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PROPRIÉTÉS QUALITATIVES D'ONDES SOLITAIRES SOLUTIONS D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES NON LINÉAIRES DISPERSIVES

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Résumé.

Ce mémoire de thèse porte sur les solutions non nulles à l'infini de certaines équations non linéaires dispersives.
On y établit des résultats d'existence de *dark solitons* pour des équations de Schrödinger non linéaires et pour des systèmes d'équations de Schrödinger couplées, en dimension 1.
On étudie la stabilité linéaire d'un type particulier de *dark solitons* des équations de Schrödinger non linéaires en dimension 1: les *black solitons*.
On étudie le problème de Cauchy pour l'équation de Schrödinger, linéaire puis non linéaire, dans les espaces de Zhidkov $X^k(\mathbb{R}^n)$.
On étudie ce même problème de Cauchy dans l'espace affine $\varphi + H^1$, pour une fonction φ (non nulle à l'infini) régulière et d'énergie finie.
On étudie le problème de Cauchy pour des équations non linéaires dispersives de type Korteweg-de Vries ou Benjamin-Ono, sur les espaces de Zhidkov $X^s(\mathbb{R})$.

Mots clefs. Equations non linéaires dispersives, Equation de Schrödinger, Dark soliton, Problème de Cauchy, Stabilité, Equation de Korteweg-de Vries, Equation de Benjamin-Ono, Onde solitaire, Inegalités de Strichartz.

Abstract.

This PhD thesis is concerned with non-zero at infinity solutions of some nonlinear dispersive equations.
We establish existence results for dark solitons of nonlinear Schrödinger equations and of system of two coupled nonlinear Schrödinger equations, in dimension 1.
We study the linear stability of a special kind of dark solitons of nonlinear Schrödinger equations: the black solitons.
We study the Cauchy problem for the linear and for the nonlinear Schrödinger equations on the Zhidkov spaces $X^k(\mathbb{R}^n)$.
We study the same initial value problem on the affine space $\varphi + H^1$, where φ is a (non -zero at infinity) regular function with finite energy.
We study the Cauchy problem for nonlinear dispersive equations which look like the Korteweg-de Vries or the Benjamin-Ono equations, on the Zhidkov spaces $X^s(\mathbb{R})$.

Key words. Nonlinear dispersive equations, Schrödinger equation, Dark soliton, Cauchy problem, Stability, Korteweg-de Vries equation, Benjamin-Ono equation, Solitary wave, Strichartz estimates.

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Introduction.

1 Les équations non linéaires dispersives.

Les équations non linéaires dispersives peuvent s'écrire de façon générique sous la forme

$$\frac{\partial u}{\partial t} = Au + F(u), \tag{1}$$

où u est une fonction réelle ou complexe d'une variable de temps $t \in \mathbb{R}$ et d'une variable d'espace $x \in \mathbb{R}^n$, A est un opérateur anti-adjoint et $F(u)$ un terme non-linéaire. Souvent (ce sera en particulier vrai pour toutes les équations non linéaires dispersives étudiées dans cette thèse), A peut être défini grâce à sa transformée de Fourier par

$$\widehat{Au}(\xi) = ip(\xi)\widehat{u}(\xi), \tag{2}$$

où la fonction à valeur réelle p est appelée symbole de l'opérateur A.

Le caractère "dispersif" de l'équation (1) n'est lié qu'à l'équation linéaire associée

$$\frac{\partial u}{\partial t} = Au. \tag{3}$$

Il signifie que les modes de Fourier $\left(e^{ix.\xi}\right)_{\xi\in\mathbb{R}^n}$ sont propagés par (3) à des vitesses dépendant de ξ. Plus précisément, en se restreignant aux cas où l'opérateur A peut être défini par (2), la relation de dispersion de (3), obtenue en recherchant les ondes planes $e^{ix.\xi-i\omega t}$ solutions de (3), s'écrit $\omega(\xi) = -p(\xi)$. (1) est dite dispersive si le déterminant de la matrice Hessienne de ω est non identiquement nul.

Du fait de la dispersion, les solutions localisées de l'équation linéaire (3) ont tendance à "s'étaler" quand t tend vers l'infini, et à converger uniformément vers 0. Vis à vis de ce comportement, le rajout d'un terme non linéaire peut avoir deux effets complètement différents. Soit il compense l'effet de la dispersion, permettant ainsi à l'équation non linéaire (1) d'avoir des solutions de type "ondes progressives", c'est à dire des profils se déplaçant sans se déformer cours du temps. A contrario, la non-linéarité peut renforcer l'effet de la dispersion. Dans ce cas, (1) n'admet pas d'onde progressive localisée. Cependant, des ondes progressives d'un autre type peuvent alors être solution de (1). Il s'agit des *dark solitons*, qui sont non nuls à l'infini.

Dans cette thèse, nous nous intéresserons exclusivement à des équations non linéaires dispersives admettant des solutions non nulles à l'infini. On considèrera principalement des équations de Schrödinger non linéaires défocalisantes, et dans une moindre mesure des équations de type Korteweg-de Vries ou Benjamin-Ono.

2 Utilisation d'équations de Schrödinger non linéaires défocalisantes en physique: quelques exemples.

L'équation de Schrödinger modélise des phénomènes physiques aussi nombreux que variés. Nous ne mentionnerons ici que deux d'entre eux, pour lesquels l'existence de *dark solitons* et leur étude présente un intérêt particulier. Ces deux exemples sont les condensats de Bose-Einstein d'une part, et d'autre part la modulation de l'amplitude d'une onde quasi-monochromatique dans un milieu non linéaire, en optique.

2.1 Condensat de Bose-Einstein et équation de Gross-Pitaevskii.

A très basse température, dans un gaz de bosons sans interaction réciproque, une partie des particules se condense dans l'état quantique d'énergie minimale, formant ce qu'on appelle un condensat de Bose-Einstein. Leur existence à été prédite par A. Einstein en 1925, tandis que leur observation expérimentale a eu lieu pour la première fois en 1995 par une équipe dirigée par E. Cornell et C. Wieman. E.P. Gross et L.P. Pitaevskii ont modélisé ce phénomène par l'équation qui porte leur nom et qui s'écrit, après adimensionnement,

$$i\frac{\partial u}{\partial t} + \Delta u + (1 - |u|^2)u \;=\; 0. \tag{4}$$

u est ici une fonction des variables $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, qui représente (à l'adimensionnement près) la fonction d'onde des bosons. $|u(t, x)|$ tend vers 1 lorsque x tend vers l'infini. Le condensat est décrit par une zone où le module de u est petit par rapport à 1.

Les très basses températures nécessaires à la formation d'un condensat sont obtenues par piégeage laser. Considérant le faisceau laser cylindrique comme un obstacle au bord duquel la fonction d'onde s'annule (voir [AB] et les références qui s'y trouvent), il est également intéressant d'étudier l'équation de Gross-Pitaevskii (4) dans le complémentaire de cet obstacle.

Parmi les équations de Schrödinger non linéaires, l'équation de Gross-Pitaevskii est la plus étudiée (c'est même cette équation que certains auteurs désignent lorsqu'ils parlent d'équation de Schrödinger non linéaire). Dans la suite, elle nous servira souvent de cas modèle.

2.2 Equations de Schrödinger non linéaires défocalisantes en optique.

On va ici montrer l'une des façons par laquelle l'équation de Schrödinger non linéaire est obtenue en optique. Il s'agit de modéliser la propagation d'une onde quasi monochromatique dans un milieu faiblement non linéaire. On part des équations de Maxwell pour un isolant parfait:

$$\mathrm{div}D = 0,$$

$$\mathrm{rot}E = -\frac{\partial B}{\partial t},$$

$$\mathrm{div}B = 0,$$

$$\mathrm{rot}H = \frac{\partial D}{\partial t},$$

où D désigne le déplacement électrique, E le champ électrique, B l'induction magnétique et H le champ magnétique. B et D s'expriment en fonction de E et H de la manière suivante.

$$B = \mu H,$$

$$D = \varepsilon_0 E + P,$$

où μ est la perméabilité magnétique du milieu, ε_0 la permittivité du vide et P le vecteur polarisation. Des équations de Maxwell, il vient

$$-\Delta E + \text{grad div}E = \text{rot rot}E = -\frac{\partial \mathrm{rot}B}{\partial t} = -\mu\frac{\partial \mathrm{rot}H}{\partial t} = -\mu\frac{\partial^2 D}{\partial t^2} = -\mu\varepsilon_0\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial^2 P}{\partial t^2}.$$

Dans un milieu isotrope, le vecteur polarisation s'exprime

$$P = \varepsilon_0 \chi_0 E,$$

où χ_0 est la susceptibilité du milieu. En particulier, E est colinéaire à D, et si χ_0 est constant, $\mathrm{div}E = 0$. E est donc solution de l'équation des ondes

$$\Delta E - \mu\varepsilon_0(1+\chi_0)\frac{\partial^2 E}{\partial t^2} = 0,$$

qui se réécrit

$$\Delta E - \frac{n^2}{c_0^2}\frac{\partial^2 E}{\partial t^2} = 0,$$

où $n = \sqrt{\frac{\mu(1+\chi_0)}{\mu_0}}$ est l'indice de réfraction du milieu et $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ la vitesse de la lumière dans le vide. Si $n \equiv n_0$ est constant, les ondes planes

$$E_0 = 2\mathfrak{Re}\left[\mathcal{E}e^{ikz-i\omega t}\overrightarrow{e}\right]$$

sont solutions de cette équation des ondes, où $\mathcal{E} \equiv \mathcal{E}_0$ est constant, $\overrightarrow{e}$ est la direction de polarisation, orthogonale à la direction de propagation $\overrightarrow{z}$ et $k\overrightarrow{z}$ le vecteur d'onde. On a alors la relation

$$\frac{c_0}{n_0} = \frac{\omega}{k}.$$

Si maintenant $n = n_0 + n_{\rm nl}(I)$, où $n_{\rm nl}$ est une petite perturbation de n_0, ne dépendant que de l'intensité lumineuse $I = |E|^2$, $\mathcal{E}$ n'est plus constante, et vérifie au premier ordre l'équation

$$2ik\frac{\partial \mathcal{E}}{\partial z} + \frac{\partial^2 \mathcal{E}}{\partial x^2} + \frac{\partial^2 \mathcal{E}}{\partial y^2} + 2k^2\frac{n_{\rm nl}}{n_0}\mathcal{E} = 0.$$

On a ici négligé $k\frac{\partial \mathcal{E}}{\partial z}$ devant $\frac{\partial^2 \mathcal{E}}{\partial z^2}$, compte tenu du fait qu'une petite variation de n ne modifie $\mathcal{E}$ que sur de grandes échelles d'espace.

La non-linéarité la plus employée dans ce cadre est la non-linéarité cubique, qui permet de modéliser le comportement de la lumière dans les milieux dits "Kerr". Dans ce cas, $n_{\rm nl} = n_3 I = n_3|\mathcal{E}|^2$, et après adimensionnement des variables, on est amené à l'étude de l'équation de Schrödinger non linéaire cubique à deux dimensions d'espace

$$i\frac{\partial u}{\partial z} + \Delta u \pm |u|^2 u = 0. \tag{5}$$

Avec un signe $+$, l'équation (5) est dite focalisante et n'admet que des ondes progressives localisées, tandis qu'avec le signe $-$, elle est dite défocalisante, et est susceptible d'admettre des *dark solitons* comme solutions. Dans ce dernier cas, on retrouve en fait l'équation de Gross-Pitaevskii. (4) est en effet équivalente à (5) avec le signe $-$, quitte à remplacer la variable d'espace dans la direction de propagation z par une variable temporelle t, et à changer u en $e^{-it}u$.

Dans ce cadre, la non-linéarité cubique n'est cependant pas la seule interessante d'un point de vue physique. Pour de fortes intensités lumineuses, certains milieux, comme certains polymères ou certains semi-conducteurs (voir [KLD]), ne peuvent plus être modélisés de façon satisfaisante par une non-linéarité "Kerr". Les physiciens considèrent alors des compétitions entre deux termes de type puissance:

$$n_{\rm nl}(I) = n_{2p+1}I^p + n_{4p+1}I^{2p}, \text{ où } \ p > 0 \text{ et } n_{2p+1}n_{4p+1} < 0.$$

Parmi ces non-linéarités, la plus fréquemment employée est la non-linéarité dite cubique-quintique

$$n_{\rm nl}(I) = n_3 I + n_5 I^2,$$

compétition entre un terme cubique focalisant ($n_3 > 0$) et un terme quintique défocalisant ($n_5 < 0$). Elle est par exemple utilisée pour les cristaux de p-toluène-sulfonate (voir [KLD]). En dehors de l'optique, sous certaines conditions supplémentaires sur les paramètres n_3 et n_5, cette non-linéarité est aussi très utilisée pour décrire certains gaz de bosons (voir [B], [BM], [BGMP] et les autres références qui s'y trouvent).

Pour de très fortes puissances, il apparaît souvent que la variation de l'indice du milieu ne peut pas dépasser une certaine valeur, d'où l'introduction de non-linéarités avec saturation, par exemple

$$n_{\rm nl}(I) = n_\infty\left(1 - \frac{1}{\left(1 + \frac{I}{I_{\rm sat}}\right)^p}\right).$$

Indépendamment du fait de modifier la non-linéarité pour mieux prendre en compte les propriétés de tel ou tel matériau, les physiciens sont aussi amenés à introduire des variantes de l'équation de Schrödinger non-linéaire. Par exemple, l'étude de l'interaction d'ondes monochromatiques dans un guide d'onde à deux dimensions mène à des systèmes du type

$$\left\{ \begin{array}{l} i\frac{\partial w}{\partial z} + r\frac{\partial^2 w}{\partial x^2} - w + v\overline{w} = 0 \\ i\sigma\frac{\partial v}{\partial z} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{w^2}{2} = 0, \end{array} \right. \tag{6}$$

où $r, s = \pm 1$ et $\sigma, \alpha > 0$ (voir [BK1],[BK2], [BDST]). Comme on le verra, pour certaines valeurs des paramètres r, s, σ, α, (6) admet pour solution des *dark solitons*, ou plus exactement un couple de deux *dark solitons*, ou bien un couple formé d'un *dark soliton* et d'un *bright soliton*.

Un autre exemple de système obtenu en optique est le système de Manakov ([SK],[Man]), qui décrit la propagation d'une lumière de polarisation quelconque dans un milieu défocalisant

$$i\frac{\partial u_\pm}{\partial z} + \frac{\partial^2 u_\pm}{\partial x^2} - (|u_+|^2 + |u_-|^2)u_\pm = 0, \tag{7}$$

où u_+ et u_- décrivent l'enveloppe du champ électrique dans deux directions transverses (z étant la direction de propagation) et orthogonales entre elles. On verra que ce système admet comme solution des *dark-bright solitons*.

2.3 Equation de Korteweg-de Vries avec données non nulles à l'infini.

L'équation de Korteweg-de Vries

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0 \tag{8}$$

permet de décrire l'écoulement d'un liquide dans un canal ([HSe],[ZG]). En particulier, en en cherchant des solutions qui tendent vers des constantes différentes en $+\infty$ et en $-\infty$, on peut modéliser des phénomènes de type mascaret ([BRS],[BoSc],[Pe]). Les mascarets sont des vagues qui peuvent remonter le courant de certaines rivières sur plusieurs kilomètres, à marée montante. Leur formation est liée a un niveau de l'eau qui devient supérieur du côté de la mer qu'en amont de la rivière.

3 Dark solitons de l'équation de Schrödinger non linéaire et de ses variantes.

La littérature des physiciens est riche en conjectures concernant l'existence ou la stabilité de *dark solitons* pour les équations sus-mentionnées. D'où le besoin de vérifier mathématiquement ces conjectures. En particulier, pour l'équation de Schrödinger non linéaire en dimension 1, on va classifier les non-linéarités suivant les différents types de *dark solitons* auxquels elles peuvent donner lieu.

Dans la suite, on considèrera l'équation de Schrödinger non linéaire

$$i\frac{\partial u}{\partial t} + \Delta u + f(|u|^2)u = 0, \tag{9}$$

où u est une fonction complexe des variables $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ désigne le laplacien sur $\mathbb{R}^n$ et $f : \mathbb{R}_+ \mapsto \mathbb{R}$ est une fonction régulière. On se donne de plus un réel $\rho_0 > 0$ (avec la terminologie issue de l'optique, introduite en 1.2.2, il s'agit de l'intensité du fond lumineux), et on suppose $f(\rho_0) = 0$. Quitte à changer u en $e^{itf(\rho_0)}u$, cette dernière condition n'est pas restrictive. On suppose de plus f défocalisante, ce qui se traduit a minima par l'hypothèse $f'(\rho_0) \leqslant 0$, et la plupart du temps par $f'(\rho_0) < 0$. On ne s'intéressera dans ce mémoire qu'à des solutions u de (9) dont l'énergie

$$\mathcal{E}(u) := \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} V(|u|^2) dx \tag{10}$$

est finie, où V est donné par:

$$V(r) := \int_r^{\rho_0} f(s) ds. \tag{11}$$

Par exemple, lorsque $\rho_0 = 1$ et $f(r) = 1 - r$, (9) n'est autre que l'équation de Gross-Pitaevskii (4).

3.1 Existence et stabilité de *dark solitons* en dimension n $\geqslant$ 2.

Les ondes progressives de l'équation de Gross-Pitaevskii.

Dans leurs travaux de 1982 [JR] et 1986 [JPR], C.A. Jones, S.J. Putterman et P.H. Roberts semblent être les premiers à s'être sérieusement posé, en dimensions 2 et 3, la question de l'existence d'ondes progressives d'énergie finie pour l'equation de Gross-Pitaevskii, c'est à dire des solutions non constantes de (4) qui s'écrivent sous la forme

$$u(x_1, \dots, x_n) = v(x_1 - ct, x_2, \dots, x_n),$$

où v est solution de

$$-ic\frac{\partial v}{\partial x_1} + \Delta v + v(1 - |v|^2) = 0, \tag{12}$$

et dont l'énergie

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} (1 - |u|^2)^2 dx$$

est finie. Dans ces articles est conjecturée l'existence d'ondes progressives pour toutes les vitesses comprises entre 0 et $\sqrt{2}$ (et seulement pour ces vitesses). Ils y décrivent aussi le comportement à l'infini de ces ondes progressives, ainsi que la structure de leurs vortex, pour des vitesses proches de 0. La vérification de leurs conjectures à nécessité de nombreux travaux de la part des mathématiciens, dont va mentionner ici les principaux résultats. Concernant l'existence d'ondes progressives en dimension 2, F. Béthuel et J.C. Saut [BeSa1], [BeSa2] ont montré par des méthodes variationnelles les résultats suivants.

Théorème 3.1.1 *([BeSa1]) En dimension $n = 2$, il existe $c_0 > 0$ tel que pour toute vitesse $c \in (0, c_0)$, l'équation (12) admet une solution v_c non constante et d'énergie finie. De plus, il existe des constantes Λ_0 et Λ_1 telles que*

$$2\pi|\log c| + \Lambda_0 \leqslant \frac{1}{2}\mathcal{E}(v_c) \leqslant 2\pi|\log c| + \Lambda_1.$$

Théorème 3.1.2 *([BeSa2]) En dimension $n = 2$, il existe une suite $c_j \in (0, \sqrt{2})$ tendant vers $\sqrt{2}$ telle que pour tout entier j, l'équation (12) (où $c = c_j$) admet une solution non constante et d'énergie finie.*

En dimension $n \geqslant 3$, également par des méthodes variationnelles, F. Béthuel, G. Orlandi et D. Smets ([BOS]) ont obtenu le résultat suivant d'existence d'ondes progressives pour des petites vitesses.

Théorème 3.1.3 *Pour $n \geqslant 3$, il existe une suite $c_j \in (0, \sqrt{2})$ tendant vers 0 telle que pour tout entier j, l'équation (12) (où $c = c_j$) admet une solution non constante et d'énergie finie.*

D. Chiron ([C]) a par la suite complété ce résultat, afin d'obtenir l'existence d'ondes progressive sur un intervalle de petites vitesses.

Théorème 3.1.4 *([C]) Pour $n \geqslant 3$, il existe $c_0 > 0$ tel que pour tout $c \in (0, c_0)$, l'équation (12) admet une solution non constante et d'énergie finie.*

En complément de ces résultats d'existence d'ondes progressives, ont été aussi obtenus des résultats de non-existence. Ainsi, en dimension $n \geqslant 2$, F. Béthuel et J.C. Saut ont montré la non-existence d'onde progressive non constante, d'énergie finie et de vitesse nulle, tandis que P. Gravejat l'a montrée pour les vitesses supersoniques, grâce à une étude fine des noyaux de convolution apparaissant dans la formulation hydrodynamique de (12). Plus précisément, on a les résultats suivants

Théorème 3.1.5 *([BeSa1]) En dimension $n \geqslant 2$, toute solution d'énergie finie de (12) avec $c = 0$ est une constante de module 1.*

Théorème 3.1.6 *([Gr2],[Gr4]) En dimension $n \geqslant 2$, toute solution d'énergie finie de (12) avec $c > \sqrt{2}$ est une constante de module 1.*
En dimension $n = 2$ ce résultat reste vrai pour $c = \sqrt{2}$.

En dehors de la question de leur existence ou de leur non-existence, on dispose de résultats donnant des éléments de description de ces ondes progressives. Notamment, A. Farina ([F]) a donné une majoration de leur module.

Théorème 3.1.7 *([F]) En dimension $n \geqslant 1$, si v_c est solution de (12), alors pour tout $x \in \mathbb{R}^n$,*

$$|v_c(x)| \leqslant \sqrt{1 + \frac{c^2}{4}}.$$

Le comportement à l'infini de ces ondes progressives est décrit par les deux résultats suivants, qui confirment des conjectures échafaudées par C.A. Jones, S.J. Putterman et P.H. Roberts ([JR],[JPR]). Le second, dû à P. Gravejat, affine le premier, dû à F. Béthuel et J.C. Saut ([BeSa1],[BeSa2]), et le généralise à des dimensions autres que 2.

Théorème 3.1.8 *([BeSa1],[BeSa2]) En dimension $n = 2$, si v est une solution d'énergie finie de (12), où $c \in (0, \sqrt{2})$, alors, $v(x)$ tend vers un complexe de module 1 lorsque x tend vers l'infini.*

Théorème 3.1.9 *([Gr1],[Gr3],[Gr5]) En dimension $n \geqslant 2$, si v est une solution d'énergie finie de (12), où $c \in (0, \sqrt{2})$, alors, à multiplication par un complexe de module 1 près, v admet le développement asymptotique*

$$v(x) \underset{x \to \infty}{=} 1 + \frac{i}{|x|^{n-1}} v_\infty \left(\frac{x}{|x|} \right) (1 + o(1)),$$

où v_∞ est une fonction régulière définie sur la sphère, connue explicitement.

Enfin, notons que A. Aftalion et X. Blanc ont montré l'existence d'ondes progressives pour l'équation de Gross-Pitaevskii (4) dans le complémentaire de la boule unité de $\mathbb{R}^2$.

Théorème 3.1.10 *([AB]) Il existe $c_0 > 0$ tel que pour tout $c \in (0, c_0)$, l'équation (12) sur $\mathbb{R}^2 \backslash \overline{B(0,1)}$, avec conditions de Dirichlet au bord de $B(0,1)$, admet une solution u_c sans vortex (c'est à dire $|u_c| > 0$ sur $\mathbb{R}^2 \backslash \overline{B(0,1)}$), où $B(0,1)$ désigne la boule unité de $\mathbb{R}^2$.*

Ce dernier résultat permet à A. Aftalion et X. Blanc de montrer l'existence de solution sans vortex pour une équation proche de l'équation de Gross-Pitaevskii, dans un domaine de $\mathbb{R}^3$ (voir [AB]).

La question de la stabilité de toutes ces ondes progressives pour l'équation de Gross-Pitaevskii en dimension $n \geqslant 2$, aussi bien sur $\mathbb{R}^n$ que sur un domaine extérieur, est à ce jour un problème ouvert.

Les bulles stationnaires.

Contrairement à ce qui se produit pour l'équation de Gross-Pitaevskii, comme l'atteste le Théorème 3.1.5, il existe des non-linéarités pour lesquelles l'équation de Schrödinger non linéaire admet des solutions non nulles à l'infini, d'énergie finie, stationnaires et non constantes.

Sous les hypothèses sur f et ρ_0 faites au début de cette section, on désigne par bulle stationnaire de (9) toute solution de (9) qui de plus est radiale, prend ses valeurs (réelles) dans l'intervalle $(0, \sqrt{\rho_0})$, est strictement croissante sur toute demi-droite issue de 0 et tend vers $\sqrt{\rho_0}$ lorsque x tend vers l'infini.

L'existence et la stabilité de ce type de solution à fait l'objet de travaux de la part de I.V. Barashenkov, A.D. Gocheva, V.G. Makhankov, I.V. Puzynin et E.Y. Panova ([BGMP], [BP], [B]) d'une part, A. de Bouard ([dB]) d'autre part.
Ces travaux, finalisés par A. de Bouard, aboutissent d'une part à une condition suffisante sur la non linéarité f assurant l'existence d'une bulle stationnaire, et d'autre part au fait que ces bulles stationnaires sont toutes instables.

Théorème 3.1.11 *([dB]) En dimension $n \geqslant 2$, si $f \in \mathcal{C}^1(\mathbb{R}_+)$ et $f'(\rho_0) < 0$, et s'il existe $\rho_1 \in [0, \rho_0)$ tel que $V(\rho_1) < 0$, alors il existe une bulle stationnaire φ solution de (9).*
De plus, si cette condition est vérifiée, alors φ est de classe $\mathcal{C}^2$, et $|\partial^\alpha(\varphi - \rho_0^{1/2})(x)|$ tend exponentiellement vers 0 quand $x \to \infty$ pour tout multi-indice α, tel que $|\alpha| \leqslant 2$.

Théorème 3.1.12 *([dB]) Soit $f \in \mathcal{C}^{m+2}(\mathbb{R}_+)$ vérifiant les hypothèses du Théorème 3.1.11, où $m > n/2$. Soit φ la bulle stationnaire de (9) donnée par le Théorème 3.1.11.*
Alors φ est instable au sens suivant: il existe $\varepsilon > 0$ tel que pour tout $\delta > 0$, il existe $u_0 \in H^m(\mathbb{R}^n)$ avec $\|u_0\|_{H^1} < \delta$ et tel que, si $v(t) = \varphi + u(t)$ est l'unique solution $v \in \varphi + \mathcal{C}(T_\star, T^\star, H^m)$ de (9) avec donnée initiale $v(0) = \varphi + u_0$, alors il existe $t_0 \in (0, T^\star)$ tel que $\|u(t_0)\|_{H^1} > \varepsilon$.

La preuve de ce résultat repose sur une analyse du spectre de l'opérateur obtenu par linéarisation de (9) au voisinage de la bulle stationnaire φ.

Les bulles progressives.

Sous des hypothèses légèrement plus restrictives sur la non-linéarité f que celles du Théorème 3.1.11 assurant l'existence de bulles stationnaires, M. Mariş [Mar3] a montré, en dimension $n \geqslant 4$, l'existence de bulles progressives de petites vitesses, solutions de l'équation (9). Ces bulles progressives sont des solutions de (9) qui s'écrivent sous la forme

$$u(x_1, \ldots, x_n) = v_c(x_1 - ct, x_2, \ldots, x_n),$$

admettant une symétrie radiale dans les variables $x_2, \ldots, x_n$.

Théorème 3.1.13 *([Mar3]) Soit $n \geqslant 4$. Il existe $c_0 > 0$ tel que pour toute vitesse $c \in (-c_0, c_0)$, l'équation (9) admet une unique bulle progressive de vitesse c notée $v_c = \sqrt{\rho_0} + u_{1c} + iu_{2c}$, où $u_{1c} \in H^1(\mathbb{R}^n)$ et $u_{2c} \in \dot{H}^1(\mathbb{R}^n)$.*
De plus, quand $c \to 0$, $u_{1c} \to \varphi - \sqrt{\rho_0}$ dans H^1, où φ est la bulle stationnaire solution de (9) donnée par le Théorème 3.1.11, et $u_{2c} \to 0$ dans $\dot{H}^1$.

Ces bulles sont obtenues comme point critique d'une fonctionnelle E_c. Les hypothèses sur la non-linéarité f requises pour la démonstration sont en particulier vérifiées pour la non-linéarité cubique-quintique $f(r) = \alpha_1 + \alpha_3 r + \alpha_5 r^2$, où $\alpha_1 < 0$, $\alpha_3 > 0$, $\alpha_5 < 0$ et $\alpha_1\alpha_5/\alpha_3^2 \in (3/16, 1/4)$, cette dernière condition étant celle sous laquelle l'équation de Schrödinger cubique-quintique admet une bulle stationnaire.

3.2 Existence et stabilité des *dark solitons* en dimension 1.

En dimension 1, les questions concernant l'existence de *dark solitons* et leur stabilité, pour des non-linéarités très variées, est relativement bien comprise, en regard des nombreux problèmes restant ouverts à ce jour, ne serait-ce qu'en ce qui concerne les ondes progressives de l'équation de Gross-Pitaevskii en dimensions supérieures ou égales à deux.

Les bulles stationnaires en dimension 1.

Concernant les bulles stationnaires en dimension 1, on dispose de résultats analogues à ceux des dimensions supérieures. On dispose même d'une condition nécessaire et suffisante portant sur la non-linéarité f, assurant leur existence.

Théorème 3.2.1 *([dB]) En dimension $n = 1$, si $f \in \mathcal{C}^1(\mathbb{R}_+)$ et $f'(\rho_0) \leqslant 0$, il existe une bulle stationnaire φ solution de (9) si et seulement si $\eta_0 := \sup\{\eta \in (0, \rho_0), V(\eta) = 0\}$ existe, $\eta_0 \in (0, \rho_0)$ et $f(\eta_0) < 0$.*
De plus, si cette condition est vérifiée et si $f'(\rho_0) < 0$, alors φ est de classe $\mathcal{C}^2$, et $\partial^\alpha(\varphi - \rho_0^{1/2})(x)$ tend exponentiellement vers 0 quand x tend vers $+\infty$ pour $\alpha \leqslant 2$.

Comme en dimension $n \geqslant 2$, les bulles stationnaires en dimension 1 sont toutes instables.

Théorème 3.2.2 *([dB]) Soit $f \in \mathcal{C}^3(\mathbb{R}_+)$ vérifiant les hypothèses du Théorème 3.2.1. Soit φ la bulle stationnaire de (9) donnée par le Théorème 3.2.1.*
Alors φ est instable au sens spécifié au Théorème 3.1.12, avec $m = 1$.

Les bulles progressives.

On appelle bulle progressive de (9), de vitesse v toute solution de (9) de la forme

$$u(t, x) = \varphi_v(x - vt),$$

dont le module $|\varphi_v|$ vérifie les propriétés caractérisant une bulle stationnaire (parité, positivité stricte sur $\mathbb{R}$, croissance stricte vers $\sqrt{\rho_0}$ sur $\mathbb{R}_+$). Notons qu'à la différence d'une bulle stationnaire, une bulle progressive n'est pas supposée rélle (et de fait, elle ne le sera jamais).

Ces bulles progressives ont été étudiées par Z. Lin ([L]). Comme pour les bulles stationnaires en dimension 1, on dispose d'une condition nécessaire et suffisante, portant sur la non-linéarité f et sur la vitesse v, assurant l'existence d'une bulle progressive de vitesse v.

Théorème 3.2.3 *([L]) Si $n = 1$ et $f \in \mathcal{C}^1(\mathbb{R}_+)$, il existe une bulle progressive φ_v de (9) de vitesse v si et seulement si les conditions suivantes sont vérifiées:*

- $f(\rho_0) = 0$
- $\eta_0 := \sup\{\eta \in (0, \rho_0), V_v(\eta) := V(\eta) - \frac{v^2(\eta-\rho_0)^2}{4\eta} = 0\}$ *existe et* $\eta_0 \in (0, \rho_0)$
- $f_v(\eta_0) := f(\eta_0) + \frac{v^2}{4}(1 - \frac{\rho_0^2}{\eta_0^2}) = -V_v'(\eta_0) < 0.$

De plus, si ces conditions sont vérifiées, φ_v est de classe $\mathcal{C}^3$. Sous la condition supplémentaire $f'(\rho_0) < 0$, φ_v ainsi que ses dérivées d'ordre 1 à 3 convergent exponentiellement vers leurs limites à l'infini.

Ce résultat appelle plusieurs commentaires. D'abord, à condition de voir les bulles stationnaires comme des bulles progressives particulières, il englobe le Théorème 3.2.1. Ensuite, il est facile de voir qu'il interdit l'existence de bulle progressive pour les vitesses supersoniques (c'est à dire les vitesses v vérifiant $v^2 > -2\rho_0 f'(\rho_0) = c^2$, où c est la vitesse du son). Enfin, sous l'hypothèse $f'(\rho_0) < 0$, pour toutes les vitesses $v \in (0, c)$, $\eta_0 \in (0, \rho_0)$ existe, et $f_v(\eta_0) \leqslant 0$. Pour des vitesses subsoniques non nulles, les seuls couples non-linéarité/vitesse pour lesquels il peut ne pas exister de bulle progressive sont ceux qui vérifient la condition $f_v(\eta_0) = 0$. Ce cas est certes pathologique, mais tout à fait possible, comme on le montre dans [Ga2].

Théorème 3.2.4 *([Ga2]) Si $f \in \mathcal{C}^1(\mathbb{R}_+)$ et $f'(\rho_0) < 0$, l'ensemble*

$$S = \{v \in (0, c), V_v'(\eta_0(v)) = 0\}$$

est fermé et ne contient aucun intervalle non trivial.
Il existe des non-linéarités f pour lesquelles S est infini.

Dans [L], Z. Lin donne également un critère de stabilité pour les bulles progressives. Ce critère était énoncé à l'origine pour des non-linéarités admettant une bulle stationnaire. Comme on le remarque dans [Ga2], il s'étend en fait à d'autres non-linéarités, pour les vitesses telles que les hypothèses du Théorème 3.2.3 sont vérifiées. Avant d'énoncer ce critère, commençons par donner la définition du moment P_v d'une bulle progressive φ_v:

$$P_v = \mathfrak{Im} \int_{-\infty}^{+\infty} \partial_x \overline{\varphi_v} \varphi_v (1 - \frac{\rho_0}{|\varphi_v|^2}) dx.$$

Le critère de stabilité de Z. Lin pour les bulles progressives s'énonce alors comme suit.

Théorème 3.2.5 *([L]) Soit $f \in C^1(\mathbb{R}_+)$ telle que $f'(\rho_0) < 0$. Soient v_1, v_2 tel que $-c < v_1 < v_2 < c$. On suppose que les hypothèses du théorème 3.2.3 assurant l'existence d'une bulle $\varphi_v = a_v e^{i\theta_v}$ sont vérifiées pour tout $v \in (v_1, v_2)$.*
Alors, pour tout $v \in (-v_1, v_2)$, cette bulle est stable si $\frac{dP_v}{dv} > 0$, et instable si $\frac{dP_v}{dv} < 0$.

Dans le Théorème 3.2.5, la stabilité est à comprendre au sens de la stabilité orbitale: pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que si la donnée initiale $\varphi^0 = a^0 e^{i\theta^0}$ vérifie

$$\inf_{s \in \mathbb{R}} (\|\tau_s {a^0}^2 - a_v^2\|_{H^1} + \|\tau_s \partial_x \theta^0 - \partial_x \theta_v\|_{L^2}) < \delta \ ,$$

alors

$$\inf_{s \in \mathbb{R}} (\|\tau_s a(t)^2 - a_v^2\|_{H^1} + \|\tau_s \partial_x \theta(t) - \partial_x \theta_v\|_{L^2}) < \varepsilon \ ,$$

pour $t \in (0, \infty)$, où on a noté $\varphi(t) = a(t)e^{i\theta(t)}$ la solution de (9) de donnée initiale $\varphi(0) = \varphi^0$, et τ_s la translation par s, à savoir $\tau_s f(x) = f(x - s)$. Le soliton est dit instable s'il n'est pas stable.

Pour établir ce résultat, Z. Lin travaille avec la formulation hydrodynamique de (9), c'est à dire qu'il en écrit les solutions sous la forme $(\rho_0 - r)^{1/2} e^{i\theta}$. Le système vérifié par (r, θ_x) a une structure Hamiltonienne, il peut donc lui appliquer la théorie sur la stabilité des ondes solitaires établie par M. Grillakis, J. Shatah et W.A. Strauss ([GSS]).

Pour résumer, étant donné une non-linéarité f et $\rho_0 > 0$, vérifiant $f(\rho_0) = 0$ et $f'(\rho_0) < 0$, pour les vitesses subsoniques non nulles, il existe le plus souvent (mais pas toujours) une bulle progressive, dont la stabilité ou l'instabilité est donnée par le critère 3.2.5. Sans tenir compte de cas très particuliers détaillés dans [Ga2], on peut séparer les non-linéarités défocalisantes (au sens où $f'(\rho_0) < 0$) en deux grandes familles, suivant le type de dark soliton auquel elles donnent lieu à vitesse nulle. Soit les conditions spécifiées au Théorème 3.2.1 sont satisfaites, et (9) admet comme solution une bulle stationnaire, qu'on sait être instable grâce au Théorème 3.1.12, soit (encore une fois, à des cas très particuliers près) V est strictement positif sur $[0, \rho_0)$. Dans ce dernier cas, un autre type de solution stationnaire est solution de (9): il s'agit d'un *black soliton*.

Les *black solitons*.

Les *black solitons* sont des solutions stationnaires de (9), rélles, impaires, qui réalisent un difféomorphisme de $\mathbb{R}$ sur $(-\sqrt{\rho_0}, \sqrt{\rho_0})$. A la différence des bulles décrites précédemment, les *black solitons* s'annulent en un point (d'où leur nom). A l'instar du Théorème 3.2.1 pour les bulles stationnaires, on dispose d'une condition nécessaire et suffisante sur la non-linéarité f assurant l'existence d'un *black soliton*.

Théorème 3.2.6 *([DMG],[Ga2]) On suppose que $f \in \mathcal{C}^1(\mathbb{R}_+)$ vérifie $f(\rho_0) = 0$. Il existe un black soliton de (9) si et seulement si*

$$V(r) > 0 \textit{ pour tout } r \in [0, \rho_0).$$

Si de plus $f'(\rho_0) < 0$, $\partial^\alpha(\sqrt{\rho_0} - \varphi)$ tend exponentiellement vers 0 lorsque x tend vers $+\infty$, pour $\alpha = 0, 1, 2$.

L'équation de Gross-Pitaevskii (4) est un exemple d'équation de Schrödinger non-linéaire admettant un *black soliton* pour solution. En effet, $V(r) = (1 - r)^2/2$, qui est strictement positif sur $[0,1)$. D'ailleurs, ses *dark solitons* sont explicitement donnés par

$$\psi_v(x) = \left(1 - \frac{v^2}{2}\right)^{1/2} \tanh\left((1 - \frac{v^2}{2})^{1/2} \frac{x}{\sqrt{2}}\right) + i\frac{v}{\sqrt{2}}. \tag{13}$$

Pour $v = 0$, on obtient le *black soliton* $\psi_0(x) = \tanh(x/\sqrt{2})$.

Compte tenu de leur point d'annulation, la stabilité des *black solitons* ne peut pas, comme pour les bulles, être étudiée sur la formulation hydrodynamique de (9), puisque l'argument et le module d'un *black soliton* présentent des singularités. Dans ([DMG]), écrivant une solution de (9) sous la forme $\varphi + u_1 + iu_2$, où φ est un *black soliton*, u_1 et u_2 sont réelles, et linéarisant au voisinage de $(u_1, u_2) = (0, 0)$, on se ramène à l'étude du problème linéaire

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \tag{14}$$

où

$$A = \begin{pmatrix} 0 & L_1 \\ -L_2 & 0 \end{pmatrix} ,$$
$$L_1 = -\partial_x^2 - f(\varphi^2) ,$$
$$L_2 = -\partial_x^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2) .$$

Avec ces notations, on montre dans ([DMG]) la condition suffisante suivante, assurant la stabilité linéaire d'un *black soliton*.

Théorème 3.2.7 *([DMG],[Ga2]) L_1 est un opérateur auto-adjoint sur L^2, de domaine H^2. Son spectre essentiel est $[0, +\infty)$, et L_1 a une unique valeur propre négative λ_0, qui est simple.*
La fonction de Vakhitov-Kolokolov

$$g(\lambda) \;\; := \;\; < (L_1 - \lambda)^{-1}\varphi', \varphi' > , \tag{15}$$

qui est bien définie pour $\lambda \in (\lambda_0, 0)$, est de classe $\mathcal{C}^1$, est croissante sur $(\lambda_0, 0)$, et $g(\lambda) \to -\infty$ lorsque $\lambda \downarrow \lambda_0$. Si la limite de $g(\lambda)$ quand $\lambda \uparrow 0$ est négative ou nulle, le spectre de A est inclus dans $i\mathbb{R}$, et φ est linéairement stable.

Pour une non-linéarité donnée, il est relativement facile de vérifier numériquement si cette condition assurant la stabilité linéaire d'un *black soliton* est vérifiée, en calculant celui-ci de manière approchée, puis en calculant la fonction de Vakhitov-Kolokolov au voisinage de 0. Il est en revanche nettement plus difficile de le montrer rigoureusement. On y parvient cependant pour l'équation de Gross-Pitaevskii, pour laquelle le *black soliton* est explicitement donné par (13), et les valeurs propres et vecteurs propres de l'opérateur L_1 sont explicitement connues. Ainsi, on obtient le résultat suivant.

Théorème 3.2.8 *([DMG]) Le black soliton $\varphi(x) = \tanh(x/\sqrt{2})$ de l'équation de Gross-Pitaevskii (4) est linéairement stable.*

On peut trouver des non-linéarités pour lesquelles la condition du Théorème 3.2.7 n'est pas vérifiée. Dans ce cas, des calculs numériques semblent indiquer que le *black soliton* est linéairement instable[1].

3.3 Existence de *dark solitons* pour des systèmes dérivés des équations de Schrödinger non linéaire.

Dans [Mar1] et [Mar2], M. Mariş montre l'existence de *dark-bright solitons*, solutions du système

$$\left\{\begin{array}{l} 2i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\frac{1}{\varepsilon^2}(|\psi|^2+\frac{1}{\varepsilon^2}|\varphi|^2-1)\psi=0 \\ 2i\delta\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}+\frac{1}{\varepsilon^2}(q^2|\psi|^2-\varepsilon^2k^2)\varphi=0, \end{array}\right. \tag{16}$$

où ε, q et k sont des constantes. Par *dark-bright soliton* on entend que ψ et φ s'écrivent sous la forme $\psi(t,x)=\tilde{\psi}(x-vt)$, $\varphi(t,x)=\tilde{\varphi}(x-vt)$, avec $|\tilde{\psi}(x)|\to 1$ et $|\tilde{\varphi}(x)|\to 0$ quand x tend vers l'infini. La méthode consiste dans un premier temps à appliquer le théorème des fonctions implicites au voisinage de la solution triviale de (16) qu'est, pour $v\neq 0$, le couple $(\psi_v(\varepsilon x-2v\varepsilon t),0)$, où ψ_v est une bulle progressive de Gross-Pitaevskii donnée par (13). Dans un second temps, M. Mariş décrit plus globalement la structure des courbes de solutions obtenues, grâce à un théorème de bifurcation de Rabinowitz.

Grâce aux conditions nécessaires et suffisantes sur la non linéarité assurant l'existence d'une bulle stationnaire (Théorème 3.2.1) ou progressive (Théorème 3.2.3) pour l'équation de Schrödinger non linéaire en dimension 1, cette méthode peut s'appliquer à des systèmes du même type que (16), mais avec des non-linéarités plus générales. On montre ainsi dans [Ga2] l'existence de *dark-bright solitons* pour le système

$$\left\{\begin{array}{l} i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2,|\varphi|^2)\psi=0 \\ i\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}+(g(|\psi|^2)+\theta)\varphi=0, \end{array}\right. \tag{17}$$

où $f,\ g\in\mathcal{C}^1(\mathbb{R}_+^2)$ vérifient

$$f(\rho_0,0)=0,$$
$$\frac{\partial f}{\partial x_1}(\rho_0,0)<0,$$
$$g(r,0)>g(\rho_0,0)\text{ pour } r\in[0,\rho_0).$$

Si la vitesse v est choisie de sorte que, notant $c=\sqrt{-2\rho_0\frac{\partial f}{\partial x_1}(\rho_0,0)}$ la vitesse du son, $v\in(-c,c)$ et il existe

$$\eta_0(v)=\sup\left\{\eta\in(0,\rho_0),\int_\eta^{\rho_0}f(s,0)ds-\frac{v^2(\eta-\rho_0)^2}{4\eta}=0\right\}<\rho_0,$$

1. A noter que, comme pour les bulles stationnaires ([dB]), lorsqu'un *black soliton* est linéairement instable, il est aussi orbitalement instable.

avec $f(\eta_0(v),0)+\frac{v^2}{4}(1-\frac{\rho_0^2}{\eta_0(v)^2})<0$, alors l'équation

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2,0)\psi=0 \tag{18}$$

admet une bulle $\widetilde{\psi_v}$ de vitesse v, et on peut appliquer le Théorème des fonctions implicites au voisinage de la solution triviale de (17) $(\widetilde{\psi_v},0)$, comme pour le système (16) étudié par M. Mariş. Notons que le système de Manakov (7) mentionné plus haut vérifie les hypothèses ci-dessus.

Dans [BK1] (voir aussi [BK2], [Ki]), A.V. Buryak et Y.S. Kivshar étudient le système (6). En particulier, ils en cherchent numériquement des solutions stationnaires. Ainsi, pour $r=-1$, $s=1$ et $\alpha>0$, ils conjecturent l'existence d'une famille de *dark-dark solitons*, avec les conditions à l'infini $w(x)\to\pm\sqrt{2\alpha}$ et $v(x)\to 1$ lorsque $x\to\pm\infty$. Pour $\alpha>8$, ces solitons sont monotones, tandis qu'ils ont des "queues oscillantes" pour $\alpha<8$. Dans [Ga4], on vérifie une partie de ces conjectures, grâce à une méthode de tir. Plus précisément, on montre le résultat suivant.

Théorème 3.3.1 *Soit $\alpha>8$. Alors le système*

$$\begin{array}{l} w''=-w+vw \\ v''=\alpha v-\frac{w^2}{2} \end{array} \tag{19}$$

admet une solution réelle $w,v\in\mathcal{C}^2(\mathbb{R})$, telle que

(i) w est impaire,

(ii) w croît strictement sur $\mathbb{R}$,

(iii) $w(x)\underset{x\to\pm\infty}{\longrightarrow}\pm\sqrt{2\alpha}$,

(iv) v est positive et paire,

(v) v croît strictement sur $\mathbb{R}_+$,

(vi) $v(x)\underset{x\to\pm\infty}{\longrightarrow}1$.

4 Le problème de Cauchy pour des équations de Schrödinger avec données non nulles à l'infini.

Du fait de l'existence de *dark soliton* solutions de certaines équations de Schrödinger non linéaires, se pose la question du problème de Cauchy dans des espaces contenant ces *dark solitons*. Savoir que le problème de Cauchy est globalement bien posé dans un "voisinage" (dans un sens à préciser) d'un *dark soliton* est en effet un renseignement précieux si on veut étudier sa stabilité, ou bien si, le sachant instable, on veut essayer de comprendre comment se manifeste cette instabilité.

On s'intéresse donc au problème de cauchy

$$\left\{\begin{array}{l} i\frac{\partial u}{\partial t}+\Delta u+f(|u|^2)u=0,\ (t,x)\in\mathbb{R}\times\Omega \\ u(0)=u_0,\ \text{où } |u_0|^2\underset{x\to\infty}{\longrightarrow}\rho_0, \end{array}\right. \tag{20}$$

où $\Omega = \mathbb{R}^n$. On suppose f régulière, telle que $f(\rho_0) = 0$ et $f'(\rho_0) < 0$. On peut par exemple choisir comme non-linéarité f l'une de celles mentionnées en 2: la non-linéarité de Gross-Pitaevskii $f(r) = 1 - r$, ou bien une non-linéarité de type "saturation", comme $f(r) = \frac{1}{(1+ar)^2} - \frac{1}{(1+a)^2}$,ou encore une non-linéarité cubique-quintique $f(r) = \alpha_1 + \alpha_3 r + \alpha_5 r^2$, où $\alpha_3 > 0$, $\alpha_5 < 0$...

Pour montrer que le problème de Cauchy (20) est globalement bien posé, il pourra être profitable de s'aider de deux des invariants de (20), conservés au moins formellement, que sont la charge

$$I(u) = \int_{\mathbb{R}^n} \left[|u(x)|^2 - \rho_0\right] dx$$

d'une part, et surtout l'énergie

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} \left[|\nabla u(x)|^2 + V(|u(x)|^2)\right]$$

d'autre part.

Historiquement, le premier résultat concernant un problème de Cauchy de type (20) semble être celui de F. Béthuel et J.C. Saut ([BeSa1]) concernant l'équation de Gross-Pitaevskii dans l'espace affine $1 + H^1$:

Théorème 4.0.2 *([BeSa1]) En dimension 1,2 et 3, le problème de Cauchy pour l'équation de Gross-Pitaevskii est globalement bien posé dans* $1 + H^1$.

Ce résultat, s'il est parfaitement satisfaisant en dimension 3, puisque les ondes progressives de l'équation de Gross-Pitaevskii appartiennent bien à l'espace $1 + H^1$, l'est moins en dimensions 1 et 2. En dimension 1 en effet, comme l'atteste leur explicitation (13), les *dark solitons* résolvant l'équation de Gross-Pitaevskii ne tendent pas vers la même constante en $-\infty$ et en $+\infty$; ils ne sont donc pas dans $1 + H^1$. En dimension 2, les ondes progressives de l'équation de Gross-Pitaevskii tendent bien vers 1 à l'infini (à multiplication par un complexe de module 1 près), mais, d'après leur développement asymptotique du Théorème 3.1.9 dû à P. Gravejat, ils ne sont pas dans $1 + L^2$. Le problème de Cauchy (20) doit donc être étudié dans un autre cadre fonctionnel.

En dimension 1, P.E. Zhidkov ([Z1],[Z2],[Z3],[Z4]) à étudié l'équation de Gross-Pitaevskii dans l'espace $X^1 = \{u \in L^\infty, u' \in L^2\}$. Dans [Ga1], on étend à une dimension quelconque la définition des espaces de Zhidkov

$$X^k(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \nabla u \in H^{k-1}(\mathbb{R}^n)\},$$

munis de leur norme naturelle. On commence par étudier le problème de Cauchy pour l'équation de Schrödinger linéaire sur $X^k(\mathbb{R}^n)$. On obtient le théorème suivant.

Théorème 4.0.3 *([Ga1],[Ga5]) L'équation de Schrödinger linéaire*

$$i\frac{\partial u}{\partial t} + \Delta u = 0 \tag{21}$$

est bien posée sur $X^k(\mathbb{R}^n)$ dès que $k > n/2$. Sous cette condition, il existe $C > 0$ tel que pour tout $t \in \mathbb{R}$, pour tout $u_0 \in X^k(\mathbb{R}^n)$,

$$\|e^{it\Delta}u_0\|_{X^k(\mathbb{R}^n)} \leqslant C(1+|t|)^\rho \|u_0\|_{X^k(\mathbb{R}^n)},$$

où

$$\rho = \begin{cases} \frac{1}{2}\frac{1}{1+\frac{1}{n}} & \textit{si } n \textit{ est pair,} \\ \frac{1}{4}\frac{1}{1+\frac{1}{2n}} & \textit{si } n \textit{ est impair.} \end{cases}$$

De plus, pour $n = 1$, $\rho = 1/6$ est optimal.

Dans [Ga1], on montre aussi que la condition $k > n/2$ est optimale, en exhibant une donnée initiale $u_0 \in L^\infty \cap H^{\lfloor \frac{n}{2} \rfloor}$, telle que la solution $u(t)$ de (21) avec donné initiale u_0, qui reste dans $H^{\lfloor \frac{n}{2} \rfloor}$ pour tout temps t, explose en norme L^∞.

La preuve du Théorème 4.0.3 consiste à montrer, grâce à des intégrations par partie, que la formule qui donne $u(t,x) = \left(e^{it\Delta}u_0\right)(t,x)$ en fonction de u_0 via le noyau de Schrödinger a bien un sens lorsque $u_0 \in X^k$, et à vérifier que $u(t)$ est continue à valeurs dans L^∞. Le fait que ∇u est continu à valeurs dans H^{k-1} est quant à lui évident. Notons qu'une preuve plus simple, donnée dans [Ga3], donne le même résultat, avec un moins bon ρ ($\rho = 1/2$). Elle consiste à passer en variables de Fourier et à montrer que $u(t) - u_0 \in \mathcal{C}(\mathbb{R}, H^k)$.

Une fois établi ce résultat concernant l'équation de Schrödinger linéaire, la théorie classique sur les équations d'évolution nous permet, dans [Ga1], de montrer que le problème de Cauchy pour l'équation non linéaire (20) est localement bien posé dans $X^k(\mathbb{R}^n)$, pour $k > n/2$. Pour obtenir un résultat d'existence globale, on suppose que la donnée initiale, en plus d'être dans $X^k(\mathbb{R}^n)$, est d'énergie finie. Ainsi en dimension 1, on obtient le résultat suivant.

Théorème 4.0.4 *([Ga1]) Soit $n = k = 1$. On suppose que f, en plus des conditions mentionnées au début de la section 4, vérifie $V(r) \geqslant C(\rho_0 - r)^2$, où $C > 0$ et V est donné par (11). On suppose que $u_0 \in X^1(\mathbb{R})$ est d'énergie $\mathcal{E}(u_0)$ finie. Alors il existe $u \in \mathcal{C}(\mathbb{R}, X^1)$, bornée dans X^1, solution de (20). De plus, l'énergie est conservée par le flot de Schrödinger.*

Une fois montrée la conservation de l'énergie, ce résultat s'obtient en remarquant que la norme H^1 de $|u|^2 - \rho_0$ est contrôlée par l'énergie. Notons cependant que la condition supplémentaire imposée à la non-linéarité est assez restrictive, puisque par exemple les non-linéarités saturées ou cubique-quintique mentionnées plus haut ne la vérifient pas.

A la suite de ces travaux, en utilisant une méthode similaire à celle employée par H. Brézis et T. Gallouët dans [BG], O. Goubet a montré que l'équation de Gross-Pitaevskii est globalement bien posée dans $X^2(\mathbb{R}^2)$:

Théorème 4.0.5 *([Go]) Le problème de Cauchy pour l'équation de Gross-Pitaevskii est globalement bien posé dans $X^2(\mathbb{R}^2)$ pour une donnée initiale u_0 dans cet espace ayant de plus une énergie $\mathcal{E}(u_0)$ finie.*

Plus récemment, P. Gérard a abordé sous un autre point de vue le problème de Cauchy (20), pour l'équation de Gross-Pitaevskii. Munissant l'espace d'énergie

$$E = \{u \in H^1_{\text{loc}}\ \nabla u \in L^2,\ 1 - |u|^2 \in L^2\}$$

d'une structure d'espace métrique complet grâce à la distance

$$d_E(u, v) = \|u - v\|_{X^1+H^1} + \||u|^2 - |v|^2\|_{L^2},$$

il montre dans [Ge] que le problème de Cauchy pour l'équation de Gross-Pitaevskii est globalement bien posé dans E, en dimensions 2 et 3, et aussi en dimension 4 pour des données initiales de petite énergie.

Théorème 4.0.6 *([Ge]) Soit $n = 2, 3$. Pour tout $u_0 \in E$, il existe une unique solution $u \in \mathcal{C}(\mathbb{R}, E)$ de l'équation de Gross-Pitaevskii (4) de donnée initiale $u(0) = u_0$. De plus,*
- *si $\Delta u_0 \in L^2(\mathbb{R}^n)$, alors $\Delta u \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$,*
- *pour tout temps t, $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$,*
- *$u - e^{it\Delta}u_0 \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$,*
- *pour tout $R > 0$ et tout $T > 0$, il existe $C > 0$ tel que pour tout $u_0, \widetilde{u_0} \in E$ tels que $\mathcal{E}(u_0) \leqslant R$ et $\mathcal{E}(\widetilde{u_0}) \leqslant R$, les solution $u, \widetilde{u}$ correspondantes vérifient*

$$\sup_{|t| \leqslant T} d_E(u(t), \widetilde{u}(t)) \leqslant C d_E(u_0, \widetilde{u_0}).$$

Pour $n = 4$, il existe $\delta > 0$ tel que pour tout $u_0 \in E$ vérifiant $\mathcal{E}(u_0) \leqslant \delta$, il existe une unique solution $u \in \mathcal{C}(\mathbb{R}, E)$ de (4) telle $\nabla u \in L^2_{loc}(\mathbb{R}, L^4(\mathbb{R}^4))$. Les propriétés de régularité et le caractère lipshitzien du flot énoncés ci-dessus pour $n = 2, 3$ sont vérifiés.

Lorsque l'on veut généraliser ce type de résultats à d'autres non-linéarités, on est confronté à une difficulté liée au fait que V n'est pas nécessairement positif. Pour l'expliquer, revenons un instant au cas plus habituel où $\rho_0 = 0$, et où u_0 est dans un espace de Sobolev H^k. Dans ce cas, pour montrer un résultat d'existence globale, on commence par montrer que la norme H^1 de u ne peut pas exploser. Sa norme L^2 est conservée grâce à l'invariant $I(u)$, tandis que, si V est positif, la norme L^2 de son gradient est majorée (par $\sqrt{\mathcal{E}_0}$). Si V n'est pas positif, mais si sa partie négative ne décroit pas trop vite, on a toujours un contrôle sur $\|\nabla u\|_{L^2}$, grâce à l'inégalité de Gagliardo-Nirenberg et à la conservation de $I(u)$. Dans le cas qui nous occupe, où $\rho_0 > 0$, ce dernier argument n'est pas valable pour au moins deux raisons. D'une part, il semble difficile d'établir la conservation de $I(u)$ autrement que formellement, et d'autre part, $I(u)$ n'est pas une norme, on ne risque donc pas de la faire apparaître dans le membre de droite d'une inégalité de Gagliardo-Nirenberg. Dans ce qui suit, on va donc montrer un résultat d'existence global sans utiliser la quantité $I(u)$.

Dans une troisième approche du problème de Cauchy (20), on travaille dans l'espace affine $\varphi + H^1$, où φ désigne une fonction régulière d'énergie finie. φ peut par exemple être la donnée initiale u_0 si celle-ci est suffisamment régulière, ou bien l'un des *dark solitons* de (9). On suppose qu'il existe $\alpha_1 \geqslant 1$ et $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ tels que la non-linéarité f satisfasse l'hypothèse

$$\exists C_0 > 0,\ A > \rho_0, \left\{ \begin{array}{l} \forall r \geqslant 1,\ \left\{ \begin{array}{ll} |f''(r)| \leqslant C_0 r^{\alpha_1 - 3} & \text{si } n = 1,2,3 \\ |f'''(r)| \leqslant C_0 r^{\alpha_1 - 4} & \text{si } n = 4 \end{array} \right. \\ \left\{ \begin{array}{l} \text{si } \alpha_1 \leqslant 3/2,\ V \text{ est minoré inférieurement} \\ \text{si } \alpha_1 > 3/2,\ \forall r \geqslant A,\ r^{\alpha_2} \leqslant C_0 V(r) \end{array} \right. \end{array} \right. . \qquad (\mathrm{H}_{\alpha_1,\alpha_2})$$

Dans cette hypothèse, la première assertion assure que la non-linéarité ne croît pas trop violemment, la seconde impose à $V(r)$ d'être "plutôt positif" pour r grand. Notons que l'hypothèse ($\mathrm{H}_{\alpha_1,\alpha_2}$) est satisfaite par l'équation de Gross-Pitaevskii pour $\alpha_1 = \alpha_2 = 2$, par la non-linéarité cubique-quintique pour $\alpha_1 = \alpha_2 = 3$ et pour la non-linéarité saturée pour $\alpha_1 = \alpha_2 = 1$. Ainsi, le principal résultat que l'on obtient dans [Ga6] peut être énoncé comme suit.

Théorème 4.0.7 *([Ga6]) Soit $n = 1,2,3$ où 4, $\rho_0 > 0$ et $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ ($k = 1$ si $n = 1$, $k = 2$ si $n = 2,3$, $k = 3$ si $n = 4$) telle que $f(\rho_0) = 0$ et $f'(\rho_0) < 0$. On suppose de plus qu'il existe $\alpha_1 \geqslant 1$, avec la condition supplémentaire $\alpha_1 < \alpha_1^*$ si $n = 3,4$ (où $\alpha_1^* = 3$ pour $n = 3$, $\alpha_1^* = 2$ pour $n = 4$), et $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ tel que (H_{α_1,α_2}) soit vraie.*
Alors pour toute fonction régulière d'énergie finie φ, c'est à dire

$$\varphi \in \mathcal{C}_b^{k+1}(\mathbb{R}^n),\ \nabla\varphi \in H^{k+1}(\mathbb{R}^n)^n,\ |\varphi|^2 - \rho_0 \in L^2(\mathbb{R}^n), \qquad (H_\varphi)$$

le problème (20) est globalement bien-posée dans $\varphi + H^1(\mathbb{R}^n)$. Plus précisément, pour tout $w_0 \in H^1(\mathbb{R}^n)$, il existe un unique $w \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ tel que $\varphi + w$ résolve (20), avec la donnée initiale $w(0) = w_0$.
Pour tout $T > 0$, le flot $w_0 \mapsto w$, $H^1 \mapsto L^\infty_T H^1$ est Lipshitzien sur les bornés de H^1.
L'énergie

$$\mathcal{E}(w) = \int_{\mathbb{R}^n} |\nabla(\varphi + w)|^2 dx + \int_{\mathbb{R}^n} V(|\varphi + w|^2) dx$$

est conservée par le flot.

