has a cap–quasi continuous representative $\widetilde{v}$ which is cap–quasi everywhere unique, in the sense that two cap–quasi continuous representatives of v are equal except on a set of null capacity.*

Proof. By density of $\mathcal{C}_c^\infty([0,T]\times\Omega)$ in W, there exists a sequence $(v^m)\subset\mathcal{C}_c^\infty([0,T]\times\Omega)$ such that (v^m) converges to v in W. We can also construct (v_m) such that

$$\sum_{m=1}^{\infty} 2^m \max\left\{\|v^{m+1}-v^m\|_W^{\frac{p}{p'}}, \|v^{m+1}-v^m\|_W^{\frac{p'}{p}}\right\} < +\infty.$$

Let then define:

$$\omega^m = \{|v^{m+1}-v^m| > 2^{-m}\}, \qquad \Omega^r = \bigcup_{m\geq r}\omega^m .$$

Since $v^{m+1} - v^m$ is continuous, ω^m is an open set; moreover, by Proposition 10.4, we have

$$\mathrm{cap}_p(\omega^m) \le C 2^m \max\left\{ \|v^{m+1} - v^m\|_W^{\frac{p}{p'}}, \|v^{m+1} - v^m\|_W^{\frac{p'}{p}} \right\}.$$

Thus we get:

$$\mathrm{cap}_p(\Omega^r) \le C \sum_{m \ge r} 2^m \max\left\{ \|v^{m+1} - v^m\|_W^{\frac{p}{p'}}, \|v^{m+1} - v^m\|_W^{\frac{p'}{p}} \right\}.$$

This proves that $\lim_{r\to\infty} \mathrm{cap}_p(\Omega^r) = 0$. Moreover for any r:

$$\forall z \notin \Omega^r, \quad \forall m \ge r, \quad |v^{m+1} - v^m|(z) \le 2^{-m},$$

hence (v^m) converges uniformly on the complement of each Ω^r and pointwise in the complement of $\bigcap_{r=1}^{\infty} \Omega^r$. Since

$$\mathrm{cap}_p\left(\bigcap_{r=1}^{\infty} \Omega^r\right) \le \mathrm{cap}_p(\Omega^r) \to 0 \qquad \text{as } r \text{ tends to infinity},$$

we have that $\mathrm{cap}_p(\bigcap_{r=1}^{\infty} \Omega^r) = 0$. Therefore the limit of v^m is defined cap–quasi everywhere and is cap–quasi continuous. Let us call $\tilde{v}$ this cap–quasi continuous representative of v, and assume that there exists another representative z of v which is cap–quasi continuous and coincides with v almost everywhere in Q. Then we have, thanks to Proposition 10.4:

$$\mathrm{cap}_p\left\{ |\tilde{v} - z| > \frac{1}{n} \right\} \le Cn \max\left\{ \|\tilde{v} - z\|_W^{\frac{p}{p'}}, \|\tilde{v} - z\|_W^{\frac{p'}{p}} \right\} = 0,$$

since $\tilde{v} - z = 0$ almost everywhere. This being true for any n, we obtain that $z = \tilde{v}$ cap–quasi everywhere, so that the cap–quasi continuous representative of v is unique up to sets of zero capacity. ∎

We can also prove the following result.

Lemma 10.3 *Let (v_n) be a sequence in W which converges to v in W, then there exists a subsequence of $(\widetilde{v}_n)$ which converges to $\widetilde{v}$ cap–quasi everywhere.*

Proof. Let us extract a subsequence of (v_n) such that

$$\sum_{n=1}^{\infty} 2^n \max\left\{ \|v_n - v\|_W^{\frac{p}{p'}}, \|v_n - v\|_W^{\frac{p'}{p}} \right\} < +\infty.$$

Thanks to Proposition 10.4 we have

$$\mathrm{cap}_p\{ |\tilde{v}_n - \tilde{v}| > 2^{-n} \} \le C 2^n \max\left\{ \|v_n - v\|_W^{\frac{p}{p'}}, \|v_n - v\|_W^{\frac{p'}{p}} \right\}. \tag{10.24}$$

Using (10.24) we can repeat the proof of Lemma 10.2, which proves that $\tilde{v}_n$ converges to $\tilde{v}$ cap–quasi everywhere. ∎

10.2.3 Measures

In the following, we denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the σ–algebra of borelian subsets of Q, and $\mathcal{M}_b^+(Q)$ will denote the subsets of nonnegative measures of $\mathcal{M}_b(Q)$.

Definition 10.5 *We define*

$$\mathcal{M}_0(Q) = \{\mu \in \mathcal{M}_b(Q) : \mu(E) = 0 \quad \textit{for every subset } E \subset Q \textit{ such that } \mathrm{cap}_p(E) = 0\}.$$

The nonnegative measures in $\mathcal{M}_0(Q)$ will be said to belong to $\mathcal{M}_0^+(Q)$.

We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality between W' and W. $W'\cap\mathcal{M}_b(Q)$ denotes the set of elements $\gamma\in W'$ such that there exists $C>0$ satisfying, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, $|\langle\langle\gamma,\varphi\rangle\rangle|\le C\|\varphi\|_{L^\infty(Q)}$; in such a case, by the Riesz representation theorem there exists a unique $\gamma^{\text{meas}}\in\mathcal{M}_b(Q)$ such that, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, $\langle\langle\gamma,\varphi\rangle\rangle=\int_Q\varphi\,d\gamma^{\text{meas}}$ (notice however that, if the knowledge of $\gamma\in W'$ entirely defines $\gamma^{\text{meas}}\in\mathcal{M}_b(Q)$, the converse is not true). We denote by $W'\cap\mathcal{M}_b^+(Q)$ the set of $\gamma\in W'\cap\mathcal{M}_b(Q)$ such that $\gamma^{\text{meas}}\in\mathcal{M}_b^+(Q)$. Now we investigate the link between measures in Q and the notion of capacity defined above. The main theorem in this sense can be obtained from the result on the "elliptic capacity" contained in [20], which also applies to this context of parabolic spaces. We rewrite thus, with the necessary adaptations to the parabolic case, the proof of G. Dal Maso.

Theorem 10.7 *Let μ belong to $\mathcal{M}_0^+(Q)$. Then there exists $\gamma\in W'\cap\mathcal{M}_b^+(Q)$ and a nonnegative function $f\in L^1(Q,d\gamma^{meas})$ such that $\mu=f\gamma^{meas}$.*

Proof. Let $\mu\in\mathcal{M}_0^+(Q)$. For any u in W, let $\tilde u$ be the cap–quasi continuous representative of u, which exists by Lemma 10.2. Since $\tilde u$ is uniquely defined up to sets of zero capacity we can define the functional $F:W\to[0,\infty]$ by

$$F(u)=\int_Q\tilde u^+\,d\mu$$

(indeed, this definition does not depend on the cap–quasi continuous representative of u, since two cap–quasi continuous representatives are equal except on a set of null capacity, that is to say μ-a.e.). Clearly F is convex, and it is also lower semicontinuous in W thanks to Lemma 10.3 and Fatou's lemma. By the separability of W', there exists then a sequence $\{a_n\}$ of real numbers and a sequence $\{\lambda_n\}$ in W' such that:

$$F(u)=\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle+a_n\}.$$

Since, for any positive t, $tF(u)=F(tu)\ge t\langle\langle\lambda_n,u\rangle\rangle+a_n$ for every n, dividing by t and letting t tend to infinity we get $F(u)\ge\langle\langle\lambda_n,u\rangle\rangle$ for all u in W. For $u=0$, we deduce that $a_n\le0$, hence

$$F(u)\ge\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle\}\ge\sup_n\{\langle\langle\lambda_n\,,\,u\rangle\rangle+a_n\}=F(u)\,. \tag{10.25}$$

By (10.25) and the definition of F, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, we have

$$\langle\langle\lambda_n,\varphi\rangle\rangle\le\int_Q\varphi^+\,d\mu\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}, \tag{10.26}$$

thus, applying this inequality to φ and $-\varphi$, we get $|\langle\langle\lambda_n,\varphi\rangle\rangle|\le\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}$, which implies that $\lambda_n\in W'\cap\mathcal{M}_b(Q)$; moreover, since $F(-\varphi)=0$ for any nonnegative $\varphi\in\mathcal{C}_c^\infty(Q)$, we have $0\le\langle\langle\lambda_n,\varphi\rangle\rangle=\int_Q\varphi\,d\lambda_n^{\text{meas}}$ for all such φ, which implies $\lambda_n^{\text{meas}}\in\mathcal{M}_b^+(Q)$ (that is to say $\lambda_n\in W'\cap\mathcal{M}_b^+(Q)$) and, applying once again (10.26) to any nonnegative $\varphi\in\mathcal{C}_c^\infty(Q)$,

$$\lambda_n^{\text{meas}}\le\mu. \tag{10.27}$$

We have thus, in particular, $\|\lambda_n^{\text{meas}}\|_{\mathcal{M}_b(Q)}\le\|\mu\|_{\mathcal{M}_b(Q)}$.
Thus the series

$$\gamma=\sum_{n=1}^\infty\frac{\lambda_n}{2^n(\|\lambda_n\|_{W'}+1)} \tag{10.28}$$

is absolutely convergent in W' and we have, for all $\varphi\in\mathcal{C}_c^\infty(Q)$,

$$\begin{aligned}|\langle\langle\gamma,\varphi\rangle\rangle| &= \left|\sum_{n=1}^\infty\frac{\langle\langle\lambda_n,\varphi\rangle\rangle}{2^n(\|\lambda_n\|_{W'}+1)}\right|\\ &\le \sum_{n=1}^\infty\frac{\|\lambda_n^{\text{meas}}\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}}{2^n}\\ &\le \|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)},\end{aligned}$$

so that $\gamma \in W' \cap \mathcal{M}_b(Q)$. The series $\sum_{n=1}^{\infty} \frac{\lambda_n^{\text{meas}}}{2^n(\|\lambda_n\|_{W'}+1)}$ strongly converges in $\mathcal{M}_b(Q)$, and we can see, applying (10.28) to functions of $\mathcal{C}_c^\infty(Q)$, that

$$\gamma^{\text{meas}} = \sum_{n=1}^{\infty} \frac{\lambda_n^{\text{meas}}}{2^n(\|\lambda_n\|_{W'}+1)}.$$

In particular, γ^{meas} is a nonnegative measure (each λ_n^{meas} is nonnegative).
Since $\lambda_n^{\text{meas}} << \gamma^{\text{meas}}$, there exists a nonnegative function $f_n \in L^1(Q, d\gamma^{\text{meas}})$ such that $\lambda_n^{\text{meas}} = f_n\gamma^{\text{meas}}$, thus (10.25) implies:

$$\int_Q \varphi\, d\mu = \sup_n \int_Q f_n\, \varphi\, d\gamma^{\text{meas}}\,, \tag{10.29}$$

for any nonnegative φ in $\mathcal{C}_c^\infty(Q)$. We also have, by (10.27), $f_n\gamma^{\text{meas}} \le \mu$, that is

$$\int_B f_n\, d\gamma^{\text{meas}} \le \mu(B)\,,$$

for any borelian subset B in Q and every n. In particular, we have

$$\int_B \sup\{f_1, f_2, \dots, f_k\}\, d\gamma^{\text{meas}} \le \mu(B)\,,$$

for any borelian subset B in Q and any $k \ge 1$. Letting k tend to infinity we deduce by the monotone convergence theorem:

$$\int_B f\, d\gamma^{\text{meas}} \le \mu(B)\,,$$

where $f = \sup_n f_n$. Then we conclude, using (10.29):

$$\int_Q \varphi\, d\mu = \sup_n \int_Q f_n\, \varphi\, d\gamma^{\text{meas}} \le \int_Q f\, \varphi\, d\gamma^{\text{meas}} \le \int_Q \varphi\, d\mu\,,$$

for any positive $\varphi \in \mathcal{C}_c^\infty(Q)$, which yields that $\mu = f\gamma^{\text{meas}}$, and since $\mu(Q) < +\infty$ it follows that $f \in L^1(Q, d\gamma^{\text{meas}})$. ■

In order to better specify the nature of a measure in $\mathcal{M}_0(Q)$, we need then to detail the structure of the dual space W'.

Lemma 10.4 *Let $g \in W'$. Then there exist $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and $g_3 \in L^{p'}(0,T;L^2(\Omega))$ such that*

$$\langle\langle g, u\rangle\rangle = \int_0^T \langle g_1, u\rangle + \int_0^T \langle u_t, g_2\rangle + \int_Q g_3 u\, dxdt \qquad \forall\, u \in W\,.$$

Moreover, we can choose (g_1, g_2, g_3) such that

$$\|g_1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2\|_{L^p(0,T;V)} + \|g_3\|_{L^{p'}(0,T;L^2(\Omega))} \le C\|g\|_{W'}, \tag{10.30}$$

with C not depending on g.

Proof. Let $E = L^p(0,T;V) \times L^{p'}(0,T;V')$ and $T : W \mapsto E$ such that $T(u) = (u, u_t)$. If we endow E with the norm

$$\|(v_1, v_2)\|_E = \|v_1\|_{L^p(0,T;V)} + \|v_2\|_{L^{p'}(0,T;V')},$$

then T is isometric from W to E. Let $G = T(W)$, with the norm of E, thus T^{-1} is defined on G. Let $g \in W'$ and let $\Phi : G \mapsto \mathbb{R}$, $\Phi(v_1, v_2) = \langle\langle g, T^{-1}(v_1, v_2)\rangle\rangle$, then Φ is a continuous linear form on G.

Hence thanks to the Hahn-Banach theorem, it can be extended to a continuous linear form on E, also denoted Φ, with $\|\Phi\|_{E'} = \|g\|_{W'}$ (since T^{-1} is isometric). There exists thus $h_1 \in (L^p(0,T;V))'$ and $h_2 \in (L^{p'}(0,T;V'))'$ such that

$$\Phi(v_1, v_2) = \langle h_1, v_1 \rangle_{(L^p(0,T;V))', L^p(0,T;V)} + \langle h_2, v_2 \rangle_{(L^{p'}(0,T;V'))', L^{p'}(0,T;V')}$$

and $\|h_1\|_{(L^p(0,T;V))'} + \|h_2\|_{(L^{p'}(0,T;V'))'} \leq C\|\Phi\|_{E'}$. But $L^p(0,T;V)$ is reflexive and $(L^p(0,T;V))' = L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega))$ (with equivalent norms), so that we can find $g_2 \in L^p(0,T;V)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $g_3 \in L^{p'}(0,T;L^2(\Omega))$ satisfying

$$\Phi(v_1, v_2) = \int_0^T \langle g_1, v_1 \rangle + \int_0^T \langle v_2, g_2 \rangle + \int_Q g_3 v_1$$

and $\|g_1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2\|_{L^p(0,T;V)} + \|g_3\|_{L^{p'}(0,T;L^2(\Omega))} \leq C(\|h_1\|_{(L^p(0,T;V))'} + \|h_2\|_{(L^{p'}(0,T;V'))'}) \leq C\|g\|_{W'}$.
Hence for all $u \in W$, $\langle\langle g, u \rangle\rangle = \Phi(T(u)) = \int_0^T \langle g_1, u \rangle + \int_0^T \langle u_t, g_2 \rangle + \int_Q g_3 u$, which concludes the proof.

■

We will need, in the following, to construct suitable smooth approximations of elements $\nu \in W' \cap \mathcal{M}_b(Q)$ which at the same time converge strongly in W' and weakly-$*$ in $\mathcal{M}_b(Q)$. As usual, we would like to start with measures ν having compact support. To this purpose, note that when θ is a regular function, since the multiplication $\varphi \to \theta\varphi$ is linear continuous from W to W, we can define the multiplication of an element $\nu \in W'$ by θ thanks to a duality method: $\theta\nu \in W'$ is defined by $\langle\langle \theta\nu, \varphi \rangle\rangle = \langle\langle \nu, \theta\varphi \rangle\rangle$.

Lemma 10.5 *Let $\nu \in W' \cap \mathcal{M}_b(Q)$ and $\theta \in \mathcal{C}_c^\infty(Q)$. We take $(\rho_n)_{n\geq 1}$ a sequence of symmetric ([4]) mollifiers in $\mathbb{R} \times \mathbb{R}^N$ and $\mu = \theta\nu \in W'$. Then $\mu \in W' \cap \mathcal{M}_b(Q)$, $\mu^{meas} = \theta\nu^{meas}$, μ^{meas} has a compact support in Q and*

$$\|\mu^{meas} * \rho_n\|_{L^1(Q)} \leq \|\mu^{meas}\|_{\mathcal{M}_b(Q)}, \qquad \mu^{meas} * \rho_n \to \mu \quad \textit{in } W'. \tag{10.31}$$

Proof.
The fact that $\mu \in W' \cap \mathcal{M}_b(\Omega)$ is quite obvious since, for all $\varphi \in \mathcal{C}_c^\infty(Q)$, $|\langle\langle \mu, \varphi \rangle\rangle| = |\langle\langle \nu, \theta\varphi \rangle\rangle| \leq C||\theta\varphi||_{L^\infty(Q)} \leq C||\theta||_{L^\infty(Q)}||\varphi||_{L^\infty(Q)}$. Moreover, by definition, one has, for all $\varphi \in \mathcal{C}_c^\infty(Q)$,

$$\int_Q \varphi \, d\mu^{\text{meas}} = \langle\langle \mu, \varphi \rangle\rangle = \langle\langle \nu, \theta\varphi \rangle\rangle = \int_Q \theta\varphi \, d\nu^{\text{meas}},$$

so that $\mu^{\text{meas}} = \theta\nu^{\text{meas}}$; thus, the measure μ^{meas} has indeed a compact support and $\mu^{\text{meas}} * \rho_n$ is well defined and is, for n large enough, a function in $\mathcal{C}_c^\infty(Q)$. By a classical result of convolution of measures, one has $\|\mu^{\text{meas}} * \rho_n\|_{L^1(Q)} \leq \|\mu^{\text{meas}}\|_{\mathcal{M}_b(Q)}$.
Let now $(g_1, g_2, g_3) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \times L^p(0,T;V) \times L^{p'}(0,T;L^2(\Omega))$ be a decomposition of ν according to Lemma 10.4. Then, for all $\varphi \in W$, one has

$$\begin{aligned}
\langle\langle \mu, \varphi \rangle\rangle &= \int_0^T \langle g_1, \theta\varphi \rangle + \int_0^T \langle (\theta\varphi)_t, g_2 \rangle + \int_Q g_3\theta\varphi \\
&= \int_0^T \langle \theta g_1, \varphi \rangle + \int_0^T \langle \varphi_t, \theta g_2 \rangle + \int_0^T \langle \theta_t\varphi, g_2 \rangle + \int_Q \theta g_3\varphi.
\end{aligned}$$

Since $\theta_t\varphi \in L^{p'}(0,T;L^2(\Omega))$ (see Remark 10.2), the term $\int_0^T \langle \theta_t\varphi, g_2 \rangle$ is in fact $\int_Q \theta_t\varphi g_2$. Moreover, since $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, there exists $G_1 \in (L^{p'}(Q))^N$ such that $g_1 = \text{div}(G_1)$, so that

$$\int_0^T \langle \theta g_1, \varphi \rangle = \int_0^T \langle \text{div}(\theta G_1), \varphi \rangle - \int_0^T \langle G_1 \cdot \nabla\theta, \varphi \rangle.$$

[4] That is to say $\rho_n(-\cdot) = \rho_n(\cdot)$.

$G_1 \cdot \nabla\theta \in L^{p'}(Q)$ and we have thus in fact

$$\int_0^T \langle \theta g_1, \varphi\rangle = \int_0^T \langle \mathrm{div}(\theta G_1), \varphi\rangle - \int_Q G_1 \cdot \nabla\theta\varphi.$$

Thus, for all $\varphi \in W$, one has

$$\langle\langle \mu, \varphi\rangle\rangle = \int_0^T \langle \mathrm{div}(\theta G_1), \varphi\rangle + \int_0^T \langle \varphi_t, \theta g_2\rangle + \int_Q \theta g_3\varphi - \int_Q G_1 \cdot \nabla\theta\varphi + \int_Q \theta_t g_2\varphi. \tag{10.32}$$

From now on, we take n large enough so that $\mathrm{Supp}(\theta) + \mathrm{Supp}(\rho_n)$ be included in a fixed compact subset K of Q. The support of $\mu^{\text{meas}} * \rho_n = (\theta\nu^{\text{meas}}) * \rho_n$ is then also contained in K; we take $\zeta \in \mathcal{C}_c^\infty(Q)$ such that $\zeta \equiv 1$ on a neighborhood of K. We also take n large enough so that $\mathrm{Supp}(\zeta) + \mathrm{Supp}(\rho_n)$ is a compact subset of Q.
By definition of the natural injection $\mathcal{C}_c^\infty(Q) \subset W'$, we have, for all $\varphi \in W$,

$$\langle\langle \mu^{\text{meas}} * \rho_n, \varphi\rangle\rangle = \int_Q \varphi\mu^{\text{meas}} * \rho_n.$$

For all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$, we have then

$$\langle\langle \mu^{\text{meas}} * \rho_n, \varphi\rangle\rangle = \int_Q \zeta\varphi\mu^{\text{meas}} * \rho_n = \int_Q (\zeta\varphi) * \rho_n \, d\mu^{\text{meas}},$$

since n has been chosen large enough so that the support of $(\zeta\varphi) * \rho_n$ is a compact subset of Q; but $(\zeta\varphi) * \rho_n \in \mathcal{C}_c^\infty(Q)$, so that, by definition and (10.32),

$$\begin{aligned}
&\langle\langle \mu^{\text{meas}} * \rho_n, \varphi\rangle\rangle \\
&= \langle\langle \mu, (\zeta\varphi) * \rho_n\rangle\rangle \\
&= \int_0^T \langle \mathrm{div}(\theta G_1), (\zeta\varphi) * \rho_n\rangle + \int_0^T \langle ((\zeta\varphi) * \rho_n)_t, \theta g_2\rangle + \int_Q \theta g_3(\zeta\varphi) * \rho_n \\
&\quad - \int_Q G_1 \cdot \nabla\theta(\zeta\varphi) * \rho_n + \int_Q \theta_t g_2(\zeta\varphi) * \rho_n.
\end{aligned}$$

We have chosen n large enough according to the supports of θ and ζ to allow us to write

$$\begin{aligned}
&\langle\langle \mu^{\text{meas}} * \rho_n, \varphi\rangle\rangle \\
&= \int_0^T \langle \mathrm{div}((\theta G_1) * \rho_n), \zeta\varphi\rangle + \int_0^T \langle (\zeta\varphi)_t, (\theta g_2) * \rho_n\rangle + \int_Q (\theta g_3) * \rho_n\zeta\varphi \\
&\quad - \int_Q (G_1 \cdot \nabla\theta) * \rho_n\zeta\varphi + \int_Q (\theta_t g_2) * \rho_n\zeta\varphi.
\end{aligned}$$

But $\zeta \equiv 1$ on a neighborhood of $\mathrm{Supp}(\theta) + \mathrm{Supp}(\rho_n)$, so that

$$\begin{aligned}
&\langle\langle \mu^{\text{meas}} * \rho_n, \varphi\rangle\rangle \\
&= \int_0^T \langle \mathrm{div}((\theta G_1) * \rho_n), \varphi\rangle + \int_0^T \langle \varphi_t, (\theta g_2) * \rho_n\rangle + \int_Q (\theta g_3) * \rho_n\varphi \\
&\quad - \int_Q (G_1 \cdot \nabla\theta) * \rho_n\varphi + \int_Q (\theta_t g_2) * \rho_n\varphi.
\end{aligned} \tag{10.33}$$

This equality has only been established for $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$, but since this space is dense in W and both sides are continuous with respect to the norm of W, this equality is still valid for all $\varphi \in W$.

We have $(\theta G_1) * \rho_n \to \theta G_1$ in $(L^{p'}(Q))^N$, $(\theta g_2) * \rho_n \to \theta g_2$ in $L^p(0,T;V)$, $(\theta g_3) * \rho_n \to \theta g_3$ in $L^{p'}(0,T;L^2(\Omega))$, $(G_1\cdot\nabla\theta)*\rho_n \to G_1\cdot\nabla\theta$ in $L^{p'}(Q)$ and $(\theta_t g_2)*\rho_n \to \theta_t g_2$ in $L^p(0,T;L^2(\Omega))$. Substracting (10.32) and (10.33), we have, for all $\varphi \in W$,

$$\begin{aligned}
&\langle\langle \mu^{\text{meas}} * \rho_n - \mu, \varphi\rangle\rangle \\
= &\int_0^T \langle \operatorname{div}((\theta G_1) * \rho_n - \theta G_1), \varphi\rangle + \int_0^T \langle \varphi_t, (\theta g_2) * \rho_n - \theta g_2\rangle + \int_Q ((\theta g_3) * \rho_n - \theta g_3)\varphi \\
&+ \int_Q (G_1\cdot\nabla\theta - (G_1\cdot\nabla\theta) * \rho_n)\varphi + \int_Q ((\theta_t g_2) * \rho_n - \theta_t g_2)\varphi \\
\le\ & \|(\theta G_1) * \rho_n - \theta G_1\|_{(L^{p'}(Q))^N} \|\nabla\varphi\|_{L^p(Q)} + \|(\theta g_2) * \rho_n - \theta g_2\|_{L^p(0,T;V)} \|\varphi_t\|_{L^{p'}(0,T;V')} \\
&+ \|(\theta g_3) * \rho_n - \theta g_3\|_{L^{p'}(0,T;L^2(\Omega))} \|\varphi\|_{L^p(0,T;L^2(\Omega))} + \|G_1\cdot\nabla\theta - (G_1\cdot\nabla\theta) * \rho_n\|_{L^{p'}(Q)} \|\varphi\|_{L^p(Q)} \\
&+ \|(\theta_t g_2) * \rho_n - \theta_t g_2\|_{L^p(0,T;L^2(\Omega))} \|\varphi\|_{L^{p'}(0,T;L^2(\Omega))} \\
\le\ & C\Big(\|(\theta G_1) * \rho_n - \theta G_1\|_{(L^{p'}(Q))^N} + \|(\theta g_2) * \rho_n - \theta g_2\|_{L^p(0,T;V)} + \|(\theta g_3) * \rho_n - \theta g_3\|_{L^{p'}(0,T;L^2(\Omega))} \\
&+ \|G_1\cdot\nabla\theta - (G_1\cdot\nabla\theta) * \rho_n\|_{L^{p'}(Q)} + \|(\theta_t g_2) * \rho_n - \theta_t g_2\|_{L^p(0,T;L^2(\Omega))} \Big) \|\varphi\|_W
\end{aligned}$$

which proves the convergence of $\mu^{\text{meas}} * \rho_n$ to μ in W'. ■

Before stating and proving the decomposition theorem for elements of $\mathcal{M}_0(Q)$, let us first make a remark on the preceding proof, that will be useful to approximate elements of $\mathcal{M}_0(Q)$ in a suitable way.

Remark 10.8 *When $L \in W'$, we say that $(G_1, g_2, g_3, h_1, h_2)$ is a pseudo-decomposition of L if $G_1 \in (L^{p'}(Q))^N$, $g_2 \in L^p(0,T;V)$, $g_3 \in L^{p'}(0,T;L^2(\Omega))$, $h_1 \in L^{p'}(Q)$, $h_2 \in L^p(0,T;L^2(\Omega))$ and, for all $\varphi \in W$,*

$$\langle\langle L, \varphi\rangle\rangle = \int_0^T \langle \operatorname{div}(G_1), \varphi\rangle + \int_0^T \langle \varphi_t, g_2\rangle + \int_Q g_3\varphi + \int_Q h_1\varphi + \int_Q h_2\varphi.$$

The proof of Lemma 10.5 states the following: if $(\operatorname{div}(G_1), g_2, g_3)$ is a decomposition of ν according to Lemma 10.4, then $(\theta G_1, \theta g_2, \theta g_3, -G_1\cdot\nabla\theta, \theta_t g_2)$ is a pseudo-decomposition of $\mu = \theta\nu$ (see (10.32)) and $((\theta G_1) * \rho_n, (\theta g_2) * \rho_n, (\theta g_3) * \rho_n, (-G_1\cdot\nabla\theta) * \rho_n, (\theta_t g_2) * \rho_n)$ is a pseudo-decomposition of $\mu^{meas} * \rho_n$ (see (10.33)).
Thus, we have proven that a pseudo-decomposition of $\mu^{meas} * \rho_n$ converges to a pseudo-decomposition of μ. It is a weaker result than the one that would state that a decomposition of $\mu^{meas} * \rho_n$ (i.e. according to Lemma 10.4) converges to a decomposition of μ, but this last result is not clear. Indeed, to compute the elements of a decomposition of $\mu^{meas} * \rho_n$ we need to start from a decomposition of μ such that each term of the decomposition has a compact support; to obtain such a property, we need to introduce the cut-off function θ (because, in Lemma 10.4, it is not clear at all that, when g has a "compact support" — notice that, in fact, this expression has no sense since g is not a distribution —, we can take (g_1, g_2, g_3) with compact supports too), and the introduction of this cut-off function θ entails the apparition of the additional term $\theta_t g_2$, which cannot in general (if $p < 2$) be put in one of the terms of a decomposition of μ according to Lemma 10.4. Moreover, when we want to represent the term in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ of a decomposition of μ as the divergence of an element of $(L^{p'}(Q))^N$ with compact support (to manipulate this term using the convolution, we need such an hypothesis on the support), the introduction of the cut-off function creates the additional term $-G_1\cdot\nabla\theta$, and finally leads to a pseudo-decomposition of μ as defined above.
Notice however that, if $p \ge 2$, the term $\theta_t g_2 \in L^p(0,T;L^2(\Omega))$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$ but the term $G_1\cdot\nabla\theta \in L^{p'}(Q)$ remains; if $p \le 2$, the term $G_1\cdot\nabla\theta$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$, but the term $\theta_t g_2$ remains. In the special case $p = 2$, both terms $G_1\cdot\nabla\theta$ and $\theta_t g_2$ can be put into the term $L^{p'}(0,T;L^2(\Omega))$ and, in this case, we have in fact proven that there exists a decomposition of $\mu^{meas} * \rho_n \in W'$ (in the sense of Lemma 10.4) which converges to a decomposition of $\mu \in W'$.

Let us now prove a decomposition result as in [12].

Theorem 10.8 *If $\mu \in \mathcal{M}_0(Q)$, then there exist $g \in W'$ and $h \in L^1(Q)$, such that $\mu = g + h$, in the sense that*

$$\int_Q \varphi \, d\mu = \langle\langle g, \varphi \rangle\rangle + \int_Q h \, \varphi \, dxdt \,, \tag{10.34}$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$.

Proof. We follow the proof of [12]. First of all, using the Hahn decomposition of μ, if $\mu \in \mathcal{M}_0(Q)$ also μ^+, $\mu^- \in \mathcal{M}_0(Q)$, hence we can assume that μ is nonnegative. Applying Theorem 10.7 above there exists $\gamma \in W' \cap \mathcal{M}_b^+(Q)$ and a nonnegative Borel function $f \in L^1(Q, d\gamma^{\text{meas}})$, such that

$$\mu(B) = \int_B f \, d\gamma^{\text{meas}}$$

for every Borel set B in Q. Now let us replace μ with a compactly supported measure. To this end, it is enough to use the fact that $\mathcal{C}_c^\infty(Q)$ is dense in $L^1(Q, d\gamma^{\text{meas}})$ since γ^{meas} is a regular measure; there exists thus a sequence $\{f_n\} \in \mathcal{C}_c^\infty(Q)$ such that f_n strongly converges to f in $L^1(Q, d\gamma^{\text{meas}})$. Without loss of generality we can assume that $\sum_0^\infty \|f_n - f_{n-1}\|_{L^1(Q, d\gamma^{\text{meas}})} < \infty$, so that, defining $\nu_n = (f_n - f_{n-1})\gamma \in W'$, we have, by Lemma 10.5, $\nu_n \in W' \cap \mathcal{M}_b(Q)$ and $\sum_0^\infty \nu_n^{\text{meas}} = \sum_0^\infty (f_n - f_{n-1})\gamma^{\text{meas}} = \mu$ converges in the strong topology of measures.

The convergence result of Lemma 10.5 applied to ν_n implies that $\rho_l * \nu_n^{\text{meas}}$ strongly converges to ν_n in W' as l tends to infinity. We can therefore extract a subsequence l_n such that $\|\rho_{l_n} * \nu_n^{\text{meas}} - \nu_n\|_{W'} \le \frac{1}{2^n}$. We have then

$$\sum_{k=0}^n \nu_k^{\text{meas}} = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}} + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) \,. \tag{10.35}$$

The first member involved in this equality, denoted hereafter by m_n, is a measure with compact support. The second term, denoted by h_n, is a function in $\mathcal{C}_c^\infty(Q)$. By letting $g_n = \sum_{k=0}^n (\nu_k - \rho_{l_k} * \nu_k^{\text{meas}}) \in W' \cap \mathcal{M}_b(Q)$, the third term of (10.35) is g_n^{meas}; moreover, we can write $g_n = \theta_n g_n$ with $\theta_n \in \mathcal{C}_c^\infty(Q)$ (indeed, take $\theta_n \equiv 1$ on a neighborhood of $\text{Supp}(f_0) \cup \cdots \cup \text{Supp}(f_n)$ and on the neighborhood of the support of the $\mathcal{C}_c^\infty(Q)$ function $\sum_{k=0}^n \rho_{k_l} * \nu_k$). (10.35) is an equality in $\mathcal{M}_b(Q)$, i.e. involving g_n^{meas} and that can be applied only with test functions in $\mathcal{C}_c^\infty(Q)$; but thanks to the preceding remarks concerning the support of the elements involved in this equality, we can in fact deduce that, for all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$, we have

$$\int_Q \varphi \, dm_n = \int_Q \varphi h_n + \langle\langle g_n, \varphi \rangle\rangle. \tag{10.36}$$

Indeed, the measures in (10.35) having compact supports, this formula can be applied to functions in $\mathcal{C}^\infty(Q)$ and since

$$\langle g_n^{\text{meas}}, \varphi \rangle = \langle g_n^{\text{meas}}, \theta_n \varphi \rangle = \langle\langle g_n, \theta_n \varphi \rangle\rangle = \langle\langle g_n, \varphi \rangle\rangle,$$

this gives (10.36).

Now, m_n is strongly convergent in $\mathcal{M}_b(Q)$ to μ. h_n strongly converges in $L^1(Q)$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}}\|_{L^1(Q)} \le \|\nu_k^{\text{meas}}\|_{\mathcal{M}_b(Q)}$ and $\sum_{k=0}^\infty \nu_k^{\text{meas}}$ is totally convergent in $\mathcal{M}_b(Q)$); we denote by h its limit. We also have that g_n is strongly convergent in W' (because $\|\rho_{l_k} * \nu_k^{\text{meas}} - \nu_k\|_{W'} \le \frac{1}{2^k}$), denoting by g its limit we get, for every $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$,

$$\langle\langle g_n, \varphi \rangle\rangle \to \langle\langle g, \varphi \rangle\rangle. \tag{10.37}$$

By convergence of h_n to h in $L^1(Q)$, and since φ is bounded, we also have

$$\int_Q h_n \varphi \to \int_Q h \varphi. \tag{10.38}$$

To prove the convergence of $\int_Q \varphi \, dm_n$ to $\int_Q \varphi \, d\mu$, we just recall that there is a natural injection

$$\begin{cases} \mathcal{M}_b(Q) \longrightarrow (\mathcal{C}_b(Q))' \\ \qquad m \longrightarrow \widetilde{m} \quad \text{defined by} \quad \widetilde{m}(f) = \int_Q f \, dm \end{cases}$$

which is linear and continuous. Thus, since m_n strongly converges in $\mathcal{M}_b(Q)$ to μ, $\widetilde{m_n}$ strongly converges in $(\mathcal{C}_b(Q))'$ to $\widetilde{\mu}$ and, since $\varphi \in \mathcal{C}_b(Q)$,

$$\int_Q \varphi \, dm_n = \widetilde{m_n}(\varphi) \to \widetilde{\mu}(\varphi) = \int_Q \varphi \, d\mu. \tag{10.39}$$

Gathering (10.36), (10.37), (10.38) and (10.39), we get (10.34). ∎

Combining Theorem 10.8 and Lemma 10.4 we deduce the following.