Pour montrer ce résultat, on cherche une solution de (20) sous la forme $\varphi + w$, et on étudie l'"équation relevée" que doit satisfaire w. La preuve peut alors se diviser en plusieurs étapes. On commence par montrer, grâce à la théorie classique sur les équations d'évolution, que l'équation relevée est localement bien posée dans un espace de Sobolev H^k pour k assez grand. Pour ces solutions régulières, on montre grâce à l'hypothèse ($\mathrm{H}_{\alpha_1,\alpha_2}$) une estimation sur la norme H^1 de w, qui entraîne que celle-ci n'explose pas sur l'intervalle d'existence dans H^k. En utilisant un argument de point fixe dans les espaces de Strichartz de la même manière que T. Kato dans [Ka2], on montre ensuite que l'équation relevée est localement bien posée dans H^1. Un argument de persistance, lui aussi obtenu grâce aux inégalités de Strichartz, ainsi que l'estimation sur la norme H^1 assurent alors que les solutions locales obtenues dans H^k sont en fait globales. La théorie globale pour les solutions H^1 s'en déduit alors aisément par régularisation.

Grâce à l'étude de l'espace d'énergie E effectuée par P. Gérard dans [Ge], on peut remarquer que tout élément u_0 de E peut s'écrire sous la forme $\varphi + w_0$, où φ est régulière d'énergie finie, et $w_0 \in H^1$. On montre ainsi que pour toute donnée initiale $u_0 \in E$, il existe une unique solution de (20) dans $u_0 + \mathcal{C}(\mathbb{R}, H^1)$. De plus, reprenant des arguments de P. Gérard dans [Ge], on montre que c'est la seule solution continue à valeurs dans E, au sens de la métrique d_E introduite par P. Gérard.

Les ondes progressives pour l'équation de Gross-Pitaevskii sur $\mathbb{R}^2 \backslash \overline{B(0,1)}$ obtenues par A. Aftalion et X. Blanc dans [AB] (voir Théorème 3.1.10), ainsi que les motivations physiques sous-jacentes, mentionnées en 2.1, montrent l'intérêt que peut présenter l'étude du problème de Cauchy (20), où Ω n'est plus l'espace $\mathbb{R}^n$ tout entier, mais un domaine extérieur inclus dans $\mathbb{R}^n$. La méthode utilisée pour montrer le Théorème 4.0.7 a l'avantage de bien s'adapter à l'étude de ce problème de Cauchy, grâce aux inégalités de Strichartz montrées par N. Burq, P. Gérard et N. Tzvetkov ([BGT]). On montre ainsi dans [Ga6] le résultat suivant.

Théorème 4.0.8 *Soit $n = 2$ où 3, et $\Omega \subset \mathbb{R}^n$ le complémentaire d'un obstacle K régulier, compact, non captant et non vide. Soit aussi $f \in \mathcal{C}^3(\mathbb{R}_+)$ telle que $f(\rho_0) = 0$ et $f'(\rho_0) < 0$. On suppose qu'il existe $\alpha_1 \geqslant 1,\ \alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ tel que (H_{α_1,α_2}) est vraie. Si $n = 3$, on suppose de plus $\alpha_1 < 2$.*
Alors, pour tout φ vérifiant

$$\varphi \in \mathcal{C}_b^\infty(\Omega),\ \nabla\varphi \in H^\infty(\Omega),\ Supp\varphi \subset \Omega \backslash (V \cap \Omega),\ |\varphi|^2 - \rho_0 \in L^2(\Omega),$$

pour tout $w_0 \in H_0^1(\Omega)$, il existe un unique $w \in \mathcal{C}(\mathbb{R}, H_0^1(\Omega))$ tel que $\varphi + w$ soit solution de (20) et $w(0) = w_0$.
Pour tout $T > 0$, le flot $w_0 \mapsto w$, $H_0^1 \mapsto L_T^\infty H_0^1$, est Lipshitzien sur les bornés de H_0^1.
L'énergie est conservée par le flot.

On a un résultat analogue dans le cas de conditions de Neumann au bord de Ω.

Comme pour le problème de Cauchy sur $\mathbb{R}^n$, toute donnée initiale u_0 dans l'espace d'énergie

$$E_D := \{u \in H^1_{\text{loc}}(\Omega), \nabla u \in L^2(\Omega), \rho_0 - |u|^2 \in L^2(\Omega), \chi u \in H_0^1(\Omega)\},$$

où χ est une fonction $\mathcal{C}^\infty$ à support compact sur $\mathbb{R}^n$ qui vaut 1 sur un voisinage de l'obstacle K, peut se décomposer sous la forme $u_0 = \varphi + w_0$, où φ est régulière d'énergie finie, et $w_0 \in H_0^1$. D'où l'existence d'une unique solution $u \in u_0 + \mathcal{C}(\mathbb{R}, H_0^1(\Omega))$ au problème de Cauchy (20) sur le domaine extérieur Ω. En revanche, l'unicité d'une solution continue à valeurs dans E_D, au sens de l'analogue pour Ω de la distance d_E introduite par P. Gérard dans le cas de $\mathbb{R}^n$, est quant à elle beaucoup moins claire (y compris pour l'équation de Schrödinger linéaire).

5 Le problème de Cauchy pour les équations de Korteweg-de Vries et Benjamin-Ono avec données non nulles à l'infini.

Dans [ILS], R. Iorio, F. Linares et M. Scialom étudient les problèmes de Cauchy pour les équations de Korteweg-de Vries (KdV) et de Benjamin-Ono (BO), soit respectivement

$$\left\{ \begin{array}{l} \partial_t u + \partial_x^3 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(0) = g \end{array} \right. \tag{22}$$

et

$$\left\{ \begin{array}{l} \partial_t u + H\partial_x^2 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(0) = g \end{array} \right. \tag{23}$$

où H designe la transformée de Hilbert, et g vérifie les conditions suivantes à l'infini:

$$\left\{ \begin{array}{rl} i) & g(x) \to C_{\pm} \text{ lorsque } x \to \pm\infty \ . \\ ii) & g' \in H^{s-1} \text{ où } s \geqslant 1 \ . \\ iii) & g - C_+ \in L^2([0,\infty)), \ g - C_- \in L^2((-\infty,0]) \ . \end{array} \right. \tag{24}$$

Grâce à une méthode de régularisation parabolique, ils établissent que les problèmes de Cauchy (22) et (23) avec une donnée initiale vérifiant (24) sont localement bien posés, pourvu que $s > 3/2$. Plutôt que d'étudier directement (22) et (23), ils en cherchent une solution sous la forme $u = \psi + w$, où ψ est régulière, c'est à dire $\psi \in \mathcal{C}^\infty$ et $\psi' \in H^\infty$, et satisfait (24), et w est continu à valeurs dans H^s. Ils étudient alors le système vérifié par w, à savoir

$$\left\{ \begin{array}{l} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(w\psi) + \psi\psi' + \psi^{(3)} = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ w(0) = g - \psi = \varphi \in H^s \end{array} \right. \tag{25}$$

au lieu de (22), et

$$\left\{ \begin{array}{l} \partial_t w + H\partial_x^2 w + w\partial_x w + \partial_x(w\psi) + \psi\psi' + H\psi^{(2)} = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ w(0) = g - \psi = \varphi \in H^s \end{array} \right. \tag{26}$$

au lieu de (23).

Précisément, ils montrent le résultat suivant.

Théorème 5.0.9 *([ILS]) Soit $\varphi \in H^s$, $s > 3/2$. Alors il existe $T = T(s, \|\varphi\|_{H^s})$ et une unique solution $w \in \mathcal{C}([0,T], H^s) \cap \mathcal{C}^1([0,T], H^{s-3})$ (resp. $\mathcal{C}^1([0,T], H^{s-2})$) solution de (25) (resp. (26)). De plus, si $\varphi_n \to \varphi$ dans H^s et w_n est la solution de (25) (resp. (26)) avec donnée initiale $w_n(0) = \varphi_n$, alors pour tout $T' \in (0,T)$, w_n est défini sur $[0,T']$ pour n assez grand, et $w_n \to w$ dans $L^\infty([0,T'], H^s)$.*

En utilisant les invariants respectivement de (KdV) et de (BO), ils en déduisent que si $s \geqslant 2$, (25) et (26) sont globalement bien posés:

Théorème 5.0.10 *Si $\varphi \in H^s$, $s \geqslant 2$, alors la solution w obtenue au Théorème 5.0.9 peut être étendue pour tout $T > 0$.*

Dans [Ga3], on établit le même type de résultats, en affaiblissant les hypothèses sur la donnée initiale g: on suppose seulement $g \in X^s$ et $s > 1$ pour (KdV), $s > 5/4$ pour (BO), où pour $s \geqslant 1$, X^s désigne l'espace de Zhidkov

$$X^s = \{f \in L^\infty(\mathbb{R}),\ f' \in H^{s-1}(\mathbb{R})\}.$$

On commence par remarquer que si $g \in X^s$, on peut la décomposer sous la forme $g = \psi + v$, où ψ a les mêmes propriétés que précédemment et $v \in H^s$. On est donc ramené à l'étude des problèmes de Cauchy (25) et (26) comme dans [ILS]. Ensuite, on adapte à ces problèmes de Cauchy la méthode utilisée par H. Koch et N. Tzvetkov dans [KoT] pour montrer que l'équation de Benjamin-Ono est localement bien posée dans H^s, $s > 5/4$. Celle-ci peut être décomposée en plusieurs étapes comme suit. D'abord, on établit des inégalités de type Strichartz pour des versions linéarisées de (25) et (26), en utilisant les inégalités de Strichartz montrées par C. Kenig, G. Ponce et L. Vega dans [KPV3]. Ensuite, on établit une estimation non linéaire, en utilisant les inégalités de Strichartz obtenues précédemment, appliquées à chaque terme de la décomposition de Littlewood-Paley d'une solution régulière de (25) ou (26). Cette estmation non linéaire permet de montrer une estimation a priori sur la norme H^s d'une solution de (25) ou (26), et on conclue par régularisation.

Cette méthode permet de traiter de la même façon les équations

$$\begin{cases} \partial_t u + -D^\alpha \partial_x u + u\partial_x u = 0\ ,\ x \in \mathbb{R},\ t \in \mathbb{R} \\ w(0) = g \in X^s, \end{cases} \tag{27}$$

où $D = (-\partial_x^2)^{1/2}$ et $\alpha \in [1, 2]$ (pour $\alpha = 1$, on retrouve (BO), tandis que pour $\alpha = 2$, (27) n'est autre que (KdV)). Ainsi, on montre le résultat suivant.

Théorème 5.0.11 *([Ga3])*
Soit $s > 3/2 - \alpha/4$, $g \in X^s$. Alors il existe $T = T(s, \|g\|_{X^s}) > 0$ et une unique solution u de (27) telle que $u \in \mathcal{C}([-T, T], X^s)$, $u - g \in \mathcal{C}([-T, T], H^s)$ et $u_x \in L^1([-T, T], L^\infty)$.
De plus, pour tout $R > 0$, l'application $g \mapsto u$ est continue depuis la boule de rayon R de X^s dans $\mathcal{C}([-T(s, R), T(s, R)], X^s)$.

Comme on montre que l'équation (26) est localement bien posée dans H^s pour $s > 5/4$, en utilisant l'invariant de l'équation de Benjamin-Ono associé à la norme $H^{3/2}$, on montre que (26) est globalement bien posée dans H^s pour $s \geqslant 3/2$. On a donc le résultat qui suit.

Théorème 5.0.12 *([Ga3]) Soit $g \in X^s$, $s \geqslant 3/2$. Alors la solution de (23) obtenue au Théorème 5.0.11 peut être étendue à $\mathbb{R}$.*

Il est intéressant de remarquer que la méthode employée par C. Kenig et K. Koenig ([KeK]) pour montrer que (BO) est localement bien posée dans H^s, $s > 9/8$, n'est pas applicable ici, quoique très voisine de celle développée par H. Koch et N. Tzvetkov dans [KoT]. En effet, l'ingrédient supplémentaire utilisé dans [KeK] par rapport à [KoT] est un argument de *local smoothing*. Il intervient dans la majoration de la quantité $\|uD_x^\beta \partial_x u\|_{L^2([0,T],L^2)}$ de la façon suivante:

$$\begin{aligned} \|uD_x^\beta \partial_x u\|_{L^2([0,T],L^2)} &= \left(\sum_j \|uD_x^\beta \partial_x u\|^2_{L^2((j,j+1)\times(0,T))}\right)^{1/2} \\ &\leqslant \left(\sum_j \|u\|^2_{L^\infty((j,j+1)\times(0,T))}\right)^{1/2} \sup_j \|D_x^\beta \partial_x u\|_{L^2((j,j+1)\times(0,T))} \,. \end{aligned} \quad (28)$$

Dans le membre de droite de (28), le second facteur est contrôlé par *local smoothing* tandis que le premier n'est fini que parce que, en temps qu'élément de L^2, "u est nulle à l'infini". Dans [Ga3], on a besoin de contrôler de la même façon le terme $\|\psi D_x^\beta \partial_x u\|_{L^2([0,T],L^2)}$. Or dès que ψ n'est pas nulle à l'infini (c'est le cas qui nous intéresse), $\left(\sum_j \|\psi\|^2_{L^\infty(j,j+1)}\right)^{1/2} = \infty$, et il n'y a pas d'espoir de contrôler $\|D_x^\beta \partial_x u\|_{L^2((j,j+1)\times(0,T))}$ par *local smoothing* autrement qu'uniformément en j.

6 Plan de la thèse.

Ce mémoire de thèse comporte 7 chapitres. Le premier fait le bilan des différents résultats préexistant dans la littérature concernant l'existence et la stabilité des *dark solitons* des équations de Schrödinger non linéaire en dimension 1, en y apportant quelques précisions (notamment en ce qui concerne les bulles progressives). Ce chapitre comporte également une application de ces résultats à la généralisation à d'autres non-linéarités d'un résultat de M. Mariş sur l'existence de *dark-bright solitons* pour un système d'équations de Schrödinger couplées.

Le second chapitre est consacré à l'étude des conditions d'existence et de stabilité des *black solitons* des équations de Schrödinger non linéaires en dimension 1. On montre notamment un critère numériquement efficient déterminant s'ils sont linéairement stables ou instables. On montre aussi grâce à ce critère la stabilité linéaire des *black solitons* de l'équation de Gross-Pitaevskii en dimension 1.

Dans le chapitre 3, on montre l'existence de *dark-dark solitons* solutions d'un système d'équations de Schrödinger couplées issu de l'optique non linéaire.

Au chapitre 4 est étudié le problème de Cauchy pour l'équation de Schrödinger, linéaire puis non linéaire, dans des espaces de fonctions non nécessairement nulles à l'infini que sont les espaces de Zhidkov. Il s'agit d'un article paru dans la revue "Advances in Differential Equations" sous le titre "Schrödinger group on Zhidkov spaces".

Le chapitre 5 est un complément du chapitre précédent. On y améliore l'estimation sur le taux de croissance du groupe de Schrödinger sur les espaces de Zhidkov établi au chapitre 5. En dimension 1, on montre que l'estimation sur ce taux de croissance est optimale.

Le chapitre 6 développe une autre approche de l'étude du problème de Cauchy pour les équations de Schrödinger non linéaires avec données non nulles à l'infini. Il s'agit de résoudre ce problème de Cauchy dans $\varphi + H^1$, où φ est une fonction régulière d'énergie finie.

Enfin, le dernier chapitre est consacré à l'étude du problème de Cauchy pour des équations de type Korteweg-de Vries ou Benjamin-Ono, avec des conditions non nulles à l'infini. Il est constitué d'un article paru dans "Advances in Differential equations" sous le titre "Korteweg-de Vries and Benjamin-Ono equations on Zhidkov spaces".

Chapitre 1

The dark solitons of the one-dimensional nonlinear Schrödinger equation

Abstract. In this paper, we first give a review of the different rigorous mathematical results that exist in the literature concerning existence and stability of dark solitons of the one-dimensional nonlinear Schrödinger equation. This results are precised with some new results, notably concerning the traveling bubbles. Namely, we prove that there exists nonlinearities for which traveling bubbles do not exist for every non-zero subsonic speed. Next, we use these results to generalize to other nonlinearities a result by M. Mariş concerning the existence of dark-bright solitons for a system of two coupled nonlinear Schrödinger equations.

1.1 Introduction

We consider here the one-dimensional nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where f is a real-valued function whose regularity properties will be specified along the paper. We will concentrate on defocusing nonlinearities. Namely, we will assume throughout this paper the existence of $\rho_0 > 0$ such that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. For such a defocusing nonlinear Schrödinger equation, there exists several kinds of solitary waves, which have the special feature of not vanishing at infinity. These solitary waves are gathered under the label "dark solitons".

Dark solitons of the nonlinear Schrödinger equation are of great interest in many physical contexts. Just for the one-dimensional nonlinear Schrödinger equation, we mention for instance the description of defectons, the theory of one-dimensional ferromagnetic and molecular chains. In both of these examples, $f(r) = -\alpha_1 + \alpha_3 r - \alpha_5 r^2$, where $\alpha_1, \alpha_3, \alpha_5$ are constants, $\alpha_3, \alpha_5 > 0$ (Cf [BGMP], [BP] and references therein). In non-linear optics, the one-dimensional nonlinear Schrödinger equation (with various defocusing nonlinearities f) is a model for the evolution of the complex amplitude envelope of the electric field in optical fibers, as well as for the self-guided beams in planar waveguides. In the very last case, both t and x are spatial variables (and "t" is often replaced by "z"). We refer to [KLD] and to the references therein for a more developed discussion about the dark solitons of the nonlinear Schrödinger equation in nonlinear optics, as well as for a list of physically relevant defocusing nonlinearities f. Moreover, we believe that a good comprehension of the dark solitons of the one-dimensional nonlinear Schrödinger equation could be useful to understand the two and three dimensional cases, which have even more physical applications.

There exists several kinds of dark solitons of (1.1). Namely, in addition to the continuous waves $\varphi(t) = \rho_0^{1/2} e^{if(\rho_0)t}$, we distinguish the kinks, which join two continuous wave backgrounds of different intensity, and the traveling bubbles, the stationary bubbles and the black solitons, which are all dark spots on a continuous wave background. Our purpose in this paper is to classify the defocusing nonlinearities f, depending on which sort of dark solitons it can give birth. We will specify the assumptions on the defocusing non-linearity f which ensure the existence of each kind of dark solitons, and we will study the stability of these solitons.

The results concerning the existence of non-kink dark solitons that are presented in this paper may be sum up as follows. Let us first precise the definition of a dark soliton, if it is not a kink.

Given $\rho_0 > 0$, and if f is such that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$, a (non-kink) dark soliton of (1.1) moving on the continuous-wave background of intensity ρ_0 (that is the sotution $\varphi \equiv \rho_0^{1/2}$ of (1.1)) is a solution of (1.1) which writes $a_v e^{i\theta_v}(x - vt)$. There is no non-trivial dark soliton (we mean by trivial dark soliton a solution such that $a_v \equiv \rho_0^{1/2}$ and θ_v is constant) if $|v|$ is larger than or equal to the sound speed $c = \sqrt{-2\rho_0 f'(\rho_0)}$. If $a_v e^{i\theta_v}$ is such a dark soliton, a_v must satisfy the ordinary differential equation

$$a_v'^{\,2} = V_v(a_v^2) \ ,$$

where $V_v(r) := \int_r^{\rho_0} f(s)ds - \frac{v^2(\rho_0 - r)^2}{4r}$. If V_v does not vanish on $[0, \rho_0)$ (a situation which may occur only at $v = 0$), the only non-trivial dark soliton on the background of intensity ρ_0 is a black soliton, which means that it vanishes at one point. If V_v admits a greatest vanishing point on $(0, \rho_0)$, and if moreover V_v vanishes at order one at this point (this last condition occurs in most of the cases, but not always), then the non-trivial dark soliton of speed v moving on the background of intensity ρ_0 is a traveling (stationary if $v = 0$) bubble, namely a lower (but non-vanishing) intensity stain on the background of intensity ρ_0. If this zero of V_v has order 2 or more, then there is no non-trivial dark soliton on the continuous wave background, but only a kink-shaped solution, which joins this background to 0.

In [Mar1] (see also [Mar2]), M. Mariş proves the existence of dark-bright solitons to the system

$$\begin{cases} 2i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \\ 2i\delta\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi \end{cases} , \tag{1.2}$$

where $q, \delta, \varepsilon, k$ are constants (the component ψ is the dark soliton, while φ is the bright soliton). His method consists in an application of the Implicit Function Theorem around the "trivial" solution $\psi(x,t) = \psi_v(x - vt)$, $\varphi(x,t) \equiv 0$, for $|v| \in (0, \frac{1}{\sqrt{2}\varepsilon})$, where

$$\psi_v(x) = \sqrt{1 - 2v^2\varepsilon^2}\tanh\left(\sqrt{1 - 2v^2\varepsilon^2}\frac{x}{\varepsilon\sqrt{2}}\right) + i\sqrt{2}v\varepsilon$$

is the traveling bubble of the Gross-Pitaevskii equation

$$2i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)\psi . \tag{1.3}$$

We generalize this last result to other nonlinearities, using the above mentioned results concerning the existence of traveling bubbles. Namely, we prove the existence of a couple of dark-bright soliton to the system

$$\begin{cases} i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + f(|\psi|^2, |\varphi|^2)\psi = 0 \\ i\delta\frac{\partial\varphi}{\partial t} + \frac{\partial^2\varphi}{\partial x^2} + g(|\psi|^2, |\varphi|^2)\varphi = 0 , \end{cases} \tag{1.4}$$

under the assumption on f which ensures that there exists a bubble solution of

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + f(|\psi|^2, 0)\psi = 0.$$

In optics, system of type (1.4) may modelize the interaction of two continuous waves of distinct frequencies. It is also a model for light pulses or 2D laser beams which take into account the polarization (one equation for each polarization). We refer to [KLD], [HSh], [SK], [Man] for more details.

The structure of this paper is as follows. In section 2, we are concerned with continuous waves. The presented results are due to P.E. Zhidkov ([Z2], [Z3], [Z4]). Section 3 is devoted to kinks, and the results are again due to P.E. Zhidkov. We study stationary bubbles in section 4. In that section, the mentioned results are due to I.V. Barashenkov and his collaborators ([BGMP]) and A. de Bouard ([dB]). In section 5, we present results which are mostly due to I.V. Barashenkov ([B]) and Z. Lin ([L]) concerning traveling bubbles. We also give necessary and sufficient conditions on the non-linearity f ensuring the existence of traveling bubbles, and we precise under which conditions the stability result of Z. Lin may be applied. Section 6 deals with black solitons. The presented results are issued from [DMG]. We sum up the existence results of dark solitons in section 7. In section 8, we use the existence result of section 5 to generalize the above mentioned result by M. Mariş (see [Mar1]) concerning the existence of a dark-bright soliton to (1.2).

1.2 The continuous waves

1.2.1 Existence

Without any regularity assumption on f, it is clear that for $a_0 \in \mathbb{R}$,

$$(t,x) \longrightarrow \varphi(t) = a_0 e^{itf(a_0^2)}$$

solves (NLS).

1.2.2 Stability

We give here the stability result for the continuous waves which is proved by P.E. Zhidkov in [Z2] and [Z4].

Theorem 1.2.1 *([Z4]) Let $f \in \mathcal{C}^2(\mathbb{R}^+)$ be a real-valued function. Let us assume the existence of $a_0 > 0$ such that $f'(a_0^2) < 0$.*
Let (thanks to the Sobolev embedding), $\delta_0 > 0$ such that $a \in H^1$ and $\|a\|_{H^1} < \delta_0$ implies $\|a\|_{L^\infty} < a_0/2$.
Then $\varphi(t) = a_0 e^{itf(a_0^2)}$ is a stable solution of (1.1) in the following sense:
For every $\varepsilon \in (0, \delta_0)$, there exists $\delta \in (0, \delta_0)$ such that if $u_0 \in X^1$ may be written under the form

$$u_0(x) = (a_0 + \bar{a}(x)) e^{i\omega(x)} \ ,$$

with $\|\bar{a}\|_{H^1} < \delta$ (and therefore $\bar{a} = |u_0| - a_0$) and $\|\omega'\|_{L^2} < \delta$, then the solution u of (1.1) with initial data u_0 is global in the Zhidkov space $X^1(\mathbb{R}) := \{u \in L^\infty(\mathbb{R}),\ u' \in L^2(\mathbb{R})\}$, does not vanish, and for every time t, it can be written as

$$u(t,x) = (a_0 + a(t,x)) e^{i(tf(a_0^2) + \omega(t,x))} \ ,$$

with $a(t) \in H^1$, $\|a(t)\|_{H^1} < \varepsilon$ and $\|\partial_x \omega(t)\|_{L^2} < \varepsilon$.

The proof of this theorem developed in [Z4] is based on the conservation of the energy

$$M(u) := \int_{-\infty}^{+\infty} \left[\frac{1}{2}|u_x|^2 - U(|u|^2) + U(a_0^2) + \frac{f(a_0^2)}{2}(|u|^2 - a_0^2) \right] dx$$

for a solution of (1.1) with initial data $u_0 \in X^1$ satisfying the assumptions of the statement, where $U(r) := \frac{1}{2}\int_0^r f(s)ds$. In [Z4], P.E. Zhidkov formally differentiates the energy to justify its conservation. For a rigorous justification of this computation, see [Ga1].

Remark that the smallness of $\|a\|_{H^1}$ and $\|\omega'\|_{L^2}$ does not imply that $\|(a_0+a)e^{i\omega} - a_0\|_{H^1}$ is small (not even that $\|(a_0 + a)e^{i\omega} - a_0\|_{L^\infty}$ is small). Thus this result does not give the stability of φ to H^1 perturbations.

The stable continuous waves are of great importance in the sequel of this paper. Indeed, the dark solitons we will study in the next sections are dark spots which join two of these stable bright backgrounds (which are the same, up to a phase change, in the non-kink case).

1.3 The kinks

Definition 1.3.1 *The kinks are solutions of (1.1) of the form*

$$u(t,x) = e^{i\omega t}\varphi(x)$$

where $\varphi(x) \to \varphi_\pm$ *as* $x \to \pm\infty$, *and* φ *is a diffeomorphism from* $\mathbb{R}$ *on* (φ_-, φ_+) *(up to a change of* x *into* $-x$, φ *will always be assumed to be non-decreasing).*

1.3.1 Existence

We first give a necessary and sufficient condition on f and $\varphi_\pm$ which ensures that there exists a kink solving (1.1), with $\varphi(x) \to \varphi_\pm$ as $x \to \pm\infty$.

Theorem 1.3.1 *([Z4]) We assume* $f \in \mathcal{C}(\mathbb{R}_+)$. *Let* $\varphi_- < \varphi_+$. *There exists a kink solving (1.1) if and only if the following conditions are satisfied:*

(a) $(-\omega + f(\varphi_\pm^2))\varphi_\pm = 0$

(b) $-\frac{\omega}{2}\varphi_-^2 + U(\varphi_-^2) = -\frac{\omega}{2}\varphi_+^2 + U(\varphi_+^2)$, *where* $U(r) = \frac{1}{2}\int_0^r f(s)ds$

(c) *for every* $s \in (\varphi_-, \varphi_+)$, $-\frac{\omega}{2}s^2 + U(s^2) < -\frac{\omega}{2}\varphi_-^2 + U(\varphi_-^2)$.

Remarks.

1. Condition (a) means that either $\varphi_\pm = 0$, or $e^{i\omega t}\varphi_\pm$ is a continuous wave.

2. In the case $\varphi_\pm \neq 0$, using (a), (b) and differentiating (c) twice, we get $f'(\varphi_\pm^2) \leqslant 0$, which is almost the assumption of Theorem 1.2.1 ensuring the stability of the bright background $e^{i\omega t}\varphi_\pm$.

3. In the case $\varphi_- < 0 < \varphi_+$, conditions (b) and (c) imply $\varphi_- = -\varphi_+$, and φ is odd.

Proof. Let us assume that $u(t,x) = e^{i\omega t}\varphi(x)$ is a kink solving (1.1), with $\varphi(x) \to \varphi_\pm$ as $x \to \pm\infty$. Then φ satisfies

$$-\omega\varphi + \varphi'' + f(\varphi^2)\varphi = 0 \tag{1.5}$$

Since φ is bounded on $\mathbb{R}$ and f is continuous, it follows that φ'' is bounded. Thus so is φ'. Letting x tend to $\pm\infty$ in (1.5), we get

$$\varphi''(x) \to \omega\varphi_\pm - f(\varphi_\pm^2)\varphi_\pm \text{ as } x \to \pm\infty.$$

This implies, together with the boundedness of φ', that $\varphi''(x)$ tends to 0 as x tends to infinity, and (a) follows. We deduce that $\varphi'(x) \to 0$ as $x \to \pm\infty$.
Multiplying (1.5) by φ' and integrating between two real numbers x_1 and x_2, we obtain

$$\frac{\varphi'(x_1)^2}{2} - \frac{\varphi'(x_2)^2}{2} - \frac{\omega}{2}(\varphi(x_1)^2 - \varphi(x_2)^2) + U(\varphi(x_1)^2) - U(\varphi(x_2)^2) = 0$$

Letting x_2 tend to $+\infty$ and x_1 to $-\infty$, we get (b). (c) follows from the fact that φ is a diffeomorphism from $\mathbb{R}$ into (φ_-, φ_+) and taking $x_1 = -\infty$, $x_2 \in \mathbb{R}$.

Conversely, let us assume that (a), (b) and (c) are satisfied. Let $\varphi_0 \in (\varphi_-, \varphi_+)$, and φ_0' given (thanks to (c)) by

$$\varphi_0' := \left(-\omega\varphi_-^2 + 2U(\varphi_-^2) + \omega\varphi_0^2 - 2U(\varphi_0^2)\right)^{1/2} .$$

Let us (again thanks to (c)) denote by φ the solution of the Cauchy problem

$$\begin{cases} \varphi'(x) = \left({\varphi_0'}^2 - \omega\varphi_0^2 + 2U(\varphi_0^2) + \omega\varphi(x)^2 - 2U(\varphi(x)^2)\right)^{1/2} \\ \varphi(0) = \varphi_0 \end{cases}$$

It is clear that the solution of this equation is global, and that $\varphi(x) \to \varphi_\pm$ as $x \to \pm\infty$. Indeed, the Cauchy-Lipshitz Theorem ensures that the unique solution of our differential equation, which can be written as

$$\varphi'(x) = \sqrt{\omega - f(\varphi_-^2) - 2f'(\varphi_-^2)\varphi_-^2}\,|\varphi(x) - \varphi_-| + o(|\varphi(x) - \varphi_-|) ,$$

with initial data $\varphi(x_0) = \varphi_-$, is $\varphi \equiv \varphi_-$. Since $\varphi(x) \to \varphi_+$ as $x \to +\infty$, there can not exist $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = \varphi_-$. Derivating, it is clear that we get the wanted kink.

1.3.2 Stability

In [Z4], P.E. Zhidkov proves for the kinks a stability result similar to that obtained for the continuous waves.

Theorem 1.3.2 *([Z4]) Let $f \in \mathcal{C}^2(\mathbb{R}^+)$ be a real-valued function. We assume that the properties (a),(b) and (c) are satisfied, and we make the supplementary assumptions*

(d) $\varphi_+ > \varphi_- > 0$

(e) $-\omega + f(\varphi_\pm^2) + 2\varphi_\pm^2 f'(\varphi_\pm^2) < 0$

Then the kink $u(t,x) = e^{i\omega t}\varphi(x)$ given by Theorem 1.3.1 is stable in the following sense: For every $\varepsilon > 0$ small enough, there exists $\delta > 0$ small enough such that if $u_0 \in X^1$ writes

$$u_0(x) = (\varphi(x) + a(x))e^{i\overline{\omega}(x)}$$

with $\||u_0| - \varphi\|_{H^1} = \|a(0)\|_{H^1} < \delta$ and $\|\overline{\omega}'\|_{L^2} < \delta$, then the solution u of (1.1) with initial data u_0 is global in time in X^1, does not vanish, and for every time t, it can be written as

$$u(t,x) = (\varphi(x) + a(t,x))e^{i(\omega t + \bar{\omega}(t,x))} ,$$

with $a(t) \in H^1$, $\|a(t)\|_{H^1} < \varepsilon$ and $\|\partial_x\bar{\omega}(t)\|_{L^2} < \varepsilon$.

Remark.
Taking (a) and (d) into account, the assumption (e) just means that $f'(\varphi_\pm^2) < 0$. The only kinks found in Theorem 1.3.1 for which Theorem 1.3.2 does not provide stability thus satisfy one of the following conditions:

- $\varphi_- = 0$ or $\varphi_+ = 0$
- $\varphi_- < 0 < \varphi_+$, which implies as we have seen $\varphi_- = -\varphi_+$. Such a soliton is called a black soliton, and will be studied below.
- $\varphi_\pm \neq 0$, and $f'(\varphi_-^2) = 0$ or $f'(\varphi_+^2) = 0$.

1.4 The stationary bubbles

Definition 1.4.1 *([BGMP]) Let $f \in C^1(\mathbb{R}_+)$, $\rho_0 > 0$ such that $f(\rho_0) = 0$. A stationary bubble of (1.1) tending to $\rho_0^{1/2}$ at infinity is a solution of (1.1) of the form*

$$u(t,x) = \varphi(x)$$

where φ is real and satisfies:

$$\begin{array}{rl} (i) & \varphi \text{ is even} \\ (ii) & \partial_x^2\varphi + f(\varphi^2)\varphi = 0 \\ (iii) & 0 < \varphi(x) < \rho_0^{1/2} \text{ for every } x \in \mathbb{R} \\ (iv) & \varphi(x) \to \rho_0^{1/2} \text{ as } x \text{ tends to infinity} \\ (v) & 0 = \varphi'(0) < \varphi'(x) \text{ for } x \in (0, +\infty). \end{array}$$

1.4.1 Existence

In [dB], A. de Bouard states and proves the following necessray and sufficient condition on the nonlinearity f ensuring the existence of a stationary bubble.

Theorem 1.4.1 *([dB]) Let $f \in C^1(\mathbb{R}_+)$, $\rho_0 > 0$, and $V(r) := -\int_{\rho_0}^r f(s)ds$. Then there exists a stationary bubble of (1.1) tending to $\rho_0^{1/2}$ at infinity if and only if the following assumptions are satisfied:*

- $f(\rho_0) = 0$
- $\eta_0 := \sup\{\eta \in (0,\rho_0), V(\eta) = 0\}$ *exists,* $\eta_0 \in (0,\rho_0)$ *and* $f(\eta_0) < 0$.

moreover, if these conditions are satisfied, if $f'(\rho_0) < 0$,[1] *then $\partial^\alpha(\varphi - \rho_0^{1/2})$ tends exponentially to 0 at infinity for $\alpha \leqslant 2$.*[2]

1.4.2 Stability

In 1989, I.V. Barashenkov, A.D. Gocheva, V.G. Makhankov and I.V. Puzynin ([BGMP]) proved the linear instability of the stationary bubbles. In 1995, Anne de Bouard ([dB]) precised this result and deduced the instability of the stationary bubbles to H^1 perturbations. More precisely, she proved the following Theorem. Note that her result was valuable in any dimension. For consistency with the rest of the paper, we restrain it here to dimension one.

Theorem 1.4.2 *([dB]) Let $\rho_0 > 0$, $f \in C^3(\mathbb{R}_+)$ such that the assumptions of Theorem 1.4.1 are satisfied and such that $f'(\rho_0) < 0$. Let φ be the stationary bubble of (1.1) tending to $\rho_0^{1/2}$ at infinity, given by Theorem 1.4.1.*
Then φ is unstable in the following sense: there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1} < \delta$ and such that, if $v(t) = u(t) + \varphi$ is the unique

1. The two other conditions already imply $f'(\rho_0) \leqslant 0$.
2. Precise asymptotics of φ and its derivatives are given in Theorem 1.5.1 below, with $v = 0$.

solution $v \in C(T_\star, T^\star, X^1)$ of (1.1) with initial data $v(0) = u_0 + \varphi$, then $u(t) \in H^1$ for $t \in (T_\star, T^\star)$, and there exists $t_0 \in (0, T^\star)$ such that $\|u(t_0)\|_{H^1} > \varepsilon$. [3]

1.5 The traveling bubbles

Definition 1.5.1 *([L]) Let $\rho_0 > 0$. A traveling bubble of (1.1) is a solution of (1.1) of the form*

$$u(t,x) = \varphi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

where $v \in \mathbb{R}$, $\varphi_v \in C^1(\mathbb{R})$, $a_v > 0$, $a_v \to \rho_0^{1/2}$, $\partial_x a_v \to 0$, $\partial_x \theta_v \to 0$ as $x \to \infty$.

1.5.1 Existence

In [L], Z. Lin gives a necessary and sufficient condition on the nonlinearity f and the speed v, under which there exists a traveling bubble solving (1.1).

Theorem 1.5.1 *([L]) Let $f \in C^1(\mathbb{R}_+)$, $\rho_0 > 0$, and $V(r) := -\int_{\rho_0}^r f(s)ds$. Then there exists a traveling bubble of (1.1), with speed v and with modulus tending to $\rho_0^{1/2}$ at infinity if and only if the following assumptions are satisfied:*

- $f(\rho_0) = 0$
- $\eta_0 := \sup\{\eta \in (0, \rho_0), V_v(\eta) := V(\eta) - \frac{v^2(\eta-\rho_0)^2}{4\eta} = 0\}$ *exists, and* $\eta_0 \in (0, \rho_0)$
- $f_v(\eta_0) := f(\eta_0) + \frac{v^2}{4}(1 - \frac{\rho_0^2}{\eta_0^2}) = -V_v'(\eta_0) < 0.$

Moreover, under these conditions, denoting by $\varphi_v = a_v e^{i\theta_v}$ this traveling bubble, a_v and θ_v' are even, and $\varphi_v \in \mathcal{C}^3(\mathbb{R})$. Under the supplementary assumption $f'(\rho_0) < 0$, φ_v converges exponentially to its limits as $x \to +\infty$, as well as its derivatives of order $1, 2, 3$. More precisely, we have the asymptotics as $x \to +\infty$ (a_v is even, thus the asymptotics as $x \to -\infty$ follow)

$$\rho_0^{1/2} - a_v(x) = \exp\left(-\sqrt{c^2 - v^2}\,x(1 + o(1))\right) ,$$

$$a_v'(x) \sim \sqrt{c^2 - v^2}(\rho_0^{1/2} - a_v(x)) ,$$

$$a_v''(x) \sim -(c^2 - v^2)(\rho_0^{1/2} - a_v(x)) ,$$

$$\theta_v'(x) \sim -\frac{v}{\rho_0^{1/2}}(\rho_0^{1/2} - a_v(x)) ,$$

$$\theta_v''(x) \sim \frac{v\sqrt{c^2 - v^2}}{\rho_0^{1/2}}(\rho_0^{1/2} - a_v(x)) .$$

3. As it was mentioned in [dB], we have in fact the instability modulo translations. Namely, $\inf_{s\in\mathbb{R}} \|v(t) - \varphi(. - s)\|_{H^1} > \varepsilon$, even if it does not look like this form of instability has to be considered (see [dB]).

Remark.

In particular, if $v^2 > -2\rho_0 f'(\rho_0) =: c^2$, there is no bubble (for instance, it is the case for every v if $f'(\rho_0) > 0$), since $V_v(r) \underset{r \to \rho_0}{\sim} \frac{c^2 - v^2}{4\rho_0}(\rho_0 - r)^2 < 0$. If $0 < v^2 < c^2$, η_0 exists, since in this case, $V(r) \geqslant 0$ in a neighborhood of ρ_0 and $V_v(r) \to -\infty$ as $r \to 0,\ r > 0$. It suffices then to see whether the condition on $f_v(\eta_0)$ is satisfied or not, in order to know if there exists a bubble.

Contrary to Z. Lin's affirmation in [L], the fact that the assumptions of Theorem 1.5.1 ensuring the existence of a bubble are satisfied for $v = 0$ (which are also the assumptions of Theorem 1.4.1 ensuring the existence of a stationary bubble), and the assumption $f'(\rho_0) < 0$ do not imply the existence of a traveling bubble for every speed $v \in (-c, c)$. It can indeed occur that there exists a speed $v_s \in (-c, c)\backslash\{0\}$ such that $f_{v_s}(\eta_0) = 0$. For this speed, we do not observe a traveling bubble, but a "traveling kink".

However, under the supplementary assumption that V'' is non-decreasing, it becomes true that there exists a traveling bubble for every $v \in (-c, c)$. Indeed, let us assume by contradiction that V'' is non-decreasing on $[0, \rho_0]$, and that there exists $v_s \in (-c, c)\backslash\{0\}$ such that $f_{v_s}(\eta_0) = -V'_{v_s}(\eta_0) = 0$. By definition of η_0, $V_{v_s}(\eta_0) = 0$, thus the Taylor formula yields, for $\eta \geqslant \eta_0$,

$$\begin{aligned} V_{v_s}(\eta) &= \int_{\eta_0}^{\eta} (\eta - t)\left(V''(t) - \frac{v_s^2 \rho_0^2}{2t^3}\right) dt \\ &> \int_{\eta_0}^{\eta} (\eta - t)\left(V''(\eta_0) - \frac{v_s^2 \rho_0^2}{2\eta_0^3}\right) dt \\ &= V''_{v_s}(\eta_0)(\eta - \eta_0)^2/2 \ . \end{aligned}$$

Now, for $\eta \in (\eta_0, \rho_0)$, $V_{v_s}(\eta) > 0$, and $V_{v_s}(\eta_0) = V'_{v_s}(\eta_0) = 0$, thus $V''_{v_s}(\eta_0) \geqslant 0$. We deduce that $V_{v_s}(\rho_0) > 0$, which is a contradiction.

We next study the regularity properties of the function η_0. As we already noticed it, $\eta_0(v)$ is well defined for $v \in (-c, c)$, $v \neq 0$. On the other side, it is clear that $\eta_0(-v) = \eta_0(v)$, in such a way that we only study η_0 on $[0, c)$.

Proposition 1.5.1 *η_0 is strictly increasing on $(0, c)$ (resp. on $[0, c)$ if $\eta_0(0)$ is well defined). η_0 is semi-continuous from above on $(0, c)$ (resp. on $[0, c)$); if $\eta_0(0)$ is not defined, $\eta_0(v) \to 0$ as $v \downarrow 0$. If there exists a traveling bubble of speed $v \in (0, c)$, then η_0 is of class $\mathcal{C}^1$ in a neighborhood of v.*

Proof. If $0 \leqslant v_1 < v_2 < c$, for every $r \in [\eta_0(v_2), \rho_0)$, $V_{v_1}(r) > V_{v_2}(r)$. Thus, by continuity of V_{v_1}, $\eta_0(v_1)$ (if it does exist) satisfies $\eta_0(v_1) < \eta_0(v_2)$. Therefore η_0 is strictly increasing.

Let $v_0 \in [0, c)$. By convention, we define $\eta_0(v_0) = 0$ if $V_{v_0} > 0$ on $[0, \rho_0)$ (which may only occur in the case $v_0 = 0$). Let $\varepsilon > 0$ be such that $\varepsilon + \eta_0(v_0) < \rho_0$. The function $r \longrightarrow V_{v_0}(r)/(r - \rho_0)^2$ is continuous on the compact set $[\eta_0(v_0) + \varepsilon, \rho_0]$, thus it reaches its infimum $\delta > 0$ on this set. Indeed, $V_{v_0}(r) > 0$ for $r \in (\eta_0(v_0), \rho_0)$, and $V_{v_0}(r)/(r - \rho_0)^2 \to$

$(c^2 - v_0^2)/4\rho_0 > 0$ as $r \to \rho_0$. For $v \in (v_0, c)$ and $r \in [\eta_0(v_0) + \varepsilon, \rho_0]$,

$$\frac{V_v(r)}{(r-\rho_0)^2} = \frac{V_{v_0}(r)}{(r-\rho_0)^2} - \frac{v^2 - v_0^2}{4r} \geqslant \delta - \frac{v^2 - v_0^2}{4(\eta_0(v_0)+\varepsilon)} \ .$$

If v is chosen such that $v^2 \leqslant v_0^2 + 2\delta(\eta_0(v_0)+\varepsilon)$, we get $V_v(r) > 0$ for every $r \in [\eta_0(v_0)+\varepsilon, \rho_0)$, and thus $\eta_0(v_0) < \eta_0(v) < \eta_0(v_0) + \varepsilon$. The semi-continuity from above of η_0 follows, as well as the fact that $\eta_0(v) \to 0$ as $v \downarrow 0$, in the case where $V_0 > 0$ on $[0, \rho_0)$.

Let $v_0 \in (0, c)$ such that there exists a traveling bubble of speed v_0. By definition of $\eta_0(v_0)$ and by continuity of V_{v_0}, $V_{v_0}(\eta_0(v_0)) = 0$. Thanks to Theorem 1.5.1, $V_{v_0}'(\eta_0(v_0)) > 0$. The map $(0, c) \times \mathbb{R}_+^* \mapsto \mathbb{R}$, $(v, r) \mapsto V_v(r)$ is clearly of class $\mathcal{C}^1$. $V_{v_0}'(\eta_0(v_0)) \neq 0$, thus, thanks to the Implicit Function Theorem, there exists a neighborhood $\mathcal{V} \in (0, c)$ of v_0, a neighborhood Ω of $(v_0, \eta_0(v_0))$ in $(0, c) \times \mathbb{R}_+^*$ and a map $\varphi : \mathcal{V} \to \mathbb{R}_+^*$ of class $\mathcal{C}^1$, such that

$$\{(v, r) \in \Omega, V_v(r) = 0\} = \{(v, \varphi(v)), v \in \mathcal{V}\} \ .$$

From the semi-continuity from above of η_0, we infer that if v is sufficiently close to v_0 and $v > v_0$, $(v, \eta_0(v)) \in \Omega$. Similarly, if $v \in \mathcal{V}$ and $v < v_0$, we get $V_v(\varphi(v)) = 0$. Thus, by definition of $\eta_0(v)$ and since η_0 is non-decreasing, $\varphi(v) \leqslant \eta_0(v) < \eta_0(v_0)$. In particular, $\eta_0(v) \to \eta_0(v_0)$ as $v \to v_0$ and $v < v_0$. Therefore there exists a neighborhood of v_0 $\mathcal{V}' \subset \mathcal{V}$ such that $(v, \eta_0(v)) \in \Omega$ as soon as $v \in \mathcal{V}'$. Since V_v is continuous, $V_v(\eta_0(v)) = 0$. Thus $\eta_0(v) = \varphi(v)$ for $v \in \mathcal{V}'$, and therefore η_0 is of class $\mathcal{C}^1$ on $\mathcal{V}'$. □

Proposition 1.5.2 *The set $S = \{v \in (0, c), V_v'(\eta_0(v)) = 0\}$ is closed and does not contain any non-trivial interval.*

Proof. We first show that the complementary set of S is open. I $V_{v_0}'(\eta_0(v_0)) \neq 0$, Theorem 1.5.1 and Proposition 1.5.1 ensure that η_0 is continuous in a neighborhood of v_0. Thus so is $v \to V_v'(\eta_0(v))$. It follows that S is closed.

Let us assume by contradiction that there exists $v_1 < v_2$ such that $[v_1, v_2] \subset S$. Then for every $v \in [v_1, v_2]$, $V_v'(\eta_0(v)) = 0$. Thanks to Proposition 1.5.1, η_0 is semi-continuous from above. Therefore there exists $\tilde{v}_2 > v_1$, $\tilde{v}_2 \leqslant v_2$ such that η_0 is continuous on $[v_1, \tilde{v}_2]$. Up to a change of v_2, we may assume that $\tilde{v}_2 = v_2$. η_0 is strictly increasing on $[v_1, v_2]$, thus is bijective from $[v_1, v_2]$ onto $[\eta_0(v_1), \eta_0(v_2)]$. For every $v \in [v_1, v_2]$, $V_v(\eta_0(v)) = V_v'(\eta_0(v)) = 0$, thus for every $r \in [\eta_0(v_1), \eta_0(v_2)]$, we have the relations

$$V(r) \quad = \quad \frac{\eta_0^{-1}(r)^2}{4} \frac{(\rho_0 - r)^2}{r} \tag{1.6}$$

and

$$V'(r) \quad = \quad \frac{\eta_0^{-1}(r)^2}{4} (1 - \frac{\rho_0^2}{r^2}) \ . \tag{1.7}$$

Let us differentiate (1.6) in $\mathcal{D}'(\eta_0(v_1), \eta_0(v_2))$:

$$V'(r) = \frac{\eta_0^{-1}(r)^2}{4} (1 - \frac{\rho_0^2}{r^2}) + \frac{(\rho_0 - r)^2}{4r} \frac{d}{dr} (\eta_0^{-1}(r)^2) \ .$$

Comparing with (1.7),

$$\frac{d}{dr}(\eta_0^{-1}(r)^2) = 0 ,$$

and thus $\eta_0^{-1}(r)^2$ is constant on $(\eta_0(v_1), \eta_0(v_2))$, which is a contradiction with the bijectivity of η_0^{-1}. □

However, the closed set S of the speeds for which there does not exist a bubble is not necessarily finite. That is what shows the following counter-example.

Let us fix $\rho_0 = 1$. Let $\alpha > 0$, $\alpha < 1/2$ such that $1/2 - 3(r-1/2) + 8(r-1/2)^2 - (1-r)^2/r > 0$ for $r \in (1/2, 1/2+\alpha]$ (a Taylor expansion at order 3 near 1/2 shows that such a α does exist). We next define a function V on $[0,1]$ as

$$V(r) = \begin{cases} \frac{(1-r)^2}{r} + (r-\frac{1}{2})^6 \sin\frac{1}{r-\frac{1}{2}} & r \in [1/4, 1/2) \\ \frac{1}{2} - 3(r-\frac{1}{2}) + 8(r-\frac{1}{2})^2 & r \in [1/2, 1/2+\alpha] \\ 4(1-r)^2 & r \in [3/4+\alpha/2, 1]. \end{cases}$$

By construction, we are ensured that V is of class $\mathcal{C}^2$ on $[1/4, 1/2+\alpha]$, and that $V(r) - (1-r)^2/r > 0$ for $r \in (1/2, 1/2+\alpha]$. We extend V to $[0,1]$ into a $\mathcal{C}^2$-function, such that $V(0) < 0$, and $V(r) - (1-r)^2/r > 0$ for every $r \in (1/2, 1]$. The sound speed is $c = \sqrt{2V''(1)} = 4$.

Proposition 1.5.3 *If V is chosen as above, the set S is not finite.*

Proof. For $n \in \mathbb{N}^*$, we define $r_n = 1/2 - 1/(2n\pi + \pi/2)$ and $s_n = 1/2 - 1/(2n\pi - \pi/2)$. r_n and s_n are two sequences which are growing to 1/2, and for every $v \in (0, c)$, we have

$$V_v(r_n) = \frac{(1-r_n)^2}{r_n}(1 - \frac{v^2}{4}) - (r_n - \frac{1}{2})^6$$

$$V_v(s_n) = \frac{(1-s_n)^2}{s_n}(1 - \frac{v^2}{4}) + (s_n - \frac{1}{2})^6 .$$

If there exists $v > 0$ such that $\eta_0(v) = s_n$, necessarily, since $V_v(\eta_0(v)) = 0$, $v = w_n$, where $w_n > 0$ is given by

$$w_n^2 = 4 + 4\frac{s_n(s_n - \frac{1}{2})^6}{(1-s_n)^2} > 4 .$$

Since η_0 is strictly increasing, $s_n = \eta_0(w_n) > \eta_0(2) = 1/2$, which is a contradiction with the fact that $s_n < 1/2$. Therefore η_0 does not take the value s_n.

Let us define $v_n > 0$ by:

$$v_n^2 = 4 - 4\frac{r_n(r_n - \frac{1}{2})^6}{(1-r_n)^2} < 4 .$$

We are going to prove that $\eta_0(v_n) = r_n$. We already know that $V_{v_n}(r_n) = 0$, and $\eta_0(v_n) < \eta_0(2) = 1/2$, thus $\eta_0(v_n) \in [r_n, 1/2)$. Now, for $r \in (r_n, 1/2)$,

$$\begin{aligned} V_{v_n}(r) &= (r_n - \frac{1}{2})^6 \frac{r_n}{(1-r_n)^2} \frac{(1-r)^2}{r} + (r - \frac{1}{2})^6 \sin(\frac{1}{r - \frac{1}{2}}) \\ &\geqslant \left((r_n - \frac{1}{2})^6 \frac{r_n}{(1-r_n)^2} - (r - \frac{1}{2})^6 \frac{r}{(1-r)^2} \right) \frac{(1-r)^2}{r} \end{aligned}$$

It is easy to see that $r \to (r - \frac{1}{2})^6 \frac{r}{(1-r)^2}$ is decreasing on an interval $[r_0, 1/2]$, which implies that for n large enough, $V_{v_n}(r) > 0$ on $(r_n, 1/2]$. This yields $\eta_0(v_n) = r_n$.

For any n, $s_n \in (r_{n-1}, r_n)$. For n large enough, η_0 takes the values r_{n-1} and r_n, but not the value s_n. This means that η_0 is not continuous on the interval $(\eta_0^{-1}(r_{n-1}), \eta_0^{-1}(r_n)) = (v_{n-1}, v_n)$. It follows that there exists $z_n \in (v_{n-1}, v_n) \cap S$. The intervals (v_{n-1}, v_n) are two by two disconnected, therefore S is not finite. $\square$

1.5.2 Stability

In 1996, I.V. Barashenkov ([B]) gave a criterion for the study of the stability of the traveling bubbles. In 2002, Z. Lin ([L]) gave a more rigorous proof of the same criterion. We mention in this section the result of [L]. We begin with the definition of the renormalized momentum, and we precise its regularity properties.

Definition 1.5.2 *The momentum of a traveling bubble φ_v of speed v, with modulus tending to $\rho_0^{1/2}$ at infinity is defined by*

$$P_v := \mathfrak{Im} \int_{-\infty}^{+\infty} \partial_x \overline{\varphi_v} \varphi_v (1 - \frac{\rho_0}{|\varphi_v|^2}) dx$$

Remark. The so-defined momentum is the opposite of this used by Z. Lin in [L].

Proposition 1.5.4 *The map $v \mapsto P_v$ is of class $\mathcal{C}^1$ on the open set Ω, where $\Omega = (-c, c) \backslash (S \cup (-S))$ if the assumptions of Theorem 1.4.1 ensuring the existence of a stationary bubble are satisfied, $\Omega = (-c, c) \backslash (S \cup (-S) \cup \{0\})$ else.*

Proof. We first give new expressions of P_v, for $v \in [v_1, v_2]$, where $[v_1, v_2] \subset \Omega$. We denote by $\varphi_v = a_v e^{i\theta_v}$, where $a_v > 0$ and θ_v is real-valued, the bubble of speed v. Using the system satisfied by (a_v, θ_v) (see [L]),

$$\begin{cases} \theta_v'(x) = \frac{1}{2} v \left(1 - \frac{\rho_0}{a_v(x)^2}\right) \\ -a_v''(x) = f_v(a_v(x)^2) a_v(x) = \left(f(a_v(x)^2) + \frac{v^2}{4} (1 - \frac{\rho_0^2}{a_v(x)^4}) \right) a_v(x) \end{cases},$$

we obtain

$$
\begin{aligned}
P_v &= \frac{i}{2}\int_{-\infty}^{\infty} \frac{\overline{\varphi_v}\varphi_v' - \varphi_v\overline{\varphi_v}'}{|\varphi_v|^2}(|\varphi_v|^2-\rho_0)dx \\
&= -\int_{-\infty}^{\infty} \theta_v'(|\varphi_v|^2-\rho_0)dx \\
&= -\frac{v}{2}\int_{-\infty}^{\infty} \frac{(\rho_0-a_v^2)^2}{a_v^2}dx \\
&= -v\int_0^{\infty} \frac{(\rho_0-a_v^2)^2}{a_v^2}dx \\
&= -v\int_{\eta_0(v)}^{\rho_0} \frac{(\rho_0-r)^2dr}{2r^{3/2}\sqrt{V_v(r)}} \\
&= -v\int_0^1 \frac{(\rho_0-\eta_0(v))^3(1-t)^2dt}{2(t\rho_0+(1-t)\eta_0(v))^{3/2}\sqrt{V_v(t\rho_0+(1-t)\eta_0(v))}} \ ,
\end{aligned}
$$

where we recall that V_v is given by

$$
V_v(r) = \int_r^{\rho_0} f_v(s)ds = V(r) - \frac{v^2(r-\rho_0)^2}{4r} \ .
$$

We next use the Theorem of differentiation of an integral in the last expression of P_v. Thanks to Taylor's formula,

$$
\begin{aligned}
V_v(\eta_0(v)+t(\rho_0-\eta_0(v))) &= t(\rho_0-\eta_0(v))V_v'(\eta_0(v)) \\
&\quad + t^2(\rho_0-\eta_0(v))^2\int_0^1 (1-s)V_v''(\eta_0(v)+ts(\rho_0-\eta_0(v)))ds \ .
\end{aligned}
$$

The second term in the right-hand side is controlled by

$$
t^2(\rho_0-\eta_0(v_1))^2\left(\|V''\|_{L^\infty(0,\rho_0)} + \frac{v_2^2\rho_0^2}{2\eta_0(v_1)^2}\right) .
$$

Let us denote by $C_1>0$ the minimum of $v\to(\rho_0-\eta_0(v))V_v'(\eta_0(v))$ on the compact set $[v_1,v_2]$. We may chose $t_0\in(0,1)$ small enough, such that $V_v(\eta_0(v)+t(\rho_0-\eta_0(v)))\geqslant tC_1/2$ for $t\in[0,t_0]$. Using once again the Taylor's formula, since $V_v(\rho_0)=V_v'(\rho_0)=0$, we have

$$
\begin{aligned}
&V_v(\eta_0(v)+t(\rho_0-\eta_0(v))) \\
&= (1-t)^2(\rho_0-\eta_0(v))^2\int_0^1(1-s)V_v''(\rho_0-s(1-t)(\rho_0-\eta_0(v)))ds \ .
\end{aligned}
$$

Now, if $s\in[0,1]$,

$$
\begin{aligned}
&V_v''(\rho_0-s(1-t)(\rho_0-\eta_0(v))) \\
&\geqslant \inf_{r\in[\rho_0-(1-t)(\rho_0-\eta_0(v_1)),\rho_0]} V''(r) - \frac{v_2^2\rho_0^2}{2(\rho_0-(1-t)(\rho_0-\eta_0(v_1)))^3} \\
&\underset{t\to1}{\longrightarrow} V''(\rho_0) - \frac{v_2^2}{2\rho_0} =: C_2 > 0
\end{aligned}
$$

Therefore there exists $t_1 \in (t_0, 1)$ such that for $t \in [t_1, 1]$, $v \in [v_1, v_2]$,

$$V_v(\eta_0(v) + t(\rho_0 - \eta_0(v))) \geqslant (1-t)^2(\rho_0 - \eta_0(v_1))^2 C_2/4 \ .$$

We deduce that

$$\frac{(\rho_0 - \eta_0(v))^3(1-t)^2}{2(t\rho_0 + (1-t)\eta_0(v))^{3/2}\sqrt{V_v(t\rho_0 + (1-t)\eta_0(v))}} \leqslant g(t)$$

where

$$g(t) := \frac{(\rho_0 - \eta_0(v_1))^3}{2\eta_0(v_1)^{3/2}} \begin{cases} \frac{\sqrt{2/C_1}}{t^{1/2}} & t \in [0, t_0] \\ \sup\limits_{(t,v)\in[t_0,t_1]\times[v_1,v_2]} \frac{1}{\sqrt{V_v(t\rho_0+(1-t)\eta_0(v))}} & t \in [t_0, t_1] \\ 2(\rho_0 - \eta_0(v_1))/\sqrt{C_2} & t \in [t_1, 1] \end{cases}$$

It is clear that $g \in L^1(0,1)$. Similar computations, together with the equality $\eta_0'(v) = \frac{v(\rho_0-\eta_0(v))^2}{2\eta_0(v)V_v'(\eta_0(v))}$ (which is obtained by differentiation of the identity $V_v(\eta_0(v)) \equiv 0$), yield the existence of $\tilde{g} \in L^1(0,1)$ such that

$$\left| \frac{d}{dv} \frac{(\rho_0 - \eta_0(v))^3(1-t)^2}{2(t\rho_0 + (1-t)\eta_0(v))^{3/2}\sqrt{V_v(t\rho_0 + (1-t)\eta_0(v))}} \right| \leqslant \tilde{g}(t) \ .$$

It follows from the theorem of differentiation of an integral with parameters that $v \mapsto P_v$ is of class $\mathcal{C}^1$ on $[v_1, v_2]$. $\square$

Next, the stability criterion of the traveling bubbles obtained by Z. Lin may be stated as follows.

Theorem 1.5.2 *([L]) Let $f \in C^1(\mathbb{R}_+)$, $\rho_0 > 0$, and $V(r) := -\int_{\rho_0}^r f(s)ds$. The assumptions of Theorem 1.5.1 ensuring the existence of a bubble are assumed to be satisfied for $v \in (-c, c)$, where $c := \sqrt{-2\rho_0 f'(\rho_0)}$ is the sound speed (we may assume for instance that V'' is non-decreasing; see the remark after Theorem 1.5.1).*
Then, for every $v \in (-c, c)$, this bubble is stable if $\frac{dP_v}{dv} > 0$, and unstable if $\frac{dP_v}{dv} < 0$. The stability shall be understood in the following sense: for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial data $\varphi_0 = a_0 e^{i\theta_0}$ satisfies

$$\inf_{s\in\mathbb{R}} (\|\tau_s a_0^2 - a_v^2\|_{H^1} + \|\tau_s \partial_x \theta_0 - \partial_x \theta_v\|_{L^2}) < \delta \ ,$$

then

$$\inf_{s\in\mathbb{R}} (\|\tau_s a(t)^2 - a_v^2\|_{H^1} + \|\tau_s \partial_x \theta(t) - \partial_x \theta_v\|_{L^2}) < \varepsilon \ ,$$

for any $t \in (0, \infty)$. We have denoted here by $\varphi(t) = a(t)e^{i\theta(t)}$ the solution of (1.1) with initial data $\varphi(0) = \varphi_0$, and by τ_s the translation by s, namely $\tau_s f(x) = f(x - s)$. The soliton is said to be unstable if it is not stable.

The assumptions of this Theorem may in fact be weakened: if the assumptions of Theorem 1.5.1 are satisfied for every v in an interval (v_1, v_2), then the stability criterion of Theorem 1.5.2 can be applied for all these values of v.

For instance, let us consider the one dimensional Gross-Pitaevskii equation

$$i\partial_t u + \partial_x^2 u + (1 - |u|^2)u = 0 \ . \tag{1.8}$$

Here we have $\rho_0 = 1$, $f(r) = 1 - r$, $V(r) = (1-r)^2/2$, and thus $V_v(r) = \frac{(1-r)^2}{2}(1 - \frac{v^2}{2r})$, $f_v(r) = 1 - r + \frac{v^2}{4}(1 - \frac{1}{r^2})$. The assumptions of Theorem 1.5.1 are not satisfied for $v = 0$, but they are for every $v \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$. The criterion remains true on both this intervals. The bubbles may in fact be explicitly computed: for $v \in (-\sqrt{2}, \sqrt{2})$, $\varphi_v(x - vt)$ solves (1.8), where

$$\varphi_v(x) = (1 - v^2/2)^{1/2} \tanh\left((1 - v^2/2)^{1/2} x/\sqrt{2}\right) + iv/\sqrt{2}. \tag{1.9}$$

P_v and his derivative with respect to v may also be explicitly computed:

$$P_v = v(2 - v^2)^{1/2} - 2\arctan(2/v^2 - 1)^{1/2} \ ,$$

$$\frac{dP_v}{dv} = 2(2 - v^2)^{1/2} > 0 \ .$$

We deduce that for the one-dimensional Gross-Pitaevskii equation, the bubbles are stable in the sense given in Theorem 1.5.2.

Let us remark that in the extreme cases $v = \pm\sqrt{2}$, (1.9) gives the trivial solutions of (1.8) which are the constants of modulus 1. The case $v = 0$ is more interesting: as it has already been said, Theorem 1.5.1 ensures that there is no stationary bubble in the sense of definition 1.5.1. However, (1.9) gives a solution of (1.8) which vanishes at $x = 0$. Such a soliton is called a black soliton, which can also been seen as a kink with $\varphi_- < 0 < \varphi_+$. Otherwise, its existence should have been predicted by Theorem 1.3.1.

1.6 The black solitons

Definition 1.6.1 *Let $\rho_0 > 0$. A black soliton of (1.1) is a solution of (1.1) of the form*

$$u(t, x) = \varphi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

where $v \in \mathbb{R}$, $\varphi_v \in \mathcal{C}^1(\mathbb{R})$, $a_v(0) = 0$, $a_v(x) > 0$ for $x \in \mathbb{R}^\star$, θ_v is of class $\mathcal{C}^1$ on $\mathbb{R}^\star$, and piecewise C^1 on $\mathbb{R}$, with $\theta_v(0_+) = \pi + \theta_v(0_-)$, and $a_v \to \rho_0^{1/2}$, $\partial_x a_v \to 0$, $\partial_x \theta_v \to 0$ as $x \to \infty$.

1.6.1 Existence

In [DMG], we have proven the following Theorem concerning the existence of a black soliton to (1.1).

Theorem 1.6.1 *We assume $f \in \mathcal{C}(\mathbb{R}_+)$. Let $\rho_0 > 0$. There exists a black soliton φ of (1.1), of speed $v \in \mathbb{R}$, with modulus tending to $\rho_0^{1/2}$ at infinity if and only if the following conditions are satisfied: (we use the same notations as in Theorem 1.5.1)*

$$\begin{array}{rl} (i) & f(\rho_0) = 0 \\ (ii) & v = 0 \\ (iii) & V(r) := \int_r^{\rho_0} f(s)ds > 0 \text{ for } r \in [0, \rho_0). \end{array}$$

Moreover, if these conditions are satisfied, up to the multiplication by a constant of modulus 1, φ is unique, odd, and may be assumed to be real, positive for $x > 0$. Under the supplementary assumption $f'(\rho_0) < 0$ ($f'(\rho_0) \leqslant 0$ is a consequence of the other assumptions), we have

$$\sqrt{\rho_0} - \varphi(x) = e^{-cx(1+o(1))} \ ,$$

$$\varphi'(x) \sim c(\sqrt{\rho_0} - \varphi(x)) \ ,$$

$$\varphi''(x) \sim -c^2(\sqrt{\rho_0} - \varphi(x))$$

as $x \to +\infty$, where $c := \sqrt{-2\rho_0 f'(\rho_0)}$ is the sound speed.

Remark. This result shows that the only black solitons of (1.1) are in fact kinks, with $\omega = 0$ (up to the multiplication by factor $e^{i\theta_+}$).

1.6.2 Stability

The study of the stability of the bubbles by Z. Lin [L] presented in the last section is based on the hydrodynamical formulation of (1.1). Namely, the bubble is written as $\varphi = (\rho_0 - r)^{1/2} e^{i\theta}$. The system satisfied by r, θ_x has a Hamiltonian structure, and Z. Lin can apply the theory of stability of solitary waves developed by M. Grillakis, J. Shatah and W.A. Strauss ([GSS]). This method fails for the black solitons, because the vanishing of φ yields singularities for r and θ. To our knowledge, the only rigorous stability result for the black solitons is displayed in [DMG], where we prove a sufficient condition on φ which ensures the linear stability of a black soliton. This condition is as follows.

We denote by φ the black soliton obtained under the conditions specified in Theorem 1.6.1. We make the the extra assumption $f'(\rho_0) < 0$. We look for solutions to (1.1) under the form $\varphi + u_1 + iu_2$, where u_1 and u_2 are real, and we linearize the obtained system near $(u_1, u_2) = (0, 0)$. The linearized system we obtain is

$$\frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ , \tag{1.10}$$

where

$$A = \begin{pmatrix} 0 & L_1 \\ -L_2 & 0 \end{pmatrix} ,$$

$$L_1 = -\partial_x^2 - f(\varphi^2) ,$$

$$L_2 = -\partial_x^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2) .$$

Under this framework, we have the following stability result:

Theorem 1.6.2 *L_1 is a self-adjoint operator on L^2 with domain H^2, its essential spectrum is $[0, +\infty)$, and L_1 has an unique negative eigenvalue λ_0 which is simple.*
The Vakhitov-Kolokolov function

$$g(\lambda) \ := \ < (L_1 - \lambda)^{-1}\varphi', \varphi' > , \tag{1.11}$$

defined for $\lambda \in (\lambda_0, 0)$, is of class $\mathcal{C}^1$, increases on $(\lambda_0, 0)$, and $g(\lambda) \to -\infty$ as $\lambda \downarrow \lambda_0$. If the limit of $g(\lambda)$ as $\lambda \uparrow 0$ is non-positive, the spectrum of A is included in $i\mathbb{R}$, and φ is linearly stable.

Given a nonlinearity, it is not easy to prove rigorously whether the condition on the sign of the limit of $g(\lambda)$ as λ goes to zero is statisfied or not. Indeed, the only case for which we know that the black soliton is linearly stable is the one-dimensional Gross-Pitaevskii equation (see [DMG]). However, it is not so difficult to check numerically whether this condition is satisfied or not. For precise examples, see [DMG].

1.7 Synthesis of the results of existence of non-kink dark solitons

In this subsection, we fix $\rho_0 > 0$, and we consider non-linearities f such that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. We are going to describe, depending on the behaviour of $V(r) := \int_r^{\rho_0} f(s)ds$ between 0 and ρ_0, which kind of solitons (traveling bubbles, stationary bubbles or black solitons) may be obtained for speeds $v \in (-c, c)$, where $c := \sqrt{-2\rho_0 f'(\rho_0)}$ is the sound speed.

1.7.1 Non-zero speeds.

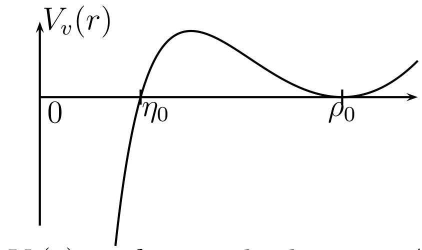
As we mentioned in section 1.5.1, for each $v \in (-c, c)\backslash\{0\}$, V_v vanishes on $(0, \rho_0)$, and η_0 is well-defined (we use the same notations that in Theorem 1.5.1), and $f_v(\eta_0) \leqslant 0$. In most cases, $f_v(\eta_0) \neq 0$ and there exists a traveling bubble of speed v. However, it may occur that for some speeds $v \in (-c, c)\backslash\{0\}$, $f_v(\eta_0) = 0$. We denote by S the set of these particular speeds.

Let us use the same notations that in Definition 1.5.1. The traveling bubbles $a_v e^{i\theta_v}$ solves the system:

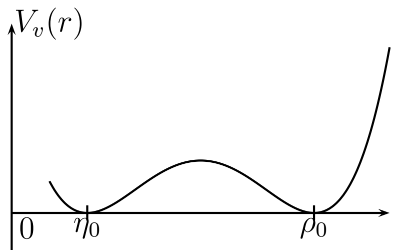
$$\theta_v' = \frac{v}{2}\left(1 - \frac{\rho_0}{a_v^2}\right) \tag{1.12}$$

$$a_v'^{\,2} = V_v(a_v^2) \tag{1.13}$$

If $v \notin S$ (that is $f_v(\eta_0) < 0$), the solution of (1.13) with initial data $a_v(x_0) = y_0 \in (\eta_0, \rho_0)$, $a_v'(x_0) > 0$ reaches η_0 at $-\infty < x_1 < x_0$, tends to ρ_0 at $+\infty$, and is symmetric with respect to x_1. On the contrary, if $v \in S$, $f_v(\eta_0) = 0$ and a_v only reaches η_0 at $x_1 = -\infty$. For such speeds, we do not have a bubble, but two solitons whose moduli behave like those of a kink, but whose phase is not constant, contrary to kinks of the Definition 1.3.1.



$V_v(r)$ in the standard case $v \notin S$



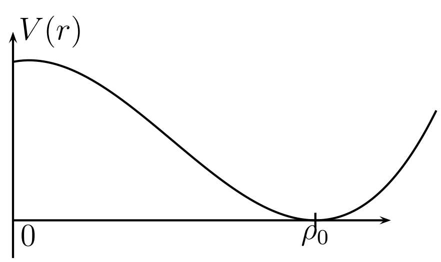
$V_v(r)$ as $v \in S$

1.7.2 $v = 0$, V vanishes on $(0, \rho_0)$.

As $v = 0$, the existence of η_0 as in Theorem 1.5.1 is not automatic any more. However, if such a η_0 exists, the analysis is the same that for non-zero speeds: either $V'(\eta_0) > 0$, and there is a stationary bubble, or $V'(\eta_0) = 0$, and we have two solutions wich are real kinks in the sense of Definition 1.3.1 (whose existence may have been deduced from Theorem 1.3.1).

1.7.3 $v = 0$, $V(r) > 0$ on $[0, \rho_0)$.

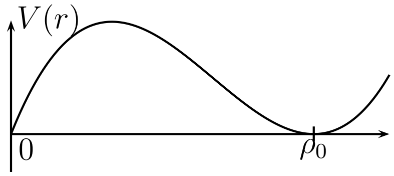
If V does not vanish on $[0, \rho_0)$, η_0 is not defined for $v = 0$. The assumptions of Theorem 1.3.1 ensuring the existence of a black soliton are satisfied in that case.



1.7.4 $v = 0$, $V(r) > 0$ on $(0, \rho_0)$, $V(0) = 0$.

If V does not vanish on $(0, \rho_0)$ and $V(0) = 0$, whatever values $V'(0) \geqslant 0$, we have the same kind of soliton as in the case $V(\eta_0) = V'(\eta_0) = 0$, namely there is no bubble for

$v = 0$, but two kinks (indeed, the assumptions of Theorem 1.3.1 are satisfied).



1.8 Dark-Bright solitons for a system of two coupled NLS equations

We are concerned with a system of two coupled NLS equations

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + f(|\psi|^2, |\varphi|^2)\psi = 0 \tag{1.14}$$

$$i\delta\frac{\partial\varphi}{\partial t} + \frac{\partial^2\varphi}{\partial x^2} + g(|\psi|^2, |\varphi|^2)\varphi = 0 \ , \tag{1.15}$$

where $\delta > 0$ and f, g are real valued functions. We are looking for "dark-bright solitons" of this system, that is solutions which can be written under the form

$$\psi(x,t) = \tilde{\psi}(x - vt) \ , \ \varphi(x,t) = e^{-i\theta t}\tilde{\varphi}(x - vt), \tag{1.16}$$

with $\theta \in \mathbb{R}$ and the boundary conditions

$$|\psi|^2 \longrightarrow \rho_0 \ , \ |\varphi|^2 \longrightarrow 0 \text{ as } x \longrightarrow \infty \ , \tag{1.17}$$

where $\rho_0 > 0$ is fixed.

In [Mar1] and [Mar2], M. Mariş found such solutions in the case of a Gross-Pitaevskii-Schrödinger system, which is up to a change of the time scale (1.14)-(1.15) with the non-linearities

$$f(x_1, x_2) = \frac{1}{\varepsilon^2}(1 - x_1 - \frac{1}{\varepsilon^2}x_2), \ g(x_1, x_2) = -\frac{q^2}{\varepsilon^2}x_1 \ .$$

We will show here that his method can be generalized to a larger class of nonlinearities.

Our assumptions on the functions $f, \ g \in \mathcal{C}^1(\mathbb{R}^2_+)$ are as follows:

(Hf1) $\quad f(\rho_0, 0) = 0$,

(Hf2) $\quad \frac{\partial f}{\partial x_1}(\rho_0, 0) < 0$.

Denoting by $c = \sqrt{-2\rho_0\frac{\partial f}{\partial x_1}(\rho_0, 0)}$ the sound speed, we also choose $v \in (-c, c)$ which satisfies the following conditions:

(Hv1) $\quad$ There exists $\eta_0(v) := \sup\left\{\eta \in (0, \rho_0), \int_\eta^{\rho_0} f(s, 0)ds - \frac{v^2(\eta-\rho_0)^2}{4\eta} = 0\right\} < \rho_0$,

(Hv2) $\quad f_v(\eta_0(v)) := f(\eta_0(v), 0) + \frac{v^2}{4}(1 - \frac{\rho_0^2}{\eta_0(v)^2}) < 0$.

Note that if $v \neq 0$, (Hv1) is automatically satisfied. From the results of the previous sections, we deduce that the assumptions (Hf1), (Hf2), (Hv1), (Hv2) ensure that there exist a stationary or traveling bubble $\psi(t,x) = \psi_v(x - vt)$ solving

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + f(|\psi|^2, 0)\psi = 0 \ , \tag{1.18}$$

with the boundary condition $|\psi_v|^2 \to \rho_0$ at infinity. We deduce that $(\psi_v, 0)$ is a trivial solution to (1.14)-(1.15) with the required boundary conditions (1.17). We write

$$\psi_v(x - vt) = a_0(1 + r_v(x - vt))e^{i\psi_{0,v}(x-vt)} \ , \tag{1.19}$$

where $a_0 = \sqrt{\rho_0}$, $r_v \in H^1(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R})$, r_v takes its values in $(-1, 0)$ and $\psi_{0,v} \in \mathcal{C}^\infty(\mathbb{R})$ is real valued.

We make the following assumption on g:

(Hg1) $$g(r, 0) > g(\rho_0, 0) \text{ for } r \in [0, \rho_0) \ .$$

We reformulate the problem (1.14)-(1.15)-(1.16) as follows: we write

$$\tilde{\psi}(x) = a_0(1 + r(x))e^{i\psi_0(x)} \ ,$$
$$\tilde{\varphi}(x) = u(x)e^{i\varphi_0(x)} \ ,$$

where ψ_0, φ_0, r, u are real-valued and $r(x) \to 0$, $u(x) \to 0$ as $x \to \infty$.

If (ψ, φ) solves (1.14)-(1.15), and if we assume moreover that $\psi_0' \to 0$ as $x \to \infty$ (which is necessary if we want ψ, and not only $|\psi|$ to have limits at $\pm\infty$) and that φ_0' is bounded, then the real functions ψ_0, φ_0, r, u must satisfy

$$\psi_0' = \tfrac{v}{2}\left(1 - \tfrac{1}{(1+r)^2}\right) \tag{1.20}$$
$$\varphi_0' = \tfrac{v}{2} \tag{1.21}$$
$$-r'' - \tfrac{v^2}{4}(1 + r)\left(1 - \tfrac{1}{(1+r)^4}\right) - f(\rho_0(1 + r)^2, u^2)(1 + r) = 0 \tag{1.22}$$
$$-u'' - g(\rho_0(1 + r)^2, u^2)u - \lambda u = 0 \ , \tag{1.23}$$

where $\lambda := v^2/4 + \delta\theta$.

We define now our functional framework, which is the same that in [Mar1] and [Mar2]. We denote

$$H := H^2_{\text{rad}} = \left\{u \in H^2(\mathbb{R}),\ u(x) = u(-x)\ ,\ \forall x \in \mathbb{R}\right\}\ ,$$
$$L := L^2_{\text{rad}} = \left\{u \in L^2(\mathbb{R}),\ u(x) = u(-x)\ ,\ \text{a.e. } x \in \mathbb{R}\right\}\ ,$$
$$V := \left\{r \in H^1(\mathbb{R}),\ \inf_{x\in\mathbb{R}} r(x) > -1\right\}\ .$$

Note that V is an open set in H^1. Once we have found r and u, we deduce easily ψ_0 and φ_0 from (1.20) and (1.21). Therefore we will look for $(r,u) \in H \cap V \times H$. As in [Mar1], we define the functionals $S : H \cap V \times H \longrightarrow L$ and $T : \mathbb{R} \times H \times H \longrightarrow L$ by

$$S(r,u) := -r'' - \frac{v^2}{4}(1+r)\left(1 - \frac{1}{(1+r)^4}\right) - f(\rho_0(1+r)^2, u^2)(1+r) \ , \tag{1.24}$$

$$T(\lambda, r, u) := -u'' - g(\rho_0(1+r)^2, u^2)u - \lambda u \ . \tag{1.25}$$

We also define the linear self-adjoint bounded from below operator on L^2, $A := -\frac{d^2}{dx^2} - g(\rho_0(1+r_v)^2, 0)$ (indeed, $A \geqslant -\sup_{r\in[0,\rho_0]} g(r,0)$).

With this new formalism, our problem reduces to find $(\lambda, r, u) \in \mathbb{R} \times H \cap V \times H$ solving $S(r,u) = 0$, $T(\lambda, r, u) = 0$. It is clear that for all $\lambda \in \mathbb{R}$, $(\lambda, r_v, 0)$ is a solution. We call such a solution a "trivial solution", as well as $(\lambda, 0, 0)$, and we look for non-trivial solutions.

1.8.1 Local curves of solutions

In a first step, following M. Mariş in [Mar1] (see also [Mar2]), we will use the Implicit Function Theorem to find non trivial solutions in a neighborhood of the trivial solution of (1.14)-(1.15) that is $(\psi_v, 0)$. We first recall a proposition which is a variant of Proposition 2.1 in [Mar2], about Schrödinger operators.

Proposition 1.8.1 *Let $q \in \mathcal{C}(\mathbb{R})$ be such that $q \not\equiv 0$, q is even and $q(x) \to 0$ as $x \to \pm\infty$. For $\lambda \leqslant 0$, we denote by $u_\lambda \in \mathcal{C}^2(\mathbb{R})$ the (unique, global and even) solution to the Cauchy problem*

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda u(x) \\ u(0) = 1, \ u'(0) = 0 \ . \end{cases}$$

We also denote by $n(\lambda)$ the number of zeroes of u_λ in $(0,\infty)$.
Then the operator $G := -\frac{d^2}{dx^2} + q(x)$, considered either as an operator on L^2 with domain H^2 or as an operator on L with domain H, is self-adjoint and satisfies the following properties:

1. *The essential spectrum of G is $\sigma_{ess}(G) = [0,\infty)$.*
2. *If $q \leqslant 0$, G has at least one negative eigenvalue.*
3. *Any eigenvalue of G is simple.*
4. *For any $\lambda < 0$ and $\varepsilon > 0$, there exists $C > 0$ such that*

$$|u_\lambda^{(m)}(x)| \leqslant Ce^{\sqrt{-\lambda+\varepsilon}|x|} \quad \textit{for} \quad m = 0,1,2.$$

   If $\lambda < 0$ is an eigenvalue of G, for any $\varepsilon \in (0,-\lambda)$, there exist positive constants C_1, C_2, M such that

$$C_1 e^{-\sqrt{-\lambda+\varepsilon}|x|} \leqslant |u_\lambda^{(m)}(x)| \leqslant C_2 e^{-\sqrt{-\lambda-\varepsilon}|x|} \ \textit{on} \ [M,\infty) \quad \textit{for} \quad m = 0,1,2.$$

5. *If $\int_0^\infty x|q(x)|dx < \infty$, then G has a finite number of negative eigenvalues.*

Proof. The assertions 1., 2., 3., 4. were proved in [Mar2]. The assumption $q \leqslant 0$ was not used (except for 2.) and the assumption $q \in L^2$ was not too. Assertion 5. is a direct consequence of v) and vi) in Proposition 2.1 of [Mar2] and Proposition 3.1 p438 in [BeSc]. □

As in [Mar1] or [Mar2], we prove the following lemmas

Lemma 1.8.1 *$d_r S(r_v, 0) : H \longmapsto L$ is invertible.*

Proof. The proof was done in [Mar1]. We recall the main arguments. We define $h_v : (-1, \infty) \longrightarrow \mathbb{R}$ by

$$h_v(x) = -f(\rho_0(1+x)^2, 0)(1+x) - \frac{v^2}{4}(1+x)\left(1 - \frac{1}{(1+x)^4}\right).$$

The domain of the operator on L^2, $B := -\frac{d^2}{dx^2} + h_v'(r_v)$ is H^2, and $d_r S(r_v, 0) = B_{|H}$.

We next show that $Ker(B) = Span(r_v')$. Indeed, we differentiate the relation $-\frac{d^2 r_v}{dx^2} + h_v(r_v) = 0$ to obtain that $r_v' \in Ker(B)$. Conversely, observing that r_v' does not vanish on $\mathbb{R}^*$ and that $\frac{d}{dx}\left(\frac{w}{r_v'}\right) = 0$ on $(-\infty, 0)$ and $(0, +\infty)$ if $w \in Ker(B)$, and using the continuity of w', we obtain $w \in Span(r_v')$. This proves that $d_r S(r_v, 0)$ is one to one, because r_v' is odd.

Let us now take $z \in L \subset (r_v')^\perp = Im(B)$, and $r \in H^2$ such that $Br = z$. Denoting $\check{r}(x) = r(-x)$, $B\check{r} = z$, $r - \check{r} = Cr_v'$ and $r - Cr_v'/2 \in H$, $B(r - Cr_v'/2) = z$. Therefore $d_r S(r_v, 0)$ is surjective from H onto L. □

Lemma 1.8.2 *The domain of the operator on L^2 $A = -\frac{d^2}{dx^2} - g(\rho_0(1+r_v)^2, 0)$ is H^2. The essential spectrum of A is $\sigma_{ess}(A) = [-g(\rho_0, 0), \infty)$. A admits at least one eigenvalue $\lambda_* < -g(\rho_0, 0)$. Let λ_* be such an eigenvalue. Then λ_* is simple. Let u_* be an associated eigenvector. Then we have*

1. *$Ker(d_u T(\lambda_*, r_v, 0)) = Span(u_*)$*

2. *$Im(d_u T(\lambda_*, r_v, 0)) = L \cap u_*^\perp$.*

Proof. The first claims are direct consequences of Proposition 1.8.1 applied to the operator $G = A + g(\rho_0, 0)$ (remark that $q = g(\rho_0, 0) - g(\rho_0(1+r_v)^2, 0) \leqslant 0$ because of assumption (Hg1)).
Since $d_u T(\lambda_*, r_v, 0) = (A - \lambda_*)_{|H}$ and $u_* \in H$, it is clear that the kernel of $d_u T(\lambda_*, r_v, 0)$ is $Span(u_*)$. It is also clear that $Im(d_u T(\lambda_*, r_v, 0)) = (A - \lambda_*)H \subset L \cap u_*^\perp$. If $z \in L \cap u_*^\perp$, let $u \in H^2$ be such that $(A - \lambda_*)u = z$. Then $(A - \lambda_*)\check{u} = z$, $(u + \check{u})/2 \in H$ and $(A - \lambda_*)(u + \check{u})/2 = z$. □

As in [Mar1], we define now $W := \left\{ r \in H, \sup_{x \in \mathbb{R}} |r(x)| < 1 \right\}$, which is an open set in H, $I := (-\alpha, \alpha)$ where $\alpha := \inf(1 + r_v) > 0$ (recall that $1 + r_v = |\psi_v|$, where ψ_v is a bubble) and $F : I \times \mathbb{R} \times W \times (H \cap u_*^\perp) \longmapsto L \times L$

$$F(s,\lambda,r,u) := \left\{ \begin{array}{ll} \begin{pmatrix} \frac{1}{s}S(r_v+sr,s(u_*+u)) \\ \frac{1}{s}T(\lambda,r_v+sr,s(u_*+u)) \end{pmatrix} & \text{if } s\neq 0, \\ \begin{pmatrix} d_rS(r_v,0).r \\ d_uT(\lambda,r_v,0).(u_*+u) \end{pmatrix} & \text{if } s=0. \end{array} \right.$$

As in [Mar1], we show that F is $\mathcal{C}^1$. $F(0,\lambda_*,0,0)=0$ and

$$d_{(\lambda,r,u)}F(0,\lambda_*,0,0)(\lambda,r,u) \quad = \quad \begin{pmatrix} d_rS(r_v,0).r \\ -\lambda u_* + d_uT(\lambda_*,r_v,0).u \end{pmatrix} .$$

It follows from lemmas 1.8.1 and 1.8.2 that $d_{(\lambda,r,u)}F(0,\lambda_*,0,0) : \mathbb{R}\times H\times(H\cap u_*^\perp) \longrightarrow L\times L$ is invertible. Thanks to the Implicit Functions Theorem, there exists $\eta>0$ and $\mathcal{C}^1$ functions

$$s \longmapsto (\lambda(s),r(s),u(s)) \in \mathbb{R}\times H\times(H\cap u_*^\perp)$$

defined on $(-\eta,\eta)$ such that $\lambda(0)=\lambda_*$, $r(0)=0$, $u(0)=0$ and

$$\begin{aligned} S(r_v+sr(s),s(u_*+u(s)))=0 \\ T(\lambda(s),r_v+sr(s),s(u_*+u(s)))=0 \ . \end{aligned}$$

This gives the only non-trivial solutions to (1.22)-(1.23) in a neighborhood of $(\lambda_*,r_v,0)$ in $\mathbb{R}\times H\times H$. Indeed, using the same arguments that in the proof of Theorem 1.7 in [CR], we prove that there exists a neighborhood U of $(\lambda_*,r_v,0)$ in $\mathbb{R}\times H\times H$ such that the solutions of $S(r,u)=0$, $T(\lambda,r,u)=0$ in U are either trivial solutions $(\lambda,r_v,0)$ or of the form $(\lambda(s),r_v+sr(s),s(u_*+u(s)))$, $s\in(-\eta,\eta)$.

The Theorem we have proved is as follows:

Theorem 1.8.1 *Let us assume that (Hf1), (Hf2), (Hg1), (Hv1), (Hv2) are satisfied. Let $\lambda_* < -g(\rho_0,0)$ be an eigenvalue of A and $u_*\in H$ an associated eigenvector.*
Then there exists $\eta>0$ and $\mathcal{C}^1$ functions

$$s \longmapsto (\lambda(s),r(s),u(s)) \in \mathbb{R}\times H\times(H\cap u_*^\perp)$$

defined on $(-\eta,\eta)$ such that $\lambda(0)=\lambda_*$, $r(0)=0$, $u(0)=0$ and

$$\begin{aligned} S(r_v+sr(s),s(u_*+u(s)))=0 \\ T(\lambda(s),r_v+sr(s),s(u_*+u(s)))=0 \ . \end{aligned}$$

Moreover, there exists a neighborhood U of $(\lambda_*,r_v,0)$ in $\mathbb{R}\times H\times H$ such that the solutions of $S(r,u)=0$, $T(\lambda,r,u)=0$ in U are either trivial solutions $(\lambda,r_v,0)$ or of the form $(\lambda(s),r_v+sr(s),s(u_*+u(s)))$, $s\in(-\eta,\eta)$.

1.8.2 Global branches of solutions

In the previous section, we have found local curves of solutions to (1.22)-(1.23) in a neighborhood of the trivial solution $(r_v, 0)$. In this section, we give a more global description of the set of solutions to (1.22)-(1.23).

In all what follows, $W : \mathbb{R} \longmapsto \mathbb{R}$ is a continuous, even function such that $W \geqslant 1$ and $W(x) \to \infty$ as $x \to \infty$, and there exists $C_W > 0$ such that for every $a, b \in \mathbb{R}$, $W(a+b) \leqslant C_W(W(a) + W(b))$. As it was remarked in [Mar2], there exist then constants $K, s > 0$ such that for $|x| \geqslant 1$, $W(x) \leqslant K|x|^s$. We define the spaces

$$L_W = \{\varphi \in L,\ W\varphi \in L^2\} \ ,$$

$$H_W = \{\varphi \in H,\ W\varphi, W\varphi', W\varphi'' \in L\} \ .$$

The main result of this section is the following theorem:

Theorem 1.8.2 *Let us assume that (Hf1), (Hf2), (Hg1), (Hv1), (Hv2) are satisfied. Let* $\mathcal{S}$ *be the set of non-trivial solutions* $(\lambda, r, u) \in \mathbb{R} \times H \cap V \times H$ *to (1.22)-(1.23). For any eigenvalue* $\lambda_m < -g(\rho_0, 0)$ *of the operator* $A = -\frac{d^2}{dx^2} - g(\rho_0(1+r_v)^2, 0)$, $\mathcal{S} \cup (\lambda_m, r_v, 0)$ *contains a maximal closed connected subset* $\mathcal{C}_m$ *in* $(-\infty, -g(\rho_0, 0)) \times H_W \times H_W$ *such that* $\mathcal{C}_m \cap \mathcal{C}_p = \emptyset$ *if* $m \neq p$ *and* $\mathcal{C}_m$ *satisfies at least one of the three properties:*

1. $\mathcal{C}_m$ *is unbounded in* $\mathbb{R} \times H \cap V \times H$.
2. *there exists a sequence* $(\lambda_n, r_n, u_n) \in \mathcal{C}_m$ *such that* $\lambda_n \to -g(\rho_0, 0)$ *as* $n \to \infty$.
3. *there exists a sequence* $(\lambda_n, r_n, u_n) \in \mathcal{C}_m$ *such that* $\inf\limits_{x \in \mathbb{R}} r_n(x) \to -1$ *as* $n \to \infty$.

Our proof of Theorem 1.8.2, which is very close to that of Theorem 3.8 in [Mar2], is based on the following Global Bifurcation Theorem of Rabinowitz.

Proposition 1.8.2 *Let* E *be a real Banach space and* $\Omega \subset \mathbb{R} \times E$ *an open set. Suppose that* $G : \Omega \longrightarrow E$ *is compact on closed, bounded subsets* $\omega \subset \Omega$ *such that* $dist(\omega, \partial\Omega) > 0$ *and is of the form* $G(a, u) = L(a, u) + H(a, u)$, *where* L *and* H *satisfy the following assumptions:*

1. $L(a, .)$ *is linear, compact for any fixed* a *and* $(a, u) \longrightarrow L(a, u)$ *is continuous and compact on closed, bounded subsets* $\omega \subset \Omega$ *such that* $dist(\omega, \partial\Omega) > 0$.
2. *For any closed, bounded subset* $\omega \subset \Omega$ *such that* $dist(\omega, \partial\Omega) > 0$, *there exists a function* ε_ω *such that* $\varepsilon_\omega(s) \to 0$ *as* $s \to 0$ *and*

$$\|H(a, u)\| \leqslant \|u\| \varepsilon_\omega(\|u\|) \qquad \textit{for any } (a, u) \in \omega \ .$$

3. *There exists* a_0 *and* $\varepsilon > 0$ *such that*
   - $(a_0, 0) \in \Omega$,
   - *for any* $a \in [a_0 - \varepsilon, a_0 + \varepsilon] \backslash \{a_0\}$, *we have* $Ker(Id - L(a, .)) = \{0\}$,
   - *if* $a_1 \in [a_0 - \varepsilon, a_0)$ *and* $a_2 \in (a_0, a_0 + \varepsilon]$, *then*

$$ind(Id - L(a_1, .), 0) \neq ind(Id - L(a_2, .), 0).$$

Let

$$\mathcal{S} := \{(a,u) \in \Omega, u \neq 0 \text{ and } u = G(a,u)\}$$

be the set of non-trivial solutions of the equation $u = G(a,u)$*. Then* $\mathcal{S} \cup \{(a_0,0)\}$ *possesses a maximal closed connected subset* $\mathcal{C}_{a_0}$ *which contains* $(a_0,0)$ *and has one of the following properties:*

1. $\mathcal{C}_{a_0}$ *is unbounded,*
2. $dist(\mathcal{C}_{a_0}, \partial\Omega) = 0$,
3. $\mathcal{C}_{a_0}$ *meets* $(a_1,0)$*, where* $a_1 \neq a_0$ *and* $Ker(Id - L(a_1,.)) \neq \{0\}$.

In order to bring our problem back to the framework of Proposition 1.8.2, we first reformulate the problem. Using the function h_v introduced in the proof of Lemma 1.8.1, (1.22)-(1.23) can be rewritten as

$$\begin{aligned} r'' &= h_v(r) + (f(\rho_0(1+r)^2,0) - f(\rho_0(1+r)^2,u^2))(1+r) \ , && (1.26)\\ u'' &= (-g(\rho_0(1+r)^2,u^2) - \lambda)u \ . && (1.27) \end{aligned}$$

If $(\lambda, r, u) \in (-\infty, -g(\rho_0,0)) \times H \cap V \times H$ solves (1.26)-(1.27), since $r_v'' = h_v(r_v)$ and $h_v'(0) = c^2 - v^2 > 0$, an easy calculation shows that $(\lambda, w = r - r_v, u) \in (-\infty, -g(\rho_0,0)) \times H \cap (V - r_v) \times H$ solves the system

$$\begin{pmatrix} w \\ u \end{pmatrix} = -\begin{pmatrix} B & 0 \\ 0 & A_\lambda \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} - \begin{pmatrix} H_1(w,u) \\ H_2(\lambda,w,u) \end{pmatrix} , \qquad (1.28)$$

where

$$\begin{aligned} B(w) &= \left(-\tfrac{d^2}{dx^2} + h_v'(0)\right)^{-1}[(h_v'(r_v) - h_v'(0))w] \, ,\\ A_\lambda(u) &= \left(-\tfrac{d^2}{dx^2} - g(\rho_0,0) - \lambda\right)^{-1}[(g(\rho_0,0) - g(\rho_0(1+r_v)^2,0))u] \, ,\\ H_1(w,u) &= \left(-\tfrac{d^2}{dx^2} + h_v'(0)\right)^{-1}[(h_v(r_v+w) - h_v(r_v) - h_v'(r_v)w \\ &\quad +(f(\rho_0(1+r_v+w)^2,0) - f(\rho_0(1+r_v+w)^2,u^2))(1+r_v+w)] \, ,\\ H_2(\lambda,w,u) &= \left(-\tfrac{d^2}{dx^2} - g(\rho_0,0) - \lambda\right)^{-1}[(g(\rho_0(1+r_v)^2,0) - g(\rho_0(1+r_v+w)^2,u^2))u] \, . \end{aligned}$$

Lemma 1.8.3 *Let* $(\lambda, r, u) \in (-\infty, -g(\rho_0,0)) \times H \cap V \times H$ *be a solution of (1.26)-(1.27). Then* $r, u \in H_W$.

Proof. Proposition 1.8.1 applied with $q(x) = g(\rho_0,0) - g(\rho_0(1+r(x))^2, u(x)^2)$ yields that if $u \not\equiv 0$, then u, u' and u'' decay faster than $e^{-\sqrt{-\lambda - g(\rho_0,0) - \varepsilon}|x|}$, for any $\varepsilon \in (0, -\lambda - g(\rho_0,0))$. Hence $u \in H_W$.

To prove that $r \in H_W$, we use the same arguments that in the proof of Lemma 3.5 in [Mar2]. The only change is that the term $u(x)^2$, which was needed to decay exponentially in [Mar2], is replaced here by

$$f(\rho_0(1+r(x))^2,0) - f(\rho_0(1+r(x))^2,u(x)^2) = -\int_0^1 \frac{\partial f}{\partial x_2}(\rho_0(1+r(x))^2, tu(x)^2)dtu(x)^2 \ .$$

But since r and u are bounded and $\frac{\partial f}{\partial x_2}$ is continuous, it is clear that this quantity also decays exponentially at infinity. □

From Lemma 1.8.3 above, we deduce that $(\lambda, r, u) \in (-\infty, -g(\rho_0, 0)) \times H \cap V \times H$ solves (1.26)-(1.27) if and only if $(\lambda, w, u) \in (-\infty, -g(\rho_0, 0)) \times H_W \cap (V - r_v) \times H_W$ solves (1.28).

We can now introduce a functional framework adapted to Proposition 1.8.2. This framework is the same that in [Mar2]. Let $E := H_W \times H_W$, $\Omega := (-\infty, -g(\rho_0, 0)) \times H_W \cap (V - r_v) \times H_W$, $L_\lambda := \begin{pmatrix} -B & 0 \\ 0 & -A_\lambda \end{pmatrix}$, $H(\lambda, w, u) := \begin{pmatrix} -H_1(w, u) \\ -H_2(\lambda, w, u) \end{pmatrix}$. In the next lemma, we verify that L_λ and H satisfy the assumptions 1. and 2. in Proposition 1.8.2.

Lemma 1.8.4

1. *For any $\lambda \in (-\infty, -g(\rho_0, 0))$, A_λ (resp. B) is linear and compact, and the map $(-\infty, -g(\rho_0, 0)) \times H_W \longmapsto H_W$, $(\lambda, u) \longmapsto A_\lambda u$ (resp. $(\lambda, w) \longmapsto Bw$) is continuous and compact on closed bounded subsets of $[d, e] \times H_W$, for $-\infty < d < e < -g(\rho_0, 0)$.*
2. *$H_1 : ((V - r_v) \cap H_W) \times H_W \longrightarrow H_W$ is continuous and compact on closed bounded subsets ω_1 of $((V - r_v) \cap H_W) \times H_W$ such that $dist(\omega_1, (H_W \backslash (V - r_v)) \times H_W) > 0$ and*

$$\|H_1(w, u)\|_{H_W} \leqslant C_{\omega_1}(\|u\|_{H_W}^2 + \|w\|_{H_W}^2)\ . \tag{1.29}$$

3. *$H_2 : (-\infty, -g(\rho_0, 0)) \times H_W \times H_W \longrightarrow H_W$ is continuous and compact on closed bounded subsets of $[d, e] \times H_W \times H_W$, for $-\infty < d < e < -g(\rho_0, 0)$, and*

$$\|H_2(\lambda, w, u)\|_{H_W} \leqslant C(\|u\|_{H_W}^2 + \|w\|_{H_W}^2)\ . \tag{1.30}$$

Proof. The proof is similar to that of Lemma 3.6 in [Mar2]. □

Next, we verify that Assumption 3. in Proposition 1.8.2 is satisfied. We begin with the following lemma, which is proved in [Mar2] (the proof is identical in our case).

Lemma 1.8.5 *For any $\lambda < -g(\rho_0, 0)$,*

1. *$Ker(Id_{H_W} + A_\lambda) \neq \{0\}$ if and only if λ is an eigenvalue of the operator $A = -\frac{d^2}{dx^2} - g(\rho_0(1 + r_v)^2, 0)$. In this case, for any $n \in \mathbb{N}^*$, $Ker(Id_{H_W} + A_\lambda)^n = Span(u_\lambda)$, where u_λ is an eigenvector of A.*
2. *If λ is not an eigenvalue of A, then $ind(Id_{H_W} + A_\lambda, 0) = (-1)^{n(\lambda)}$, where $n(\lambda)$ is the number of eigenvalues of A less than λ.*

Corollary 1.8.1 *If $\lambda_* < -g(\rho_0, 0)$ is an eigenvalue of A, then there exists $\varepsilon > 0$ such that*

1. *$(\lambda_*, 0, 0) \in (-\infty, -g(\rho_0, 0)) \times H_W \cap (V - r_v) \times H_W$,*
2. *if $\lambda \in [\lambda_* - \varepsilon, \lambda_* + \varepsilon] \backslash \{\lambda_*\}$, then $Ker(Id - L_\lambda) = \{0\}$,*
3. *if $\lambda_1 \in [\lambda_* - \varepsilon, \lambda_*)$ and $\lambda_2 \in (\lambda_*, \lambda_* + \varepsilon]$, then $ind(Id - L_{\lambda_1}, 0) \neq ind(Id - L_{\lambda_2}, 0)$.*

Proof. Assertion 1. is true because $r_v \in V$.
A admits an eigenvalue $\lambda_* < -g(\rho_0, 0)$ by Lemma 1.8.2. λ_* is isolated in the spectrum of A, therefore there exists $\varepsilon > 0$ such that $[\lambda_* - \varepsilon, \lambda_* + \varepsilon] \cap \sigma(A) = \{\lambda_*\}$. By Lemma 1.8.1 and Proposition 1.8.1 , it is easy to see that $w, u \in H_W$ and $w + Bw = 0$, $u + A_\lambda u = 0$ if and only if $w, u \in H$ and $w = 0$, $Au = \lambda u$. This yields Assertion 2.

As in [Mar2], it follows from a result of Leray and Schauder and Lemma 1.8.5 that

$$\begin{aligned} ind\left[\begin{pmatrix} Id_{H_W} + B & 0 \\ 0 & Id_{H_W} + A_{\lambda_1} \end{pmatrix}, 0\right] &= ind(Id_{H_W} + B, 0)ind(Id_{H_W} + A_{\lambda_1}, 0) \\ \neq ind(Id_{H_W} + B, 0)ind(Id_{H_W} + A_{\lambda_2}, 0) &= ind\left[\begin{pmatrix} Id_{H_W} + B & 0 \\ 0 & Id_{H_W} + A_{\lambda_1} \end{pmatrix}, 0\right]. \end{aligned}$$

□

Let $\tilde{\mathcal{S}}_0 := \{(\lambda, w, u) \in \Omega, (w, u) \neq (0, 0) \text{ solves } (1.28)\}$. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_N < -g(\rho_0, 0)$ be the eigenvalues of A (which are in finite number because of Proposition 1.8.1 and because $r_v \in H_W$ implies $\int_{\mathbb{R}} |x|(g(\rho_0(1 + r_v)^2, 0) - g(\rho_0, 0))dx < \infty$). We apply Proposition 1.8.2: for $m \in \{1 \ldots N\}$, $\tilde{\mathcal{S}}_0 \cup \{(\lambda_m, 0, 0)\}$ possesses a maximal closed subset $\mathcal{D}_m$ which contains $(\lambda_m, 0, 0)$ and has at least one of the following properties:

1. $\mathcal{D}_m$ is unbounded in $\mathbb{R} \times H_W \times H_W$.
2. there exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $\lambda_n \to -g(\rho_0, 0)$.
3. there exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $\inf\limits_{x \in \mathbb{R}} (w_n(x) + r_v(x)) \to -1$.
4. $\mathcal{D}_m$ meets $(\lambda_p, 0, 0)$ with $m \neq p$.

Let us next define $\mathcal{C}_m = \mathcal{D}_m + (0, r_v, 0)$. As in [Mar2], we can show that there is a neighborhood of $(-\infty, -g(\rho_0, 0)) \times \{(-r_v, 0)\}$ in $\mathbb{R} \times E$ which is not intersected by any of the $\mathcal{D}_m$ sets. Thus the $\mathcal{C}_m$ sets do no contain any of the trivial solutions to (1.22)-(1.23) of type $(\lambda, 0, 0)$, which means that $\mathcal{C}_m \subset \mathcal{S} \cup \cup_p \{(\lambda_p, r_v, 0)\}$. The Alternative 4 can be eliminated as it is by M. Mariş in [Mar2]. The argument is the continuity on $(\widetilde{\mathcal{S}}_0 + (0, r_v, 0)) \cup \cup_p \{(\lambda_p, r_v, 0)\}$ of the integer-valued function which maps (λ, r, u) onto the number of negative eigenvalues smaller than λ of the operator $-\frac{d^2}{dx^2} - g(\rho_0(1 + r)^2, u^2)$. Since this number is not the same for $(\lambda_p, r_v, 0)$ and $(\lambda_m, r_v, 0)$ as soon as $m \neq p$, the Assertion 4. is not possible.

On the contrary, it does not seem to be possible here to eliminate the alternative 3., which was shown not to occur in the particular case studied in [Mar2].

Chapitre 2

The black solitons of the 1D NLS equation

Ce chapitre est le fruit d'une collaboration avec Laurent di Menza.

Abstract. In this paper, we give a sufficient condition for the linear stability of a black soliton solution to a one-dimensional nonlinear Schrödinger equation. We compute numerically the black soliton and evaluate the limit at 0 of the Vakhitov-Kolokolov function. When the condition for linear stability is not satisfied, numerical computations suggest that the black soliton is linearly unstable. In the Gross-Pitaevskii case, we prove rigorously the linear stability of the black soliton. Finally, we check the dynamical stability of these solutions solving the partial differential equation with a finite differences algorithm.

2.1 Introduction

We consider here the Nonlinear Schrödinger equation in dimension 1:

$$\begin{cases} i\partial_t u + \partial_x^2 u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 \end{cases}, \tag{2.1}$$

where f is a real valued function and $|u_0(x)|^2 \to \rho_0$ as $x \to \infty$. We assume that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. In nonlinear optics, this equation is a model for the evolution of the complex amplitude envelope of a electric field in optical fibers, or for the self-guided beams in planar waveguides. We refer to [KLD] for a list of physically relevant nonlinearities f. Equation (2.1) also appears in the description of defectons or in the theory of one-dimensional ferromagnetic or molecular chains (see [BGMP], [BP], [B]).

In this paper, we focus on stationary solutions $u(t,x) = \varphi(x)$ to (2.1). Thus we are faced with the second-order differential equation

$$\varphi'' + f(|\varphi|^2)\varphi = 0. \tag{2.2}$$

In [dB] (see also [BGMP], [BP]), A. de Bouard studies one special kind of these solutions: the stationary bubbles, which may exist under some conditions on the nonlinearity f. Namely, defining the potential V as $V(r) = \int_r^{\rho_0} f(s)ds$, if V vanishes in the interval $(0, \rho_0)$ and if the largest zero of V in this interval is of multiplicity one, there does exist a stationary bubble solving (2.1), that is a real, strictly positive, even function, growing on $\mathbb{R}_+$, tending to $\rho_0^{1/2}$ at infinity.

Now, if V is positive on $[0, \rho_0)$, (2.1) admits an other kind of stationary solutions: the black solitons, which are real, odd functions tending to $\pm\rho_0^{1/2}$ at $\pm\infty$. For example, the Gross-Pitaevskii equation, which is (2.1) with $f(r) = 1-r$, $\rho_0 = 1$, has $V(r) = (1-r)^2/2 > 0$ on $[0, \rho_0)$ The terminology of black soliton comes from optics, where they describe black spots on bright backgrounds (see [KLD], [KiK], [MS]).

In [dB], the stationary bubbles solving (2.1) are shown to be all unstable. The proof is based on a subtle analysis of the spectrum of the operator obtained by linearization of (2.1) near the bubble. On the contrary, it was first thought by physicists that the black solitons were all stable (see [KLD], [MS]). However, Y.S. Kivshar and W. Krolikowski are the first who observed numerically unsatble black solitons ([KiK]). In the same paper, they justify by variational arguments a criterion determining if a dark soliton (and in particular, a black soliton) is stable or not. We recall that a dark soliton solving (2.1) is a solution of the form $u(t,x) = u_v(x - vt)$, where $v \in \mathbb{R}$ is the speed of the soliton, and $|u_v(x)|^2 \to \rho_0$ as $x \to \infty$ (see [Ga2]). This criterion reduces to the analysis of the sign of the derivative dP_v/dv, where P_v is the momentum of the dark soliton of speed v, given by

$$P_v = \mathfrak{Im} \int_{-\infty}^{+\infty} \partial_x \overline{u_v} u_v \left(1 - \frac{\rho_0}{|u_v|^2}\right) dx.$$

Namely, u_v is stable if $dP_v/dv > 0$, unstable if $dP_v/dv < 0$. A rigorous proof of this criterion has been done by Z. Lin ([L]). However, the proof is based on the hydrodynamical form of Equation (2.1). A solution to (2.1) is written as $u = (\rho_0 - r)^{1/2} e^{i\theta}$, and Z. Lin makes the analysis on the system satisfied by (r, θ_x), which has a Hamiltonian structure. This analysis is valid for non-zero speeds (or for stationary bubbles), because the dark soliton is in that case a traveling bubble, which in particular means that $|u_v(x)| > 0$ for $x \in \mathbb{R}$. This approach breaks down for the black solitons, which vanish at one point.

The goal of this paper is to fill as far as we can this gap concerning the study of the stability of the black solitons. We revisit the spectral analysis which was done by A. de Bouard on the stationary bubbles. Denoting by φ a black soliton solving (2.1), we look for solutions to (2.1) under the form $\varphi + u_1 + iu_2$, where u_1 and u_2 are real valued. The linearization of (2.1) near $(u_1, u_2) = (0, 0)$ writes

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.3}$$

where

$$A = \begin{pmatrix} 0 & L_1 \\ -L_2 & 0 \end{pmatrix},$$

$$L_1 = -\partial_x^2 - f(\varphi^2) \ ,$$
$$L_2 = -\partial_x^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2) \ .$$

We will see that $f(\varphi^2)$ and $\varphi^2 f'(\varphi^2)$ converge exponentially at infinity. It follows as in [dB] that the essential spectrum $\sigma_e(A)$ of A is contained in $i\mathbb{R}$.

Under this framework, we establish the following sufficient condition for the linear stability of the black solitons [1].

Theorem 2.1.1 *L_1 is a self-adjoint operator on L^2 with domain H^2, its essential spectrum is $[0, +\infty)$, and L_1 has an unique negative eigenvalue λ_0 which is simple.*
The Vakhitov-Kolokolov function

$$g(\lambda) \ := \ < (L_1 - \lambda)^{-1}\varphi', \varphi' > \ , \tag{2.4}$$

defined for $\lambda \in (\lambda_0, 0)$, is of class $\mathcal{C}^1$, increases on $(\lambda_0, 0)$, and $g(\lambda) \to -\infty$ as $\lambda \downarrow \lambda_0$. If the limit of $g(\lambda)$ as $\lambda \uparrow 0$ is non-positive, the spectrum of A is included in $i\mathbb{R}$, and φ is linearly stable.

As it will be seen, it is possible to verify numerically if the condition of Theorem 2.1.1 ensuring the linear stability of a black soliton is satisfied or not. Given a nonlinearity f such that there exists a black soliton solving (2.1), it seems to be more difficult to prove rigorously the linear stability of a black soliton. We can however do it for the black soliton of the Gross-Pitaevskii equation, thanks to the explicit knowledge of the black soliton φ, of the negative eigenvalue λ_0 of the operator L_1 and of the associated eigenvector:

Theorem 2.1.2 *The black soliton $\varphi(x) = \tanh(x/\sqrt{2})$ of the Gross-Pitaevskii equation is linearly stable.*

This paper is organized as follows. In section 2, we give a precise definition of the black solitons, and we give a necessary and sufficient condition on the nonlinearity f ensuring that there exists a black soliton. Section 3 is devoted to the proof of Theorem 2.1.1. In section 4, we show that Theorem 2.1.1 is numerically efficient. That is, given a nonlinearity f, we verify numerically if the condition in Theorem 2.1.1 is satisfied or not. Theorem 2.1.2 is proven in section 5. Finally, we numerically study in section 6 the dynamical stability of the black solitons, for the pure power defocusing equation as well as for the saturated equation.

2.2 Existence of a black soliton

We first give a precise definition of a black soliton. Even if that definition does not a priori excludes the case where black solitons are traveling at a non-zero speed v, we will

1. The notion of linear stability we consider here is in fact spectral stability. That is, φ is said to be linearly stable if the spectrum $\sigma(A)$ is a subset of $\{\lambda, \mathfrak{Re}\lambda \leqslant 0\}$, and linearly unstable in the other case.

see thereafter that black solitons may only be stationary solutions (ie, $v = 0$).

Definition 2.2.1 *Let $\rho_0 > 0$. A black soliton of (2.1) is a solution of (2.1) of the form*

$$u(t,x) = \varphi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

where $v \in \mathbb{R}$, $\varphi_v \in C^1(\mathbb{R})$, $a_v(0) = 0$, $a_v(x) > 0$ for $x \in \mathbb{R}^\star$, θ_v is of class C^1 on $\mathbb{R}^\star$, and piecewise $\mathcal{C}^1$ on $\mathbb{R}$, with $\theta_v(0_+) = \pi + \theta_v(0_-)$, and $a_v \to \rho_0^{1/2}$, $\partial_x a_v \to 0$, $\partial_x \theta_v \to 0$ as $x \to \pm\infty$.

Next, we give a necessary and sufficient condition on the nonlinearity f which ensure the existence of a black soliton to (2.1).

Theorem 2.2.1 *We assume $f \in \mathcal{C}(\mathbb{R}_+)$. Let $\rho_0 > 0$. There exists a black soliton φ of (2.1), of speed $v \in \mathbb{R}$, with modulus tending to $\rho_0^{1/2}$ at infinity if and only if the following conditions are satisfied:*
(i) $f(\rho_0) = 0$
(ii) $v = 0$
(iii) $V(r) := \int_r^{\rho_0} f(s)ds > 0$ *for* $r \in [0, \rho_0)$.
Moreover, if these conditions are satisfied, up to the multiplication by a constant of modulus 1, φ is unique, odd, and may be assumed to be real, positive for $x > 0$. Under the supplementary assumption $f'(\rho_0) < 0$ ($f'(\rho_0) \leqslant 0$ is a consequence of the other assumptions),

$$\sqrt{\rho_0} - \varphi(x) = e^{-cx(1+o(1))} \ ,$$

$$\varphi'(x) \sim c(\sqrt{\rho_0} - \varphi(x)) \ ,$$

$$\varphi''(x) \sim -c^2(\sqrt{\rho_0} - \varphi(x))$$

as $x \to +\infty$, where $c := \sqrt{-2\rho_0 f'(\rho_0)}$ is the sound speed.

Proof. Let us assume the existence of a black soliton $\varphi_v = a_v e^{i\theta_v} = ae^{i\theta}$ of (2.1), with modulus tending to $\rho_0^{1/2}$ at infinity.
$\varphi_v' = (a' + ia\theta')e^{i\theta}$ tends to 0 as $x \to \pm\infty$, and φ_v solves

$$-iv\varphi_v' + \varphi_v'' + f(|\varphi_v|^2)\varphi_v = 0, \tag{2.5}$$

thus $|\varphi_v''(x)| \to |f(\rho_0)|\rho_0^{1/2} =: l_0$ as $x \to \pm\infty$, and therefore $f(\rho_0) = 0$. Indeed, let us assume by contradiction that $l_0 > 0$. There exists $A > 0$ such that $x \geqslant A$ implies $3l_0/4 \leqslant |\varphi_v''(x)| \leqslant 5l_0/4$. φ_v solves (2.5), thus, since φ_v' and φ_v'' are bounded, φ_v and φ_v' are uniformly continuous, and so is φ_v''. It follows that there exists $\eta > 0$ such that $|x_1 - x_2| \leqslant \eta$ implies $|\varphi_v''(x_1) - \varphi_v''(x_2)| \leqslant l_0/4$. If $x_1, x_2 \geqslant A$ and $|x_1 - x_2| = \eta$,

$$\begin{aligned} &|\varphi_v'(x_1)| + |\varphi_v'(x_2)| \geqslant |\varphi_v'(x_1) - \varphi_v'(x_2)| \\ &= \ |\int_{x_1}^{x_2} \varphi_v''(x)dx| \geqslant |\int_{x_1}^{x_2} (|\varphi_v''(x_1)| - l_0/4)dx| \geqslant l_0\eta/2 \ . \end{aligned} \tag{2.6}$$

Let $\varepsilon \in (0, l_0\eta/4)$ and $B \geqslant A$ such that $x \geqslant B$ implies $|\varphi_v'(x)| \leqslant \varepsilon$. For $x_1, x_2 \geqslant B$ and $|x_1 - x_2| = \eta$, (2.6) yields a contradiction.

For $x \in \mathbb{R}^\star$, $\varphi'(x) = (a'(x) + ia(x)\theta'(x))e^{i\theta(x)}$. Letting x tend to $0_\pm$ in this equality and using the regularity properties of φ and θ, it follows that $\varphi'(0_\pm) = a'(0_\pm)e^{i\theta(0_\pm)}$. Since $\theta_v(0_+) = \pi + \theta_v(0_-)$, we get $a'(0_-) = -a'(0_+)$.
Moreover, an elementary computation (see for instance [L]) yields that for $x \in \mathbb{R}^\star$,

$$\theta_x(x) = \frac{1}{2}v\left(1 - \frac{\rho_0}{a(x)^2}\right) \tag{2.7}$$

$$-a_{xx}(x) = f_v(a(x)^2)a(x), \tag{2.8}$$

where $f_v(r) := f(r) + \frac{v^2}{4}(1 - \frac{\rho_0^2}{r^2})$. We also define $V_v(r) := \int_r^{\rho_0} f_v(s)ds = V(r) - \frac{v^2(r-\rho_0)^2}{4r}$. The multiplication of (2.8) by a_x and its integration between x_1 and x_2, where $x_1,\ x_2 \in \mathbb{R}^\star_\pm$ yields

$$-a_x(x_1)^2 + a_x(x_2)^2 = -V_v(a(x_1)^2) + V_v(a(x_2)^2)\ .$$

Letting x_2 tend to $\pm\infty$ (depending on the sign of x_1), we get

$$-a'(x)^2 = -V_v(a(x)^2),\ \ x \in \mathbb{R}^\star\ .$$

Therefore $V_v(r) \geqslant 0$ for $r \in [0, \rho_0)$. In particular, $v = 0$.
Since $a \geqslant 0$ and $a(0) = 0$, we have $a'(0_+) \geqslant 0$. Now, if $a'(0_+) = 0$, $a \equiv 0$ and φ would not be a black soliton. Thus, in a neighborhood of 0 in $\mathbb{R}_+$, a is the solution of the Cauchy Problem

$$\left\{\begin{array}{l} a'(x) = V(a(x)^2)^{1/2} \\ a(0) = 0 \end{array}\right. \quad . \tag{2.9}$$

Therefore a is strictly increasing as long as $V(a^2) > 0$. Let us assume by contradiction that there exists $r \in (0, \rho_0)$ such that $V(r) = 0$. Let $r_0 := \min\{r \in (0, \rho_0),\ V(r) = 0\} \in (0, \rho_0)$. Since $V \geqslant 0$ on $(0, \rho_0)$, we must have $f(r_0) = -V'(r_0) = 0$. The intermediate value theorem ensures that there exists $x_0 \in \mathbb{R}^\star_+$ such that $a(x_0) = r_0^{1/2}$, and then $a \equiv r_0^{1/2}$, which is a contradiction to the fact that φ is a black soliton.

On the other side, if the assumptions of the statement are satisfied, we easily construct a black soliton $\varphi = ae^{i\theta}$ with modulus tending to $\rho_0^{1/2}$ at infinity, in the following way: we show that the solution a to (2.9) is global ($a \in \mathcal{C}^1(\mathbb{R}_+)$). Indeed, let us denote by $[0, T^*)$ the maximal interval of existence of the solution to (2.9). If there exists $x_0 > 0$ such that $a(x_0) = \rho_0^{1/2}$, $a'(x_0) = 0$ and the uniqueness in the Cauchy-Lipshitz Theorem yields $a \equiv \rho_0^{1/2}$, which is a contradiction with $a(0) = 0$. Thus $0 \leqslant a(x) < \rho_0^{1/2}$ for $x \in [0, T^*)$, $T^* = +\infty$, and $a(x) \uparrow \rho_0^{1/2}$ as $x \to +\infty$. Next, we extend a to $\mathbb{R}$ such that a is even, and we choose $\theta \equiv \theta_+ \in \mathbb{R}$ on $\mathbb{R}^\star_+$, $\theta \equiv \theta_- = \pi + \theta_+$ on $\mathbb{R}^\star_-$.

From now on, we assume $f'(\rho_0) < 0$. It follows from (2.9) that

$$a'(x) \underset{x\to\infty}{\sim} c(\sqrt{\rho_0} - a(x))\ , \tag{2.10}$$

which may be integrated as

$$\sqrt{\rho_0} - a(x) \underset{x\to\infty}{=} e^{-cx(1+o(1))} \tag{2.11}$$

We inject (2.11) into $a''(x) = -f(a(x)^2)a(x)$ to obtain the other asymptotics. $\square$

2.3 A sufficient condition for linear stability

In this section, we prove Theorem 2.1.1.

Let us denote by φ the soliton obtained under the conditions (i)-(ii)-(iii) of Theorem 2.2.1, with the extra assumption $f'(\rho_0) < 0$.

In our study of the linear stability of φ, we use the same notations as in [dB]. The only difference is that here, φ denotes a black soliton, whereas it denotes a stationary bubble in [dB]. As we mentioned it in the introduction, the linearization of (2.1) near the black soliton φ yields to the system (2.3).