Theorem 10.9 *Let $\mu \in \mathcal{M}_0(Q)$, then there exists a decomposition (f, g_1, g_2) of μ in the sense that $f \in L^1(Q)$, $g_1 \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and*

$$\int_Q \varphi \, d\mu = \int_Q f \, \varphi \, dxdt + \int_0^T \langle g_1, \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g_2 \rangle \, dt \,, \qquad \forall \varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)\,.$$

Of course, there are infinitely many possible different decompositions of the same measure $\mu \in \mathcal{M}_0(Q)$, so the following lemma will be useful for further purposes.

Lemma 10.6 *Let $\mu \in \mathcal{M}_0(Q)$, and let (f, g_1, g_2) and $(\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ be two different decompositions of μ according to Theorem 10.9. Then we have $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$ in distributional sense, $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T]; L^1(\Omega))$ and $(g_2 - \tilde{g}_2)(0) = 0$.*

Proof. By assumption we have :

$$\int_Q (\tilde{f} - f)\varphi \, dxdt + \int_0^T \langle (\tilde{g}_1 - g_1), \varphi \rangle \, dt = - \int_0^T \langle \varphi_t, g_2 - \tilde{g}_2 \rangle \, dt \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)\,, \tag{10.40}$$

which implies, in particular, that $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$ in distributional sense. Thus $g_2 - \tilde{g}_2 \in L^p(0,T; W_0^{1,p}(\Omega))$ and $(g_2 - \tilde{g}_2)_t \in L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))$. By Theorem 1.1 in [63] it follows that $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T]; L^1(\Omega))$. Since

$$\int_0^T \langle \varphi_t, g_2 - \tilde{g}_2 \rangle \, dt + \int_0^T \langle (g_2 - \tilde{g}_2)_t, \varphi \rangle \, dt = - \int_\Omega (g_2 - \tilde{g}_2)(0)\varphi(0) \, dx$$

for all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$ such that $\varphi(T) = 0$, we deduce from (10.40) (since $(g_2 - \tilde{g}_2)_t = \tilde{f} - f + \tilde{g}_1 - g_1$) that

$$\int_\Omega (g_2 - \tilde{g}_2)(0)\varphi(0) \, dx = 0$$

for all $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)$ such that $\varphi(T) = 0$. Choosing $\varphi = (T - t)\psi$, with $\psi \in \mathcal{C}_c^\infty(\Omega)$ implies that $(g_2 - \tilde{g}_2)(0) = 0$. ∎

We will now state and prove, thanks to what has been done in the proof of Theorem 10.8 and Remark 10.8, an approximation result concerning elements of $\mathcal{M}_0(Q)$, which will allow us to obtain additional regularity results on the renormalized solution of (10.1).

Proposition 10.5 *Let $\mu \in \mathcal{M}_0(Q)$. Then there exist a decomposition $(f, \mathrm{div}(G_1), g_2)$ of μ in the sense of Theorem 10.9 and an approximation μ_n of μ satisfying:*

$$\begin{aligned}
&\mu_n \in \mathcal{C}_c^\infty(Q)\,, \quad \|\mu_n\|_{\mathcal{M}_b(Q)} \le C\,,\\
&\int_Q \mu_n \varphi \, dxdt = \int_Q \varphi \, f_n \, dxdt + \int_0^T \langle \mathrm{div}(G_1^n), \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g_2^n \rangle \, dt \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega)\,,\\
&f_n \in \mathcal{C}_c^\infty(Q)\,, \quad f_n \to f \qquad \textit{strongly in } L^1(Q),\\
&G_1^n \in (\mathcal{C}_c^\infty(Q))^N\,, \quad G_1^n \to G_1 \qquad \textit{strongly in } (L^{p'}(Q))^N,\\
&g_2^n \in \mathcal{C}_c^\infty(Q)\,, \quad g_2^n \to g_2 \qquad \textit{strongly in } L^p(0,T;V).
\end{aligned}$$

Proof. We will prove that there exists a decomposition $(f, \mathrm{div}(G_1), g_2)$ of μ such that, for all $\varepsilon > 0$, we can find $\mu_\varepsilon \in \mathcal{C}_c^\infty(Q)$ satisfying $\|\mu_\varepsilon\|_{L^1(Q)} \leq C$,

$$\int_Q \mu_\varepsilon \varphi \, dxdt = \int_Q \varphi \, f_\varepsilon \, dxdt + \int_0^T \langle \mathrm{div}(G_1^\varepsilon), \varphi\rangle \, dt - \int_0^T \langle \varphi_t, g_2^\varepsilon\rangle \, dt \,, \quad \forall \varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)\,,$$

with $f_\varepsilon \in \mathcal{C}_c^\infty(Q)$ such that $\|f_\varepsilon - f\|_{L^1(Q)} \leq C\varepsilon$, $G_1^\varepsilon \in (\mathcal{C}_c^\infty(Q))^N$ such that $\|G_1^\varepsilon - G_1\|_{(L^{p'}(Q))^N} \leq C\varepsilon$ and $g_2^\varepsilon \in \mathcal{C}_c^\infty(Q)$ such that $\|g_2^\varepsilon - g_2\|_{L^p(0,T;V)} \leq C\varepsilon$ (with C not depending on ε).
We use the notations of the proof of Theorem 10.8.
Recalling that $\nu_k = (f_k - f_{k-1})\gamma$, we take $\zeta_k \in \mathcal{C}_c^\infty(Q)$ such that $\zeta_k \equiv 1$ on a neighborhood of $\mathrm{Supp}(f_k - f_{k-1})$; there exists $C(\zeta_k)$ only depending on ζ_k such that,

if $E \in \{(L^{p'}(Q))^N \,,\ L^p(0,T;V)\,,\ L^{p'}(0,T;L^2(\Omega))\}$ and $h\in E$, then $\|\zeta_k h\|_E \leq C(\zeta_k)\|h\|_E$,
if $H \in (L^{p'}(Q))^N$ then $\|H\cdot\nabla\zeta_k\|_{L^{p'}(Q)} \leq C(\zeta_k)\|H\|_{(L^{p'}(Q))^N}$,
if $h \in L^p(0,T,L^2(\Omega))$, then $\|(\zeta_k)_t h\|_{L^p(0,T;L^2(\Omega))} \leq C(\zeta_k)\|h\|_{L^p(0,T;L^2(\Omega))}$.

Instead of the l_k chosen in the proof of Theorem 10.8, we take here l_k such that $\|\rho_{l_k} * \nu_k^{\mathrm{meas}} - \nu_k\|_{W'} \leq 1/(2^k(C(\zeta_k)+1))$ and $\zeta_k \equiv 1$ on a neighborhood of $\mathrm{Supp}(\rho_{l_k} * \nu_k^{\mathrm{meas}})$. With this choice and taking $(\mathrm{div}(B_1^k), b_2^k, b_3^k)$ a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$ as in Lemma 10.4, satisfying moreover

$$\|B_1^k\|_{(L^{p'}(Q))^N} + \|b_2^k\|_{L^p(0,T;V)} + \|b_3^k\|_{L^{p'}(0,T;L^2(\Omega))} \leq C\|\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}\|_{W'}$$

with C not depending on k (this is possible thanks to (10.30)), we notice that

$$\begin{array}{l} \sum_{k\geq 1}\zeta_k B_1^k \text{ converges in } (L^{p'}(Q))^N\,,\ \sum_{k\geq1}\zeta_k b_2^k \text{ converges in } L^p(0,T;V)\,,\\ \sum_{k\geq1}\zeta_k b_3^k \text{ converges in } L^{p'}(0,T;L^2(\Omega))\,,\ \sum_{k\geq1} B_1^k\cdot\nabla\zeta_k \text{ converges in } L^{p'}(Q)\,,\\ \sum_{k\geq1}(\zeta_k)_t b_2^k \text{ converges in } L^p(0,T;L^2(\Omega)). \end{array} \tag{10.41}$$

We denote by G_1, $-g_2$, f_1, f_2 and f_3 the respective limits of these terms; notice that the last three convergences imply in particular the convergence in $L^1(Q)$.
Since $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}})$ in W' (by choice of ζ_k and l_k) and $(\mathrm{div}(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$, $(\zeta_k B_1^k, \zeta_k b_2^k, \zeta_k b_3^k, -B_1^k\cdot\nabla\zeta_k, (\zeta_k)_t b_2^k)$ is a pseudo-decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\mathrm{meas}}$ (see Remark 10.8).
Thus, by (10.36), for all $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$,

$$\begin{aligned} \int_Q \varphi\, dm_n &= \int_Q \varphi h_n + \int_0^T \langle \mathrm{div}\left(\sum_{k=0}^n \zeta_k B_1^k\right), \varphi\rangle + \int_0^T \langle \varphi_t, \sum_{k=0}^n \zeta_k b_2^k\rangle \\ &\quad + \int_0^T \sum_{k=0}^n \zeta_k b_3^k \varphi + \int_Q \sum_{k=0}^n (-B_1^k\cdot\nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (\zeta_k)_t b_2^k \varphi, \end{aligned}$$

and, by the convergences of m_n to μ, of h_n to h and (10.41), we deduce that

$$\int_Q \varphi\, d\mu = \int_Q (h + f_1 - f_2 + f_3)\varphi + \int_0^T \langle \mathrm{div}(G_1), \varphi\rangle - \int_0^T \langle \varphi_t, g_2\rangle,$$

i.e. that $(f = h + f_1 - f_2 + f_3, \mathrm{div}(G_1), g_2)$ is a decomposition of μ in the sense of Theorem 10.9.
We fix now $\varepsilon > 0$ and take n large enough (in fact $n = n_\varepsilon$ is fixed in dependence of ε hereafter) so that

$$\left\|\sum_{k=0}^n \zeta_k B_1^k - G_1\right\|_{(L^{p'}(Q))^N} \leq \varepsilon\,, \tag{10.42}$$

$$\left\|\sum_{k=0}^n \zeta_k b_2^k + g_2\right\|_{L^p(0,T;V)} \leq \varepsilon\,, \tag{10.43}$$

$$\left\|h_n + \sum_{k=0}^n \zeta_k b_3^k - \sum_{k=0}^n (B_1^k\cdot\nabla\zeta_k) + \sum_{k=0}^n (\zeta_k)_t b_2^k - f\right\|_{L^1(Q)} \leq \varepsilon\,. \tag{10.44}$$

Since $\nu_k - \rho_{l_k} * \nu_k^{\text{meas}} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$ and $(\operatorname{div}(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{\text{meas}}$, we also know that, for j large enough, $((\zeta_k B_1^k) * \rho_j, (\zeta_k b_2^k) * \rho_j, (\zeta_k b_3^k) * \rho_j, (-B_1^k \cdot \nabla\zeta_k) * \rho_j, ((\zeta_k)_t b_2^k) * \rho_j)$ is a pseudo-decomposition of $(\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_j \in \mathcal{C}_c^\infty(Q)$ (see Remark 10.8). We take j_n such that, for all $k \in [0, n]$,

$$\|(\zeta_k B_1^k) * \rho_{j_n} - \zeta_k B_1^k\|_{(L^{p'}(Q))^N} \leq \frac{\varepsilon}{n+1}, \tag{10.45}$$

$$\|(\zeta_k b_2^k) * \rho_{j_n} - \zeta_k b_2^k\|_{L^p(0,T;V)} \leq \frac{\varepsilon}{n+1}, \tag{10.46}$$

$$\begin{aligned} &\|(\zeta_k b_3^k) * \rho_{j_n} - \zeta_k b_3^k\|_{L^1(Q)} + \|(B_1^k \cdot \nabla\zeta_k) * \rho_{j_n} - B_1^k \cdot \nabla\zeta_k\|_{L^1(Q)} \\ &\quad + \|((\zeta_k)_t b_2^k) * \rho_{j_n} - (\zeta_k)_t b_2^k\|_{L^1(Q)} \leq \frac{\varepsilon}{n+1} \end{aligned} \tag{10.47}$$

Define $G_1^\varepsilon = \sum_{k=0}^n (\zeta_k B_1^k) * \rho_{j_n} \in (\mathcal{C}_c^\infty(Q))^N$; we have, by (10.42) and (10.45), $\|G_1^\varepsilon - G_1\|_{(L^{p'}(Q))^N} \leq 2\varepsilon$. Let $g_2^\varepsilon = -\sum_{k=0}^n (\zeta_k b_2^k) * \rho_{j_n} \in \mathcal{C}_c^\infty(Q)$; we have, by (10.43) and (10.46), $\|g_2^\varepsilon - g_2\|_{L^p(0,T;V)} \leq 2\varepsilon$. If $f_\varepsilon = h_n + \sum_{k=0}^n (\zeta_k b_3^k) * \rho_{j_n} - \sum_{k=0}^n (B_1^k \cdot \nabla\zeta_k) * \rho_{j_n} + \sum_{k=0}^n ((\zeta_k)_t b_2^k) * \rho_{j_n} \in \mathcal{C}_c^\infty(Q)$, we have, by (10.44) and (10.47), $\|f_\varepsilon - f\|_{L^1(Q)} \leq 2\varepsilon$.
Define now $\mu_\varepsilon = f_\varepsilon + \operatorname{div}(G_1^\varepsilon) + (g_2^\varepsilon)_t \in \mathcal{C}_c^\infty(Q)$; it remains to prove that $\|\mu_\varepsilon\|_{L^1(Q)} \leq C$ with C not depending on ε. To see this, we recall that $((\zeta_k B_1^k) * \rho_{j_n}, (\zeta_k b_2^k) * \rho_{j_n}, (\zeta_k b_3^k) * \rho_{j_n}, (-B_1^k \cdot \nabla\zeta_k) * \rho_{j_n}, ((\zeta_k)_t b_2^k) * \rho_{j_n})$ is a pseudo-decomposition of $(\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_{j_n}$ so that

$$\mu_\varepsilon = h_n + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}) * \rho_{j_n} = h_n + \left(\sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})\right) * \rho_{j_n} = h_n + g_n^{\text{meas}} * \rho_{j_n}.$$

Since, by (10.35), $g_n^{\text{meas}} = m_n - h_n$, we deduce that $\|\mu_\varepsilon\|_{L^1(Q)} \leq 2\|h_n\|_{L^1(Q)} + \|m_n\|_{\mathcal{M}_b(Q)}$. Since $\{h_n\}$ converges in $L^1(Q)$ and $\{m_n\}$ converges in $\mathcal{M}_b(Q)$, $\{\|h_n\|_{L^1(Q)}\}$ and $\{\|m_n\|_{\mathcal{M}_b(Q)}\}$ are bounded, which imply the desired majoration on $\|\mu_\varepsilon\|_{L^1(Q)}$. ■

10.3 The IBV problem with data in $\mathcal{M}_0(Q)$.

Let us turn to the study of initial boundary value problems with data taken in $\mathcal{M}_0(Q)$. We start by introducing the following nonlinear monotone operators.
Let $a :]0, T[\times\Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function (i.e., $a(\cdot,\cdot,\xi)$ is measurable on Q for every ξ in $\mathbf{R}^N$, and $a(t, x, \cdot)$ is continuous on $\mathbf{R}^N$ for almost every (t, x) in Q), such that the following holds:

$$a(t, x, \xi) \cdot \xi \geq \alpha\, |\xi|^p, \quad p > 1, \tag{10.48}$$

$$|a(t, x, \xi)| \leq \beta\, [b(t, x) + |\xi|^{p-1}], \tag{10.49}$$

$$[a(t, x, \xi) - a(t, x, \eta)] \cdot (\xi - \eta) > 0, \tag{10.50}$$

for almost every (t, x) in Q, for every ξ, η in $\mathbf{R}^N$, with $\xi \neq \eta$, where α and β are two positive constants, and b is a nonnegative function in $L^{p'}(Q)$.
Let us define the differential operator

$$A(u) = -\operatorname{div}(a(t, x, \nabla u)), \qquad u \in L^p(0, T; W_0^{1,p}(\Omega)).$$

Under assumptions (10.48), (10.49) and (10.50), A is a coercive and pseudomonotone operator acting from the space $L^p(0, T; W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0, T; W^{-1,p'}(\Omega))$, hence for $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, (10.1) has a unique solution in $\widetilde{W}$ (see Definition 10.2) in the weak sense (see [51]).

10.3.1 Variational case

Let us justify the interest of W', giving the following existence and uniqueness theorem (this theorem could also be stated with right-hand sides in $\widetilde{W}'$ with no major modification in the proof).

Theorem 10.10 *Let g belong to W', and let $u_0 \in L^2(\Omega)$. Assume that (10.48)–(10.50) hold true. Then there exists a unique solution u of*

$$\begin{cases} u_t + A(u) = g & \textit{in }]0,T[\times\Omega, \\ u = 0 & \textit{on }]0,T[\times\partial\Omega, \\ u(0) = u_0 & \textit{in } \Omega, \end{cases} \tag{10.51}$$

in the sense that $u \in L^p(0,T;V)$ and satisfies

$$-\int_Q \langle\varphi_t, u\rangle\, dt - \int_\Omega u_0\varphi(0)\, dx + \int_Q a(t,x,\nabla u)\nabla\varphi\, dxdt = \langle\langle g, \varphi\rangle\rangle, \tag{10.52}$$

for all $\varphi \in W$ with $\varphi(T) = 0$.

Remark 10.9 *Since $g \in W'$, there exists*

$$g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))\,,\ g_2 \in L^p(0,T;V) \text{ and } g_3 \in L^{p'}(0,T;L^2(\Omega))$$

such that

$$\langle\langle g, \varphi\rangle\rangle = \int_0^T \langle g_1, \varphi\rangle - \int_0^T \langle\varphi_t, g_2\rangle + \int_Q g_3\varphi\,, \quad \forall\varphi \in W.$$

For any such decomposition, we deduce that u satisfying (10.52) is such that $(u-g_2)_t = -A(u)+g_1+g_3 \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$, so that $u - g_2 \in W \subset \mathcal{C}([0,T];L^2(\Omega))$ and, returning to (10.52), we find $(u-g_2)(0) = u_0$.
Moreover, for any two solutions u and v of (10.52), we have $u - v = u - g_2 - (v - g_2) \in W$ and $(u-v)(0) = 0$.

Proof of Theorem 10.10. We take $(g_1, -g_2, g_3)$ a decomposition of g according to Lemma 10.4. Let $(g_1^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_1 in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, $(g_2^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_2 in $L^p(0,T;V)$ and $(g_3^n)_{n\geq 1} \in \mathcal{C}_c^\infty(Q)$ strongly converge to g_3 in $L^{p'}(0,T;L^2(\Omega))$ (the existence of such sequences is a consequence of Lemma 10.8 and Remark 10.17). Thanks to [51], there exists a solution u_n of

$$\begin{cases} u_t^n + A(u^n) = g_1^n + g_3^n + (g_2^n)_t & \text{in }]0,T[\times\Omega, \\ u^n = 0 & \text{on }]0,T[\times\partial\Omega, \\ u^n(0) = u_0 & \text{in } \Omega, \end{cases}$$

in the sense that $u^n \in \widetilde{W}$ and

$$\int_\Omega (u^n - g_2^n)(t)\varphi(t)\, dx - \int_0^t \langle\varphi_t, u^n - g_2^n\rangle ds - \int_\Omega u_0\varphi(0)\, dx$$
$$+ \int_0^t\int_\Omega a(s,x,\nabla u^n)\nabla\varphi\, dxds = \int_0^t \langle g_1^n, \varphi\rangle ds + \int_0^t\int_\Omega g_3^n\varphi\, dxds$$

for all $\varphi \in W$ and $t \in [0,T]$. Note that since $g_2^n \in \mathcal{C}_c^\infty(Q)$, we have $(u^n - g_2^n)(0) = u^n(0) = u_0$. Using $u^n - g_2^n$ as test function, and integrating by parts, we find

$$\int_\Omega \frac{(u^n - g_2^n)^2(t)}{2} - \int_\Omega \frac{u_0^2}{2} + \int_0^t\int_\Omega a(s,x,\nabla u^n)\nabla(u^n - g_2^n)\, dxds$$
$$= \int_0^t \langle g_1^n, u^n - g_2^n\rangle ds + \int_0^t\int_\Omega g_3^n(u^n - g_2^n)\, dxds$$

thus, using (10.48), (10.49) and Young's inequality,

$$\begin{aligned}
&\int_\Omega \frac{(u^n-g_2^n)^2(t)}{2}\,dx + \int_0^t\int_\Omega |\nabla u_n|^p\,dxds\\
&\leq\; C\Big(\|g_1^n\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|g_3^n\|^2_{L^{p'}(0,T;L^2(\Omega))} + \|u_0\|^2_{L^2(\Omega)}\\
&\qquad +||b||^{p'}_{L^{p'}(Q)}\Big) + \frac{1}{4T^{\frac{2}{p}}}\|u^n-g_2^n\|^2_{L^p(0,T;L^2(\Omega))}\\
&\leq\; C\Big(\|g_1^n\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|g_2^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|g_3^n\|^2_{L^{p'}(0,T;L^2(\Omega))} + \|u_0\|^2_{L^2(\Omega)}\\
&\qquad +\|b\|^{p'}_{L^{p'}(Q)}\Big) + \frac{T^{\frac{2}{p}}}{4T^{\frac{2}{p}}}\|u^n-g_2^n\|^2_{L^\infty(0,T;L^2(\Omega))}
\end{aligned}$$

which implies

$$\|u^n-g_2^n\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u^n\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C\,. \tag{10.53}$$

Thanks to the equation, we deduce from this that $(u^n-g_2^n)_t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$ so that, in fact, $(u^n-g_2^n)$ is bounded in W. There exists thus $w\in W$ such that, up to a subsequence, $u^n-g_2^n\to w$ weakly in W. But, from (10.53), (u^n) is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and converges thus, up to a subsequence, weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to a function u. Since $g_2^n\to g_2$ in $L^p(0,T;W_0^{1,p}(\Omega))$, this implies that $u^n-g_2^n\to u-g_2$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ so that $w=u-g_2\in W\subset \mathcal{C}([0,T];L^2(\Omega))$; note also that, since $u-g_2\in W$ and $g_2\in L^p(0,T;V)$, one has $u\in L^p(0,T;V)$.
Moreover, $A(u^n)$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, thus (up to subsequences) it converges weakly to an element f in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Using the equation in the sense of the distributions, we have $(u-g_2)_t+f=g_1+g_3$, hence, since $u-g_2\in W$,

$$-\int_0^T\langle\varphi_t,u-g_2\rangle dt - \int_\Omega (u-g_2)(0)\varphi(0)\,dx = \int_0^T\langle g_1-f,\varphi\rangle dt + \int_Q g_3\varphi\,dxdt.$$

for all $\varphi\in W$ such that $\varphi(T)=0$. On the other hand the equation implies, passing to the limit in n, that, with $\varphi\in W$ such that $\varphi(T)=0$,

$$-\int_0^T\langle\varphi_t,u-g_2\rangle dt - \int_\Omega u_0\varphi(0)\,dx = \int_0^T\langle g_1-f,\varphi\rangle dt + \int_Q g_3\varphi\,dxdt$$

so that $(u-g_2)(0)=u_0$. Now using $(u^n-g_2^n)-(u-g_2)$ as test function (note that $((u^n-g_2^n)-(u-g_2))(0)=0$), one has

$$\begin{aligned}
&\int_\Omega \frac{((u^n-g_2^n)-(u-g_2))^2(T)}{2}\,dx + \int_0^T\langle (u-g_2)_t,(u^n-g_2^n)-(u-g_2)\rangle dt\\
&\quad+\int_Q [a(t,x,\nabla u^n)-a(t,x,\nabla u)]\nabla(u^n-u)\,dxdt + \int_Q a(t,x,\nabla u)\nabla(u^n-u)\,dxdt\\
&\quad+\int_Q a(t,x,\nabla u^n)\nabla(g_2-g_2^n)\,dxdt\\
&=\int_0^T\langle g_1^n,(u^n-g_2^n)-(u-g_2)\rangle dt + \int_Q g_3^n[(u^n-g_2^n)-(u-g_2)]\,dxdt\,.
\end{aligned}$$

Since the second term and last four terms converge to 0, thanks to the positivity of the first one and to (10.50), one gets

$$\lim_{n\to\infty}\int_Q [a(t,x,\nabla u^n)-a(t,x,\nabla u)]\nabla(u^n-u)\,dxdt = 0$$

hence, using the standard monotonicity argument (see Lemma 5 in [13]), one has the convergence almost everywhere of ∇u^n to ∇u and the strong convergence of $a(t,x,\nabla u^n)$ to $a(t,x,\nabla u)$ in $(L^{p'}(Q))^N$. This proves the existence of a solution.
For uniqueness, let us suppose there are two solutions u and v, thanks to Remark 10.9, $u-v\in W$ so that, subtracting the two equations, one can choose $u-v$ as test function, obtaining:

$$\int_\Omega \frac{(u-v)^2(t)}{2}\,dx + \int_0^t\int_\Omega [a(t,x,\nabla u)-a(t,x,\nabla v)]\nabla(u-v)\,dxdt = 0, \quad \forall t\in]0,T[\,,$$

thus $u=v$ using (10.50). ∎

10.3.2 Definition and properties of renormalized solutions

Now we want to deal with the general problem (10.1) when μ is a measure which does not charge sets of zero capacity. In virtue of Theorem 10.8, this means that we consider measure data which split in a term of W' and a term in $L^1(Q)$. It is then well known that, if dealing with L^1 data, the concept of solution in the sense of distributions of problems like (10.1) may turn out to be not convenient in order to prove uniqueness of solutions. Moreover, we will deal with functions that may not belong to Sobolev spaces, so that we need to give a suitable definition of "gradient" for functions that enjoy some properties. To this purpose, if $k>0$, we define

$$T_k(s)=\max(-k,\min(k,s))\,, \qquad \forall s\in\mathbf{R}\,,$$

the truncature at levels k and $-k$, and $\Theta_k(s)=\int_0^s T_k(t)\,dt$. One has $\Theta_k(s)\ge 0$.
The truncations will provide very useful for defining a good class of solutions, as in [4].

Definition 10.6 *Let u be a measurable function on Q such that $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Then (see [4], Lemma 2.1) there exists a unique measurable function $v: Q\to\mathbf{R}^N$ such that*

$$\nabla T_k(u) = v\,\chi_{\{|u|<k\}}\,, \quad \text{almost everywhere in } Q, \text{ for every } k>0.$$

We will define the gradient of u as the function v, and we will denote it by $v=\nabla u$. If u belongs to $L^1(0,T;W_0^{1,1}(\Omega))$, then this gradient coincides with the usual gradient in distributional sense.

Let us introduce the definition of renormalized solution of (10.1).

Definition 10.7 *Let $\mu\in\mathcal{M}_0(Q)$. A measurable function u is a renormalized solution of (10.1) if there exists a decomposition (f,g_1,g_2) of μ such that*

$$u-g_2\in L^\infty(0,T;L^1(\Omega))\,,\ T_k(u-g_2)\in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k>0, \tag{10.54}$$

$$\lim_{n\to\infty}\int_{\{n\le|u-g_2|\le n+1\}} |\nabla u|^p\,dxdt = 0\,, \tag{10.55}$$

and, for every $S\in W^{2,\infty}(\mathbf{R})$ such that S' has compact support,

$$\begin{aligned}(S(u-g_2))_t - \operatorname{div}(a(t,x,\nabla u)S'(u-g_2)) + S''(u-g_2)a(t,x,\nabla u)\nabla(u-g_2) =\\ = S'(u-g_2)f + G_1S''(u-g_2)\nabla(u-g_2) - \operatorname{div}(G_1S'(u-g_2))\end{aligned} \tag{10.56}$$

in the sense of distributions (where $g_1=-\operatorname{div}(G_1)$) and

$$S(u-g_2)(0)=S(u_0) \text{ in } L^1(\Omega). \tag{10.57}$$

Remark 10.10 *Note that the distributional meaning of each term in (10.56) is well defined thanks to the fact that $T_k(u-g_2)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$ and since S' has compact support. Indeed, by taking M such that $\mathrm{Supp}(S')\subset]-M,M[$, since $S'(u-g_2)=S''(u-g_2)=0$ as soon as $|u-g_2|\geq M$, we can replace, everywhere in (10.56), $\nabla(u-g_2)$ by $\nabla(T_M(u-g_2))\in(L^p(Q))^N$ and ∇u by $\nabla(T_M(u-g_2))+\nabla g_2\in(L^p(Q))^N$.*
We also have, for all S as above, $S(u-g_2)=S(T_M(u-g_2))\in L^p(0,T;W_0^{1,p}(\Omega))$; thus, by the equation (10.56), $(S(u-g_2))_t$ belongs to the space $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, which implies that $S(u-g_2)$ belongs to $\mathcal{C}([0,T];L^1(\Omega))$ (again see [63]). Thus condition (10.57) makes sense. Furthermore, since $(S(u-g_2))_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$ we can use, as test functions in (10.56), not only functions in $\mathcal{C}_c^\infty(Q)$ but also functions in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$.
Finally, observe also that condition (10.55) is equivalent to

$$\lim_{n\to\infty}\int\limits_{\{n\leq|u-g_2|\leq n+1\}}|\nabla(u-g_2)|^p\,dxdt=0\,,$$

since $g_2\in L^p(0,T;W_0^{1,p}(\Omega))$ and $u-g_2$ is almost everywhere finite.

Remark 10.11 *The initial condition $S(u-g_2)(0)=S(u_0)$ is the renormalized version of the requirement that $(u-g_2)(0)=u_0$. However, it also expresses, in a weak sense, that $u(0)=u_0$, as written in (10.1). This is due to the fact that μ is a measure on the σ-algebra of the borelians of the open set Q, which implies that μ is taken in a way that it does not charge the sets at $t=0$. More precisely, if $\xi_\varepsilon(t)=(\frac{\varepsilon-t}{\varepsilon})^+$, for any $\varphi\in\mathcal{C}_c^\infty(\Omega)$ we have, by Lebesgue's theorem,*

$$\lim_{\varepsilon\to0}\int_Q\varphi\,\xi_\varepsilon\,d\mu=0\,.$$

It follows then for any decomposition of μ

$$\lim_{\varepsilon\to0}\int_Q f\xi_\varepsilon\,\varphi\,dxdt+\int_0^T\langle g_1,\varphi\rangle\xi_\varepsilon\,dt+\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega g_2\,\varphi\,dxdt=0\,,$$

which implies, by the time regularity of f and g_1,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega g_2\,\varphi\,dxdt=0\,,\quad\forall\varphi\in\mathcal{C}_c^\infty(\Omega)\,.\tag{10.58}$$

Note that (10.58) is a weak expression of the fact that $g_2(0)=0$, so that $(u-g_2)(0)=u(0)$ in some weak sense thanks to the fact that the measure μ is defined in the interior of Q.
On the other hand, it would also be possible to consider measures μ on the σ-algebra of borelians of $[0,T)\times\Omega$, hence μ would charge the level $t=0$. However, this case easily reduces to the previous one. Indeed, we can split $\mu=\mu_Q+\mu_i$, where $\mu_i=\mu_{|\{t=0\}}$ is the restriction of μ to $t=0$ (i.e. $\mu_i(E)=\mu(E\cap(\{t=0\}\times\Omega))$ for any set E) and μ_Q is the restriction to the open set Q. In this case problem (10.1) is equivalent to problem

$$\begin{cases}u_t+A(u)=\mu_Q & \text{in }]0,T[\times\Omega,\\ u=0 & \text{on }]0,T[\times\partial\Omega,\\ u(0)=u_0+\mu_i & \text{in }\Omega.\end{cases}\tag{10.59}$$

If μ is a measure which does not charge sets of zero capacity we have by Theorem 10.5 that $\mu_i\in L^1(\Omega)$, and the study of (10.59) reduces to the one we can do for measures μ only defined on the interior of Q.

Remark 10.12 *As we have already noticed, when u is a renormalized solution of (10.1) and S is as in Definition 10.7, we have $S(u-g_2)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and $(S(u-g_2))_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$; this allows, thanks to (ii) in Lemma 10.1, to prove that $S(u-g_2)$ has a cap–quasi continuous representative.*

In order to deal with the renormalized formulation, we will often make use of the following auxiliary functions of real variable.

Definition 10.8 *We define:*

$$\theta_n(s) = T_1(s - T_n(s)), \quad h_n(s) = 1 - |\theta_n(s)|, \quad S_n(s) = \int_0^s h_n(r)dr, \qquad \forall s \in \mathbf{R}.$$

Let us first prove that the formulation of renormalized solution does not depend on the decomposition of μ. This essentially relies on Lemma 10.6 proved before.

Proposition 10.6 *Let u be a renormalized solution of (10.1). Then u satisfies (10.54), (10.55), (10.56) and (10.57) for every decomposition (f, g_1, g_2) of μ.*

Proof. Assume that u satisfies the conditions of Definition 10.7 for (f, g_1, g_2), and let $(\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ be a different decomposition of μ. In the following we write $\tilde{g}_1 = -\mathrm{div}(\tilde{G}_1)$. Note that since, by Lemma 10.6, $g_2 - \tilde{g}_2 \in \mathcal{C}([0,T]; L^1(\Omega))$ we have $u - \tilde{g}_2 \in L^\infty(0,T; L^1(\Omega))$, hence it is also almost everywhere finite. First of all we prove that $T_k(u - \tilde{g}_2) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$. To do this, we let $S = S_n$ in (10.56), where S_n is defined in Definition 10.8, and we choose as test function $T_k(S_n(u - g_2) + g_2 - \tilde{g}_2)$, which belongs to $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. Using Lemma 10.6 we have:

$$\begin{aligned}
&\int_0^T \langle (S_n(u - g_2) + g_2 - \tilde{g}_2)_t, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \rangle \, dt \\
&\qquad + \int_Q S_n'(u - g_2) a(t, x, \nabla u) \nabla T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\quad = - \int_Q S_n''(u - g_2) a(t, x, \nabla u) \nabla(u - g_2) \, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\qquad + \int_Q ((S_n'(u - g_2) - 1) f + \tilde{f}) \, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\qquad + \int_Q ((S_n'(u - g_2) - 1) G_1 + \tilde{G}_1) \nabla T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\qquad + \int_Q S_n''(u - g_2) G_1 \nabla(u - g_2) T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt.
\end{aligned} \tag{10.60}$$

Since by (10.49)

$$\begin{aligned}
&\Big| - \int_Q S_n''(u - g_2) a(t, x, \nabla u) \nabla(u - g_2) \, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\qquad + \int_Q S_n''(u - g_2) G_1 \nabla(u - g_2) T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \Big| \\
&\quad \le Ck \int_{\{n \le |u - g_2| \le n+1\}} (|\nabla u|^p + |\nabla g_2|^p + |G_1|^{p'} + |b|^{p'}) \, dxdt,
\end{aligned}$$

thanks to (10.55) and the fact that $u - g_2$ is almost everywhere finite, we get

$$\begin{aligned}
\lim_{n \to \infty} &\Big| - \int_Q S_n''(u - g_2) a(t, x, \nabla u) \nabla(u - g_2) \, T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \\
&\qquad + \int_Q S_n''(u - g_2) G_1 \nabla(u - g_2) T_k(S_n(u - g_2) + g_2 - \tilde{g}_2) \, dxdt \Big| = 0.
\end{aligned}$$

Let us denote by $\omega(n)$ quantities going to zero as n tends to infinity. Integrating the first term of (10.60) in time, using that $k|s| \geq \Theta_k(s) \geq 0$, $(g_2 - \tilde{g}_2)(0) = 0$ and $0 \leq S'_n(s) \leq 1$, we obtain

$$\begin{aligned}
&\int_Q S'_n(u-g_2)a(t,x,\nabla u)\nabla T_k(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt \\
&\quad \leq k\left(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right) \\
&\qquad + \int_Q \left((S'_n(u-g_2)-1)G_1 + \tilde{G}_1\right)\nabla T_k(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt + \omega(n)\,.
\end{aligned}$$

Setting $E_n = \{|S_n(u-g_2)+g_2-\tilde{g}_2| \leq k\}$ we have:

$$\begin{aligned}
&\int_{E_n} [S'_n(u-g_2)]^2 a(t,x,\nabla u)\nabla u\,dxdt \\
&\leq k\left(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right) + \int_{E_n} (|G_1| + |\tilde{G}_1|)\,S'_n(u-g_2)|\nabla u|\,dxdt \\
&\quad + \int_{E_n} S'_n(u-g_2)|a(t,x,\nabla u)|\,(|\nabla\tilde{g}_2| + |\nabla g_2|)\,dxdt \\
&\quad + \int_{E_n} [S'_n(u-g_2)]^2|a(t,x,\nabla u)||\nabla g_2|\,dxdt + 2\int_Q (|G_1| + |\tilde{G}_1|)\,(|\nabla\tilde{g}_2| + |\nabla g_2|)\,dxdt + \omega(n)\,.
\end{aligned}$$

Young's inequality then implies, using also (10.48), (10.49) and $S'_n(s) \leq S'_n(s)^2 + \chi_{\{n\leq|s|\leq n+1\}}$:

$$\begin{aligned}
&\int_{E_n} [S'_n(u-g_2)]^2|\nabla u|^p\,dxdt \\
&\leq Ck(\|\tilde{f}\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) + C\int_Q (|G_1|^{p'} + |\tilde{G}_1|^{p'} + |\nabla\tilde{g}_2|^p + |\nabla g_2|^p + |b|^{p'})\,dxdt \\
&\quad + C\int_{\{n\leq|u-g_2|\leq n+1\}} |\nabla u|^p\,dxdt + \omega(n)\,.
\end{aligned}$$

Using $S'_n(s)^p \leq S'_n(s) \leq S'_n(s)^2 + \chi_{\{n\leq|s|\leq n+1\}}$ (because $0 \leq S'_n \leq 1$) and the fact that g_2 belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce from the preceding inequality that, for all $n \geq 1$,

$$\int_Q \chi_{E_n}|\nabla(S_n(u-g_2))|^p \leq C.$$

Since $\nabla(T_k(S_n(u-g_2)+g_2-\tilde{g}_2)) = \chi_{E_n}\nabla(S_n(u-g_2)+g_2-\tilde{g}_2)$ and since g_2, $\tilde{g}_2 \in L^p(0,T;W_0^{1,p}(\Omega))$, this implies that $v_n = T_k(S_n(u-g_2)+g_2-\tilde{g}_2)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and converges, up to a subsequence, to v weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, thus also in $\mathcal{D}'(Q)$; but $v_n \to T_k(u-\tilde{g}_2)$ a.e. in Q and is bounded by k, so that $v_n \to T_k(u-\tilde{g}_2)$ in $\mathcal{D}'(Q)$. We have then $T_k(u-\tilde{g}_2) = v \in L^p(0,T;W_0^{1,p}(\Omega))$, for all $k > 0$.