Let us first inspect the spectra of the operators L_1 and L_2. As for the stationary bubbles studied in [dB], the assumption $f'(\rho_0) < 0$ ensures that $\sigma_e(L_1) = [0, +\infty)$ and $\sigma_e(L_2) = [-2\rho_0 f'(\rho_0), +\infty)$. The fact that φ is a solution of (2.1) writes $L_1\varphi = 0$. The differentiation of this equality yields $L_2\varphi' = 0$. φ admits exactly one zero, whereas φ' does not vanish. Therefore Sturm's theory (see [BeSc]) ensures that 0 is the lowest eigenvalue of L_2 and that it is simple. It also suggest that L_1 has an unique strictly negative eigenvalue which is simple. That is what we prove in the next lemma[2].

Lemma 2.3.1 *L_1 has an unique strictly negative eigenvalue λ_0, which is simple. The associated eigenvector u_0 may be chosen positive.*

Proof of Lemma 2.3.1 If $v \in H^2$,

$$< L_1 v, v > = \int_{\mathbb{R}} |\partial_x v|^2 - f(\varphi^2)v^2 =: Q(v,v) \ ,$$

where Q is defined on H^1. Since $\sigma_e(L_1) = [0, +\infty)$, the existence of $v \in H^1$ such that $Q(v,v) < 0$ will imply the existence of a negative eigenvalue λ_0 of L_1. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ supported in $[-2, 2]$, with $\chi(0) = 1$. For $\delta > 0$, we define the function φ_δ on $\mathbb{R}$ by

$$\varphi_\delta(x) := \begin{cases} |\varphi(x)| & \text{if } |x| \geqslant \delta \\ \varphi(\delta) & \text{if } -\delta \leqslant x \leqslant \delta \end{cases} ,$$

and for $n, \delta > 0$,

$$\varphi_{\delta,n}(x) := \varphi_\delta(x)\chi(x/n) \ .$$

2. Note that in the case of the stationary bubbles studied in [dB], φ was positive while φ' vanished exactly once on $\mathbb{R}$. Thus L_1 was positive, while L_2 had an unique negative eigenvalue. This explains why in the sequel, the roles of L_1 and L_2 are upturned in comparison with [dB]

It is clear that $\varphi_{\delta,n} \in H^1$, and

$$\begin{aligned} Q(\varphi_{\delta,n},\varphi_{\delta,n}) &= \int_{\mathbb{R}} \Big[\varphi_\delta'(x)^2\chi(x/n)^2 + \frac{2}{n}\varphi_\delta(x)\varphi_\delta'(x)\chi(x/n)\chi'(x/n) \\ &\qquad + \frac{1}{n^2}\varphi_\delta(x)^2\chi'(x/n)^2 - f(\varphi(x)^2)\varphi_\delta(x)^2\chi(x/n)^2 \Big] dx \\ &\underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \left[\varphi_\delta'(x)^2 - f(\varphi(x)^2)\varphi_\delta(x)^2\right] dx \ . \end{aligned}$$

In spite of the fact that $\varphi_\delta \notin H^1$, we will denote this very last quantity by $Q(\varphi_\delta, \varphi_\delta)$.

$$\begin{aligned} Q(\varphi_\delta,\varphi_\delta) &= \int_{|x|\geqslant\delta} (\varphi'(x)^2 - f(\varphi(x)^2)\varphi(x)^2)dx + \int_{|x|\leqslant\delta} -f(\varphi(x)^2)\varphi(\delta)^2 dx \\ &= \underbrace{\int_{\mathbb{R}} (\varphi'(x)^2 - f(\varphi(x)^2)\varphi(x)^2)dx}_{=0} \\ &\qquad + \int_{-\delta}^{\delta} \left[-\varphi'(x)^2 + f(\varphi(x)^2)(\varphi(x)^2 - \varphi(\delta)^2)\right] dx \\ &\underset{\delta\to 0}{=} -2\delta\varphi'(0)^2 + o(\delta) \ , \end{aligned}$$

where we have used the fact that $L_1\varphi = 0$. Thus we can choose $\delta > 0$ small enough such that $Q(\varphi_\delta,\varphi_\delta) < 0$, and then n large enough such that $Q(\varphi_{\delta,n},\varphi_{\delta,n}) < 0$. Therefore L_1 has a smallest eigenvalue $\lambda_0 < 0$. It is a classical result that λ_0 is simple, and that there exists a positive associated eigenvector u_0. Let us assume by contradiction that L_1 has an other eigenvalue $\lambda_1 \in (\lambda_0, 0)$, and let u_1 be an associated eigenvector. Then u_1 vanishes at $x_0 \in \mathbb{R}$ (because $u_1 \perp u_0 > 0$), and the Sturm-Liouville theory implies that φ, which solves $L_1\varphi = 0$ vanishes in both intervals $(-\infty, x_0)$ and $(x_0, +\infty)$, which is a contradiction, because φ vanishes only at 0. $\square$

As we already mentioned it in the introduction, the essential spectrum of A is purely imaginary. Moreover, if λ is an eigenvalue of A with associated eigenvector (u_1, u_2), $-\lambda$ is also an eigenvalue, with associated eigenvector $(-u_1, u_2)$. Thus the linear instability of φ is equivalent to the existence of an eigenvalue of A with non-zero real part. If λ ($\neq 0$) is such an eigenvalue and if (u_1, u_2) is an associated eigenvector, we have

$$\left\{ \begin{array}{rcl} L_1 u_2 &=& \lambda u_1 \\ -L_2 u_1 &=& \lambda u_2 \end{array} \right. .$$

In particular, since L_2 is self-adjoint, $u_2 \in (\ker L_2)^\perp = (\varphi')^\perp$ and $u_1 \in -\lambda \tilde{L_2}^{-1} u_2 + Vect(\varphi')$, where we have denoted by $\tilde{L_2}$ the restriction of L_2 to $(\varphi')^\perp$ (it is clear that $\tilde{L_2}$ is invertible). Therefore, taking the scalar product with u_2 (which is orthogonal to φ'), and using the fact that $\tilde{L_2}$ is a self-adjoint coercive operator, we obtain

$$-\lambda^2 = \frac{< L_1 u_2, u_2 >}{< \tilde{L_2}^{-1} u_2, u_2 >} \ .$$

We deduce that $-\lambda^2 \in \mathbb{R}$, thus $\lambda \in \mathbb{R} \cup i\mathbb{R}$. Therefore, if λ is an eigenvalue of A with non-zero real part, $\lambda \in \mathbb{R}^*$, and the quantity

$$\inf_{v \in (\varphi')^\perp \cap D(L_1), v \neq 0} \frac{< L_1 v, v >}{< \tilde{L_2}^{-1} v, v >}$$

is strictly negative. Since $\tilde{L_2}^{-1}$ is positive, so is

$$\alpha := \inf_{v \in (\varphi')^\perp \cap D(L_1), v \neq 0} \frac{< L_1 v, v >}{\|v\|_{L^2}^2}.$$

We next prove that if $\alpha < 0$, the infimum α is reached. Indeed, let $(v_n)_n$ be a sequence of $(\varphi')^\perp \cap D(L_1) = (\varphi')^\perp \cap H^2$ such that $\|v_n\|_{L^2} = 1$ and $< L_1 v_n, v_n > \to \alpha$.

$$\int |\partial_x v_n|^2 dx \;=\; < L_1 v_n, v_n > + \int f(\varphi^2)|v_n|^2 dx \leqslant \sup_n < L_1 v_n, v_n > + \|f\|_{L^\infty},$$

thus $(v_n)_n$ is bounded in H^1. We can extract a subsequence (also denoted by v_n, for convenience), such that $v_n \rightharpoonup v_*$ in H^1. We have $< v_*, \varphi' > = \lim_n < v_n, \varphi' > = 0$, and the fast decrease of $f(\varphi^2)$ at infinity ensures that $\int f(\varphi^2)|v_n|^2 \to \int f(\varphi^2)|v_*|^2$. On the other side, $\int |\partial_x v_*|^2 dx \leqslant \liminf \int |\partial_x v_n|^2 dx$, thus

$$< L_1 v_*, v_* > = \int |\partial_x v_*|^2 - \int f(\varphi^2)|v_*|^2 \leqslant \liminf \int |\partial_x v_n|^2 - \lim \int f(\varphi^2)|v_n|^2 = \alpha \ .$$

Moreover, $\|v_*\|_{L^2} \leqslant \liminf \|v_n\|_{L^2} = 1$. Assume by contradiction that $\|v_*\|_{L^2} < 1$. Then

$$\frac{< L_1 v_*, v_* >}{\|v_*\|_{L^2}^2} \leqslant \frac{\alpha}{\|v_*\|_{L^2}^2} < \alpha < 0 \ ,$$

which is a contradiction with the definition of α. Therefore $\|v_*\|_{L^2} = 1$, $\frac{< L_1 v_*, v_* >}{\|v_*\|_{L^2}^2} = \alpha$, and the infimum α is reached at v_*. Hence v_* satisfies the Euler equation

$$L_1 v_* = \alpha v_* + \beta \varphi' \ , \quad \text{where } \beta \in \mathbb{R} \ . \tag{2.12}$$

It can be easily seen that $\alpha > \lambda_0$. Indeed, it is clear that $\alpha \geqslant \lambda_0$, and if $\alpha = \lambda_0$, (2.12) implies

$$0 = < v_*, (L_1 - \lambda_0) u_0 > = < (L_1 - \lambda_0) v_*, u_0 > = \beta < \varphi', u_0 > \ ,$$

where u_0 denotes a positive eigenvector of L_1 associated with the eigenvalue λ_0. Since φ' and u_0 are both positive, it follows that $\beta = 0$. (2.12) with $\beta = 0$ implies that v_* is colinear to u_0, because λ_0 is a simple eigenvalue of L_1. This is a contradiction, because $u_0 \notin (\varphi')^\perp$.

Since $\sigma(L_1) \cap (\lambda_0, 0) = \emptyset$, we can define on the interval $(\lambda_0, 0)$ the Vakhitov-Kolokolov fonction g by

$$g(\lambda) \;:=\; < (L_1 - \lambda)^{-1} \varphi', \varphi' > \ . \tag{2.13}$$

We have just seen that if φ is linearly unstable, the infimum $\alpha \in (0, \rho_0)$ of $< L_1 v, v > / \|v\|_{L^2}^2$ is reached at some $v_* \in (\varphi')^\perp$ which satisfies the Euler equation (2.12). This last equation can be rewritten as $v_* = \beta (L_1 - \alpha)^{-1} \varphi'$, and thus $g(\alpha) = 0$.

Now, g is of class C^1 on $(\lambda_0, 0)$ and $g'(\lambda) = \|(L^1 - \lambda)^{-1} \varphi'\|^2 > 0$, thus g is increasing on $(\lambda_0, 0)$. Moreover, if we denote by Π the orthogonal projection on $u_0^\perp$,

$$(L_1 - \lambda)^{-1} = \frac{1}{\lambda_0 - \lambda} \frac{< u_0, . >}{\|u_0\|^2} u_0 + \Pi (L_1 - \lambda)^{-1} \Pi \ ,$$

and thus

$$g(\lambda) = \frac{1}{\lambda_0 - \lambda} \frac{< u_0, \varphi' >^2}{\|u_0\|^2} + < (L_1 - \lambda)^{-1} \Pi \varphi', \Pi \varphi' > \ . \tag{2.14}$$

Letting λ decrease to λ_0, the first term of the right-hand side tends to $-\infty$, whereas the second one remains bounded. It follows that $g(\lambda) \to -\infty$ as $\lambda \downarrow \lambda_0$, and to study if g vanishes or not on $(\lambda_0, 0)$, it suffices to study the sign of the limit of $g(\lambda)$ as $\lambda \uparrow 0$. Theorem 2.1.1 follows. □

2.4 Numerical check of linear stability

First, it is possible to find an approximate value of the black soliton using a differential solver, since $\varphi(0) = 0$ and the value $\varphi'(0)$ is explicitly known. In fact, if we multiply (2.2) by φ' and integrate between 0 and infinity, it can be shown that

$$\varphi'(0) = \sqrt{V(0)}.$$

For instance, for the pure power defocusing equation

$$i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + (1 - |u|^{2\sigma}) u = 0, \tag{2.15}$$

which corresponds to $f(r) = 1 - r^\sigma$, we have

$$\varphi'(0) = \sqrt{\frac{\sigma}{\sigma + 1}}.$$

It is sufficient to compute the approximate solution of (2.2) on a sufficiently large domain in such a way that the derivative almost vanishes at the boundary. For some prescribed λ, the Vakhitov-Kolokolov function is given by

$$g(\lambda) =< (L_1 - \lambda I_d)^{-1} \varphi', \varphi' >:=< \psi, \varphi' >,$$

where $\psi \in L^2(\mathbb{R})$ satisfies the differential equation $(L_1 - \lambda I_d)\psi = \varphi'$. This equation can be solved with use of finite differences, seeking the approximate values ψ_j of the solution at gridpoints $x_j = jh$ where $j = -J, \cdots, J$. In this case, the discretized linear system can be written as $A\Psi = \Phi$, where $\Psi = (\psi_{-J}, \psi_{-J+1}, \cdots, \psi_{J-1}, \psi_J)^T \in \mathbb{R}^{2J+1}$, $\Phi = (\varphi_{-J}, \varphi_{-J+1}, \cdots, \varphi_{J-1}, \varphi_J)^T$ and the matrix A is given by

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} + \mathrm{diag}\Big(q_1(x_j) - \lambda, -J \leqslant j \leqslant J\Big).$$

The function g is then given by $g(\lambda) =< \Psi, \Phi >$. In Figures 2.1 and 2.2 are respectively plotted the profiles of black solitons obtained for different σ ($\sigma = 0.5$, 1, 2 and 3) and for each case the Vakhitov-Kolokolov function with respect to log $(-\lambda)$.

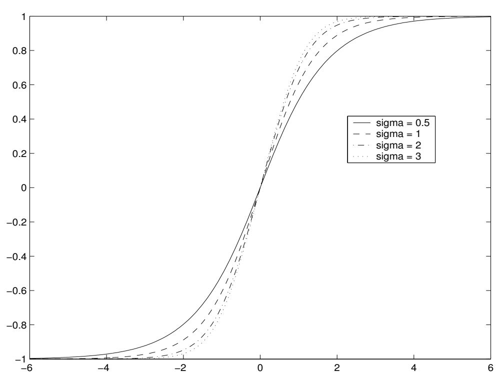


FIG. 2.1 – *Black solitons for different values σ.*

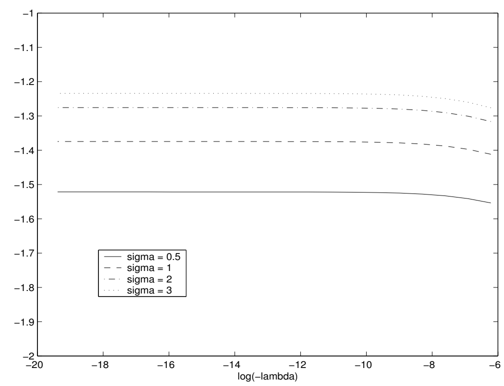


FIG. 2.2 – *g with respect to log$(-\lambda)$ for different values σ.*

It can be observed that the function g is always strictly negative when λ tends to zero and admits at each time a strictly negative limit at zero. This numerically checks the linear stability in this case. It can be also observed that this limit seems to be an increasing function of σ. Nevertheless, when σ becomes large, it has been found that the limit value $l(\sigma)$ when λ tends to zero is always strictly negative. As indicated in Figure 2.3, it seems that this $l := \lim\limits_{\sigma\to\infty} l(\sigma)$ exists and remains strictly negative (we have $l = -1.133...$). This result suggests the linear stability of black solitons for any nonnegative value of σ.

We then focus on the case of the saturated equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \Big(\frac{1}{(1+a|u|^2)^2} - \frac{1}{(1+a)^2}\Big)u = 0, \tag{2.16}$$

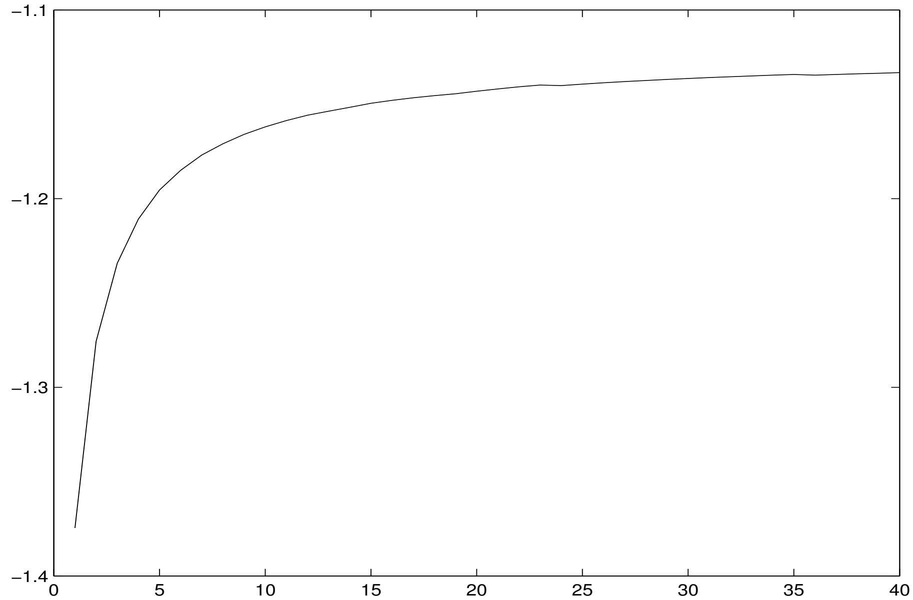


FIG. 2.3 – *Plot of* $l(\sigma)$ *versus* σ.

where a is a prescribed positive parameter. Here, the nonlinear term still vanishes for $|u|^2 = 1$ and remains positive between the two values 0 and 1. The value at the origin of the black soliton derivative is given in this case by the relation

$$\varphi'(0) = \frac{\sqrt{a}}{1+a}$$

and it is still possible here to compute the soliton and the Vakhitov-Kolokolov function. In Figures 2.4 are shown the profiles of black solitons for different values of a. It can be noticed that contrary to the first equation investigated, the black soliton derivative at zero is no more a monotonous function of the parameter (in fact, the maximum value obtained is reached for $a_* = 1$). In figure 2.5 is plotted the value $g(\lambda_0)$ for $\lambda_0 = -10^{-5}$ for different values of a. It clearly indicates that the black soliton is linearly stable when $a < a_c \simeq 7.4725$ and may suggest instability when $a > a_c$. This kind of instability threshold has been already observed in the literature (see [KiK]).

2.5 The linear stability of the black soliton of the 1D Gross-Pitaevskii equation

In the cubic defocusing case $f(r) = 1 - r$, the black soliton φ is explicitely known:

$$\varphi(x) = \tanh \frac{x}{\sqrt{2}} \ .$$

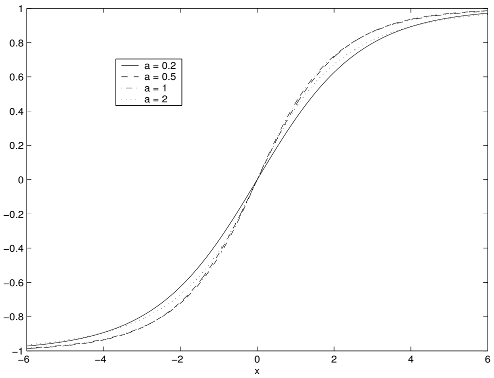


FIG. 2.4 – *Black solitons for different a.*

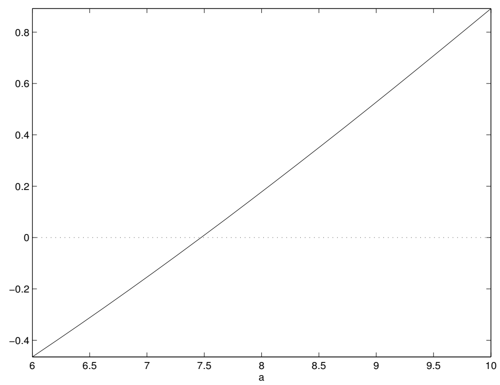


FIG. 2.5 – *$g(\lambda_0)$ versus a.*

Thus

$$\varphi'(x) = \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}} \ ,$$

$$f(\varphi(x)^2) = \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} \ ,$$

and

$$L := L_1 = -\frac{d^2}{dx^2} - \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} \ .$$

We also know that the lowest eigenvalue of L is $\lambda_0 = -1/2$, and an associated eigenvector is

$$u_0(x) = \frac{1}{\cosh \frac{x}{\sqrt{2}}} \ .$$

The explicit knowledge of this data enable us to prove the linear stability of the black soliton of the Gross-Pitaevskii equation stated in Theorem 2.1.2.

Proof of Theorem 2.1.2. Let $\lambda \in (-1/2, 0)$, and

$$v := (L - \lambda)^{-1}\varphi' \in H^2 \subset L^\infty \ .$$

Then it is clear that v is even, $v(x) \to 0$ as $x \to \infty$, and v solves

$$v''(x) \ = \ -\frac{v(x)}{\cosh^2 \frac{x}{\sqrt{2}}} - \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}} - \lambda v(x) \ . \tag{2.17}$$

For every integere $n \geqslant 1$, let us define

$$u_n := \int_{-\infty}^{+\infty} \frac{v(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx$$

and

$$a_n := \int_{-\infty}^{+\infty} \frac{1}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx \ .$$

Note that $u_1 = \sqrt{2} g(\lambda)$. The strategy of the proof is as follows. Thanks to an induction relation between u_n and u_{n+1} as well as an asymptotic expansion of u_n as n goes to infinity, we first establish a link between u_1 and $v(0)$. In particular, this relation implies that $v(0) > 0$ as soon as $u_1 \geqslant 0$. Finally, we show that the condition $v(0) > 0$ is inconsistent with the fact that $v(x) \to 0$ as $x \to \infty$.

Multiplying (2.17) by $1/\cosh^{2n} \frac{x}{\sqrt{2}}$ and summing over $\mathbb{R}$, we get

$$\int_{-\infty}^{+\infty} \frac{v''(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx \quad = \quad -u_{n+1} - \frac{1}{\sqrt{2}} a_{n+1} - \lambda u_n \ . \tag{2.18}$$

Using integration by parts, the left hand side of (2.18) can be expressed as

$$\int_{-\infty}^{+\infty} \frac{v''(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx \quad = \quad 2n^2 u_n - (2n+1) n u_{n+1} \ . \tag{2.19}$$

From (2.18) and (2.19) we infer the following induction relation:

$$u_{n+1} \quad = \quad \frac{1 + \frac{\lambda}{2n^2}}{1 + \frac{1}{2n} - \frac{1}{2n^2}} u_n + \frac{1}{\sqrt{2}} \frac{a_{n+1}}{2n^2 + n - 1} \ . \tag{2.20}$$

(2.20) yields a relation between u_{n+1} and u_1:

$$u_{n+1} \quad = \quad \prod_{l=1}^{n} \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) u_1 + \frac{1}{\sqrt{2}} \sum_{k=1}^{n} \prod_{l=k+1}^{n} \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) \frac{a_{k+1}}{2k^2 + k - 1} . \tag{2.21}$$

We will next compute asymptotic expansions of each term in (2.21). We begin with the term containing u_1. First remark that the product $P_\lambda := \prod_{l=1}^{\infty} \left(1 + \frac{\lambda}{2l^2}\right)$ converges. Thus

$$\prod_{l=1}^{n} \left(1 + \frac{\lambda}{2l^2} \right) = P_\lambda + o(1) \ ,$$

with $P_\lambda \in (0,1)$ (because $\lambda \in (-1/2, 0)$). On the other side, there exists a constant $P > 0$ such that

$$\prod_{l=1}^{n} \left(1 + \frac{1}{2l} - \frac{1}{2l^2} \right) = P \sqrt{n} (1 + o(1)) \ .$$

Therefore

$$\prod_{l=1}^{n} \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) \quad = \quad \frac{P_\lambda}{P \sqrt{n}} (1 + o(1)) \ . \tag{2.22}$$

We begin the computation of the asymptotic expansions of the two others terms in (2.21) with the following lemma:

Lemma 2.5.1 *Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$, bounded, even, with $\varphi(0) \neq 0$. Then*

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x} dx \underset{n\to\infty}{=} \frac{\varphi(0)}{\sqrt{n}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} dy (1 + o(1)) \ .$$

Proof. We write

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x} dx = 2 \int_0^\infty \varphi(x) e^{-2n \ln \cosh x} dx \ ,$$

and we make the change of variables $y = 2n \ln \cosh x$, such that

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x} dx &= 2 \int_0^\infty \frac{\varphi(\mathrm{arccosh} e^{\frac{y}{2n}})}{\sqrt{e^{\frac{y}{n}} - 1}} e^{-y} e^{\frac{y}{2n}} \frac{dy}{2n} \\ &= \frac{1}{\sqrt{n}} \int_0^\infty \tilde{\varphi}(\frac{y}{n}) \frac{e^{-y}}{\sqrt{y}} dy \ , \end{aligned}$$

where for $y > 0$,

$$\tilde{\varphi}(y) := \varphi(\mathrm{arccosh} e^{\frac{y}{2}}) \frac{e^{\frac{y}{2}} \sqrt{y}}{\sqrt{e^y - 1}}$$

and $\tilde{\varphi}(0) = \varphi(0)$. $\tilde{\varphi}$ is continuous on $\mathbb{R}_+$, and the dominated convergence theorem yields

$$\int_0^\infty \tilde{\varphi}(\frac{y}{n}) \frac{e^{-y}}{\sqrt{y}} dy \underset{n\to\infty}{\longrightarrow} \int_0^\infty \varphi(0) \frac{e^{-y}}{\sqrt{y}} dy \ ,$$

and the lemma follows. □

We apply this lemma on the one side with $\varphi(x) = v(\sqrt{2}x)$, on the other side with $\varphi(x) = 1$. We obtain respectively

$$u_n = \sqrt{2} \int_{-\infty}^\infty \frac{v(\sqrt{2}x)}{\cosh^{2n} x} dx = \frac{\sqrt{2} v(0)}{\sqrt{n}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} dy (1 + o(1)) \tag{2.23}$$

and

$$a_n = \sqrt{2} \int_{-\infty}^\infty \frac{dx}{\cosh^{2n} x} = \frac{\sqrt{2}}{\sqrt{n}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} dy (1 + o(1)) \ . \tag{2.24}$$

We next give an asymptotic expansion of the last term in (2.21). We rewrite it under the form

$$\begin{aligned} &\frac{1}{\sqrt{2}} \sum_{k=1}^n \prod_{l=k+1}^n \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) \frac{a_{k+1}}{2k^2 + k - 1} \\ &= \frac{1}{\sqrt{2}} \left(\prod_{l=1}^n \frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) \sum_{k=1}^n \frac{a_{k+1}}{\prod_{l=1}^k \left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}} \right) (2k^2 + k - 1)} \end{aligned}$$

The series in the right hand side converges, and we will denotes its sum by S_λ. Indeed, it follows from (2.24) and (2.22) that

$$\frac{a_{k+1}}{\prod_{l=1}^{k}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)(2k^2+k-1)} = \frac{\sqrt{2}P\sqrt{k}}{2\sqrt{k}P_\lambda}\int_0^\infty \frac{e^{-y}}{\sqrt{y}}dy\frac{1}{k^2}(1+o(1)) = O(\frac{1}{k^2}) \ .$$

Therefore, using once again (2.22), we get

$$\frac{1}{\sqrt{2}}\sum_{k=1}^{n}\prod_{l=k+1}^{n}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)\frac{a_{k+1}}{2k^2+k-1} \quad = \quad \frac{P_\lambda S_\lambda}{P\sqrt{2n}}(1+o(1)) \ . \tag{2.25}$$

We inject (2.23), (2.22) and (2.25) into (2.21), and we obtain the relation

$$\sqrt{2}v(0)\int_0^\infty \frac{e^{-y}}{\sqrt{y}} = \frac{P_\lambda}{P}(u_1+\frac{S_\lambda}{\sqrt{2}}) \ . \tag{2.26}$$

For $\lambda \in (-1/2, 0)$, it is easy to see that $S_\lambda > S_0 \in (0, +\infty)$. Therefore, if we assume by contradiction that $u_1 > -S_0/2\sqrt{2} > -S_\lambda/2\sqrt{2}$, (2.26) implies $v(0) > 0$.

In the sequel, $\lambda \in (-1/2, 0)$ is fixed, and we assume by contradiction that $u_1 > -S_0/2\sqrt{2}$, and thus that $v(0) > 0$. We will show that this implies that $v(x) \to -\infty$ as $x \to +\infty$, which is a contradiction, because we know that $v(x) \to 0$ as $x \to +\infty$.

Let us first define

$$w(x) := v(x)\cosh\frac{x}{\sqrt{2}} \ , \quad \text{for} \quad x \in \mathbb{R}. \tag{2.27}$$

Then for $x \in \mathbb{R}$, we have

$$v(x) = \frac{w(x)}{\cosh\frac{x}{\sqrt{2}}} \ ,$$

$$v'(x) = \frac{w'(x)}{\cosh\frac{x}{\sqrt{2}}} - \frac{1}{\sqrt{2}}\frac{\tanh\frac{x}{\sqrt{2}}}{\cosh\frac{x}{\sqrt{2}}}w(x) \ ,$$

$$v''(x) = \frac{w''(x)}{\cosh\frac{x}{\sqrt{2}}} - \frac{2}{\sqrt{2}}\frac{\tanh\frac{x}{\sqrt{2}}}{\cosh\frac{x}{\sqrt{2}}}w'(x) - \frac{w(x)}{\cosh^3\frac{x}{\sqrt{2}}} + \frac{1}{2}\frac{w(x)}{\cosh\frac{x}{\sqrt{2}}} \ .$$

Since v satisfies (2.17), it follows that w solves

$$w''(x) = \sqrt{2}\tanh\frac{x}{\sqrt{2}}w'(x) - \frac{1}{\sqrt{2}\cosh\frac{x}{\sqrt{2}}} - (\frac{1}{2}+\lambda)w(x). \tag{2.28}$$

Remark that $w(0) = v(0) > 0$, w is even and $w'(0) = 0$. Evaluating (2.28) at $x = 0$ and using the fact that $\lambda > -1/2$, we thus have

$$w''(0) = -\frac{1}{\sqrt{2}} - (\frac{1}{2}+\lambda)w(0) < 0 \ .$$

Therefore there exists $\eta_0 > 0$ such that if $x \in [0, \eta_0]$, $w''(x) < 0$ and $w(x) > 0$. Thus we can define

$$x_1 := \sup\{x > 0, w(y) > 0 \text{ and } w''(y) < 0 \text{ for every } y \in [0, x]\} \ .$$

If $x_1 = +\infty$, we have $w''(y) < 0$ on $[0, \infty)$, w' is strictly decreasing on $\mathbb{R}_+$, and for $x \geqslant 1$, $w'(x) \leqslant w'(1) < w'(0) = 0$. Therefore for $x \geqslant 1$, $w(x) \leqslant w'(1)(x-1) + w(1)$. In particular, $w(x) < 0$ for x large enough, which is a contradiction with the assumption $x_1 = +\infty$.

Thus $x_1 < \infty$. It is clear that $w(x_1) = 0$ or $w''(x_1) = 0$. Indeed, if it was not the case, thanks to the continuity of w and w'', there would exist a neighbourhood of x_1 in $\mathbb{R}_+$ on which $w > 0$ and $w'' < 0$, which would yield a contradiction with the definition of x_1. We next prove that $w''(x_1) < 0$, and thus $w(x_1) = 0$. Indeed,

$$\begin{aligned} w''(x_1) &= \sqrt{2}\tanh\frac{x_1}{\sqrt{2}} w'(x_1) - \frac{1}{\sqrt{2}\cosh\frac{x_1}{\sqrt{2}}} - (\frac{1}{2} + \lambda) w(x_1) \\ &\leqslant -\frac{1}{\sqrt{2}\cosh\frac{x_1}{\sqrt{2}}} < 0 \ , \end{aligned}$$

because w' is strictly decreasing on $[0, x_1]$ (thus $w'(x_1) < w'(0) = 0$), $1/2 + \lambda > 0$ and $w(x_1) \geqslant 0$. Therefore $v(x_1) = w(x_1) = 0$, and $v'(x_1) = \frac{w'(x_1)}{\cosh\frac{x_1}{\sqrt{2}}} < 0$. It follows that there exists $\eta_1 > 0$ such that $-1/\sqrt{2} < v(x) < 0$ for $x \in (x_1, x_1 + \eta_1]$. Thus we can define

$$x_2 := \sup\{x > x_1, -1/\sqrt{2} < v(y) \leqslant 0 \text{ for } y \in [x_1, x)\} \geqslant x_1 + \eta_1 > x_1 \ .$$

For $x \in [x_1, x_2)$, since $-\lambda > 0$, $v(x) \leqslant 0$ and $1/\sqrt{2} + v(x) > 0$, we have

$$v''(x) = -\frac{1}{\cosh^2\frac{x}{\sqrt{2}}}\left(\frac{1}{\sqrt{2}} + v(x)\right) - \lambda v(x) < 0 \ .$$

Therefore v' is decreasing on $[x_1, x_2)$, and $v(x) \leqslant v'(x_1)(x - x_1)$ if $x \in [x_1, x_2)$. $v'(x_1) < 0$, thus $x_2 < \infty$, $v(x_2) = -1/\sqrt{2}$, $v'(x_2) < v'(x_1) < 0$.

From now on, we distinguish the two cases:

1. $x_2 \geqslant x_\lambda := \sqrt{2}\text{arccosh}\frac{1}{\sqrt{-\lambda}}$,
2. $x_2 < x_\lambda$.

In the first case, for $x \geqslant x_2$, $\frac{1}{\cosh^2\frac{x}{\sqrt{2}}} + \lambda \leqslant 0$. $v'(x_2) < 0$, therefore there exists $\eta_2 > 0$, $v'(x) < 0$ for $x \in [x_2, x_2 + \eta_2]$, and the definition of $x_3 > x_2$ as

$$x_3 := \sup\{x > x_2, \ v'(y) < 0 \text{ for } y \in (x_2, x)\} > x_2$$

makes sense. v is decreasing on $[x_2, x_3)$, thus $v(x) \leqslant -1/\sqrt{2}$ for every $x \in [x_2, x_3)$. It follows that for $x \in [x_2, x_3)$, $-\left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda\right) v(x) \leqslant 0$, and

$$v''(x) = -\frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} - \left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda\right) v(x) \leqslant -\frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} < 0 \ .$$

Therefore v' is decreasing on $[x_2, x_3)$. This implies that $x_3 = +\infty$, and for $x \geqslant x_2$, $v(x) \leqslant v(x_2) + v'(x_2)(x - x_2)$, thus $v(x) \to -\infty$ as $x \to +\infty$.

We are now concerned with the second case $x_2 < x_\lambda$. We introduce

$$x_4 := \sup\{x \in (x_2, x_\lambda), v'(y) < 0 \text{ for } y \in [x_2, x)\}$$

and the open set $\Omega := \{x \in (x_2, x_4), v''(x) > 0\}$. Ω is the disjointed union of its connex parts $\omega_i = (a_i, b_i)$ for $i \in I$, where I is a finite or countable set. For $x \in \omega_i$, $v''(x) > 0$, and thus integrating twice, for $x \in \omega_i$,

$$v'(x) \geqslant v'(a_i) \ ,$$

$$v(x) - v(a_i) \geqslant v'(a_i)(x - a_i) \ .$$

Using (2.17) and the fact that $x < x_\lambda$, it follows that

$$v''(x) \leqslant \left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda\right) (-v(a_i) - v'(a_i)(x - a_i)) - \frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} \ ,$$

which may be integrated between a_i and x. After an integration by parts, we obtain

$$\begin{aligned} v'(x) \ &\leqslant \ v'(a_i) - (\sqrt{2} v(a_i) + 1)(\tanh \frac{x}{\sqrt{2}} - \tanh \frac{a_i}{\sqrt{2}}) - \lambda v(a_i)(x - a_i) \\ &-\lambda \frac{v'(a_i)}{2} (x - a_i)^2 - v'(a_i)\sqrt{2} \tanh \frac{x}{\sqrt{2}} (x - a_i) + v'(a_i) \int_{a_i}^{x} \sqrt{2} \tanh \frac{y}{\sqrt{2}} dy \ . \end{aligned} \tag{2.29}$$

For $x \in (x_2, x_4)$, $v'(x) < 0$, thus $v(x) < v(x_2) = -1/\sqrt{2}$. In particular, $v'(a_i) < 0$ and $v(a_i) + 1/\sqrt{2} < 0$ (indeed, $a_i > x_2$ because $v''(x_2) = \lambda/\sqrt{2} < 0$ while $v''(a_i) = 0$). For $x \in \omega_i$, it follows from (2.29), the mean value theorem and the inequality $\tanh \frac{x}{\sqrt{2}} \leqslant 1$ that

$$\begin{aligned} v'(x) \ &\leqslant \ v'(a_i) - \underbrace{\left[(v(a_i) + \frac{1}{\sqrt{2}}) \frac{1}{\cosh^2 \frac{a_i}{\sqrt{2}}} + \lambda v(a_i)\right]}_{= -v''(a_i) = 0.} (x - a_i) \\ &\qquad -\lambda \frac{v'(a_i)}{2} (x - a_i)^2 - v'(a_i)\sqrt{2} \int_{a_i}^{x} (1 - \tanh \frac{y}{\sqrt{2}}) dy \\ &= \ v'(a_i) \left[1 - \sqrt{2} \int_{a_i}^{x} (1 - \tanh \frac{y}{\sqrt{2}}) dy - \frac{\lambda (x - a_i)^2}{2}\right] \\ &\leqslant \ v'(a_i) \left[1 - \sqrt{2} \int_{x_2}^{\infty} (1 - \tanh \frac{y}{\sqrt{2}}) dy\right] = v'(a_i)(1 - 2\ln(1 + e^{-\sqrt{2} x_2})). \end{aligned} \tag{2.30}$$

We provisionnaly admit the fact that $(1 - 2\ln(1 + e^{-\sqrt{2}x_2})) > 0$. Let us assume by contradiction that $v'(x_4) = 0$.

If $x_4 \notin \overline{\Omega}$, there exists $\eta > 0$ such that $v''(x) \leqslant 0$ for $x \in [x_4 - \eta, x_4]$, and thus $0 = v'(x_4) \leqslant v'(x_4 - \eta) < 0$, which is a contradiction.

Thus $x_4 \in \overline{\Omega}$. Let $(c_n)_n$ be a sequence in Ω such that $c_n \to x_4$. For each n, $c_n \in \omega_{i(n)}$, and $c_n < b_{i(n)} \leqslant x_4$. Thus $b_{i(n)} \to x_4$. If $b_{i(n)}$ was constant (equal to $x_4 = b_{i_0}$) for $n \geqslant n_0$, we would have by (2.30)

$$0 = v'(x_4) = v'(b_{i_0}) \leqslant v'(a_{i_0})(1 - 2\ln(1 + e^{-\sqrt{2}x_2})) < 0 ,$$

which is a contradiction. Thus $b_{i(n)}$ takes infinitely many values. We next remark that if $y \in (0, x_\lambda)$ satisfies $v''(y) = 0$, it follows from (2.17) that

$$v(y) = -\frac{1}{\sqrt{2}(1 + \lambda \cosh^2 \frac{y}{\sqrt{2}})} =: f(y) .$$

Here, we have $0 = v''(b_{i(n)}) \to v''(x_4)$, thus $v''(x_4) = 0$, and $v(x_4) = f(x_4)$. Next,

$$f'(x_4) = \lambda \frac{\sinh \frac{x_4}{\sqrt{2}} \cosh \frac{x_4}{\sqrt{2}}}{\left(1 + \lambda \cosh^2 \frac{x_4}{\sqrt{2}}\right)^2} < 0 = v'(x_4) .$$

This implies that there exists $\eta_4 > 0$ such that $v(x) < f(x)$ for $x \in [x_4 - \eta_4, x_4)$. For n large enough, $b_{i(n)} \in [x_4 - \eta_4, x_4)$, $v''(b_{i(n)}) = 0$, thus $v(b_{i(n)}) = f(b_{i(n)})$. This is a contradiction.

If $x_4 < x_\lambda$, the definition of x_4 implies that we must have $v'(x_4) = 0$, which we have just seen to be impossible. Thus $x_4 = x_\lambda$ and $v'(x_4) < 0$. Then, we conclude as in the case $x_2 \geqslant x_\lambda$, with x_2 replaced by x_4.

To complete the proof, it remains to prove the estimate

$$2\ln\left(1 + e^{-\sqrt{2}x_2}\right) \quad < \quad 1 , \tag{2.31}$$

in the case $x_2 < x_\lambda$, which will be done by bounding x_2 from below. For $x \in [0, x_2]$, $\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} > 0$, and we have seen that v is decreasing on $[0, x_2]$, thus $v(x) \leqslant v_0 = v(0)$ for $x \in [0, x_2]$, and

$$\begin{aligned} v''(x) &= -\left(\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}}\right) v(x) - \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}} \\ &\geqslant -\left(\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}}\right) v_0 - \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}} \\ &\geqslant -\left(v_0 + \frac{1}{\sqrt{2}}\right) \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} , \end{aligned}$$

because $-\lambda v_0 > 0$. Integrating this inequality between 0 and $x \in [0, x_2]$, we get

$$|v'(x)| = -v'(x) \leqslant (1 + v_0\sqrt{2}) \tanh \frac{x}{\sqrt{2}} \leqslant (1 + v_0\sqrt{2}) \ .$$

Using the mean value Theorem, it follows that

$$\frac{1}{\sqrt{2}} + v_0 = |v(x_2) - v(0)| \leqslant (1 + v_0\sqrt{2})x_2 \ .$$

Therefore $x_2 \geqslant 1/\sqrt{2}$, and

$$2\ln(1 + e^{-\sqrt{2}x_2}) \leqslant 2\ln(1 + e^{-1}) < 1 \ .$$

□

2.6 Dynamical stability

In this section, we wish to investigate the dynamical stability of the black solitons. For this purpose, we use the finite differences in both time and space.

2.6.1 Description of the numerical method

Space and time steps h and δt being given, all the computations will be made in a rectangular grid at spatial points $x_j = jh$ and at discrete times $t_n = n\delta t$. Let us define u_j^n as the approximate value of $u(x_j, t_n)$. For any given smooth function w, we also define the operators L_x as

$$(L_x w)_j = \frac{1}{h^2}\Big(w_{j+1} - 2w_j + w_{j-1}\Big) = \Big(\frac{\partial^2 w}{\partial x^2}\Big)_j + \mathcal{O}(\Delta x^2)$$

that can be seen as the discretization of the differential operator $\partial^2/\partial x^2$ with use of Taylor formulas. We then consider the symmetric Crank-Nicolson discretization of (2.1) : setting $u^{n+1/2} = (u^n + u^{n+1})/2$, the system writes

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + (L_x u^{n+1/2})_j + f(|u_j^{n+1/2}|^2)u_j^{n+1/2} = 0$$

for $n \geqslant 0$ and $|j| \leqslant J$, with J given. This is a nonlinear algebraic discrete system that is solved at each time step using a fixed point method. This way of discretizing the system enables us to have the conservation of the discrete first invariant in the numerical domain

$$I_n = \sum_{j=-J}^{J} \Big(|u_j^n|^2 - 1\Big) = I_0, \quad \forall\, n \geqslant 0.$$

The computations will be made in a numerical domain that is chosen in such a way that the boundary will not generate artificial reflections of the numerical solution, setting $u_{\pm J}^n = \pm 1$.

2.6.2 The boundary problem

When dealing with asymptotic behaviour of time evolution problems, numerics may become delicate since for computational cost reasons, computations have to be made in a bounded interval of $\mathbb{R}$. In this case, an inappropriate boundary condition can lead to artificial reflection and may cause a wrong approximation of the exact solution. In all the cases that will be investigated, the solutions under study are assumed to tend to different limit values at $\pm\infty$. The most natural way to proceed is to set these values for the numerical approximation at each extremity of the computational domain. We show here that even in the linear case

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.32}$$

with boundary conditions -1 and 1 at $-\infty$ and ∞, this treatment can seriously affect the solution asymptotics. In order to avoid such problems, a numerical trick consists in adding a diffusion term in (2.32), that will come into play near the boundary. We thus replace (2.32) by

$$(i - \alpha(x))\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.33}$$

where the function α vanishes at the interior of the numerical domain. This means that the original problem is turned into a diffusive-like equation where the diffusion term only plays a role close to the boundary. Consequently, waves that could reflect on the boundary are absorbed. The term α is referred in the literature as a sponge factor. Such a function is plotted in Figure 2.6. Remark that since we deal with a modified equation, conservation of invariants for initial problem no more holds.

We now show the results obtained for the numerical resolution of (2.32) considered with the initial data $u_0(x) = \tanh x$, for the following choice of grid parameters: $h = 0.1$, $J = 200$, $\delta t = 0.1$ and a sponge factor given by

$$\alpha(x) = \exp(-((|x| - L)/2)^2),$$

where $L = Jh$. In Figures 2.7 and 2.8, we view the solution computed with the sponge factor compared to both solution computed with nonhomogeneous Dirichlet conditions $u(t, -L) = -1$, $u(t, L) = 1$ and solution considered in the larger domain $]-60, 60[$. It can be observed that the first solution is a much better approximation of the reference one than the solution obtained with Dirichlet conditions.

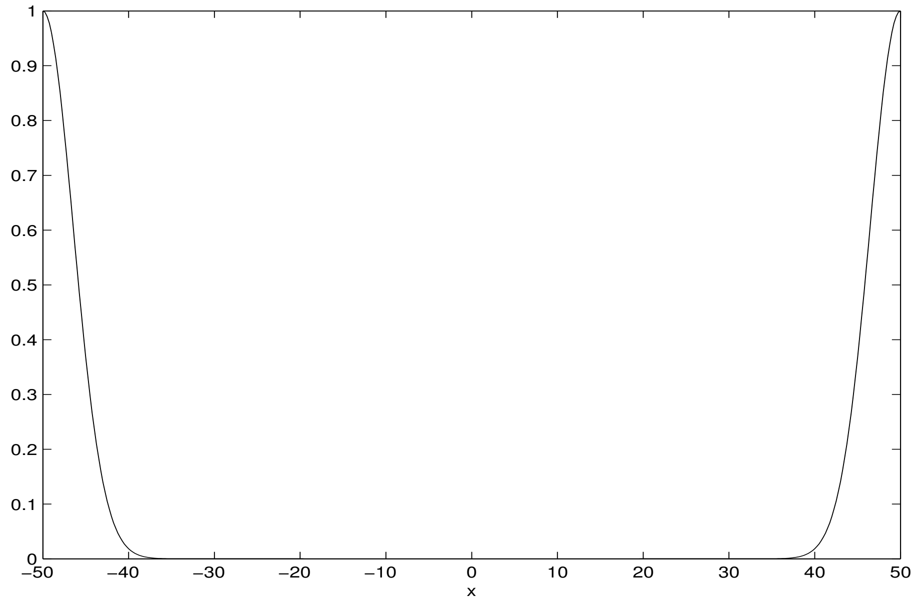


FIG. 2.6 – *Plot of α.*

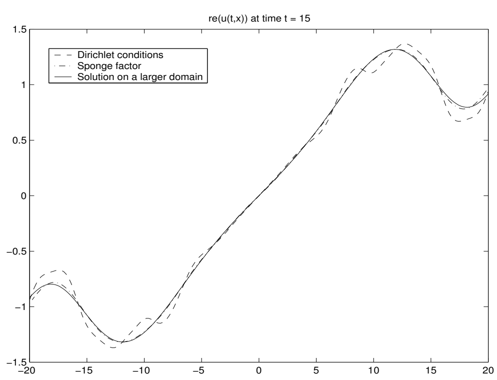


FIG. 2.7 – *Profile of the real part of the solutions of (2.32) with Dirichlet boundary conditions, with sponge factor and computed on the larger domain $[-60, 60]$, plot on the small domain $]-20, 20[$, $T = 15$.*

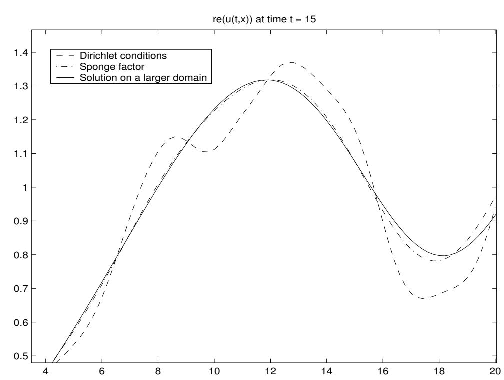


FIG. 2.8 – *Profile of the real part of the solutions of (2.32) with Dirichlet boundary conditions, with sponge factor and computed on the larger domain $[-60, 60]$, $T = 15$, zoom near the right boundary.*

2.6.3 The pure power defocusing equation

Since the stability condition stated in Theorem 2.1.1 deals with linear stability, we first perform computations for the linearized equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + f(\varphi^2)u + 2\varphi^2 f'(\varphi^2)\mathfrak{Re}u = 0, \tag{2.34}$$

with $f(r) = 1 - r^\sigma$ (which is just an other way to write system (2.3)) in order to check if a stability threshold occurs. We start from the initial condition

$$\varepsilon(x) = e^{-0.01x^2}, \tag{2.35}$$

that can be seen as a perturbation of zero. Note that due to the linearity of the equation, this perturbation is not needed to be small. We numerically solved equation (2.34) and plotted the evolution of the maximal amplitude in order to look for the generation of a unstable mode. For the sake of clarity, we have preferred to view the profile of $k(t) := \log(1 + \|u(t)\|_{L^\infty})/t$: if k tends to zero as t is large, then no exponential growth is obtained and the black soliton can be considered as linearly stable. If k tends to some positive constant λ, then instability occurs with the corresponding growth rate given by λ. We have computed the evolution of k for different values of σ ($\sigma = 1$, $\sigma = 2$ and $\sigma = 5$) with parameters $J = 400$, $h = 0.125$, $\delta t = 5$ and $N = 1000$. Figure 2.9 indicates that k globally decreases to 0, in spite of numerical oscillations : there is no positive eigenvalue for the linearized operator, which illustrates linear stability for any value of σ.

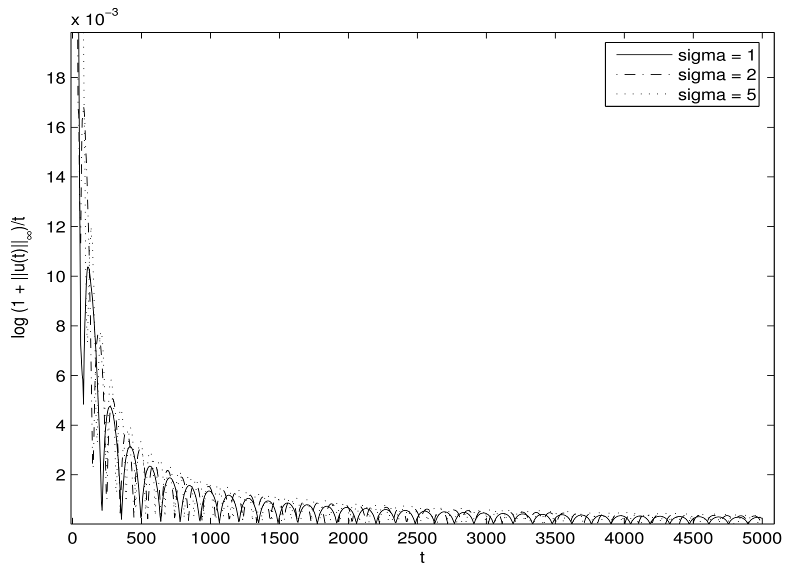


FIG. 2.9 – *Plot of k versus time, $\sigma = 1$, $\sigma = 2$ and $\sigma = 5$.*

We then numerically solved problem (2.15), injecting the black soliton numerically calculated as initial data. We took the following parameters : $\Delta x = 0.05$, $J = 400$, $\Delta t = 0.05$ in order to investigate long time asymptotics. Our first experiments showed the dynamic stability of the solution as the persistence of the soliton structure through time. For this aim, we plot in Figure 2.10 the evolution of the L^2 norm of the error between the black soliton and the numerical solution with respect to time, for different values of nonlinear exponent σ ($\sigma = 1$, 2 and 3). It can be observed that this difference is bounded through time, which is a grant of stability.

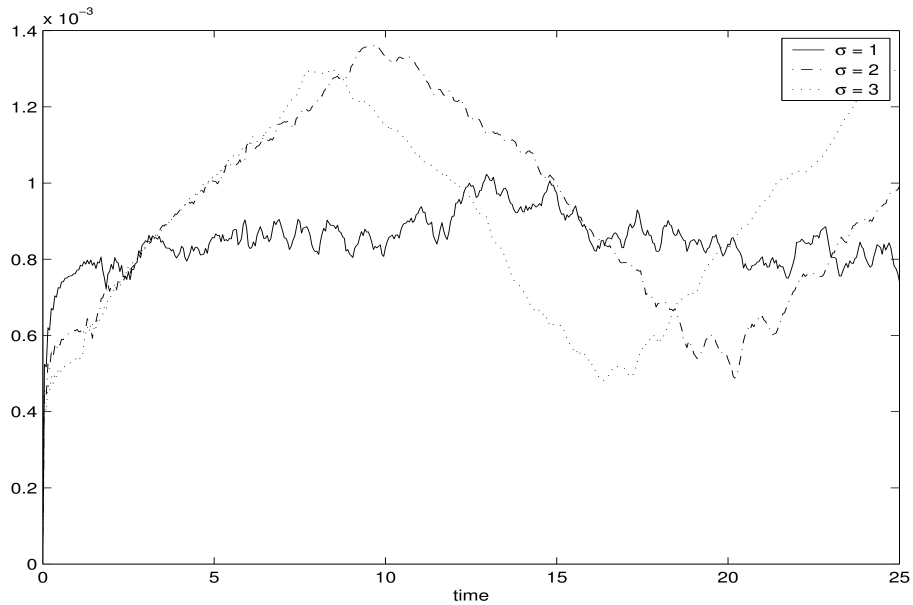


FIG. 2.10 – *Plot of the L^2 error between the finite differences solution and the exact black soliton versus time, $\sigma = 1$, $\sigma = 2$ and $\sigma = 3$.*

We then perturb the initial data with an oscillatory function $\varepsilon(x) = (1 + 0.5e^{-x^2} \sin 5x)$ (that is we start with the initial data $u_0 = \varepsilon\varphi$ instead of φ). Of course, the numerical domain has to be large enough in order to avoid the generation of reflection waves caused by the boundary condition $u^n_{\pm J} = \pm 1$ that could become non-adapted for the computation at late stages. In our test, we set $J = 800$, $\Delta x = \Delta t = 0.05$ and for simulations performed until final time $T = 5$. In Figures 2.11 is plotted the solution profile at successive times $t = 1$, $t = 3$ and $t = 5$. It can be observed a propagation to the right of the wavetrain, with a similar effect to the left side for negative x. Since its length seems quite constant through time, the global L^2 error between the numerical solution and the black soliton will remain bounded. Since it is worth investigating the behaviour at the left of the perturbed wave, we also define the *local* L^2 error, that is computed around the origin (in our simulations, we chose the domain $D = [-10, 10]$). In Figure 2.12 are plotted the two errors and it can be noticed that once the perturbation has left D, then the approximate solution mimics the soliton profile and the L^2 error becomes small, while the global error stays bounded since the traveling perturbation keeps the same amplitude. This effect could be referred as *local* asymptotic stability, meaning that in compact domains of the real line, the black soliton attracts the solutions for large times.

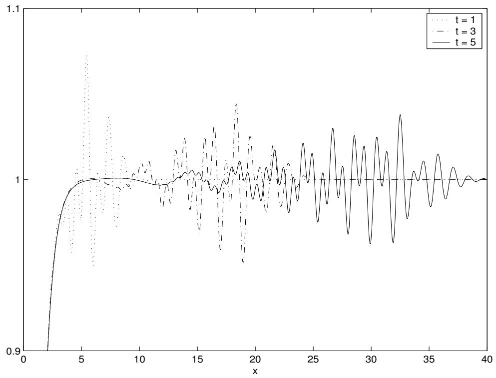


FIG. 2.11 – *Snapshots of the perturbed solution at different times ($t = 1$, $t = 3$ and $t = 5$) around the black soliton profile, $\sigma = 1$.*

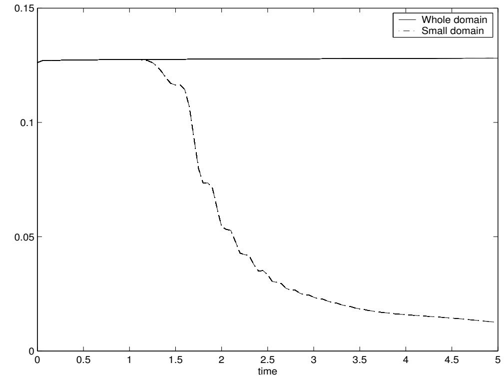


FIG. 2.12 – *Evolution of global and local L^2 error between the perturbed solution and the black soliton versus time.*

2.6.4 The saturated equation

As in the previous section, we first investigate the linearized equation (2.34), injecting as initial data the same perturbation as before. We used the following parameters : $J = 400$, $h = 0.075$, $\delta t = 10$ and $N = 2000$. The numerical experiments have shown that initial perturbation (2.35) leads us to stability for $a < a_c$ and instability for $a > a_c$ (see Figure 2.13, where k has been plotted in the cases $a = 6$, $a = 7$, $a = 8$ and $a = 9$).

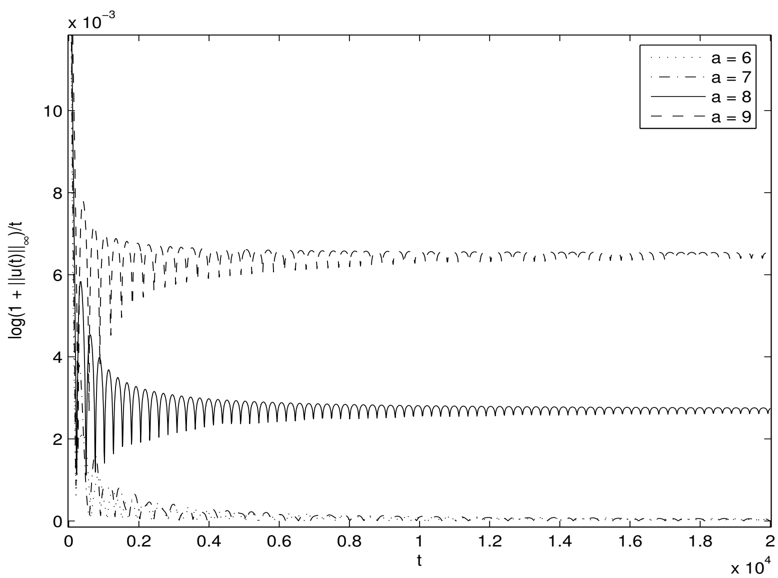


FIG. 2.13 – *Plot of k versus time, $a = 6$, $a = 7$, $a = 8$ and $a = 9$.*

We also solved equation (2.16) starting from an Gaussian modulated initial perturbation

$$\varepsilon(x) = q(1+i)e^{-0.01x^2}\cos x. \tag{2.36}$$

For any value of a, the numerical solution seems to travel with a velocity depending on a. This may suggest that the black soliton is not stable. However, we translate the solution in such a way that the real part changes its sign at the origin and we study the L^∞ error with respect to the black soliton. We found that taking $a < a_c$ leads us to a bounded variation of the error, whereas for $a > a_c$ the error seems to increase with time. In our computations, we chose $\delta t = 1$, $J = 400$, $h = 0.25$, $q = 0.01$ in the cases $a = 5$ and $a = 8$ (see Figures 2.14 and 2.16). In these plots, it can be noticed jumps of the error that are caused by the discrete traveling jump of the vanishing point from one grid cell to another one and the large value of the derivative of the black soliton at $x = 0$. In the case $a = 5$ (see Figure 2.15), one can observe that the soliton profile travels to the right with no alteration, contrary to the solution computed with $a = 8$, where the black soliton profile is partially lost at the left side (see Figure 2.17).

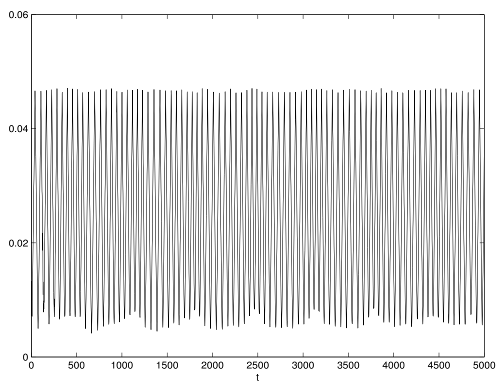


FIG. 2.14 – *Plot of the error between the black soliton and the translated solution of (2.16), $a = 5$.*

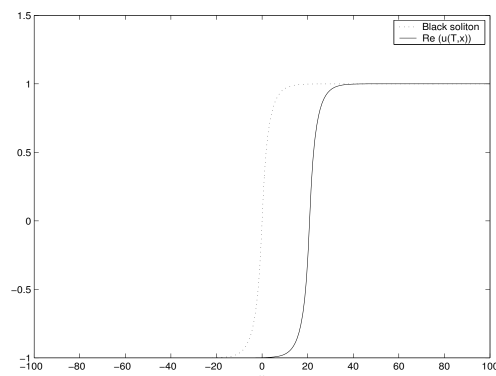


FIG. 2.15 – *Profile of the real part of the solution at final time $T = 5000$ compared with the black soliton, $a = 5$.*

In order to see more precisely the influence of parameter a, we view in Figure 2.18 on the same plot the errors between the black soliton and the translated solution of (2.16) obtained for different values of a: $a = 6$, $a = 7$, $a = 8$ and $a = 9$. We have taken the same values of the grid parameters and we chose a larger amplitude $q = 0.03$ for the initial perturbation (2.35). As in the previous test, this figure suggests the occurrence of a threshold value for $a > 7$.