Similarly we prove that (10.55) holds true for $\tilde{g}_2$ as well : we choose $S = S_n$ and test function $\theta_h(S_n(u-g_2)+g_2-\tilde{g}_2)$ in (10.56). Reasoning as above we obtain, setting $F_n = \{h \leq |S_n(u-g_2)+g_2-\tilde{g}_2| \leq h+1\}$:

$$\begin{aligned}
&\int_{F_n} [S'_n(u-g_2)]^2 a(t,x,\nabla u)\nabla u\,dxdt \\
&\leq \int_Q ((S'_n(u-g_2)-1)f + \tilde{f})\theta_h(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt + \int_\Omega\int_0^{S_n(u_0)} \theta_h(r)\,dr\,dx \\
&\quad + \int_{F_n} S'_n(u-g_2)(|G_1| + |\tilde{G}_1|)|\nabla u|\,dxdt + \int_{F_n} S'_n(u-g_2)|a(t,x,\nabla u)|\,(|\nabla\tilde{g}_2| + |\nabla g_2|)\,dxdt \\
&\quad + \int_{F_n} [S'_n(u-g_2)]^2|a(t,x,\nabla u)||\nabla g_2|\,dxdt + 2\int_{F_n} (|G_1| + |\tilde{G}_1|)\,(|\nabla\tilde{g}_2| + |\nabla g_2|)\,dxdt + \omega(n)\,.
\end{aligned}$$

As before, thanks to Young's inequality, (10.49) and by properties of S_n we get:

$$\begin{aligned}
&\int_{F_n} [S_n'(u-g_2)]^2|\nabla u|^p \,dxdt \\
&\leq \; C\int_Q (|f|+|\tilde{f}|)|\theta_h(S_n(u-g_2)+g_2-\tilde{g}_2)|\,dxdt + \int_\Omega\int_0^{S_n(u_0)} \theta_h(r)\,dr\,dx \\
&\quad +C\int_{F_n}(|G_1|^{p'}+|\tilde{G}_1|^{p'}+|\nabla\tilde{g}_2|^p+|\nabla g_2|^p+|b|^{p'})\,dxdt \\
&\quad +C\int_{\{n\leq|u-g_2|\leq n+1\}} |\nabla u|^p\,dxdt+\omega(n)\,.
\end{aligned}$$

Letting n tend to infinity, using (10.55) and since χ_{F_n} converges to $\chi_{\{h\leq|u-\tilde{g}_2|\leq h+1\}}$ we obtain:

$$\begin{aligned}
\int_{\{h\leq|u-\tilde{g}_2|\leq h+1\}} |\nabla u|^p\,dxdt &\leq \int_{\{|u_0|>h\}} |u_0|\,dx \\
&+ \int_{\{|u-\tilde{g}_2|\geq h\}} \left(|f|+|\tilde{f}|+|G_1|^{p'}+|\tilde{G}_1|^{p'}+|\nabla\tilde{g}_2|^p+|\nabla g_2|^p+|b|^{p'}\right)\,dxdt\,,
\end{aligned}$$

which yields, as h tends to infinity (recall that $u-\tilde{g}_2$ is almost everywhere finite),

$$\lim_{h\to\infty}\int_{\{h\leq|u-\tilde{g}_2|\leq h+1\}} |\nabla u|^p\,dxdt=0\,.$$

We are left with the proof that the renormalized equation (10.56) and the initial condition (10.57) hold with $\tilde{g}_2$ as well. To this aim, we take $S=S_n$ in (10.56), we choose a function S such that S' has compact support and we take $S'(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi$ as test function in (10.56), with $\varphi\in\mathcal{C}_c^\infty(Q)$. By Lemma 10.6 we get:

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g_2)+g_2-\tilde{g}_2)_t, S'(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi\rangle\,dt \\
&\quad+\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla\varphi\,S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt \\
&\quad+\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla(S'(S_n(u-g_2)+g_2-\tilde{g}_2))\varphi\,dxdt \\
&\quad+\int_Q S_n''(u-g_2)\,a(t,x,\nabla u)\nabla(u-g_2)\,S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,\varphi\,dxdt \\
&=\int_Q ((S_n'(u-g_2)-1)f+\tilde{f})\,S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,\varphi\,dxdt \\
&\quad+\int_Q ((S_n'(u-g_2)-1)G_1+\tilde{G}_1)\nabla\varphi\,S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt \\
&\quad+\int_Q ((S_n'(u-g_2)-1)G_1+\tilde{G}_1)\nabla(S'(S_n(u-g_2)+g_2-\tilde{g}_2))\varphi\,dxdt \\
&\quad+\int_Q S_n''(u-g_2)\,G_1\nabla(u-g_2)S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,\varphi\,dxdt\,.
\end{aligned} \tag{10.61}$$

We will now pass to the limit in each term of this equation.

To handle the first one, we write $\langle (S_n(u-g_2)+g_2-\tilde{g}_2)_t, S'(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi\rangle = \langle (S(S_n(u-g_2)+g_2-\tilde{g}_2))_t,\varphi\rangle$, so that, by definition of the derivative in $\mathcal{D}'(Q)$, this term passes to the limit thanks to the dominated convergence theorem:

$$\int_0^T \langle (S(S_n(u-g_2)+g_2-\tilde{g}_2))_t,\varphi\rangle = -\int_0^T S(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi_t \longrightarrow -\int_0^T S(u-\tilde{g}_2)\varphi_t.$$

To handle the other terms, we take M such that $\mathrm{Supp}(S')\subset[-M,M]$. Since $S_n(x)-1\le x\le S_n(x)+1$ for all $x\in[-n-1,n+1]$, one has

$$\mathrm{Supp}\left(S_n'(u-g_2)\,S'(S_n(u-g_2)+g_2-\tilde{g}_2)\right)\subset\{|u-g_2|\le n+1, |u-\tilde{g}_2|\le M+1\};$$

thus, in each of the integrals on Q of (10.61), ∇u can be replaced by $V=\nabla(T_{M+1}(u-\tilde{g}_2)+\tilde{g}_2)\in(L^p(Q))^N$; we can then pass to the limit with the help of the dominated convergence theorem. Since $u-\tilde{g}_2=T_{M+1}(u-\tilde{g}_2)$ whenever $S'(u-\tilde{g}_2)\neq 0$ or $S''(u-\tilde{g}_2)\neq 0$, we can then replace V by ∇u in each limit term.
Indeed, since $S_n'\to 1$ and is bounded by 1, we have

$$\begin{aligned}&\int_Q S_n'(u-g_2)\,a(t,x,\nabla u)\nabla\varphi\, S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt\\ =\ &\int_Q S_n'(u-g_2)\,a(t,x,V)\nabla\varphi\, S'(S_n(u-g_2)+g_2-\tilde{g}_2)\,dxdt\\ \longrightarrow\ &\int_Q a(t,x,V)\nabla\varphi\, S'(u-\tilde{g}_2)\,dxdt=\int_Q a(t,x,\nabla u)\nabla\varphi\, S'(u-\tilde{g}_2)\,dxdt.\end{aligned}$$

For the third term of (10.61), we write $\nabla(S'(S_n(u-g_2)+g_2-\tilde{g}_2))=S''(S_n(u-g_2)+g_2-\tilde{g}_2)(S_n'(u-g_2)\nabla(u-g_2)+\nabla(g_2-\tilde{g}_2))=S''(S_n(u-g_2)+g_2-\tilde{g}_2)(S_n'(u-g_2)(V-\nabla g_2)+\nabla(g_2-\tilde{g}_2))$ with V, ∇g_2, $\nabla\tilde{g}_2\in(L^p(Q))^N$ so that this term tends to

$$\int_Q S''(u-\tilde{g}_2)a(t,x,V)(V-\nabla\tilde{g}_2)\varphi\,dxdt=\int_Q S''(u-\tilde{g}_2)a(t,x,\nabla u)\nabla(u-\tilde{g}_2)\varphi\,dxdt.$$

The fourth term tends to 0, because $S_n''\to 0$ and, in this term, $a(t,x,\nabla u)\nabla(u-g_2)=a(t,x,V)(V-\nabla g_2)\in L^1(Q)$. A straight application of the dominated convergence theorem show that the fifth term tends to $\int_Q \tilde{f}S'(u-\tilde{g}_2)\varphi$ and that the sixth term tends to $\int_Q \tilde{G}_1\nabla\varphi S'(u-\tilde{g}_2)$.
To study the convergence of the seventh term, we write, as above, $\nabla(S'(S_n(u-g_2)+g_2-\tilde{g}_2)=S''(S_n(u-g_2)+g_2-\tilde{g}_2)(S_n'(u-g_2)(V-\nabla g_2)+\nabla(g_2-\tilde{g}_2))$ so that, again thanks to the dominated convergence theorem, the limit of this term is

$$\int_Q S''(u-\tilde{g}_2)\tilde{G}_1(V-\nabla\tilde{g}_2)\varphi\,dxdt=\int_Q S''(u-\tilde{g}_2)\tilde{G}_1\nabla(u-\tilde{g}_2)\varphi\,dxdt.$$

Since $\nabla(u-g_2)=V-\nabla g_2\in(L^p(Q))^N$ in the last term of (10.61), we see that this term tends to 0 as $n\to\infty$. Gathering all the preceding convergences, we see that u satisfies (10.56) with $\tilde{g}_2$ instead of g_2.
To get back the initial condition with $\tilde{g}_2$ instead of g_2, we take $\varphi=(T-t)\psi$ with $\psi\in\mathcal{C}_c^\infty(\Omega)$, and we use, as before, (10.56) with $S=S_n$ and the test function $S'(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$; this gives (10.61). Now, however, since $\varphi(0)\neq 0$, the integration by parts in time in the first term of (10.61) gives

$$\int_0^T\langle (S(S_n(u-g_2)+g_2-\tilde{g}_2))_t,\varphi\rangle=-\int_\Omega S(S_n(u-g_2)(0)+(g_2-\tilde{g}_2)(0))\varphi(0)-\int_Q S(S_n(u-g_2)+g_2-\tilde{g}_2)\varphi_t.$$

Since $S_n(u-g_2)(0)=S_n(u_0)$ and $(g_2-\tilde{g}_2)(0)=0$, we have $S(S_n(u-g_2)(0)+(g_2-\tilde{g}_2)(0))=S(S_n(u_0))$ so that the first term of (10.61) tends now to

$$-\int_\Omega S(u_0)\varphi(0)-\int_Q S(u-\tilde{g}_2)\varphi_t.$$

The other terms tend to the same limits as before and we get thus

$$\begin{aligned} -\int_\Omega S(u_0)\varphi(0) - \int_Q S(u-\tilde g_2)\varphi_t + \int_Q a(t,x,\nabla u)\nabla\varphi\, S'(u-\tilde g_2)\,dxdt \\ + \int_Q S''(u-\tilde g_2)a(t,x,\nabla u)\nabla(u-\tilde g_2)\varphi\,dxdt \\ = \int_Q \tilde f S'(u-\tilde g_2)\varphi + \int_Q \tilde G_1\nabla\varphi S'(u-\tilde g_2) + \int_Q S''(u-\tilde g_2)\tilde G_1\nabla(u-\tilde g_2)\varphi\,dxdt. \end{aligned} \tag{10.62}$$

On the other hand, since $S(u-\tilde g_2)\in L^p(0,T;W_0^{1,p}(\Omega))$ satisfies (10.56) (with $\tilde g_2$ instead of g_2), we have $(S(u-\tilde g_2))_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, so that $S(u-\tilde g_2)\in\mathcal{C}([0,T];L^1(\Omega))$ and we can use φ as a test function in (10.56) written with $\tilde g_2$; this gives

$$\begin{aligned} -\int_\Omega S(u-\tilde g_2)(0)\varphi(0) - \int_Q S(u-\tilde g_2)\varphi_t + \int_Q a(t,x,\nabla u)\nabla\varphi\, S'(u-\tilde g_2)\,dxdt \\ + \int_Q S''(u-\tilde g_2)a(t,x,\nabla u)\nabla(u-\tilde g_2)\varphi\,dxdt \\ = \int_Q \tilde f S'(u-\tilde g_2)\varphi + \int_Q \tilde G_1\nabla\varphi S'(u-\tilde g_2) + \int_Q S''(u-\tilde g_2)\tilde G_1\nabla(u-\tilde g_2)\varphi\,dxdt. \end{aligned} \tag{10.63}$$

From (10.62) and (10.63) we deduce that $\int_\Omega S(u-\tilde g_2)(0)\psi = \int_\Omega S(u_0)\psi$ for all $\psi\in\mathcal{C}_c^\infty(\Omega)$, that is to say $S(u-\tilde g_2)(0) = S(u_0)$. ∎

Remark 10.13 *It should be noted that the definition of renormalized solution is not restricted to the case that μ is a measure, since (10.54)–(10.57) make sense whenever $f\in L^1(Q)$, $g_1\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2\in L^p(0,T;V)$. Thus the definition of renormalized solution makes sense also if $\mu\in L^1(Q)+W'$, without being necessarily a measure. In this case (f,g_1,g_2) is a decomposition of μ in $L^1(Q)+W'$. Note also that the conclusion of Lemma 10.6 is still true if $\mu\in L^1(Q)+W'$, hence the result of Proposition 10.6 would remain true in this case too.*

10.3.3 Proof of existence and uniqueness theorems

We can now start the proof of the existence result for problem (10.1). Following a standard approach, we obtain the existence of a solution as limit of nonsingular approximating problems. To this purpose, let μ_n be an approximation of μ given by Proposition 10.5, and let $\{u_{0n}\}\in L^\infty(\Omega)$ converge to u_0 strongly in $L^1(\Omega)$. Then by classical results (see for instance [51]) there exists a unique solution u_n in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ of the Cauchy–Dirichlet problem:

$$\begin{cases} (u_n)_t - \operatorname{div}(a(t,x,\nabla u_n)) = \mu_n & \text{in }]0,T[\times\Omega, \\ u_n = 0 & \text{on }]0,T[\times\partial\Omega, \\ u_n(0) = u_{0n} & \text{in } \Omega. \end{cases} \tag{10.64}$$

Moreover, from Proposition 10.5, u_n satisfies:

$$\begin{aligned} \int_0^t \langle (u_n - g_2^n)_t, \varphi\rangle\,ds + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla\varphi\,dxds = \int_0^t\int_\Omega f_n\varphi\,dxds \\ + \int_0^t \langle g_1^n,\varphi\rangle\,ds\,, \qquad \forall\varphi\in L^p(0,T;V)\,,\ \forall t\in[0,T]\,. \end{aligned} \tag{10.65}$$

Let us begin by getting *a priori* estimates on u_n.

Proposition 10.7 *Let u_n be the solution of (10.64). Then we have:*

$$\begin{aligned}
&\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C\,,\\
&\int_Q |\nabla T_k(u_n)|^p\,dxdt \le C\,k\;,\\
&\|u_n - g_2^n\|_{L^\infty(0,T;L^1(\Omega))} \le C\,,\\
&\int_Q |\nabla T_k(u_n - g_2^n)|^p\,dxdt \le C\,(k+1)\,,\\
&\lim_{h\to\infty}\left(\sup_n \int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt\right) = 0\,,\quad \forall k>0\,.
\end{aligned}\tag{10.66}$$

Moreover there exists a measurable function $u : Q \to \mathbf{R}$ such that $T_k(u)$ and $T_k(u-g_2)$ belong to $L^p(0,T;W_0^{1,p}(\Omega))$, u and $u-g_2$ belong to $L^\infty(0,T;L^1(\Omega))$ and, up to a subsequence, for any $k>0$:

$$\begin{aligned}
&u_n \to u \qquad \text{a.e. in } Q,\\
&T_k(u_n - g_2^n) \to T_k(u-g_2) \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q.
\end{aligned}\tag{10.67}$$

Finally, we have

$$\lim_{h\to\infty}\int_{\{h\le|u-g_2|\le h+k\}} |\nabla u|^p\,dxdt = 0\,,\quad \forall k>0\,.\tag{10.68}$$

Proof. First of all, we choose $T_k(u_n)$ as test function in (10.64) and we integrate in $]0,t[$ to get:

$$\int_\Omega \Theta_k(u_n)(t)\,dx + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla T_k(u_n)\,dxds = \int_0^t \mu_n\,T_k(u_n)\,dxds + \int_\Omega \Theta_k(u_{0n})\,dx\,,$$

which yields, from (10.48) and the fact that $\|u_{0n}\|_{L^1(\Omega)}$ and $\|\mu_n\|_{L^1(Q)}$ are bounded:

$$\int_\Omega \Theta_k(u_n)(t)\,dx + \int_0^t\int_\Omega |\nabla T_k(u_n)|^p\,dxds \le Ck\,.$$

Since $\Theta_k(s)\ge 0$, for $t=T$ we get that $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. If $k=1$, we also get:

$$\int_\Omega \Theta_1(u_n)(t) \le C \qquad \forall t\in[0,T]\,,$$

which implies, since $\Theta_1(s)\ge |s|-1$, that

$$\int_\Omega |u_n(t)|\,dx \le C \qquad \forall t\in[0,T]\,.$$

Taking the supremum on $]0,T[$ we obtain the estimate of u_n in $L^\infty(0,T;L^1(\Omega))$. Similarly we can get the estimates on $u_n - g_2^n$: let us choose $T_k(u_n - g_2^n)$ as test function in (10.65). Integrating by parts (recall that g_2^n has compact support, so that $u^n(0) - g_2^n(0) = u^n(0) = u_{0n}$) and using (10.48) this gives:

$$\begin{aligned}
&\int_\Omega \Theta_k(u_n - g_2^n)(t)\,dx + \alpha\int_0^t\int_\Omega |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxds\\
&\le \int_\Omega \Theta_k(u_{0n})\,dx + \int_Q f_n\,T_k(u_n-g_2^n)\,dxdt + \int_0^t\int_\Omega G_1^n\nabla u_n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds\\
&\quad - \int_0^t\int_\Omega G_1^n\nabla g_2^n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds + \int_0^t\int_\Omega a(s,x,\nabla u_n)\nabla g_2^n\,\chi_{\{|u_n-g_2^n|\le k\}}dxds\,.
\end{aligned}$$

Using assumption (10.49) and by means of Young's inequality we obtain:

$$\int_\Omega \Theta_k(u_n - g_2^n)(t)\,dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxdt \le k\int_Q |f_n|\,dxdt$$
$$+ C\int_Q |G_1^n|^{p'}\,dxdt + C\int_Q |\nabla g_2^n|^p\,dxdt + C\int_Q |b(t,x)|^{p'}dxdt + k\int_\Omega |u_{0n}|\,dx\,.$$

Since G_1^n is bounded in $L^{p'}(Q)$, g_2^n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f_n is bounded in $L^1(Q)$ and u_{0n} is bounded in $L^1(\Omega)$, we obtain

$$\int_\Omega \Theta_1(u_n - g_2^n)(t)\,dx \le C \qquad \forall t \in]0,T[\,,$$

which implies the estimate of $u_n - g_2^n$ in $L^\infty(0,T;L^1(\Omega))$, and also

$$\int_Q |\nabla u_n|^p\,\chi_{\{|u_n-g_2^n|\le k\}}dxdt \le C\,(k+1)\,,$$

which yields that $T_k(u_n - g_2^n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k > 0$ (recall that g_2^n itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). Now, let $\psi(s) = T_k(s - T_h(s))$ and take $\psi(u_n - g_2^n)$ as test function in (10.65). Reasoning as above, using that $\psi'(s) = \chi_{\{h\le|s|\le h+k\}}$ and applying Young's inequality we obtain:

$$\int\limits_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt \le Ck\int\limits_{\{|u_{0n}|>h\}} |u_{0n}|\,dx + \; Ck\int\limits_{\{|u_n-g_2^n|>h\}} |f_n|\,dxdt + \; C\int\limits_{\{|u_n-g_2^n|>h\}} (|G_1^n|^{p'} + |\nabla g_2^n|^p + |b(x,t)|^{p'})\,dxdt\,.$$

Since $u_n - g_2^n$ is bounded in $L^\infty(0,T;L^1(\Omega))$ we have

$$\lim_{h\to\infty}\left(\sup_n \operatorname{meas}\{|u_n - g_2^n| > h\}\right) = 0\,,$$

then by means of the equi–integrability of the sequences f_n, $|G_1^n|^{p'}$ and $|\nabla g_2^n|^p$ in $L^1(Q)$ we deduce that:

$$\lim_{h\to\infty}\left(\sup_n \int\limits_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^p\,dxdt\right) = 0\,, \tag{10.69}$$

for every $k > 0$.

We are going to prove now that, up to subsequences, u_n converges almost everywhere in Q towards a measurable function u. To this aim, let $\mathcal{T}_k(s)$ be a $C^2(\mathbf{R})$, nondecreasing function such that $\mathcal{T}_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\mathcal{T}_k(s) = \operatorname{sgn}(s)k$ for $|s| > k$. If we multiply pointwise equation (10.64) by $\mathcal{T}_k'(u_n - g_2^n)$ (equivalently if we choose $\mathcal{T}_k'(u_n - g_2^n)\psi$ as test function in (10.65) with $\psi \in \mathcal{C}_c^\infty(Q)$) we get that:

$$\begin{aligned}&(\mathcal{T}_k(u_n - g_2^n))_t - \operatorname{div}(a(t,x,\nabla u_n)\mathcal{T}_k'(u_n - g_2^n))\\ &\quad + a(t,x,\nabla u_n)\nabla(u_n - g_2^n)\mathcal{T}_k''(u_n - g_2^n)\\ &= \mathcal{T}_k'(u_n - g_2^n)f_n - \operatorname{div}(G_1^n\mathcal{T}_k'(u_n - g_2^n)) + \mathcal{T}_k''(u_n - g_2^n)G_1^n\nabla(u_n - g_2^n)\,.\end{aligned} \tag{10.70}$$

Observe that thanks to the fact that $\mathcal{T}_k'$ has compact support and since $|\nabla u_n|^p\chi_{\{|u_n-g_2^n|\le k\}}$ is bounded in $L^1(Q)$ we deduce from (10.49) that $a(t,x,\nabla u_n)\nabla(u_n - g_2^n)\mathcal{T}_k''(u_n - g_2^n)$ is bounded in $L^1(Q)$ and so is $\mathcal{T}_k''(u_n-g_2^n)G_1^n\nabla(u_n-g_2^n)$ (since G_1^n is bounded in $(L^{p'}(Q))^N$). Similarly, we have that $a(t,x,\nabla u_n)\mathcal{T}_k'(u_n-g_2^n)$ as well as $G_1^n\mathcal{T}_k'(u_n-g_2^n)$ is bounded in $(L^{p'}(Q))^N$, so that we conclude from (10.70) that $(\mathcal{T}_k(u_n-g_2^n))_t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. Since we have just proven that $\mathcal{T}_k(u_n - g_2^n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ a classical compactness result (see [69]) allows us to deduce that $\mathcal{T}_k(u_n-g_2^n)$ is compact in $L^1(Q)$. Thus, for a subsequence, it also converges in measure. Let then $\sigma > 0$ and, given $\varepsilon > 0$, let us fix h such that, for every n, meas $\{|u_n - g_2^n| > \frac{h}{2}\} \le \varepsilon$ (this is possible as a consequence of the estimate of $u_n - g_2^n$ in $L^\infty(0,T;L^1(\Omega))$). Since $\mathcal{T}_h(u_n - g_2^n)$ converges in measure, for n and m sufficiently large we have:

$$\text{meas}\{|\mathcal{T}_h(u_n - g_2^n) - \mathcal{T}_h(u_m - g_2^m)| > \sigma\} \le \varepsilon\,.$$

On the other hand we have, by definition of $\mathcal{T}_k$:

$$\begin{aligned}\text{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\} &\le \text{meas}\{|u_n - g_2^n| > \frac{h}{2}\}\\ + \text{meas}\{|u_m - g_2^m| > \frac{h}{2}\} &+ \text{meas}\{|\mathcal{T}_h(u_n - g_2^n) - \mathcal{T}_h(u_m - g_2^m)| > \sigma\}\,,\end{aligned}$$

hence the choice of h implies, for n and m sufficiently large,

$$\text{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\} \le 3\varepsilon\,,$$

so that $u_n - g_2^n$ is a Cauchy sequence in measure. Up to subsequences, we deduce that $u_n - g_2^n$ almost everywhere converges in Q, and since g_2^n strongly converges to g_2 in $L^p(0,T;W_0^{1,p}(\Omega))$, there exists a measurable function u such that u_n almost everywhere converges to u and $T_k(u_n - g_2^n)$ weakly converges to $T_k(u - g_2)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. The estimates previously obtained on u_n also imply that $u \in L^\infty(0,T;L^1(\Omega))$ (indeed, use Fatou's lemma on the estimate of (u_n) in $L^\infty(0,T;L^1(\Omega))$) and that $T_k(u_n)$ weakly converges to $T_k(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$.
Let us prove (10.68). Let $\psi(s) = T_k(s - T_h(s))$; one has

$$\int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt = \int\limits_{\{h\le|u_n - g_2^n|\le h+k\}} |\nabla(u_n - g_2^n)|^p\,dxdt \le \int_Q |\nabla T_{h+k}(u_n - g_2^n)|^p\,dxdt \le C,$$

hence $\psi(u_n - g_2^n)$ converges (up to subsequences) weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and almost everywhere in Q to $\psi(u - g_2)$. Thus

$$\int\limits_{\{h\le|u-g_2|\le h+k\}} |\nabla(u - g_2)|^p\,dxdt = \int_Q |\nabla\psi(u - g_2)|^p\,dxdt \le \liminf_{n\to\infty} \int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt$$

Moreover

$$\int_Q |\nabla\psi(u_n - g_2^n)|^p\,dxdt \le C \int\limits_{\{h\le|u_n-g_2^n|\le h+k\}} (|\nabla u_n|^p + |\nabla g_2^n|^p)\,dxdt$$

Hence, using (10.69), one gets

$$\lim_{h\to\infty} \int\limits_{\{h\le|u-g_2|\le h+k\}} |\nabla(u - g_2)|^p\,dxdt = 0$$

as h tends to ∞, and (10.68) follows. ■

Remark 10.14 *In the proof of Proposition 10.7 we did not use the fact that the approximating sequence μ_n converging to μ is bounded in $L^1(Q)$ but for the first two estimates on u_n. The estimates concerning $u_n - g_2^n$ in (10.66) as well as (10.67) and (10.68) only needed the "separate" approximations of f, g_1, g_2 in the respective functional spaces. In particular, they hold true if μ belongs to $L^1(Q) + W'$, being (f, g_1, g_2) a decomposition of μ.*

Next we prove the strong convergence of $T_k(u_n - g_2^n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. To obtain this result, we use the same technique as in [63] adapted to the sequence $u_n - g_2^n$.
We need then to recall the following definition of a time–regularization of $T_k(u)$, which was first introduced in [50], then used in several papers afterwards (see particularly [23], [9]). Let z_ν be a sequence of functions such that:

$$
\begin{aligned}
& z_\nu \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,, \qquad \|z_\nu\|_{L^\infty(\Omega)} \le k\,,\\
& z_\nu \to T_k(u_0) \quad \text{a.e. in } \Omega \text{ as } \nu \text{ tends to infinity,}\\
& \frac{1}{\nu}\|z_\nu\|^p_{W_0^{1,p}(\Omega)} \to 0 \qquad \text{as } \nu \text{ tends to infinity.}
\end{aligned}
$$

Then, for fixed $k > 0$, and $\nu > 0$, we denote by $T_k(u)_\nu$ the unique solution of the problem

$$
\begin{cases}
\frac{\partial T_k(u)_\nu}{\partial t} = \nu(T_k(u) - T_k(u)_\nu) & \text{in the sense of distributions,}\\
T_k(u)_\nu(0) = z_\nu & \text{in } \Omega.
\end{cases}
\tag{10.71}
$$

Then $T_k(u)_\nu$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and $\frac{\partial T_k(u)_\nu}{\partial t}$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, and it can be proved (see also [50]) that, up to a subsequence,

$$
\begin{aligned}
& T_k(u)_\nu \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q,\\
& \|T_k(u)_\nu\|_{L^\infty(Q)} \le k \qquad \forall \nu > 0\,.
\end{aligned}
\tag{10.72}
$$

Proposition 10.8 *Let u_n be the solution of (10.64), where μ_n is given by Proposition 10.5. Then there exists a measurable function u in Q and a subsequence, not relabeled, such that:*

$$
T_k(u_n - g_2^n) \to T_k(u - g_2) \qquad \textit{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \textit{ for any } k > 0.
$$

Proof.
We take a subsequence such that $u_n \to u$ almost everywhere in Q. Let us denote, throughout what follows, $v_n = u_n - g_2^n$, and $v = u - g_2$. By Proposition 10.7 we know that $v \in L^\infty(0,T;L^1(\Omega))$ (in particular it is almost everywhere finite), $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$ and

$$
T_k(v_n) \to T_k(v) \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q \text{ for any } k > 0. \tag{10.73}
$$

We take a subsequence of $T_k(v)_\nu$, the approximation of $T_k(v)$ defined in (10.71), such that $T_k(v)_\nu \to T_k(v)$ almost everywhere in Q (this subsequence only depends on v and k, i.e. quantities that will not vary in the following proof). For $h > 2k$, we then introduce the function

$$
w_n = T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\,.
$$

The use of w_n as test function to prove the strong convergence of truncations was first introduced in the stationary case in [52], then adapted to parabolic equations in [63]. The advantage in working with w_n is that, since

$$
\nabla w_n = \nabla(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\chi_{E_n}\,,
$$

with $E_n = \{|v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu| \le 2k\}$, in particular we have $\nabla w_n = 0$ if $|v_n| > h + 4k$. Thus the estimate on $T_k(v_n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ appearing in Proposition 10.7 implies that w_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, then by the almost everywhere convergence of v_n to v we deduce:

$$
w_n \to T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \,\text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q. \tag{10.74}
$$

In the following we set $M = h+4k$, moreover we will denote by $\omega(n,\nu,h)$ all quantities (possibly different) such that

$$
\lim_{h\to+\infty}\ \lim_{\nu\to+\infty}\ \limsup_{n\to+\infty} |\omega(n,\nu,h)| = 0\,, \tag{10.75}
$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then ν, and finally h. Similarly we will write only $\omega(n)$, or $\omega(n,\nu)$, to mean that the limits are made only on the specified parameters. Choosing w_n as test function in (10.65) we have:

$$\int_0^T \langle (v_n)_t, w_n\rangle\, dt + \int_Q a(t,x,\nabla u_n)\nabla w_n\, dxdt = \int_Q f_n\, w_n\, dxdt + \int_0^T \langle g_1^n, w_n\rangle\, dt\,. \tag{10.76}$$

Then from (10.74) we obtain:

$$\lim_{n\to\infty}\int_Q f_n\, w_n\, dxdt = \int_Q f\, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\, dxdt\,,$$
$$\lim_{n\to\infty}\int_0^T \langle g_1^n, w_n\rangle\, dt = \int_0^T \langle g_1, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\rangle\, dt\,.$$

Moreover we have that $T_k(v)_\nu$ converges to $T_k(v)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ and almost everywhere in Q as ν tends to infinity, so that

$$\lim_{\nu\to\infty}\int_Q f\, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\, dxdt = \int_Q f\, T_{2k}(v - T_h(v))\, dxdt\,,$$
$$\lim_{\nu\to\infty}\int_0^T \langle g_1, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\rangle\, dt = \int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle\, dt\,.$$

By means of Lebesgue's theorem we can conclude

$$\lim_{h\to\infty}\int_Q f\, T_{2k}(v - T_h(v))\, dxdt = 0\,.$$

Moreover, since

$$\int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle\, dt = \int_Q G_1 \nabla v\, \chi_{\{h\le |v|\le h+2k\}}\, dxdt\,,$$

Hölder's inequality implies

$$\left|\int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle\, dt\right| \le \|G_1\|_{(L^{p'}(Q))^N} \left(\int_{\{h\le |u-g_2|\le h+2k\}} |\nabla(u-g_2)|^p\, dxdt\right)^{\frac{1}{p}}\,.$$

Then thanks to (10.68) we obtain:

$$\lim_{h\to\infty}\int_0^T \langle g_1, T_{2k}(v - T_h(v))\rangle\, dt = 0\,.$$

Thus, recalling the notation introduced in (10.75), we have proven that

$$\int_Q f_n\, w_n\, dxdt + \int_0^T \langle g_1^n, w_n\rangle\, dt = \omega(n,\nu,h)\,. \tag{10.77}$$

Let us estimate the second term in (10.76). Since $\nabla w_n = 0$ if $|v_n| > M = h + 4k$ we have:

$$\int_Q a(t,x,\nabla u_n)\nabla w_n\, dxdt = \int_Q a(t,x,\nabla u_n \chi_{\{|v_n|\le M\}})\nabla w_n\,.$$

Next we split the integral in the sets $\{|v_n| \le k\}$ and $\{|v_n| > k\}$ so that we have, recalling that $E_n = \{|v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu| \le 2k\}$ and $h \ge 2k$:

$$\begin{aligned}\int_Q a(t,x,\nabla u_n)\nabla w_n\,dxdt &= \int_Q a(t,x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v)_\nu)\,dxdt \\ &+ \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt \\ &- \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)_\nu\,\chi_{E_n}\,dxdt\,.\end{aligned} \tag{10.78}$$

Let us denote (A), (B) and (C) the three terms of the right hand side in (10.78). Let us estimate (B). Clearly we have

$$\begin{aligned}&\left|\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt\right| \\ &\qquad \le \int_{\{h\le|v_n|\le h+4k\}} |a(t,x,\nabla u_n)|\,|\nabla(u_n - g_2^n)|\,dxdt\,,\end{aligned}$$

and using (10.49) and Young's inequality we get:

$$\begin{aligned}&\int_{\{h\le|v_n|\le h+4k\}} |a(t,x,\nabla u_n)|\,|\nabla(u_n - g_2^n)|\,dxdt \\ &\le C\int_{\{h\le|v_n|\le h+4k\}} |\nabla u_n|^p\,dxdt + C\int_{\{h\le|v_n|\le h+4k\}} |\nabla g_2^n|^p\,dxdt + C\int_{\{h\le|v_n|\le h+4k\}} |b(x,t)|^{p'}\,dxdt\,.\end{aligned}$$