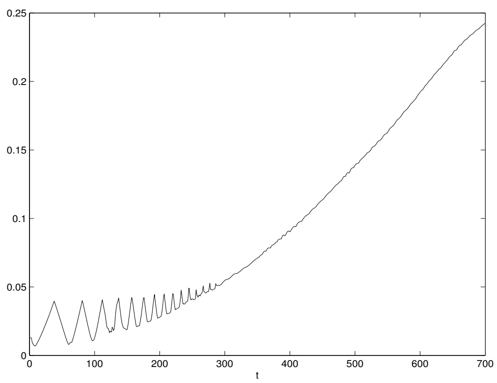


FIG. 2.16 – *Plot of the error between the black soliton and the translated solution of (2.16), $a = 8$.*

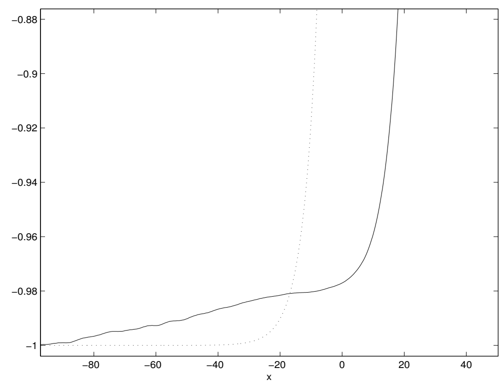


FIG. 2.17 – *Zoomed profile of the real part of the solution at final time $T = 700$ compared with the black soliton, $a = 8$.*

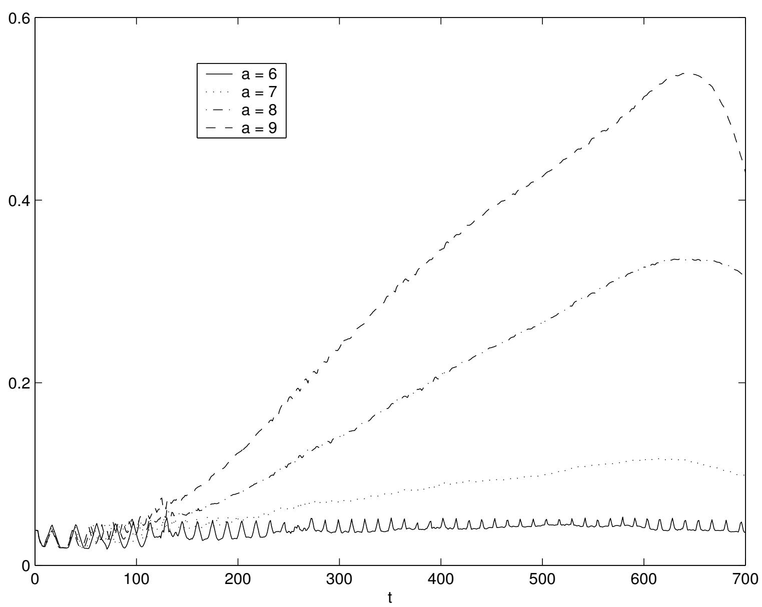


FIG. 2.18 – *Plot of the error between the black soliton and the translated solution of (2.16) $a = 6$, $a = 7$, $a = 8$ and $a = 9$.*

Chapitre 3

Existence of dark-dark soliton to a system of two coupled NLS equations

Abstract. We prove the existence of stationary dark-dark solitons with monotonic tails to a system of two coupled nonlinear Schrödinger equations.

We are concerned with the system

$$\begin{cases} i\frac{\partial w}{\partial t} + r\frac{\partial^2 w}{\partial x^2} - w + \overline{w}v = 0, \\ i\sigma\frac{\partial v}{\partial t} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2}w^2 = 0, \end{cases} \tag{3.1}$$

with $r, s = \pm 1$ and $\alpha > 0$.

In [BK1] (see also [BK2], [Ki]), A.V. Buryak and Y.S. Kivshar derive this system from Maxwell equations with $\chi^{(2)}$ nonlinear susceptibility. They found numerically real stationnary solutions to (3.1), for different values of r, s and α. Namely, for $r = -1$, $s = 1$ and $\alpha > 0$, they found a continuous family of dark solitons, with $w(x) \longrightarrow \pm\sqrt{2\alpha}$ and $v(x) \longrightarrow 1$ as $x \to \pm\infty$, and with monotonic (resp. non-monotonic) tails for $\alpha > 8$ (resp. $\alpha < 8$). For $r = s = -1$, they found a discrete set of values of α for which there exists dark solitons to (3.1). For $r = 1$, $s = \pm 1$, they found bright solitons to (3.1). We give here a rigorous proof of the existence of dark solitons of (3.1) with monotonic tails, in the case $r = -1$, $s = 1$. We use a shooting method inspired by [BLP]. Our main result is as follows:

Theorem 3.0.1 *Let $\alpha > 8$. Then the system*

$$w'' = -w + vw \tag{3.2}$$

$$v'' = \alpha v - \frac{w^2}{2} \tag{3.3}$$

admits a real solution $w, v \in \mathcal{C}^2(\mathbb{R})$, such that

(i) w is odd,

(ii) w is strictly increasing on $\mathbb{R}$,

(iii) $w(x) \underset{x\to\pm\infty}{\longrightarrow} \pm\sqrt{2\alpha}$,

(iv) v *is positive and even,*
(v) v *is strictly increasing on* $\mathbb{R}_+$*,*
(vi) $v(x) \underset{x\to\pm\infty}{\longrightarrow} 1$.

Proof. If $w, v \in \mathcal{C}^2(\mathbb{R})$ solve (3.2)-(3.3) and satisfy (i)-(vi), multiplying (3.2) by w', (3.3) by v', making the difference and integrating, we obtain the equality

$$w'(x)^2 - v'(x)^2 = (1 - v(x)) \left[\alpha(1 + v(x)) - w(x)^2\right]. \tag{3.4}$$

Thus, taking into account that $w(0) = 0$, $v'(0) = 0$ and $v(0) = v_0 \in (0, 1)$, we must have $w'(0)^2 = \alpha(1 - v_0^2)$. For $v_0 \in (0, 1)$, we will consider the Cauchy Problem

$$\begin{pmatrix} v \\ v' \\ w \\ w' \end{pmatrix}' = \begin{pmatrix} v' \\ \alpha v - \frac{w^2}{2} \\ w' \\ -w(1-v) \end{pmatrix} \quad , \quad \begin{pmatrix} v \\ v' \\ w \\ w' \end{pmatrix}(0) = \begin{pmatrix} v_0 \\ 0 \\ 0 \\ \sqrt{\alpha(1 - v_0^2)} \end{pmatrix}. \tag{3.5}$$

Given $v_0 \in (0, 1)$, the solution of the Cauchy Problem (3.5) exists on a maximal interval $[0, x^*(v_0))$. In the sequel, we will always denote this solution by (v, w). Thus we are looking for $v_0 \in (0, 1)$ such that $x^*(v_0) = \infty$ and v, w satisfy (i)-(vi). We define the following four subsets of $(0, 1)$:

$$E_v := \{v_0 \in (0,1) | \exists x_0 \in (0, x^*(v_0)), \begin{cases} v(x_0) = 1 \\ v(x) < 1 \text{ on } [0, x_0) \\ v'(x) > 0 \text{ on } (0, x_0] \\ w(x) < \sqrt{\alpha(1 + v(x))} \text{ on } [0, x_0] \\ w'(x) > 0 \text{ on } [0, x_0] \end{cases} \},$$

$$E_w := \{v_0 \in (0,1) | \exists x_0 \in (0, x^*(v_0)), \begin{cases} w(x_0) = \sqrt{\alpha(1 + v(x_0))} \\ v(x) < 1 \text{ on } [0, x_0] \\ v'(x) > 0 \text{ on } (0, x_0] \\ w(x) < \sqrt{\alpha(1 + v(x))} \text{ on } [0, x_0) \\ w'(x) > 0 \text{ on } [0, x_0] \end{cases} \},$$

$$E_{v'} := \{v_0 \in (0,1) | \exists x_0 \in (0, x^*(v_0)), \begin{cases} v'(x_0) = 0 \\ v(x) < 1 \text{ on } [0, x_0] \\ v'(x) > 0 \text{ on } (0, x_0) \\ w(x) < \sqrt{\alpha(1 + v(x))} \text{ on } [0, x_0] \\ w'(x) > 0 \text{ on } [0, x_0] \end{cases} \},$$

$$E_{w'} := \{v_0 \in (0,1) | \exists x_0 \in (0, x^*(v_0)), \begin{cases} w'(x_0) = 0 \\ v(x) < 1 \text{ on } [0, x_0] \\ v'(x) > 0 \text{ on } (0, x_0] \\ w(x) < \sqrt{\alpha(1 + v(x))} \text{ on } [0, x_0] \\ w'(x) > 0 \text{ on } [0, x_0) \end{cases} \}.$$

We split the proof in four steps:

1. the sets $E_v,\ E_w,\ E_{v'},\ E_{w'}$ are disjoint two by two.
2. E_v and $E_{v'}$ are non empty.
3. if $v_0 \notin E_v \cup\ E_w \cup\ E_{v'} \cup\ E_{w'}$, then $x^*(v_0)=\infty$ and v,w satisfy the conditions (i)-(vi).
4. the sets $E_v,\ E_w,\ E_{v'},\ E_{w'}$ are open.

Since $(0,1)$ is connex, it is clear that the theorem follows from these assertions.

1st step. It is quite clear that $E_v,\ E_w,\ E_{v'},\ E_{w'}$ are disjoint two by two. We show for instance that $E_v \cap E_w = \emptyset$. We argue by contradiction. If $v_0 \in E_v \cap E_w$, let $x_{0,v}$ be as in the definition of E_v, $x_{0,w}$ as in the definition of E_w. Since $v(x_{0,v})=1$ and $v(x)<1$ on $[0,x_{0,w}]$, $x_{0,v}>x_{0,w}$. In a similar way, since $w(x_{0,w})=\sqrt{\alpha(1+v(x_{0,w}))}$ and $w(x)<\sqrt{\alpha(1+v(x))}$ on $[0,x_{0,v}]$, $x_{0,w}>x_{0,v}$, which is a contradiction. The proof of the emptnyness of the intersection of any two sets chosen among $E_v,\ E_w,\ E_{v'}$ and $E_{w'}$ is identical.

2nd step. For any $v_0 \in (0,1)$, we define

$$x_0(v_0) := \sup\{x>0, x<x^*(v_0) \mid \begin{cases} v(y)<1 \\ w(y)^2<\alpha(1+v(y)) \\ v'(y)>0 \\ w'(y)>0 \end{cases} \text{for } y\in(0,x)\}.$$

We first remark that $x_0(v_0)$ is defined as the supremum of a non-empty set. Indeed, $v(0)<1$, $w'(0)>0$, $\alpha(1+v(0))>w(0)^2=0$, therefore by continuity, $1-v$, w' and $\alpha(1+v)-w^2$ remain strictly positive in a neighborhood of 0 in $[0,x^*(v_0))$. Moreover, $v'(0)=0$, $v''(0)=\alpha v_0>0$, therefore $v'(x)>0$ for $x>0$ and x small. Thus for $y>0$ small enough, the four inequalities in the definition of $x_0(v_0)$ are satisfied.

We will show that if $v_0 \in (0,1)$ is chosen next to 1 (respectively next to 0), then $x_0(v_0)<x^*(v_0)$ and $v_0\in E_v$ (respectively $v_0 \in E_{v'}$). We begin with several estimations on v, w and their derivates on $[0,x_0(v_0)]$.

$w \geqslant 0$ and $v<1$ on $[0,x_0(v_0))$, therefore $w''=-w(1-v)\leqslant 0$ on $[0,x_0(v_0)]$. Thus, integrating twice this inequality, we obtain that

$$\forall x\in[0,x_0(v_0)] \quad , \quad 0\leqslant w(x)\leqslant \sqrt{\alpha(1-v_0^2)}x \ . \tag{3.6}$$

Therefore, for $x\in[0,x_0(v_0)]$,

$$v''(x)=\alpha v(x)-\frac{w(x)^2}{2} \ \geqslant\ \alpha v_0-\frac{\alpha(1-v_0^2)}{2}x^2, \tag{3.7}$$

$$v'(x)\ \geqslant\ \alpha x(v_0-\frac{(1-v_0^2)}{6}x^2), \tag{3.8}$$

and

$$v(x) \;\geqslant\; v_0 + \frac{\alpha v_0}{2}x^2 - \frac{\alpha(1-v_0^2)}{24}x^4 \;. \tag{3.9}$$

Let $x_1(v_0)$ be the smallest positive root of the polynom $1 - v_0 - \frac{\alpha v_0}{2}x^2 + \frac{\alpha(1-v_0^2)}{24}x^4$ (it does exist for v_0 sufficiently close to 1). Then

$$x_1(v_0) \underset{v_0 \uparrow 1}{\sim} \sqrt{\frac{2}{\alpha}}(1-v_0)^{1/2} \;.$$

In particular, we deduce that

$$x_0(v_0) \leqslant x_1(v_0) \underset{v_0 \uparrow 1}{\longrightarrow} 0. \tag{3.10}$$

From (3.6) and (3.9), we infer that for $x \in [0, x_0(v_0)]$,

$$\begin{aligned} w''(x) &= -w(x)(1-v(x)) \\ &\geqslant -\sqrt{\alpha(1-v_0^2)}\,x(1 - v_0 - \frac{\alpha v_0}{2}x^2 + \frac{\alpha(1-v_0^2)}{24}x^4) \;, \end{aligned} \tag{3.11}$$

and, integrating (3.11),

$$w'(x) \;\geqslant\; \sqrt{\alpha(1-v_0^2)}\left(1 - \frac{1-v_0}{2}x^2 + \frac{\alpha v_0}{8}x^4 - \frac{\alpha(1-v_0^2)}{144}x^6\right). \tag{3.12}$$

For any fixed v_0, the right hand side is a decreasing function of x on $[0, x_1(v_0)]$, therefore for $x \in [0, x_0(v_0)]$,

$$w'(x) \geqslant \sqrt{\alpha(1-v_0^2)}\underbrace{\left(1 - \frac{1-v_0}{2}x_1(v_0)^2 + \frac{\alpha v_0}{8}x_1(v_0)^4 - \frac{\alpha(1-v_0^2)}{144}x_1(v_0)^6\right)}_{\longrightarrow 1 \text{ as } v_0 \to 1}. \tag{3.13}$$

In particular, if v_0 is choosen close enough to 1,

$$\forall x \in [0, x_0(v_0)] \quad , \quad w'(x) > 0 \;. \tag{3.14}$$

In a similar way, we infer from (3.8) that if v_0 is sufficiently close to 1,

$$\forall x \in (0, x_0(v_0)] \quad , \quad v'(x) > 0 \;. \tag{3.15}$$

From (3.9) and (3.6), we deduce that for $x \in [0, x_0(v_0)]$,

$$\alpha(1+v(x)) - w(x)^2 \;\geqslant\; \alpha(1 + v_0 + \frac{\alpha v_0}{2}x^2 - \frac{\alpha(1-v_0^2)}{24}x^4) - \alpha(1-v_0^2)x^2. \tag{3.16}$$

Now, since $x_0(v_0)$ tends to 0 as $v_0 \uparrow 1$, the infimum of the right hand side of (3.16) as x describes the interval $[0, x_0(v_0)]$ tends to $2\alpha > 0$ as $v_0 \uparrow 1$. Therefore, using (3.15), (3.16) and (3.14), if v_0 is choosen close enough to 1, we can ensure that $v' > 0$, $\alpha(1+v) > w^2$ and $w' > 0$ on $(0, x_0(v_0)]$, which implies $v(x_0(v_0)) = 1$ (if it was not the case, using the continuity of v, w, v' and w', we would obtain a contradiction with the definition of $x_0(v_0)$). Hence v_0 belongs to E_v, which is the a non-empty set.

We next prove that $E_{v'} \neq \emptyset$. More precisely, we will show that if v_0 is sufficiently close to 0, then $v_0 \in E_{v'}$. For every $v_0 \in (0,1)$ and $x \in [0, x_0(v_0)]$, $v''(x) = \alpha v(x) - \frac{w(x)^2}{2} \leqslant \alpha v(x)$. Integrating this inequality, we deduce that

$$\forall x \in [0, x_0(v_0)] \quad , \quad v(x) \leqslant v_0 \cosh \sqrt{\alpha} x \; . \tag{3.17}$$

Integrating (3.12), we get

$$w(x) \quad \geqslant \quad \sqrt{\alpha(1-v_0^2)}x \left(1 - \frac{1-v_0}{6}x^2 + \frac{\alpha v_0}{40}x^4 - \frac{\alpha(1-v_0^2)}{1008}x^6\right) . \tag{3.18}$$

We use (3.17) and (3.18) to bound v'' from above. For $x \in [0, x_0(v_0)]$,

$$v''(x) \leqslant \alpha v_0 \cosh \sqrt{\alpha} x - \frac{\alpha(1-v_0^2)}{2}x^2 \left(1 - \frac{1-v_0}{6}x^2 + \frac{\alpha v_0}{40}x^4 - \frac{\alpha(1-v_0^2)}{1008}x^6\right)^2 ,$$

which can be integrated into

$$v'(x) \quad \leqslant \quad \sqrt{\alpha} v_0 \sinh \sqrt{\alpha} x - Q_{v_0}(x) \; , \tag{3.19}$$

where

$$Q_{v_0}(x) := \frac{\alpha(1-v_0^2)}{2} \int_0^x y^2 \left(1 - \frac{1-v_0}{6}y^2 + \frac{\alpha v_0}{40}y^4 - \frac{\alpha(1-v_0^2)}{1008}y^6\right)^2 dy \; .$$

Since $Q_0(x) \sim \frac{\alpha}{6}x^3$ as $x \to 0$, there exists $x^* > 0$ such that $Q_0(x) > 0$ for every $x \in [0, x^*]$. Let $x_2 < x^*$. We observe that

$$\sqrt{\alpha} v_0 \sinh \sqrt{\alpha} x_2 - Q_{v_0}(x_2) \underset{v_0 \to 0}{\longrightarrow} -Q_0(x_2) < 0 \; ,$$

and that if $v_0 \in (0,1)$ is fixed,

$$\sqrt{\alpha} v_0 \sinh \sqrt{\alpha} x - Q_{v_0}(x) \underset{x \to 0}{\sim} \alpha v_0 x > 0 \; .$$

Therefore, by the intermediate value theorem, there exists $v_0(x_2) > 0$ such that if $v_0 < v_0(x_2)$, there exists $x_3(v_0) \in (0, x_2]$ which solves

$$\sqrt{\alpha} v_0 \sinh \sqrt{\alpha} x_3(v_0) = Q_{v_0}(x_3(v_0)) \; .$$

We let x_2 and (up to a change of the function $v_0(x_2)$) $v_0(x_2)$ tend to 0. Then $v_0 \to 0$ and $x_2 \geqslant x_3(v_0) \to 0$. The right hand side in (3.19) vanishes at $x_4(v_0) \leqslant x_3(v_0)$, with $x_4(v_0) \geqslant x_0(v_0)$.

We infer from (3.12) that $\lim_{v_0\to 0}\left(\inf_{x\in[0,x_0(v_0)]} w'(x)\right) \geqslant \sqrt{\alpha} > 0$, thus if we choose v_0 small enough,

$$\forall x \in [0, x_0(v_0)] \quad , \quad w'(x) > 0 \ . \tag{3.20}$$

It follows from (3.17) that for $x \in [0, x_0(v_0)]$, $v(x) \leqslant v_0 \cosh \sqrt{\alpha} x_0(v_0) \to 0$ as $v_0 \to 0$. Then, if we take v_0 sufficiently close to 0, we will have

$$\forall x \in [0, x_0(v_0)] \quad , \quad v(x) < 1 \ . \tag{3.21}$$

Finally, we use (3.6) to get, for $x \in [0, x_0(v_0)]$,

$$\alpha(1 + v(x)) - w(x)^2 \geqslant \alpha(1 + v_0) - \alpha(1 - v_0^2)x_0(v_0) \underset{v_0\to 0}{\longrightarrow} \alpha > 0 \ . \tag{3.22}$$

Thus, thanks to (3.20), (3.21) and (3.22), we can choose v_0 small enough such that $w' > 0$, $v < 1$ and $\alpha(1+v) > w^2$ on $[0, x_0(v_0)]$. This yields $x_0(v_0) = x_4(v_0)$ and $v_0 \in E_{v'} \neq \emptyset$.

3$^{\text{rd}}$ step. Let $v_0 \in (0, 1)$, $v_0 \notin E_v \cup E_w \cup E_{v'} \cup E_{w'}$. Two situations may occur: $x_0(v_0) = x^*(v_0)$ or $x_0(v_0) < x^*(v_0)$. In the first case, if $x_0(v_0) = x^*(v_0)$ was finite, then (v, w, v', w') would blow up at $x_0(v_0)$, which is impossible, as it can easily be seen. Thus $x_0(v_0) = x^*(v_0) = +\infty$, and we obtain in this case a soliton like in the Theorem.

We next prove by contradidtion that the second situation may in fact not occur. We assume v_0 be such that $x_0(v_0) < x^*(v_0)$. In particular, $x_0(v_0)$ is finite. It follows from the definition of $x_0(v_0)$ that at least one of the four following inequalities is violated:

$$v(x_0(v_0)) < 1,\ w(x_0(v_0))^2 < \alpha(1 + v(x_0(v_0))),\ v'(x_0(v_0)) > 0,\ w'(x_0(v_0)) > 0 \ .$$

By continuity of $v,\ w,\ v',\ w'$, the only way for one of these inequalities to be false, is to be an equality. Moreover, if only one of these four inequalities was false, we see easily that v_0 would belong to one of the sets $E_v,\ E_w,\ E_{v'},\ E_{w'}$. Therefore at least two of them are false. Thanks to (3.4), it follows that we are in one of the following situations:

1. $v(x_0) = 1$, $w'(x_0) = 0$ and $v'(x_0) = 0$,
2. $v(x_0) = 1$, $w(x_0) = \sqrt{\alpha(1 + v(x_0))} = \sqrt{2\alpha}$ and $v'(x_0) = w'(x_0) \geqslant 0$,
3. $w'(x_0) = 0$, $v'(x_0) = 0$ and $w(x_0) = \sqrt{\alpha(1 + v(x_0))}$.

In the first case, since $v(x) < 1$ for $x < x_0$, we have

$$0 \geqslant v''(x_0) = \alpha v(x_0) - w(x_0)^2/2 \geqslant \alpha v(x_0) - \alpha(1 + v(x_0))/2 = \alpha(v(x_0) - 1)/2 = 0 \ ,$$

and then $w(x_0) = \sqrt{\alpha(1 + v(x_0))} = \sqrt{2\alpha}$. The "uniqueness part" in the Cauchy-Lipshitz theorem implies that $(v, w) \equiv (1, \sqrt{2\alpha})$, which is a contradiction.

In the second case, let us define $\beta := v'(x_0) = w'(x_0) \geqslant 0$. If $\beta = 0$, we are brought to the first case. If $\beta > 0$, then $v(x) = 1 + \beta(x - x_0) + o(x - x_0)$ and $w(x) = \sqrt{2\alpha} + \beta(x -$

$x_0) + o(x - x_0)$ as x is close to 0. For $x < x_0$, we know that $\alpha(1+v(x)) - w(x)^2 > 0$. Near x_0, we have also

$$\alpha(1+v(x)) - w(x)^2 = \sqrt{\alpha}\beta(\sqrt{\alpha} - 2\sqrt{2})(x - x_0) + o(x - x_0) \ .$$

Therefore, since $\alpha > 8$, $\alpha(1+v(x)) - w(x)^2 < 0$ for x in a neighborhood of x_0 and $x < x_0$, which is a contradiction.

In the third case, $w''(x_0) = (v(x_0) - 1)\sqrt{\alpha(1+v(x_0))}$ and $v''(x_0) = \alpha\frac{v(x_0)-1}{2}$. Thus

$$\alpha(1+v(x)) - w(x)^2 = \alpha(v(x_0) - 1)(\frac{\alpha}{4} - (1 + v(x_0)))(x - x_0)^2 + o((x - x_0)^2). \quad (3.23)$$

Since $v(x_0) \leqslant 1$ and $\alpha > 8$, $\alpha/4 - (1 + v(x_0)) > 0$. For $x < x_0$, we know that $\alpha(1+v(x)) - w(x)^2 > 0$, therefore it follows from (3.23) that $v(x_0) - 1 \geqslant 0$, thus $v(x_0) = 1$, and we conclude to a contradiction as in the other cases.

4th step. Let us next prove that E_v is open. Let $v_0 \in E_v$. We denote by (v_{v_0}, w_{v_0}) the solution of the Cauchy Problem (3.5). Since $v_0 \in E_v$, we have $v_{v_0}(x_0(v_0)) = 1$, $v'_{v_0}(x_0(v_0)) > 0$, $w_{v_0}(x_0(v_0)) < \sqrt{\alpha(1 + v_{v_0}(x_0(v_0)))}$ and $w'_{v_0}(x_0(v_0)) > 0$. Therefore there exists $x_5(v_0) > x_0(v_0)$ such that $v'_{v_0} > 0$, $w_{v_0} < \sqrt{\alpha(1 + v_{v_0})}$ and $w'_{v_0} > 0$ on $[x_0(v_0), x_5(v_0)]$. Since $v''_{v_0}(0) = \alpha v_0 > 0$, there exists $x_6(v_0) > 0$ such that $v''_{v_0} = \alpha v - w^2/2 > 0$ on $[0, x_6(v_0)]$.

The map $v_0 \mapsto (v_0, 0, 0, \sqrt{\alpha(1 - v_0^2)})$, $(0, 1) \mapsto \mathbb{R}^4$ is continuous and the map

$$(v, dv, w, dw) \mapsto (dv, \alpha v - w^2/2, dw, -w(1 - v)), \ \mathbb{R}^4 \mapsto \mathbb{R}^4$$

is Lipshitz-continuous. Thus a classical result of continuity of the flow ensures that there exists $\delta > 0$, $K > 0$ such that if $v_1 \in (0, 1)$, $|v_0 - v_1| < \delta$, then $x^*(v_1) > x_5(v_0)$ and

$$\|v_{v_1} - v_{v_0}\| + \|v'_{v_1} - v'_{v_0}\| + \|w_{v_1} - w_{v_0}\| + \|w'_{v_1} - w'_{v_0}\| \leqslant K|v_0 - v_1| \quad (3.24)$$

where (v_{v_1}, w_{v_1}) is the solution of the Cauchy Problem (3.5) with v_0 replaced by v_1, and $\|.\|$ denotes the norm of the space $L^\infty(0, x_5(v_0))$. Let $\tilde{\delta} > 0$ be such that $\tilde{\delta} \leqslant \delta$,

$$K\tilde{\delta} \leqslant \frac{1}{2} \inf_{x \in [0, x_5(v_0)]} w'_{v_0}(x) \ , \quad (3.25)$$

$$K\tilde{\delta} \leqslant \frac{1}{2} \inf_{x \in [x_6(v_0), x_5(v_0)]} v'_{v_0}(x) \ , \quad (3.26)$$

$$K\tilde{\delta}(\alpha + \|w_{v_0}\| + \frac{K\tilde{\delta}}{2}) \leqslant \frac{1}{2} \inf_{x \in [0, x_6(v_0)]} \left(\alpha v_{v_0}(x) - \frac{w_{v_0}(x)^2}{2}\right) \ , \quad (3.27)$$

$$K\tilde{\delta}(1 + \sqrt{\alpha}) \leqslant \frac{1}{2} \inf_{x \in [0, x_5(v_0)]} (\sqrt{\alpha(1 + v_{v_0}(x))} - w_{v_0}(x)) \quad (3.28)$$

and

$$K\tilde{\delta} \leqslant \frac{1}{2}(v_{v_0}(x_5(v_0)) - 1) \ . \tag{3.29}$$

If $v_1 \in (0,1)$ satisfies $|v_0 - v_1| < \tilde{\delta}$, it follows from (3.24) and (3.25) that

$$w'_{v_1} \geqslant w'_{v_0} - \|w'_{v_1} - w'_{v_0}\| \geqslant \frac{1}{2}\inf_{x\in[0,x_5(v_0)]} w'_{v_0}(x) > 0 \ \text{ on } [0, x_5(v_0)] \ ,$$

from (3.26) we infer that

$$v'_{v_1} \geqslant v'_{v_0} - \|v'_{v_1} - v'_{v_0}\| \geqslant \frac{1}{2}\inf_{x\in[x_6(v_0),x_5(v_0)]} v'_{v_0}(x) > 0 \ \text{ on } [x_6(v_0), x_5(v_0)] \ ,$$

on $[0, x_6(v_0)]$, (3.27) yields

$$\begin{aligned} v''_{v_1} = \alpha v_{v_1} - \frac{w_{v_1}^2}{2} &\geqslant \alpha v_{v_0} - \frac{w_{v_0}^2}{2} - \|\alpha v_{v_1} - \frac{w_{v_1}^2}{2} - (\alpha v_{v_0} - \frac{w_{v_0}^2}{2})\| \\ &\geqslant \alpha v_{v_0} - \frac{w_{v_0}^2}{2} - \alpha K\tilde{\delta} - \frac{1}{2}(2\|w_{v_0}\| + K\tilde{\delta})K\tilde{\delta} \\ &\geqslant \inf_{x\in[0,x_6(v_0)]} \left(\alpha v_{v_0}(x) - \frac{w_{v_0}(x)^2}{2}\right) > 0 \ , \end{aligned}$$

and thus $v'_{v_1}(x) > 0$ on $(0, x_6(v_0)]$. (3.28) implies that

$$\begin{aligned} \sqrt{\alpha(1+v_{v_1})} - w_{v_1} &\geqslant \sqrt{\alpha(1+v_{v_0})} - w_{v_0} - K\tilde{\delta}(1+\sqrt{\alpha}) \\ &\geqslant \frac{1}{2}\inf_{x\in[0,x_5(v_0)]}(\sqrt{\alpha(1+v_{v_0}(x))} - w_{v_0}(x)) > 0 \ . \end{aligned}$$

Finally, it follows from (3.29) that

$$v_{v_1}(x_5(v_0)) \geqslant \frac{1 + v_{v_0}(x_5(v_0))}{2} > 1 \ .$$

Therefore $x_0(v_1) < x_5(v_0)$, $v_{v_1}(x_0(v_1)) = 1$ and $v_1 \in E_v$.

The proof of the openness of $E_{v'}$, E_w and $E_{w'}$ is rather similar, and we will not give the details here. We just mention that to prove the openness of $E_{w'}$, it is usefull to remark that if $v_0 \in E_{w'}$, $w''(x_0(v_0)) = -w(x_0(v_0))(1 - v(x_0(v_0))) < 0$, and in order to prove that $E_{v'}$ is open, we can see that $v''(x_0(v_0)) < 0$ as soon as $v_0 \in E_{v'}$. Indeed, it is clear that $v''(x_0(v_0)) \leqslant 0$, because $v(x) < 1$ for $x < x_0(v_0)$. For the same reason, if $v''(x_0(v_0))$ was equal to 0, we would have

$$\begin{aligned} 0 \leqslant v'''(x_0(v_0)) &= \alpha v'(x_0(v_0)) - w(x_0(v_0))w'(x_0(v_0)) \\ &= -(ww')(x_0(v_0)) < 0 \ , \end{aligned}$$

which is a contradiction.

Remark. A byproduct of the proof is that the sets E_w and $E_{w'}$ are empty. More precisely, denoting by $v_c := v(0) \in (0,1)$, where(v,w) is the dark-dark soliton given by Theorem 3.0.1, we have in fact $E_v = (v_c, 1)$, while $E_{v'} = (0, v_c)$.

Chapitre 4

Schrödinger group on Zhidkov spaces

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Résumé. On considère le problème de Cauchy pour l'équation de Schrödinger non linéaire sur $\mathbb{R}^n$, dans des espaces de fonctions non (nécessairement) nulles à l'infini. Ce problème se pose par exemple dans l'étude de la stabilité des "dark solitons". On montre que l'opérateur de Schrödinger génère un groupe sur les espaces de Zhidkov $X^k(\mathbb{R}^n)$ pour $k > n/2$, puis que le problème de Cauchy pour SNL est localement bien posé dans ces mêmes espaces de Zhidkov. On montre aussi la conservation des invariants classiques, et, dans certains cas, on en déduit que le problème de Cauchy est globalement bien posé.

Abstract. We consider the Cauchy problem for nonlinear Schrödinger equations on $\mathbb{R}^n$ with non zero boundary condition at infinity, a situation which occurs in stability studies of dark solitons. We prove that the Schrödinger operator generates a group on Zhidkov spaces $X^k(\mathbb{R}^n)$ for $k > n/2$, and that the Cauchy problem for NLS is locally well-posed on the same Zhidkov spaces. We justify the conservation of classical invariants which implies in some cases the global well-posedness of the Cauchy problem.

4.1 Introduction

This paper is devoted to the Cauchy problem for the Nonlinear Schrödinger equation (NLS)

$$\begin{cases} iu_t + \Delta u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0) = u_0 \end{cases} \tag{4.1}$$

with non zero boundary condition at infinity. Such boundary conditions occur in the "defocusing" case (e.g. $f(|u|^2) = 1 - |u|^2$), and are pertinent to many physical contexts. In nonlinear optics, the so-called dark soliton (see [KLD]) is a solution of (4.1) of the form

$u(x,t) = u_v(x - vt)$. For instance, for $n = 1$ and $f(r) = 1 - r$, we can compute that, for $v \in (-\sqrt{2}, \sqrt{2})$, $u_v(x - vt)$ solves (4.1), where u_v is given by:

$$u_v(x) = \sqrt{1 - \frac{v^2}{2}} \tanh\left(\sqrt{1 - \frac{v^2}{2}} \frac{x}{\sqrt{2}}\right) + i\frac{v}{\sqrt{2}} \tag{4.2}$$

The Gross-Pitaevskii equation (see [BeSa1], [BeSa2] and references therein)

$$iu_t + \Delta u + (1 - |u|^2)u = 0$$

with the boundary condition $u \to 1$ as $|x| \to \infty$ is a model for Superfluid Helium II at a temperature near zero and for Bose-Einstein condensation. More generally, NLS with the boundary condition $|u| \to \rho_0$ as $|x| \to \infty$ where ρ_0 is a positive constant such that $f(\rho_0^2) = 0$ occurs in several physical contexts (see [BGMP]), an especially interesting particular case being the cubic-quintic "$\psi^3 - \psi^5$" NLS.

These nonlinear Schrödinger equations possess solitons or solitary waves (see [BeSa1], [dB], [Mar3], [KLD]) and it is natural to study the Cauchy problem in spaces the solitary waves belongs to. Of course they can not be the usual Sobolev spaces $H^s(\mathbb{R}^n)$ because of the boundary condition at infinity. A possibility would be to work in the affine space $1 + H^s(\mathbb{R}^n)$, when $u \to 1$ as $|x| \to \infty$, and this actually was done in [BeSa1]. This approach obviously fails when only $|u|$ (but not u) tends to a constant at infinity, as it is the case for the dark soliton (4.2). Also, the solitary wave φ could be only slowly decaying at infinity, implying that $\varphi - 1 \notin L^2(\mathbb{R}^n)$ (see [Gr3] for the travelling wave of Gross-Pitaevskii equation in the 2-dimensional case).

In [Z1] Zhidkov introduces in the one-dimensional case the spaces X^k (with k a natural number), which consist of functions defined on $\mathbb{R}$, bounded and uniformly continuous, with derivatives up to order k in L^2, and proved that the Cauchy problem for NLS is locally well-posed in X^k.

Our aim here is to complete and generalize Zhidkov results. We introduce the Zhidkov spaces $X^k(\mathbb{R}^n)$ in higher dimensions and prove that the linear Schrödinger equation defines a strongly continuous group on $X^k(\mathbb{R}^n)$ if and only if $k > n/2$, and consequently that the Cauchy problem for NLS is locally well-posed in $X^k(\mathbb{R}^n)$ if $k > n/2$. We also justify rigorously the conservation of natural invariants of the NLS yielding some global well-posedness in $X^k(\mathbb{R}^n)$ for some defocusing NLS. A byproduct is the complete justification of the result of Zhiwu Lin in [L] that gives a criterion of stability for dark solitons of a class of NLS equations.

This paper is organized as follows. In section 2, we define the Zhidkov spaces $X^k(\mathbb{R}^n)$ and state some useful properties of these spaces. In section 3, we prove that the linear Schrödinger equation is well posed in $X^k(\mathbb{R}^n)$ if and only if $k > n/2$. In section 4, we show that the Cauchy problem for NLS is locally well-posed in $X^k(\mathbb{R}^n)$, if $k > n/2$. In section 5, we introduce the renormalized energy for (4.1) and prove its conservation, for $n = 1$ or 2, under some hypothesis on f and u_0, and we show that in dimension 1, it implies the globalness of the solution of (4.1) in a defocusing case.

Notations Throughout this paper, C denotes a constant that can change from line to line.
If j is a positive integer and u is a map of class C^j from $\mathbb{R}^n$ into $\mathbb{C}$, and $x, v_1, ..., v_j \in \mathbb{R}^n$, we denote by $D^j u(x)(v_1...v_j)$ the j^{th} differential of u at x applied to $(v_1, ..., v_j)$.
If E is a Banach space, we denote by $C_b(\mathbb{R}, E)$ the space of continuous bounded functions from $\mathbb{R}$ into E.

4.2 Some properties of Zhidkov spaces

Definition 4.2.1 *Let $n, k \in \mathbb{N}^*$. We define the space $X^k(\mathbb{R}^n)$ as the closure for the norm*

$$\|u\|_{X^k(\mathbb{R}^n)} := \|u\|_{L^\infty} + \sum_{1 \leqslant |\alpha| \leqslant k} \|\partial^\alpha u\|_{L^2}$$

of the space $\{u \in C^k(\mathbb{R}^n)$, bounded, uniformly continuous, with $\nabla u \in H^{k-1}\}$.

Note that a function in X^k is uniformly continuous.

Proposition 4.2.1 *Let $n, k \in \mathbb{N}^*$. Then $X^{k+1}(\mathbb{R}^n)$ is dense in $X^k(\mathbb{R}^n)$.*

Proof. Let $u \in X^k(\mathbb{R}^n)$, $(\rho_l)_{l \geqslant 1}$ a mollifier sequence (i.e. $\rho_l \in C_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \rho_l = 1$, $\rho_n \geqslant 0$, *Supp* $\rho_l \subset B(0, 1/l)$). Then $\rho_l * u$ is a sequence in X^{k+1} that converges to u in X^k. □

We show now a regularity result for functions in $X^k(\mathbb{R}^n)$, for $k > n/2$.

Proposition 4.2.2 *Let $n \in \mathbb{N}^*$, let $k = \lfloor n/2 \rfloor + 1$, and $p \in \mathbb{R}$ such that*

$$\begin{cases} p > n & \text{if } n \text{ is even} \\ p = 2n & \text{if } n \text{ is odd} \end{cases} .$$

Then $\nabla u \in L^p(\mathbb{R}^n)$ for $u \in X^k(\mathbb{R}^n)$ and there exists $C > 0$ such that

$$|u(x) - u(y)| \leqslant C|x - y|^{1-n/p} \|\nabla u\|_{L^p}, \ x, y \in \mathbb{R}^n \ . \tag{4.3}$$

Proof. By our choice of p and k, Sobolev's embedding implies that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leqslant C \|\nabla u\|_{H^{k-1}(\mathbb{R}^n)} \leqslant C \|u\|_{X^k(\mathbb{R}^n)}, \ u \in X^k(\mathbb{R}^n) \ . \tag{4.4}$$

Following the proof of Morrey's theorem given in [B], we can show that there exists a constant $C > 0$ that depends only on n and p such that for any compact set $K \subset \mathbb{R}^n$, and any cube $Q = [-r, r]^n$ containing K,

$$|u(x) - u(y)| \leqslant C|x - y|^{1-n/p} \|\nabla u\|_{L^p(Q)}, \ x, y \in K, \ u \in H^k_{loc}(\mathbb{R}^n) \ . \tag{4.5}$$

Since $X^k(\mathbb{R}^n) \subset H^k_{loc}(\mathbb{R}^n)$, (4.4) and (4.5) prove the announced result. □

Remark . In our proof, we did not use the fact that the elements of $X^k(\mathbb{R}^n)$ are uniformly continuous. Therefore, for $k > n/2$, an equivalent definition for X^k could be

$$X^k(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n),\ \nabla u \in H^{k-1}(\mathbb{R}^n)\}\ .$$

In particular, for $k > n/2$, $H^k(\mathbb{R}^n) \subset X^k(\mathbb{R}^n)$.

4.3 The Schrödinger group on $X^k(\mathbb{R}^n)$

In this section, we prove that if $k > n/2$, the Schrödinger operator defines a group on $X^k(\mathbb{R}^n)$. More precisely,

Theorem 4.3.1 *Let $n \in \mathbb{N}^*$, $k > n/2$ and $u_0 \in X^k(\mathbb{R}^n)$. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the quantity*

$$S(t)u_0(x) = \begin{cases} e^{-in\pi/4}\pi^{-n/2}\lim\limits_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}u_0(x+2\sqrt{t}z)dz & \text{if } t \geqslant 0 \\ e^{in\pi/4}\pi^{-n/2}\lim\limits_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(-i-\varepsilon)|z|^2}u_0(x+2\sqrt{-t}z)dz & \text{if } t \leqslant 0 \end{cases} \tag{4.6}$$

makes sense, and the family of operators $(S(t))_{t\in\mathbb{R}}$ defines a strongly continuous group on $X^k(\mathbb{R}^n)$.
Moreover, there exists a constant $C > 0$ that depends only on n and k, such that for every $u_0 \in X^k(\mathbb{R}^n)$ and $t \in \mathbb{R}$,

$$\|S(t)u_0\|_{X^k} \leqslant C(1+|t|^\rho)\|u_0\|_{X^k} \tag{4.7}$$

where

$$\rho = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ 1/4 & \text{if } n \text{ is odd} \end{cases} .$$

For convenience, we also denote $S(t)u_0(x)$ by $u(t,x)$.

All the computations below will be performed with $t \geqslant 0$. The case $t \leqslant 0$ is similar. Before starting the proof of the theorem itself, we need to prove some technical lemmas.

Lemma 4.3.1 *Let $n, k \geqslant 1$, $u_0 \in X^k(\mathbb{R}^n)$, $\beta > 0$, $x \in \mathbb{R}^n$ and $t > 0$.*
We define $g : (\beta,\infty) \to \mathbb{C}$ by

$$g(r) = \int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv \tag{4.8}$$

then $g \in X^k(\beta,\infty)$ (the definition of which is clear), and for all $j \in \{1...k\}$, $g^{(j)} \in L^2(\beta,\infty,r^{n-1}dr)$ with

$$\|g^{(j)}\|_{L^2(\beta,\infty,r^{n-1}dr)} \leqslant (2\sqrt{t})^{j-n/2}|\mathbb{S}^{n-1}|^{1/2}\|u_0\|_{X^k(\mathbb{R}^n)}\ . \tag{4.9}$$

Proof. For any $r \in (\beta, \infty)$, $|g(r)| \leqslant |\mathbb{S}^{n-1}| \|u_0\|_{L^\infty}$, hence $g \in L^\infty(\beta, \infty)$.

Let $\varepsilon > 0$. Since u_0 is uniformly continuous, there exists some $\delta > 0$ such that $|y - z| \leqslant \delta$ implies $|u_0(y) - u_0(z)| \leqslant \varepsilon$. Let r_1, r_2 be such that $|r_1 - r_2| \leqslant \delta/(2\sqrt{t})$. Then we get: $|g(r_1) - g(r_2)| \leqslant |\mathbb{S}^{n-1}|\varepsilon$ and g is uniformly continuous.

For $j \in \{1...k\}$, the Cauchy-Schwarz inequality and a change of variables yield:

$$\begin{aligned}\int_\beta^\infty |g^{(j)}(r)|^2 r^{n-1} dr &\leqslant (2\sqrt{t})^{2j} |\mathbb{S}^{n-1}| \int_\beta^\infty \int_{\mathbb{S}^{n-1}} |D^j u_0(x + 2\sqrt{t} r v)(v...v)|^2 r^{n-1} dv dr \\ &= (2\sqrt{t})^{2j} |\mathbb{S}^{n-1}| \int_{2\sqrt{t}\beta}^\infty \int_{\mathbb{S}^{n-1}} |D^j u_0(x + rv)(v...v)|^2 \frac{r^{n-1} dv dr}{(2\sqrt{t})^n} \\ &\leqslant (2\sqrt{t})^{2j-n} |\mathbb{S}^{n-1}| \|u_0\|^2_{X^k(\mathbb{R}^n)}\end{aligned}$$

which is the announced inequality. $\square$

We state next two elementary lemmas which are straightforward consequences of the Leibniz formula

Lemma 4.3.2 *Let $k \in \mathbb{N}$. There exists constants $(b_{l,k})_{0 \leqslant l \leqslant k}$ such that*

$$\left(\frac{d}{dr}\left(\frac{1}{r}.\right)\right)^k = \sum_{l=0}^k b_{l,k} \frac{1}{r^{2k-l}} \frac{d^l}{dr^l} \ . \tag{4.10}$$

Lemma 4.3.3 *There exists constants $(a_{k,j})_{0 \leqslant j \leqslant k}$ such that*

$$\left(\frac{d}{dr}\left(\frac{1}{r}.\right)\right)^k (r^{n-1}.) = \sum_{j=0}^k a_{k,j} \frac{1}{r^{2k-n+1-j}} \frac{d^j}{dr^j}. \tag{4.11}$$

We are now ready to prove theorem 4.3.1. We fix $n \in \mathbb{N}^*$, and we introduce a function $\chi \in C^\infty(\mathbb{R}^n)$ such that:

- χ is radial
- χ increases along any half-line issued at 0
- $\chi \equiv 0$ on $\{x, |x| \leqslant 1\}$
- $\chi \equiv 1$ on $\{x, |x| \geqslant 2\}$

For any $\beta > 0$, we define $\chi_\beta = \chi(./\beta)$. For convenience, we will also use the notation $\chi_\beta(|.|) = \chi_\beta(.)$.

Proof of theorem 4.3.1. We assume first that $k = \lfloor n/2 \rfloor + 1$. The first step of the proof is to show that the limit in formula (4.6) is well defined. Let us fix $t > 0$, $\beta > 0$ and $\varepsilon > 0$. We split the integral in (4.6) into two parts:

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x+2\sqrt{t}z)dz \;\; &= \;\; \underbrace{\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))u_0(x+2\sqrt{t}z)dz}_{(I)} \\
&\quad + \underbrace{\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x+2\sqrt{t}z)dz}_{(II)}
\end{aligned}
$$

We first consider the term (I):

$$
\left| e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))u_0(x+2\sqrt{t}z)\right| \leqslant \|u_0\|_{L^\infty}\mathbf{1}_{B(0,2\beta)}(z)
$$

Then by Lebesgue's theorem, the limit as $\varepsilon \to 0$ of (I) exists and

$$
\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))u_0(x+2\sqrt{t}z)dz\right| \;\; \leqslant \;\; \|u_0\|_{L^\infty}|B(0,1)|(2\beta)^n \; . \tag{4.12}
$$

We now consider the term (II). We compute it by using polar coordinates, and with the notations of lemma 4.3.1, we integrate by parts (which is justified by the regularity results on g proved in lemma 4.3.1) using lemma 4.3.3:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x+2\sqrt{t}z)dz \\
&= \int_0^\infty e^{(i-\varepsilon)r^2}\chi_\beta(r)r^{n-1}\left(\int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv\right)dr \\
&= \int_\beta^\infty \left(\frac{1}{2(i-\varepsilon)r}\frac{d}{dr}.\right)^k\left(e^{(i-\varepsilon)r^2}\right)\chi_\beta(r)r^{n-1}\left(\int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv\right)dr \\
&= \left(\frac{-1}{2(i-\varepsilon)}\right)^k \int_\beta^\infty e^{(i-\varepsilon)r^2}\sum_{j=0}^k a_{k,j}\frac{1}{r^{2k-n+1-j}}\frac{d^j}{dr^j}\left[\chi_\beta(r)g(r)\right]dr \\
&= \left(\frac{-1}{2(i-\varepsilon)}\right)^k \sum_{j=0}^k a_{k,j}\sum_{l=0}^j \binom{j}{l}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}dr
\end{aligned} \tag{4.13}
$$

where we have used the Leibniz formula in the last equality. We will now apply Lebesgue's theorem to each term of this sum.

For $l=0$, $j\in\{0...k\}$, we have:

$$
\begin{aligned}
\left|e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\chi_\beta^{(j)}(r)r^{n-1}\right| \;\; &\leqslant \;\; \frac{|\mathbb{S}^{n-1}|\|u_0\|_{L^\infty}}{r^{2k-n+1}}\begin{cases} \|\chi\|_{L^\infty} & \text{if } j=0 \\ \frac{r^j}{\beta^j}\|\chi^{(j)}\|_{L^\infty} & \text{if } j\in\{1...k\} \text{ and } r\leqslant 2\beta \\ 0 & \text{if } j\in\{1...k\} \text{ and } r>2\beta \end{cases} \\
&\leqslant \;\; \frac{|\mathbb{S}^{n-1}|\|u_0\|_{L^\infty}}{r^{2k-n+1}}2^j\|\chi^{(j)}\|_{L^\infty}
\end{aligned}
$$

Since $k > n/2$, $\int_\beta^\infty dr/r^{2k-n+1} = \beta^{n-2k}/(2k-n) < \infty$ and we can pass to the limit as $\varepsilon \to 0$ by Lebesgue's theorem. We obtain

$$\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\chi_\beta^{(j)}(r)r^{n-1}dr\right| \leqslant \frac{|\mathbb{S}^{n-1}|\|u_0\|_{L^\infty}2^j\|\chi^{(j)}\|_{L^\infty}}{(2k-n)\beta^{2k-n}} . \qquad (4.14)$$

For $l \geqslant 1$, $j \in \{l...k\}$, we have:

$$\left|e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}\right| \leqslant \|\chi^{(j-l)}\|_{L^\infty}\frac{1}{\beta^{j-l}}\frac{1}{r^{2k-j}}|g^{(l)}(r)|r^{n-1} .$$

Notice that $2(2k-j)-(n-1) > 1$, so that $r \to 1/r^{2k-j} \in L^2(\beta,\infty,r^{n-1}dr)$. The Cauchy-Schwarz inequality together with (4.9) lead to

$$\int_\beta^\infty \frac{1}{r^{2k-j}}|g^{(l)}(r)|r^{n-1}dr \leqslant \frac{1}{(4k-2j-n)^{1/2}\beta^{2k-j-n/2}}(2\sqrt{t})^{l-n/2}|\mathbb{S}^{n-1}|^{1/2}\|u_0\|_{X^k} .$$

So Lebesgue's theorem can be applied, and

$$\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}dr\right| \leqslant \frac{|\mathbb{S}^{n-1}|^{1/2}\|\chi^{(j-l)}\|_{L^\infty}}{(4k-2j-n)^{1/2}}\|u_0\|_{X^k}\frac{(2\sqrt{t})^{l-n/2}}{\beta^{2k-l-n/2}}. \qquad (4.15)$$

We fix now $\beta = 1$. There is a positive constant C such that for $j \in \{1...k\}$,

$$(2\sqrt{t})^{j-n/2} \leqslant C(t^{(1-n/2)/2}+t^{(k-n/2)/2}).$$

Therefore, using (4.12),(4.14) and (4.15), there exists a positive constant C such that for all $u_0 \in X^k$, for all $t > 0$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leqslant C(1+t^{(1-n/2)/2}+t^{(k-n/2)/2})\|u_0\|_{X^k(\mathbb{R}^n)} \qquad (4.16)$$

The second step of the proof consists in proving that $u(t) \to u_0$ in L^∞ as $t \to 0$. We first introduce some definitions.

Definition 4.3.1 *For $l \in \{0...k\}$ and $h \in L^2(\beta,\infty,r^{n-1}dr)$, we define*

$$T_h^l := \int_\beta^\infty e^{(i-\varepsilon)r^2}h(r)g^{(l)}(r)r^{n-1}dr$$

and we will say that such a quantity is "of type $T^{(l)}$".
If h can be written $h(r) = \chi_\beta^{(p)}(r)/r^q$ with $2q-n > 0$, we will say that the "order" of T_h^l in β is $p+q-n/2$.
If h is given as a linear combination of such terms, we define the "order" of T_h^l as the lowest order of non-zero monomials in the expression of h (remark that our "definition" of the order of T_h^l depends on the decomposition of h).

Let $\alpha \in (0, 1/(2n+1))$, and $m > 0$ such that $m \geqslant (n/2-1)/\alpha$. The following technical lemma gives a new expression of the term (II). It consists in doing as much integration by parts as necessary, in order to express (II) as a sum of terms of orders $\geqslant m$, which we are able to estimate in an appropriate way (see estimate (4.19) below.

Lemma 4.3.4 *(II) can be written as a linear combination of terms of type $T^{(l)}$ with $l \in \{0...k\}$, such that for $l \in \{1...k-1\}$, the order of the terms of type $T^{(l)}$ in this linear combination is $\geqslant m$.*

Proof. Let $l \in \{1...k-1\}$, $p, q \in \mathbb{N}$ such that $2q - n > 0$ and $h(r) = \chi_\beta^{(p)}(r) r^q$. We transform T_h^l by integrations by parts, as in (4.13). After some computations, we get (this is actually (4.13) where we have replaced k by $k-l$, g by $g^{(l)}$ and χ_β by $\chi_\beta^{(p)}(r)/r^q$):

$$T_h^l = \left(\frac{-1}{2(i-\varepsilon)}\right)^{k-l} \sum_{j=0}^{k-l} a_{k-l,j} \sum_{c=0}^{j} \binom{j}{c} \sum_{b=0}^{j-c} \binom{j-c}{b} (-q)...(-q-(j-c-b-1)) \\ \times \int_\beta^\infty \frac{e^{(i-\varepsilon)r^2} \chi_\beta^{(p+b)}(r) g^{(l+c)}(r)}{r^{2(k-l)+q-c-b}} r^{n-1} dr \ . \qquad (4.17)$$

Since $2(2(k-l)+q-c-b)-n \geqslant 2(k-l)+2q-n$, $k \geqslant l$ and $2q-n>0$, we have written T_h^l as a linear combination of terms of types $T^{(l)}, T^{(l+1)}...T^{(k)}$. Moreover, the order of the terms of type $T^{(l)}$ (that correspond to $c=0$) in this sum is:

$$p + b + 2(k-l) + q - b - n/2 = \underbrace{p+q-n/2}_{\text{order of } T_h^l} + 2\underbrace{(k-l)}_{\geqslant 1} \ .$$

The conclusion of this computation is that passing from the canonical expression of T_h^l to its new expression, the order of type $T^{(l)}$ terms increases at least by 2.

We now use the above calculation to show the result by induction on $l \in \{0...k-1\}$. Let us consider, for $l \in \{0...k-1\}$, the induction hypothesis H_l: "(II) can be written as a sum of terms of type $T^{(\gamma)}$, $0 \leqslant \gamma \leqslant k$, so that if $1 \leqslant \gamma \leqslant l$, the term of type $T^{(\gamma)}$ in this sum has order $\geqslant m$."
Formula (4.13) implies that

$$(II) = \sum_{l=0}^{k} T_{h^{(l)}}^l, \text{ whith } h^{(l)}(r) = \sum_{j=l}^{k} a_{kj} \binom{j}{l} \frac{\chi_\beta^{(j-l)}(r)}{r^{2k-j}} \ .$$

We have $2(2k-j)-n \geqslant 2k-n > 0$, so that H_0 is true thanks to (4.13).
Let us now take $l \in \{1...k-1\}$ and suppose H_{l-1}. The induction hypothesis implies that (II) can be written under the form: $(II) = \sum_\gamma^k \lambda_\gamma T_{h_\gamma}^\gamma$ where $\lambda_\gamma \in \mathbb{C}$, h_γ is a linear combination of $\chi_\beta^{(p)}(r)/r^q$, and for $\gamma \in \{1...l-1\}$, $T_{h_\gamma}^\gamma$'s order is at least m. Applying the former calculation to the term $T_{h_l}^l$, we get a new expression of (II) where the terms of type $T^{(\gamma)}$

with $\gamma \leqslant l-1$ are unchanged (in particular, the new expression still verifies H_{l-1}), the order of the term of type $T^{(l)}$ has increased by $2(k-l) \geqslant 2$. We can start this process again, as long as it is necessary (but with a finite number of steps) to ensure that the term of type $T^{(l)}$ is of order $\geqslant m$, and hence H_l is true.
So we have proved that H_{k-1} is true, which is the result of the lemma. □

We give now an upper bound on $|T_h^l|$, for $l \in \{1...k\}$, with $h(r) = \chi_\beta^{(p)}(r)/r^q$ and $2q-n>0$:

$$\begin{aligned} |T_h^l| &\leqslant \frac{\|\chi^{(p)}\|_\infty}{\beta^p} \int_\beta^\infty \frac{1}{r^q} |g^{(l)}(r)| r^{n-1} dr \\ &\leqslant \underbrace{\frac{\|\chi^{(p)}\|_\infty}{\sqrt{2q-n}} \|u_0\|_{X^k} |\mathbb{S}^{n-1}|^{1/2}}_{=:C} \frac{(2\sqrt{t})^{l-n/2}}{\beta^{p+q-n/2}} \end{aligned} \tag{4.18}$$

where we have used the Cauchy-Schwarz inequality and lemma 4.3.1. If we assume that $\beta \geqslant 1$ and T_h^l's order is at least m, then

$$|T_h^l| \leqslant C \frac{(2\sqrt{t})^{l-n/2}}{\beta^m} . \tag{4.19}$$

We write β on the form $\beta = \tilde{\beta}/(2\sqrt{t})^\alpha$, for $t \leqslant 1$ and $\tilde{\beta} > 2^\alpha$. Let us fix $\delta > 0$. The choice of m and the fact that $l \geqslant 1$ ensure that $l - n/2 + \alpha m \geqslant 0$.
We choose $\tilde{\beta} > 2^\alpha$ large enough such that for all $t \leqslant 1$,

$$\frac{(2\sqrt{t})^{l-n/2+\alpha m}}{\tilde{\beta}^m} \leqslant \delta \ , \ l \in \{1...k\}$$

and

$$\frac{|\mathbb{S}^{n-1}| \|u_0\|_\infty 2^j \|\chi^{(j)}\|_{L^\infty}}{(2k-n)\tilde{\beta}^{2k-n}} (2\sqrt{t})^{\alpha(2k-n)} \leqslant \delta \ , \ j \in \{0...k\} \ .$$

The property H_{k-1} proven in Lemma 4.3.4, (4.19) and (4.14) imply that there exists a constant $C>0$ (which does not depend on δ) such that

$$\forall \varepsilon > 0, \left| \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_{\tilde{\beta}/(2\sqrt{t})^\alpha}(z) u_0(x+2\sqrt{t}z) dz \right| \leqslant C\delta . \tag{4.20}$$

Proposition 4.2.2 with $p = 2n$ yields

$$\begin{aligned} &\left| \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1-\chi_\beta(z)) u_0(x+2\sqrt{t}z) dz - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1-\chi_\beta(z)) u_0(x) dz \right| \\ &\leqslant \int_{\mathbb{R}^n} (1-\chi_\beta(z)) |u_0(x+2\sqrt{t}z) - u_0(x)| dz \\ &\leqslant \int_{|z| \leqslant 2\beta} C(2\sqrt{t}2\beta)^{1/2} \|\nabla u_0\|_{L^p} dz \\ &\leqslant C(2\sqrt{t})^{1/2} \beta^{n+1/2} = C(2\sqrt{t})^{1/2-\alpha(n+1/2)} \tilde{\beta}^{n+1/2} . \end{aligned} \tag{4.21}$$

By the choice of α, $1/2-\alpha(n+1/2)>0$. So we can take $t_0<1$ such that

$$\forall t\leqslant t_0,\quad C(2\sqrt{t})^{1/2-\alpha(n+1/2)}\tilde{\beta}^{n+1/2}\ \leqslant\ \delta \tag{4.22}$$

Therefore, for $t\leqslant t_0$,

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}u_0(x+2\sqrt{t}z)dz-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}u_0(x)dz\right|\\
&\leqslant\ \left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x+2\sqrt{t}z)dz\right|+\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x)dz\right|\\
&\quad+\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))(u_0(x+2\sqrt{t}z)-u_0(x))dz\right|\\
&\leqslant\ C\delta\ +\ C\delta\ +\ \delta
\end{aligned} \tag{4.23}$$

by (4.20), (4.20) applied to $z\to u_0(x)$ instead of u_0 and (4.22).
Inequality (4.23) and the well-known identity

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}dz=\pi^{n/2}e^{in\pi/4}$$

imply that $u(t)\to u_0=u(0)$ in L^∞ as $t\to 0$, $t>0$. We could prove in the same way the left continuity at 0, and this, combined with the group property $S(t+s)=S(t)S(s),\ t,s\in\mathbb{R}$ (which is easy to verify) shows that for all $u_0\in X^k$, $S(.)u_0\in C(\mathbb{R},L^\infty(\mathbb{R}^n))$.

We consider now the general case $k>n/2$. For any multi-index α such that $1\leqslant|\alpha|\leqslant k$, for any $t>0$ (we have a similar formula for $t<0$),

$$\partial^\alpha u(t)=\frac{e^{-in\pi/4}}{\pi^{n/2}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\partial^\alpha u_0(x+2\sqrt{t}z)dz\ .$$

Here, $\partial^\alpha u_0\in L^2$. Since (4.6) defines classically a unitary group on L^2, the map $t\to\partial^\alpha u(t)$ belongs to $C(\mathbb{R},L^2(\mathbb{R}^n))$, with $\|\partial^\alpha u(t)\|_{L^2}=\|\partial^\alpha u_0\|_{L^2}$, $t\in\mathbb{R}$.
Therefore we can conclude that for all $u_0\in X^k$, $t\to S(t)u_0\in C(\mathbb{R},X^k(\mathbb{R}^n))$. Moreover, since for all $u_0\in X^k$, $S(.)u_0$ is bounded on $[-1,1]$, the Banach-Steinhaus theorem implies that $\|S(t)\|_{\mathcal{L}(X^k)}$ is bounded on $[-1,1]$. Combining this with (4.16), we get (4.7). $\square$

We give now the infinitesimal generator of $S(t)$.