Thanks to the equi–integrability of $|\nabla g_2^n|^p$, using (10.66) and that $\text{meas}\{h \le |v_n| \le h+k\}$ converges to zero as h tends to infinity uniformly with respect to n we obtain:

$$\lim_{h\to\infty}\limsup_{n\to\infty}\left|\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,dxdt\right| = 0\,,$$

that is $(B) = \omega(n,h)$. For (C), let us remark that, since $\nabla u_n\chi_{\{|v_n|\le M\}}$ is bounded in $L^p(Q)$, (10.49) implies that $|a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})|$ is bounded in $L^{p'}(Q)$. The almost everywhere convergence of v_n to v implies that $|\nabla T_k(v)|\chi_{\{|v_n|>k\}}$ strongly converges to zero in $L^p(Q)$, so that we have

$$\lim_{n\to\infty}\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)\,\chi_{E_n}\,dxdt = 0\,.$$

Thus we get

$$\begin{aligned}&\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)_\nu\,\chi_{E_n}\,dxdt \\ &\qquad = \omega(n) + \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(T_k(v)_\nu - T_k(v))\,\chi_{E_n}\,dxdt\,.\end{aligned}$$

Using that $|a(t,x,\nabla u_n\chi_{\{|v_n|\leq M\}})|$ is bounded in $L^{p'}(Q)$, applying Hölder's inequality and thanks to (10.72) we also have

$$\int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\leq M\}})\nabla(T_k(v)_\nu - T_k(v))\,\chi_{E_n}\,dxdt = \omega(n,\nu)\,,$$

therefore we conclude:

$$(C) = \int_{\{|v_n|>k\}} a(t,x,\nabla u_n\chi_{\{|v_n|\leq M\}})\nabla T_k(v)_\nu\,\chi_{E_n}\,dxdt = \omega(n,\nu)\,.$$

We have then obtained from (10.78), using that (B) and (C) converge to 0:

$$\int_Q a(t,x,\nabla u_n)\nabla w_n\,dxdt = \int_Q a(t,x,\nabla u_n\chi_{\{|v_n|\leq k\}})\nabla(v_n - T_k(v)_\nu)\,dxdt + \omega(n,\nu,h)\,. \tag{10.79}$$

Putting together (10.77), (10.79) and (10.76) we have:

$$\int_0^T \langle (v_n)_t, w_n\rangle\,dt + \int_Q a(t,x,\nabla u_n\chi_{\{|v_n|\leq k\}})\nabla(v_n - T_k(v)_\nu)\,dxdt = \omega(n,\nu,h)\,.$$

As far as the first term is concerned, that is

$$\int_0^T \langle (v_n)_t, T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\rangle\,dt\,,$$

we can apply Lemma 2.1 in [63] to the function v_n, using the fact that u_{0n} and z_ν strongly converge to u_0 and to $T_k(u_0)$ respectively in $L^1(\Omega)$. This lemma, based on the monotonicity properties of the time–regularization $T_k(v)_\nu$, gives that

$$\int_0^T \langle (v_n)_t, w_n\rangle\,dt \geq \omega(n,\nu,h)\,,$$

hence we finally have:

$$\int_Q a(t,x,\nabla u_n\chi_{\{|v_n|\leq k\}})\nabla(v_n - T_k(v)_\nu)\,dxdt \leq \omega(n,\nu,h)\,. \tag{10.80}$$

Without loss of generality, we can assume that k is such that $\chi_{\{|v_n|\leq k\}}$ almost everywhere converges to $\chi_{\{|v|\leq k\}}$ (in fact this is true for almost every k, see also Lemma 3.2 in [9]). Then, the strong convergence of g_2^n in $L^p(0,T;W_0^{1,p}(\Omega))$ and (10.49) imply that $a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\leq k\}})$ strongly converges to $a(t,x,\nabla(g_2 + T_k(v))\chi_{\{|v|\leq k\}})$ in $L^{p'}(Q)^N$. Since

$$\begin{aligned}\int_Q & a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\leq k\}})\nabla(v_n - T_k(v)) \\ &= \int_Q a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\leq k\}})\nabla(T_k(v_n) - T_k(v))\,dxdt\,,\end{aligned}$$

the weak convergence of $T_k(v_n)$ to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ allows to conclude that:

$$\lim_{n\to\infty}\int_Q a(t,x,\nabla(g_2^n + T_k(v))\chi_{\{|v_n|\leq k\}})\nabla(v_n - T_k(v))\,dxdt = 0\,,$$

hence we obtain from (10.80), using also the strong convergence of $T_k(v)_\nu$ to $T_k(v)$ as ν tends to infinity:

$$\lim_{n\to\infty}\int_Q \big[a(t,x,\nabla u_n\chi_{\{|v_n|\le k\}}) \\ -a(t,x,\nabla(g_2^n+T_k(v))\chi_{\{|v_n|\le k\}})\big]\,(\nabla u_n-\nabla(g_2^n+T_k(v)))\,dxdt=0\,. \tag{10.81}$$

Using that $\chi_{\{|v_n|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ and that g_2^n strongly converges to g_2 in $L^p(0,T;W_0^{1,p}(\Omega))$, through the standard monotonicity argument which relies on (10.50) (see Lemma 5 in [13]) we can deduce from (10.81) that

$$\nabla u_n\chi_{\{|v_n|\le k\}}\to\nabla(g_2+T_k(v))\chi_{\{|v|\le k\}}=\nabla u\chi_{\{|v|\le k\}}\qquad\text{a.e. in } Q,$$

and then that $a(t,x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla u_n$ strongly converges to $a(t,x,\nabla u\chi_{\{|v|\le k\}})\nabla u$ in $L^1(Q)$. Finally, together with (10.48) this proves that the sequence $|\nabla u_n|^p\chi_{\{|u_n-g_2^n|\le k\}}$ is equi–integrable in Q, which as a consequence of Vitali's theorem and since g_2^n strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ yields

$$T_k(u_n-g_2^n)\to T_k(u-g_2)\qquad\text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

In fact, since we have proved it for almost every k the result holds true for any k as well. ■

The proof of the existence of a renormalized solution will easily follow from the previous estimates and compactness results.

Theorem 10.11 *Assume that (10.48), (10.49), (10.50) hold true, and let $\mu\in\mathcal{M}_0(Q)$, $u_0\in L^1(\Omega)$. Then there exists a renormalized solution u of problem (10.1) in the sense of Definition 10.7. Moreover u belongs to $L^\infty(0,T;L^1(\Omega))$ and $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$.*

Remark 10.15 *We already remarked that the definition of renormalized solution does not make use of the fact that μ is a measure (only its decomposition in $L^1(Q)+W'$ is needed), in particular all the regularity asked on renormalized solutions concerns the difference $u-g_2$. However, due to the fact that μ is a measure (and can be approximated by sequences bounded in $L^1(Q)$) we have found a solution u with the additional regularity properties $u\in L^\infty(0,T;L^1(\Omega))$ and $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Last one in particular says that $|\nabla u|^p\chi_{\{|u|\le k\}}\in L^1(Q)$, which is not at all contained in the request $|\nabla u|^p\chi_{\{|u-g_2|\le k\}}\in L^1(Q)$ for renormalized solutions. Actually, this regularity result is consistent with the first existence result found in [10].*

Proof. Let u_n be the sequence of solutions of (10.64), where μ_n and u_{0n} approximate μ and u_0 respectively in the sense specified above, and let $u\in L^\infty(0,T;L^1(\Omega))$ be such that the results of Proposition 10.7 and Proposition 10.8 hold true. Then we have that

$$\begin{aligned} &u_n\to u \qquad \text{a.e. in } Q,\\ &T_k(u_n-g_2^n)\to T_k(u-g_2)\qquad\text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ for any } k>0 \text{ and a.e. in } Q. \end{aligned} \tag{10.82}$$

Let $S\in W^{2,\infty}(\mathbf{R})$ be such that S' has compact support, and take $S'(u_n-g_2^n)\varphi$ as test function in (10.65), with $\varphi\in\mathcal{C}_c^\infty(Q)$. Then we have:

$$\begin{aligned} &-\int_Q\varphi_t\,S(u_n-g_2^n)\,dxdt+\int_Q a(t,x,\nabla u_n)\nabla\varphi\,S'(u_n-g_2^n)\,dxdt\\ &+\int_Q S''(u_n-g_2^n)a(t,x,\nabla u_n)\nabla(u_n-g_2^n)\,\varphi\,dxdt=\int_Q f_nS'(u_n-g_2^n)\varphi\,dxdt\\ &+\int_Q G_1^n\nabla\varphi\,S'(u_n-g_2^n)\,dxdt+\int_Q S''(u_n-g_2^n)G_1^n\,\nabla(u_n-g_2^n)\,\varphi\,dxdt\,. \end{aligned} \tag{10.83}$$

Since $\mathrm{Supp}(S')$ is compact there exists $M > 0$ such that $a(t,x,\nabla u_n)S'(u_n - g_2^n) = a(t,x,\nabla T_M(u_n - g_2^n) + \nabla g_2^n)S'(u_n - g_2^n)$, so that (10.82), the strong convergence of g_2^n in $L^p(0,T;W_0^{1,p}(\Omega))$ and assumption (10.49) imply that

$$a(t,x,\nabla u_n)S'(u_n - g_2^n) \to a(t,x,\nabla u)S'(u - g_2) \qquad \text{strongly in } (L^{p'}(Q))^N.$$

Similarly we have that

$$S''(u_n - g_2^n)a(t,x,\nabla u_n)\nabla(u_n - g_2^n) \to S''(u - g_2)a(t,x,\nabla u)\nabla(u - g_2) \qquad \text{strongly in } L^1(Q)$$

and

$$S''(u_n - g_2^n)\nabla(u_n - g_2^n) \to S''(u - g_2)\nabla(u - g_2) \qquad \text{strongly in } (L^p(Q))^N.$$

Therefore, by means of (10.82) and the dominated convergence theorem, we can pass to the limit in (10.83) as n tends to infinity obtaining:

$$\begin{aligned} &-\int_Q \varphi_t S(u-g_2)\,dxdt + \int_Q a(t,x,\nabla u)\nabla\varphi S'(u-g_2)\,dxdt \\ &\quad + \int_Q S''(u-g_2)a(t,x,\nabla u)\nabla(u-g_2)\,\varphi\,dxdt = \int_Q fS'(u-g_2)\varphi\,dxdt \\ &\qquad + \int_Q G_1\nabla\varphi\, S'(u-g_2)\,dxdt + \int_Q S''(u-g_2)G_1\,\nabla(u-g_2)\,\varphi\,dxdt\,. \end{aligned} \tag{10.84}$$

Thus u satisfies (10.56), while (10.55) is (10.68) with $k = 1$ and has been proved in Proposition 10.7. Finally, passing to the limit (thanks to (10.82)) in (10.83) written in distributional sense we have

$$(S(u_n - g_2^n))_t \quad \text{is strongly convergent in } L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q),$$

and since $S(u_n - g_2^n)$ strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ we deduce (see Theorem 1.1 in [63]) that

$$S(u_n - g_2^n) \to S(u - g_2) \qquad \text{strongly in } \mathcal{C}([0,T];L^1(\Omega)).$$

In particular, being $S(u_n - g_2^n)(0) = S(u_{0n})$ we get that $S(u - g_2)(0) = S(u_0)$ in $L^1(\Omega)$. This concludes the proof that u is a renormalized solution of (10.1).

■

Here we prove the uniqueness of the renormalized solution of (10.1)

Theorem 10.12 *Assume (10.48), (10.49), (10.50). Let $\mu \in \mathcal{M}_0(Q)$, then there exists a unique renormalized solution of (10.1).*

Proof. Let u_1, u_2 be two renormalized solutions of (10.1), let (f, g_1, g_2) be a decomposition of μ, so that u_1 and u_2 both satisfy (10.56). Note that the same decomposition of μ can be used for both equations of u_1 and u_2 thanks to Proposition 10.6. Let S_n be as defined in Definition 10.8, in particular we have that $S_n(u_1 - g_2)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ as well as $S_n(u_2 - g_2)$. We choose then $T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))$ as test function in both the equations solved by u_1 and u_2. In the following we write $v_1 = u_1 - g_2$

and $v_2 = u_2 - g_2$; subtracting the equations then we have:

$$
\begin{aligned}
&\int_0^T \langle (S_n(v_1) - S_n(v_2))_t, T_k(S_n(v_1) - S_n(v_2))\rangle \, dt \\
&\quad + \int_Q [S_n'(v_1)a(t,x,\nabla u_1) - S_n'(v_2)a(t,x,\nabla u_2)] \cdot \nabla T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&= \int_Q f\,(S_n'(v_1) - S_n'(v_2))\, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q G_1\,(S_n'(v_1) - S_n'(v_2))\, \nabla T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q [S_n''(v_1)G_1\nabla v_1 - S_n''(v_2)G_1\nabla v_2]\, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \\
&\quad + \int_Q [S_n''(v_2)a(t,x,\nabla u_2)\nabla v_2 - S_n''(v_1)a(t,x,\nabla u_1)\nabla v_1]\, T_k(S_n(v_1) - S_n(v_2)) \, dxdt \,.
\end{aligned}
\tag{10.85}
$$

Let us denote by (A)–(F) the six integrals above, we study the behaviour of each as n tends to infinity. To this purpose, let us recall that by definition of S_n we have that $S_n'(s)$ converges to 1 for every s in $\mathbf{R}$. This is enough to conclude by means of Lebesgue's theorem that

$$\lim_{n\to\infty} (C) = 0\,.$$

Let us study the limit of (E) now. We have $(E) = (E_1) + (E_2)$, where

$$(E_1) = \int_Q S_n''(v_1)G_1\nabla v_1\, T_k(S_n(v_1) - S_n(v_2))\, dxdt\,.$$

Since (E_2) has the same form of (E_1) with the roles of v_1 and v_2 interchanged, it is enough to deal with (E_1). Recalling that $S_n''(s) = -\text{sign}(s)\chi_{\{n\le|s|\le n+1\}}$, we have:

$$|(E_1)| \le k \int\limits_{\{n\le|v_1|\le n+1\}} |G_1|\,|\nabla v_1|\, dxdt\,,$$

so that, using Hölder's inequality we get:

$$|(E_1)| \le k\|G_1\|_{L^{p'}(Q)} \left(\int\limits_{\{n\le|u_1-g_2|\le n+1\}} |\nabla u_1 - \nabla g_2|^p\, dxdt \right)^{\frac{1}{p}}.$$

Thus by (10.55) written for u_1 we get that (E_1) converges to zero as n tends to infinity. The same is true for (E_2), hence we deduce:

$$\lim_{n\to\infty} (E) = 0\,.$$

The term (F) can be dealt with in the same way. First we write $(F) = (F_1) + (F_2)$, with

$$(F_1) = \int_Q S_n''(v_2)a(t,x,\nabla u_2)\nabla v_2\, T_k(S_n(v_1) - S_n(v_2))\, dxdt\,.$$

Clearly, by symmetry between (F_1) and (F_2) it is enough to prove that (F_1) tends to zero. To this goal, using again the properties of S_n'' and (10.49) we have:

$$|(F_1)| \le \beta\, k \int\limits_{\{n\le|v_2|\le n+1\}} |\nabla v_2|\,(|b(x,t)| + |\nabla u_2|^{p-1})\, dxdt\,,$$

which yields, by Young's inequality:

$$|(F_1)| \le C \left(\int\limits_{\{n\le|u_2-g_2|\le n+1\}} (|\nabla g_2|^p + |b(x,t)|^{p'})\, dxdt + \int\limits_{\{n\le|u_2-g_2|\le n+1\}} |\nabla u_2|^p\, dxdt \right).$$

Using that $u_2 - g_2$ is almost everywhere finite and thanks to (10.55) written for u_2 we conclude that (F_1) converges to zero, and (F_2) as well, so that

$$\lim_{n\to\infty} (F) = 0\,.$$

As regards (D) note that, since $S_n'(v_1) - S_n'(v_2) = 0$ in $\{|v_1| \le n\,, |v_2| \le n\} \cup \{|v_1| > n+1\,, |v_2| > n+1\}$ we can split the integral as follows:

$$\begin{aligned}
&\int\limits_{\{|S_n(v_1)-S_n(v_2)|\le k\}} G_1\left(S_n'(v_1) - S_n'(v_2)\right) \nabla(S_n(v_1) - S_n(v_2))\chi_{\{|v_1|\le n\}}\ \chi_{\{|v_2|>n\}}\, dxdt \\
&\quad + \int\limits_{\{|S_n(v_1)-S_n(v_2)|\le k\}} G_1\left(S_n'(v_1) - S_n'(v_2)\right) \nabla(S_n(v_1) - S_n(v_2))\chi_{\{n<|v_1|\le n+1\}}\, dxdt \\
&\quad + \int\limits_{\{|S_n(v_1)-S_n(v_2)|\le k\}} G_1\left(S_n'(v_1) - S_n'(v_2)\right) \nabla(S_n(v_1) - S_n(v_2))\chi_{\{|v_2|\le n+1\}}\ \chi_{\{|v_1|>n+1\}}\, dxdt\,.
\end{aligned} \tag{10.86}$$

We call (D_1)–(D_3) the three integrals in (10.86). Using the properties of S_n and S_n' (recall that $S_n(t) = t$ if $|t| \le n$, that S_n is nondecreasing and $\mathrm{Supp}(S_n') \subset [-n-1, n+1]$) we have:

$$|(D_1)| \le \int\limits_{\{n-k\le|u_1-g_2|\le n\}} |G_1|\,|\nabla(u_1 - g_2)|\, dxdt + \int\limits_{\{n\le|u_2-g_2|\le n+1\}} |G_1|\,|\nabla(u_2 - g_2)|\, dxdt\,.$$

Applying Hölder's inequality and using property (10.55) for renormalized solutions we easily get that (D_1) converges to zero as n tends to infinity. Similarly, since $|S_n(t)| > n-k$ implies $|t| > n-k$ we have:

$$|(D_2)| \le \int\limits_{\{n\le|u_1-g_2|\le n+1\}} |G_1|\,|\nabla(u_1 - g_2)|\, dxdt + \int\limits_{\{n-k\le|u_2-g_2|\le n+1\}} |G_1|\,|\nabla(u_2 - g_2)|\, dxdt\,.$$

Again, Hölder 's inequality together with (10.55) allow to deduce that (D_2) converges to zero as well. The term (D_3) is dealt with in the same way (using that $S_n'(t) = 0$ if $|t| > n+1$), so that we finally get that

$$\lim_{n\to\infty} (D) = 0\,.$$

We deal with (B) splitting it as below:

$$\begin{aligned}
(B) = &\int\limits_{\{|v_1|\le n, |v_2|\le n\}} [a(t,x,\nabla u_1) - a(t,x,\nabla u_2)] \cdot \nabla T_k(u_1 - u_2)\, dxdt \\
&+ \int\limits_{\left\{ {|S_n(v_1)-S_n(v_2)|\le k \atop |v_1|\le n\,,\ |v_2|>n} \right\}} [S_n'(v_1)a(t,x,\nabla u_1) - S_n'(v_2)a(t,x,\nabla u_2)] \cdot \nabla(S_n(v_1) - S_n(v_2))\, dxdt \\
&+ \int\limits_{\left\{ {|S_n(v_1)-S_n(v_2)|\le k \atop |v_1|>n} \right\}} [S_n'(v_1)a(t,x,\nabla u_1) - S_n'(v_2)a(t,x,\nabla u_2)] \cdot \nabla(S_n(v_1) - S_n(v_2))\, dxdt\,.
\end{aligned}$$

Let us set (B_1)–(B_3) the three integrals above. Since $\{|S_n(v_1)-S_n(v_2)| \le k\,, |v_1| > n\} \subset \{|v_1| > n\,, |v_2| > n-k\}$, we have, using that $S_n'(t) = 0$ if $|t| > n+1$:

$$
\begin{aligned}
|(B_3)| \le & \int_{\{n \le |u_1-g_2| \le n+1\}} |a(t,x,\nabla u_1)|\,|\nabla(u_1-g_2)|\,dxdt \\
& + \int_{\{n \le |u_1-g_2| \le n+1\}} |a(t,x,\nabla u_1)|\,|\nabla(u_2-g_2)|\chi_{\{n-k\le|u_2-g_2|\le n+1\}}\,dxdt \\
& + \int_{\{n \le |u_1-g_2| \le n+1\}} |a(t,x,\nabla u_2)|\,|\nabla(u_1-g_2)|\chi_{\{n-k\le|u_2-g_2|\le n+1\}}\,dxdt \\
& + \int_{\{n-k \le |u_2-g_2| \le n+1\}} |a(t,x,\nabla u_2)|\,|\nabla(u_2-g_2)|\,dxdt\,.
\end{aligned} \tag{10.87}
$$

Using assumption (10.49), Young's inequality and the condition (10.55) for renormalized solutions, we can conclude as we did before that all the four terms in the right hand side of (10.87) converge to zero. Thus we get that (B_3) converges to zero. Changing the roles of u_1 and u_2, the same arguments prove that (B_2) also converges to zero as n tends to infinity. Thus we conclude, using Fatou's lemma in (B_1):

$$\liminf_{n\to\infty}(B) \ge \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla T_k(u_1-u_2)\,dxdt\,.$$

In the term (A) of (10.85) we can integrate using that $S_n(v_1)$ and $S_n(v_2)$ belong to $\mathcal{C}([0,T];L^1(\Omega))$ and $S_n(v_1)(0) = S_n(v_2)(0) = S_n(u_0)$. We then obtain:

$$(A) = \int_\Omega \Theta_k(S_n(v_1) - S_n(v_2))(T)\,dx\,,$$

where $\Theta_k(s) = \int_0^s T_k(t)\,dt$, and since Θ_k is nonnegative we conclude that $(A) \ge 0$. Putting together the results obtained on (A)–(F) we obtain from (10.85), as n tends to infinity:

$$\int_{\{|u_1-u_2|\le k\}} (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla(u_1-u_2)\,dxdt \le 0\,,$$

and then, letting k tend to infinity:

$$\int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla(u_1-u_2)\,dxdt \le 0\,.$$

The strict monotonicity assumption (10.50) then implies that $u_1 = u_2$ almost everywhere in Q. ∎

Remark 10.16 *In fact, the proof of the uniqueness of renormalized solutions does not need the strict monotonicity assumption (10.50) but only that*

$$(a(t,x,\xi) - a(t,x,\eta))\cdot(\xi-\eta) \ge 0 \qquad \forall(\xi,\eta)\in\mathbb{R}^N\,.$$

This can be seen performing the same proof as in Theorem 10.12 above in the interval $]0,t[$*, with* $t < T$*. Using that the term* (A) *is not only nonnegative as we already remarked but indeed*

$$\liminf_{n\to\infty}(A) \ge \int_\Omega \Theta_k(u_1-u_2)(t)\,dx\,,$$

we can obtain

$$\int_\Omega \Theta_k(u_1-u_2)(t)\,dx \le 0 \qquad \forall t\in]0,T[\,,$$

hence it follows that $u_1 = u_2$.

10.3.4 Data in $L^1 + W'$.

It is possible to extend the result on existence and uniqueness of renormalized solutions to data which belong to $L^1 + W'$, without being necessarily measures. In fact, let $\mu \in L^1(Q) + W'$, then a renormalized solution of (10.1) is defined exactly as in Definition 10.7, where f, g_1, g_2 is a decomposition of μ in $L^1(Q) + W'$, moreover this definition does not depend on the decomposition of μ (see Remark 10.13). Then all the results proved in the previous section apply without any change except for the first two estimates of Proposition 10.7 for which we used the fact that μ was a bounded measure (see Remark 10.14). Thus, we obtain the following result.

Theorem 10.13 *Let $\mu \in L^1(Q) + W'$, and let $u_0 \in L^1(\Omega)$. Assume that hypotheses (10.48), (10.49), (10.50) hold true. Then there exists a unique renormalized solution of problem (10.1) in the sense of Definition 10.7.*

10.4 Appendix: Proof of the density theorem

10.4.1 The case of compactly supported functions

Lemma 10.7 *Let $u \in W$ have a compact support in Q and $(\rho_n)_{n\geq 1}$ be a smoothing kernel. Then, for n large enough (depending on the support of u), $u * \rho_n$ is well defined, is in $\mathcal{C}_c^\infty(Q)$ and $u * \rho_n \to u$ in W as $n \to \infty$.*

Proof. The fact that $u * \rho_n$ is well defined and is in $\mathcal{C}_c^\infty(Q)$ for n large enough is a classical convolution result. It is still classical, since $u \in L^p(Q) \cap L^p(0,T;L^2(\Omega))$, that $u * \rho_n \to u$ in $L^p(Q) \cap L^p(0,T;L^2(\Omega))$. Moreover, in the sense of distributions, $\nabla(u * \rho_n) = \nabla u * \rho_n$ so that, since $\nabla u \in (L^p(Q))^N$, one has $\nabla(u * \rho_n) \to \nabla u$ in $(L^p(Q))^N$.
Thus, $u * \rho_n \to u$ in $L^p(0,T;V)$ and it remains to prove the convergence of the time derivative.
To see this, we take $v_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $v_2 \in L^{p'}(0,T;L^2(\Omega))$ such that $u_t = v_1 + v_2$. We have $u = \theta u$ for some $\theta \in \mathcal{C}_c^\infty(Q)$ so that $u_t = \theta_t u + \theta u_t = \theta v_1 + (\theta v_2 + \theta_t u)$ with $\theta v_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\theta v_2 + \theta_t u \in L^{p'}(0,T;L^2(\Omega))$ (because, u being in W, it is also in $\mathcal{C}([0,T];L^2(\Omega))$); moreover, θv_1 and $\theta v_2 + \theta_t u$ have compact supports in Q. Denote $w_1 = \theta v_1$ and $w_2 = \theta v_2 + \theta_t u$.
We have then, in the sense of distributions, $(u * \rho_n)_t = u_t * \rho_n = w_1 * \rho_n + w_2 * \rho_n$ for n large enough. Since $w_2 \in L^{p'}(0,T;L^2(\Omega))$, we have $w_2 * \rho_n \to w_2$ in $L^{p'}(0,T;L^2(\Omega))$. For the convergence of $w_1 * \rho_n$, write $v_1 = \operatorname{div}(V_1)$ for some $V_1 \in (L^{p'}(Q))^N$; we have $w_1 = \operatorname{div}(\theta V_1) - V_1 \cdot \nabla\theta$ with $\theta V_1 \in (L^{p'}(Q))$ and $V_1 \cdot \nabla\theta \in L^{p'}(Q)$ having compact supports in Q, so that $w_1 * \rho_n = \operatorname{div}((\theta V_1) * \rho_n) - (V_1 \cdot \nabla\theta) * \rho_n$; since $(\theta V_1) * \rho_n \to \theta V_1$ in $(L^{p'}(Q))^N$ and $(V_1 \cdot \nabla\theta) * \rho_n \to V_1 \cdot \nabla\theta$ in $L^{p'}(Q)$, this gives the convergence of $w_1 * \rho_n$ to w_1 in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.
We have thus proven that $(u * \rho_n)_t \to w_1 + w_2 = u_t$ in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p'}(0,T;L^2(\Omega)) = L^{p'}(0,T;V')$, and this concludes the proof. ■

This technique of approximation is however limited to compactly supported elements of W; for general elements of W, we must find another kind of approximation by regular functions.

10.4.2 The general case

We prove the density of $\mathcal{C}_c^\infty([0,T] \times \Omega)$ in W, that is Theorem 10.4. To prove this density result, we will use two main tools: some results coming from the vector-valued integral and Sobolev space theory and the following theorem, which states a density result in spaces of functions on Ω. Let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$.

Theorem 10.14 *If Ω is a bounded open subset of $\mathbb{R}^N$ and $1 < p < \infty$, then $\mathcal{C}_c^\infty(\Omega)$ is dense in V.*

Proof of Theorem 10.14.
Let $u \in V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$.
Let $S \in \mathcal{C}^\infty(\mathbb{R})$ such that $S(s) = s$ when $|s| \leq 1$ and $S'(s) = 0$ when $|s| \geq 2$. We define, for $n \geq 1$, $S_n(s) = nS(\frac{s}{n})$; notice that $S_n(s) \to s$ and $S_n'(s) = S'(\frac{s}{n}) \to 1$ when $n \to \infty$; moreover, $|S_n(s)| \leq \|S_n'\|_{L^\infty(\mathbb{R})}|s|$ and $\|S_n'\|_{L^\infty(\mathbb{R})} \leq \|S'\|_{L^\infty(\mathbb{R})}$.
$S_n(u) \to u$ on Ω and is dominated by $\|S'\|_{L^\infty(\mathbb{R})}|u| \in L^p(\Omega)\cap L^2(\Omega)$; the convergence thus also happens in $L^p(\Omega)\cap L^2(\Omega)$. Moreover, $\nabla(S_n(u)) = S_n'(u)\nabla u \to \nabla u$ on Ω and is dominated by $\|S'\|_{L^\infty(\mathbb{R})}|\nabla u| \in L^p(\Omega)$, which proves that $\nabla(S_n(u)) \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$. Thus, $S_n(u) \to u$ in V as $n \to \infty$.
Let $(\varphi_m)_{m\geq 1} \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi_m \to u$ in $W_0^{1,p}(\Omega)$ (by definition of $W_0^{1,p}(\Omega)$, such a sequence exists); we can suppose, up to a subsequence, that $\varphi_m \to u$ and $\nabla\varphi_m \to \nabla u$ a.e. on Ω. We have, for all $n \geq 1$, $S_n(\varphi_m) \in \mathcal{C}_c^\infty(\Omega)$ and $S_n(\varphi_m) \to S_n(u)$ a.e. on Ω when $m \to \infty$; since $(S_n(\varphi_m))_{m\geq 1}$ is bounded in $L^\infty(\Omega)$ (by $\|S_n\|_{L^\infty(\mathbb{R})}$) and Ω is of finite measure, this implies that $S_n(\varphi_m) \to S_n(u)$ in $L^q(\Omega)$ for all $q < \infty$, and in particular in $L^p(\Omega)$ and in $L^2(\Omega)$. We also have $\nabla(S_n(\varphi_m)) = S_n'(\varphi_m)\nabla\varphi_m \to S_n'(u)\nabla u = \nabla(S_n(u))$ a.e. on Ω and $|\nabla(S_n(\varphi_m))| \leq \|S'\|_{L^\infty(\mathbb{R})}|\nabla\varphi_m|$; this last inequality tells us that $(\nabla(S_n(\varphi_m)))_{m\geq 1}$ is equi-integrable in $(L^p(\Omega))^N$ (because $(\nabla\varphi_m)_{m\geq 1}$ is equi-integrable in this space, since it converges) and thus that $\nabla(S_n(\varphi_m)) \to \nabla(S_n(u))$ in $(L^p(\Omega))^N$ as $m \to \infty$.
We have proven that $S_n(\varphi_m) \to S_n(u)$ in V as $m \to \infty$. Take then $m_n \geq 1$ such that $\|S_n(\varphi_{m_n}) - S_n(u)\|_V \leq 1/n$; since $S_n(u) \to u$ in V, we deduce that $S_n(\varphi_{m_n}) \to u$ in V and this concludes the proof of this theorem. ■

The results coming from the vector-valued integral and Sobolev space theory we will use here are, for the most part, very intuitive when one recalls the same results for scalar-valued integral and Sobolev spaces. We will thus only give the ideas of the reasoning that lead to the use of Theorem 10.14, and refer the interested reader to [32].
One of these results, however, is a little bit tricky; it comes from the density of simple functions in $L^{p'}(0,T;B)$, but it is not easy to explain without going further into the theory (and, especially, without explaining the concept of μ-mesurability, which we do not want here). We will thus state it, without proof, in the following lemma.

Lemma 10.8 *Let B be a Banach space and D be a dense subset in B. If $1 \leq q < \infty$, then the set*

$$S(D) = \left\{ \sum_{i=1}^n d_i\varphi_i\,,\ n \geq 1\,,\ d_i \in D\,,\ \varphi_i \in \mathcal{C}^\infty([0,T];\mathbb{R}) \right\}$$

is dense in $L^q(0,T;B)$.

Remark 10.17 *In fact, the result of this lemma is still true if we take the functions φ_i in $\mathcal{C}_c^\infty(]0,T[;\mathbb{R})$ (see [32]).*

Let us now give the ideas that lead from Lemma 10.8 and Theorem 10.14 to Theorem 10.4.

Proof of Theorem 10.4. Let $u \in W$, that is to say $u \in L^p(0,T;V)$ such that $u_t \in L^{p'}(0,T;V')$. We want to find $(v_n)_{n\geq 1} \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ such that $v_n \to u$ in $L^p(0,T;V)$ and $(v_n)_t \to u_t$ in $L^{p'}(0,T;V')$.
Step 1: define $\widetilde{u} :]-T,2T[\to V$ almost everywhere by:

$$\widetilde{u}(t) = \begin{cases} u(-t) & \text{if } t \in]-T,0[, \\ u(t) & \text{if } t \in]0,T[, \\ u(2T-t) & \text{if } t \in]T,2T[. \end{cases}$$

One has $\widetilde{u} \in L^p(-T,2T;V)$. Moreover, since we have made two even reflections, it is easy (as for the classical Sobolev spaces) to see that $\widetilde{u}_t \in L^{p'}(-T,2T;V')$ with

$$\widetilde{u}_t(t) = \begin{cases} -u_t(-t) & \text{if } t \in]-T,0[, \\ u_t(t) & \text{if } t \in]0,T[, \\ -u_t(2T-t) & \text{if } t \in]T,2T[. \end{cases}$$

Define $\overline{u} \in L^p(\mathbb{R};V)$ as the extension of $\widetilde{u}$ by 0 outside $]-T,2T[$ and take $(\rho_n)_{n\geq 1}$ a smoothing kernel on $\mathbb{R}$ such that $\mathrm{Supp}(\rho_n) \subset]-T,T[$. Let $\overline{u}_n = \overline{u} * \rho_n \in L^p(\mathbb{R};V)$ (the convolution product is defined exactly as for scalar-valued integral, and the same results as in the scalar-valued case hold in the vector-valued case). One has $\overline{u}_n \in \mathcal{C}^\infty(\mathbb{R};V) \subset \mathcal{C}^\infty(\mathbb{R};V')$ (since $V \hookrightarrow V'$) and $\overline{u}_n \to \overline{u}$ in $L^p(\mathbb{R};V)$; thus, $u_n = (\overline{u}_n)_{|]0,T[} \in \mathcal{C}^\infty([0,T];V) \subset \mathcal{C}^\infty([0,T];V')$ and $u_n \to u$ in $L^p(0,T;V)$. Moreover, since $\widetilde{u}_t \in L^p(-T,2T;V')$, one can verify that, by defining $v \in L^{p'}(\mathbb{R};V)$ as the extension of $\widetilde{u}_t$ by 0 outside $]-T,2T[$, we have $(\overline{u}_n)_t = v * \rho_n$ in $\mathcal{C}^\infty(\mathbb{R};V')$. Thus, $(u_n)_t = (v*\rho_n)_{|]0,T[} \to v_{|]0,T[} = u_t$ in $L^{p'}(0,T;V')$.
We thus have found $(u_n)_{n\geq 1} \in \mathcal{C}^\infty([0,T];V)$ such that $u_n \to u$ in $L^p(0,T;V)$ and $(u_n)_t \to u_t$ in $L^{p'}(0,T;V')$.

Step 2: to approximate u in W, we thus just need to approximate in W a given function $v \in \mathcal{C}^\infty([0,T];V)$. Let v be such a function, and let $D = \mathcal{C}_c^\infty(\Omega)$. According to Theorem 10.14, D is a dense subset of V. Since $v' \in \mathcal{C}^\infty([0,T];V) \subset L^{p'}(0,T;V)$, using Lemma 10.8, there exists $(w_n)_{n\geq 1} \in S(D)$ which converges to v' in $L^{p'}(0,T;V)$, and thus also in $L^{p'}(0,T;V')$. Moreover, in V, one has $v(t) = v(0) + \int_0^t v'(s)\,ds$. Define $W_n(t) = \int_0^t w_n(s)\,ds$; since $w_n \to v'$ in $L^{p'}(0,T;V)$, one has $W_n \to \int_0^\cdot v'(s)\,ds = v - v(0)$ in $L^\infty(0,T;V)$, and thus in $L^p(0,T;V)$. Taking $(d_n)_{n\geq 1} \in D$ which converges to $v(0) \in V$ in V, the functions $v_n = d_n + W_n$ converge to v in $L^p(0,T;V)$ and the derivatives of these functions, $v_n' = W_n' = w_n$, converges to v' in $L^{p'}(0,T;V')$.
By noticing that $v_n(t) = d_n + \int_0^t w_n(s)\,ds \in S(D)$, we have proven that v is approximable in W by a sequence of functions in $S(D)$. Since $S(D) \subset \mathcal{C}_c^\infty([0,T]\times\Omega)$, this concludes the proof. ■

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Annexe A

Continuité höldérienne des solutions d'une équation elliptique

Résumé: Nous prouvons ici, par les outils de De Giorgi [25], le théorème de Stampacchia [70] (étendu aux conditions au bord mixtes, par les méthodes de [29]) concernant la continuité höldérienne des solutions d'EDP elliptiques.

A.1 Introduction

A.1.1 Notations Générales

N est un entier supérieur ou égal à 2. Lorsque $N = 2$, N_* désigne un réel fixé dans $]2, \infty[$; lorsque $N \geq 3$, on pose $N_* = N$. On notera $2^* = \frac{2N_*}{N_*-2}$.
Le produit scalaire euclidien de deux vecteurs $(a, b) \in \mathbb{R}^N$ est noté $a \cdot b$ et la norme induite $|\cdot|$; pour $\rho > 0$, B_ρ désigne la boule ouverte de centre 0 et de rayon ρ dans $\mathbb{R}^N$. La mesure de Lebesgue d'une partie $E \subset \mathbb{R}^N$ mesurable est notée $|E|$.

Ω est un ouvert borné de $\mathbb{R}^N$; on note d un majorant du diamètre de Ω.
Soit $p \in [1, N]$ et $q \leq \frac{Np}{N-p}$ lorsque $p < N$, ou $q < +\infty$ lorsque $p = N$. En prenant $a \in \Omega$, l'injection de Sobolev $W_0^{1,p}(a + B_d) \hookrightarrow L^q(a + B_d)$ et l'inégalité de Poincaré dans $W_0^{1,p}(a + B_d)$ nous donnent $C_S(N, d, p, q)$ tel que, pour tout $\varphi \in W_0^{1,p}(a + B_d)$,

$$||\varphi||_{L^q(a+B_d)} \leq C_S(N, d, p, q)||\,|\nabla\varphi|\,||_{L^p(a+B_d)}$$

($C_S(N, d, p, q)$ ne dépend pas de a grâce à l'invariance du problème par translation). Mais, pour tout $\varphi \in W_0^{1,p}(\Omega)$, puisque $\Omega \subset a + B_d$, l'extension $\widetilde{\varphi}$ de φ à $a + B_d$ par 0 hors de Ω est dans $W_0^{1,p}(a + B_d)$; ainsi, on déduit de l'inégalité précédente que, pour tout $\varphi \in W_0^{1,p}(\Omega)$,

$$||\varphi||_{L^q(\Omega)} \leq C_S(N, d, p, q)||\,|\nabla\varphi|\,||_{L^p(\Omega)}. \tag{A.1}$$

Lorsque Γ est une partie mesurable de $\partial\Omega$, $W_\Gamma^{1,p}(\Omega)$ est l'espace des fonctions de $W^{1,p}(\Omega)$ dont la trace est nulle sur Γ; on le munit de la même norme que $W^{1,p}(\Omega)$.

A.1.2 L'Equation

Nous prenons maintenant Γ_d une partie mesurable de $\partial\Omega$ et, en notant $H^1_{\Gamma_d}(\Omega)$ l'espaces des fonctions de $H^1(\Omega)$ dont la trace sur Γ_d est nulle, nous nous intéressons aux solutions de

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi + \int_\Omega \varphi\mathbf{v}\cdot\nabla u = \langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{A.2}$$

A.1.3 Matrice de Diffusion

L'équation considérée est elliptique, ce qui signifie que la matrice de diffusion vérifie:

$$\begin{array}{c} A:\Omega\to M_N(\mathbb{R}) \text{ est mesurable,} \\ \exists\alpha_A>0 \text{ tel que } A(x)\xi\cdot\xi\ge\alpha_A|\xi|^2 \text{ pour presque tout } x\in\Omega, \text{ pour tout } \xi\in\mathbb{R}^N, \\ \exists\Lambda_A \text{ tel que } ||A(x)|| := \sup\{|A(x)\xi|\,,\ \xi\in\mathbb{R}^N\,,\ |\xi|=1\}\le\Lambda_A \text{ pour presque tout } x\in\Omega. \end{array} \tag{A.3}$$

A.1.4 Terme de convection

Le coefficient de convection vérifie:

$$\mathbf{v}\in(L^{N_*}(\Omega))^N. \tag{A.4}$$

Lorsque $q\in[1,\infty]$, l'espace $(L^q(\Omega))^N$ est muni de la norme $||F||_{(L^q(\Omega))^N} = ||\,|F|\,||_{L^q(\Omega)}$. $B(\Omega,q,R)$ désigne la boule fermée, dans $(L^q(\Omega))^N$, de centre 0 et de rayon R.
Afin de préciser les dépendances des différentes constantes vis-à-vis des données du problème, on se donnera $\chi\ge 0$ (précisé ultérieurement, selon que l'on étudiera la continuité à l'intérieur ou au bord) et on supposera que

$$\exists r>N\,,\ \exists\Lambda>0 \text{ tel que } \mathbf{v}\in B(\Omega,N_*,\chi)+B(\Omega,r,\Lambda) \tag{A.5}$$

(remarquons que, pour tout $\mathbf{v}\in(L^{N_*}(\Omega))^N$ et tout $\chi>0$, il existe $r>N$ — on peut prendre $r=\infty$ — et $\Lambda\ge 0$ tel que $\mathbf{v}$ satisfasse (A.5); cependant, ce Λ ne dépend pas que de la norme de $\mathbf{v}$ dans $(L^{N_*}(\Omega))^N$).
Pour tout ce qui concerne les résultats à l'intérieur de l'ouvert Ω, le χ apparaissant dans (A.5) peut d'ores et déjà être donné; il suffira qu'il vérifie

$$\chi\in\left[0,\frac{\alpha_A}{C_S(N,d,2,2^*)}\right[. \tag{A.6}$$

A.1.5 Second Membre

On supposera le second membre de (A.2) plus régulier que strictement nécessaire:

$$\exists p\in]N,\infty[\,,\ \exists\mathcal{L}\in(W^{1,p'}_{\Gamma_d}(\Omega))' \text{ tel que } L=\mathcal{L}_{|H^1_{\Gamma_d}(\Omega)}. \tag{A.7}$$

On notera alors Λ_L une borne supérieure de $||\mathcal{L}||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$.

A.1.6 Conditions au bord

$\Gamma_f\subset\partial\Omega$ est mesurable et vérifie: $\sigma(\Gamma_d\cap\Gamma_f)=0$, $\partial\Omega=\Gamma_d\cup\Gamma_f$.
Afin d'obtenir la continuité holdérienne jusqu'au bord des solutions de (A.2), nous devrons supposer une hypothèse sur Γ_d et Γ_f. On définit donc les deux types suivants de cartes locales de $\partial\Omega$:

$$\begin{array}{c} O \text{ est un ouvert de } \mathbb{R}^N, \\ h:O\to B:=\{x\in\mathbb{R}^N\mid |x|<1\} \text{ est un homéomorphisme bilipschitzien,} \\ h(O\cap\Omega)=B_+:=\{x\in B\mid x_N>0\}, \\ h(O\cap\partial\Omega)=B^{N-1}:=\{x\in\partial B_+\mid x_N=0\} \end{array} \tag{A.8}$$

et

$$\begin{array}{c} O \text{ est un ouvert de } \mathbb{R}^N, \\ h : O \to B \text{ est un homéomorphisme bilipschitzien,} \\ h(O \cap \Omega) = B_{++} := \{x \in B \mid x_N > 0\,,\ x_{N-1} > 0\}, \\ h(O \cap \Gamma_f) = \Gamma_1 := \{x \in \partial B_{++} \mid x_{N-1} = 0\}, \\ h(O \cap \Gamma_d) = \Gamma_2 := \{x \in \partial B_{++} \mid x_N = 0\}. \end{array} \tag{A.9}$$

L'hypothèse sur Γ_d et Γ_f est la suivante:

$$\begin{array}{c} \text{Il existe un nombre fini de } (O_i, h_i)_{i \in [1,m]} \text{ tels que} \\ \partial\Omega \subset \cup_{i=1}^m O_i \text{ et, pour tout } i \in [1,m],\ (O_i, h_i) \text{ est de l'un des types suivants:} \\ \left| \begin{array}{ll} \text{(D)} & O_i \cap \partial\Omega = O_i \cap \Gamma_d \text{ et } (O_i, h_i) \text{ satisfait (A.8)} \\ \text{(F)} & O_i \cap \partial\Omega = O_i \cap \Gamma_f \text{ et } (O_i, h_i) \text{ satisfait (A.8)} \\ \text{(DF)} & (O_i, h_i) \text{ satisfait (A.9).} \end{array} \right. \end{array} \tag{A.10}$$

Pour établir des résultat de continuité jusqu'au bord de Ω, il faudra que le χ apparaissant dans (A.5) vérifie (A.6) ainsi que

$$\chi < \inf_{i \in [1,m]} \frac{\alpha_A}{4||Jh_i||_{L^\infty(O_i \cap \Omega)}||\,||(h_i^{-1})'||\,||^2_{L^\infty(h_i(O_i \cap \Omega))}||Jh_i^{-1}||^{1-1/N_*}_{L^\infty(h_i(O_i \cap \Omega))}||\,||h_i'||\,||_{L^\infty(O_i \cap \Omega)} C_S(N,2,2,2^*)} \tag{A.11}$$

(cette estimation est un peu forte; on pourrait faire plus fin en découpant $\mathbf{v}$ dans chaque carte $O_i \cap \Omega$).

A.1.7 Le Théorème

Le résultat principal prouvé ici est le suivant.

Théorème A.1 *Soit $M \geq 0$. Sous les hypothèses (A.3)—(A.7), (A.10) et (A.11), il existe $\kappa > 0$ ne dépendant que de $(\Omega, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$ [1] et C ne dépendant que de $(\Omega, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, M)$ tels que, si u vérifie (A.2) et $||u||_{L^2(\Omega)} \leq M$, alors $u \in \mathcal{C}^{0,\kappa}(\Omega)$ et $||u||_{\mathcal{C}^{0,\kappa}(\Omega)} \leq C$.*

A.2 L'équation sans second membre

Nous prenons ici U un ouvert inclus dans Ω et nous nous intéressons aux solutions $v \in H^1(U)$, de

$$\forall \varphi \in H_0^1(U)\,,\ \int_U A\nabla v \cdot \nabla\varphi + \int_U \varphi \mathbf{v} \cdot \nabla v = 0. \tag{A.12}$$

Nous notons, pour $v : U \to \mathbb{R}$ mesurable, $x_0 \in \Omega$ et $R \in]0, \mathrm{dist}(x_0, \mathbb{R}^N \backslash U)[$, $\mathrm{supess}(v, x_0 + B_R)$ et $\mathrm{infess}(v, x_0 + B_R)$ les bornes supérieure et inférieure essentielles de v sur $x_0 + B_R$. Lorsque ces bornes sont finies, on note $\omega(v, x_0, R) = \mathrm{supess}(v, x_0 + B_R) - \mathrm{infess}(v, x_0 + B_R)$ le module de continuité essentielle de v sur $x_0 + B_R$.
Nous cherchons à prouver la proposition suivante, clef du résultat de continuité höldérienne du théorème A.1.

Proposition A.1 *Sous les hypothèses (A.3)—(A.5) et (A.6), il existe $C > 0$ et $\alpha \in]0,1[$ ne dépendant que de*

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda)$$

tel que, si $v \in H^1(\Omega)$ vérifie (A.12), alors, pour tout $x_0 \in \Omega$ et tout $R > 0$ tel que $x_0 + B_{2R} \subset U$, on a

$$||v||_{L^\infty(x_0 + B_R)} \leq \frac{C}{R^{\frac{N}{2}}} ||v||_{L^2(U)} \quad \text{et} \quad \omega(v, x_0, R/4) \leq \alpha\omega(v, x_0, R).$$

[1] Une dépendance par rapport à Ω prend en compte une dépendance par rapport à N et aux cartes $(O_i, h_i)_{i \in [1,m]}$.

A.2.1 Résultats préliminaires

Nous démontrons d'abord quelques lemmes techniques qui nous serviront dans la preuve de la proposition A.1.

Les deux premiers lemmes sont assez intuitifs et simples.

Lemme A.1 *Soit* $(f,g) \in \mathcal{C}^1(\mathbb{R})$ *vérifiant* $f' \in L^\infty(\mathbb{R})$ *et* $(g,g') \in L^\infty(\mathbb{R})$. *Si* $v \in H^1(\Omega) \cap L^\infty(\Omega)$ *et* $w \in H^1(\Omega)$, *alors* $f(v)g(w) \in H^1(\Omega)$ *et* $\nabla(f(v)g(w)) = f(v)\nabla(g(w)) + g(w)\nabla(f(v))$.

Preuve du lemme A.1
La mesurabilité de $f(v)g(w)$ est évidente et, puisque $v \in L^\infty(\Omega)$, f est continue (donc bornée sur les compacts) et g est bornée, on a $f(v)g(w) \in L^\infty(\Omega)$.
f et g étant régulières, on a $(f(v), g(w)) \in H^1(\Omega)$, donc $f(v)g(w) \in W^{1,1}(\Omega)$ avec $\nabla(f(v)g(w)) = f(v)\nabla(g(w)) + g(w)\nabla(f(v))$; or g est bornée, donc $g(w) \in L^\infty(\Omega)$ et, v étant bornée, $f(v) \in L^\infty(\Omega)$ (f est continue). Ainsi, puisque $(\nabla(f(v)), \nabla(g(w))) \in (L^2(\Omega))^N$, on en déduit que $f(v)\nabla(g(w)) + g(w)\nabla(f(v)) \in (L^2(\Omega))^N$, ce qui conclut la preuve de ce lemme. ∎

Lemme A.2 *Il existe* $\mathcal{K}_0$ *ne dépendant que de* N *tel que, pour tous* $0 < \rho_1 < \rho_2 < \infty$, *il existe* $\eta \in \mathcal{C}_c^\infty(B_{\rho_2})$ *vérifiant:*

$$\eta \geq 0 \ \textit{sur}\ \mathbb{R}^N, \quad ||\eta||_{L^\infty(\mathbb{R}^N)} \leq 1, \quad \eta \equiv 1 \ \textit{sur}\ B_{\rho_1} \quad \textit{et} \quad ||\,|\nabla\eta|\,||_{L^\infty(\mathbb{R}^N)} \leq \frac{\mathcal{K}_0}{\rho_2 - \rho_1}.$$

Preuve du lemme A.2
Soit $\theta \in \mathcal{C}_c^\infty(B_1)$, positive et d'intégrale égale à 1; un tel choix de θ ne dépend que de N. On note, pour $s > 0$, $\theta_s(x) = s^{-N}\theta(x/s)$; pour tout $s > 0$, $\theta_s \in \mathcal{C}_c^\infty(B_s)$ est positive et d'intégrale égale à 1.
En notant $\overline{\rho} = (\rho_1 + \rho_2)/2$ et $d = (\rho_2 - \rho_1)/2 > 0$, on pose $\eta = \mathbf{1}_{B_{\overline{\rho}}} * \theta_d$.
η est de classe $\mathcal{C}^\infty$ et à support dans $\overline{B_{\overline{\rho}}} + \mathrm{supp}(\theta_d)$; $\mathrm{supp}(\theta_d)$ étant un compact inclus dans B_d, le support de η est un compact inclus dans $\overline{B_{\overline{\rho}}} + B_d \subset B_{\overline{\rho}+d}$; or $\overline{\rho} + d = \rho_2$. On a donc $\eta \in \mathcal{C}_c^\infty(B_{\rho_2})$.
η est clairement positive sur $\mathbb{R}^N$ (puisque $\mathbf{1}_{B_{\overline{\rho}}}$ et θ_d sont positives).
Par l'inégalité de Young, $||\eta||_{L^\infty(\mathbb{R}^N)} \leq ||\mathbf{1}_{B_{\overline{\rho}}}||_{L^\infty(\mathbb{R}^N)} ||\theta_d||_{L^1(\mathbb{R}^N)} = 1$.
Prenons $x \in B_{\rho_1}$; on a alors, par définition de d, $x + B_d \subset B_{\overline{\rho}}$, donc $\mathrm{supp}(\theta_s(x - \cdot)) \subset x + B_d \subset B_{\overline{\rho}}$; ainsi

$$\eta(x) = \int_{\mathbb{R}^N} \mathbf{1}_{B_{\overline{\rho}}}(y)\theta_d(x-y)\,dy = \int_{B_{\overline{\rho}}} \theta_d(x-y)\,dy = \int_{\mathbb{R}^N} \theta_d(x-y)\,dy = 1.$$

Puisque $\nabla\eta = \mathbf{1}_{B_{\overline{\rho}}} * \nabla\theta_d$, on a, par Young et pour tout $i \in [1, N]$,

$$||D_i\eta||_{L^\infty(\mathbb{R}^N)} \leq ||\mathbf{1}_{B_{\overline{\rho}}}||_{L^\infty(\mathbb{R}^N)} ||D_i\theta_d||_{L^1(\mathbb{R}^N)}.$$

Or $D_i\theta_d(x) = d^{-1}d^{-N}D_i\theta(x/d)$, donc $||D_i\theta_d||_{L^1(\mathbb{R}^N)} = d^{-1}||D_i\theta||_{L^1(\mathbb{R}^N)}$ et

$$||D_i\eta||_{L^\infty(\mathbb{R}^N)} \leq d^{-1}||D_i\theta||_{L^1(\mathbb{R}^N)}.$$

En notant donc $\mathcal{K}_0 = 2\sum_{i=1}^N ||D_i\theta||_{L^1(\mathbb{R}^N)}$ ($\mathcal{K}_0$ ne dépend que de N), on en déduit, puisque $|\nabla\eta| \leq \sum_{i=1}^N |D_i\eta|$, que $||\,|\nabla\eta|\,||_{L^\infty(\mathbb{R}^N)} \leq \frac{\mathcal{K}_0}{2}d^{-1}$, c'est à dire $||\,|\nabla\eta|\,||_{L^\infty(\mathbb{R}^N)} \leq \mathcal{K}_0/(\rho_2 - \rho_1)$. ∎

Le lemme suivant est un résultat classique dû à l'injection compacte de $H_0^1(U)$ dans $L^p(U)$ lorsque $p < 2N/(N-2)$.

Lemme A.3 *Si* $r > N$ *alors, pour tout* $\varepsilon > 0$, *il existe* $\mathcal{K}(N,d,r,\varepsilon)$ *tel que, pour tout* $\varphi \in H_0^1(U)$,

$$||\varphi||_{L^{\frac{2r}{r-2}}(U)} \leq \varepsilon||\,|\nabla\varphi|\,||_{L^2(U)} + \mathcal{K}(N,d,r,\varepsilon)||\varphi||_{L^2(U)}.$$

Preuve du lemme A.3
Etape 1: on commence par montrer que le résultat est vérifié avec $U = B_d$.
La démonstration se fait par l'absurde. Fixons donc $r > N$ et $\varepsilon > 0$ et supposons qu'un tel $\mathcal{K}(N, d, r, \varepsilon)$ n'existe pas; dans ce cas, pour tout $n \geq 1$, on pourrait trouver $\varphi_n \in H_0^1(B_d)$ tel que

$$||\varphi_n||_{L^{\frac{2r}{r-2}}(B_d)} > \varepsilon|| \, |\nabla\varphi_n| \, ||_{L^2(B_d)} + n||\varphi_n||_{L^2(B_d)}.$$

En posant $\psi_n = \varphi_n/||\varphi_n||_{L^{\frac{2r}{r-2}}(B_d)} \in H_0^1(B_d)$, on aurait donc, pour tout $n \geq 1$,

$$||\psi_n||_{L^{\frac{2r}{r-2}}(B_d)} = 1, \tag{A.13}$$

$$|| \, |\nabla\psi_n| \, ||_{L^2(B_d)} \leq \frac{1}{\varepsilon}, \tag{A.14}$$

$$||\psi_n||_{L^2(B_d)} \leq \frac{1}{n}. \tag{A.15}$$

Or $2r/(r-2) < 2N/(N-2)$ donc $H_0^1(B_d)$ s'injecte compactement dans $L^{\frac{2r}{r-2}}(B_d)$. Comme, par (A.14) et (A.15), $(\psi_n)_{n\geq 1}$ est bornée dans $H_0^1(B_d)$, on peut alors extraire de cette suite une sous-suite, encore notée $(\psi_n)_{n\geq 1}$, telle que $\psi_n \to \psi$ dans $L^{\frac{2r}{r-2}}(B_d)$.
En passant à la limite $n \to \infty$ dans (A.13), on constate que $||\psi||_{L^{\frac{2r}{r-2}}(B_d)} = 1$; ψ n'est donc pas la fonction nulle. Mais, puisque $2r/(r-2) > 2$, la convergence dans $L^{\frac{2r}{r-2}}(B_d)$ implique la convergence dans $L^2(B_d)$, ce qui donne, en faisant $n \to \infty$ dans (A.15), $||\psi||_{L^2(B_d)} = 0$, ce qui est une contradiction puisque l'on a vu que $\psi \neq 0$.

Etape 2: passage à U quelconque inclus dans Ω.
Soit $r > N$ et $\varepsilon > 0$.
On prend $a \in U$ et $\varphi \in H_0^1(U)$; comme $\varphi(\cdot - a) \in H_0^1(U - a)$ et $U - a \subset B_d$, l'extension $\widetilde{\varphi}$ de $\varphi(\cdot - a)$ à B_d par 0 hors de $U - a$ est dans $H_0^1(B_d)$.
On a donc, avec le $\mathcal{K}(N, d, r, \varepsilon)$ obtenu dans l'étape 1,

$$||\widetilde{\varphi}||_{L^{\frac{2r}{r-2}}(B_d)} \leq \varepsilon|| \, |\nabla\widetilde{\varphi}| \, ||_{L^2(B_d)} + \mathcal{K}(N, d, r, \varepsilon)||\widetilde{\varphi}||_{L^2(B_d)}.$$

Compte tenu de la définition de $\widetilde{\varphi}$, on déduit de cette inégalité:

$$||\varphi||_{L^{\frac{2r}{r-2}}(U)} \leq \varepsilon|| \, |\nabla\varphi| \, ||_{L^2(U)} + \mathcal{K}(N, d, r, \varepsilon)||\varphi||_{L^2(U)},$$

ce qui conclut la démonstration de ce lemme. ■

Lorsque E est une partie mesurable de $\mathbb{R}^N$ de mesure non-nulle et $f \in L^1(E)$, $\overline{f}^E$ désigne la moyenne de f sur E.
Le lemme suivant sert à estimer, dans l'inégalité de Poincaré moyenne exprimée sur une boule, la dépendance de la constante par rapport au rayon de la boule.

Lemme A.4 *Il existe $\mathcal{K}_1$ ne dépendant que de (N, N_*) tel que, pour tout $\rho > 0$ et tout $\varphi \in H^1(B_\rho)$,*

$$||\varphi - \overline{\varphi}^{B_\rho}||_{L^{2^*}(B_\rho)} \leq \mathcal{K}_1 \rho^{1-\frac{N}{N_*}} || \, |\nabla\varphi| \, ||_{L^2(B_\rho)}.$$

Remarque A.1 *Lorsque $N \geq 3$, $1 - \frac{N}{N_*} = 0$.*

Preuve du lemme A.4
Soit $\psi \in H^1(B_1)$ définie par $\psi(x) = \varphi(\rho x)$. On a

$$\overline{\psi}^{B_1} = \frac{1}{|B_1|} \int_{B_1} \varphi(\rho x)\,dx = \frac{1}{\rho^N |B_1|} \int_{B_\rho} \varphi(y)\,dy = \overline{\varphi}^{B_\rho}.$$

Par l'inégalité de Poincaré moyenne et l'injection de Sobolev sur B_1, il existe $\mathcal{K}_1$ ne dépendant que de (N, N_*) tel que

$$||\psi - \overline{\psi}^{B_1}||_{L^{2^*}(B_1)} \le \mathcal{K}_1 ||\,|\nabla\psi|\,||_{L^2(B_1)}. \tag{A.16}$$

Or, par changement de variable,

$$\int_{B_1} |\psi(x) - \overline{\psi}^{B_1}|^{2^*}\,dx = \rho^{-N} \int_{B_\rho} |\varphi(y) - \overline{\varphi}^{B_\rho}|^{2^*}\,dy \tag{A.17}$$

et, puisque $\nabla\psi(x) = \rho\nabla\varphi(\rho x)$,

$$\int_{B_1} |\nabla\psi(x)|^2\,dx = \rho^{-N}\rho^2 \int_{B_\rho} |\nabla\varphi(y)|^2\,dy. \tag{A.18}$$

(A.16), (A.17) et (A.18) donnent donc

$$\rho^{-\frac{N}{2^*}} ||\varphi - \overline{\varphi}||_{L^{2^*}(B_\rho)} \le \mathcal{K}_1 \rho^{1-\frac{N}{2}} ||\,|\nabla\varphi|\,||_{L^2(B_\rho)};$$

comme $1 - \frac{N}{2} + \frac{N}{2^*} = 1 - \frac{N}{N_*}$, on en déduit le résultat du lemme. ■

Nous prouvons maintenant une serie de lemmes spécifiquement liés à la preuve de la proposition A.1.

Lemme A.5 *Soit $(\rho, \sigma) \in]0, \infty[$. Si $\varphi \in H^1(B_{\rho+\sigma})$ vérifie*

$$||\,|\nabla\varphi|\,||_{L^2(B_\rho)} \le \frac{C}{\sigma} ||\varphi||_{L^2(B_{\rho+\sigma})},$$

alors il existe $\mathcal{K}_2$ ne dépendant que de (C, N, N_*) tel que

$$||\varphi||_{L^{2^*}(B_\rho)} \le \mathcal{K}_2 \rho^{1-\frac{N}{N_*}} \left(\frac{1}{\sigma} + \frac{1}{\rho}\right) ||\varphi||_{L^2(B_{\rho+\sigma})}.$$

Preuve du lemme A.5
Par le lemme A.4, on a

$$||\varphi - \overline{\varphi}^{B_\rho}||_{L^{2^*}(B_\rho)} \le \mathcal{K}_1 \rho^{1-\frac{N}{N_*}} ||\,|\nabla\varphi|\,||_{L^2(B_\rho)} \le \frac{C\mathcal{K}_1 \rho^{1-\frac{N}{N_*}}}{\sigma} ||\varphi||_{L^2(B_{\rho+\sigma})},$$

soit

$$||\varphi||_{L^{2^*}(B_\rho)} \le |B_\rho|^{\frac{1}{2^*}} |\overline{\varphi}^{B_\rho}| + \frac{C\mathcal{K}_1 \rho^{1-\frac{N}{N_*}}}{\sigma} ||\varphi||_{L^2(B_{\rho+\sigma})}. \tag{A.19}$$

Or $\overline{\varphi}^{B_\rho} \le |B_\rho|^{-1/2} ||\varphi||_{L^2(B_\rho)} \le |B_\rho|^{-1/2} ||\varphi||_{L^2(B_{\rho+\sigma})}$, donc, par (A.19),

$$||\varphi||_{L^{2^*}(B_\rho)} \le \left(\frac{C\mathcal{K}_1 \rho^{1-\frac{N}{N_*}}}{\sigma} + |B_\rho|^{\frac{1}{2^*}-\frac{1}{2}}\right) ||\varphi||_{L^2(B_{\rho+\sigma})}.$$

Comme $|B_\rho|^{\frac{1}{2^*}-\frac{1}{2}} = |B_1|^{\frac{1}{2^*}-\frac{1}{2}} \rho^{\frac{N}{2^*}-\frac{N}{2}} = C_0 \rho^{-\frac{N}{N_*}}$ avec C_0 ne dépendant que de (N, N_*), on en déduit

$$||\varphi||_{L^{2^*}(B_\rho)} \le \sup(C\mathcal{K}_1, C_0) \rho^{1-\frac{N}{N_*}} \left(\frac{1}{\sigma} + \frac{1}{\rho}\right) ||\varphi||_{L^2(B_{\rho+\sigma})}$$

c'est à dire le résultat souhaité. ■

Lemme A.6 *Soit* $\varphi : \mathbb{R}^N \to [0, \infty]$ *mesurable et* $R \in]0, R_0]$*. S'il existe* $C > 0$ *tel que, pour tous* $(\rho, \sigma) \in]0, \infty[$ *vérifiant* $\rho + \sigma \leq R$ *et pour tout* $k \geq 1$*, on a*

$$||\varphi^k||_{L^{2^*}(B_\rho)} \leq C\rho^{1-\frac{N}{N_*}} \left(\frac{1}{\sigma} + \frac{1}{\rho}\right) ||\varphi^k||_{L^2(B_{\rho+\sigma})},$$

alors il existe $\mathcal{K}_3$ *ne dépendant que de* (C, N, N_*, R_0) *tel que*

$$||\varphi||_{L^\infty(B_{R/2})} \leq \frac{\mathcal{K}_3}{R^{\frac{N}{2}}} ||\varphi||_{L^2(B_R)}.$$

Preuve du lemme A.6
Notons $\alpha = \frac{2^*}{2} = \frac{N_*}{N_*-2} > 1$. En prenant $k \geq 1$ et en notant $q = 2k$, on a donc, pour tout $q \geq 2$ et tous $(\rho, \sigma) \in]0, \infty[$ tels que $\rho + \sigma \leq R$,

$$||\varphi||_{L^{\alpha q}(B_\rho)} \leq \left(C\rho^{1-\frac{N}{N_*}} \left(\frac{1}{\sigma} + \frac{1}{\rho}\right)\right)^{\frac{2}{q}} ||\varphi||_{L^q(B_{\rho+\sigma})}. \tag{A.20}$$

Posons, pour $n \geq 0$, $q_n = 2\alpha^n$, $\rho_n = \frac{R}{2} + \frac{R}{2^{n+1}}$ et $\sigma_n = \frac{R}{2^{n+1}}$. On a, pour tout $n \geq 1$, $\rho_n + \sigma_n = \rho_{n-1} \leq R$ et

$$\frac{1}{\sigma_n} + \frac{1}{\rho_n} = \frac{\rho_n + \sigma_n}{\sigma_n \rho_n} \leq \frac{R}{\frac{R}{2^{n+1}} \times \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) R} \leq \frac{2^{n+1}}{\left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) R} \leq \frac{2^{n+2}}{R}.$$

En appliquant alors (A.20) avec $q = q_{n-1}$, $\rho = \rho_n$ et $\sigma = \sigma_n$ (pour un $n \geq 1$), on trouve donc

$$||\varphi||_{L^{q_n}(B_{\rho_n})} \leq \left(\frac{2^{n+2} C\rho_n^{1-\frac{N}{N_*}}}{R}\right)^{\frac{1}{\alpha^{n-1}}} ||\varphi||_{L^{q_{n-1}}(B_{\rho_{n-1}})}.$$

En notant $a_n = ||\varphi||_{L^{q_n}(B_{\rho_n})}$, on obtient donc, puisque $\rho_n \leq R$ et $1 - \frac{N}{N_*} \geq 0$, pour tout $n \geq 1$,

$$a_n \leq 2^{\frac{n}{\alpha^{n-1}}} \times \left(\frac{4C}{R^{\frac{N}{N_*}}}\right)^{\frac{1}{\alpha^{n-1}}} \times a_{n-1},$$

ce qui donne, par récurrence, pour tout $n \geq 1$,

$$a_n \leq 2^{\sum_{i=1}^n \frac{i}{\alpha^{i-1}}} \times \left(\frac{4C}{R^{\frac{N}{N_*}}}\right)^{\sum_{i=1}^n \frac{1}{\alpha^{i-1}}} \times a_0.$$

Comme $\alpha > 1$, la série $\sum_{i\geq 1} \frac{i}{\alpha^{i-1}}$ converge; on a donc, pour tout $n \geq 1$,

$$a_n \leq C_1 \left(\frac{4C}{R^{\frac{N}{N_*}}}\right)^{\sum_{i=1}^n \frac{1}{\alpha^{i-1}}} a_0$$

avec C_1 ne dépendant que de N_* (rappelons que α ne dépend que de N_*).
De plus, $\sum_{i\geq 1} 1/\alpha^{i-1} = \alpha/(\alpha - 1) = N_*/2$ donc:

i) Si $4C/R^{N/N_*} < 1$, $(4C/R^{N/N_*})^{\sum_{i=1}^n \frac{1}{\alpha^{i-1}}} \leq 1 \leq R_0^{N/2}/R^{N/2}$ (rappelons que $R \leq R_0$),

ii) Si $4C/R^{N/N_*} \geq 1$, $(4C/R^{N/N_*})^{\sum_{i=1}^n \frac{1}{\alpha^{i-1}}} \leq (4C/R^{N/N_*})^{N_*/2} = (4C)^{N_*/2}/R^{N/2}$.

Dans tous les cas, il existe C_2 ne dépendant que de (C, N, N_*, R_0) tel que, pour tout $n \geq 1$,

$$a_n \leq \frac{C_1 C_2}{R^{\frac{N}{2}}} a_0.$$

Comme, pour tout $n \geq 1$, $a_n \geq ||\varphi||_{L^{q_n}(B_{R/2})}$ (car $\rho_n \geq R/2$) et, puisque $q_n \to \infty$ lorsque $n \to \infty$, $\lim_{n\to\infty} ||\varphi||_{L^{q_n}(B_{R/2})} = ||\varphi||_{L^\infty(B_{R/2})}$, passer à la limite $n \to \infty$ dans l'inégalité précédente donne le résultat du lemme (avec $\mathcal{K}_3 = C_1 C_2$). ■

Remarque A.2 *Par translation, les résultats des lemmes A.2, A.4, A.5 et A.6 restent valables (sans changer les constantes) lorsque les boules ne sont plus centrées en* 0 *mais en n'importe quel point de* $\mathbb{R}^N$.

A.2.2 Preuve de la proposition A.1

Rappelons qu'une fonction $f : \mathbb{R} \to \mathbb{R}$ est dite "de Stampacchia" si elle est continue, $\mathcal{C}^1$ par morceaux et vérifie $f' \in L^\infty(\mathbb{R})$. Une telle fonction a la propriété principale que, pour tout $\varphi \in H^1(\Omega)$, $f(\varphi) \in H^1(\Omega)$ et $\nabla(f(\varphi)) = f'(\varphi)\nabla\varphi$ (cette propriété est en fait aussi vérifiée pour toute fonction lipschitzienne f).

Lemme A.7 *Sous les hypothèses (A.3)—(A.6), il existe C ne dépendant que de*

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda)$$

tel que, si $v \in H^1(U)$ vérifie (A.12), alors, pour toute fonction de Stampacchia convexe positive f, pour tout $x_0 \in \Omega$ et tous $(\rho, \sigma) \in]0, \infty[$ vérifiant $x_0 + B_{\rho+\sigma} \subset U$, on a

$$||\,|\nabla f(v)|\,||_{L^2(x_0+B_\rho)} \leq \frac{C}{\sigma} ||f(v)||_{L^2(x_0+B_{\rho+\sigma})}.$$

Preuve du lemme A.7
On suppose pour commencer que f est convexe positive de classe $\mathcal{C}^2$ et vérifie $(f', f'') \in L^\infty(\mathbb{R})$.
Soit $x_0 \in \Omega$ et $(\rho, \sigma) \in]0, \infty[$ tels que $x_0 + B_{\rho+\sigma} \subset U$. Par lemme A.2, il existe $\eta \in \mathcal{C}_c^\infty(x_0 + B_{\rho+\sigma})$ positive telle que $||\eta||_{L^\infty(\mathbb{R}^N)} \leq 1$, $\eta \equiv 1$ sur $x_0 + B_\rho$ et $||\,|\nabla\eta|\,||_{L^\infty(\mathbb{R}^N)} \leq \mathcal{K}_0/\sigma$, où $\mathcal{K}_0$ ne dépend que de N.
On note, pour $n \geq 0$, $T_n(s) = \min(n, \max(s, -n))$ la troncature au niveau n.
Par le lemme A.1, $f(T_n(v))f'(v) \in H^1(U)$, donc, puisque $\eta^2 \in \mathcal{C}_c^\infty(U)$, la fonction $\varphi = \eta^2 f(T_n(v))f'(v)$ appartient à $H_0^1(U)$. En utilisant cette fonction dans (A.12), on obtient

$$\int_U A\nabla v \cdot \nabla(f(T_n(v))f'(v)\eta^2 + \int_U A\nabla v \cdot \nabla(f'(v))f(T_n(v))\eta^2 + 2\int_U A\nabla v \cdot \nabla\eta f'(v)f(T_n(v))\eta$$
$$+ \int_U \eta^2 f(T_n(v))f'(v)\mathbf{v} \cdot \nabla v = 0.$$

Traitons chacun de ces termes séparément.

i) $A\nabla v \cdot \nabla(f'(v)) = f''(v)A\nabla v \cdot \nabla v \geq 0$ puisque, f étant convexe, $f'' \geq 0$; donc, f étant positive, $A\nabla v \cdot \nabla(f'(v))f(T_n(v))\eta^2 \geq 0$.

ii) Hors de $\{|v| \leq n\}$, $\nabla(f(T_n(v))) = f'(T_n(v))\mathbf{1}_{\{|v|\leq n\}}\nabla v = 0$ et, sur $\{|v| \leq n\}$, $v = T_n(v)$, donc

$$\begin{aligned} A\nabla v \cdot \nabla(f(T_n(v)))f'(v)\eta^2 &= A\nabla(T_n(v)) \cdot \nabla(f(T_n(v)))f'(T_n(v))\eta^2 \\ &= A\nabla(f(T_n(v))) \cdot \nabla(f(T_n(v)))\eta^2 \\ &\geq \alpha_A|\nabla(f(T_n(v)))|^2\eta^2. \end{aligned}$$

iii) $|A\nabla v \cdot \nabla\eta f'(v) f(T_n(v))\eta| \leq \Lambda_A |\eta\nabla(f(v))|\,|f(T_n(v))\nabla\eta|$.

iv) $|\eta^2 f(T_n(v)) f'(v)\mathbf{v}\cdot\nabla v| \leq |\eta f(T_n(v))\mathbf{v}|\,|\eta\nabla(f(v))|$.