Theorem 4.3.2 *Let $k>n/2$. The generator of the group $(S(t))_{t\in\mathbb{R}}$ on $X^k(\mathbb{R}^n)$ defined in theorem 4.3.1 is $i\Delta$, its domain is $X^{k+2}(\mathbb{R}^n)$.*

Proof. We denote by $A=S'(0)$ the generator of $(S(t))_t$.
We split the proof into three steps. In the first step, we show that if $u_0\in X^{k+4}(\mathbb{R}^n)$, $(S(t)u_0-u_0)/t\ \underset{t\to 0}{\longrightarrow}\ i\Delta u_0$ in $X^k(\mathbb{R}^n)$, and therefore $(X^{k+4}(\mathbb{R}^n),i\Delta)\subset(D(A),A)$. In the second step, we prove that $(X^{k+2}(\mathbb{R}^n),i\Delta)\subset(D(A),A)$. We conclude in the third step.

1^st^ step. Let $u_0 \in X^{k+4}(\mathbb{R}^n) \subset C^4(\mathbb{R}^n)$. We want to show that

$$\frac{S(t)u_0 - u_0}{t} \underset{t\to 0}{\longrightarrow} i\Delta u_0 \quad \text{in } X^k \ . \tag{4.24}$$

We will prove (4.24) for $t \to 0,\ t > 0$. The proof is similar in the case $t \to 0,\ t < 0$.

By Taylor's formula

$$u_0(x + 2\sqrt{t}z) - u_0(x) = \nabla u_0(x).2\sqrt{t}z + \int_0^1 (1-s)D^2u_0(x + 2\sqrt{t}z)(2\sqrt{t}z, 2\sqrt{t}z)ds \ ,$$

the fact that

$$\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\nabla u_0(x).2\sqrt{t}zdz = 0 \ ,$$

and

$$i\Delta u_0(x) = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{e^{-in\pi/4}}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}D^2u_0(x)(2\sqrt{t}z, 2\sqrt{t}z)dz \ ,$$

we obtain

$$\begin{aligned}&\left(\frac{S(t)u_0 - u_0}{t} - i\Delta u_0\right)(x)\\ &\quad = 4\frac{e^{-in\pi/4}}{\pi^{n/2}}\lim_{\varepsilon\to 0}\int_0^1 (1-s)\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(D^2u_0(x + 2s\sqrt{t}z) - D^2u_0(x))(z,z)dzds \ .\end{aligned}$$

Next,

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(D^2u_0(x + 2s\sqrt{t}z) - D^2u_0(x))(z,z)dz\\ &\quad = \left(\frac{-1}{2(i-\varepsilon)}\right)^2\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\Delta^2u_0(x + 2s\sqrt{t}z)(2s\sqrt{t})^2dz\\ &\qquad -\frac{1}{2(i-\varepsilon)}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(\Delta u_0(x + 2s\sqrt{t}z) - \Delta u_0(x))dz \ ,\end{aligned}$$

hence

$$\begin{aligned}\left(\frac{S(t)u_0 - u_0}{t} - i\Delta u_0\right)(x) &= -4\left[t\int_0^1 (1-s)s^2S(ts^2)\Delta^2u_0(x)ds\right.\\ &\qquad \left.+\frac{1}{2i}\int_0^1 (1-s)(S(ts^2)\Delta u_0 - \Delta u_0)(x)ds\right].\end{aligned} \tag{4.25}$$

Since $k > n/2$, $\Delta^2 u_0 \in H^k(\mathbb{R}^n) \subset X^k(\mathbb{R}^n)$. Therefore $(S(ts^2)\Delta^2u_0)_{t,s\in[0,1]}$ is bounded in X^k and the first term in the right hand side of (4.25) tends to 0 in X^k as $t \to 0$. Since $\Delta u_0 \in H^{k+2}(\mathbb{R}^n) \subset X^k(\mathbb{R}^n)$, $S(ts^2)\Delta u_0 - \Delta u_0 \underset{t\to 0}{\longrightarrow} 0$ in X^k, uniformly in $s \in [0,1]$. Thus (4.24) has been proven.

Therefore, if $u_0 \in X^{k+4}(\mathbb{R}^n)$, $u_0 \in D(A)$ and $Au_0 = i\Delta u_0$.

2$^{\text{nd}}$ step. Let $u_0 \in X^{k+2}(\mathbb{R}^n)$. Thanks to Proposition 4.2.1, there exists a sequence $(v_0^l)_{l\in\mathbb{N}} \subset X^{k+4}(\mathbb{R}^n)$ such that $v_0^l \to u_0$ in $X^{k+2} \subset X^k$. Hence $i\Delta v_0^l \to i\Delta u_0$ in $H^k \subset X^k$. Therefore $(u_0, i\Delta u_0)$ belongs to the closure of $\{(v_0, i\Delta v_0),\ v_0 \in X^{k+4}(\mathbb{R}^n)\}$ in $X^k \times X^k$, and since the infinitesimal generator of a strongly continuous semigroup is a closed operator (see corollary 2.5 in [Pa] p5), this implies that $u_0 \in D(A)$ and $Au_0 = i\Delta u_0$, i.e. $(X^{k+2}(\mathbb{R}^n), i\Delta) \subset (D(A), A)$.

3$^{\text{rd}}$ step. Let $u_0 \in D(A)$. We want to show that $Au_0 = i\Delta u_0$ in $\mathcal{D}'$. Since $X^{k+2}(\mathbb{R}^n)$ is dense in $X^k(\mathbb{R}^n)$, there exists a sequence $(v_0^l)_{l\in\mathbb{N}} \in X^{k+2}(\mathbb{R}^n)$ that tends to u_0 in $X^k(\mathbb{R}^n)$, and then

$$Av_0^l = i\Delta v_0^l \underset{l\to\infty}{\longrightarrow} i\Delta u_0 \text{ in } \mathcal{D}' \ .$$

Let us show that $Av_0^l \to Au_0$ in $\mathcal{D}'$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned} &< Av_0^l, \varphi >_{\mathcal{D}',\mathcal{D}} = < \varphi, Av_0^l >_{(X^k)',X^k} = < A^*\varphi, v_0^l >_{(X^k)',X^k} \\ &\qquad \to \ < A^*\varphi, u_0 >_{(X^k)',X^k} = < \varphi, Au_0 >_{(X^k)',X^k} = < Au_0, \varphi >_{\mathcal{D}',\mathcal{D}} \ . \end{aligned} \tag{4.26}$$

To justify this calculation, we need to prove that $\varphi \in D(A^*)$. Since the Schrödinger group is continuous in the Schwartz space $\mathcal{S}$, one has for any $u \in D(A)$,

$$\begin{aligned} < Au, \varphi >_{\mathcal{D}',\mathcal{D}} &= \lim_{t\to 0} \left\langle \frac{S(t)-id}{t} u, \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} u(y) \left(\frac{S(-t)-id}{t} \overline{\varphi} \right)(y) dy = - < i\Delta\varphi, u >_{\mathcal{D}',\mathcal{D}} \end{aligned}$$

and then

$$| < S'(0)u, \varphi >_{\mathcal{D}',\mathcal{D}} | \leqslant \|\Delta\varphi\|_{L^1} \|u\|_{X^k},\ u \in D(A) \ .$$

It follows that $\varphi \in D(A^*)$ and (4.26) is justified.
Therefore, $Au_0 = i\Delta u_0$ in $\mathcal{D}'$. Now, $u_0 \in X^k$, $Au_0 = i\Delta u_0 \in X^k$, hence $\Delta \nabla u_0 \in \left(H^{k-1}\right)^n$ and since $\nabla u_0 \in \left(H^{k-1}\right)^n$, this yields $\nabla u_0 \in \left(H^{k+1}\right)^n$ and we conclude that $u_0 \in X^{k+2}$. $\square$

Finally, we show that the assumption $k > n/2$ we made in theorem 4.3.1 is sharp. More precisely, we have:

Proposition 4.3.1 *Let us take $n \in \mathbb{N}^*$. For $x \in \mathbb{R}^n$, we define*

$$u_0(x) = \frac{e^{-i|x|^2}}{(1+|x|^2)^{n/2} \log \sqrt{2+|x|^2}} \ .$$

Then $u_0 \in X^{\lfloor n/2 \rfloor}(\mathbb{R}^n)$, but if we define $u(t,x)$ by formula (4.6), $u(1/4, 0) = \infty$.

Proof. It is clear that $u_0 \in L^\infty(\mathbb{R}^n)$ and is uniformly continuous (because ∇u_0 is bounded). Let α be a multi-index with $1 \leqslant |\alpha| \leqslant \lfloor n/2 \rfloor$. The "worst term" in $\partial^\alpha u_0$ is the one obtained by $|\alpha|$ successive differentiations of the factor $e^{-i|x|^2}$. This term is

$$(-2ix)^\alpha e^{-i|x|^2}(1+|x|^2)^{-n/2}/\log\sqrt{2+|x|^2}.$$

Plainly,

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \frac{(-2ix)^\alpha e^{-i|x|^2}}{(1+|x|^2)^{n/2}\log\sqrt{2+|x|^2}} \right|^2 dx &\leqslant 2^{|\alpha|} \int_{\mathbb{R}^n} \frac{(|x|^2)^{|\alpha|}}{(1+|x|^2)^n \log^2\sqrt{2+|x|^2}} dx \\
&= 2^{|\alpha|}|\mathbb{S}^{n-1}| \int_0^\infty \frac{r^{2|\alpha|+n-1}dr}{(1+r^2)^n \log^2\sqrt{2+r^2}} .
\end{aligned}$$

The right hand side term in the above inequality is finite, because $2n-(2|\alpha|+n-1) \geqslant 1$ and $\int^\infty \frac{dr}{r\log^2 r} < \infty$. Therefore $u_0 \in X^{\lfloor n/2 \rfloor}(\mathbb{R}^n)$. Moreover, since $u_0 \in L^2$, $u_0 \in H^{\lfloor n/2 \rfloor}(\mathbb{R}^n)$, and $u \in C(\mathbb{R}, H^{\lfloor n/2 \rfloor})$ is a solution of the linear Schrödinger equation.

For $t = 1/4,\ x = 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x+2\sqrt{t}z)dz &= \int_{\mathbb{R}^n} \frac{e^{-\varepsilon|y|^2}dy}{(1+|y|^2)^{n/2}\log\sqrt{2+|y|^2}} \\
&= |\mathbb{S}^{n-1}| \int_0^\infty \frac{e^{-\varepsilon r^2} r^{n-1}dr}{(1+r^2)^{n/2}\log\sqrt{2+r^2}}
\end{aligned}$$

which tends to $+\infty$ as $\varepsilon \to 0$, by the monotone convergence theorem and since $\int^\infty \frac{dr}{r\log r} = \infty$.

An amusing fact is that for $(t,x) \neq (1/4,0)$, $u(t,x)$ is well defined by formula (4.6) and then u is a continuous function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n \backslash (1/4,0)$. $\square$

Remark The "ill-posedness" result in proposition 4.3.1 pertains to the dispersive blow-up phenomena described in [BoSa1] for KdV type equations and in [BoSa2] for general dispersive equations.

4.4 Local existence for NLS in $X^k(\mathbb{R}^n)$

We consider now the Cauchy problem (4.1) with a more general nonlinearity:

$$\begin{cases} iu_t + \Delta u + F(u) = 0 \\ u(0) = u_0 \in X^k(\mathbb{R}^n) \end{cases} \tag{4.27}$$

where $k > n/2$ and $F : X^k(\mathbb{R}^n) \to X^k(\mathbb{R}^n)$ is of class C^1. We have shown in section 3 that the Schrödinger group $S(t)$ defines a strongly continuous group on $X^k(\mathbb{R}^n)$. By a classical fixed point argument one obtains the local well-posedness of the Cauchy problem (4.27). Namely, we have:

Theorem 4.4.1 *Let $M > 0$. Then there exists $T_+(M) > 0$ and $T_-(M) < 0$ such that for all $u_0 \in X^k(\mathbb{R}^n)$ with $\|u_0\|_{X^k} \leqslant M$, there is an unique mild solution $u \in C([T_-(M), T_+(M)], X^k)$ of (4.27). We recall that a mild solution of (4.27) is a solution of the integral equation*

$$u(t) = S(t)u_0 + i\int_0^t S(t-s)F(u(s))ds \ , \ t \in [T_-, T_+] \ . \tag{4.28}$$

We also recall (see [Pa] or [CH]) that if u solves (4.28), it is a solution of (4.27) in the space $C(\mathbb{R}, H^{k-2}_{loc})$, and that if $u_0 \in D(S'(0)) = X^{k+2}$, $u \in C^1([T_-, T_+], X^k) \cap C([T_-, T_+], X^{k+2})$ is the classical solution of (4.27).

Proof. It is a direct application of proposition 4.3.3 in [CH]. □

We have furthermore

Theorem 4.4.2 *For every $u_0 \in X^k(\mathbb{R}^n)$, there exists $T_*(u_0) \in [-\infty, 0)$ and $T^*(u_0) \in (0, +\infty]$ such that*

- *there exists a maximal solution $u \in C(T_*(u_0), T^*(u_0), X^k)$ which is the unique solution of (4.28), for all $T_\pm$, $T_*(u_0) < T_- < 0 < T_+ < T^*(u_0)$,*
- *either $T^*(u_0) = +\infty$ or $\|u(t)\|_{X^k} \underset{t \uparrow T^*(u_0)}{\longrightarrow} +\infty$,*
- *either $T_*(u_0) = -\infty$ or $\|u(t)\|_{X^k} \underset{t \downarrow T_*(u_0)}{\longrightarrow} +\infty$.*

Proof. It is a direct application of theorem 4.3.4 in [CH]. □

Theorem 4.4.3 *If $u_0 \in X^{k+2}(\mathbb{R}^n)$, the solution $u \in C(T_*(u_0), T^*(u_0), X^k)$ of (4.28) given by theorems 4.4.1 and 4.4.2 is a classical solution of (4.27) in X^k, which means that $u \in C(T_*(u_0), T^*(u_0), X^{k+2}) \cap C^1(T_*(u_0), T^*(u_0), X^k)$.*

Proof. It is a direct application of theorem 1.5 in [Pa], because $F : X^k \to X^k$ is of class C^1, and because the domain of $i\Delta$ (which is the generator of the group $S(t)$) on X^k is X^{k+2}. □

Here is a typical example where Theorem 4.4.1 applies.

Proposition 4.4.1 *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be of class C^{k+1}. Then $F : X^k(\mathbb{R}^n) \to X^k(\mathbb{R}^n)$ defined by $F(u) = f(|u|^2)u$ is of class C^1.*

Proof. Let $u \in X^k(\mathbb{R}^n)$. It is clear that $F(u) \in L^\infty$.

For any multi-index α such that $1 \leqslant |\alpha| \leqslant k$, $\partial^\alpha F(u)$ can be written as a linear combination of terms of type $u^a \overline{u}^b f^{(l)}(|u|^2)\partial^{\alpha_1} u ... \partial^{\alpha_r} u \partial^{\beta_1}\overline{u}...\partial^{\beta_s}\overline{u}$, where a, b, r , s, l are integers and α_i, β_j are multi-indices such that $l \leqslant |\alpha| \leqslant k$ and $\sum |\alpha_i| + \sum |\beta_j| \leqslant |\alpha| \leqslant k$.

By Sobolev's embeddings and generalized Hölder's inequality, it can be shown easily that $\partial^\alpha F(u) \in L^2$, and hence $F(u) \in X^k$.
The proof that for $u \in X^k$, $F'(u)$ maps X^k into X^k and that F' is continuous is similar. $\square$

We end up this paragraph by proving the persistency of higher regularity.

Proposition 4.4.2 *Let $n \in \mathbb{N}^*$, $k > n/2 + 1$, and $u_0 \in X^k(\mathbb{R}^n)$.*
We suppose $f \in C^{k+1}(\mathbb{R}_+)$. For $l \in \{\lfloor n/2 \rfloor + 1 ... k\}$, we denote by $u \in C(T_*(l), T^*(l), X^l)$ the solution of (4.28) with $F(u) = f(|u|^2)u$, where $]T_*(l), T^*(l)[$ is the maximal existence interval of u in $X^l(\mathbb{R}^n)$.
Then
$$T^* := T^*(\lfloor n/2 \rfloor + 1) = ... = T^*(k)$$
and similarly, $T_* := T_*(\lfloor n/2 \rfloor + 1) = ... = T_*(k)$.

Proof. We make the proof for T^*, it is similar for T_*.
A priori, for $l > \lfloor n/2 \rfloor + 1$, $T^* \geqslant T^*(l)$. We suppose by contradiction that $T^* > T^*(l)$. For $t \in]T_*, T^*[$,
$$u(t) = S(t)u_0 + i \int_0^t S(t-s)\left(f(|u(s)|^2)u(s)\right) ds$$
hence for $t \in]T_*(l), T^*(l)[$,
$$\|u(t)\|_{X^l} \leqslant \|S(t)\|_{\mathcal{L}(X^l, X^l)} \|u_0\|_{X^l} + \int_0^t \|S(t-s)\|_{\mathcal{L}(X^l, X^l)} \|f(|u(s)|^2)u(s)\|_{X^l} ds$$
We know by (4.7) that $\|S(t)\|_{\mathcal{L}(X^l, X^l)}$ is bounded on $[0, T^*(l)]$, and that u is continuous from $[0, T^*(l)]$ into $X^{\lfloor n/2 \rfloor + 1}(\mathbb{R}^n)$, so there exists $M > 0$ such that for all $t \in [0, T^*(l)]$, $\|u(t)\|_{X^{\lfloor n/2 \rfloor + 1}} \leqslant M$. In particular, $\|u(t)\|_{L^\infty} \leqslant M$, which implies that there exists $\tilde{M} > 0$ such that $\|f(|u(s)|^2)u(s)\|_{X^l} \leqslant \tilde{M} \|u(s)\|_{X^l}$. Finally,
$$\|u(t)\|_{X^l} \leqslant C\|u_0\|_{X^l} + \int_0^t C\tilde{M} \|u(s)\|_{X^l} ds$$
and Gronwall's lemma concludes as usual that $\|u(t)\|_{X^l}$ can not blow up at $T^*(l)$, which is a contradiction with the definition of $T^*(l)$. $\square$

4.5 Conserved quantities and global well-posedness of NLS in $X^k(\mathbb{R}^n)$

We first show the conservation of the renormalized energy for (4.1), for a "regular" initial data in $X^k(\mathbb{R}^n)$, in the case $n = 1$ or 2.

Proposition 4.5.1 *Let $n = k = 1, 2$, $u_0 \in X^{k+2}(\mathbb{R}^n)$, $f \in C^{k+1}(\mathbb{R}_+)$.*
We denote by $u \in C(T_*, T^*, X^{k+2}(\mathbb{R}^n)) \cap C^1(T_*, T^*, X^k(\mathbb{R}^n))$ the solution to the Cauchy problem (4.1) obtained in theorem 4.4.3, where (T_*, T^*) is its maximal existence interval.
Let $V(r) := -\int^r f(s)ds$. We assume that $\int_{\mathbb{R}^n} V(|u_0(x)|^2)dx$ converges (in a sense we will precise in the proof).
Then for all $t \in (T_*, T^*)$, $\int_{\mathbb{R}^n} V(|u(t,x)|^2)dx$ converges (in the same sense), and the energy is conserved:

$$\int_{\mathbb{R}^n} \left[|\nabla u(t,x)|^2 + V(|u(t,x)|^2)\right] dx = \int_{\mathbb{R}^n} \left[|\nabla u_0(x)|^2 + V(|u_0(x)|^2)\right] dx \tag{4.29}$$

proof. Let us multiply (4.1) by $\bar{u_t}$ and take the real part:

$$2\mathfrak{Re}(\Delta u \bar{u_t}) = \partial_t V(|u|^2) \ .$$

We fix $t \in (T_*, T^*)$, $x \in \mathbb{R}^n$ and integrate over $[0, t]$:

$$2\mathfrak{Re} \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds \quad = \quad V(|u(t,x)|^2) - V(|u_0(x)|^2) \tag{4.30}$$

We choose a non-increasing function $\theta \in C_c^\infty(\mathbb{R})$ such that:

$$\theta(x) = \begin{cases} 1 & if \quad x \leqslant 1 \\ 0 & if \quad x \geqslant 2 \end{cases} .$$

For $R > 0$, we define $\theta_R(x) := \theta(|x|/R)$, $x \in \mathbb{R}^n$. We have then:

$$\|\nabla \theta_R\|_{L^2(\mathbb{R}^n)} = R^{n/2-1} \left(\int_{\mathbb{R}^n} |\theta'(|y|)|^2 dy \right)^{1/2} .$$

The last integral is finite because $\theta'(|.|) \in C_c^\infty(\mathbb{R}^n)$. In particular, $\{\nabla \theta_R\}_{R \geqslant 1}$ is bounded in $L^2(\mathbb{R}^n)$.

We now multiply (4.30) by $\theta_R(x)$ and integrate over $\mathbb{R}^n$:

$$\begin{aligned} &2\mathfrak{Re} \int_{\mathbb{R}^n} \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds\theta_R(x)dx \\ &= \int_{\mathbb{R}^n} V(|u(t,x)|^2)\theta_R(x)dx - \int_{\mathbb{R}^n} V(|u_0(x)|^2)\theta_R(x)dx \ . \end{aligned} \tag{4.31}$$

Using Fubini's theorem and an integration by parts, we calculate the left hand side of (4.31):

$$\begin{aligned} 2\mathfrak{Re} \int_{\mathbb{R}^n} \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds\theta_R(x)dx \quad = \quad &-\int_{\mathbb{R}^n} \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right) \theta_R(x)dx \\ &-2\mathfrak{Re} \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x)\nabla\theta_R(x)\bar{u_t}(s,x)dxds \end{aligned}$$

For convenience, we assume $t > 0$. Since $\nabla\theta_R$ is supported in $\{x, |x| \geqslant R\}$, the Cauchy Schwarz inequality implies

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x) \nabla\theta_R(x) \bar{u_t}(s,x) dx ds \right| \\
&\qquad \leqslant \sup_{s\in[0,t]} \|u_t(s)\|_{L^\infty} \left(\int_0^t \|\nabla u(s)\|_{L^2(|x|\geqslant R)} ds \right) \|\nabla\theta_R\|_{L^2(\mathbb{R}^n)} \ ,
\end{aligned}$$

and this last quantity converges to 0 as $R \to \infty$, because $\|\nabla\theta_R\|_{L^2}$ is bounded. Moreover, $\int_{\mathbb{R}^n} \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right) \theta_R(x) dx$ clearly converges to $\|\nabla u(t)\|_{L^2}^2 - \|\nabla u_0\|_{L^2}^2$ as $R \to \infty$. Hence, if we assume that

$$\int_{\mathbb{R}^n} V(|u_0(x)|^2) dx := \lim_{R\to\infty} \int_{\mathbb{R}^n} V(|u_0(x)|^2) \theta_R(x) dx$$

exists (it a priori depends on the choice of θ), taking the limit in (4.31), for all $t \in (T_*, T^*)$, we obtain that

$$\int_{\mathbb{R}^n} V(|u(t,x)|^2) dx := \lim_{R\to\infty} \int_{\mathbb{R}^n} V(|u(t,x)|^2) \theta_R(x) dx$$

exists, and the energy is conserved. □

Remark This proof does not seem to work for $n \geqslant 3$, because we have used the existence of $\{\theta_R\}_{R\geqslant 1} \subset C_c^\infty(\mathbb{R}^n)$, with $\theta_R(x) \equiv 1$ for $|x| \leqslant R$ and $\nabla\theta_R$ bounded in $L^2(\mathbb{R}^n)$, and it can be shown that such a sequence does not exist for $n \geqslant 3$.

Next, we give a variant of theorem 4.5.1 in dimension 1, that will be useful later.

Proposition 4.5.2 *Let $\varphi,\ \varphi_0 : \mathbb{R} \to \mathbb{R}$ non-increasing functions of class C^∞, such that*

$$\varphi(x) = \varphi_0(x) \equiv \left\{ \begin{array}{lll} 1 & if & x \leqslant 0 \\ 0 & if & x \geqslant 1 \end{array} \right. .$$

For $R > 0$, we define $\varphi_R^+(x) = \varphi(x-R)\varphi_0(-x)$ and $\varphi_R^-(x) = \varphi_R^+(-x)$.
Then if $\lim_{R\to\infty} \int_{\mathbb{R}} V(|u_0(x)|^2)\varphi_R^\pm(x) dx$ exists, so does $\lim_{R\to\infty} \int_{\mathbb{R}} V(|u(t,x)|^2)\varphi_R^\pm(x) dx$.

Proof. It suffices to replace θ_R by $\varphi_R^\pm$ in the proof of theorem 4.5.1. We obtain in the same way that

$$\begin{aligned}
&\int_{\mathbb{R}} \left[V(|u(t,x)|^2) - V(|u_0(x)|^2) + \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2 \right) \right] \varphi_R^\pm(x) dx \\
&\quad = \ \mp 2\mathfrak{Re} \int_0^t \int_{\mathbb{R}} \nabla u(s,x) (\nabla\varphi(\pm x - R)\varphi_0(\mp x) - \varphi(\pm x - R)\nabla\varphi_0(\mp x)) \overline{u_t}(s,x) dx ds.
\end{aligned}$$

Passing to the limit as $R \to \infty$, the fact that $(\nabla\varphi(\pm x - R))_{R\geqslant 0}$ is bounded in L^2 and the assumption that $\int_{\mathbb{R}} V(|u_0(x)|^2)\varphi_R^{\pm}(x)dx$ converges as $R \to \infty$ imply that the limit as R goes to ∞ of $\int_{\mathbb{R}} V(|u(t,x)|^2)\varphi_R^{\pm}(x)dx$ exists, with

$$\begin{aligned}\lim_{R\to\infty}\int_{\mathbb{R}} V(|u(t,x)|^2)\varphi_R^{\pm}(x)dx &- \lim_{R\to\infty}\int_{\mathbb{R}} V(|u_0(x)|^2)\varphi_R^{\pm}(x)dx \\ = \ -\int_{\mathbb{R}} &\left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right)\varphi_0(\mp x)dx \\ &\pm 2\mathfrak{Re}\int_0^t\int_{\mathbb{R}} \nabla u(s,x)\nabla\varphi_0(\mp x)\overline{u_t}(s,x)dxds \ .\end{aligned}$$

Remark that the limit $\lim\limits_{R\to\infty}\int_{\mathbb{R}} V(|u(t,x)|^2)\varphi_R^{\pm}(x)dx$ depends on φ_0 but not on φ if the limit $\lim\limits_{R\to\infty}\int_{\mathbb{R}} V(|u_0(x)|^2)\varphi_R^{\pm}(x)dx$ does not. □

We want now to improve Proposition 4.5.1, to make it work for $u_0 \in X^k(\mathbb{R}^n)$, with $k > n/2$, and not only $k > n/2+2$. The price to pay is an extra assumption on the nonlinearity f:

Theorem 4.5.1 *Let $n = k = 1$ or 2.*
We suppose that $f \in C^{k+1}(\mathbb{R}_+)$ and that there exists some $\rho_0 > 0$ such that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. We define

$$V(r) := -\int_{\rho_0}^{r} f(s)ds \ .$$

If $n = 1$, we assume that $\{r,\ V(r) = 0\}$ is discrete, and if $n = 2$, we assume that V is non-negative on $\mathbb{R}_+$.
Let $0 < C_1 < 1 < C_2$, and $\delta_0 > 0$ ($\delta_0 < \rho_0$) such that

$$|r-\rho_0| \leqslant \delta_0 \Rightarrow C_1\frac{V''(\rho_0)}{2}(r-\rho_0)^2 \leqslant V(r) \leqslant C_2\frac{V''(\rho_0)}{2}(r-\rho_0)^2$$

(the assumptions on f ensure that such a δ_0 does exist).
Finally we assume that $u_0 \in X^k(\mathbb{R}^n)$, $x \to V(|u_0(x)|^2) \in L^1(\mathbb{R}^n)$ and there exists $0 < \delta_1 < \delta_0$ and $A > 0$ such that $|x| \geqslant A$ implies $||u_0(x)|^2 - \rho_0| \leqslant \delta_1$.
Then, denoting (T_*, T^*) the maximal existence interval of the solution of (4.1) associated to u_0, the energy

$$E(u(t)) \ := \ \int_{\mathbb{R}^n}\left[|\nabla u(t,x)|^2 + V(|u(t,x)|^2)\right]dx \tag{4.32}$$

is finite and conserved for all $t \in (T_*, T^*)$.

Examples. We give some examples for which Theorem 4.5.1 is valid

– the cubic defocusing NLS equation:

$$iu_t + \Delta u + (\rho_0 - |u|^2)u = 0, \quad x \in \mathbb{R}^n, \ n = 1 \text{ or } 2. \tag{4.33}$$

In this case, $f(r) = \rho_0 - r$, $V(r) = (\rho_0 - r)^2/2 \geqslant 0$, and the assumptions of theorem 4.5.1 are verified. In fact, here, the assumption of the existence of δ_1 and A can be relaxed, because it is a consequence of $V(|u_0|^2) \in L^1$.

– more generally, the "pure power case":

$$iu_t + \Delta u + \alpha(\rho_0^p - |u|^{2p})u = 0, \quad x \in \mathbb{R}^n, \ n = 1 \text{ or } 2. \tag{4.34}$$

where $\alpha > 0$ and $p \geqslant 1/2$ if $n = 1$, $p \geqslant 1$ if $n = 2$. Here, $f(r) = \alpha(\rho_0^p - r^p)$, $V(r) \geqslant 0$ and $V(r) = 0$ if and only if $r = \rho_0$. The assumption $f \in C^{k+1}(\mathbb{R}^+)$ in Theorem 4.5.1 is verified only if $p = 1$ or $p \geqslant 2$ in the one-dimensional case, and if $p = 1, 2$ or $p \geqslant 3$ in the case $n = 2$. However, in other cases, $X^k \ni u \to f(|u|^2)u \in X^k$ is of class C^1, and it suffices for the conclusion of Theorem 4.5.1.

– the cubic-quintic NLS equation:

$$iu_t + \Delta u - \alpha_1 u + \alpha_3 u|u|^2 - \alpha_5 u|u|^4 = 0, \ x \in \mathbb{R} \tag{4.35}$$

where $\alpha_1, \ \alpha_3, \ \alpha_5$ are positive constant such that $3/16 < \alpha_1\alpha_5/\alpha_3^2 < 1/4$. In this case, using some scale transformations (see [BGMP]), (4.35) can be rewritten as

$$iu_t + \Delta u + (|u|^2 - \rho_0)(2a + \rho_0 - 3|u|^2)u = 0 \tag{4.36}$$

with $0 < a < \rho_0$. Here, $f(r) = (r - \rho_0)(2a + \rho_0 - 3r)$ and $V(r) = (r - \rho_0)^2(r - a)$, and the assumptions of theorem 4.5.1 are verified. They are also verified for other values of the parameters $\alpha_1, \ \alpha_3, \ \alpha_5$ (for instance $a \leqslant 0$), but this seems to be less interesting from a physical point of view (see [BGMP]).

Proof of theorem 4.5.1. Let us take a mollifier sequence $(\rho_l)_{l\geqslant 1}$ (with $\int \rho_l = 1$, *Supp* $\rho_l \subset B(0, 1/l)$ and $\rho_l \geqslant 0$).

In a first step, we will control $|\rho_0 - |\rho_l * u_0(x)|^2|$ for l large and $|x| \geqslant A$.

$$|\rho_0 - |\rho_l * u_0(x)|^2| \quad \leqslant \quad (\sqrt{\rho_0} + \|u_0\|_\infty) \begin{cases} |\rho_l * u_0(x)| - \rho_0 & \text{if } |\rho_l * u_0(x)| \geqslant \sqrt{\rho_0} \quad \text{(case 1)} \\ \rho_0 - |\rho_l * u_0(x)| & \text{if } |\rho_l * u_0(x)| < \sqrt{\rho_0} \quad \text{(case 2)} \end{cases}$$

In case 1, we have:

$$\begin{aligned} 0 \leqslant |\rho_l * u_0(x)| - \sqrt{\rho_0} \quad &\leqslant \quad \int_{\mathbb{R}^n} \rho_l(x - y)(|u_0(y)| - \sqrt{\rho_0})dy \\ &\leqslant \quad \int_{\mathbb{R}^n} \rho_l(x - y)||u_0(y)| - \sqrt{\rho_0}|dy \ . \end{aligned} \tag{4.37}$$

In case 2, note that for $|x| \geqslant A$,

$$|u_0(x)|^2 \geqslant \rho_0 - |\rho_0 - |u_0(x)|^2| \geqslant \rho_0 - \delta_1 > 0 \ . \tag{4.38}$$

Let $\alpha := \sqrt{\rho_0 - \delta_1}$ and $v \in \mathbb{C}$ such that $|v| = 1$ and $v\overline{u_0(x)} \in i\mathbb{R}$. For $y \in \mathbb{R}^n$, the decomposition of $u_0(y)$ on the $\mathbb{R}$-basis $(u_0(x), v)$ of $\mathbb{C}$ can be written as

$$u_0(y) = \mathfrak{Re}[u_0(y)\overline{u_0(x)}]\frac{u_0(x)}{|u_0(x)|^2} + P(u_0(y))v \ .$$

Then, because of Pythagoras' theorem,

$$0 \leqslant \sqrt{\rho_0} - |\rho_l * u_0(x)| \quad \leqslant \quad \sqrt{\rho_0} - \left|\left(\int_{\mathbb{R}^n} \rho_l(x-y)\mathfrak{Re}[u_0(y)\overline{u_0(x)}]dy\right)\frac{u_0(x)}{|u_0(x)|^2}\right|$$

We choose p as in Proposition 4.2.2, and $l_0 \in \mathbb{N}^*$ such that

$$|x-y| \leqslant 1/l_0 \text{ implies } |u_0(x) - u_0(y)| \leqslant C|x-y|^{1-n/p}\|\nabla u_0\|_{L^p} \leqslant \alpha/2 \ .$$

So for $l \geqslant l_0$ and $y \in B(x, 1/l)$, we have

$$\begin{aligned} \mathfrak{Re}[u_0(y)\overline{u_0(x)}] &= |u_0(x)|^2 + \mathfrak{Re}[(u_0(y) - u_0(x))\overline{u_0(x)}] \\ &\geqslant |u_0(x)|^2 - |u_0(y) - u_0(x)\|u_0(x)| \geqslant \alpha^2/2 > 0 \ , \end{aligned}$$

and therefore, for $l \geqslant l_0$,

$$\begin{aligned} & 0 \leqslant \sqrt{\rho_0} - |\rho_l * u_0(x)| \\ &\leqslant \sqrt{\rho_0} - \int_{\mathbb{R}^n} \rho_l(x-y)\mathfrak{Re}[u_0(y)\overline{u_0(x)}]dy\frac{1}{|u_0(x)|} \\ &= \sqrt{\rho_0} - |u_0(x)| + \frac{1}{|u_0(x)|}\int_{\mathbb{R}^n} \rho_l(x-y)\mathfrak{Re}[(u_0(x) - u_0(y))\overline{u_0(x)}]dy \\ &\leqslant |\sqrt{\rho_0} - |u_0(x)|| + \int_{\mathbb{R}^n} \rho_l(x-y)|u_0(x) - u_0(y)|dy \ . \end{aligned} \tag{4.39}$$

Let $\varepsilon > 0$. Since $\rho_l * u_0 \to u_0$ in L^∞, there exists $l_1 \geqslant l_0$ such that $l \geqslant l_1$ implies $\||\rho_l * u_0|^2 - |u_0|^2\|_{L^\infty} \leqslant \delta_0 - \delta_1$ and then for $|x| \geqslant A$, $||\rho_l * u_0(x)|^2 - \rho_0| \leqslant \delta_0$. Therefore, for $|x| \geqslant A$ and $l \geqslant l_1$, we have:

$$0 \ \leqslant \ V(|\rho_l * u_0(x)|^2) \ \leqslant \ C_2\frac{V''(\rho_0)}{2}||\rho_l * u_0(x)|^2 - \rho_0|^2 \ . \tag{4.40}$$

Let $B \geqslant A+1$. (4.37), (4.39) and (4.40) imply, for $l \geqslant l_1$:

$$
\begin{aligned}
&\int_{|x|\geqslant B} V(|\rho_l * u_0(x)|^2)dx \\
&\leqslant C_2\frac{V''(\rho_0)}{2}(\sqrt{\rho_0}+\|u_0\|_\infty)^2 \\
&\quad \times \int_{|x|\geqslant B}\Bigg[\left(\int_{\mathbb{R}^n}\rho_l(x-y)||u_0(y)|-\sqrt{\rho_0}|dy\right)^2 \mathbf{1}_{|\rho_l*u_0(x)|-\sqrt{\rho_0}\geqslant 0} \\
&\quad + \left(|\sqrt{\rho_0}-|u_0(x)||+\int_{\mathbb{R}^n}\rho_l(x-y)|u_0(y)-u_0(x)|dy\right)^2 \mathbf{1}_{|\rho_l*u_0(x)|-\sqrt{\rho_0}\leqslant 0}\Bigg]dx \\
&\leqslant C_2\frac{V''(\rho_0)}{2}(\sqrt{\rho_0}+\|u_0\|_\infty)^2\int_{|x|\geqslant B}\Bigg[\int_{\mathbb{R}^n}\rho_l(x-y)||u_0(y)|-\sqrt{\rho_0}|^2dy \\
&\quad + 2||u_0(x)|-\sqrt{\rho_0}|^2+2\left(\int_{\mathbb{R}^n}\rho_l(x-y)|u_0(y)-u_0(x)|dy\right)^2\Bigg]dx\ . \qquad (4.41)
\end{aligned}
$$

We will now control each integral in the right hand side of (4.41). We begin by the first one:

$$
\begin{aligned}
\int_{|x|\geqslant B}\int_{\mathbb{R}^n}\rho_l(x-y)||u_0(y)|-\sqrt{\rho_0}|^2dydx &\leqslant \int_{|x|\geqslant B}\int_{|y|\geqslant B-1/l}\rho_l(x-y)\frac{||u_0(y)|^2-\rho_0|^2}{\rho_0}dydx \\
&\leqslant \frac{1}{\rho_0}\int_{|y|\geqslant B-1/l}||u_0(y)|^2-\rho_0|^2dy \\
&\leqslant \frac{2}{C_1\rho_0V''(\rho_0)}\int_{|y|\geqslant B-1/l}V(|u_0|^2)dy\ . \qquad (4.42)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\int_{|x|\geqslant B}2|\sqrt{\rho_0}-|u_0(x)||^2dx &\leqslant \frac{2}{\rho_0}\int_{|x|\geqslant B}|\rho_0-|u_0(x)|^2|^2dx \\
&\leqslant \frac{4}{C_1\rho_0V''(\rho_0)}\int_{|x|\geqslant B}V(|u_0|^2)dx\ . \qquad (4.43)
\end{aligned}
$$

It is a bit more difficult to find an upper bound to the third one. We provisionally admit the following lemma:

Lemma 4.5.1 *Let $n \in \mathbb{N}^*$, $k > n/2$ and $u_0 \in X^k(\mathbb{R}^n)$. Then for all (x,y), the function $f:[0,1]\to\mathbb{C}$, $t \to u_0(x+t(y-x))$ is absolutely continuous.*

Thanks to this lemma, we can write:

$$
|u_0(y)-u_0(x)| = |\int_0^1 (y-x).\nabla u_0(x+t(y-x))dt|\ ,
$$

and

$$\begin{aligned}
&\int_{|x|\geqslant B}\left(\int_{\mathbb{R}^n}\rho_l(x-y)|u_0(y)-u_0(x)|dy\right)^2dx\\
\leqslant\ & \int_{|x|\geqslant B}\int_{\mathbb{R}^n}\rho_l(x-y)|y-x|^2\int_0^1|\nabla u_0(x+t(y-x))|^2dtdydx\\
\leqslant\ & \frac{1}{l^2}\int_{|x|\geqslant B}\int_0^1\int_{\mathbb{R}^n}\rho_l(\frac{x-\tilde{y}}{t})|\nabla u_0(\tilde{y})|^2\frac{d\tilde{y}}{t^n}dxdt\\
\leqslant\ & \frac{1}{l^2}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho_l(\tilde{x})|\nabla u_0(\tilde{y})|^2d\tilde{x}d\tilde{y}dt=\frac{\|\nabla u_0\|_{L^2}}{l^2} \qquad (4.44)
\end{aligned}$$

(here we have made successively the changes of variables $\tilde{y}=ty+(1-t)x$ and $\tilde{x}=\frac{x-\tilde{y}}{t}$). By possibly enlarging B one may assume that

$$(\sqrt{\rho_0}+\|u_0\|_\infty)^2\frac{C_2}{C_1\rho_0}\left[\int_{|y|\geqslant B-1}V(|u_0|^2)dy+2\int_{|x|\geqslant B}V(|u_0|^2)dx\right]\ \leqslant\ \varepsilon/2\ . \qquad (4.45)$$

We also choose $l_2\geqslant l_1$ such that

$$2C_2\frac{V''(\rho_0)}{2}(\sqrt{\rho_0}+\|u_0\|_\infty)^2\frac{\|\nabla u_0\|_{L^2}}{l_2^2}\ \leqslant\ \varepsilon/2\ . \qquad (4.46)$$

Combining (4.41), (4.42), (4.43), (4.44), (4.45), (4.46), we obtain finally, for $l\geqslant l_2$,

$$\int_{|x|\geqslant B}V(|(\rho_l*u_0)(x)|^2)dx\leqslant\varepsilon\ .$$

In particular, $V(|(\rho_l*u_0)|^2)\in L^1(\mathbb{R}^n)$. By possibly enlarging B one may assume that

$$\int_{|x|\geqslant B}V(|u_0(x)|^2)dx\leqslant\varepsilon\ .$$

Moreover, there exists $l_3\geqslant l_2$ such that for $l\geqslant l_3$,

$$\int_{|x|\leqslant B}|V(|(\rho_l*u_0)(x)|^2)-V(|u_0(x)|^2)|dx\leqslant\varepsilon$$

because $\rho_l*u_0\to u_0$ in L^∞. The last three inequalities and the triangular inequality show that for $l\geqslant l_3$,

$$\int_{\mathbb{R}^n}|V(|(\rho_l*u_0)(x)|^2)-V(|u_0(x)|^2)|dx\leqslant 3\varepsilon$$

and therefore $V(|\rho_l*u_0|^2)\to V(|u_0|^2)$ in L^1.

For $l\in\mathbb{N}^*$, we denote by $u_l(t)$ the solution of (4.1) with initial data $u_l(0)=\rho_l*u_0$, and we denote by $(T_*(l),T^*(l))$ its maximal existence interval. Theorem 4.5.1 ensures that for $t\in(T_*(l),T^*(l))$

$$\int_{\mathbb{R}^n}\left[|\nabla u_l(t)|^2+V(|u_l(t)|^2)\right]dx\ =\ \int_{\mathbb{R}^n}\left[|\nabla\rho_l*u_0|^2+V(|\rho_l*u_0|^2)\right]dx\ . \qquad (4.47)$$

Let $T_* < \tilde{T}_1 < \tilde{T}_2 < T^*$. By continuity with respect to the initial data (see for example [CH]), there exists $K > 0,\ \delta > 0$ such that $\|\rho_l * u_0 - u_0\|_{X^k} \leqslant \delta$ implies $T^*(l) > \tilde{T}_2,\ T_*(l) < \tilde{T}_1$, and

$$\|u_l(t) - u(t)\|_{X^k} \leqslant K\|\rho_l * u_0 - u_0\|_{X^k},\ \text{for } t \in [\tilde{T}_1, \tilde{T}_2]\ .$$

In particular, since $\rho_l * u_0 \to u_0$ in X^k, we have that $\nabla u_l(t) \to \nabla u(t)$ in L^2. Moreover, $\nabla\rho_l * u_0 = \rho_l * \nabla u_0 \to \nabla u_0$ in L^2 and we have already shown that $V(|\rho_l * u_0|^2) \to V(|u_0|^2)$ in L^1. In order to take the limit as $l \to \infty$ in (4.47), it remains to take care of the term $\int_{\mathbb{R}^n} V(|u_l(t)|^2)dx$. We distinguish the cases $n = 2$ and $n = 1$.

In the case $n = 2$ one has $V \geqslant 0$ and for $l \geqslant l_3$, $x \to V(|u_l(t)|^2)$ is a L^1 function and $\int_{\mathbb{R}^n} V(|u_l(t)|^2)dx$ has a limit as $l \to \infty$. We apply Fatou's lemma to the sequence $(V(|u_l(t,.)|^2))_{l\in\mathbb{N}}$. We already know that it is bounded in L^1, and the continuity with respect to the initial data ensures that $u_l(t) \to u(t)$ in X^k. Therefore $V(|u_l(t)|^2) \to V(|u(t)|^2)$ in L^∞. Thus $V(|u(t,.)|^2) \in L^1$ and

$$\int V(|u(t,x)|^2)dx \leqslant \liminf_{l\to\infty} \int V(|u_l(t,x)|^2)\ .$$

We can now pass to the limit in (4.47):

$$\int_{\mathbb{R}^n} \left[|\nabla u(t)|^2 + V(|u(t)|^2)\right] dx \quad\leqslant\quad \int_{\mathbb{R}^n} \left[|\nabla u_0|^2 + V(|u_0|^2)\right] dx\ . \tag{4.48}$$

By reversing time, we get the inverse inequality, and the conservation of the energy (4.29) has been proved.

In the case $n = 1$, we also show that $V(|u_l(t,.)|^2) \in L^1$. For $l \geqslant l_3$, we know that $V(|\rho_l * u_0|^2) \in L^1(\mathbb{R})$. Hence for all choice of φ as in Proposition 4.5.2,

$$\lim_{R\to\infty} \int_{\mathbb{R}} V(|\rho_l * u_0(x)|^2)\varphi_R^{\pm}(x)dx = \int_{\mathbb{R}} V(|\rho_l * u_0(x)|^2)\varphi_0(\mp x)dx\ ,$$

and Proposition 4.5.2 ensures that $\lim_{R\to\infty} \int_{\mathbb{R}} V(|u_l(t,x)|^2)\varphi_R^{\pm}(x)dx$ exists and does not depend on the choice of φ. This implies that $V(|u_l(t,x)|^2) \to 0$ as $x \to \pm\infty$. Indeed, we have the following lemma (we will prove it later):

Lemma 4.5.2 *Let $f : \mathbb{R} \to \mathbb{R}$ an uniformly continuous function such that for all φ as in Proposition 4.5.2, $\lim_{R\to\infty} \int_{\mathbb{R}} f(x)\varphi_R^{\pm}(x)dx$ exists. Then $f(x) \underset{x\to\pm\infty}{\to} 0$.*

Since $\{r, V(r) = 0\}$ is discrete and $u_l(t) \in C_b(\mathbb{R})$, there exists $r^l_{\pm}(t)$ such that

$$V(r^l_{\pm}(t)) = 0 \quad \text{and} \quad |u_l(t,x)|^2 \underset{x\to\infty}{\to} r^l_{\pm}(t)\ .$$

Let us show that $r^l_{\pm}$ is continuous on $]T_*(l), T^*(l)[$. Let $t \in]T_*(l), T^*(l)[$, h so that $t + h \in]T_*(l), T^*(l)[$, $\varepsilon > 0$ and $x \in \mathbb{R}$. One has

$$|r^l_{\pm}(t+h) - r^l_{\pm}(t)| \leqslant |r^l_{\pm}(t+h) - u_l(t+h,x)| + |u_l(t+h,x) - u_l(t,x)| + |u_l(t,x) - r^l_{\pm}(t)|\ .$$

$u_l \in C(]T_*(l),T^*(l)[,X^k) \subset C(]T_*(l),T^*(l)[,L^\infty)$, thus we can choose h small enough in order that

$$\|u_l(t+h)-u_l(t)\|_{L^\infty} \leqslant \varepsilon/3 \ .$$

We also choose $|x|$ large enough such that

$$|r^l_\pm(t+h)-u_l(t+h,x)| \leqslant \varepsilon/3 \quad \text{and} \quad |u_l(t,x)-r^l_\pm(t)| \leqslant \varepsilon/3 \ .$$

Hence $|r^l_\pm(t+h)-r^l_\pm(t)| \leqslant \varepsilon$. Therefore $r^l_\pm$ is continuous with value in a discrete set, which means that it is constant. For $l \geqslant l_3$, the fact that $\||\rho_l * u_0|^2 - |u_0|^2\|_{L^\infty} \leqslant \delta_0 - \delta_1$ and the assumption on u_0 imply that $r^l_\pm(0) = \rho_0$, hence

$$\forall l \geqslant l_3,\ \forall t \in]T_*(l),T^*(l)[,\ |u_l(t,x)|^2 \underset{x\to\pm\infty}{\longrightarrow} \rho_0 \ .$$

We choose $l_4 \geqslant l_3$ such that for $l \geqslant l_4$,

$$\begin{aligned}
&\||u_l(t)|^2-|u(t)|^2\|_{L^\infty} \\
&\quad\leqslant\ \|u_l(t)-u(t)\|_{L^\infty}(\|u_l(t)-u(t)\|_{L^\infty}+2\|u(t)\|_{L^\infty}) \\
&\quad\leqslant\ K\|\rho_l * u_0 - u_0\|_{X^k}(K\|\rho_l * u_0-u_0\|_{X^k} + 2\sup_{t\in[\tilde{T}_1,\tilde{T}_2]}\|u(t)\|_{X^k}) \leqslant \frac{\delta_0-\delta_1}{2}.
\end{aligned}$$

Let $D>0$ such that $|x| \geqslant D$ implies $||u_{l_4}(t,x)|^2-\rho_0| \leqslant \delta_1$. Then for $l \geqslant l_4,\ |x| \geqslant D$,

$$\begin{aligned}
&||u_l(t,x)|^2-\rho_0| \\
&\quad\leqslant\ ||u_l(t,x)|^2-|u(t,x)|^2| + ||u(t,x)|^2-|u_{l_4}(t,x)|^2| + ||u_{l_4}(t,x)|^2-\rho_0| \\
&\quad\leqslant\ 2\frac{\delta_0-\delta_1}{2}+\delta_1 = \delta_0
\end{aligned}$$

This implies that $V(|u_l(t,x)|)$ is non-negative on $\{x, |x| \geqslant D\}$, and now (4.47) yields:

$$\begin{aligned}
&\int_{|x|\geqslant D} V(|u_l(t,x)|^2)dx \\
&\quad=\ \int_{\mathbb{R}} \left[|\nabla \rho_l * u_0|^2 + V(|\rho_l * u_0|^2)\right] dx - \int_{\mathbb{R}} |\nabla u_l(t,x)|^2 dx - \int_{|x|\leqslant D} V(|u_l(t,x)|^2)dx \\
&\underset{l\to\infty}{\longrightarrow}\ \int_{\mathbb{R}} \left[|\nabla u_0|^2 + V(|u_0|^2)\right] dx - \int_{\mathbb{R}} |\nabla u(t,x)|^2 dx - \int_{|x|\leqslant D} V(|u(t,x)|^2)dx \ ,
\end{aligned}$$

and $\left(V(|u_l(t,.)|^2)\mathbf{1}_{\{|x|\geqslant D\}}\right)_{l\geqslant l_4}$ is bounded in L^1. We apply Fatou's lemma to this sequence, and we can conclude similarly to the case $n=2,\ V \geqslant 0$. $\square$

To complete the proof of theorem 4.5.1, it just remains to prove lemmas 4.5.1 and 4.5.2.

Proof of lemma 4.5.2. We argue by contradiction. Assume that there exists $\varepsilon > 0$ such that for all $A > 0$, there exists $x > A$ such that $|f(x)| > \varepsilon$. Since f is uniformly continuous, there exists $\delta \in (0, 2)$, such that $|x - y| < \delta$ implies $|f(x) - f(y)| \leqslant \varepsilon/2$. We may thus construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to \infty$ and $|f(y)| \geqslant \varepsilon/2$ as soon as $|y - x_n| \leqslant \delta$. We may assume moreover that the intervals $]x_n - \delta, x_n + \delta[$ are disjoint, and that for instance $f(x_n) > 0$. We choose φ as in proposition 4.5.2, namely

$$\varphi(x) = \begin{cases} 1 & if \quad x \leqslant 0 \\ 0 & if \quad x \geqslant \delta/2 \end{cases} .$$

Then

$$\begin{aligned} \left| \int_{\mathbb{R}} f(y)\varphi_{x_n-\delta}(y)dy - \int_{\mathbb{R}} f(y)\varphi_{x_n+\delta/2}(y)dy \right| &= \int_{\mathbb{R}} f(y)(\varphi_{x_n+\delta/2}(y) - \varphi_{x_n-\delta}(y))dy \\ &\geqslant \int_{x_n-\delta/2}^{x_n+\delta/2} f(y)dy \geqslant \delta\varepsilon/2 \end{aligned}$$

and this is a contradiction. □

Proof of lemma 4.5.1. Let us fix $x, y \in \mathbb{R}^n$ with $x \neq y$. It suffices to show that $f : t \in [0, 1] \to u_0(x + t(y - x))$ is absolutely continuous. By definition of X^k, u_0 is the limit in X^k of a sequence $(v_l)_{l\in\mathbb{N}}$ of functions of class C^k. Therefore the functions $f_l : t \in [0, 1] \to v_l(x + t(y - x))$ are absolutely continuous.

We are reduced to prove that the moduli of absolute continuity of f_l's are uniformly bounded. Indeed, if we assume this fact, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all positive integer m, and for all choice of $(\alpha_j, \beta_j)_{1\leqslant j\leqslant m}$ with $0 \leqslant \alpha_1 < \beta_1 < ... < \alpha_m < \beta_m \leqslant 1$ and $\sum_{j=1}^{m}(\beta_j - \alpha_j) \leqslant \delta$, we have for all l,

$$\sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| \leqslant \varepsilon/2 .$$

Then

$$\sum_{j=1}^{m} |f(\beta_j) - f(\alpha_j)| \leqslant \sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| + \sum_{j=1}^{m} [|f(\beta_j) - f_l(\beta_j)| + |f(\alpha_j) - f_l(\alpha_j)|]$$

choosing l large enough, $\|f - f_l\|_\infty \leqslant \frac{\varepsilon}{4m}$. We infer that

$$\sum_{j=1}^{m} |f(\beta_j) - f(\alpha_j)| \leqslant \varepsilon ,$$

and therefore f is absolutely continuous.

We now prove that the moduli of absolute continuity of f_l's are uniformly bounded. Since $v_l \to u_0$ in X^k, v_l is bounded in X^k. We choose m, α_j, β_j as above. For all l, we have:

$$\begin{aligned}
\sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| &= \sum_{j=1}^{m} \left| \int_{\alpha_j}^{\beta_j} f_l'(s)ds \right| \leqslant \int_{\overset{m}{\underset{j=1}{\cup}}]\alpha_j,\beta_j[} |f_l'(s)|ds \\
&\leqslant \left(\int_{\overset{m}{\underset{j=1}{\cup}}]\alpha_j,\beta_j[} ds \right)^{1/2} \left(\int_{\overset{m}{\underset{j=1}{\cup}}]\alpha_j,\beta_j[} |f_l'(s)|^2 ds \right)^{1/2} \\
&\leqslant \left(\sum_{j=1}^{m} (\beta_j - \alpha_j) \right)^{1/2} \left(\int_0^1 |f_l'(s)|^2 ds \right)^{1/2} . \qquad (4.49)
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_0^1 |f_l'(s)|^2 ds &= \int_0^1 |(y-x).\nabla v_l(x + s(y-x))|^2 ds \\
&\leqslant |y-x| \int_0^{|y-x|} \left| \nabla v_l \left(x + s\frac{y-x}{|y-x|} \right) \right|^2 ds .
\end{aligned}$$

Since $\nabla u_0 \in H^{k-1}(\mathbb{R}^n)$, the trace theorem ensures that the mapping that sends a function of $H^{k-1}(\mathbb{R}^n)$ on its trace on a line D of $\mathbb{R}^n$ is continuous from $H^{k-1}(\mathbb{R}^n)$ to $H^{(k-1)-(n-1)/2}(D) \subset L^2(D)$ because $k-1-(n-1)/2 \geqslant \lfloor n/2 \rfloor - (n-1)/2 \geqslant 0$. Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned}
\left(\int_0^1 |f_l'(s)|^2 ds \right)^{1/2} &\leqslant |y-x|^{1/2} \|\nabla v_l{}_{\ |\{x+t\frac{y-x}{|y-x|}, t\in\mathbb{R}\}}\|_{L^2(\{x+t\frac{y-x}{|y-x|}, t\in\mathbb{R}\})} \\
&\leqslant C|y-x|^{1/2} \|\nabla v_l\|_{H^{k-1}(\mathbb{R}^n)} \\
&\leqslant C|y-x|^{1/2} \sup_l \|v_l\|_{X^k(\mathbb{R}^n)} . \qquad (4.50)
\end{aligned}$$

Finally, (4.49) and (4.50) ensure that f_l's absolute continuity moduli are uniformly bounded. $\square$

Remark. The only reason for which Theorem 4.5.1 in the case $V \geqslant 0$ is not valid for $n \geqslant 3$ is that we did not prove Proposition 4.5.1 for $n \geqslant 3$. The whole proof of Theorem 4.5.1 with $V \geqslant 0$ would be valid for $n \geqslant 3$ if Proposition 4.5.1 was.

We justify next the conservation of the momentum, in the one dimensional case.

Theorem 4.5.2 *Let $n = k = 1$. The assumptions on the nonlinearity f are as in Theorem 4.5.1.*
Let $u_0 \in X^1(\mathbb{R})$ such that $V(|u_0|^2) \in L^1$ and $|u_0| \geqslant a > 0$.
Let $0 \in (\tilde{T}_*, \tilde{T}^*) \subset (T_*, T^*)$ be the maximal interval on which $|u(t,x)| > 0$, $t \in (\tilde{T}_*, \tilde{T}^*)$,

$x \in \mathbb{R}$.
Then $\forall t \in (\tilde{T}_*, \tilde{T}^*)$, $P(u(t)) = P(u_0)$, *where the renormalized momentum is given by:*

$$P(u) \quad = \quad \mathfrak{Im} \int_{-\infty}^{\infty} \frac{\overline{u}_x u}{|u|^2} (|u|^2 - \rho_0) \tag{4.51}$$

Proof. It follows from the proof of Theorem 4.5.1 that for $t \in (\tilde{T}_*, \tilde{T}^*)$, $|u(t)|^2 - \rho_0 \in L^2$ and that $|u(t)| \geqslant a(t)$, where $a(t) \in \mathbb{R}_+^*$ (because $|u(t,x)|^2 \underset{x\to\infty}{\longrightarrow} \rho_0$ and $|u(t,x)| > 0$). Moreover, $u_x(t,.) \in L^2$. Hence $P(u(t))$ is well defined.

If $u_0 \in X^3(\mathbb{R})$, the formal proof of the conservation of P on $(\tilde{T}_*, \tilde{T}^*)$ (see [L], [GSS]) is valid, since $u \in C(\tilde{T}_*, \tilde{T}^*, X^3(\mathbb{R})) \cap C^1(\tilde{T}_*, \tilde{T}^*, X^1(\mathbb{R}))$.

Proceeding as in the proof of theorem 4.5.1, we approach $u_0 \in X^1$ by $\rho_l * u_0 \in X^3$. Let us fix $t \in (\tilde{T}_*, \tilde{T}^*)$. Since $u_l(t) \to u(t)$ in X^1, for l large enough we have $|u_l(t)| \geqslant a(t)/2 > 0$, and then $(\tilde{T}_*, \tilde{T}^*) \subset (\tilde{T}_*(l), \tilde{T}^*(l))$. It follows from the proof of Theorem 4.5.1 that $(|u_l(t)|^2 - \rho_0) \to (|u(t)|^2 - \rho_0)$ in L^2 and $\partial_x u_l(t) \to \partial_x u(t)$ in L^2. Hence $P(u_l(t)) \to P(u(t))$, and the conservation of the momentum, which is true for u_l, is also true for u. $\square$

Remark. The above analysis fills a gap in the proof of Theorem 1.1 in [L]. Namely, Zhiwu Lin proves a criterion of stability for the traveling bubbles solution of NLS in the one dimensional case, for a nonlinearity f verifying:

(1) $f(\rho_0) = 0,\ \eta_0 = \sup\{\eta,\ 0 < \eta < \rho_0,\ V(\eta) = 0\}$ exists,
$0 < \eta_0 < \rho_0,\ f(\eta_0) < 0$
(2) $f'(\rho_0) < 0$

This proof consists in applying Theorem 3 in [GSS] to the hydrodynamical problem corresponding to (4.1) (i.e. with the complex unknown u replaced by $(r, v) = (\rho_0 - |u|^2, \partial_x \arg u)$). Theorem 4.5.1 and 4.5.2 ensure that Assumption 1 of Theorem 3 in [GSS] is verified. Namely, in a neighbourhood of the soliton, our results imply the following facts which were not discussed in [L]:

- the local existence for the hydrodynamical Cauchy problem with $(r, v) \in H^1 \times L^2$ (and not only $X^1 \times L^2$) if the energy is finite at initial time
- the conservation of energy and momentum.

In dimension 1, we prove next that the conservation of energy implies a global existence result for a solution of (4.1) in X^1.

Theorem 4.5.3 *Let* $n = k = 1$. *The assumptions on* f *and* u_0 *are as in theorem 4.5.1, and we assume moreover that there exists some* $C > 0$ *such that* $V(r) \geqslant C(\rho_0 - r)^2$.
Then $u \in C_b(\mathbb{R}, X^k)$.

Proof. We define the energy at initial time

$$E_0 = \int_{\mathbb{R}^n} \left[|\nabla u_0(x)|^2 + V(|u_0(x)|^2) \right] dx \ .$$

We know by Theorem 4.5.1 that the energy is conserved for $t \in (T_*, T^*)$ and that if T^* is finite, $\|u(t)\|_{X^k} \to \infty$ as $t \to T^*$ (and we have a similar result by replacing T^* by T_*) (see [CH]). The conservation of the energy and the fact that $V \geqslant 0$ imply that for all t, $\int_{\mathbb{R}} |\partial_x u(t)|^2 dx \leqslant E_0$. To prove that $\|u(t)\|_{X^k}$ can not blow up in finite time, it suffices then to show that $\|u(t)\|_{L^\infty}$ can not blow up in finite time. By the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$,

$$\begin{aligned} \|u(t)\|_{L^\infty}^2 &\leqslant \rho_0 + \||u(t)|^2 - \rho_0\|_{L^\infty} \\ &\leqslant \rho_0 + C\||u(t)|^2 - \rho_0\|_{H^1} \\ &= \rho_0 + C\sqrt{\||u(t)|^2 - \rho_0\|_{L^2}^2 + \|\partial_x(|u(t)|^2 - \rho_0)\|_{L^2}^2} \end{aligned}$$

We now use the additional assumption that $V(r) \geqslant C(1-r)^2$:

$$\begin{aligned} \|u(t)\|_{L^\infty}^2 &\leqslant \rho_0 + C\sqrt{E_0 + 4\|u(t)\|_{L^\infty}^2 \int |\partial_x u(t)|^2 dx} \\ &\leqslant \rho_0 + C\sqrt{E_0} + 2C\|u(t)\|_{L^\infty}\sqrt{E_0} \ . \end{aligned}$$

Therefore $\|u(t)\|_{L^\infty}$ can not blow up in finite time, thus $(u(t))_{t\in\mathbb{R}}$ is global. Moreover, $\|u(t)\|_{L^\infty}$ is bounded on $\mathbb{R}$ and therefore $(u(t))_{t\in\mathbb{R}}$ is bounded in $X^1(\mathbb{R})$. $\square$

Example. In NLS with a pure defocusing power (4.34), with $n = 1$ and $p \geqslant 1$, we have $V(r) \geqslant \alpha\rho_0^{p-1}(\rho_0 - r)^2/2$, hence the assumptions of Theorem 4.5.3 are verified.

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Chapitre 5

The growth rate of the Schrödinger group on Zhidkov spaces

Abstract. We give an upper bound on the growth rate of the Schrödinger group on Zhidkov spaces. In dimension 1, we prove that this bound is sharp.

5.1 Introduction

This paper is devoted to the study of the linear Schrödinger equation

$$\left\{ \begin{array}{l} i\frac{\partial u}{\partial t} + \Delta u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0) = u_0 \end{array} \right. \tag{5.1}$$

with non-zero boundary conditions at infinity. In [Ga1], we proved that (5.1) is well-posed on the Zhidkov spaces

$$X^k(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n), \nabla u \in H^{k-1}(\mathbb{R}^n)\},$$

under the condition $k > n/2$. Moreover, we gave a superior bound on the growth rate of the Schrödinger group on $X^k(\mathbb{R}^n)$, which will be denoted by $S(t)$. More precisely, we proved that there exists a constant $C > 0$ such that

$$\|S(t)\|_{\mathcal{L}(X^k(\mathbb{R}^n),X^k(\mathbb{R}^n))} \leqslant C(1 + |t|^\rho), \tag{5.2}$$

where

$$\rho = \left\{ \begin{array}{ll} 1/2 & \text{if } n \text{ is even} \\ 1/4 & \text{if } n \text{ is odd.} \end{array} \right. \tag{5.3}$$

The goal of this paper is to decrease as far as we can the upper bound on the growth rate of $S(t)$ given by (5.3), and to find the sharp exponent when it is possible .

We will first give a much more simpler proof of the well-posedness of (5.1) in $X^k(\mathbb{R}^n)$, which however yields to a worst estimation on the growth rate of $S(t)$ ($1/2$ instead of the exponent ρ given by (5.3)). It consists in seeing the well-posedness of (5.1) in $X^k(\mathbb{R}^n)$ as a corollary of the following result:

Theorem 5.1.1 *For every $u_0 \in Y := \{u \in \mathcal{S}'(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n)\}$,*

$$e^{it\Delta}u_0 \in u_0 + \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)),$$

and for every $u_0 \in Y$, for every $t \in \mathbb{R}$,

$$\|e^{it\Delta}u_0 - u_0\|_{H^1} \leqslant 2(1 + |t|^{1/2})\|\nabla u_0\|_{L^2}. \tag{5.4}$$

Moreover, the growth rate $1/2$ in (5.4) is sharp. Namely, for every $\rho < 1/2$, every $C > 0$, there exists a time $\tau > 0$ and $u_{0\tau} \in Y$ such that

$$\|e^{i\tau\Delta}u_{0\tau} - u_{0\tau}\|_{H^1} > C(1 + \tau^\rho)\|\nabla u_{0\tau}\|_{L^2}.$$

The sharpness of $\rho = 1/2$ in (5.4) does not imply the sharpness of $\rho = 1/2$ in (5.2) (hopefully, since it would be a contradiction with (5.2)-(5.3) for odd dimensions). Indeed, we will improve our demonstration of the well-posedness of (5.1) in $X^k(\mathbb{R}^n)$ given in [Ga1], and show the following result.

Theorem 5.1.2 *Let $k > d/2$. There exists a constant $C > 0$ such that for every $u_0 \in X^k(\mathbb{R}^n)$,*

$$\|S(t)u_0\|_{X^k(\mathbb{R}^n)} \leqslant C(1 + |t|^\rho)\|u_0\|_{X^k(\mathbb{R}^n)}, \tag{5.5}$$

where

$$\rho = \begin{cases} \frac{1}{2}\frac{1}{1+\frac{1}{n}} & \textit{if } n \textit{ is even} \\ \frac{1}{4}\frac{1}{1+\frac{1}{2n}} & \textit{if } n \textit{ is odd.} \end{cases} \tag{5.6}$$

Moreover, for $n = 1$, $\rho = 1/6$ is sharp.

Remark. The sharpness result in Theorem 5.1.2 should be understood as follows: for every $C > 0$, every $\rho < 1/6$, there exists $\tau > 0$ and $u_{0\tau} \in X^k(\mathbb{R}^n)$ such that

$$\|S(\tau)u_{0\tau}\|_{X^k(\mathbb{R}^n)} \geqslant \|S(\tau)u_{0\tau}\|_{L^\infty(\mathbb{R}^n)} > C(1 + \tau^\rho)\|u_{0\tau}\|_{X^k(\mathbb{R}^n)}.$$

The next section is devoted to the proof of Theorem 5.1.1, while Theorem 5.1.2 is proven in the following one.

Notations. If $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$ denotes the Fourier transform of f.
In all this paper, C denotes a harmless positive constant which can change from line to line.

5.2 The Schrödinger propagator on Y

In this section, we prove Theorem 5.1.1. We first prove (5.4), after what we show the sharpness of exponent 1/2.

5.2.1 Proof of inequality (5.4).

Let $u_0 \in Y$, and $u(t) = e^{it\Delta}u_0$ the solution to (5.1) with initial data u_0. Then for every $t \neq 0$,

$$\begin{aligned} \widehat{u}(t,\xi) - \widehat{u_0}(\xi) &= (e^{-it|\xi|^2} - 1)\widehat{u_0}(\xi) \\ &= \sum_{j=1}^{n} \frac{(e^{-it|\xi|^2} - 1)\xi_j}{\sqrt{|t|}|\xi|^2}\sqrt{|t|}\xi_j \hat{u_0}(\xi). \end{aligned} \tag{5.7}$$

$\xi_j \widehat{u_0} \in L^2$ because $\nabla u_0 \in L^2$, and it follows from the mean value theorem that

$$\left| \frac{(e^{-it|\xi|^2} - 1)}{\sqrt{|t|}|\xi|} \right| \leqslant 2.$$

Thus by the Cauchy-Schwarz inequality and the Plancherel theorem, we get that $u(t) - u_0 \in L^2$, and

$$\|u(t) - u_0\|_{L^2} \leqslant 2\sqrt{|t|}\|\nabla u_0\|_{L^2}.$$

Since $e^{it\Delta}$ is isometric on L^2, it is clear that

$$\|\nabla(u(t) - u_0)\|_{L^2} \leqslant 2\|\nabla u_0\|_{L^2},$$

and (5.4) follows. □

From this result we easily recover the well-posedness of (5.1) on $X^k(\mathbb{R}^n)$, if $k > n/2$.

Corollary 5.2.1 *If $k > n/2$ and $u_0 \in X^k(\mathbb{R}^n)$, then $u(t) = S(t)u_0 = e^{it\Delta}u_0 \in \mathcal{C}(\mathbb{R}, X^k)$, with the estimate*

$$\|S(t)u_0\|_{X^k(\mathbb{R}^n)} \leqslant C(1 + |t|^{1/2})\|u_0\|_{X^k(\mathbb{R}^n)},$$

for some positive constant C.

Proof. Indeed, we have just seen that $u(t) - u_0 \in \mathcal{C}(\mathbb{R}, H^k)$, with

$$\|u(t) - u_0\|_{H^k} \leqslant 2(1 + \sqrt{|t|})\|\nabla u_0\|_{H^{k-1}}.$$

Since $k > n/2$, we have by Sobolev $H^k \subset X^k$ with continuous embedding, and the announced estimate is true. □

5.2.2 Proof of the sharpness of exponent $1/2$ in (5.4)

We are going to prove the following fact: for every $\rho < 1/2$, there exists $t_\rho > 0$, $C_\rho > 0$ such that for every $\tau \geqslant t_\rho$, there exists $u_{0\tau} \in Y$ such that

$$\|u_\tau(\tau) - u_{0\tau}\|_{L^2(\mathbb{R}^n)} \quad > \quad C_\rho \tau^\rho \|\nabla u_{0\tau}\|_{L^2(\mathbb{R}^n)}, \tag{5.8}$$

where $u_\tau(t) = e^{it\Delta}u_{0\tau}$. Let us first show that (5.8) implies the wanted result. Let $\rho < 1/2$, $C > 0$ and $\tilde{\rho} \in (\rho, 1/2)$. If (5.8) is true, then for every $\tau \geqslant t_{\tilde{\rho}}$ large enough, since $\tilde{\rho} > \rho$, there exists $u_{0\tau}$ such that

$$\|u_\tau(\tau) - u_{0\tau}\|_{L^2} > C_{\tilde{\rho}}\tau^{\tilde{\rho}}\|\nabla u_{0\tau}\|_{L^2} \geqslant C(1+\tau^\rho)\|\nabla u_{0\tau}\|_{L^2},$$

which exactly means that $1/2$ is sharp in (5.4). Let us next prove (5.8).

Let $\rho < 1/2$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a radial function such that $0 \leqslant \chi \leqslant 1$ and

$$\chi(x) \equiv \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases} \tag{5.9}$$

For $\tau > 0$, we define $\xi(\tau) = 1/\tau^{1-\rho}$, as well as $u_{0\tau} \in \mathcal{S}(\mathbb{R}^n)$, by

$$\widehat{u_{0\tau}}(\xi) = m(\tau)\chi\left(\frac{\xi}{\xi(\tau)}\right),$$

where $m(\tau) > 0$ is chosen such that $\|\nabla u_{0\tau}\|_{L^2} = 1$, which yields

$$m(\tau) = \frac{1}{\sqrt{\int_{|\eta|\leqslant 2}|\eta|^2\chi(\eta)^2 d\eta}}\frac{1}{\xi(\tau)^{1+n/2}} = \frac{C}{\xi(\tau)^{1+n/2}}.$$

Then, using the Plancherel theorem, a Taylor expansion and the definition of $u_{0\tau}$,

$$\begin{aligned}
\|u_\tau(\tau) - u_{0\tau}\|_{L^2}^2 &= \int_{\mathbb{R}^n} |e^{-i\tau|\xi|^2} - 1|^2 |\widehat{u_{0\tau}}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n} \left| -i\tau|\xi|^2 - \tau^2|\xi|^4 \int_0^1 (1-s)e^{-is\tau|\xi|^2}ds \right|^2 |\widehat{u_{0\tau}}(\xi)|^2 d\xi \\
&= \tau^2 m(\tau)^2 \int_{\mathbb{R}^n} |\xi|^4 \chi\left(\frac{\xi}{\xi(\tau)}\right)^2 d\xi
\end{aligned}$$
$$+\tau^2 m(\tau)^2 \int_{\mathbb{R}^n} |\xi|^4 \left[2\mathfrak{Re}\left(i\tau|\xi|^2 \int_0^1 (1-s)e^{-is\tau|\xi|^2}ds \right) + \tau^2|\xi|^4 \left| \int_0^1 (1-s)e^{-is\tau|\xi|^2}ds \right|^2 \right] \chi\left(\frac{\xi}{\xi(\tau)}\right)^2 d\xi$$
$$=: \ A_0(\tau) + B_0(\tau).$$

We will see that $A_0(\tau) \gg |B_0(\tau)|$ for τ large. On the one side, passing in polar coordinates and taking into account the definitions of $m(\tau)$ and $\xi(\tau)$,

$$
\begin{aligned}
A_0(\tau) &\geqslant \tau^2 m(\tau)^2 \int_{|\xi|\leqslant\xi(\tau)} |\xi|^4 d\xi = \tau^2 m(\tau)^2 \int_0^{\xi(\tau)} r^4 \int_{\mathbb{S}^{n-1}} dv r^{n-1} dr \\
&= \frac{|\mathbb{S}^{n-1}|C^2}{n+4} \tau^2 \xi(\tau)^2 \\
&= C\tau^2\xi(\tau)^2 = C\tau^{2\rho}. \qquad (5.10)
\end{aligned}
$$

On the other side, our choice of $m(\tau)$ and $\xi(\tau)$ imply

$$
\begin{aligned}
|B_0(\tau)| &\leqslant \tau^2 m(\tau)^2 \int_{|\xi|\leqslant 2\xi(\tau)} \left[2\tau|\xi|^6 + \tau^2|\xi|^8\right] d\xi \\
&\leqslant \tau^2 m(\tau)^2 |\mathbb{S}^n| (2\xi(\tau))^n [2\tau|\xi(\tau)|^6 + \tau^2|\xi(\tau)|^8] \\
&\leqslant C(\tau^{4\rho-1} + \tau^{6\rho-2}) = o(\tau^{2\rho}), \qquad (5.11)
\end{aligned}
$$

where we have used that $\rho < 1/2$. (5.8) follows from (5.10) and (5.11), and because $\|\nabla u_0\|_{L^2} = 1$. This completes the proof of Theorem 5.1.1. □

5.3 Growth rate of the Schrödinger propagator on X^k.

This section is devoted to the proof of Theorem 5.1.2. First, we revisit and improve the proof of the well-posedness of (5.1) on $X^k(\mathbb{R}^n)$ we gave in [Ga1], in order to verify (5.5)-(5.6). Then, we show that the growth rate is 1/6 in dimension 1.

5.3.1 Proof of (5.5)-(5.6).

Given $u_0 \in X^k(\mathbb{R}^n)$, the proof of (5.2)-(5.3) we gave in [Ga1] consists in giving a sense to the limit as $\varepsilon \to 0$ of

$$
I_\varepsilon := \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x + 2\sqrt{t}z) dz
$$

(for convenience, we will assume from now on $t > 0$). Up to the multiplication by a harmless constant, this limit is showed to be the solution $u(t)$ to (5.1) with initial data u_0.

Using the radial cut-off function χ introduced in (5.9), we write $\psi = 1 - \chi$, and we define, for $\beta > 0$, $\chi_\beta(z) = \chi(z/\beta)$, $\psi_\beta(z) = \psi(z/\beta)$. For convenience, we will also use the notation $\chi_\beta(|.|) = \chi_\beta(.)$ and $\psi_\beta(|.|) = \psi_\beta(.)$.

We split I_ε into two parts: $I_\varepsilon = A_\varepsilon + B_\varepsilon$, where

$$
A_\varepsilon = \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_\beta(z) u_0(x + 2\sqrt{t}z) dz
$$

and

$$
B_\varepsilon = \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \psi_\beta(z) u_0(x + 2\sqrt{t}z) dz.
$$

Using Lebesgue's convergence theorem, it is easy to see that A_ε has a limit as ε goes to 0, and that

$$\left|\lim_{\varepsilon\to 0} A_\varepsilon\right| \leqslant \|u_0\|_{L^\infty}(2\beta)^n|\mathcal{B}(0,1)|, \tag{5.12}$$

where $\mathcal{B}(0,1)$ denotes the unit ball in $\mathbb{R}^n$. Next, passing in polar coordinates, defining

$$g(r) = \int_{\mathbb{S}^{n-1}} u_0(x + 2\sqrt{t}rv)dv$$

and integrating by parts, we have (see [Ga1] for more details)

$$\begin{aligned} B_\varepsilon &= \int_\beta^\infty e^{(i-\varepsilon)r^2}\psi_\beta(r)r^{n-1}g(r)dr \\ &= \left(\frac{-1}{2(i-\varepsilon)}\right)^k \sum_{j=0}^k \sum_{l=0}^j a_{k,j}\binom{j}{l}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\psi_\beta^{(j-l)}(r)r^{n-1}dr, \end{aligned}$$

where the $a_{k,j}$ depend only on integers k and j. We next apply Lebesgue's theorem to each term of this sum. For $l=0$, $j\in\{0\dots k\}$, we get

$$\left|e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\psi_\beta^{(j)}(r)r^{n-1}\right| \leqslant \frac{|\mathbb{S}^{n-1}|\|u_0\|_{L^\infty}}{r^{2k-n+1}}2^j\|\psi^{(j)}\|_{L^\infty},$$

and therefore the limit of these terms as ε tends to 0 exists, and

$$\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\psi_\beta^{(j)}(r)r^{n-1}dr\right| \leqslant \frac{|\mathbb{S}^{n-1}|\|u_0\|_{L^\infty}2^j\|\psi^{(j)}\|_{L^\infty}}{(2k-n)\beta^{2k-n}}. \tag{5.13}$$

For $l\geqslant 1$, $j\in\{l\dots k\}$, using

$$\|g^{(j)}\|_{L^2(\beta,\infty,r^{n-1}dr)} \leqslant (2\sqrt{t})^{j-n/2}|\mathbb{S}^{n-1}|^{1/2}\|u_0\|_{X^k(\mathbb{R}^n)}$$

(which has been established in [Ga1]), we similarly obtain

$$\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\psi_\beta^{(j-l)}(r)r^{n-1}dr\right| \leqslant \frac{|\mathbb{S}^{n-1}|^{1/2}\|\psi^{(j-l)}\|_{L^\infty}}{(4k-2j-n)^{1/2}}\|u_0\|_{X^k}\frac{(2\sqrt{t})^{l-\frac{n}{2}}}{\beta^{2k-l-\frac{n}{2}}}. \tag{5.14}$$

Thanks to (5.12), (5.13) and (5.14), I_ε as a limit as ε tends to 0, and

$$\left|\lim_{\varepsilon\to 0} I_\varepsilon\right| \leqslant C\|u_0\|_{X^k}\left(\beta^n + \frac{1}{\beta^{2k-n}} + \frac{\sqrt{t}^{-n/2}}{\beta^{2k-n/2}}(\beta\sqrt{t} + (\beta\sqrt{t})^k)\right).$$

We next choose $\beta = t^\gamma$ (and not $\beta=1$ as we did in [Ga1]), where

$$\gamma = \frac{1}{2}\frac{k-\frac{n}{2}}{k+\frac{n}{2}} > 0.$$

Taking into account this choice of γ, it follows that

$$\begin{aligned}\left|\lim_{\varepsilon\to 0} I_\varepsilon\right| &\leqslant C\|u_0\|_{X^k}\left(t^{\gamma n}+t^{-\gamma(2k-n)}+t^{\gamma(1-2k+n/2)+1/2-n/4}+t^{\gamma(-k+n/2)+k/2-n/4}\right)\\ &\leqslant C\|u_0\|_{X^k}(1+t^\rho),\end{aligned}\tag{5.15}$$

where $\rho=\gamma n$. When $k=\left\lceil\frac{n}{2}\right\rceil$, we obtain the ρ given in (5.6). The rest of the proof is similar to that of Theorem 3.1 in [Ga1]. In particular, using the Banach-Steinhaus Theorem, we deduce that for every integer $k>n/2$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|u_0\|_{X^k(\mathbb{R}^n)}(1+t^\rho).$$

$\square$

5.3.2 Proof of the sharpness of (5.6) in dimension 1.

Let $k\geqslant 1$. For $\tau\geqslant 1$, let us define

$$u_{0\tau}(x)=e^{-\frac{i|x|^2}{4\tau}}\psi\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^{2(k+1)/3}}{x^{k+1}}.$$

Then $u_{0\tau}\in L^\infty$, with $\|u_{0\tau}\|_{L^\infty}=\sup_{r\geqslant 0}\frac{\psi(r)}{r^{k+1}}$. Next, the derivatives from order 1 to k of $u_{0\tau}$ are in L^2, with L^2 norm uniformly bounded with respect to τ. Indeed, it is easy to see by induction on $m\in\{1\dots k\}$ that $u_{0\tau}^{(m)}$ is a linear combination of terms which look like

$$Q(x)=e^{-\frac{i|x|^2}{4\tau}}\psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^\alpha}{x^q},$$

where $q\geqslant k+1-m$ (the only term with $q=k+1-m$ is obtained by differentiating m times the exponential factor in $u_{0\tau}$). $q\geqslant 1$ because $m\leqslant k$, and since moreover $x\mapsto\psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)$ is uniformly bounded on $\mathbb{R}$ and supported in $[\tau^{2/3},\infty)$, it follows that $u_{0\tau}^{(m)}\in L^2$, for $m\in\{1\dots k\}$. Next, we easily compute

$$\|Q\|_{L^2} = \tau^{\alpha-2q/3+1/3}\left\|\frac{\psi^{(p)}(x)}{x^q}\right\|_{L^2},\tag{5.16}$$

and

$$Q'(x) = -\frac{ix}{2\tau}Q(x)+e^{-\frac{i|x|^2}{4\tau}}\psi^{(p+1)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^{\alpha-2/3}}{x^q}-qe^{-\frac{i|x|^2}{4\tau}}\psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^\alpha}{x^{q+1}}\tag{5.17}$$

The L^2 norms of the three terms in the right hand side of (5.17) are respectively

$$\frac{1}{2}\tau^{\alpha-1-2(q-1)/3+1/3}\left\|\frac{\psi^{(p)}(x)}{x^{q-1}}\right\|_{L^2},$$

$$\tau^{\alpha-2/3-2q/3+1/3}\left\|\frac{\psi^{(p+1)}(x)}{x^q}\right\|_{L^2}$$

and

$$q\tau^{\alpha-2(q+1)/3+1/3}\left\|\frac{\psi^{(p)}(x)}{x^{q+1}}\right\|_{L^2}.$$

Therefore there exists a constant $C>0$ such that $\|Q'\|_{L^2}\leqslant C\tau^{-1/3}\|Q\|_{L^2}$. Thus, for $m\in\{1\dots k\}$, $\|u_{0\tau}^{(m)}\|_{L^2}\leqslant C\tau^{-1/3}\|u_{0\tau}^{(m-1)}\|_{L^2}$. Since $u_{0\tau}\in L^2$ and $\|u_{0\tau}\|_{L^2}=C\tau^{1/3}$, we get the announced result: $u_{0\tau}\in X^k$ and $\|u_{0\tau}\|_{X^k}\leqslant C$, where C does not depend on τ.

On the other side, since $z\mapsto\chi(z)$ and $z\mapsto\psi(2z/\tau^{1/6})$ have disjoint supports for τ large enough,

$$\int_{-\infty}^{\infty}e^{(i-\varepsilon)z^2}\chi(z)u_{0\tau}(2\sqrt{\tau}z)dz \;=\; \int_{-\infty}^{\infty}e^{(i-\varepsilon)z^2}\chi(z)e^{-iz^2}\psi\left(\frac{2z}{\tau^{1/6}}\right)\frac{\tau^{\frac{k+1}{6}}}{2^{k+1}z^{k+1}}dz=0, \tag{5.18}$$

while

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{-\infty}^{\infty}e^{(i-\varepsilon)z^2}\psi(z)u_{0\tau}(2\sqrt{\tau}z)dz &= \lim_{\varepsilon\to0}\int_{-\infty}^{\infty}e^{(i-\varepsilon)z^2}\psi(z)e^{-iz^2}\psi\left(\frac{2z}{\tau^{1/6}}\right)\frac{\tau^{\frac{k+1}{6}}}{2^{k+1}z^{k+1}}dz\\ &= \tau^{1/6}\int_{-\infty}^{\infty}\frac{\psi(2r)}{2^{k+1}r^{k+1}}dr. \end{aligned}\tag{5.19}$$

As a conclusion, for τ large enough, denoting by u_τ the solution to (5.1) with initial data $u_{0\tau}$,

$$\|u_\tau(\tau)\|_{X^k}\geqslant\|u_\tau(\tau)\|_{L^\infty}\geqslant|u_\tau(\tau,0)|=C\tau^{1/6}\geqslant C\tau^{1/6}\|u_{0\tau}\|_{X^k}.$$

This completes the proof of the sharpness of the exponent 1/6 in (5.6). □

Remark. We can also prove this sharpness result by using the initial conditions

$$u_{0\tau}(x)=e^{-\frac{ix^2}{4\tau}\chi\left(\frac{x}{\tau^{2/3+\delta}}\right)}$$

or

$$u_{0\tau}(x)=e^{-\frac{ix^2}{4\tau}}\chi\left(\frac{x}{\tau^{2/3-\delta}}\right),$$

and letting the parameter $\delta>0$ tend to 0. In this two cases, taking the notations of section 3.1, the predominant part in I_ε is A_ε (on the contrary, in the proof we developed here, $B_\varepsilon(\tau)\gg A_\varepsilon(\tau)$ as τ is large).

Chapitre 6

The Cauchy Problem for defocusing Nonlinear Schrödinger Equations with non-vanishing initial data at infinity

Abstract. For rather general nonlinearities, we prove that defocusing nonlinear Schrödinger equations in $\mathbb{R}^n$ ($n \leqslant 4$), with non-vanishing initial data at infinity u_0, are globally well-posed in $u_0 + H^1$. The same result holds in an exterior domain in $\mathbb{R}^n$, $n = 2, 3$.

6.1 Introduction

This paper is devoted to the study of the Cauchy problem for defocusing nonlinear Schrödinger equations in dimensions $n \leqslant 4$:

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \end{cases}, \tag{6.1}$$

where $\Omega = \mathbb{R}^n$. The initial data u_0 has the boundary condition

$$|u_0(x)|^2 \to \rho_0 \text{ as } x \to \infty , \tag{6.2}$$

where $\rho_0 > 0$ denotes the light intensity of the background. The real-valued function f is assumed to be defocusing. Namely, f satisfies the following assumption:

$$f(\rho_0) = 0 \text{ and } f'(\rho_0) < 0. \tag{H_f}$$

Under the same condition (H_f) on f, we also study the Cauchy Problem (6.1) where Ω is an exterior domain in $\mathbb{R}^n$, $n = 2, 3$, with a data u_0 which satisfies the same condition at infinity (6.2).

Equation (6.1) with $\Omega = \mathbb{R}^n$ admits many particular solutions with the boundary condition (6.2). These solutions may be gathered under the label "dark solitons". For general nonlinearities, let us mention for instance the stationary and the travelling bubbles. A. de Bouard [dB] gave a necessary and sufficient condition on the nonlinearity ensuring the existence of a stationary bubble, in any dimension. She also proved that the stationary bubbles are all unstable (see also [BGMP], [BP]). Z. Lin [L] studied the travelling bubbles in dimension one. He gave a criterion on the variation of the momentum with respect to the speed which determinates if these bubbles are stable or not. The Gross-Pitaevskii equation, which is (6.1) with $f(r) = 1 - r$ (here, $\rho_0 = 1$), has been the object of a deeper study. F. Bethuel and J.C. Saut [BeSc] proved the existence of travelling waves for the Gross-Pitaevskii equation for small non-zero speeds, in dimension two. Similar results have been obtained by D. Chiron [C] in dimension three and more.

The existence of all these dark solitons makes relevant the study of the Cauchy problem (6.1) with condition (6.2) at infinity. For example, it is a preliminary to the study of their stability when it is not known whether these solitons are stable or not[1]. For unstable dark solitons, the study of the Cauchy Problem (6.1) gives informations on the way this unstability occurs: for instance, a global well posedness result prohibits blowing-up.

A first step in the study of this Cauchy Problem has been done in [BeSc], where it was shown that the Gross-Pitaevskii equation is globally well-posed in $1 + H^1(\mathbb{R}^n)$ for $n = 2, 3$. However, this point of vue is not relevant in all cases. Indeed, in dimension one, most of the travelling bubbles have different limits at $+\infty$ and $-\infty$. Moreover, it was shown by P. Gravejat [Gr5] that the two dimensional travelling waves for the Gross-Pitaevskii equation do not belong to the space $1 + H^1$ (they do not even belong to $1 + L^2$), in spite of the fact that they tend to 1 at infinity (up to the multiplication by a constant of modulus 1).

As a consequence, we need to find a more appropriate framework to study the Cauchy Problem. In [Ga1] (see also the works of P.E. Zhidkov [Z1], [Z2], [Z3], [Z4]), we worked in the Zhidkov spaces

$$X^k(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n), \nabla u \in H^{k-1}(\mathbb{R}^n)\}.$$

We proved some global well-posedness results for (6.1) in dimension one, with condition (6.2) at infinity. However, we assumed the potential

$$V(r) = \int_r^{\rho_0} f(s)ds$$

to be positive, a condition which is not satisfied for all the nonlinearities for which there exists a stationary bubble (see [dB]). In the Gross-Pitaevskii case, as for the existence of travelling waves, the Cauchy problem has been the object of deeper investigations. Using a Brezis-Gallouët method, O. Goubet [Go] proved the global well-posedness for the Gross-Pitaevskii equation in $X^2(\mathbb{R}^2)$, if the initial data has finite energy. More recently, P. Gérard

1. As far as we know, the only dark solitons for which a stability result has been established are the stationary bubbles (see [dB]) which are known to be unstable, the travelling bubbles in dimension 1 (see [L]) and the black solitons in dimension 1, which are stationary solutions to (6.1)-(6.2) vanishing at one point, in the contrary to the bubbles (see [DMG], [Ga2]).

[Ge] obtained a global well-posedness result for the Gross-Pitaevskii equation in dimension two and three in the energy space

$$\{u \in H^1_{\mathrm{loc}}, \nabla u \in L^2, 1 - |u|^2 \in L^2\}.$$

In this paper, we generalize this results to a larger class of nonlinearities. In particular, the potential is not assumed to be positive[2]. Our main result is as follows.

Theorem 6.1.1 *Let $n = 1, 2, 3$ or 4, $\rho_0 > 0$, and $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ ($k = 1$ if $n = 1$, $k = 2$ if $n = 2, 3$, $k = 3$ if $n = 4$) which satisfies (H_f). We assume moreover that there exists $\alpha_1 \geqslant 1$, with the supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ (where $\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$) , and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$ such that*

$$\exists C_0 > 0,\ A > \rho_0, \left\{ \begin{array}{l} \forall r \geqslant 1, \left\{ \begin{array}{ll} |f''(r)| \leqslant C_0 r^{\alpha_1 - 3} & \text{if } n = 1,2,3 \\ |f'''(r)| \leqslant C_0 r^{\alpha_1 - 4} & \text{if } n = 4 \end{array} \right. \\ \left\{ \begin{array}{l} \text{if } \alpha_1 \leqslant 3/2,\ V \text{ is bounded from below} \\ \text{if } \alpha_1 > 3/2,\ \forall r \geqslant A,\ r^{\alpha_2} \leqslant C_0 V(r) \end{array} \right. \end{array} \right. . \qquad (H_{\alpha_1,\alpha_2})^3$$

Then for any regular function of finite energy φ, which means

$$\varphi \in \mathcal{C}_b^{k+1}(\mathbb{R}^n),\ \nabla\varphi \in H^{k+1}(\mathbb{R}^n)^n,\ |\varphi|^2 - \rho_0 \in L^2(\mathbb{R}^n), \qquad (H_\varphi)$$

equation (6.1) is globally well-posed in $\varphi + H^1(\mathbb{R}^n)$. Namely, for every $w_0 \in H^1(\mathbb{R}^n)$, there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\varphi + w$ solves (6.1), with the initial data $w(0) = w_0$.
For any $T > 0$, the flow map $w_0 \mapsto w$, $H^1 \mapsto \mathcal{C}([0,T], H^1)$ is Lipshitz continuous on the bounded sets of H^1.
The energy

$$\mathcal{E}(w) = \int_{\mathbb{R}^n} |\nabla(\varphi + w)|^2 dx + \int_{\mathbb{R}^n} V(|\varphi + w|^2) dx$$

is conserved by the flow.

For a very large class of defocusing nonlinearities, assumption ($\mathrm{H}_{\alpha_1,\alpha_2}$) is satisfied for some α_1, α_2 as required in Theorem 6.1.1. We give here some examples.

Examples.
1. The pure powers: $f(r) = (\rho_0^p - r^p)$ where p is a positive integer if $n = 1$ or 2, $p = 1$ if $n = 3$. In that case, $V(r) \geqslant 0$ on $\mathbb{R}_+$, $V(r) \sim \frac{1}{p+1} r^{p+1}$ as $r \to \infty$ and $f''(r) = -p(p-1)r^{p-2}$. Thus ($\mathrm{H}_{\alpha_1,\alpha_2}$) is satisfied for $\alpha_1 = \alpha_2 = p + 1$.

2. In the usual "0 at infinity" case, the boundedness of the H^1 norm may be deduced from the conservation of the energy, the conservation of the charge and a Gagliardo-Nirenberg inequality. In our case, the analoguous to the charge is the quantity $\int(|u|^2 - \rho_0)$, the conservation of which is not so clear.

3. Remark that the first condition in ($\mathrm{H}_{\alpha_1,\alpha_2}$) implies $|V(r)| \lesssim r^{\alpha_1}$ for $r \geqslant 1$. Thus, in the case $\alpha_1 > 3/2$, ($\mathrm{H}_{\alpha_1,\alpha_2}$) may be satisfied only if $\alpha_2 \leqslant \alpha_1$, so that $\alpha_2 \in [\alpha_1 - \frac{1}{2}, \alpha_1]$. Remark also that in the case $\alpha_1 \leqslant 3/2$, α_2 plays no role.

2. Saturated nonlinearity: $f(r) = \frac{1}{(1+ar)^2} - \frac{1}{(1+a)^2}$. In this case, $f''(r) = \frac{6a^2}{(1+ar)^4}$ and $V(r) = \frac{a(1-r)^2}{(1+ar)(1+a)^2} \geqslant 0$. In particular, $(\mathrm{H}_{1,\alpha_2})$ is satisfied for any α_2.
3. The cubic-quintic case: $f(r) = (r-\rho_0)(2a+\rho_0-3r)$, where $0 < a < \rho_0$. Then $V(r) = (r-\rho_0)^2(r-a)$ and $f''(r) = -6$. In the contrary to the two previous examples, V is not positive on $\mathbb{R}_+$, but $(\mathrm{H}_{\alpha_1,\alpha_2})$ is satisfied for $\alpha_1 = \alpha_2 = 3$. Thus Theorem 6.1.1 applies in dimensions one and two (in dimension three, Theorem 6.1.3 below applies).

The global well-posedness for an initial data in the energy space

$$E = \{u \in H^1_{\mathrm{loc}}, \nabla u \in L^2, \rho_0 - |u|^2 \in L^2\}.$$

is a consequence of Theorem 6.1.1 and of the following proposition, which directly follows from the results of P. Gérard in [Ge].

Proposition 6.1.1 *Let $u \in E$. Then there exists $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n) \cap E$ such that $\nabla\varphi \in H^\infty(\mathbb{R}^n)^n$ and $w \in H^1(\mathbb{R}^n)$ such that $u = \varphi + w$.*

From this Proposition we deduce:

Theorem 6.1.2 *Under the same assumptions that in Theorem 6.1.1, for any $u_0 \in E$, there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $u := u_0 + w$ solves (6.1).*

Proof. Given $u_0 \in E$, let $u_0 = \varphi + w_0$ be a decomposition as in Proposition 6.1.1. Thanks to Theorem 6.1.1, there exists an unique $\tilde{w} \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\varphi + \tilde{w}$ solves (6.1). Therefore $w = \tilde{w} - w_0$ is the unique element of $\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $u := u_0 + w$ solves (6.1). □

In particular, the solution $u = \varphi + w$ given by Theorem 6.1.1 does not depend on the choice of the decomposition of $u_0 \in E$ into $\varphi + w_0$.

In the critical case $\alpha_1 = \alpha_1^*$, we obtain the following local result.

Theorem 6.1.3 *Under the same assumptions on f and φ that in Theorem 6.1.1, if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$, there exists $R > 0$ and $T > 0$ such that, for $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, there exists a unique $w \in \mathcal{C}([0,T], H^1)$ such that $\varphi + w$ solves (6.1).*
For that T, the flow $w_0 \mapsto w$ is locally Lipshitz continuous from the ball of radius R in H^1 into $\mathcal{C}([0,T], H^1)$.
The energy is conserved on $[0,T]$.

Similarly to the sub-critical case, we deduce from Theorem 6.1.3 and Proposition 6.1.1 the following result.

Theorem 6.1.4 *Under the same assumptions on f that in Theorem 6.1.3, if $u_0 \in E$ satisfies (H_φ), there exists $T(u_0) > 0$ and a unique $w \in \mathcal{C}([0,T], H^1)$ such that $u_0 + w$ solves (6.1).*

It was shown in [Ge] that in dimensions two and three, the Gross-Pitaevskii equation is globally well-posed in the energy space E endowed with a structure of complete metric space by the distance

$$d_E(u,v) = \|u-v\|_{X^1+H^1} + \||u|^2 - |v|^2\|_{L^2}.$$

It is quite clear that for any $T>0$ and $u_0 \in E$, $u_0 + \mathcal{C}([0,T],H^1)$ is strictly included in $\mathcal{C}([0,T],E)$. In particular, P. Gérard obtained in [Ge] the uniqueness of the solutions to (6.1) in a bigger space than in Theorem 6.1.2. Using some of the arguments developed in [Ge] we get the uniqueness in the energy space for other non-linearities than Gross-Pitaevskii. More precisely, we have the following result.

Theorem 6.1.5 *Let $n=2,3,4$. Under the assumptions of Theorem 6.1.1, let $T>0$, $u_0 \in E$ and $u \in \mathcal{C}([0,T],E)$ be a solution of (6.1) with $u(0)=u_0$. Then $u - u_0 \in \mathcal{C}([0,T],H^1)$, and therefore u is the solution of (6.1) given by Theorem 6.1.2.*

Remarks.

1. In Theorem 6.1.3, $R = R(\varphi)$ depends on φ. For general $u_0 \in E$, it is not clear whether we can find a function φ which satisfies (H_φ) and such that $w_0 = u_0 - \varphi \in H^1$ has H^1-norm less than $R(\varphi)$. That is why we need to assume that u_0 satisfies (H_φ) in Theorem 6.1.4.

2. In dimension 4, the Gross-Pitaevskii equation is critical (that is $\alpha_1 = \alpha_1^*$). In [Ge], P. Gérard proved that the four-dimensional Gross-Pitaevskii equation is globally well-posed in the energy space E, provided the initial condition u_0 has small energy. In Theorem 6.1.3, we prove that for critical non-linearities (and in particular for Gross-Pitaevskii in dimension 4), (6.1) is locally well-posed in $u_0 + H^1$, for any regular initial condition u_0 in the energy space, without the smallness assumption on the energy. However, we do not obtain any global well-posedness result, and we only prove the local Lipshitz continuity of the flow on small intervals of time. The missing argument is a persistency result. The reason of this missing is that for general non-linearities (in particular when the potential V is non-positive), the conservation of the energy does not imply the conservation of the smallness of w in H^1.

Our proof of Theorem 6.1.1 consists in looking for a solution of (6.1) under the form $\varphi + w$. Thus the equation satisfied by w writes

$$\begin{cases} i\frac{\partial w}{\partial t} + \Delta w = F(w(t)) \\ w(0) = w_0 \end{cases}, \tag{6.3}$$

where

$$F(w) = -\Delta\varphi - f(|\varphi + w|^2)(\varphi + w). \tag{6.4}$$

We prove that (6.3) is locally well-posed in $H^k(\mathbb{R}^n)$, $k=1$ for $n=1$, $k=2$ for $n=2,3$ and $k=3$ for $n=4$, and we give estimations for the H^1-norm of w on the interval of existence in H^k. Next, for $n = 2, 3$ or 4, using Strichartz inequalities, we prove that (6.3) is locally well-posed in H^1, and globally in H^k. Finally, we approximate our (local) H^1 solution by the (global) H^k solution, and we deduce that its H^1-norm may not blow up on bounded intervals of time.

Using the Strichartz inequalities obtained by N. Burq, P. Gérard and N. Tzvetkov in [BGT], the same method gives similar results, in dimensions two and three, when $\mathbb{R}^n$ is replaced by an exterior domain $\Omega = \mathbb{R}^n \backslash K$, with either Dirichlet or Neumann boundary conditions. More precisely, we consider the initial value problem

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta_D u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \in E_D \end{cases}, \tag{6.5}$$

where the initial condition u_0 belongs to the energy space with Dirichlet boundary conditions

$$E_D := \{u \in H^1_{\text{loc}}(\Omega), \nabla u \in L^2(\Omega), \rho_0 - |u|^2 \in L^2(\Omega), \chi u \in H^1_0(\Omega)\}.$$

Here, $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\chi \equiv 1$ in a neighborhood V of the obstacle K. We also consider

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta_N u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \in E_N \end{cases}, \tag{6.6}$$

where the initial condition u_0 belongs to the energy space with Neumann boundary conditions

$$E_N := \{u \in H^1_{\text{loc}}(\Omega), \nabla u \in L^2(\Omega), \rho_0 - |u|^2 \in L^2(\Omega), \chi u \in H^1_N(\Omega)\}.$$

The result we prove is as follows.

Theorem 6.1.6 *Let $n = 2$ or 3, and $\Omega \subset \mathbb{R}^n$ be the exterior domain of a smooth, compact, non-trapping, non-empty obstacle K, and $f \in \mathcal{C}^3(\mathbb{R}_+)$ which satisfies (H_f). We assume moreover that there exists $\alpha_1 \geqslant 1$, $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ such that (H_{α_1,α_2}) is true. If $n = 3$, we assume moreover $\alpha_1 < 2$.*
Then, for every $u_0 \in E_D$ (resp. E_N), there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1_0(\Omega))$ (resp. $\mathcal{C}(\mathbb{R}, H^1_N(\Omega))$) such that $u_0 + w$ solves (6.5) (resp. (6.6)).
Given $\psi \in E_D$ (resp. E_N), for any $T > 0$, the flow map $w_0 \mapsto w$, $H^1_0 \mapsto \mathcal{C}([0,T], H^1_0)$ (resp $H^1_N \mapsto \mathcal{C}([0,T], H^1_N)$), where $w(0) = w_0$ and $\psi + w$ solves (6.5) (resp. (6.6)), is Lipshitz continuous on the bounded sets of H^1_0 (resp H^1_N).
The energy is conserved by the flow.

In the critical case $n = 3$, $\alpha_1 = 2$, we obtain:

Theorem 6.1.7 *Let $\Omega \subset \mathbb{R}^3$ be the exterior domain of a smooth, compact, non-trapping, non-empty obstacle K, and $f \in \mathcal{C}^3(\mathbb{R}_+)$ which satisfies (H_f). We assume moreover that there exists $\alpha_2 \in [3/2, 2]$ such that (H_{2,α_2}) is true.*
Then, for every φ satisfying

$$\varphi \in \mathcal{C}^\infty_b(\Omega), \ \nabla\varphi \in H^\infty(\Omega), \ \mathit{Supp}\varphi \subset \Omega \backslash (V \cap \Omega), \ |\varphi|^2 - \rho_0 \in L^2(\Omega)$$

(in particular, $\varphi \in E_D \cap E_N$), there exists $R > 0$ and $T > 0$ such that for every $w_0 \in H_0^1$ (resp. H_N^1) with $\|w_0\|_{H^1(\Omega)} \leqslant R$, there exists an unique $w \in \mathcal{C}([0,T], H_0^1(\Omega))$ (resp. $\mathcal{C}([0,T], H_N^1(\Omega))$) such that $w(0) = w_0$ and $\varphi + w$ solves (6.5) (resp. (6.6)).
For that T, the flow $w_0 \mapsto w$ is locally Lipshitz continuous from the ball of radius R in $H_0^1(\Omega)$ (resp. $H_N^1(\Omega)$) into $\mathcal{C}([0,T], H_0^1(\Omega))$ (resp. $\mathcal{C}([0,T], H_N^1(\Omega))$).
The energy is conserved by the flow.

Remark. In the case of an exterior domain, we only obtain uniqueness results in spaces like $u_0 + \mathcal{C}([0,T], H_0^1)$, and not in $\mathcal{C}([0,T], E_D)$ (the continuity in E_D should be understood in the sense of the analogous to the distance d_E for an exterior domain). Indeed, even for the linear Schrödinger equation, the well-posedness in the energy space is not that clear.

Notations. If $m \in [0, \infty]$, $\mathcal{C}_b^m(\mathbb{R}^n)$ denotes the space of bounded functions of class $\mathcal{C}^m$ on $\mathbb{R}^n$.
We denote $H^\infty(\mathbb{R}^n) = \bigcap\limits_{s \geqslant 0} H^s(\mathbb{R}^n)$.
The notation $A \lesssim B$ means that there exists a harmless constant $C > 0$ such that $A \leqslant CB$.
If $T > 0$, $p, q \geqslant 1$, $L_T^p L^q$ denotes the Banach space $L^p([0,T], L^q)$ equipped with its natural norm.
If $p \in [1, \infty]$, we denote by $p' = \frac{p}{p-1}$ its conjugate exponent.

The structure of this paper is as follows. In section 2, we prove that (6.3) is globally well-posed in a space $H^k(\mathbb{R}^n)$ with k large. In section 3, we give an estimation on the H^1 norm of this solution on its maximal interval of existence in H^k. In section 4, thanks to a fixed point argument in $\mathcal{C}([0,T], H^1)$ and Strichartz estimates, we prove that (6.3) is locally well-posed in H^1. In section 5, we prove a persistence result and obtain the global well-posedness of equation (6.3) in H^1, in the sub-critical case. Section 6 is devoted to the proofs of Proposition 6.1.1 and Theorem 6.1.5. In that section, most of the arguments are due to P. Gérard (see [Ge]). In section 7, we adapt the method to the case of an exterior domain in $\mathbb{R}^n$, $n = 2, 3$. Section 8 is devoted to the proof of some technical lemmas concerning the $L^p + L^q$ spaces, stated and used in section 6.

6.2 Local theory for regular solutions

Lemma 6.2.1 *We assume $(n,k) = (1,1)$, $(2,2)$, $(3,2)$ or $(4,3)$, $f \in \mathcal{C}^k(\mathbb{R}_+)$ satisfies (H_f), φ satisfies (H_φ). Then F maps $H^k(\mathbb{R}^n)$ into itself.*

Proof. Let $w \in H^k(\mathbb{R}^n)$ and $\varphi \in L^\infty$. Then $\varphi + w \in L^\infty$ because of the Sobolev embedding $H^k(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. Since f and f' are continuous, it follows that $f(|\varphi + w|^2)$, $f'(|\varphi + w|^2) \in$

L^∞. Next, we write

$$\begin{aligned} f(|\varphi+w|^2)(\varphi+w) &= (|\varphi|^2-\rho_0)\int_0^1 f'(\rho_0+s(|\varphi|^2-\rho_0))ds(\varphi+w) \\ &\quad +2\mathfrak{Re}\left[w\int_0^1 \overline{\varphi+sw}f'(|\varphi+sw|^2)ds\right](\varphi+w)\ . \end{aligned} \tag{6.7}$$

Using (H_φ), it is easy to see that the right-hand side in (6.7) belongs to H^k. Thus $F(w)\in H^k$, because $\Delta\varphi\in H^k$. □

Lemma 6.2.2 *We assume $(n,k)=(1,1),\ (2,2),\ (3,2)$ or $(4,3)$, $f\in\mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), φ satisfies (H_φ). Then $F: H^k(\mathbb{R}^n)\mapsto H^k(\mathbb{R}^n)$ is locally Lipshitz continuous.*

Proof. Let us take $R>0$ and $w_1,w_2\in H^k$ such that $\|w_1\|_{H^k},\|w_2\|_{H^k}\leqslant R$. Then

$$\begin{aligned} F(w_1)-F(w_2) &= \int_0^1\Big[f(|\varphi+w_1+s(w_2-w_1)|^2)(w_2-w_1) \\ &\quad +2\mathfrak{Re}\left[(w_2-w_1)\overline{\varphi+w_1+s(w_2-w_1)}\right] \\ &\quad \times f'(|\varphi+w_1+s(w_2-w_1)|^2)(\varphi+w_1+s(w_2-w_1))\Big]ds. \end{aligned} \tag{6.8}$$

Next, for all $x\in\mathbb{R}$,

$$|\varphi(x)+w_1(x)+s(w_2(x)-w_1(x))|\leqslant\|\varphi\|_{L^\infty}+2CR\ ,$$

where C is the norm of the continuous Sobolev embedding $H^k\subset L^\infty$. Thus there exists a constant $C(R)>0$ such that $\|f^{(\alpha)}(|\varphi+w_1+s(w_2-w_1)|^2)\|_{L^\infty}\leqslant C(R)$, for $\alpha=0,..,k+1$. Using again Sobolev embeddings $H^k(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)$, as well as $H^1(\mathbb{R}^n)\subset L^4(\mathbb{R}^n)$ for $n=2$ or 3, $H^1(\mathbb{R}^4)\subset L^4(\mathbb{R}^4)$ and $H^2(\mathbb{R}^4)\subset L^p(\mathbb{R}^4)$ for every $p\in[2,\infty)$, it follows from (6.8) and its differentiation that

$$\|F(w_1)-F(w_2)\|_{H^k}\leqslant\tilde{C}(R)\|w_1-w_2\|_{H^k},$$

where $\tilde{C}(R)$ only depends on R, and not on w_1,w_2. □

Once these two lemmas have been established, we can apply the classical results of the theory of nonlinear evolution equations (see for instance [Pa], Theorems 6.1.4 and 6.1.5, and [CH]). We deduce the following local well-posedness result:

Theorem 6.2.1 *$(n,k)=(1,1),\ (2,2),\ (3,2)$ or $(4,3)$, $f\in\mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), φ satisfies (H_φ). For every $w_0\in H^k(\mathbb{R}^n)$, there exists $T^*(w_0)>0$ such that (6.3) has a unique mild solution $w\in\mathcal{C}([0,T^*),H^k(\mathbb{R}^n))$, which means that $w(t)=e^{it\Delta}w_0-i\int_0^t e^{i(t-s)\Delta}F(w(s))ds$, where $e^{it\Delta}$ denotes the Schrödinger group. If $T^*<\infty$, then $\|w(t)\|_{H^k}\uparrow+\infty$ as $t\uparrow T^*$. Moreover, the map $T^*: H^k\mapsto\mathbb{R}_+^*$ is semi continuous from below, and if $w_0\in H^{k+2}(\mathbb{R}^n)$, w is a classical solution to (6.3), which means that $w\in\mathcal{C}([0,T^*),H^{k+2}(\mathbb{R}^n))\cap\mathcal{C}^1((0,T^*),H^k(\mathbb{R}^n))$.*

6.3 Estimate on the H^1 norm for regular solutions

We next prove under the supplementary assumption ($\mathrm{H}_{\alpha_1,\alpha_2}$) for some $\alpha_1 \geqslant 1$, $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ that the norm of $w(t)$ in $H^1(\mathbb{R}^n)$ (where w is the solution of (6.3) given by Theorem 6.2.1) can not blow up on $[0, T^*(w_0))$. In particular, in the one-dimensional case, this result and Theorem 6.2.1 imply that w is global. Namely, for every $w_0 \in H^1(\mathbb{R})$, $T^*(w_0) = +\infty$, and Theorem 6.1.1 is proven in the case $n = 1$. We first prove that the energy is conserved on $[0, T^*(w_0))$.

Lemma 6.3.1 *Let $(n,k) = (1,1),\ (2,2),\ (3,2)$ or $(4,3)$, $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), φ satisfies (H_φ), $w_0 \in H^k(\mathbb{R}^n)$. Then for every $t \in [0, T^*(w_0))$, the energy*

$$\mathcal{E}(t) \ := \ \|\nabla\varphi + \nabla w(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} V(|\varphi(x) + w(t,x)|^2)dx \tag{6.9}$$

is conserved: $\mathcal{E}(t) \equiv \mathcal{E}(0) =: \mathcal{E}_0$, where $V(r) := \int_r^{\rho_0} f(s)ds$.

Proof. It suffices to prove Lemma 6.3.1 for $w_0 \in H^{k+2}$. Indeed, once this is established, the lower semi-continuity of T^*, the continuity of the flow $w_0 \mapsto w(t)$ from H^k into H^k for every $t < T^*$ (see [CH]), the density of H^{k+2} into H^k, and the continuity of the map $H^k \ni w \mapsto V(|\varphi + w|^2) \in L^1$ imply the Lemma in all its generality. Let us first verify that

$$\left(\begin{array}{ccc} w & \mapsto & V(|\varphi + w|^2) \\ H^k(\mathbb{R}^n) & \mapsto & L^1 \end{array} \right.$$

is continuous. We write

$$\begin{aligned} V(|\varphi + w|^2) \ &= \ V(|\varphi|^2) - \int_0^1 2\mathfrak{Re}\left[w\overline{\varphi + sw}\right] \\ &\quad \times \left(f(|\varphi|^2) + \int_0^1 2\mathfrak{Re}\left[sw\overline{\varphi + s\tau w}\right] f'(|\varphi + s\tau w|^2)d\tau \right) ds. \end{aligned} \tag{6.10}$$

We have already seen in the proof of Lemma 6.2.1 that $f(|\varphi|^2) \in L^2$. Thus, using the Cauchy Schwarz inequality and the Sobolev embedding $H^k \subset L^\infty$, it follows that the last term in the right-hand side of (6.10) is continuous from H^k into L^1. In order to prove that $V(|\varphi|^2) \in L^1$, we just write

$$V(|\varphi|^2) \ = \ (|\varphi|^2 - \rho_0)^2 \int_0^1 s \int_0^1 -f'(\rho_0 + s\tau(|\varphi|^2 - \rho_0))d\tau ds,$$

and we use the assumption $|\varphi|^2 - \rho_0 \in L^2$, as well as the boundedness of φ and the continuity of f'.

We next prove the Lemma for $w_0 \in H^{k+2}$, which will be assumed from now on. Let $t \in [0, T^*(w_0))$. Let us multiply (6.3) by $\overline{\partial_t w(t)}$, sum over $\mathbb{R}^n$ and take the real part. We get

$$-\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla(\varphi + w(t))(x)|^2 dx - \int_{\mathbb{R}^n} \frac{d}{dt} V(|\varphi(x) + w(t,x)|^2)dx = 0\ .$$

Next, $w \mapsto V(|\varphi + w|^2) \in \mathcal{C}^1(H^k, L^1)$, as is shown by the following expansion, which is obtained by the Taylor formula. For every $w, \delta w \in H^k$,

$$\begin{aligned} V(|\varphi + w + \delta w|^2) &= V(|\varphi + w|^2) - 2\mathfrak{Re}\left[\delta w \overline{\varphi + w}\right] f(|\varphi + w|^2) \\ &-4\int_0^1 \int_0^1 \mathfrak{Re}\left[\delta w \overline{\varphi + w}\right] \mathfrak{Re}\left[s\delta w \overline{\varphi + w + s\tau\delta w}\right] f'(|\varphi + w + s\tau\delta w|^2) d\tau ds \\ &+2\int_0^1 s|\delta w|^2 f(|\varphi + w + s\delta w|^2) ds. \end{aligned} \tag{6.11}$$

Since moreover $w \in \mathcal{C}^1((0,T^*), H^k)$, the map $t \mapsto V(|\varphi + w(t)|^2)$, $[0, T^*(w_0)) \mapsto L^1(\mathbb{R}^n)$ is of class $\mathcal{C}^1$. Thus

$$\int_{\mathbb{R}^n} \frac{d}{dt} V(|\varphi(x) + w(t,x)|^2) dx = \frac{d}{dt} \int_{\mathbb{R}^n} V(|\varphi(x) + w(t,x)|^2) dx \ ,$$

and the lemma is proved. $\square$

Lemma 6.3.2 *$(n,k) = (1,1)$, $(2,2)$, $(3,2)$ or $(4,3)$, $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), φ satisfies (H_φ). Let us write V as $V = V_+ - V_-$, where $V_+, V_- \geqslant 0$ and V_- is assumed to be bounded. Then there exists a constant $C_1 > 0$ such that for every $w_0 \in H^k(\mathbb{R}^n)$, $t \in [0, T^*)$, we have*

$$\|\nabla\varphi + \nabla w(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} V_+(|\varphi(x) + w(t,x)|^2) dx \ \leqslant \ C_1(1 + \|w(t)\|_{L^2}^2) \ . \tag{6.12}$$

Proof. Thanks to Lemma 6.3.1, it is clear that the left hand side in (6.12) equals

$$\mathcal{E}_0 + \int_{\mathbb{R}^n} V_-(|\varphi(x) + w(t,x)|^2) dx \ .$$

Next, the definition of V and (H_f) imply $V(\rho_0) = 0$, $V'(\rho_0) = -f(\rho_0) = 0$ and $V''(\rho_0) = -f'(\rho_0) > 0$. It follows that there exists $C_2 > 0$ and $\delta > 0$ (for convenience, we assume $\delta < \rho_0$) such that $V(r) \geqslant C_2(\rho_0 - r)^2$ for every $r \in [\rho_0 - \delta, \rho_0 + \delta]$. In particular, $V_- \equiv 0$ on $[\rho_0 - \delta, \rho_0 + \delta]$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} V_-(|\varphi(x) + w(t,x)|^2) dx \ &\leqslant \ \|V_-\|_{L^\infty} \left|\{x, |\varphi + w(t)|^2 \leqslant \rho_0 - \delta\}\right| \\ &+ \|V_-\|_{L^\infty} \left|\{x, |\varphi + w(t)|^2 \geqslant \rho_0 + \delta\}\right| . \end{aligned} \tag{6.13}$$

Next, using the triangle inequality, $\{x, |\varphi + w|^2 \leqslant \rho_0 - \delta\}$ is a subset of $\{x, |w| \geqslant |\varphi| - (\rho_0 - \delta)^{1/2}\}$, which is itself included in the union of $\{x, |w| \geqslant |\varphi| - (\rho_0 - \delta)^{1/2} \geqslant \frac{\rho_0^{1/2} - (\rho_0 - \delta)^{1/2}}{2}\}$ and $\{x, |\varphi| - (\rho_0 - \delta)^{1/2} \leqslant \frac{\rho_0^{1/2} - (\rho_0 - \delta)^{1/2}}{2}\}$. Similarly, $\{x, |\varphi + w(t)|^2 \geqslant \rho_0 + \delta\} \subset \{x, |w(t)| \geqslant$

$(\rho_0+\delta)^{1/2} - |\varphi| \geqslant \frac{(\rho_0+\delta)^{1/2}-\rho_0^{1/2}}{2}\} \cup \{x, |\varphi| \geqslant \frac{(\rho_0+\delta)^{1/2}+\rho_0^{1/2}}{2}\}$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} V_-(|\varphi(x)+w(t,x)|^2)dx \\
&\quad\leqslant \; \|V_-\|_{L^\infty}\left(\int_{\mathbb{R}^n}\frac{4|w(t,x)|^2dx}{(\rho_0^{1/2}-(\rho_0-\delta)^{1/2})^2} + \left|\left\{x, |\varphi| \leqslant \frac{\rho_0^{1/2}+(\rho_0-\delta)^{1/2}}{2}\right\}\right|\right. \\
&\quad\left. + \int_{\mathbb{R}^n}\frac{4|w(t,x)|^2dx}{((\rho_0+\delta)^{1/2}-\rho_0^{1/2})^2} + \left|\left\{x, |\varphi| \geqslant \frac{\rho_0^{1/2}+(\rho_0+\delta)^{1/2}}{2}\right\}\right|\right) .
\end{aligned}$$

The result follows, since (H_φ) implies $|\varphi(x)|^2 \to \rho_0$ as $x \to \infty$. □

We next use the assumption $(\mathrm{H}_{\alpha_1,\alpha_2})$ to control the L^2-norm of w. It will then follow that its H^1-norm remains bounded on bounded intervals, because of Lemma 6.3.2.