En rassemblant ces termes, on trouve donc

$$\begin{aligned}\alpha_A||\,|\eta\nabla(f(T_n(v))|\,||^2_{L^2(U)} \quad \leq \quad & 2\Lambda_A||\,|\eta\nabla(f(v))|_{L^2(U)}||\,|f(T_n(v))\nabla\eta|\,||_{L^2(U)} \\ & +||\,|\eta f(T_n(v))\mathbf{v}|\,||_{L^2(U)}||\,|\eta\nabla(f(v))|\,||_{L^2(U)}.\end{aligned} \tag{A.21}$$

Lorsque $n\to\infty$, $f(T_n(v))\to f(v)$ en étant majorée par $\mathrm{Lip}(f)|v|+|f(0)|\in L^{2^*}(U)\subset L^2(U)$; donc, puisque $\eta\mathbf{v}\in(L^{N_*}(U))^N$ et $\nabla\eta\in(L^\infty(U))^N$ le théorème de convergence dominée donne $\eta f(T_n(v))\mathbf{v}\to\eta f(v)\mathbf{v}$ dans $(L^2(U))^N$ et $f(T_n(v))\nabla\eta\to f(v)\nabla\eta$ dans $(L^2(U))^N$.
De plus, $\nabla(f(T_n(v)) = f'(T_n(v))\nabla v\to f'(v)\nabla v=\nabla(f(v))$ lorsque $n\to\infty$ (rappelons que f' est continue) en étant majorée par $||f'||_{L^\infty(\mathbb{R})}|\nabla v|\in L^2(U)$; par convergence dominée, on a donc $\eta\nabla(f(T_n(v)))\to\eta\nabla(f(v))$ dans $L^2(U)$.
Le passage à la limite $n\to\infty$ dans (A.21) donne donc, après simplification par $||\,|\eta\nabla(f(v))|\,||_{L^2(U)}<\infty$,

$$\alpha_A||\,|\eta\nabla(f(v))|\,||_{L^2(U)}\leq 2\Lambda_A||\,|f(v)\nabla\eta|\,||_{L^2(U)}+||\,|\eta f(v)\mathbf{v}|\,||_{L^2(U)}. \tag{A.22}$$

Or $\nabla\eta\equiv 0$ hors de $x_0+B_{\rho+\sigma}$ et $||\,|\nabla\eta|\,||_{L^\infty(\mathbb{R}^N)}\leq\mathcal{K}_0/\sigma$, donc

$$||\,|f(v)\nabla\eta|\,||_{L^2(U)}\leq\frac{\mathcal{K}_0}{\sigma}||f(v)||_{L^2(x_0+B_{\rho+\sigma})}.$$

Comme $\mathbf{v}\in B(\Omega,N_*,\chi)+B(\Omega,r,\Lambda)$, $\eta=0$ hors de $x_0+B_{\rho+\sigma}$, $|\eta|\leq 1$ sur $\mathbb{R}^N$ et $\eta f(v)\in H^1_0(U)$, on a, pour tout $\varepsilon>0$, par le lemme A.3 et l'inégalité (A.1) (d est aussi un majorant de $U\subset\Omega$),

$$\begin{aligned}||\,|\eta f(v)\mathbf{v}|\,||_{L^2(U)} \quad \leq \quad & \chi||\eta f(v)||_{L^{2^*}(U)}+\Lambda||\eta f(v)||_{L^{\frac{2r}{r-2}}(U)} \\ \leq \quad & \chi C_S(N,d,2,2^*)||\,|\nabla(\eta f(v))|\,||_{L^2(U)}+\Lambda\varepsilon||\,|\nabla(\eta f(v))|\,|||_{L^2(U)} \\ & +\Lambda\mathcal{K}(N,d,r,\varepsilon)||\eta f(v)||_{L^2(U)} \\ \leq \quad & \chi C_S(N,d,2,2^*)||\,|\eta\nabla(f(v))|\,||_{L^2(U)}+\chi C_S(N,d,2,2^*)||\,|f(v)\nabla\eta|\,||_{L^2(U)} \\ & +\Lambda\varepsilon||\,|\eta\nabla(f(v))|\,|||_{L^2(U)}+\Lambda\varepsilon||\,|f(v)\nabla\eta|\,|||_{L^2(U)} \\ & +\Lambda\mathcal{K}(N,d,r,\varepsilon)||\eta f(v)||_{L^2(U)} \\ \leq \quad & (\chi C_S(N,d,2,2^*)+\Lambda\varepsilon)||\,|\eta\nabla(f(v))|\,||_{L^2(U)} \\ & +\frac{(\chi C_S(N,d,2,2^*)+\Lambda\varepsilon)\mathcal{K}_0}{\sigma}||f(v)||_{L^2(x_0+B_{\rho+\sigma})} \\ & +\Lambda\mathcal{K}(N,d,r,\varepsilon)||f(v)||_{L^2(x_0+B_{\rho+\sigma})}.\end{aligned}$$

Puisque χ vérifie (A.6), on peut trouver $\varepsilon>0$ ne dépendant que de $(\chi,C_S(N,d,2,2^*),\Lambda,\alpha_A)$, i.e. uniquement de $(d,\alpha_A,N_*,\chi,\Lambda)$, tel que $\chi C_S(N,d,2,2^*)+\varepsilon\Lambda<\alpha_A$; on a alors, dans (A.22),

$$\begin{aligned}&(\alpha_A-\chi C_S(N,d,2,2^*)-\varepsilon\Lambda)||\,|\eta\nabla(f(v))|\,||_{L^2(U)} \\ &\leq \left(\frac{2\Lambda_A\mathcal{K}_0}{\sigma}+\frac{(\chi C_S(N,d,2,2^*)+\Lambda\varepsilon)\mathcal{K}_0}{\sigma}+\Lambda\mathcal{K}(N,d,r,\varepsilon)\right)||f(v)||_{L^2(x_0+B_{\rho+\sigma})} \\ &\leq \left(\frac{2\Lambda_A\mathcal{K}_0}{\sigma}+\frac{(\chi C_S(N,d,2,2^*)+\Lambda\varepsilon)\mathcal{K}_0}{\sigma}+\frac{\Lambda\mathcal{K}(N,d,r,\varepsilon)d}{\sigma}\right)||f(v)||_{L^2(x_0+B_{\rho+\sigma})}\end{aligned}$$

(car $\sigma\leq d$).
Puisque $\eta\equiv 1$ sur x_0+B_ρ et $\alpha_A-\chi C_S(N,d,2,2^*)-\varepsilon\Lambda>0$, on en déduit donc $||\,|\nabla(f(v))|\,||_{L^2(x_0+B_\rho)}\leq\frac{C}{\sigma}||f(v)||_{L^2(x_0+B_{\rho+\sigma})}$ avec

$$C=\frac{2\Lambda_A\mathcal{K}_0+(\chi C_S(N,d,2,2^*)+\Lambda\varepsilon)\mathcal{K}_0+\Lambda\mathcal{K}(N,d,r,\varepsilon)d}{\alpha_A-\chi C_S(N,d,2,2^*)-\varepsilon\Lambda}$$

ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda)$, ce qui est le résultat du lemme lorsque $f \in \mathcal{C}^2(\mathbb{R})$ est convexe positive et $(f', f'') \in L^\infty(\mathbb{R})$.

Supposons maintenant que f n'est plus forcément de classe $\mathcal{C}^2$, mais juste de Stampacchia convexe positive.
Soit $(\theta_n)_{n\geq 1}$ un noyau régularisant sur $\mathbb{R}$ (i.e. pour tout $n \geq 1$, $\theta_n \in \mathcal{C}_c^\infty(]-1/n, 1/n[)$, $\theta_n \geq 0$ sur $\mathbb{R}$ et $\int_{\mathbb{R}} \theta_n = 1$).
Pour tout $n \geq 1$, la fonction $f_n = f * \theta_n$ est de classe $\mathcal{C}^2$, positive (car f et θ_n sont positifs) et convexe (car f est convexe et θ_n est positive). De plus, puisque $f'_n = f' * \theta_n$ (comme f'_n et $f' * \theta_n$ sont deux fonctions continues qui coincident — c'est trivial par dérivation sous l'intégrale — hors des points — en nombre fini — de discontinuité de f', ces fonctions coincident partout) et $f''_n = f' * \theta'_n$, on a $(f'_n, f''_n) \in L^\infty(\mathbb{R})$ (avec $||f'_n||_{L^\infty(\mathbb{R})} \leq ||f'||_{L^\infty(\mathbb{R})} ||\theta_n||_{L^1(\mathbb{R})} = ||f'||_{L^\infty(\mathbb{R})}$ et $||f''_n||_{L^\infty(\mathbb{R})} \leq ||f'||_{L^\infty(\mathbb{R})} ||\theta'_n||_{L^\infty(\mathbb{R})}$). Ainsi, par le raisonnement précédent, on a, pour tout $n \geq 1$,

$$|| \, |\nabla(f_n(v))| \, ||_{L^2(x_0+B_\rho)} \leq \frac{C}{\sigma} ||f_n(v)||_{L^2(x_0+B_{\rho+\sigma})} \tag{A.23}$$

avec le même C que précédemment.
Mais, lorsque $n \to \infty$, puisque $f_n \to f$ sur $\mathbb{R}$ et $||f'_n||_{L^\infty(\mathbb{R})} \leq ||f'||_{L^\infty(\mathbb{R})}$, $f_n(v) \to f(v)$ en étant majorée par $||f'||_{L^\infty(\mathbb{R})}|v| + \sup_{n\geq 1} |f_n(0)| \in L^2(U)$ ($\sup_{n\geq 1} |f_n(0)| < \infty$ car $f_n(0) \to f(0)$); donc, par convergence dominée, $f_n(u) \to f(u)$ dans $L^2(U)$.
Hors des points $\{s_1, \ldots, s_k\}$ de discontinuité de f', on a $f'_n = f' * \theta_n \to f'$; comme $\nabla v = 0$ presque partout sur $\{x \in U \mid v(x) \in \{s_1, \ldots, s_k\}\}$, on en déduit que $f'_n(v)\nabla u \to f'(u)\nabla u$ presque partout sur U, en étant majorée par $||f'||_{L^\infty(\mathbb{R})}|\nabla v| \in L^2(U)$. Par le théorème de convergence dominée, on a donc $\nabla(f_n(v)) = f'_n(v)\nabla v \to f'(v)\nabla v = \nabla(f(v))$ dans $L^2(U)$.
Ainsi, en passant à la limite dans (A.23), on en déduit le résultat du lemme pour f. ■

Corollaire A.1 *Sous les hypothèses (A.3)—(A.6), il existe C ne dépendant que de*

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda)$$

tel que, si $v \in H^1(U)$ vérifie (A.12), alors, pour toute fonction de Stampacchia convexe positive f, pour tout $x_0 \in U$ et tout $R > 0$ tel que $x_0 + B_R \subset U$,

$$||f(v)||_{L^\infty(x_0+B_{R/2})} \leq \frac{C}{R^{\frac{N}{2}}} ||f(v)||_{L^2(x_0+B_R)}.$$

Preuve du corollaire A.1
Soit $k \geq 1$; on pose, pour tout $n \geq 1$,

$$g_n(s) = \begin{cases} s^k & \text{si } 0 < s \leq n, \\ n^k + kn^{k-1}(s-n) & \text{si } s > n, \\ 0 & \text{si } s \leq 0. \end{cases}$$

$g_n \in \mathcal{C}^1([0, \infty[)$ est croissante, positive, convexe et vérifie $g'_n \in L^\infty(\mathbb{R}^+)$. Puisque f est positive, $g_n \circ f$ est donc positive, continue, $\mathcal{C}^1$ par morceaux sur $\mathbb{R}$ et $g_n \circ f' = g'_n(f)f' \in L^\infty(\mathbb{R})$. De plus, g_n étant convexe croissante sur $\mathbb{R}^+$ et f étant convexe positive sur $\mathbb{R}$, $g_n \circ f$ est convexe sur $\mathbb{R}$.
Par le lemme A.7, on a donc, pour tout $x_0 \in U$, pour tout $R > 0$ tel que $x_0 + B_R \subset U$ et tous (ρ, σ) tel que $\rho + \sigma \leq R$ (ce qui implique $x_0 + B_{\rho+\sigma} \subset U$),

$$|| \, |\nabla(g_n \circ f(v))| \, ||_{L^2(x_0+B_\rho)} \leq \frac{C_0}{\sigma} ||g_n \circ f(v)||_{L^2(x_0+B_{\rho+\sigma})}$$

avec C_0 ne dépendant que de $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$. On déduit du lemme A.5 qu'il existe $\mathcal{K}_2$ ne dépendant que de (C_0,N,N_*) tel que, pour tout (ρ,σ) tel que $\rho+\sigma\le R$,

$$||g_n\circ f(v)||_{L^{2^*}(x_0+B_\rho)}\le \mathcal{K}_2\rho^{1-\frac{N}{N_*}}\left(\frac{1}{\sigma}+\frac{1}{\rho}\right)||g_n\circ f(v)||_{L^2(x_0+B_{\rho+\sigma})}$$

et ce pour tout $n\ge 1$.
Or $0\le g_n(s)\le s^k$ pour tout $s\ge 0$ (car $g_n(s)=\int_0^s g_n'(t)\,dt$ et $g_n'(t)=kt^{k-1}\mathbf{1}_{]0,n[}+kn^{k-1}\mathbf{1}_{[n,\infty[}\le kt^{k-1}$ pour $t\ge 0$); on en déduit donc, puisque f est positive,

$$||g_n\circ f(v)||_{L^{2^*}(x_0+B_\rho)}\le \mathcal{K}_2\rho^{1-\frac{N}{N_*}}\left(\frac{1}{\sigma}+\frac{1}{\rho}\right)||f(v)^k||_{L^2(x_0+B_{\rho+\sigma})}.$$

Comme $g_n(s)\to s^k$ lorsque $s\ge 0$, le lemme de Fatou nous permet de voir, en faisant $n\to\infty$ dans l'inégalité précédente,

$$||f(v)^k||_{L^{2^*}(x_0+B_\rho)}\le \mathcal{K}_2\rho^{1-\frac{N}{N_*}}\left(\frac{1}{\sigma}+\frac{1}{\rho}\right)||f(v)^k||_{L^2(x_0+B_{\rho+\sigma})}. \qquad (A.24)$$

Par le lemme A.6 et (A.24), il existe donc de $\mathcal{K}_3$ ne dépendant que de $(\mathcal{K}_2,N,N_*,d)$ (car un R tel que $x_0+B_R\subset U$ vérifie forcément $R\le d$), i.e. uniquement de $(d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$, tel que

$$||f(v)||_{L^\infty(x_0+B_{R/2})}\le \frac{\mathcal{K}_3}{R^{\frac{N}{2}}}||f(v)||_{L^2(x_0+B_R)},$$

c'est à dire le résultat voulu. ■

Lemme A.8 *Sous les hypothèses (A.3)—(A.6), il existe* $\beta\in]0,1[$ *ne dépendant que de*

$$(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$$

tel que, pour tout $x_0\in U$ *et* $R>0$ *vérifiant* $x_0+B_R\subset U$*, si* $v\in H^1(U)$ *vérifie (A.12),* $v\ge 0$ *presque partout sur* x_0+B_R *et* $|\{x\in x_0+B_{R/2}\mid v(x)\ge 1\}|\ge\frac{1}{2}|B_{R/2}|$*, alors* $v\ge\beta$ *presque partout sur* $x_0+B_{R/4}$.

Preuve du lemme A.8
Soit $\varepsilon>0$ et $g:[0,\infty[\to]-\infty,-\varepsilon[$ définie par

$$g(s)=\begin{cases}-1-\varepsilon & \text{si } s\ge 1\\ (1-s)^3-1-\varepsilon & \text{si } 0\le s\le 1.\end{cases}$$

$g\in\mathcal{C}^2([0,\infty[)$ est inférieure à $-\varepsilon$ sur $[0,\infty[$; $f=-\ln(-g)+\ln(1+\varepsilon)$ est donc dans $\mathcal{C}^2([0,\infty[)$. On a:

$$f\ge 0 \text{ sur } [0,\infty[, \qquad (A.25)$$
$$(f',f'')\in L^\infty([0,\infty[), \qquad (A.26)$$
$$f''\ge f'^2 \text{ sur } [0,\infty[. \qquad (A.27)$$

(A.25) vient du fait que, sur $[0,\infty[$, $-1-\varepsilon\le g$, ce qui implique $\ln(-g)\le\ln(1+\varepsilon)$, donc $-\ln(-g)\ge -\ln(1+\varepsilon)$. (A.26) et (A.27) viennent des des formules $f'=-g'/g$ et $f''=-g''/g+g'^2/g^2=-g''/g+f'^2$ et du fait que $(g',g'')\in L^\infty([0,\infty[)$, $g''\ge 0$ sur $[0,\infty[$ et $g\le-\varepsilon$ sur $[0,\infty[$.
On étend f à $\mathbb{R}$ en posant, pour $s<0$, $f(s)=f(0)+sf'(0)$. Cette extension fait de f une fonction de classe $\mathcal{C}^1$ positive sur $\mathbb{R}$ (car $f(0)=-\ln(\varepsilon)+\ln(1+\varepsilon)\ge 0$ et $f'(0)=-(-3)/(-\varepsilon)<0$).
Sur $[0,\infty[$, puisque $f''\ge f'^2\ge 0$, f' est croissante; comme $f'_{|]-\infty,0]}=f'(0)$, on en déduit la croissance de f' sur $\mathbb{R}$. f est donc convexe sur $\mathbb{R}$.

f étant $\mathcal{C}^2$ sur $[0,\infty[$ et sur $]-\infty,0]$ avec une dérivée seconde bornée sur $\mathbb{R}$ (elle est nulle sur $]-\infty,0]$), f' est de Stampacchia.

Soit $\eta \in \mathcal{C}_c^\infty(x_0+B_R)$ telle que $0 \le \eta \le 1$ sur $\mathbb{R}^N$, $\eta \equiv 1$ sur $x_0+B_{R/2}$ et $|\nabla\eta| \le 2\mathcal{K}_0/R$ sur $\mathbb{R}^N$ (avec $\mathcal{K}_0$ ne dépendant que de N, cf lemme A.2).
f' étant de Stampacchia, on peut prendre $\varphi = f'(v)\eta^2$ dans (A.12); on obtient alors

$$2\int_U A\nabla v\cdot\nabla\eta f'(v)\eta + \int_U A\nabla v\cdot\nabla(f'(v))\eta^2 + \int_U \eta^2 f'(v)\mathbf{v}\cdot\nabla v = 0. \tag{A.28}$$

Or, presque partout sur U, $A\nabla v\cdot\nabla(f'(v)) = A\nabla v\cdot\nabla v f''(v)$. Puisque $v\ge 0$ là où η n'est pas nulle, on déduit de cette égalité et de (A.27), $A\nabla v\cdot\nabla(f'(v))\eta^2 \ge \alpha_A|\nabla v|^2(f'(v))^2\eta^2 = \alpha_A|\nabla(f(v))|^2\eta^2$ presque partout sur U.
Injectée dans (A.28), cette inégalité donne

$$\alpha_A||\,|\eta\nabla(f(v))|\,||^2_{L^2(U)} \le 2\Lambda_A||\,|\eta\nabla(f(v))|\,||_{L^2(U)}||\,|\nabla\eta|\,||_{L^2(U)} + ||\,|\eta\nabla(f(v))|\,||_{L^2(x_0+B_{2R})}||\,|\mathbf{v}|\,||_{L^2(x_0+B_R)},$$

soit, en simplifiant par $||\,|\eta\nabla(f(v))|\,||_{L^2(U)} < \infty$,

$$\alpha_A||\,|\eta\nabla(f(v))|\,||_{L^2(U)} \le \frac{4\Lambda_A\mathcal{K}_0}{R}R^{\frac{N}{2}} + \chi|B_R|^{\frac12-\frac{1}{N_*}} + \Lambda|B_R|^{\frac12-\frac1r}. \tag{A.29}$$

Or, pour tout $q\in]0,\infty[$, $|B_R|^{\frac12-\frac1q} = |B_1|^{\frac12-\frac1q}R^{\frac N2-\frac Nq} = C_0R^{1-\frac Nq}R^{\frac N2-1}$ avec C_0 ne dépendant que de (N,q). Lorsque $q\ge N$, on a $1-\frac Nq \ge 0$, donc $|B_R|^{\frac12-\frac1q} \le C_0 d^{1-\frac Nq}R^{\frac N2-1} \le C_1R^{\frac N2-1}$ où C_1 ne dépend que de (N,d,q). On trouve donc, en utilisant cette majoration pour $q=N_*$ et $q=r$ dans (A.29), et puisque $\eta\equiv 1$ sur $x_0+B_{R/2}$,

$$||\,|\nabla(f(v))|\,||_{L^2(x_0+B_{R/2})} \le C_2R^{\frac N2-1} \tag{A.30}$$

avec C_2 ne dépendant que de $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$.

Lorsque $v\ge 1$, on a $f(v)=0$ donc, en notant $E=\{x\in x_0+B_{R/2}\mid v(x)\ge 1\}$,

$$\begin{aligned}
|\overline{f(v)}^{x_0+B_{R/2}}| &\le |B_{R/2}|^{-1}\int_{(x_0+B_{R/2})\setminus E}|f(v)| \\
&\le \frac{(|x_0+B_{R/2}|-|E|)^{1-1/2^*}}{|B_{R/2}|}||f(v)||_{L^{2^*}(x_0+B_{R/2})} \\
&\le \frac{(\frac12|B_{R/2}|)^{1-1/2^*}}{|B_{R/2}|}||f(v)||_{L^{2^*}(x_0+B_{R/2})} \\
&\le \left(\frac12\right)^{1-1/2^*}|B_{R/2}|^{-1/2^*}||f(v)||_{L^{2^*}(x_0+B_{R/2})}.
\end{aligned}$$

Par le lemme A.4 et (A.30), on en déduit donc

$$\begin{aligned}
||f(v)||_{L^{2^*}(x_0+B_{R/2})} &\le \left|\overline{f(v)}^{x_0+B_{R/2}}\right||B_{R/2}|^{\frac{1}{2^*}} + \mathcal{K}_1\left(\frac R2\right)^{1-\frac{N}{N_*}}||\,|\nabla(f(v))|\,||_{L^2(x_0+B_{R/2})} \\
&\le \left(\frac12\right)^{1-\frac{1}{2^*}}||f(v)||_{L^{2^*}(x_0+B_{R/2})} + 2^{\frac{N}{N_*}-1}C_2\mathcal{K}_1R^{\frac N2-\frac{N}{N_*}}
\end{aligned}$$

($\mathcal{K}_1$ ne dépend que de (N,N_*)), soit

$$\left(1-\left(\frac12\right)^{1-\frac{1}{2^*}}\right)||f(v)||_{L^{2^*}(x_0+B_{R/2})} \le 2^{\frac{N}{N_*}-1}C_2\mathcal{K}_1R^{\frac N2-\frac{N}{N_*}} \tag{A.31}$$

avec $1-(1/2)^{1-1/2^*}>0$ ne dépendant que de N_*.
f étant une fonction de Stampacchia convexe positive, par le corollaire A.1, on a

$$||f(v)||_{L^\infty(x_0+B_{R/4})} \leq \frac{C_3}{R^{\frac{N}{2}}}||f(v)||_{L^2(x_0+B_{R/2})},$$

où C_3 ne dépend que de $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$. Donc, par (A.31),

$$\begin{aligned}||f(v)||_{L^\infty(x_0+B_{R/4})} &\leq \frac{C_3}{R^{\frac{N}{2}}}|B_R|^{\frac{1}{2}-\frac{1}{2^*}}||f(u)||_{L^{2^*}(x_0+B_{R/2})}\\ &\leq C_4 R^{\frac{N}{2}-\frac{N}{2^*}-\frac{N}{N_*}},\end{aligned}$$

avec $C_4>0$ ne dépendant que de $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$.
Comme $\frac{N}{2}-\frac{N}{2^*}-\frac{N}{N_*}=\frac{N}{2}-N\left(\frac{1}{2}-\frac{1}{N_*}\right)-\frac{N}{N_*}=0$, on a en fait prouvé que

$$||f(v)||_{L^\infty(x_0+B_{R/4})} \leq C_4.$$

Soit $F=\{x\in x_0+B_{R/4}\mid v(x)\leq 1\}$; puisque $f=-\ln((\cdot-1)^3+1+\varepsilon)+\ln(1+\varepsilon)$ sur $[0,1]$ et $v\geq 0$ presque partout sur $x_0+B_{R/4}$, on a, presque partout sur F, $f(v)=-\ln((v-1)^3+1+\varepsilon)+\ln(1+\varepsilon)\leq C_4$. Cette égalité étant vérifiée pour tout $\varepsilon>0$ (et C_4 ne dépendant pas de ε), on en déduit que, presque partout sur F, $(v-1)^3\geq -1+e^{-C_4}$. v étant inférieure à 1 sur F, cela implique donc, presque partout sur F, $|v-1|\leq(1-e^{-C_4})^{1/3}$, d'où $v\geq 1-(1-e^{-C_4})^{1/3}$.
Posons $\beta=1-(1-e^{-C_4})^{1/3}\in]0,1[$. β ne dépend que de $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda)$, on a prouvé que $v\geq\beta$ presque partout sur F; comme, sur $(x_0+B_{R/4})\backslash F$, on a $v\geq 1>\beta$, le lemme est prouvé. ■

Nous pouvons maintenant démontrer la proposition clef de cette section.
Preuve de la proposition A.1
L'estimation sur $||v||_{L^\infty(x_0+B_R)}$ découle du corollaire (A.1) appliqué à $f=Id$; on déduit de cette estimation que, pour $\rho\leq R$, $\mathrm{supess}(v,x_0+B_\rho)$ et $\mathrm{infess}(v,x_0+B_\rho)$ sont finis.
Si $\mathrm{supess}(v,x_0+B_R)=\mathrm{infess}(v,x_0+B_R)$, alors v est égale presque partout à une constante sur x_0+B_R et il n'y a donc rien à prouver; on suppose donc que $\omega(v,x_0,R)\neq 0$.

Soit

$$v_1=2\left(1+\frac{2v-\mathrm{supess}(u,x_0+B_R)-\mathrm{infess}(v,x_0+B_R)}{\mathrm{supess}(v,x_0+B_R)-\mathrm{infess}(v,x_0+B_R)}\right)$$

et

$$v_2=2\left(1-\frac{2v-\mathrm{supess}(u,x_0+B_R)-\mathrm{infess}(v,x_0+B_R)}{\mathrm{supess}(v,x_0+B_R)-\mathrm{infess}(v,x_0+B_R)}\right).$$

v_1 et v_2 ayant été construites à partir de v en ajoutant et en multipliant par des constantes, elles vérifient (A.12). Presque partout sur x_0+B_R, v_1 et v_2 sont positives et, puisque $v_1+v_2\equiv 2$, on a $\{x\in x_0+B_{R/2}\mid v_1\geq 1\}\cup\{x\in x_0+B_{R/2}\mid v_2\geq 1\}=x_0+B_{R/2}$; il existe donc $i\in\{1,2\}$ tel que $|\{x\in x_0+B_{R/2}\mid v_i\geq 1\}|\geq\frac{1}{2}|B_{R/2}|$.
Avec β donné dans le lemme A.8 ($\beta\in]0,1[$ ne dépend que de $(N,d,\alpha_A,\Lambda_A,N_*,r,\Lambda)$), on a donc $v_i\geq\beta$ presque partout sur $x_0+B_{R/4}$.

i) Si $i=1$, cela implique, presque partout sur $x_0+B_{R/4}$,

$$2v\geq\mathrm{supess}(v,x_0+B_R)+\mathrm{infess}(v,x_0+B_R)+\left(\frac{\beta}{2}-1\right)(\mathrm{supess}(v,x_0+B_R)-\mathrm{infess}(v,x_0+B_R)).$$

On en déduit que

$$2\mathrm{infess}(v,x_0+B_{R/4})\geq\mathrm{supess}(v,x_0+B_R)+\mathrm{infess}(v,x_0+B_R)+\left(\frac{\beta}{2}-1\right)\omega(v,x_0,R),$$

ce qui donne

$$\begin{aligned}&2(\mathrm{supess}(v, x_0 + B_{R/4}) - \mathrm{infess}(v, x_0 + B_{R/4}))\\ \leq\ & 2(\mathrm{supess}(v, x_0 + B_{R/4}) - \mathrm{supess}(v, x_0 + B_R)) + \left(2 - \frac{\beta}{2}\right)\omega(v, x_0, R).\end{aligned}$$

Mais $\mathrm{supess}(v, x_0 + B_{R/4}) - \mathrm{supess}(v, x_0 + B_R) \leq 0$, et on a finalement $2\omega(v, x_0, R/4) \leq (2 - \frac{\beta}{2})\omega(v, x_0, R)$, c'est à dire le résultat voulu avec $\alpha = 1 - \beta/4$.

ii) Si $i = 2$, alors on a, presque partout sur $x_0 + B_{R/4}$,

$$\left(1 - \frac{\beta}{2}\right)\omega(v, x_0, R) + \mathrm{supess}(v, x_0 + B_R) + \mathrm{infess}(v, x_0 + B_R) \geq 2v,$$

ce qui implique

$$2\mathrm{supess}(v, x_0 + B_{R/4}) \leq \mathrm{supess}(v, x_0 + B_R) + \mathrm{infess}(v, x_0 + B_R) + \left(1 - \frac{\beta}{2}\right)\omega(v, x_0, R),$$

d'où

$$\begin{aligned}&2(\mathrm{supess}(v, x_0 + B_{R/4}) - \mathrm{infess}(v, x_0 + B_{R/4}))\\ \leq\ & 2(\mathrm{infess}(v, x_0 + B_R) - \mathrm{infess}(v, x_0 + B_{R/4})) + \left(2 - \frac{\beta}{2}\right)\omega(v, x_0, R).\end{aligned}$$

Comme $\mathrm{infess}(v, x_0+B_R) - \mathrm{infess}(v, x_0+B_{R/4}) \leq 0$, on en déduit $2\omega(v, x_0, R/4) \leq (2-\frac{\beta}{2})\omega(v, x_0, R)$, c'est à dire le résultat souhaité avec $\alpha = 1 - \beta/4$.

■

A.3 A l'intérieur de Ω

A.3.1 Le problème de Dirichlet sur des petits ouverts

Commençons par un lemme de Stampacchia.

Lemme A.9 *Soit $F : \mathbb{R}^+ \to \mathbb{R}^+$ une fonction décroissante. S'il existe $\alpha > 0$, $\beta > 1$ et $C > 0$ tels que*

$$\forall h > k \geq 0\,,\ F(h) \leq \frac{C^\alpha}{(h-k)^\alpha} F(k)^\beta$$

et si

$$H = 2^{\frac{1}{\alpha}} C F(0)^{\frac{\beta-1}{\alpha}} \sum_{i=0}^{\infty} \frac{1}{\left(2^{\frac{\beta-1}{\alpha}}\right)^i},$$

alors $F(H) = 0$.

Preuve du lemme A.9
Si $F(0) = 0$, le lemme est trivial; on suppose donc que $F(0) > 0$.
Soit $h_0 = 0$ et, pour $n \geq 1$,

$$h_n = 2^{\frac{1}{\alpha}} C F(0)^{\frac{\beta-1}{\alpha}} \sum_{i=0}^{n-1} \frac{1}{\left(2^{\frac{\beta-1}{\alpha}}\right)^i}.$$

On constate par récurrence sur n que l'on a, pour tout $n \geq 0$, $F(h_n) \leq F(0)/2^n$; en effet, c'est évident pour $n = 0$ et, lorsque $n \geq 1$, puisque

$$h_n - h_{n-1} = \frac{2^{\frac{1}{\alpha}} C F(0)^{\frac{\beta-1}{\alpha}}}{\left(2^{\frac{\beta-1}{\alpha}}\right)^{n-1}},$$

on a

$$\begin{aligned} F(h_n) &\leq \frac{C^\alpha}{(h_n - h_{n-1})^\alpha} F(h_{n-1})^\beta \\ &\leq \frac{C^\alpha 2^{(n-1)(\beta-1)}}{2C^\alpha F(0)^{\beta-1}} \times \frac{F(0)^\beta}{2^{(n-1)\beta}} \\ &\leq \frac{F(0)}{2^n}. \end{aligned}$$

F étant décroissante, on en déduit que, pour tout $n \geq 0$, puisque $h_n \leq H$, $F(H) \leq F(h_n) \leq F(0)/2^n$; en faisant tendre n vers l'infini dans cette dernière inégalité, on obtient $F(H) \leq 0$ ce qui conclut la démonstration du lemme. ■

Par un léger abus de notation, lorsque U est un ouvert inclus dans Ω, on peut considérer $\mathcal{L} \in (W^{1,p'}_{\Gamma_d}(\Omega))'$ comme un élément de $W^{-1,p}(U)$; en effet, si $\varphi \in W^{1,p'}_0(U)$, on note $\widetilde{\varphi} \in W^{1,p'}_{\Gamma_d}(\Omega)$ son extension par 0 hors de Ω et on pose alors $\langle \widetilde{\mathcal{L}}, \varphi \rangle_{(W^{1,p'}_0(U))', W^{1,p'}_0(U)} = \langle \mathcal{L}, \widetilde{\varphi} \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)}$. Etant donné que l'extension par 0 hors de U est linéaire continue $W^{1,p'}_0(U) \to W^{1,p'}_{\Gamma_d}(\Omega)$, ceci définit bien un $\widetilde{\mathcal{L}} \in W^{-1,p}(U)$ ($\widetilde{\mathcal{L}}$ est en fait l'image de $\mathcal{L}$ par l'application duale de l'extension à Ω par 0 hors de U). On notera dans la suite, par abus, $\widetilde{\mathcal{L}} = \mathcal{L}$; de même, $L \in (H^1_0(U))'$ désignera aussi la restriction de $\mathcal{L}$ à $H^1_0(U)$.

Lemme A.10 *Sous les hypothèses (A.3)—(A.7), il existe $\delta > 0$ ne dépendant que de*

$$(N, d, \alpha_A, N_*, \chi, r, \Lambda)$$

tel que, pour tout ouvert $U \subset \Omega$ de mesure inférieure à δ, le problème

$$\left\{ \begin{array}{l} w \in H^1_0(U)\,, \\ \displaystyle\int_U A\nabla w \cdot \nabla\varphi + \int_U \varphi \mathbf{v} \cdot \nabla w = \langle L, \varphi \rangle_{(H^1_0(U))', H^1_0(U)}\,,\ \forall \varphi \in H^1_0(U) \end{array} \right. \tag{A.32}$$

admet une unique solution. Il existe de plus C ne dépendant que de $(N, d, \alpha_A, N_*, \chi, r, \Lambda, p, \Lambda_L)$ tel que cette solution w vérifie $||w||_{L^\infty(U)} \leq C|U|^{\frac{1}{N}-\frac{1}{p}}$.

Preuve du lemme A.10
Soit U un ouvert inclus dans Ω. On a, par (A.5), pour tout $\varphi \in H^1_0(U)$,

$$\begin{aligned} &\int_U A\nabla\varphi \cdot \nabla\varphi + \int_U \varphi \mathbf{v} \cdot \nabla\varphi \\ &\geq \alpha_A || \, |\nabla\varphi| \, ||^2_{L^2(U)} - || \, |\varphi \mathbf{v}| \, ||_{L^2(U)} || \, |\nabla\varphi| \, ||_{L^2(U)} \\ &\geq \alpha_A || \, |\nabla\varphi| \, ||^2_{L^2(U)} - \chi ||\varphi||_{L^{2^*}(U)} || \, |\nabla\varphi| \, ||_{L^2(U)} - \Lambda ||\varphi||_{L^{\frac{2r}{r-2}}(U)} || \, |\nabla\varphi| \, ||_{L^2(U)}. \end{aligned}$$

Or d est aussi un majorant du diamètre de U, donc (A.1) est valable avec Ω remplacé par U. De plus, puisque $\frac{2r}{r-2} < \frac{2N}{N-2}$, on peut prendre $q \in]\frac{2r}{r-2}, \frac{2N}{N-2}[$ (le choix d'un tel q ne dépend que de r et N); on déduit donc de l'inégalité précédente et de (A.1) que

$$\int_U A\nabla\varphi \cdot \nabla\varphi + \int_U \varphi \mathbf{v} \cdot \nabla\varphi \tag{A.33}$$

$$\geq (\alpha_A - \chi C_S(N, d, 2, 2^*)) || \, |\nabla\varphi| \, ||^2_{L^2(U)} - \Lambda |U|^{\frac{r-2}{2r} - \frac{1}{q}} ||\varphi||_{L^q(U)} || \, |\nabla\varphi| \, ||_{L^2(U)}$$

$$\geq (\alpha_A - \chi C_S(N, d, 2, 2^*) - \Lambda |U|^{\frac{r-2}{2r} - \frac{1}{q}} C_S(N, d, 2, q)) || \, |\nabla\varphi| \, ||^2_{L^2(x_0 + B_R)}. \tag{A.34}$$

Or $\frac{r-2}{2r}-\frac{1}{q}>0$ donc, puisque χ vérifie (A.6), il existe $\delta>0$ ne dépendant que de $(\alpha_A,\chi,N,d,N_*,\Lambda,r,q)$, i.e. uniquement de $(N,d,\alpha_A,N_*,\chi,r,\Lambda)$, tel que, pour tout U vérifiant $|U|\le\delta$,

$$\begin{aligned}&\alpha_A-\chi C_S(N,d,2,2^*)-\Lambda|U|^{\frac{r-2}{2r}-\frac{1}{q}}C_S(N,d,2,q)\\ &\quad\ge\ \alpha_A-\chi C_S(N,d,2,2^*)-\Lambda\delta^{\frac{r-2}{2r}-\frac{1}{q}}C_S(N,d,2,q)=C_0>0.\end{aligned}\tag{A.35}$$

Pour tout U ouvert de mesure inférieure à δ, la forme bilinéaire apparaissant dans (A.32) est donc coercitive sur $H_0^1(U)$; l'existence et l'unicité d'une solution à (A.32) est alors la conséquence directe du théorème de Lax-Milgram.

Prouvons maintenant l'estimation sur $||w||_{L^\infty(U)}$.
Soit, pour $k\ge0$, $S_k(t)=t-T_k(t)$ (i.e. S_k est continue, affine par morceaux, nulle sur $[-k,k]$ et de dérivée égale à 1 hors de $[-k,k]$). Comme $\nabla(S_k(w))=\mathbf{1}_{\{|w|\ge k\}}\nabla w$, on a $\nabla w=\nabla(S_k(w))$ là où $S_k(w)\neq0$; ainsi, en prenant $\varphi=S_k(w)\in H_0^1(U)$ dans (A.32), on obtient

$$\begin{aligned}\langle L,S_k(w)\rangle_{(H_0^1(U))',H_0^1(U)} &= \int_U A\nabla w\cdot\nabla S_k(w)+\int_U S_k(w)\mathbf{v}\cdot\nabla v\\ &= \int_U A\nabla(S_k(w))\cdot\nabla(S_k(w))+\int_U S_k(w)\mathbf{v}\cdot\nabla(S_k(w)).\end{aligned}\tag{A.36}$$

Comme $|U|\le\delta$, (A.36), (A.34) et (A.35) donnent donc

$$\begin{aligned}C_0||\,|\nabla(S_k(w))|\,||_{L^2(U)}^2 &\le \langle L,S_k(w)\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\\ &= \langle\mathcal{L},S_k(w)\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))',W^{1,p'}_{\Gamma_d}(\Omega)}\\ &\le \Lambda_L||S_k(w)||_{W^{1,p'}(U)},\end{aligned}\tag{A.37}$$

avec C_0 ne dépendant que de $(N,d,\alpha_A,N_*,\chi,r,\Lambda)$.
Comme $S_k(w)\in W_0^{1,p'}(U)$, on a, par l'inégalité de Poincaré et puisque d est un majorant du diamètre de U,

$$||S_k(w)||_{W^{1,p'}(U)}=||S_k(w)||_{L^{p'}(U)}+||\,|\nabla(S_k(w))|\,||_{L^{p'}(U)}\le(d+1)||\,|\nabla(S_k(w))|\,||_{L^{p'}(U)}.\tag{A.38}$$

De plus, $\nabla(S_k(w))=0$ hors de $E_k=\{|w|\ge k\}$, donc

$$||\,|\nabla(S_k(w))|\,||_{L^{p'}(U)}\le|E_k|^{\frac{1}{p'}-\frac{1}{2}}||\,|\nabla(S_k(w))|\,||_{L^2(U)}.\tag{A.39}$$

(A.37), (A.38) et (A.39) donnent donc

$$C_0||\,|\nabla(S_k(w))|\,||_{L^2(U)}\le(1+d)\Lambda_L|E_k|^{\frac{1}{2}-\frac{1}{p}}.\tag{A.40}$$

Soit $h>k$; comme $|S_k(w)|\ge h-k$ sur E_h, on a, par (A.1) et puisque $\nabla(S_k(w))=0$ hors de E_k,

$$\begin{aligned}(h-k)|E_h|^{\frac{N-1}{N}} &\le ||S_k(w)||_{L^{\frac{N}{N-1}}(U)}\\ &\le C_S(N,d,1,\frac{N}{N-1})||\,|\nabla(S_k(w))|\,||_{L^1(U)}\\ &\le C_S(N,d,1,\frac{N}{N-1})|E_k|^{\frac{1}{2}}||\,|\nabla(S_k(w))|\,||_{L^2(U)}.\end{aligned}$$

Utilisée dans (A.40), cette inégalité donne donc, pour tout $h>k\ge0$,

$$|E_h|\le\frac{C_1^\alpha}{(h-k)^\alpha}|E_k|^\beta\tag{A.41}$$

avec C_1 ne dépendant que de $(N, d, \alpha_A, N_*, \chi, r, \Lambda, \Lambda_L)$, $\alpha = \frac{N}{N-1} > 0$ et $\beta = (1-\frac{1}{p})\frac{N}{N-1} > 1$ (rappelons que $p > N$).
Le lemme A.9 nous donne alors

$$H_0 = 2^{\frac{1}{\alpha}} C_1 \sum_{i=0}^{\infty} \frac{1}{\left(2^{\frac{\beta-1}{\alpha}}\right)^i},$$

ne dépendant que de (C_1, α, β), i.e. ne dépendant que de $(N, d, \alpha_A, N_*, \chi, r, \Lambda, p, \Lambda_L)$, tel que

$$|E_{H_0|E_0|^{\frac{\beta-1}{\alpha}}}| = 0,$$

ce qui signifie $|w| \le H_0|E_0|^{\frac{\beta-1}{\alpha}}$ presque partout sur U. Comme $\frac{\beta-1}{\alpha} = \frac{1}{N} - \frac{1}{p}$ et $|E_0| = |U|$, cela conclut la preuve de ce lemme. ■

A.3.2 Continuité höldérienne sur les compacts de Ω

Tout d'abord un petit lemme technique, puis le résultat proprement dit.

Lemme A.11 *Soit K un compact de Ω, $R_0 < \mathrm{dist}(K, \partial\Omega)/2$ et $u : \Omega \to \mathbb{R}$ essentiellement bornée par C sur K. S'il existe $\delta \in]0,1[$ et C' tels que, pour tout $x \in K$ et tout $n \ge 0$,*

$$\omega\left(u, x, \frac{R_0}{4^n}\right) \le C'\delta^n,$$

alors il existe $\kappa > 0$ ne dépendant que de δ et C'' ne dépendant que de (C, δ, C', R_0) tels que $u \in \mathcal{C}^{0,\kappa}(K)$ et $||u||_{\mathcal{C}^{0,\kappa}(K)} \le C''$.

Remarque A.3 *Le fait que l'on parle de $\omega(u, x, R_0/4^n)$ suppose que u est essentiellement bornée sur $x + B_{R_0/4^n}$ pour tout $x \in K$ et tout $n \ge 0$.*

Preuve du lemme A.11
Soit $R \in]0, R_0]$; il existe $n \ge 0$ tel que $R \in]R_0/4^{n+1}, R_0/4^n]$.
On a alors $R_0/(4R) \le 4^n$, soit $n\ln(4) \ge \ln(R_0/4) - \ln(R)$, d'où $n \ge (\ln(4))^{-1}\ln(R_0/4) - (\ln(4))^{-1}\ln(R)$; puisque $\delta \in]0,1[$, cela donne

$$n\ln(\delta) \le \ln(\delta)(\ln(4))^{-1}\ln(R_0/4) - \ln(\delta)(\ln(4))^{-1}\ln(R) = \ln((R_0/4)^{\ln(\delta)/\ln(4)}) + \ln(R^{-\ln(\delta)/\ln(4)}).$$

Ainsi, pour un tel n,

$$\delta^n \le \left(\frac{R_0}{4}\right)^{\frac{\ln(\delta)}{\ln(4)}} \times R^{-\frac{\ln(\delta)}{\ln(4)}} = C_0 R^\kappa,$$

avec C_0 ne dépendant que de (δ, R_0) et $\kappa = -\ln(\delta)/\ln(4) > 0$ ne dépendant que de δ.
De plus, puisque $R \le R_0/4^n$, on a $\omega(u, x, R) \le \omega(u, x, R_0/4^n)$, donc

$$\omega(u, x, R) \le C'C_0R^\kappa = C_1R^\kappa, \tag{A.42}$$

où C_1 ne dépend que de (δ, C', R_0).

Cette inégalité permet de voir que, pour tout $x \in K$,

$$\lim_{R\to 0} \frac{1}{|B_R|}\int_{x+B_R} u(t)\,dt$$

existe dans $\mathbb{R}$; pour cela, on voit que lorsque $R \le R_0$, on a

$$\mathrm{infess}(u, x, R) \le \frac{1}{|B_R|}\int_{x+B_R} u(t)\,dt \le \mathrm{supess}(u, x, R).$$

Mais, par (A.42), $\mathrm{supess}(u,x,R)-\mathrm{infess}(u,x,R)=\omega(u,x,R)\to 0$ lorsque $R\to 0$, et $\lim_{R\to 0}\mathrm{supess}(u,x,R)$ existe dans $\mathbb{R}$ (car $R\to \mathrm{supess}(u,x,R)$ est croissante et bornée, lorsque $R\leq R_0$, par $||u||_{L^\infty(x_0+B_{R_0})}$ — on a supposé, en parlant de $\omega(u,x,R_0)$, que cette dernière quantité était finie). On en déduit que $\lim_{R\to 0}\mathrm{infess}(u,x,R)$ existe dans $\mathbb{R}$ et vaut $\lim_{R\to 0}\mathrm{supess}(u,x,R)$. Par le théorème des gendarmes, $\lim_{R\to 0}\frac{1}{|B_R|}\int_{x+B_R}u$ existe donc aussi dans $\mathbb{R}$.
Comme cette limite vaut, pour presque tout $x\in K$ (en tous les points de Lebesgue de u), $u(x)$, quitte à changer u sur un ensemble de mesure nulle on peut supposer que

$$u(x)=\lim_{R\to 0}\frac{1}{|B_R|}\int_{x+B_R}u(t)\,dt \quad \text{pour tout } x\in K.$$

Dès que $R\leq R_0$, on a pour tout $y\in(x+B_R)\cap K$ et tout $\rho<R-||y-x||$, $y+B_\rho\subset x+B_R$, donc

$$\mathrm{infess}(u,x,R)\leq\frac{1}{|B_\rho|}\int_{y+B_\rho}u\leq\mathrm{supess}(u,x,R)$$

ce qui donne, en faisant $\rho\to 0$ (rappelons que $y\in K$),

$$\mathrm{infess}(u,x,R)\leq u(y)\leq\mathrm{supess}(u,x,R)$$

c'est à dire, par (A.42),

$$\sup_{(x+B_R)\cap K}u-\inf_{(x+B_R)\cap K}u\leq\omega(u,x,R)\leq C_1R^\kappa$$

La continuité höldérienne de u sur K est maintenant aisée à voir. En effet, pour tous $(x,y)\in K^2$,

i) Si $|x-y|<R_0$, alors, en prenant $R\in]|x-y|,R_0]$, $|u(x)-u(y)|\leq\sup_{x+B_R}u-\inf_{x+B_R}u\leq C_1R^\kappa$ c'est à dire, en faisant $R\to|x-y|$, $|u(x)-u(y)|\leq C_1|x-y|^\kappa$,

ii) Si $|x-y|\geq R_0$, $|u(x)-u(y)|\leq 2||u||_{L^\infty(K)}\leq\frac{2||u||_{L^\infty(K)}}{R_0^\kappa}|x-y|^\kappa\leq\frac{2C}{R_0^\kappa}|x-y|^\kappa$.

En notant donc $C_2=\sup(C_1,2C/R_0^\kappa)$, qui ne dépend que de (C,δ,C',R_0), on a, pour tous $(x,y)\in K^2$, $|u(x)-u(y)|\leq C_2|x-y|^\kappa$; associée à la majoration supposée de $||u||_{L^\infty(K)}$ par C, cette inégalité donne le résultat du lemme. ■

Proposition A.2 *Soit K un compact de Ω et $M>0$. Sous les hypothèses (A.3)—(A.7), il existe*

$\kappa>0$ *ne dépendant que de* $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda,p)$

et

C *ne dépendant que de* $(N,d,\alpha_A,\Lambda_A,N_*,\chi,r,\Lambda,p,\Lambda_L,K,M)$

tels que, si $u\in H^1(\Omega)$ vérifie $||u||_{L^2(\Omega)}\leq M$ et

$$\int_\Omega A\nabla u\cdot\nabla\varphi+\int_\Omega\varphi\mathbf{v}\cdot\nabla u=\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\ ,\ \forall\varphi\in H^1_0(\Omega)\,, \tag{A.43}$$

alors $u\in\mathcal{C}^{0,\kappa}(K)$ et $||u||_{\mathcal{C}^{0,\kappa}(K)}\leq C$.

Preuve de la proposition A.2
Prenons le δ donné par le lemme A.10 (δ ne dépend que de $(N,d,\alpha_A,N_*,\chi,r,\Lambda)$) et $R_0<\mathrm{dist}(K,\partial\Omega)/2$ tel que $|B_{2R_0}|\leq\delta$ (R_0 ne dépend que de K et δ, i.e. de $(N,d,\alpha_A,N_*,\chi,r,\Lambda,K)$).

Etape 0: préliminaires.

Soit $x \in K$ et $R \leq R_0$. Par le lemme A.10, et puisque $x + B_{2R} \subset \Omega$, il existe $w_{x,R}$ solution de (A.32) lorsque $U = x + B_{2R}$. $w_{x,R}$ vérifie de plus

$$||w_{x,R}||_{L^\infty(x+B_{2R})} \leq C_0|B_{2R}|^{\frac{1}{N}-\frac{1}{p}} \leq C_1 R^{1-\frac{N}{p}}, \tag{A.44}$$

avec C_0 et C_1 ne dépendant que de $(N, d, \alpha_A, N_*, \chi, r, \Lambda, p, \Lambda_L)$.
Soit $\varphi \in H_0^1(x + B_{2R})$; en notant $\widetilde{\varphi} \in H_0^1(\Omega)$ l'extension de φ à Ω par 0 hors de Ω, on a

$$\begin{aligned}
\int_{x+B_{2R}} A\nabla u \cdot \nabla\varphi + \int_{x+B_{2R}} \varphi \mathbf{v} \cdot \nabla u &= \int_\Omega A\nabla u \cdot \nabla\widetilde{\varphi} + \int_\Omega \widetilde{\varphi}\mathbf{v} \cdot \nabla u \\
&= \langle L, \widetilde{\varphi}\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} \\
&= \langle L, \varphi\rangle_{(H^1_0(x+B_{2R}))', H^1_0(x+B_{2R})}
\end{aligned}$$

(avec le petit abus de notation concernant $L \in (H_0^1(x + B_{2R}))'$ dont nous avons déja parlé).
Ainsi, $v_{x,R} = u - w_{x,R} \in H^1(x + B_{2R})$ vérifie (A.12) avec $U = x + B_{2R}$.
De la proposition A.1, on déduit l'existence de $\alpha \in]0, 1[$ et $C_2 > 0$ ne dépendant que de

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda)$$

tels que

$$||v_{x,R}||_{L^\infty(x+B_R)} \leq C_2 R^{-\frac{N}{2}} ||v_{x,R}||_{L^2(x+B_{2R})} \tag{A.45}$$

$$\omega(v_{x,R}, x, R/4) \leq \alpha\omega(v_{x,R}, x, R). \tag{A.46}$$

Or, par (A.44),

$$\begin{aligned}
||v_{x,R}||_{L^2(x+B_{2R})} &\leq ||u||_{L^2(x+B_{2R})} + ||w_{x,R}||_{L^2(x+B_{2R})} \\
&\leq ||u||_{L^2(\Omega)} + |B_{2R}|^{\frac{1}{2}} ||w_{x,R}||_{L^\infty(x+B_{2R})} \\
&\leq M + |B_{2R_0}|^{\frac{1}{2}} C_1 R_0^{1-\frac{N}{p}} = C_3,
\end{aligned}$$

où C_3 ne dépend que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$. (A.45) peut donc se re-écrire

$$||v_{x,R}||_{L^\infty(x+B_R)} \leq C_2 C_3 R^{-\frac{N}{2}}. \tag{A.47}$$

Etape 1: borne essentielle de u sur K.
De (A.44) et (A.47) appliqués à $R = R_0$, on déduit, puisque $u = w_{x,R_0} + v_{x,R_0}$ sur $x + B_{R_0}$,

$$||u||_{L^\infty(x+B_{R_0})} \leq C_1 R_0^{1-\frac{N}{p}} + C_2 C_3 R_0^{-\frac{N}{2}} = C_4, \tag{A.48}$$

avec C_4 ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$.
Mais on a $||u||_{L^\infty(K)} \leq \sup_{x\in K} ||u||_{L^\infty(x+B_{R_0})}$. En effet, pour voir cela, on commence par recouvrir K par un nombre fini de boules $(x_i + B_{R_0})_{i\in[1,l]}$ (un nombre dénombrable serait en fait suffisant); si on avait $||u||_{L^\infty(K)} > \sup_{i\in[1,l]} ||u||_{L^\infty(x_i+B_{R_0})}$, alors il existerait un ensemble A de mesure non nulle, inclus dans K, tel que $|u| > \sup_{i\in[1,l]} ||u||_{L^\infty(x_i+B_{R_0})}$ sur A; A étant de mesure non nulle et inclus dans K, il existe $i \in [1, l]$ tel que $A \cap (x_i + B_{R_0})$ soit de mesure non-nulle; on aurait alors $|u| > ||u||_{L^\infty(x_i+B_{R_0})}$ sur un ensemble inclus dans $x_i + B_{R_0}$ et de mesure non-nulle, ce qui est une contradiction.
On en déduit donc

$$||u||_{L^\infty(K)} \leq C_4.$$

Etape 2: continuité höldérienne de u.

Comme $u = v_{x,R} + w_{x,R}$ sur $x + B_{R/4}$, il est aisé de constater que $\omega(u,x,R/4) \leq \omega(v_{x,R},x,R/4) + \omega(w_{x,R},x,R/4)$.
Or, pour tout $\rho \leq 2R$,

$$\omega(w_{x,R},x,\rho) \leq 2||w_{x,R}||_{L^\infty(x+B_\rho)} \leq 2||w_{x,R}||_{L^\infty(x+B_{2R})} \leq 2C_1 R^{1-\frac{N}{p}}, \tag{A.49}$$

donc

$$\omega(u,x,R/4) \leq \omega(v_{x,R},x,R/4) + 2C_1 R^{1-\frac{N}{p}}.$$

De plus, par (A.46), $\omega(v_{x,R},x,R/4) \leq \alpha\omega(v_{x,R},x,R)$; mais, puisque $v_{x,R} = u - w_{x,R}$ sur $x + B_R$, par (A.49), on a $\omega(v_{x,R},x,R) \leq \omega(u,x,R) + \omega(w_{x,R},x,R) \leq \omega(u,x,R) + 2C_1 R^{1-\frac{N}{p}}$, d'où

$$\omega(u,x,R/4) \leq \alpha\omega(u,x,R) + 2C_1 R^{1-\frac{N}{p}} + 2\alpha C_1 R^{1-\frac{N}{p}} = \alpha\omega(u,x,R) + C_5 R^\beta, \tag{A.50}$$

avec $\beta = 1 - \frac{N}{p}$ et C_5 ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L)$.

Soit $n \geq 0$; par (A.50) appliqué à $R = R_0/4^n \in]0, R_0]$, on a

$$\omega\left(u,x,\frac{R_0}{4^{n+1}}\right) \leq \alpha\omega\left(u,x,\frac{R_0}{4^n}\right) + \frac{C_5 R_0^\beta}{(4^\beta)^n} = \alpha\omega\left(u,x,\frac{R_0}{4^n}\right) + \frac{C_6}{\zeta^n}$$

avec $\zeta > 1$ ne dépendant que de (N,p) et C_6 ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K)$.
Ceci nous permet de voir par récurrence sur n que, pour tout $n \geq 1$,

$$\omega\left(u,x,\frac{R_0}{4^n}\right) \leq \alpha^n\omega(u,x,R_0) + \sum_{i=0}^{n-1} \frac{C_6\alpha^{n-1-i}}{\zeta^i} \tag{A.51}$$

(cette formule est aussi vérifiée pour $n = 0$ à condition de poser $\sum_{i=0}^{-1} = 0$).
En posant $\gamma = \sup(\alpha, 1/\zeta) \in]0,1[$ (rappelons que $\alpha \in]0,1[$ et $\zeta > 1$) qui ne dépend que de

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p),$$

on a, pour tout $i \in [0, n-1]$, puisque $n - 1 - i \geq 0$ et $i \geq 0$,

$$\frac{\alpha^{n-1-i}}{\zeta^i} = \alpha^{n-1-i} \times \left(\frac{1}{\zeta}\right)^i \leq \gamma^{n-1-i}\gamma^i = \gamma^{n-1},$$

donc (A.51) nous donne, pour tout $n \geq 0$, et puisque $\omega(u,x,R_0) \leq 2||u||_{L^\infty(x+B_{R_0})} \leq 2C_4$ (cf (A.48)),

$$\omega\left(u,x,\frac{R_0}{4^n}\right) \leq 2C_4\alpha^n + nC_6\gamma^{-1}\gamma^n. \tag{A.52}$$

Prenons $\widetilde{\gamma} \in]\gamma, 1[$ (un tel choix ne dépend que de γ, i.e. de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$); on a alors

$$n\gamma^n = n\left(\frac{\gamma}{\widetilde{\gamma}}\right)^n \widetilde{\gamma}^n.$$

Or $\gamma/\widetilde{\gamma} < 1$, donc la suite $(n(\gamma/\widetilde{\gamma})^n)_{n\geq 0}$ est bornée par C_7 ne dépendant que de $(\gamma, \widetilde{\gamma})$, i.e. de

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$$

Ainsi, $n\gamma^n \leq C_7\widetilde{\gamma}^n$ et on a, par (A.52), pour tout $n \geq 0$,

$$\omega\left(u,x,\frac{R_0}{4^n}\right) \leq 2C_4\alpha^n + C_6 C_7\gamma^{-1}\widetilde{\gamma}^n,$$

soit, en posant $\delta = \sup(\alpha, \widetilde{\gamma}) \in]0,1[$ (δ ne dépend que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$), pour tout $n \geq 0$,

$$\omega\left(u, x, \frac{R_0}{4^n}\right) \leq C_8 \delta^n, \tag{A.53}$$

avec C_8 ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$.
Par le lemme A.11, et puisque u est essentiellement bornée sur K par C_4 ne dépendant que de

$$(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M),$$

u est égale presque partout sur K à une fonction continue et il existe $\kappa > 0$ ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$ et C_9 ne dépendant que de $(N, d, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$ tels que $u \in \mathcal{C}^{0,\kappa}(K)$ avec $||u||_{\mathcal{C}^{0,\kappa}(K)} \leq C_9$, ce qui conclut cette démonstration. ■

A.4 Continuité höldérienne près du bord de Ω

On commence par prouver la continuité dans une carte locale, puis on regroupera toutes les cartes et l'intérieur.

Proposition A.3 *Soit $M \geq 0$. Sous les hypothèses (A.3)—(A.7), (A.10) et (A.11), pour tout $i \in [1, m]$ et tout compact K de O_i, il existe $\kappa > 0$ ne dépendant que de $(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$ et C ne dépendant que de $(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$ tels que, si u vérifie (A.2) et $||u||_{L^2(\Omega)} \leq M$, alors $u \in \mathcal{C}^{0,\kappa}(\Omega \cap K)$ et $||u||_{\mathcal{C}^{0,\kappa}(\Omega\cap K)} \leq C$.*

Preuve de la proposition A.3
On commence par transporter le problème dans une partie de B, puis il faudra différencier selon que (O_i, h_i) vérifie (D), (F) ou (DF).
On note $O_{\rm tr} = h_i(O_i \cap \Omega)$, $\Gamma = h_i(O_i \cap \Gamma_d)$ et $\Gamma_0 = \partial O_{\rm tr} \cap \partial B$.

Etape 1: transport dans $O_{\rm tr}$.
Soit $q \in [1, \infty[$ et E^q l'espace vectoriel des fonction de $W^{1,q}_\Gamma(O_{\rm tr})$ à support compact dans B, muni de la norme de $W^{1,q}(O_{\rm tr})$. L'application

$$\left\{ \begin{array}{ccl} E^q & \longrightarrow & F^q = \{\theta \in W^{1,q}_{O_i\cap\Gamma_d}(O_i \cap \Omega) \mid \theta \text{ est à support compact dans } O_i\} \\ \varphi & \longrightarrow & \varphi \circ h_i \end{array} \right.$$

est linéaire continue bijective, de norme ne dépendant que de (q, h_i) (F^q est muni de la norme de $W^{1,q}(O_i \cap \Omega)$). De plus, par un lemme classique, l'extension par 0 hors de $O_i \cap \Omega$ est une application linéaire continue $F^q \to W^{1,q}_{\Gamma_d}(\Omega)$ de norme égale à 1.
La composée de ces deux applications est donc une application linéaire continue

$$T_i^q : E^q \to W^{1,q}_{\Gamma_d}(\Omega)$$

dont la norme ne dépend que de (q, h_i).

Soit $\varphi \in E^2$ et $\phi = T_i^2 \varphi \in H^1_{\Gamma_d}(\Omega)$. L'équation satisfaite par u nous donne

$$\begin{aligned} \int_{O_i\cap\Omega} A\nabla u \cdot \nabla(\varphi \circ h_i) + \int_{O_i\cap\Omega} \varphi \circ h_i \mathbf{v} \cdot \nabla u &= \int_\Omega A\nabla u \cdot \nabla\phi + \int_\Omega \phi \mathbf{v} \cdot \nabla u \\ &= \langle L, \phi \rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} \\ &= \langle (T_i^2)^* L, \varphi \rangle_{(E^2)', E^2}, \end{aligned} \tag{A.54}$$

où $(T_i^2)^* : (H^1_{\Gamma_d}(\Omega))' \to (E^2)'$ est la transposée de T_i^2.

En posant $u_{\rm tr}=u_{|O_i\cap\Omega}\circ h_i^{-1}\in H^1_\Gamma(O_{\rm tr})$ (car $u_{|O_i\cap\Omega}\in H^1_{O_i\cap\Gamma_d}(O_i\cap\Omega)$), on a, puisque $||u||_{L^2(\Omega)}\le M$,

$$||u_{\rm tr}||_{L^2(O_{\rm tr})}\le ||Jh_i||^{1/2}_{L^\infty(O_i\cap\Omega)}M. \tag{A.55}$$

Sur $O_i\cap\Omega$, $u=u_{\rm tr}\circ h_i$, donc $\nabla u=(h_i')^T\nabla u_{\rm tr}\circ h_i$; on en déduit que, sur $O_i\cap\Omega$,

$$A\nabla u\cdot\nabla(\varphi\circ h_i)=A((h_i')^T\nabla u_{\rm tr}\circ h_i)\cdot((h_i')^T\nabla\varphi\circ h_i)=h_i'A(h_i')^T(\nabla u_{\rm tr}\circ h_i)\cdot(\nabla\varphi\circ h_i)$$

et

$$\varphi\circ h_i\mathbf{v}\cdot\nabla u=\varphi\circ h_i\mathbf{v}\cdot((h_i')^T\nabla u_{\rm tr}\circ h_i)=\varphi\circ h_ih_i'\mathbf{v}\cdot(\nabla u_{\rm tr}\circ h_i);$$

par le théorème de changement de variable ($h_i^{-1}:O_{\rm tr}\to O_i\cap\Omega$ est un homéomorphisme localement lipschitzien), on a donc

$$\begin{aligned}&\int_{O_i\cap\Omega}A\nabla u\cdot\nabla(\varphi\circ h_i)+\int_{O_i\cap\Omega}\varphi\circ h_i\mathbf{v}\cdot\nabla u\\ &=\int_{O_{\rm tr}}(|Jh_i^{-1}|(h_i'A(h_i')^T)\circ h_i^{-1})\nabla u_{\rm tr}\cdot\nabla\varphi+\int_{O_{\rm tr}}\varphi(|Jh_i^{-1}|(h_i'\mathbf{v})\circ h_i^{-1})\cdot\nabla u_{\rm tr}.\end{aligned} \tag{A.56}$$

On a, pour tout $\theta\in E^2\subset E^{p'}$, et puisque $T_i^{p'}\theta=T_i^2\theta$,

$$\langle (T_i^{p'})^*\mathcal{L},\theta\rangle_{(E^{p'})',E^{p'}}=\langle\mathcal{L},T_i^{p'}\theta\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))',W^{1,p'}_{\Gamma_d}(\Omega)}=\langle L,T_i^2\theta\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}=\langle (T_i^2)^*L,\theta\rangle_{(E^2)',E^2}.$$

Cela signifie donc que $(T_i^2)^*L=((T_i^{p'})^*\mathcal{L})_{|E^2}$.
Comme, pour tout $q\in]1,\infty[$, E^q est dense dans $W^{1,q}_{\Gamma\cup\Gamma_0}(O_{\rm tr})$, les fonctions $(T_i^2)^*L\in(E^2)'$ et $(T_i^{p'})^*\mathcal{L}\in(E^{p'})'$ s'étendent en respectivement en

$$L_{\rm tr}\in H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr})\quad\text{et}\quad\mathcal{L}_{\rm tr}\in W^{1,p'}_{\Gamma\cup\Gamma_0}(O_{\rm tr})\text{ telles que }L_{\rm tr}=\mathcal{L}_{{\rm tr}|H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr})}. \tag{A.57}$$

Remarquons aussi au passage que

$$\begin{aligned}||\mathcal{L}_{\rm tr}||_{(W^{1,p'}_{\Gamma\cup\Gamma_0}(O_{\rm tr}))'}&=||(T_i^p)^*\mathcal{L}||_{(E^{p'})'}\\ &\le ||(T_i^p)^*||_{\mathcal{L}((W^{1,p'}_{\Gamma_d}(\Omega))',(E^{p'})')}||\mathcal{L}||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}\\ &\le ||T_i^p||_{\mathcal{L}(E^{p'},W^{1,p'}_{\Gamma_d}(\Omega))}\Lambda_L=\Lambda_{L,{\rm tr}},\end{aligned} \tag{A.58}$$

avec $\Lambda_{L,{\rm tr}}$ ne dépendant que de (p,h_i,Λ_L).

On obtient finalement, par (A.54) et (A.56), pour tout $\varphi\in E^2$,

$$\int_{O_{\rm tr}}(|Jh_i^{-1}|(h_i'A(h_i')^T)\circ h_i^{-1})\nabla u_{\rm tr}\cdot\nabla\varphi+\int_{O_{\rm tr}}\varphi(|Jh_i^{-1}|(h_i'\mathbf{v})\circ h_i^{-1})\cdot\nabla u_{\rm tr}=\langle L_{\rm tr},\varphi\rangle_{(H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr}))',H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr})}.$$

E^2 étant dense dans $H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr})$, on déduit de cette équation que $u_{\rm tr}$ vérifie

$$\left\{\begin{array}{l}u_{\rm tr}\in H^1_\Gamma(O_{\rm tr})\,,\\ \displaystyle\int_{O_{\rm tr}}A_{\rm tr}\nabla u_{\rm tr}\cdot\nabla\varphi+\int_{O_{\rm tr}}\varphi\mathbf{v}_{\rm tr}\nabla u_{\rm tr}=\langle L_{\rm tr},\varphi\rangle_{(H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr}))',H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr})}\,,\ \forall\varphi\in H^1_{\Gamma\cup\Gamma_0}(O_{\rm tr}),\end{array}\right. \tag{A.59}$$

où $A_{tr}=|Jh_i^{-1}|(h_i'A(h_i')^T)\circ h_i^{-1}=\frac{h_i'A(h_i')^T}{|Jh_i|}\circ h_i^{-1}$ (car $Jh_i^{-1}=(Jh_i\circ h_i^{-1})^{-1}$) et $\mathbf{v}_{\rm tr}=|Jh_i^{-1}|(h_i'\mathbf{v})\circ h_i^{-1}$.

Avant de passer à la suite, étudions un peu $A_{\rm tr}$ et $\mathbf{v}_{\rm tr}$.
On a, pour presque tout $x\in O_{\rm tr}$,

$$||A_{\rm tr}(x)||\le ||Jh_i^{-1}||_{L^\infty(O_{\rm tr})}||h_i'(h_i^{-1}(x))||\,||(h_i')^T(h_i^{-1}(x))||\,||A(h_i^{-1}(x))||,$$

soit, puisque h_i^{-1} transporte les ensembles de mesure nulle sur des ensembles de mesure nulle et, pour $H\in M_N(\mathbb{R})$, $||H^T||\le ||H||$, pour presque tout $x\in O_{\rm tr}$,

$$||A_{\rm tr}(x)||\le ||Jh_i^{-1}||_{L^\infty(O_{\rm tr})}||\,||h_i'||\,||^2_{L^\infty(O_i\cap\Omega)}\Lambda_A=\Lambda_{A,{\rm tr}} \tag{A.60}$$

avec $\Lambda_{A,{\rm tr}}$ ne dépendant que de (h_i,Λ_A).
De plus, pour presque tout $x\in O_{\rm tr}$ et tout $\xi\in\mathbb{R}^N$, on a

$$|(h_i')^T(h_i^{-1}(x))\xi|\ge \frac{|\xi|}{||((h_i^{-1})')^T(x)||}\ge \frac{|\xi|}{||\,||(h_i^{-1})'||\,||_{L^\infty(O_{\rm tr})}},$$

donc

$$\begin{aligned}
A_{\rm tr}(x)\xi\cdot\xi &= \frac{h_i'A(h_i')^T}{|Jh_i|}\circ h_i^{-1}(x)\xi\cdot\xi\\
&= \frac{A(h_i^{-1}(x))(h_i')^T(h_i^{-1}(x))\xi\cdot(h_i')^T(h_i^{-1}(x))\xi}{|Jh_i(h_i^{-1}(x))|}\\
&\ge \frac{\alpha_A}{||Jh_i||_{L^\infty(O_i\cap\Omega)}}|(h_i')^T(h_i^{-1}(x))\xi|^2\\
&\ge \frac{\alpha_A}{||Jh_i||_{L^\infty(O_i\cap\Omega)}||\,||(h_i^{-1})'||\,||^2_{L^\infty(O_{\rm tr})}}|\xi|^2:=\alpha_{A,{\rm tr}}|\xi|^2.
\end{aligned}\tag{A.61}$$

Soit $q\in[1,\infty[$ et $f\in(L^q(O_i\cap\Omega))^N$; on a, pour presque tout $x\in O_{\rm tr}$,

$$|Jh_i^{-1}(x)|\,|(h_i'f)\circ h_i^{-1}(x)|\le ||Jh_i^{-1}||^{1-1/q}_{L^\infty(O_{\rm tr})}||\,||h_i'||\,||_{L^\infty(O_i\cap\Omega)}|Jh_i^{-1}(x)|^{1/q}|f|\circ h_i^{-1}(x),$$

ce qui implique, par changement de variable,

$$||\,|Jh_i^{-1}|\,|(h_i'f)\circ h_i^{-1}|\,||_{L^q(O_{\rm tr})}\le ||Jh_i^{-1}||^{1-1/q}_{L^\infty(O_{\rm tr})}||\,||h_i'||\,||_{L^\infty(O_i\cap\Omega)}||f||_{L^q(O_i\cap\Omega)}$$

(cette égalité est en fait aussi valable avec $q=\infty$). Comme $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ avec $\mathbf{v}_1\in B(\Omega,N_*,\chi)$ et $\mathbf{v}_2\in B(\Omega,r,\Lambda)$, cette inégalité appliquée à $(q,f)=(N_*,\mathbf{v}_{1|O_i\cap\Omega})$ et à $(q,f)=(r,\mathbf{v}_{2|O_i\cap\Omega})$ nous permet de voir que

$$\mathbf{v}_{\rm tr}\in B(O_{\rm tr},N_*,\chi_{\rm tr})+B(O_{\rm tr},r,\Lambda_{\rm tr}), \tag{A.62}$$

où $\chi_{\rm tr}=||Jh_i^{-1}||^{1-1/N_*}_{L^\infty(O_{\rm tr})}||\,||h_i'||\,||_{L^\infty(O_i\cap\Omega)}\chi$ et $\Lambda_{\rm tr}=||Jh_i^{-1}||^{1-1/r}_{L^\infty(O_{\rm tr})}||\,||h_i'||\,||_{L^\infty(O_i\cap\Omega)}\Lambda$.

Soit K un compact de O_i et $K_0=h_i(K)$ (c'est un compact de B).
Les trois étapes suivantes consistent à montrer, dans chaque cas (F), (D) et (DF), que $u_{\rm tr}$ est höldérienne sur $K_0\cap O_{\rm tr}$. On utilisera pour cela des techniques de réflexion et la proposition A.2.

Etape 2: (O_i,h_i) est du type (F).
Dans ce cas, $O_{\rm tr}=B_+$ et $\Gamma=\emptyset$. Nous appelons τ la symétrie par rapport à la dernière variable de $\mathbb{R}^N$ (i.e. $\tau(x_1,\dots,x_N)=(x_1,\dots,x_{N-1},-x_N)$); nous confondrons, dans la suite, τ et sa matrice dans la base canonique.

Définissons $u_{\rm sy}$, $A_{\rm sy}$ et $\mathbf{v}_{\rm sy}$ presque partout sur B par:

- $u_{\rm sy}=u_{\rm tr}$ sur B_+ et $u_{\rm sy}=u_{\rm tr}\circ\tau$ sur $B_-=\tau(B_+)$;

- $A_{\rm sy} = A_{\rm tr}$ sur B_+ et $A_{\rm sy} = \tau A_{\rm tr} \circ \tau\tau$ sur B_-;
- $\mathbf{v}_{\rm sy} = \mathbf{v}_{\rm tr}$ sur B_+ et $\mathbf{v}_{\rm sy} = \tau\mathbf{v}_{\rm tr} \circ \tau$ sur B_-.

Par un résultat classique, $u_{\rm sy}$ est dans $H^1(B)$ et on a $\nabla u_{\rm sy} = \nabla u$ sur B_+, $\nabla u_{\rm sy} = \tau\nabla u_{\rm tr} \circ \tau$ sur B_-; τ étant de jacobien égal à 1 en valeur absolue, (A.55) nous donne

$$||u_{\rm sy}||_{L^2(B)} \le 2||Jh_i||^{1/2}_{L^\infty(O_i\cap\Omega)} M = M_0 \tag{A.63}$$

(M_0 ne dépend que de (h_i, M)). τ étant une isométrie, on a, pour presque tout $x \in B$ et tout $\xi \in \mathbb{R}^N$, par (A.60) et (A.61),

$$||A_{\rm sy}(x)|| \le \Lambda_{A,{\rm tr}} \quad \text{et} \quad A_{\rm sy}(x)\xi\cdot\xi \ge \alpha_{A,{\rm tr}}|\xi|^2. \tag{A.64}$$

Enfin, (A.62) donne

$$\mathbf{v}_{\rm sy} \in B(B, N_*, 2\chi_{\rm tr}) + B(B, r, 2\Lambda_{\rm tr}). \tag{A.65}$$

Puisque, pour $q \in [1,\infty[$ et $\varphi \in W^{1,q}_0(B)$, on a $\varphi_{|B_+} + \varphi_{|B_-} \circ \tau \in W^{1,q}_{\Gamma_0}(B_+)$ (rappelons que l'on a ici $O_{\rm tr} = B_+$ et $\Gamma_0 = \partial B_+ \cap \partial B$); on peut donc définir $L_{\rm sy} \in (H^1_0(B))'$ et $\mathcal{L}_{\rm sy} \in (W^{1,p'}_0(B))'$ par

$$\langle L_{\rm sy}, \varphi\rangle_{(H^1_0(B))', H^1_0(B)} = \langle L_{\rm tr}, \varphi_{|B_+} + \varphi_{|B_-}\circ\tau\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)}$$

et

$$\langle \mathcal{L}_{\rm sy}, \varphi\rangle_{(W^{1,p'}_0(B))', W^{1,p'}_0(B)} = \langle \mathcal{L}_{\rm tr}, \varphi_{|B_+} + \varphi_{|B_-}\circ\tau\rangle_{(W^{1,p'}_{\Gamma_0}(B_+))', W^{1,p'}_{\Gamma_0}(B_+)}.$$

On a alors

$$L_{\rm sy} = \mathcal{L}_{{\rm sy}|H^1_0(B)} \tag{A.66}$$

et $||\mathcal{L}_{\rm sy}||_{(W^{1,p'}_0(B))'} \le 2\Lambda_{L,{\rm tr}}$ ($\Lambda_{L,{\rm tr}}$ est donné dans (A.58)).

Nous allons montrer que $u_{\rm sy}$ vérifie une équation du genre (A.43).
Soit $\varphi \in H^1_0(B)$; en utilisant $\varphi_{|B_+} + \varphi_{|B_-}\circ\tau \in H^1_{\Gamma_0}(B_+)$ dans (A.59), on a

$$\begin{aligned}
&\langle L_{\rm sy}, \varphi\rangle_{(H^1_0(B))', H^1_0(B)}\\
&= \langle L_{\rm tr}, \varphi_{|B_+} + \varphi_{|B_-}\circ\tau\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)}\\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr}\cdot\nabla\varphi_{|B_+} + \int_{B_+}\varphi_{|B_+}\mathbf{v}_{\rm tr}\cdot\nabla u_{\rm tr} + \int_{B_+} A_{\rm tr}\nabla u_{\rm tr}\cdot\nabla(\varphi_{|B_-}\circ\tau)\\
&\quad + \int_{B_+}\varphi_{|B_-}\circ\tau\mathbf{v}_{\rm tr}\cdot\nabla u_{\rm tr}\\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr}\cdot\nabla\varphi + \int_{B_+}\varphi\mathbf{v}_{\rm tr}\cdot\nabla u_{\rm tr} + \int_{B_-} A_{\rm tr}\circ\tau(\nabla u_{\rm tr}\circ\tau)\cdot(\tau\nabla\varphi) + \int_{B_-}\varphi\mathbf{v}_{\rm tr}\circ\tau\cdot\nabla u_{\rm tr}\circ\tau\\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr}\cdot\nabla\varphi + \int_{B_+}\varphi\mathbf{v}_{\rm tr}\cdot\nabla u_{\rm tr} + \int_{B_-}(\tau A_{\rm tr}\circ\tau\tau)\nabla(u_{\rm tr}\circ\tau)\cdot\nabla\varphi\\
&\quad + \int_{B_-}\varphi(\tau\mathbf{v}_{\rm tr}\circ\tau)\cdot\nabla(u_{\rm tr}\circ\tau)\\
&= \int_B A_{\rm sy}\nabla u_{\rm sy}\cdot\nabla\varphi + \int_B \varphi\mathbf{v}_{\rm sy}\cdot\nabla u_{\rm sy}.
\end{aligned}$$

Cette équation, vérifiée pour tout $\varphi \in H^1_0(B)$, montre que $u_{\rm sy}$ vérifie (A.43) lorsque Ω est remplacé par B, Γ_d par $\partial\Omega$, A par $A_{\rm sy}$, $\mathbf{v}$ par $\mathbf{v}_{\rm sy}$ et L par $L_{\rm sy}$.

Par la proposition A.2, K_0 étant un compact de B, $u_{\rm sy}$ vérifiant (A.63), $A_{\rm sy}$ vérifiant (A.64), $\mathbf{v}_{\rm sy}$ vérifiant (A.65) avec, par (A.6) et (A.11),

$$0 \le 2\chi_{\rm tr} < \frac{\alpha_{A,{\rm tr}}}{C_S(N, d_0, 2, 2^*)}$$

(où $d_0 = 2$ est un majorant du diamètre de B), et $L_{\rm sy}$ vérifiant (A.66), il existe $\kappa > 0$ ne dépendant que de N, d_0, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_{\rm tr}$, r, $2\Lambda_{\rm tr}$ et p, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p),$$

et C_0 ne dépendant que de N, d_0, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_{\rm tr}$, r, $2\Lambda_{\rm tr}$, p, $2\Lambda_{L,{\rm tr}}$, K_0 et M_0, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M),$$

tel que $u_{\rm sy} \in \mathcal{C}^{0,\kappa}(K_0)$ et $||u_{\rm sy}||_{\mathcal{C}^{0,\kappa}(K_0)} \le C_0$.
Puisque $u_{\rm sy} = u_{\rm tr}$ sur B_+, on en déduit que $u_{\rm tr} \in \mathcal{C}^{0,\kappa}(K_0 \cap B_+)$ et $||u_{\rm tr}||_{\mathcal{C}^{0,\kappa}(K_0\cap B_+)} \le C_0$.

Etape 3: (O_i, h_i) est de type (D).
La symétrie effectuée est alors cette fois impaire.
On a $O_{\rm tr} = B_+$, $\Gamma = B^{N-1}$ et $\Gamma \cup \Gamma_0 = \partial B_+$. On note toujours τ la symétrie par rapport à la dernière variable de $\mathbb{R}^N$.

Définissons $u_{\rm sy}$, $A_{\rm sy}$ et $\mathbf{v}_{\rm sy}$ presque partout sur B par:

- $u_{\rm sy} = u_{\rm tr}$ sur B_+ et $u_{\rm sy} = -u_{\rm tr} \circ \tau$ sur $B_- = \tau(B_+)$;
- $A_{\rm sy} = A_{\rm tr}$ sur B_+ et $A_{\rm sy} = \tau A_{\rm tr} \circ \tau\tau$ sur B_-;
- $\mathbf{v}_{\rm sy} = \mathbf{v}_{\rm tr}$ sur B_+ et $\mathbf{v}_{\rm sy} = \tau \mathbf{v}_{\rm tr} \circ \tau$ sur B_-.

Puisque $u_{\rm tr} \in H^1_{B^{N-1}}(B_+)$, $u_{\rm sy}$ est dans $H^1(B)$ (on peut prouver ceci en vérifiant, grâce à une intégration par parties dans B_+ et dans B_- et au fait que $u_{{\rm tr}|B^{N-1}} = 0$, que la dérivée au sens des distributions de $u_{\rm sy}$ est la fonction de $L^2(B)$ égale à $\nabla u_{\rm tr}$ sur B_+ et a $-\tau\nabla u_{\rm tr} \circ \tau$ sur B_-). (A.55) nous donne

$$||u_{\rm sy}||_{L^2(B)} \le 2||Jh_i||^{1/2}_{L^\infty(O_i\cap\Omega)} M = M_0 \tag{A.67}$$

(M_0 ne dépend que de (h_i, M)). Comme précédemment, on a, par (A.60) et (A.61),

$$||A_{\rm sy}(x)|| \le \Lambda_{A,{\rm tr}} \quad \text{et} \quad A_{\rm sy}(x)\xi\cdot\xi \ge \alpha_{A,{\rm tr}}|\xi|^2 \quad \text{pour presque tout } x \in B \text{ et tout } \xi \in \mathbb{R}^N, \tag{A.68}$$

et, par (A.62),

$$\mathbf{v}_{\rm sy} \in B(B, N_*, 2\chi_{\rm tr}) + B(B, r, 2\Lambda_{\rm tr}). \tag{A.69}$$

Pour tout $q \in [1,\infty[$ et tout $\varphi \in W^{1,q}_0(B)$, on a $\varphi_{|B_+} - \varphi_{|B_-} \circ \tau \in W^{1,q}_0(B_+)$; l'appartenance de cette fonction à $W^{1,q}(B_+)$ est évidente. Pour la nullité de la trace, on sait que, τ étant un homéomorphisme bilipschitzien entre B_- et B_+, en notant f la trace de $\varphi_{|B_-}$ sur B_-, la trace sur ∂B_+ de $\varphi_{|B_-} \circ \tau$ est égale à $f \circ \tau$; ainsi, si g désigne la trace de $\varphi_{|B_+}$ sur ∂B_+, la trace de $\varphi_{|B_+} - \varphi_{|B_-} \circ \tau$ est $g - f \circ \tau$; or cette fonction est nulle sur Γ_0 car $f = 0$ sur $\partial B \cap \partial B_-$ et $g = 0$ sur $\partial B \cap \partial B_+$ (puisque $\varphi \in H^1_0(B)$) et aussi nulle sur B^{N-1}, puisque $f \circ \tau_{|B^{N-1}} = f = g$ (les traces de $\varphi_{|B_+}$ et de $\varphi_{|B_-}$ sur B^{N-1} sont égales).
On peut donc définir $L_{\rm sy} \in (H^1_0(B))'$ et $\mathcal{L}_{\rm sy} \in (W^{1,p'}_0(B))'$ par

$$\langle L_{\rm sy}, \varphi\rangle_{(H^1_0(B))', H^1_0(B)} = \langle L_{\rm tr}, \varphi_{|B_+} - \varphi_{|B_-} \circ \tau\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)}$$

et

$$\langle \mathcal{L}_{\rm sy}, \varphi\rangle_{(W^{1,p'}_0(B))', W^{1,p'}_0(B)} = \langle \mathcal{L}_{\rm tr}, \varphi_{|B_+} - \varphi_{|B_-} \circ \tau\rangle_{(W^{1,p'}_{\Gamma_0}(B_+))', W^{1,p'}_{\Gamma_0}(B_+)}.$$

On a alors

$$L_{\rm sy} = \mathcal{L}_{{\rm sy}|H^1_0(B)} \tag{A.70}$$

et $||\mathcal{L}_{\rm sy}||_{(W^{1,p'}_0(B))'} \le 2\Lambda_{L,{\rm tr}}$ ($\Lambda_{L,{\rm tr}}$ est donné dans (A.58)).

Nous allons montrer ici aussi que $u_{\rm sy}$ vérifie une équation du genre (A.43).
Soit $\varphi \in H_0^1(B)$. En prenant $\varphi_{|B_+} - \varphi_{|B_-} \circ \tau \in H_0^1(B_+)$ dans (A.59), on a

$$\begin{aligned}
&\langle L_{\rm sy}, \varphi\rangle_{(H_0^1(B))', H_0^1(B)} \\
&= \langle L_{\rm tr}, \varphi_{|B_+} - \varphi_{|B-} \circ \tau\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr} \cdot \nabla \varphi_{|B_+} + \int_{B_+} \varphi_{|B_+} \mathbf{v}_{\rm tr} \cdot \nabla u_{\rm tr} - \int_{B_+} A_{\rm tr}\nabla u_{\rm tr} \cdot \nabla(\varphi_{|B_-} \circ \tau) \\
&\quad - \int_{B_+} \varphi_{|B_-} \circ \tau \mathbf{v}_{\rm tr} \cdot \nabla u_{\rm tr} \\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr} \cdot \nabla \varphi + \int_{B_+} \varphi \mathbf{v}_{\rm tr} \cdot \nabla u_{\rm tr} - \int_{B_-} A_{\rm tr} \circ \tau (\nabla u_{\rm tr} \circ \tau) \cdot (\tau \nabla \varphi) - \int_{B_-} \varphi \mathbf{v}_{\rm tr} \circ \tau \cdot \nabla u_{\rm tr} \circ \tau \\
&= \int_{B_+} A_{\rm tr}\nabla u_{\rm tr} \cdot \nabla \varphi + \int_{B_+} \varphi \mathbf{v}_{\rm tr} \cdot \nabla u_{\rm tr} + \int_{B_-} (\tau A_{\rm tr} \circ \tau \tau) \nabla(-u_{\rm tr} \circ \tau) \cdot \nabla \varphi \\
&\quad + \int_{B_-} \varphi(\tau \mathbf{v}_{\rm tr} \circ \tau) \cdot \nabla(-u_{\rm tr} \circ \tau) \\
&= \int_B A_{\rm sy} \nabla u_{\rm sy} \cdot \nabla \varphi + \int_B \varphi \mathbf{v}_{\rm sy} \cdot \nabla u_{\rm sy}.
\end{aligned}$$

Cette équation, vérifiée pour tout $\varphi \in H_0^1(B)$, montre que $u_{\rm sy}$ vérifie (A.43) lorsque Ω est remplacé par B, Γ_d par $\partial\Omega$, A par $A_{\rm sy}$, $\mathbf{v}$ par $\mathbf{v}_{\rm sy}$ et L par $L_{\rm sy}$.

Comme dans l'étape 2, par la proposition A.2, puisque K_0 est un compact de B, $u_{\rm sy}$ vérifie (A.67), $A_{\rm sy}$ vérifie (A.68), $\mathbf{v}_{\rm sy}$ vérifie (A.69) avec, par (A.6) et (A.11),

$$0 \le 2\chi_{\rm tr} < \frac{\alpha_{A,{\rm tr}}}{C_S(N, d_0, 2, 2^*)}$$

(où $d_0 = 2$ est un majorant du diamètre de B), et $L_{\rm sy}$ vérifie (A.70), il existe $\kappa > 0$ ne dépendant que de N, d_0, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_{\rm tr}$, r, $2\Lambda_{\rm tr}$, et p, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p),$$

et C_0 ne dépendant que de N, d_0, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_{\rm tr}$, r, $2\Lambda_{\rm tr}$, p, $2\Lambda_{L,{\rm tr}}$, K_0 et M_0, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M),$$

tel que $u_{\rm sy}$ est κ-höldérienne sur K_0 avec $||u_{\rm sy}||_{\mathcal{C}^{0,\kappa}(K_0)} \le C_0$; $u_{\rm tr}$ étant égale à $u_{\rm sy}$ sur B_+, on en déduit que $u_{\rm tr} \in \mathcal{C}^{0,\kappa}(K_0 \cap B_+)$ et $||u_{\rm tr}||_{\mathcal{C}^{0,\kappa}(K_0 \cap B_+)} \le C_0$.

Etape 4: (O_i, h_i) est de type (DF).
On a ici $O_{\rm tr} = B_{++}$ et $\Gamma = \Gamma_2$. Il faut effectuer une symétrie par rapport à l'hyperplan $x_{N-1} = 0$ pour se ramener au cas (D).

Soit τ' la réflexion par rapport à $x_{N-1} = 0$ (i.e. $\tau'(x) = (x_1, \ldots, x_{N-2}, -x_{N-1}, x_N)$). On définit u_1, A_1 et $\mathbf{v}_1$ presque partout sur B_+ par

- $u_1 = u_{\rm tr}$ sur B_{++}, $u_1 = u_{\rm tr} \circ \tau'$ sur $B_{-+} = \tau'(B_{++})$;
- $A_1 = A_{\rm tr}$ sur B_{++}, $A_1 = \tau' A_{\rm tr} \circ \tau' \tau'$ sur B_{-+};
- $\mathbf{v}_1 = \mathbf{v}_{\rm tr}$ sur B_{++}, $\mathbf{v}_1 = \tau' \mathbf{v}_{\rm tr} \circ \tau'$ sur B_{-+};

Par (A.55), on a $u_1 \in H^1_{B^{N-1}}(B_+)$ (rappelons que $u_{\rm tr} \in H^1_\Gamma(B_{++})$) et

$$||u_1||_{L^2(B_+)} \le 2||Jh_i||^{1/2}_{L^\infty(O_i\cap\Omega)} M = ||Jh_i||^{1/2}_{L^\infty(O_i\cap\Omega)} M_1 \tag{A.71}$$

(M_1 ne dépend que de (h_i, M)). Par (A.60) et (A.61), on a

$$||A_1(x)|| \le \Lambda_{A,{\rm tr}} \quad \text{et} \quad A_1(x)\xi\cdot\xi \ge \alpha_{A,{\rm tr}}|\xi|^2 \quad \text{pour presque tout } x \in B_+ \text{ et tout } \xi \in \mathbb{R}^N. \tag{A.72}$$

Enfin, par (A.62),

$$\mathbf{v}_1 \in B(B_+, N_*, \chi_1) + B(B_+, r, \Lambda_1), \tag{A.73}$$

avec $\chi_1 = 2\chi_{\rm tr}$ et $\Lambda_1 = 2\Lambda_{\rm tr}$.

Pour $q \in [1,\infty[$ et $\varphi \in W^{1,q}_0(B_+)$, on a $\varphi_{|B_{++}} + \varphi_{|B_{-+}} \circ \tau' \in W^{1,q}_{\Gamma\cup\Gamma_0}(B_{++})$, donc on peut définir $L_1 \in (H^1_0(B_+))'$ et $\mathcal{L}_1 \in (W^{1,p'}_0(B_+))'$ par

$$\langle L_1, \varphi\rangle_{(H^1_0(B_+))', H^1_0(B_+)} = \langle L_{\rm tr}, \varphi_{|B_{++}} + \varphi_{|B_{-+}} \circ \tau'\rangle_{(H^1_{\Gamma\cup\Gamma_0}(B_{++}))', H^1_{\Gamma\cup\Gamma_0}(B_{++})}$$

et

$$\langle \mathcal{L}_1, \varphi\rangle_{(W^{1,p'}_0(B_+))', W^{1,p'}_0(B_+)} = \langle \mathcal{L}_{\rm tr}, \varphi_{|B_{++}} + \varphi_{|B_{-+}} \circ \tau'\rangle_{(W^{1,p'}_{\Gamma\cup\Gamma_0}(B_{++}))', W^{1,p'}_{\Gamma\cup\Gamma_0}(B_{++})},$$

et on a

$$L_1 = \mathcal{L}_{1|H^1_0(B_+)} \tag{A.74}$$

avec $||\mathcal{L}_1||_{(W^{1,p'}_0(B_+))'} \le 2\Lambda_{L,{\rm tr}} = \Lambda_{L,1}$ ($\Lambda_{L,1}$ ne dépend que de (h_i, p, Λ_L)).

Soit $\varphi \in H^1_0(B_+)$; en utilisant $\varphi_{|B_{++}} + \varphi_{|B_{-+}} \circ \tau' \in H^1_{\Gamma\cup\Gamma_0}(B_{++})$ dans (A.59), on voit avec le même genre de calculs que dans l'étape 1 que $u_1 \in H^1_{B^{N-1}}(B_+)$ vérifie

$$\int_{B_+} A_1 \nabla u_1 \cdot \nabla\varphi + \int_{B_+} \varphi \mathbf{v}_1 \cdot \nabla u_1 = \langle L_1, \varphi\rangle_{(H^1_0(B_+))', H^1_0(B_+)}.$$

u_1 vérifie donc (A.59) lorsque $O_{\rm tr}$ est remplacé par B_+, Γ par B^{N-1}, $A_{\rm tr}$ par A_1, $\mathbf{v}_{\rm tr}$ par $\mathbf{v}_1$ et $L_{\rm tr}$ par L_1. Avec ces changements de notation, u_1 vérifie (A.55) avec M remplacé par M_1 (propriété (A.71)), A_1 vérifie (A.60) et (A.61) (propriété (A.72)), L_1 vérifie (A.57) et (A.58) avec $\mathcal{L}_{\rm tr}$ remplacé par $\mathcal{L}_1$ et $\Lambda_{L,{\rm tr}}$ remplacé par $\Lambda_{L,1}$ (propriété (A.74)) et $\mathbf{v}_1$ vérifie (A.62) avec $\chi_{\rm tr}$ remplacé par χ_1 et $\Lambda_{\rm tr}$ remplacé par Λ_1 (propriété (A.73)).
Par (A.6) et (A.11), on constate que

$$0 \le 2\chi_1 < \frac{\alpha_{A,{\rm tr}}}{C_S(N, 2, 2, 2^*)}.$$

On peut donc appliquer le raisonnement de l'étape 3 en changeant tous les indices "tr" en indices "1", et on en déduit qu'il existe $\kappa > 0$ ne dépendant que de N, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_1$, r, $2\Lambda_1$ et p, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p),$$

et C_0 ne dépendant que de N, $\alpha_{A,{\rm tr}}$, $\Lambda_{A,{\rm tr}}$, N_*, $2\chi_1$, r, $2\Lambda_1$, p, $2\Lambda_{L,1}$, K_0 et M_1, i.e. ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M),$$

tel que $u_1 \in \mathcal{C}^{0,\kappa}(K_0 \cap B_+)$ et $||u_1||_{\mathcal{C}^{0,\kappa}(K_0\cap B_+)} \le C_0$; comme $u_{\rm tr} = u_1$ sur B_{++}, on en déduit que $u_{\rm tr} \in \mathcal{C}^{0,\kappa}(K_0 \cap B_{++})$ et que $||u_{\rm tr}||_{\mathcal{C}^{0,\kappa}(K_0\cap B_{++})} \le C_0$.

Etape 5: conclusion.

On a donc trouvé, dans l'étape 2, 3 ou 4 (selon que (O_i, h_i) est de type (F), (D) ou (DF)), $\kappa > 0$ ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p),$$

et C_0 ne dépendant que de

$$(h_i, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, K, M)$$

tels que $u_{\rm tr} \in \mathcal{C}^{0,\kappa}(K_0 \cap O_{\rm tr})$ et que $||u_{\rm tr}||_{\mathcal{C}^{0,\kappa}(K_0\cap O_{\rm tr})} \le C_0$ (rappelons que $K_0 = h_i(K)$). Or $u = u_{\rm tr} \circ h_i$ sur $O_i \cap \Omega$; u est donc continue sur $K \cap \Omega$ et

$$||u||_{L^\infty(K\cap\Omega)} \le ||u_{\rm tr}||_{L^\infty(h_i(K\cap\Omega))} = ||u_{\rm tr}||_{L^\infty(K_0\cap O_{\rm tr})} \le C_0. \tag{A.75}$$

De plus, h_i étant lipschitzienne sur O_i, on a, pour tous $(x, y) \in K \cap \Omega$, puisque $((h_i(x), h_i(y)) \in K_0 \cap O_{\rm tr}$,

$$|u(x) - u(y)| = |u_{\rm tr}(h_i(x)) - u_{\rm tr}(h_i(y))| \le C_0|h_i(x) - h_i(y)|^\kappa \le C_0 \mathrm{Lip}(h_i)^\kappa |x - y|^\kappa. \tag{A.76}$$

(A.75) et (A.76) permettent de voir que $u \in \mathcal{C}^{0,\kappa}(K \cap \Omega)$ et donnent une estimation de la norme de u dans cet espace, ce qui conclut la démonstration. ■

A.5 Preuve du théorème A.1

La preuve du théorème est maintenant un simple agencement des propositions A.2 et A.3

Preuve du théorème A.1
Lorsque les hypothèses du théorème A.1 sont satisfaites, alors celles des propositions A.2 et A.3 le sont aussi.

On note $\kappa_0 > 0$ le κ donné par la proposition A.2 et $\kappa_i > 0$ le κ donné par la proposition A.3 appliquée à $i \in [1, m]$. On pose $\kappa = \inf(\kappa_0, \ldots, \kappa_m) > 0$; κ ne dépend que de $(\Omega, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p)$.

Pour U ouvert de $\mathbb{R}^N$ et $s > 0$, on note $K_s(U) = \{x \in \Omega \mid \mathrm{dist}(x, \mathbb{R}^N \backslash U) \ge 1/s\}$; les $(K_s(U))_{s>0}$ sont des compacts de U et l'union croissante des intérieurs des $(K_s(U))_{s>0}$ est U.
Pour uniformiser les notations, O_0 désignera Ω.
Puisque $\overline{\Omega} \subset \cup_{i=0}^m O_i$, on a $\overline{\Omega} \subset \bigcup_{s>0} \cup_{i=0}^m \mathrm{int}(K_s(O_i))$; $\overline{\Omega}$ étant un compact de $\mathbb{R}^N$, on peut extraire de ce recouvrement ouvert un recouvrement fini; cela signifie, la suite $(\cup_{i=0}^m \mathrm{int}(K_s(O_i)))_{s>0}$ étant croissante, qu'il existe $s_0 > 0$ tel que $\overline{\Omega} \subset \cup_{i=0}^m \mathrm{int}(K_{s_0}(O_i))$ (le choix de s_0 ne dépend que de Ω).
En notant C_0 le C donné par la proposition A.2 appliquée à $K = K_{2s_0}(O_0) = K_{2s_0}(\Omega)$ et, pour $i \in [1, m]$, C_i le C donné par la proposition A.3 appliquée à i et $K = K_{2s_0}(O_i)$, on pose $H = \sup_{i\in[0,m]} C_i$; puisque le choix de s_0 ne dépend que de Ω, H ne dépend que de $(\Omega, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, M)$.

Par les propositions A.2 et A.3, u étant (presque partout égale à une fonction) continue sur chaque ouvert $\mathrm{int}(K_{s_0}(O_i)) \cap \Omega$ ($i \in [0, m]$) et Ω étant la réunion de ces ouverts, u est (presque partout égale à une fonction) continue sur Ω.
De plus, puisque pour tout $i \in [0, m]$, $\sup_{K_{s_0}(O_i)\cap\Omega} |u| \le \sup_{K_{2s_0}(O_i)\cap\Omega} |u| \le H$, on a

$$\sup_\Omega |u| \le H. \tag{A.77}$$

Soit $(x, y) \in \Omega$. Notons $i \in [0, m]$ un indice tel que $x \in K_{s_0}(O_i)$ (un tel indice existe par choix de s_0).
Si $y \in K_{2s_0}(O_i)$, alors par la proposition correspondante (proposition A.2 si $i = 0$, proposition A.3 si $i > 0$), on a $|u(x) - u(y)| \le H|x - y|^{\kappa_i} \le H|x - y|^{\kappa_i-\kappa}|x - y|^\kappa$; or, d étant un majorant du diamètre de Ω, on a $|x - y| \le d$ et $\kappa_i - \kappa \ge 0$, donc $|x - y|^{\kappa_i-\kappa} \le d^{\kappa_i-\kappa}$. On a donc, pour tout $y \in K_{2s_0}(O_i)$, $|u(x) - u(y)| \le H d^{\kappa_i-\kappa}|x - y|^\kappa$.

Si ce n'est pas le cas, on a $\mathrm{dist}(y, \mathbb{R}^N \backslash O_i) < 1/(2s_0)$, donc, par 1-lipschitziannité de la fonction distance,

$$|x-y| \geq \mathrm{dist}(x, \mathbb{R}^N \backslash O_i) - \mathrm{dist}(y, \mathbb{R}^N \backslash O_i) \geq \frac{1}{s_0} - \frac{1}{2s_0} = \frac{1}{2s_0};$$

on déduit alors de (A.77) que $|u(x) - u(y)| \leq 2H \leq 2H(1/2s_0)^{-\kappa}|x-y|^{\kappa}$.
En posant donc $C = \sup(\sup_{i \in [0,m]}(Hd^{\kappa_i - \kappa}), 2H(2s_0)^{\kappa})$, qui ne dépend que de

$$(\Omega, \alpha_A, \Lambda_A, N_*, \chi, r, \Lambda, p, \Lambda_L, M),$$

on a, pour tous $(x, y) \in \Omega$,

$$|u(x) - u(y)| \leq C|x-y|^{\kappa}. \tag{A.78}$$

Le résultat du théorème découle alors de (A.77) et (A.78). ∎

Etude de certaines Equations aux Dérivées Partielles

Résumé

La première partie de ce travail concerne les équations elliptiques non coercitives. Nous prouvons, tout d'abord dans un cadre linéaire, l'existence et l'unicité d'une solution faible dans l'espace d'énergie habituel $H^1(\Omega)$ pour une classe d'équations de convection-diffusion pour lesquelles le terme de convection provoque la perte de coercitivité. Nous prouvons des résultats de régularité höldérienne sur les solutions de ces équations qui permettent ensuite de résoudre ces mêmes équations avec un second membre mesure. Nous étendons aussi les résultats d'existence et d'unicité d'une solution dans des cas variationnels non-linéaires non-coercitifs et nous étudions, pour une équation elliptique linéaire non-coercitive, la convergence d'un schéma volumes finis.

La deuxième partie concerne l'unicité des solutions à des problèmes elliptiques non-linéaires avec seconds membres mesure.

La troisième partie aborde la question de la condition d'hyperbolicité des systèmes du premier ordre à coefficients constants. Nous prouvons une CNS pour qu'un tel système ait une solution pour toute condition initiale de type Riemann (condition initiale naturelle dans l'étude des discrétisations numériques de ces systèmes). A l'aide d'un système particulier, nous étudions ensuite la différence entre notre CNS et les diverses conditions d'hyperbolicité de la littérature, puis nous prouvons que la solution d'un système hyperbolique n'est pas toujours stable par rapport au flux.

La quatrième partie rassemble quelques autres travaux. Le premier concerne la densité dans $W^{1,p}(\Omega)$ des fonctions régulières satisfaisant une condition de Neumann. Le second est l'étude d'une discrétisation EF mixtes—VF pour un écoulement diphasique à travers un milieu poreux. Le troisième et dernier est l'étude des mesures sur $]0,T[\times\Omega$ ne chargeant pas les boréliens de capacité parabolique nulle et l'application de cette étude à la résolution d'une équation parabolique non-linéaire avec second membre mesure.

Mots-clés: Equations elliptiques non coercitives, Termes de convection, Solutions par dualité, Régularité höldérienne, Données mesures, Unicité, Stabilité, Hyperbolicité, Condition initiale de type Riemann, Densité dans les espaces de Sobolev, Schéma éléments finis mixtes—volumes finis, Capacité parabolique et mesures, Solutions renormalisées.

Study of some Partial Differential Equations

Abstract

The first part of this work concerns non-coercive elliptic equations. We first prove existence and uniqueness of a weak solution in the usual energy space $H^1(\Omega)$ for a class of linear convection-diffusion equations in which the convection term entails the loss of coercivity. We prove Hölder regularity results for the solutions of these equations, and this allows us to solve the same equations with a measure right-hand side. We also extend the existence and uniqueness results to the variational nonlinear noncoercive case. We study then, for a linear noncoercive elliptic equation, the convergence of a finite volume scheme.

The second part concerns the uniqueness of solutions to nonlinear elliptic problems with a measure right-hand side.

In the third part, we study the hyperbolicity condition for first order systems with constant coefficients. We prove a necessary and sufficient condition for such a system to have solutions for any initial condition of Riemann type (a natural initial condition in the study of numerical schemes for such systems). Thanks to a particular system, we study the difference between our condition and the several hyperbolicity conditions of the literature, and we then prove that the solution of a hyperbolic system is not always stable with respect to the flux.

The fourth part gathers some other works. The first work concerns the density in $W^{1,p}(\Omega)$ of regular functions satisfying a Neumann condition. The second is the study of a Mixed Finite Element—Finite Volume scheme for a two-fluids flow through a porous media. The third and last is the study of measures on $]0,T[\times\Omega$ that do not charge sets of null parabolic capacity and the application of this study to a nonlinear parabolic equations with measure right-hand side.

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