Lemma 6.3.3 *Let us assume that (H_f), (H_φ) and (H_{α_1,α_2}) are satisfied, for some $\alpha_1 \geqslant 1$, $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$. Then there exists a constant $C_3 > 0$ such that for every $t \in [0, T^*)$, we have*

$$\|w(t)\|^2_{L^2(\mathbb{R}^n)} \;\;\leqslant\;\; (1+\|w_0\|^2_{L^2(\mathbb{R}^n)})e^{C_3 t}. \tag{6.14}$$

Proof. Since $C_0 > 0$, in the case $\alpha_1 > 3/2$, $(\mathrm{H}_{\alpha_1,\alpha_2})$ implies $V(r) > 0$, and thus $V_-(r) \equiv 0$ for $r \geqslant A$. Therefore V_- is bounded, as it is required to apply Lemma 6.3.2. This is also true if $\alpha_1 \leqslant 3/2$. As in the proof of Lemma (6.3.1), the study may be reduced to the case $w_0 \in H^{k+2}$. Under this assumption, let us multiply (6.3) by $\overline{w(t)}$, sum over $\mathbb{R}^n$ and take the imaginary part. We get

$$\frac{d}{dt}\|w(t)\|^2_{L^2} \;\;=\;\; -2\mathfrak{Im}\int_{\mathbb{R}^n}\left[\Delta\varphi + f(|\varphi+w(t)|^2)\varphi\right]\overline{w(t)}dx,$$

where

$$f(|\varphi+w(t)|^2) = f(|\varphi|^2) + \int_0^1 2\mathfrak{Re}[w(t)\overline{\varphi+sw(t)}]f'(|\varphi+sw(t)|^2)ds\ .$$

For any $\beta \geqslant 0$ we denote by $A_\beta \geqslant 1$ a constant such that for every $a, b > 0$, $(a+b)^\beta \leqslant A_\beta(a^\beta + b^\beta)$. From the first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$, we deduce the existence of $C_0' > 0$ such that for every $r \geqslant 0$, $r^{1/2}|f'(r)| \leqslant C_0'(1 + r^{\alpha_1 - 3/2})$ if $\alpha_1 > 3/2$ and $r^{1/2}|f'(r)| \leqslant C_0'$ if

$\alpha_1 \leqslant 3/2$. Then, in the case $\alpha_1 > 3/2$,

$$\begin{aligned}
&\frac{d}{dt}\|w(t)\|_{L^2}^2 \\
&\leqslant \; 2(\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^\infty}\|f(|\varphi|^2)\|_{L^2})\|w(t)\|_{L^2} \\
&\quad + \int_{\mathbb{R}^n}\int_0^1 4|w(t,x)|^2|\varphi(x)|C_0'(1+|\varphi(x)+sw(t,x)|^{2(\alpha_1-3/2)})dsdx \\
&\leqslant \; 2(\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^\infty}\|f(|\varphi|^2)\|_{L^2})\|w(t)\|_{L^2} \\
&\quad +4\|\varphi\|_{L^\infty}C_0'A_{2\alpha_1-3}\left(\|w(t)\|_{L^2}^2(1+\|\varphi\|_{L^\infty}^{2\alpha_1-3}) + \|w(t)\|_{L^{2\alpha_1-1}}^{2\alpha_1-1}\right) \\
&\leqslant \; C_4(1+\|w(t)\|_{L^2}^2 + \|w(t)\|_{L^{2\alpha_1-1}}^{2\alpha_1-1}),
\end{aligned} \tag{6.15}$$

where C_4 is a positive constant. When $\alpha_1 \leqslant 3/2$, we similarly obtain

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 \;\leqslant\; C_4(1+\|w(t)\|_{L^2}^2).$$

If $\alpha_1 \leqslant 3/2$, the result follows by the Gronwall lemma with $C_3 = C_4$. If $\alpha_1 > 3/2$, it remains to control the $L^{2\alpha_1-1}$ norm of $w(t)$. In the sequel, $\alpha_1 > 3/2$ is assumed. Using the second assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$[4], the assumption $2\alpha_1 - 1 \leqslant 2\alpha_2$ and Lemma 6.3.2 we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}|w(t,x)|^{2\alpha_1-1}dx \\
&= \int_{\{x,|\varphi+w|\leqslant A^{1/2}\}}|w(t,x)|^2|w(t,x)|^{2\alpha_1-3}dx + \int_{\{x,|\varphi+w|\geqslant A^{1/2}\}}|w(t,x)|^{2\alpha_1-1}dx \\
&\leqslant \int_{\{x,|w(t,x)|\leqslant A^{1/2}+\|\varphi\|_{L^\infty}\}}|w(t,x)|^2|w(t,x)|^{2\alpha_1-3}dx \\
&\quad +A_{2\alpha_1-1}\int_{\{x,|\varphi+w|\geqslant A^{1/2}\}}(|\varphi|^{2\alpha_1-1}+|\varphi+w|^{2\alpha_1-1})dx \\
&\leqslant (A^{1/2}+\|\varphi\|_{L^\infty})^{2\alpha_1-3}\int_{\mathbb{R}^n}|w(t,x)|^2dx + A_{2\alpha_1-1}\|\varphi\|_{L^\infty}^{2\alpha_1-1}\int_{\{x,|\varphi+w|\geqslant A^{1/2}\}}dx \\
&\quad +A_{2\alpha_1-1}C_0\int_{\{x,|\varphi+w|\geqslant A^{1/2}\}}V_+(|\varphi+w|^2)dx \\
&\leqslant (A^{1/2}+\|\varphi\|_{L^\infty})^{2\alpha_1-3}\|w(t)\|_{L^2}^2 + \frac{A_{2\alpha_1-1}\|\varphi\|_{L^\infty}^{2\alpha_1-1}}{(A^{1/2}-\|\varphi\|_{L^\infty})^2}\|w(t)\|_{L^2}^2 \\
&\quad +A_{2\alpha_1-1}C_0C_1(1+\|w(t)\|_{L^2}^2)\ .
\end{aligned} \tag{6.16}$$

Next, concatenating the estimations (6.15) and (6.16), there exists a constant $C_3 > 0$ such that

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 \;\leqslant\; C_3(1+\|w(t)\|_{L^2}^2)\ .$$

4. Up to a change of A, we may assume in the sequel $A > \max(\rho_0, \|\varphi\|_{L^\infty}^2, 1)$.

We conclude by the Gronwall lemma. $\square$

In the one-dimensional case, the global well-posedness of (6.3) in H^1 is a straightforward consequence of Theorem 6.2.1, Lemma 6.3.2 and Lemma 6.3.3. More precisely, we have proven:

Theorem 6.3.1 *If $n=1$, under the conditions (H_f), (H_{α_1,α_2}), (H_φ), (6.3) is globally well posed in $H^1(\mathbb{R})$. Namely, for every $w_0 \in H^1(\mathbb{R})$, $T^*(w_0)=+\infty$.*

Remarks.

1. In the one-dimensional case, the Lipshitz continuity of the flow from the bounded sets of H^1 into $\mathcal{C}([0,T],H^1)$ for any $T>0$ may be obtained by classical methods (see [Pa], [CH]), so that we will drop the proof.

2. In the case $V \geqslant 0$, Lemma 6.3.1 gives a better information than Lemma 6.3.2. Indeed, it says that for every $t \geqslant 0$, $\|\nabla w(t)\|_{L^2} \leqslant \mathcal{E}_0^{1/2} + \|\nabla\varphi\|_{L^2}$. Coming back to the examples presented in the introduction, this implies that $\|\nabla w(t)\|_{L^2}$ remains bounded on $\mathbb{R}$, for the pure powers and for the saturated nonlinearities.

3. In the one-dimensional case, if φ_v is one of the traveling bubbles studied by Zhiwu Lin or a black soliton, φ_v satisfies the assumption (H_φ). Thus (6.1) is globally well-posed in $\varphi_v + H^1$. In the cases where φ_v is unstable, one can not expect to prove instability by blow-up and the mechanism of instability seems unknown so far.

6.4 Local theory for H^1 solutions

In this section, we prove that (6.3) is locally well-posed in $H^1(\mathbb{R}^n)$, for $n=2,3$ or 4, in both sub-critical and critical cases. Namely, we prove that for every $w_0 \in H^1(\mathbb{R}^n)$ (small in H^1 if $n=3,4$ and $\alpha_1=\alpha_1^*$), there exists $T>0$ and a unique solution $w \in \mathcal{C}([0,T],H^1(\mathbb{R}^n))$ of

$$w = e^{it\Delta}w_0 - i\int_0^t e^{i(t-s)\Delta}F(w(s))ds. \tag{6.17}$$

We employ a fixed point argument for the map

$$\Phi(w) = e^{it\Delta}w_0 - i\int_0^t e^{i(t-s)\Delta}F(w(s))ds. \tag{6.18}$$

in the space

$$X_T = L_T^\infty H^1 \cap L_T^{p_0}W^{1,q_0}$$

equipped with its natural norm $\|w\|_{X_T} = \|w\|_{L_T^\infty H^1} + \|w\|_{L_T^{p_0}W^{1,q_0}}$, where (p_0,q_0) is an admissible pair. A pair $(p,q)\in[2,\infty]$ is said to be admissible if

$$\frac{2}{p}+\frac{n}{q}=\frac{n}{2},\ (p,q)\neq(2,\infty).$$

We fix $(p_0,q_0)=(2,6)$ for $n=3$, $(p_0,q_0)=(2,4)$ for $n=4$, while for $n=2$, q_0 may be choosen large, but finite (and thus $p_0>2$ is close to 2). We will use the Strichartz estimates which we recall now (for a proof, we refer to [KeT]).

Proposition 6.4.1 *For every admissible pairs (p,q) and $(\tilde p,\tilde q)$, for every $v_0\in L^2(\mathbb{R}^n)$ and $f\in L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))$,*

$$\|e^{it\Delta}v_0\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\lesssim\|v_0\|_{L^2(\mathbb{R}^n)} \tag{6.19}$$

and

$$\|\int_{-\infty}^t e^{i(t-\tau)\Delta}f(\tau)d\tau\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\lesssim\|f\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}. \tag{6.20}$$

The result we next prove is as follows.

Theorem 6.4.1 *We assume that f satisfies (H_{α_1,α_2}) for some $\alpha_1\geqslant 1$, with the supplementary condition $\alpha_1\leqslant\alpha_1^*$ if $n=3,4$ (where $\alpha_1^*=3$ if $n=3$, $\alpha_1^*=2$ if $n=4$), and some $\alpha_2\in[\alpha_1-1/2,\alpha_1]$.*

If $n=2$, or $n=3,4$ and $\alpha_1<\alpha_1^*$, then for every $R>0$, there exists $T(R)>0$ such that for every $w_0\in H^1$ with $\|w_0\|_{H^1}\leqslant R$, there exists an unique $w\in X_{T(R)}$ solving (6.17). Moreover, $w\in\mathcal{C}([0,T(R)],H^1)$.
If for some $T>0$, $\tilde w\in\mathcal{C}([0,T],H^1)$ solves (6.17) then $\tilde w\in X_T$, and $\tilde w$ is the unique solution to (6.17) in $\mathcal{C}([0,T],H^1)$.
The flow map is locally Lipshitz continuous.

For $n=3,4$ and $\alpha_1=\alpha_1^*$, there exists $R>0$ and $T>0$ such that for every $w_0\in H^1$ with $\|w_0\|_{H^1}\leqslant R$, there exists an unique $w\in X_T$ solving (6.17). It is the unique solution in $\mathcal{C}([0,T],H^1)$. The flow map $w_0\mapsto w$, $H^1\mapsto X_T$ (with the same small T) is locally Lipshitz continuous.

The next four lemmas give the estimates which will enable us to apply the fixed point argument.

Lemma 6.4.1 *Let $T>0$ and $w\in X_T$. Then*

$$\begin{aligned}\|\Phi(w)\|_{L^\infty_TL^2}+\|\Phi(w)\|_{L^{p_0}_TL^{q_0}} &\leqslant \|w_0\|_{L^2}+CT(1+\|w\|_{L^\infty_TL^2})\\ &\quad +CT^{1/p'}(\|w\|^2_{L^\infty_TH^1}+\|w\|^{\max(2,2\alpha_1-1)}_{L^\infty_TH^1}),\end{aligned} \tag{6.21}$$

where C is a positive constant.

Proof. We first decompose F in the following way:

$$\begin{aligned} F(w) &= -\Delta\varphi - f(|\varphi+w|^2)(\varphi+w) \\ &= \left.-\Delta\varphi - f(|\varphi|^2)\varphi - f(|\varphi|^2)w - 2\mathfrak{Re}[w\overline{\varphi}]f'(|\varphi|^2)\varphi \right\} =: F_1(w) \\ &\quad \left.\begin{array}{l} -2\int_0^1 \mathfrak{Re}[w\overline{\varphi+sw}]f'(|\varphi+sw|^2)dsw - 2|w|^2\varphi\int_0^1 sf'(|\varphi+sw|^2)ds \\ -4\mathfrak{Re}[w\overline{\varphi}]\varphi\int_0^1\int_0^1 \mathfrak{Re}[sw\overline{\varphi+s\tau w}]f''(|\varphi+s\tau w|^2)d\tau ds. \end{array}\right\} =: F_2(w) \end{aligned}$$

The Strichartz inequalities (6.19) and (6.20) yield

$$\|\Phi(w)\|_{L^\infty_T L^2} + \|\Phi(w)\|_{L^{p_0}_T L^{q_0}} \leqslant \|w_0\|_{L^2} + C\|F_1(w)\|_{L^1_T L^2} + C\|F_2(w)\|_{L^{p'}_T L^{q'}}, \quad (6.22)$$

where (p,q) is any admissible pair and $C>0$. For the sequel, we fix

$$(p',q') = \begin{cases} (4/3,4/3) & \text{if } n=2, \\ (2,6/5) & \text{if } n=3, \\ (2,4/3) & \text{if } n=4. \end{cases} \quad (6.23)$$

On the one hand,

$$\|F_1(w)\|_{L^1_T L^2} \lesssim T(1+\|w\|_{L^\infty_T L^2}), \quad (6.24)$$

while on the other hand, using the first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$,

$$|F_2(w)| \lesssim |w|^2(1+|w|)^{\max(0,2\alpha_1-3)}. \quad (6.25)$$

Thus,

$$\begin{aligned} \|F_2(w)\|_{L^{p'}_T L^{q'}} &\lesssim \|w\|^2_{L^{2p'}_T L^{2q'}} + \|w\|^{\max(2,2\alpha_1-1)}_{L^{p'\max(2,2\alpha_1-1)}_T L^{q'\max(2,2\alpha_1-1)}} \\ &\lesssim T^{1/p'}\|w\|^2_{L^\infty_T H^1} + T^{1/p'}\|w\|^{\max(2,2\alpha_1-1)}_{L^\infty_T H^1}, \end{aligned} \quad (6.26)$$

because of the Hölder inequality in time and Sobolev embeddings. Note that $q'\max(2,2\alpha_1-1)$ is finite if $n=2$, and is not larger than 6 if $n=3$, than 4 if $n=4$, thanks to $\alpha_1 \leqslant \alpha_1^*$. Concatenating (6.22), (6.24) and (6.26), we obtain (6.21) (the constant C may have change). □

We next prove the same kind of estimation for $\nabla F(w)$.

Lemma 6.4.2 *There exists $\theta \geqslant 0$, with $\theta = 0$ only if $n=3,4$ and $\alpha_1 = \alpha_1^*$, such that for every $T>0$ and $w \in X_T$,*

$$\begin{aligned} \|\nabla\Phi(w)\|_{L^\infty_T L^2} + \|\nabla\Phi(w)\|_{L^{p_0}_T L^{q_0}} &\leqslant \|\nabla w_0\|_{L^2} + CT(1+\|\nabla w\|_{L^\infty_T L^2}) \\ &+C(1+\|\nabla w\|_{L^\infty_T L^2})(T^{1/p'}\|w\|_{L^\infty_T H^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{X_T}), \end{aligned} \quad (6.27)$$

where $C>0$.

Proof. Let us first write

$$\begin{aligned}
\nabla F(w) &= -\nabla\Delta\varphi - f(|\varphi+w|^2)\nabla(\varphi+w) - 2\mathfrak{Re}[\nabla(\varphi+w)\overline{\varphi+w}](\varphi+w)f'(|\varphi+w|^2)\\
&= \left.-\nabla\Delta\varphi - f(|\varphi|^2)\nabla(\varphi+w) - 2\mathfrak{Re}[\nabla(\varphi+w)\overline{\varphi}]f'(|\varphi|^2)\varphi \quad\right\} =: G_1(w)\\
&\quad \left.\begin{aligned}
&-2\int_0^1 \mathfrak{Re}[w\overline{\varphi+sw}]f'(|\varphi+sw|^2)ds\nabla(\varphi+w)\\
&-4\mathfrak{Re}[\nabla(\varphi+w)\overline{\varphi}]\varphi\int_0^1 \mathfrak{Re}[w\overline{\varphi+sw}]f''(|\varphi+sw|^2)ds\\
&-2\mathfrak{Re}[\nabla(\varphi+w)\overline{w}](\varphi+w)f'(|\varphi+w|^2)\\
&-2\mathfrak{Re}[\nabla(\varphi+w)\overline{\varphi}]wf'(|\varphi+w|^2).
\end{aligned}\right\} =: G_2(w)
\end{aligned}$$

Next, thanks to the Strichartz inequalities (6.19) and (6.20),

$$\|\nabla\Phi(w)\|_{L^\infty_T L^2} + \|\nabla\Phi(w)\|_{L^{p_0}_T L^{q_0}} \leqslant \|\nabla w_0\|_{L^2} + C\|G_1(w)\|_{L^1_T L^2} + C\|G_2(w)\|_{L^{p'}_T L^{q'}}, \tag{6.28}$$

with the same choice of p', q' as in (6.23), and $C > 0$. Next,

$$\|G_1(w)\|_{L^1_T L^2} \lesssim T(1 + \|\nabla w\|_{L^\infty_T L^2}) \tag{6.29}$$

and

$$|G_2(w)| \lesssim |\nabla(\varphi+w)||w|(1+|w|)^{\max(0,2\alpha_1-3)}, \tag{6.30}$$

which implies, thanks to the Hölder inequality and Sobolev embeddings,

$$\begin{aligned}
\|G_2(w)\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\varphi+w)\|_{L^\infty_T L^2}\left(\|w\|_{L^{p'}_T L^\beta} + \|w\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}L^{\beta\max(1,2\alpha_1-2)}}\right)\\
&\lesssim (1+\|\nabla w\|_{L^\infty_T L^2})\left(T^{1/p'}\|w\|_{L^\infty_T H^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{L^s_T W^{1,r}}\right),
\end{aligned} \tag{6.31}$$

where $1/q' = 1/2 + 1/\beta$ (our choice of q' gives $\beta = 4$ if $n = 2$ or 4, $\beta = 3$ if $n = 3$), $\theta = \frac{1}{p'} - \frac{\max(1,2\alpha_1-2)}{s}$, and the pair (s, r) is chosen such that:

- $(s, r) = (\infty, 2)$ if $\frac{1}{2} - \frac{1}{n} \leqslant \frac{1}{\beta\max(1,2\alpha_1-2)}$
- else, $r > 2$ and
  - (i) $\frac{2}{s} + \frac{n}{r} = \frac{n}{2}$ (which means that (s, r) is an admissible pair),
  - (ii) $\frac{1}{r} - \frac{1}{n} \leqslant \frac{1}{\beta\max(1,2\alpha_1-2)}$ (which gives the Sobolev embedding $W^{1,r} \subset L^{\beta\max(1,2\alpha_1-2)}$),
  - (iii) $\frac{1}{p'\max(1,2\alpha_1-2)} \geqslant \frac{1}{s}$ (that is $\theta \geqslant 0$).

Such a choice of s and r is possible if and only if

$$\frac{n}{2} - 1 \leqslant \left(\frac{n}{\beta} + \frac{2}{p'}\right)\frac{1}{\max(1, 2\alpha_1 - 2)}. \tag{6.32}$$

Indeed, if (6.32) is true, it suffices to choose $n/r \in [\frac{n}{2} - \frac{2}{p'\max(1,2\alpha_1-2)}, 1 + \frac{n}{\beta\max(1,2\alpha_1-2)}]$. Moreover, if (6.32) is a strict inequality, r and s may be chosen in such a way that $\theta > 0$.

For $n = 2$, (6.32) is always satisfied and is strict, while for $n = 3$ or 4, (6.32) is equivalent to $\alpha_1 \leqslant \alpha_1^*$, and is strict if and only if $\alpha_1 < \alpha_1^*$.
Since $r \in [2, q_0]$ (taking q_0 large enough, this can always be assumed for $n = 2$, while for $n = 3, 4$, $n/r \geqslant n/2 - 1 = n/q_0$), and (s, r) is an admissible pair, we obtain by interpolation

$$\|w\|_{L^s_T W^{1,r}} \lesssim \|w\|^{\tilde{\theta}}_{L^\infty_T H^1} \|w\|^{1-\tilde{\theta}}_{L^{p_0}_T W^{1,q_0}} \lesssim \|w\|_{X_T}, \tag{6.33}$$

where $\tilde{\theta} \in [0, 1]$. We deduce (6.27) from (6.28), (6.29), (6.31) and (6.33). □

In the next two lemmas, we evaluate $\Phi(w_1) - \Phi(w_2)$ in X_T, provided $w_1, w_2 \in X_T$.

Lemma 6.4.3 *There exists $\theta_0 > 0$ and $\theta_1 \geqslant 0$ (with $\theta_1 = 0$ only if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$) such that for every $T > 0$ and $w_1, w_2 \in L^\infty_T H^1 \subset L^{p_0}_T L^{q_0}$,*

$$\begin{aligned}
&\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2} + \|\Phi(w_1) - \Phi(w_2)\|_{L^{p_0}_T L^{q_0}} \\
&\quad \lesssim \quad T\|w_1 - w_2\|_{L^\infty_T L^2} + (\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}}) \\
&\qquad \times \left(T^{\theta_0}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + T^{\theta_1}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1, 2\alpha_1 - 2)}\right)
\end{aligned} \tag{6.34}$$

Proof. First,

$$\begin{aligned}
&F(w_1) - F(w_2) \\
&= \left.2\mathfrak{Re}[(w_2 - w_1)\overline{\varphi}]f'(|\varphi|^2)\varphi + f(|\varphi|^2)(w_2 - w_1) \right\}\delta_1(w_1, w_2) \\
&\left.\begin{aligned}
&+4\mathfrak{Re}[(w_2 - w_1)\overline{\varphi}] \int_0^1 \int_0^1 \mathfrak{Re}[(w_1 + s(w_2 - w_1))\overline{\varphi + \tau(w_1 + s(w_2 - w_1))}] \\
&\qquad\qquad \times f''(|\varphi + \tau(w_1 + s(w_2 - w_1))|^2) ds d\tau \varphi \\
&+2\int_0^1 \mathfrak{Re}[(w_2 - w_1)\overline{\varphi + w_1 + s(w_2 - w_1)}] f'(|\varphi + w_1 + s(w_2 - w_1)|^2) ds w_2 \\
&+2\int_0^1 \mathfrak{Re}[(w_2 - w_1)\overline{w_1 + s(w_2 - w_1)}] f'(|\varphi + w_1 + s(w_2 - w_1)|^2) ds \varphi \\
&+2\int_0^1 \mathfrak{Re}[w_1\overline{\varphi + sw_1}] f'(|\varphi + sw_1|^2) ds (w_2 - w_1).
\end{aligned}\right\}\delta_2(w_1, w_2)
\end{aligned}$$

Next, with p', q' as in (6.23), we get by the non-homogeneous Strichartz estimate (6.20)

$$\begin{aligned}
&\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2} + \|\Phi(w_1) - \Phi(w_2)\|_{L^{p_0}_T L^{q_0}} \\
&\quad \leqslant \quad \|\delta_1(w_1, w_2)\|_{L^1_T L^2} + \|\delta_2(w_1, w_2)\|_{L^{p'}_T L^{q'}}.
\end{aligned} \tag{6.35}$$

It can easily be seen that

$$\|\delta_1(w_1, w_2)\|_{L^1_T L^2} \quad \lesssim \quad T\|w_1 - w_2\|_{L^\infty_T L^2}\ , \tag{6.36}$$

and

$$|\delta_2(w_1,w_2)| \lesssim |w_1-w_2|(|w_1|+|w_2|)(1+|w_1|+|w_2|)^{\max(0,2\alpha_1-3)} . \qquad (6.37)$$

Thus, by Hölder and Sobolev,

$$\begin{aligned} \|\delta_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} &\lesssim \|w_1-w_2\|_{L_T^{p'}L^{2q'}}(\|w_1\|_{L_T^\infty L^{2q'}}+\|w_2\|_{L_T^\infty L^{2q'}}) \\ &+\|w_1-w_2\|_{L_T^{p'}L^{q_1}}\||w_1|+|w_2|\|^{\max(1,2\alpha_1-2)}_{L_T^\infty L^{q_2\max(1,2\alpha_1-2)}}, \end{aligned} \qquad (6.38)$$

where $1/q'=1/q_1+1/q_2$, with $(q_1,q_2)=(2,\beta)$ if $n=2$ or $n=3$ and $\alpha_1\leqslant 2$, whereas $q_2=q_0/\max(1,2\alpha_1-2)$ if n=3 and $\alpha_1>2$ or $n=4$. Thus, by Sobolev and Hölder in time,

$$\begin{aligned} \|\delta_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} &\lesssim T^{\frac{1}{p'}-\frac{1}{p_3}}\|w_1-w_2\|_{L_T^{p_3}L^{2q'}}(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1}) \\ &+T^{\frac{1}{p'}-\frac{1}{p_2}}\|w_1-w_2\|_{L_T^{p_2}L^{q_1}}\||w_1|+|w_2|\|^{\max(1,2\alpha_1-2)}_{L_T^\infty H^1}, \end{aligned} \qquad (6.39)$$

where $(p_3,2q')$ and (p_2,q_1) are admissible pairs (q_1,q_2 have been chosen in sort that $2\leqslant q_1\leqslant q_0$). Moreover, $1/p'-1/p_3>0$, and our choice of q_2 ensures that $1/p'-1/p_2>0$ as soon as $n=2$ or $n=3,4$ and $\alpha_1<\alpha_1^*$. An interpolation argument yields the announced result as in the proof of Lemma 6.4.2. □

We next estimate $\nabla(\Phi(w_1)-\Phi(w_2))$ in $L_T^\infty L_2\cap L_T^{p_0}L^{q_0}$.

Lemma 6.4.4 *There exists $\theta_2,\theta_3>0$, with $\theta_2=0$ and $\theta_3=0$ only if $n\neq 2$ and $\alpha_1=\alpha_1^*$, such that for every $T>0$, $w_1,w_2\in X_T$,*

$$\begin{aligned} \|\nabla\Phi(w_1)-\nabla\Phi(w_2)\|_{L_T^\infty L^2}+\|\nabla\Phi(w_1)-\nabla\Phi(w_2)\|_{L_T^{p_0}L^{q_0}} &\lesssim T\|\nabla(w_1-w_2)\|_{L_T^\infty L^2} \\ +T^{1/p'}(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})^{\max(1,2\alpha_1-2)}\|w_1-w_2\|_{L_T^\infty H^1} & \\ +T^{\theta_2}\|w_1-w_2\|_{X_T}(\|w_1\|^{\max(1,2\alpha_1-2)}_{X_T}+\|w_2\|^{\max(1,2\alpha_1-2)}_{X_T}) & \\ +T^{\theta_3}\|w_1-w_2\|_{X_T}(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})(\|w_1\|^{\max(0,2\alpha_1-3)}_{X_T}+\|w_2\|^{\max(0,2\alpha_1-3)}_{X_T}). & \end{aligned} \qquad (6.40)$$

Proof. We first write

$$
\begin{aligned}
&\nabla F(w_1)-\nabla F(w_2)\\
&=2\mathfrak{Re}[\nabla(w_2-w_1)\overline{\varphi}]f'(|\varphi|^2)\varphi+f(|\varphi|^2)\nabla(w_2-w_1) && \Big\}\ \gamma_1(w_1,w_2)\\
&\quad\left.\begin{aligned}
&+2\mathfrak{Re}[\nabla(w_2-w_1)\overline{\varphi}]\textstyle\int_0^1 2\mathfrak{Re}[w_2\overline{\varphi+sw_2}]f''(|\varphi+sw_2|^2)ds\varphi\\
&+2\mathfrak{Re}[\nabla(w_2-w_1)\overline{\varphi+w_2}]f'(|\varphi+w_2|^2)w_2\\
&+2\mathfrak{Re}[\nabla(w_2-w_1)\overline{w_2}]f'(|\varphi+w_2|^2)\varphi\\
&+2\textstyle\int_0^1\mathfrak{Re}[w_1\overline{\varphi+sw_1}]f'(|\varphi+sw_1|^2)ds\nabla(w_2-w_1)
\end{aligned}\right\}\ \gamma_2(w_1,w_2)\\
&\quad\left.\begin{aligned}
&+2\textstyle\int_0^1\mathfrak{Re}[\nabla(\varphi+w_1)\overline{w_2-w_1}]f'(|\varphi+w_1+s(w_2-w_1)|^2)(\varphi+w_1+s(w_2-w_1))ds\\
&+4\textstyle\int_0^1\mathfrak{Re}[\nabla(\varphi+w_1)\overline{\varphi+w_1+s(w_2-w_1)}]\mathfrak{Re}[(w_2-w_1)\overline{\varphi+w_1+s(w_2-w_1)}]\\
&\qquad\qquad\times f''(|\varphi+w_1+s(w_2-w_1)|^2)(\varphi+w_1+s(w_2-w_1))ds\\
&+2\textstyle\int_0^1\mathfrak{Re}[\nabla(\varphi+w_1)\overline{\varphi+w_1+s(w_2-w_1)}]f'(|\varphi+w_1+s(w_2-w_1)|^2)(w_2-w_1)ds\\
&+2\textstyle\int_0^1\mathfrak{Re}[(w_2-w_1)\overline{\varphi+w_1+s(w_2-w_1)}]f'(|\varphi+w_1+s(w_2-w_1)|^2)ds\nabla(\varphi+w_2).
\end{aligned}\right\}\ \gamma_3(w_1,w_2)
\end{aligned}
$$

The non-homogeneous Strichartz inequality (6.20) implies

$$
\begin{aligned}
&\|\nabla(\Phi(w_1)-\Phi(w_2))\|_{L^\infty_T L^2}+\|\nabla(\Phi(w_1)-\Phi(w_2))\|_{L^{p_0}_T L^{q_0}}\\
&\quad\leqslant\ \|\gamma_1(w_1,w_2)\|_{L^1_T L^2}+\|\gamma_2(w_1,w_2)\|_{L^{p'}_T L^{q'}}+\|\gamma_3(w_1,w_2)\|_{L^{p'}_T L^{q'}},
\end{aligned}
\tag{6.41}
$$

where (p',q') is given by (6.23). Next,

$$
\|\gamma_1(w_1,w_2)\|_{L^1_T L^2}\ \lesssim\ T\|\nabla(w_1-w_2)\|_{L^\infty_T L^2},
\tag{6.42}
$$

while

$$
|\gamma_2(w_1,w_2)|\ \lesssim\ |\nabla(w_2-w_1)|(|w_1|+|w_2|)(1+|w_1|+|w_2|)^{\max(0,2\alpha_1-3)}
\tag{6.43}
$$

and

$$
|\gamma_3(w_1,w_2)|\ \lesssim\ (|\nabla\varphi|+|\nabla w_1|+|\nabla w_2|)|w_1-w_2|(1+|w_1|+|w_2|)^{\max(0,2\alpha_1-3)}.
\tag{6.44}
$$

Using the Hölder inequality, we get

$$
\begin{aligned}
\|\gamma_2(w_1,w_2)\|_{L^{p'}_T L^{q'}}\ &\lesssim\ \|\nabla(w_1-w_2)\|_{L^\infty_T L^2}\\
&\times\left(\|w_1\|_{L^{p'}_T L^\beta}+\|w_2\|_{L^{p'}_T L^\beta}+\||w_1|+|w_2|\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}L^{\beta\max(1,2\alpha_1-2)}}\right)
\end{aligned}
\tag{6.45}
$$

and

$$
\begin{aligned}
&\|\gamma_3(w_1,w_2)\|_{L^{p'}_T L^{q'}}\ \lesssim\ (\|\nabla\varphi\|_{L^2}+\|\nabla w_1\|_{L^\infty_T L^2}+\|\nabla w_2\|_{L^\infty_T L^2})\\
&\quad\times\Big(\|w_1-w_2\|_{L^{p'}_T L^\beta}+\|(w_1-w_2)|w_1|^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^\beta}+\|(w_1-w_2)|w_2|^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^\beta}\Big)
\end{aligned}
\tag{6.46}
$$

Next, the Hölder inequality in time and the Sobolev embeddings ensure that for $j=1,2$,

$$\|w_j\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_j\|_{L_T^\infty H^1}$$

and

$$\|w_j\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}L^{\beta\max(1,2\alpha_1-2)}} \lesssim T^{\theta_2}\|w_j\|^{\max(1,2\alpha_1-2)}_{L_T^s W^{1,r}},$$

with the same choice of $s,\ r$ and $\theta_2=\theta$ we did to get (6.31). It follows then from (6.45) as in the proof of Lemma 6.4.2 that

$$\begin{aligned}\|\gamma_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} &\lesssim \|\nabla(w_1-w_2)\|_{L_T^\infty L^2}\\ &\times\Big(T^{1/p'}(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})+T^{\theta_2}(\|w_1\|^{\max(1,2\alpha_1-2)}_{X_T}+\|w_2\|^{\max(1,2\alpha_1-2)}_{X_T})\Big)\end{aligned}\tag{6.47}$$

Using the same arguments, we also have

$$\|w_1-w_2\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_1-w_2\|_{L_T^\infty H^1}\tag{6.48}$$

and, if $\beta\max(1,2\alpha_1-2)\leqslant q_0$, for $j=1,2$,

$$\|(w_1-w_2)|w_j|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\begin{cases}\|w_1-w_2\|_{L_T^\infty H^1} & \text{if } 2\alpha_1-3\leqslant 0\\ \|w_1-w_2\|_{L_T^\infty L^{\frac{\beta}{1-\varepsilon}}}\|w_j\|^{2\alpha_1-3}_{L_T^\infty L^{\frac{\beta(2\alpha_1-3)}{\varepsilon}}} & \text{if } 2\alpha_1-3>0,\end{cases}$$

where $\varepsilon=\beta(2\alpha_1-3)/q_0$ (note that $2\alpha_1-3>0$ and $\beta(2\alpha_1-2)\leqslant q_0$ imply $\varepsilon\in(0,1)$). In that case, it follows from the Sobolev embeddings that

$$\|(w_1-w_2)|w_j|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_1-w_2\|_{L_T^\infty H^1}\|w_j\|^{\max(0,2\alpha_1-3)}_{L_T^\infty H^1}.\tag{6.49}$$

Next, if $\beta\max(1,2\alpha_1-2)>q_0$, since $\beta\leqslant q_0$, we have $2\alpha_1-3>0$ and the Hölder inequality yields

$$\|(w_1-w_2)|w_j|^{2\alpha_1-3}\|_{L_T^{p'}L^\beta} \lesssim \|w_1-w_2\|_{L_T^{p_1}L^{q_1}}\|w_j\|^{2\alpha_1-3}_{L_T^{p_2(2\alpha_1-3)}L^{q_2(2\alpha_1-3)}},$$

where $(p_1,q_1)=(2\alpha_1-2)(p',\beta)$ and $(p_2,q_2)=\frac{2\alpha_1-2}{2\alpha_1-3}(p',\beta)$. Then,

$$\|(w_1-w_2)|w_j|^{2\alpha_1-3}\|_{L_T^{p'}L^\beta} \lesssim \begin{cases} T^{\frac{1}{p'}-\frac{2\alpha_1-2}{s}}\|w_1-w_2\|_{L_T^sW^{1,r}}\|w_j\|^{2\alpha_1-3}_{L_T^sW^{1,r}} & \text{if } q_1>q_0\\ T^{1/p'}\|w_1-w_2\|_{L_T^\infty H^1}\|w_j\|^{2\alpha_1-3}_{L_T^\infty H^1} & \text{if } q_1\leqslant q_0.\end{cases}\tag{6.50}$$

If $n=2$, it is possible to choose q_0 large enough, such that $q_1\leqslant q_0$. If $n=3,4$ and $q_1>q_0$, r and s are chosen such that

(i) $\frac{2}{s}+\frac{n}{r}=\frac{n}{2}$,

(ii) $\frac{1}{r} - \frac{1}{n} \leqslant \frac{1}{q_1} < \frac{1}{q_0} = \frac{1}{2} - \frac{1}{n}$ (thus $r > 2$),

(iii) $\frac{1}{p_1} - \frac{1}{s} \geqslant 0$.

These conditions may be satisfied if and only if $\frac{n}{2} - \frac{2}{p_1} \leqslant 1 + \frac{n}{q_1}$, which is true if $\alpha_1 \leqslant 3$ for $n = 3$, $\alpha_1 \leqslant 2$ for $n = 4$. Moreover, as in the proof of Lemma 6.4.2, if $n = 2$ or $n = 3,4$ and $\alpha_1 < \alpha_1^*$, s and r may be chosen in such a way that $\theta_3 := \frac{1}{p'} - \frac{2\alpha_1 - 2}{s} > 0$.

As in the proof of Lemma 6.4.2, it follows from an interpolation argument and from (6.46), (6.48), (6.49) and (6.50) that

$$\begin{aligned} \|\gamma_3(w_1, w_2)\|_{L_T^{p'} L^{q'}} &\lesssim T^{1/p'}(1 + \|w_1\|_{L_T^\infty H^1} + \|w_2\|_{L_T^\infty H^1})^{\max(1, 2\alpha_1 - 2)} \|w_1 - w_2\|_{L_T^\infty H^1} \\ &+ T^{\theta_3}(1 + \|w_1\|_{L_T^\infty H^1} + \|w_2\|_{L_T^\infty H^1}) \|w_1 - w_2\|_{X_T} (\|w_1\|_{X_T}^{\max(0, 2\alpha_1 - 3)} + \|w_2\|_{X_T}^{\max(0, 2\alpha_1 - 3)}). \quad (6.51) \end{aligned}$$

for some $\theta_3 \geqslant 0$, with $\theta_3 = 0$ only if $n = 3,4$ and $\alpha_1 = \alpha_1^*$. Concatenating (6.41), (6.42), (6.47) and (6.51), we obtain the announced result. □

The fixed point argument in the sub-critical case. The last four lemmas enable us to apply a fix-point argument to Φ in X_T. We first consider the cases $n = 2$ and $n = 3,4$ with $\alpha_1 < \alpha_1^*$. Let us take $R > 0$ and $w_0 \in H^1$, $\|w_0\|_{H^1} \leqslant R$. Let B_{R+1} be the ball of radius $R + 1$ in X_T, for some $T > 0$. Thanks to Lemmas 6.4.1 and 6.4.2, since $\theta > 0$ in (6.27), Φ maps B_{R+1} into itself as soon as T is chosen small enough. Since $\theta_0, \theta_1, \theta_2, \theta_3 > 0$, Lemmas 6.4.3 and 6.4.4 then ensure that Φ defines a contraction on B_{R+1}, taking if necessary T even smaller. Then, existence and uniqueness of a solution to (6.17) in X_T follows from a fixed point argument. Revisiting all the arguments above with $X_T = L_T^\infty H^1 \cap L_T^{p_0} W^{1,q_0}$ replaced by $\mathcal{C}([0,T], H^1) \cap L_T^{p_0} W^{1,q_0}$, we deduce that this solution belongs to $\mathcal{C}([0,T], H^1)$.

The fixed point argument in the critical case. Let us now take care of the case $n = 3$ or 4, $\alpha_1 = \alpha_1^*$. Since $\theta = \theta_1 = \theta_2 = \theta_3 = 0$ in Lemmas 6.4.2, 6.4.3 and 6.4.4, the argument we used for $\alpha_1 < \alpha_1^*$ breaks down. However, since $\max(1, 2\alpha_1 - 2) > 1$, using Lemmas 6.4.1 and 6.4.2, Φ maps B_{2R} into itself for every $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, provided R and T are small enough. In a similar way, taking R and T even smaller if necessary, Φ defines a contraction on B_{2R}, thanks to Lemmas 6.4.3 and 6.4.4, and because $\max(0, 2\alpha_1 - 3) > 0$.

In order to complete the proof of Theorem 6.4.1, it remains to show the uniqueness in $\mathcal{C}([0,T], H^1)$, as well as the Lipshitz-continuity of the flow. We first prove this in the case $n = 2$ or $n = 3,4$ and $\alpha_1 < \alpha_1^*$.

Proof of Theorem 6.1.1: the uniqueness. Let $T > 0$ and $w_1, w_2 \in \mathcal{C}([0,T], H^1)$ be two solutions to (6.17) with initial data $w_1(0) = w_2(0) = w_0 \in H^1$. Then by Lemma 6.4.3, for $\tilde{T} \leqslant \min(T, 1)$, defining $\tilde{\theta} := \min(1, \theta_0, \theta_1) > 0$,

$$
\begin{aligned}
&\|w_1 - w_2\|_{L^\infty_{\tilde T} L^2} + \|w_1 - w_2\|_{L^{p_0}_{\tilde T} L^{q_0}} \\
&\quad \leqslant C\tilde T^{\tilde\theta} \left(\|w_1 - w_2\|_{L^\infty_{\tilde T} L^2} + \|w_1 - w_2\|_{L^{p_0}_{\tilde T} L^{q_0}} \right) \\
&\quad \times \left(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + (\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}\right).
\end{aligned}
$$

Since $\tilde\theta > 0$, we can choose $\tilde T$ small enough, in such a way that

$$
C\tilde T^{\tilde\theta} \left(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + (\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}\right) < 1,
$$

which implies that $w_1 \equiv w_2$ on $[0, \tilde T]$. Since $\tilde T$ only depends on $\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}$, we can reiterate this argument on small intervals of length $\tilde T$ until the whole interval $[0, T]$ is recovered. This proves the uniqueness of a solution to (6.17) in $\mathcal{C}([0,T], H^1)$.

We next prove that if for some $T > 0$, $w \in \mathcal{C}([0,T], H^1)$ solves (6.17), then $w \in X_T$. Let $T > 0$ and $w \in \mathcal{C}([0,T], H^1)$ be a solution to (6.17). Let us define $R_0 := \|w\|_{L^\infty_T H^1}$. From the contraction argument developped above, we deduce that there exists $T(R_0) > 0$ such that for every data in the ball of radius R_0 in H^1, there exists an unique solution in $X_{T(R_0)}$ with this initial condition. It is the unique solution in $\mathcal{C}([0, T(R_0)], H^1)$ with that initial data. Thanks to this argument, for every $k \in \mathbb{N}$ such that $I_k := [kT(R_0), (k+1)T(R_0)] \cap [0,T] \neq \emptyset$, there exists $w_k \in \mathcal{C}(I_k, H^1) \cap L^{p_0}(I_k, W^{1,q_0})$ which solves (6.17), with $w_k(kT(R_0)) = w(kT(R_0))$. The uniqueness of the solution in $\mathcal{C}(I_k, H^1)$ implies that w coincides with w_k on I_k. In particular, $w_{|I_k} \in L^{p_0}(I_k, W^{1,q_0})$, and thus $w \in X_T$.

Next, we prove the local Lipshitz continuity of the flow, in the sub-critical case.

Proof of the local Lipschitz continuity of the flow in the sub-critical case. We first define $R = \|w\|_{L^\infty_T H^1} + 1$. The contraction argument we employed above ensures that there exists $T(R) \in (0,1)$ such that for every data $\tilde{w_0}$ in the ball of radius R in H^1, there exists an unique solution to (6.17) (with w_0 replaced by $\tilde{w_0}$) in $X_{T(R)}$. This solution has been obtained by a contraction argument in the ball of radius $R+1$ in $X_{T(R)}$. In particular, its $X_{T(R)}$-norm is less than $R+1$. Thus, if $w_{0,k} \in H^1$ satisfies $\|w_{0,k} - w(kT(R))\|_{H^1} \leqslant 1$ for some $k \in \mathbb{N}$ such that $kT(R) \leqslant T$, there exists $w_k \in X_{T(R)}$ solving (6.17) (with w_0 replaced by $w_{0,k}$). Defining $\theta_4 = \min(\theta_0, \theta_1, \theta_2, \theta_3) > 0$, slightly modified versions of Lemmas 6.4.3 and 6.4.4 yield

$$
\begin{aligned}
&\|w_k - w(kT(R) + .)\|_{X_{T(R)}} \\
&\quad \leqslant C\|w_{0,k} - w(kT(R))\|_{H^1} + C\|w_k - w(kT(R) + .)\|_{X_{T(R)}} T(R)^{\theta_4} (1 + R^{\max(1,2\alpha_1-1)}),
\end{aligned}
$$

where $C > 0$. Up to a change of $T(R)$, one may assume that

$$
CT(R)^{\theta_4}(1 + R^{\max(1,2\alpha_1-1)}) \leqslant 1/2,
$$

in such a way that

$$\|w_k - w(kT(R)+.)\|_{X_{T(R)}} \leqslant 2C\|w_{0,k} - w(kT(R))\|_{H^1}.$$

If $\tilde{w}_0$ satisfies $\|\tilde{w}_0 - w_0\|_{H^1} \leqslant (1/\max(1,2C))^{\lceil \frac{T}{T(R)} \rceil}$, a solution $\tilde{w}$ to (6.17) with w_0 replaced by $\tilde{w}_0$ may be constructed step by step by this argument, recovering $[0,T]$ by intervals of length $T(R)$. We deduce that $T^*(\tilde{w}_0) \geqslant T^*(w_0) \geqslant T$. Moreover,

$$\|w - \tilde{w}\|_{X_T} \lesssim \|\tilde{w}_0 - w_0\|_{H^1},$$

which completes the proof of the local Lipshitz continuity of the flow if $n = 2$ or $n = 3,4$ and $\alpha_1 < \alpha_1^*$.

Proof of Theorem 6.1.3: the uniqueness. In the critical case $n = 3,4$ and $\alpha_1 = \alpha_1^*$, let as before be T and R small enough such that Φ defines a contraction on B_{2R}. Let $w_1, w_2 \in \mathcal{C}([0,T],H^1)$ be two solutions to (6.17) with the same initial condition $w_0 \in H^1$, which satisfies $\|w_0\|_{H^1} \leqslant R$. Defining $\tilde{\theta} = \min(1,\theta_0)$, Lemma 6.4.3 provides, replacing T by $\min(T,1)$,

$$\begin{aligned}\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}} \quad \leqslant \quad & C(\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}}) \\ & \times(T^{\tilde{\theta}}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + (4R)^{2\alpha_1^* - 2}).\end{aligned}$$

Taking T and R even smaller if necessary, one may assume that

$$C(T^{\tilde{\theta}}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + (4R)^{2\alpha_1^* - 2}) < 1,$$

which implies that $w_1 \equiv w_2$ on $[0,T]$.

Proof of Theorem 6.1.3: the local Lipschitz continuity of the flow. Let $w \in X_T$ be a solution to (6.17), with $\|w_0\|_{H^1} \leqslant R/2$. Let $\tilde{w}_0 \in H^1$ be such that $\|w_0 - \tilde{w}_0\|_{H^1} \leqslant R/2$. Previous results ensure that there exists $\tilde{w} \in X_T$ which is a solution to (6.17) with w_0 replaced by w_0. Then, taking $\check{\theta} = \min(1, 1/p', \theta_0)$, modified versions of Lemmas 6.4.3 and 6.4.4 yield

$$\begin{aligned}\|w - \tilde{w}\|_{X_T} \quad \leqslant \quad & C\|w_0 - \tilde{w}_0\|_{H^1} + CT^{\check{\theta}}\|w - \tilde{w}\|_{X_T}(1 + \|w\|_{L^\infty_T H^1} + \|\tilde{w}\|_{L^\infty_T H^1})^{2\alpha_1^* - 2} \\ & + C\|w - \tilde{w}\|_{X_T}(\|w\|_{X_T} + \|\tilde{w}\|_{X_T})^{2\alpha_1^* - 2} \\ & + C\|w - \tilde{w}\|_{X_T}(1 + \|w\|_{L^\infty_T H^1} + \|\tilde{w}\|_{L^\infty_T H^1})(\|w\|_{X_T} + \|\tilde{w}\|_{X_T})^{2\alpha_1^* - 3} \\ \leqslant \quad & C\|w_0 - \tilde{w}_0\|_{H^1} \\ & + C\|w - \tilde{w}\|_{X_T}(T^{\check{\theta}}(1 + 4R)^{2\alpha_1^* - 2} + (4R)^{2\alpha_1^* - 2} + (1 + 4R)(4R)^{2\alpha_1^* - 3}).\end{aligned} \tag{6.52}$$

Choosing T and R even smaller if necessary, one may assume that

$$C(T^{\check{\theta}}(1 + 4R)^{2\alpha_1^* - 2} + (4R)^{2\alpha_1^* - 2} + (1 + 4R)(4R)^{2\alpha_1^* - 3}) < 1/2.$$

Under this condition, (6.52) induces the Lipshitz continuity of the flow on small intervals of time.

6.5 The global well-posedness

As it was remarked at the end of section 3, Theorem 6.1.1 has already been proved in the one-dimensional case. This section is devoted to the proof of a persistency result which, once combined with the results of the previous sections, will give the global well-posedness of (6.17) in H^1, for dimensions $n = 2, 3, 4$, in the sub-critical case. In the following, $(n, m) = (2, 2), (3, 2)$ or $(4, 3)$.

Let $w_0 \in H^m$, and $T_m^*(w_0) > 0$ be the maximal time of existence of a solution w to (6.3) in $H^m(\mathbb{R}^n)$, given by Theorem 6.2.1, in such a way that $\|w(t)\|_{H^m} \underset{t \uparrow T_m^*(w_0)}{\longrightarrow} \infty$ if $T_m^*(w_0)$ is finite. We also define

$$T_1^*(w_0) = \sup\{T > 0, \text{there exists a solution to (6.17) in } X_T\}.$$

Since $\mathcal{C}([0, T], H^m) \subset X_T$, it is clear that $T_m^*(w_0) \leqslant T_1^*(w_0)$. Let us assume by contradiction that $T_m^*(w_0) < T_1^*(w_0)$. In particular, $T_m^*(w_0) < \infty$. The uniqueness result in Theorem 6.4.1 ensures that w is the restriction to $[0, T_m^*(w_0))$ of a function which lives in $X_{T_1^*(w_0)-\varepsilon}$, for every $\varepsilon \in (0, T_1^*(w_0) - T_m^*(w_0))$. The results of section 3 ensure that $\|w(t)\|_{H^1}$ remains bounded on $[0, T_m^*(w_0))$. Therefore $\sum\limits_{2 \leqslant |\alpha| \leqslant m} \|\partial^\alpha w(t)\|_{L^2} \to \infty$ as $t \uparrow T_m^*(w_0)$.

Let us differentiate (6.3) twice, in directions x_j and x_k. Using also the Taylor formula, it follows that $\partial_{j,k}^2 w$ solves

$$\begin{aligned}
&i\partial_t \partial_{j,k}^2 w + \Delta \partial_{j,k}^2 w \\
&\quad = \left.\begin{aligned}&-\Delta\partial_{j,k}^2 \varphi - f(|\varphi|^2)\partial_{j,k}^2(\varphi + w) - 2\mathfrak{Re}\left[\partial_{j,k}^2(\varphi + w)\overline{\varphi}\right] f'(|\varphi|^2)\varphi\end{aligned}\right\} g_0(t) \\
&\qquad \left.\begin{aligned}
&-2\int_0^1 \mathfrak{Re}\left[w\overline{\varphi + sw}\right] f'(|\varphi + sw|^2) ds \partial_{j,k}^2(\varphi + w) \\
&-2\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\varphi + w)\overline{w}\right] f'(|\varphi + sw|^2)(\varphi + sw) ds \\
&-4\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\varphi + w)\overline{\varphi + sw}\right] \mathfrak{Re}\left[w\overline{\varphi + sw}\right] f''(|\varphi + sw|^2)(\varphi + sw) ds \\
&-2\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\varphi + w)\overline{\varphi + sw}\right] f'(|\varphi + sw|^2) w ds
\end{aligned}\right\} g_1(t) \\
&\qquad \left.\begin{aligned}
&-2\mathfrak{Re}\left[\partial_k(\varphi + w)\overline{\varphi + w}\right] \partial_j(\varphi + w) f'(|\varphi + w|^2) \\
&-2\mathfrak{Re}\left[\partial_j(\varphi + w)\overline{\varphi + w}\right] \partial_k(\varphi + w) f'(|\varphi + w|^2) \\
&-2\mathfrak{Re}\left[\partial_j(\varphi + w)\overline{\partial_k(\varphi + w)}\right] (\varphi + w) f'(|\varphi + w|^2) \\
&-4\mathfrak{Re}\left[\partial_j(\varphi + w)\overline{\varphi + w}\right] \mathfrak{Re}\left[\partial_k(\varphi + w)\overline{\varphi + w}\right] f''(|\varphi + w|^2)(\varphi + w).
\end{aligned}\right\} g_2(t)
\end{aligned}$$

It follows from the Strichartz estimates (6.19) and (6.20) that for $T < T_m^*(w_0)$,

$$\|\partial_{j,k}^2 w\|_{L_T^\infty L^2} \leqslant \|\partial_{j,k}^2 w_0\|_{L^2} + \|g_0\|_{L_T^1 L^2} + \|g_1\|_{L_T^{p'} L^{q'}} + \|g_2\|_{L_T^{p'} L^{q'}}, \tag{6.53}$$

where p', q' are given by (6.23). First,

$$\|g_0\|_{L_T^1 L^2} \lesssim T(1 + \|\partial_{j,k}^2 w\|_{L_T^\infty L^2}) . \tag{6.54}$$

Next,

$$|g_1(s)| \lesssim |w|(1+|w|^{\max(0,2\alpha_1-3)})|\partial^2_{j,k}(\varphi+w)|,$$

while

$$\begin{aligned} |g_2(s)| &\lesssim (1+|w|^{\max(0,2\alpha_1-3)})|\partial_j(\varphi+w)||\partial_k(\varphi+w)| && (6.55)\\ &\lesssim (1+|w|^{\max(1,2\alpha_1-3)})|\partial_j(\varphi+w)||\partial_k(\varphi+w)|. && (6.56)\end{aligned}$$

Then, using arguments developped in the proof of Lemma 6.4.2 to control $\|G_2(w)\|_{L^{p'}_T L^{q'}}$, we obtain

$$\begin{aligned} \|g_1\|_{L^{p'}_T L^{q'}} &\lesssim \|\partial^2_{j,k}(\varphi+w)\|_{L^\infty_T L^2}(\|w\|_{L^{p'}_T L^\beta} + \|w\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)} L^{\beta\max(1,2\alpha_1-2)}})\\ &\lesssim \|\partial^2_{j,k}(\varphi+w)\|_{L^\infty_T L^2}(T^{1/p'}\|w\|_{L^\infty_T H^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{X_T}), && (6.57)\end{aligned}$$

where β and θ are the same as in Lemma 6.4.2. Using (6.56), we also have

$$\begin{aligned} \|g_2\|_{L^{p'}_T L^{q'}} &\lesssim T^{1/p'}\|\partial_j(\varphi+w)\|_{L^\infty_T L^{2q'}}\|\partial_k(\varphi+w)\|_{L^\infty_T L^{2q'}}\\ &+\|\partial_j(\varphi+w)\partial_k(\varphi+w)|w|^{\max(1,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}} && (6.58)\end{aligned}$$

From now on, we distinguish the cases $n=2$, $n=3$ and $n=4$. For $n=2$, thanks to Hölder, Sobolev and Gagliardo-Nirenberg inequalities, (6.58) yields

$$\begin{aligned} \|g_2\|_{L^{4/3}_T L^{4/3}} &\lesssim T^{3/4}(1+\|w\|_{L^\infty_T H^1})^{3/2}(1+\|\Delta w\|_{L^\infty_T L^2})^{1/2}\\ &+T^{3/4}(1+\|w\|_{L^\infty_T H^1})(1+\|\Delta w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-3)}_{L^\infty_T H^1}. && (6.59)\end{aligned}$$

For $n=3$, the same arguments yield

$$\begin{aligned} \|g_2\|_{L^2_T L^{6/5}} &\lesssim T^{1/2}(1+\|w\|_{L^\infty_T H^1})^{3/2}(1+\|\Delta w\|_{L^\infty_T L^2})^{1/2}\\ &+T^{\frac12-\frac{\max(1,2\alpha_1-3)}{s}}(1+\|w\|_{L^\infty_T H^1})(1+\|\Delta w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-3)}_{L^s_T W^{1,r}}, && (6.60)\end{aligned}$$

where s and r are chosen in such a way that

(i) $\frac{2}{s}+\frac{3}{r}=\frac{3}{2}$,

(ii) $\frac{1}{r}-\frac{1}{3}\leqslant\frac{1}{6\max(1,2\alpha_1-3)}$,

(iii) $\frac{1}{2}-\frac{\max(1,2\alpha_1-3)}{s}>0$.

These conditions may be satisfied, provided $\alpha_1 < 3$.

For $n = 4$, we deduce from (6.55) and the Gagliardo-Nirenberg inequality that if $\alpha_1 \leqslant 3/2$,

$$\begin{aligned} \|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2} \|\partial_j(\varphi + w)\|_{L^\infty_T L^{8/3}} \|\partial_k(\varphi + w)\|_{L^\infty_T L^{8/3}} \\ &\lesssim T^{1/2}(1 + \|w\|_{L^\infty_T H^1})(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}), \end{aligned} \tag{6.61}$$

while if $\alpha_1 > 3/2$ (and $\alpha_1 < 2$),

$$\begin{aligned} \|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}(1 + \|w\|_{L^\infty_T H^1})(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad + \|\partial_j(\varphi + w)\partial_k(\varphi + w)|w|^{2\alpha_1 - 3}\|_{L^2_T L^{4/3}}. \end{aligned} \tag{6.62}$$

Let us fix $\varepsilon := \frac{1}{2\alpha_1 - 3} - 1 > 0$. Then by Hölder,

$$\begin{aligned} &\|\partial_j(\varphi + w)\partial_k(\varphi + w)|w|^{2\alpha_1-3}\|_{L^2_T L^{4/3}} \\ &\leqslant \||\nabla(\varphi + w)|^{\frac{1}{1+\varepsilon}}\|_{L^2_T L^{4(1+\varepsilon)}} \||\nabla(\varphi + w)|^{\frac{1+2\varepsilon}{1+\varepsilon}}\|_{L^\infty_T L^{4\frac{1+\varepsilon}{1+3\varepsilon}}} \||w|^{2\alpha_1-3}\|_{L^\infty_T L^{4(1+\varepsilon)}} \\ &= \|\nabla(\varphi + w)\|^{\frac{1}{1+\varepsilon}}_{L_T^{\frac{2}{1+\varepsilon}} L^4} \|\nabla(\varphi + w)\|^{\frac{1+2\varepsilon}{1+\varepsilon}}_{L^\infty_T L^{4\frac{1+2\varepsilon}{1+3\varepsilon}}} \|w\|^{2\alpha_1-3}_{L^\infty_T L^4} \end{aligned} \tag{6.63}$$

Next, the Hölder inequality in time yields

$$\|\nabla(\varphi + w)\|^{\frac{1}{1+\varepsilon}}_{L_T^{\frac{2}{1+\varepsilon}} L^4} \lesssim T^{\frac{1}{1+\varepsilon}(\frac{1+\varepsilon}{2} - \frac{1}{2})} \|\nabla(\varphi + w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4} = T^{\frac{\varepsilon}{2(1+\varepsilon)}} \|\nabla(\varphi + w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4}. \tag{6.64}$$

It follows from the Gagliardo-Nirenberg inequality that

$$\|\nabla(\varphi + w)\|^{\frac{1+2\varepsilon}{1+\varepsilon}}_{L^\infty_T L^{4\frac{1+2\varepsilon}{1+3\varepsilon}}} \lesssim \|\nabla(\varphi + w)\|^{\frac{\varepsilon}{1+\varepsilon}}_{L^\infty_T L^2} \|\Delta(\varphi + w)\|_{L^\infty_T L^2}, \tag{6.65}$$

and by Sobolev

$$\|w\|^{2\alpha_1-3}_{L^\infty_T L^4} \lesssim \|w\|^{2\alpha_1-3}_{L^\infty_T H^1}. \tag{6.66}$$

We deduce from (6.62), (6.63), (6.64), (6.65) and (6.66) that if $\alpha_1 > 3/2$,

$$\begin{aligned} \|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}(1 + \|w\|_{L^\infty_T H^1})(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad + T^{\frac{\varepsilon}{2(1+\varepsilon)}} \|\nabla(\varphi + w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4} \|\nabla(\varphi + w)\|^{\frac{\varepsilon}{1+\varepsilon}}_{L^\infty_T L^2} \|\Delta(\varphi + w)\|_{L^\infty_T L^2} \|w\|^{2\alpha_1-3}_{L^\infty_T H^1} \end{aligned} \tag{6.67}$$

Concatenating (6.53), (6.54), (6.57), as well as (6.59), (6.60), (6.61) or (6.67), and summing over the indices of length 2, we get

$$
\begin{aligned}
\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2} &\lesssim \sum_{|\alpha|=2} \|\partial^\alpha w_0\|_{L^2} + T(1+\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}) \\
&\quad +(T^{1/p'}\|w\|_{L^\infty_{T^*_k(w_0)}H^1} + T^\theta \|w\|^{\max(1,2\alpha_1-2)}_{X_{T^*_k(w_0)}})(1+\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}) \\
&\quad + \begin{cases} T^{1/p'}(1+\|w\|_{L^\infty_{T^*_k(w_0)}H^1})^{3/2}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{1/2} & \\ +T^{\tilde\theta}(1+\|w\|_{L^\infty_{T^*_k(w_0)}H^1})\|w\|^{\max(1,2\alpha_1-3)}_{X_{T^*_k(w_0)}} & \text{if } n=2,3 \\ +T^{\check\theta}(1+\|w\|_{X_{T^*_k(w_0)}})\|w\|^{\max(0,2\alpha_1-3)}_{X_{T^*_k(w_0)}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) & \text{if } n=4, \end{cases}
\end{aligned}
\tag{6.68}
$$

where $\tilde\theta, \check\theta > 0$. Thus there exists a small $T_0 > 0$ depending only on $\|w\|_{X_{T^*_k(w_0)}} < \infty$ such that

$$
\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_{T_0} L^2} \leqslant C(\|w\|_{X_{T^*_k(w_0)}}) + C\sum_{|\alpha|=2} \|\partial^\alpha w_0\|_{L^2},
$$

where $C>0$ and $C(\|w\|_{X_{T^*_k(w_0)}})$ only depends on $\|w\|_{X_{T^*_k(w_0)}}$. Recovering $[0, T^*_k(w_0)]$ by a finite number of intervals of length T_0, it follows that $\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}$ remains bounded as $T \uparrow T^*_k(w_0)$. This is a contradiction in dimensions $n=2$ and $n=3$.

For $n=4$, we need to control the derivatives of order 3 of w. Denoting by $\partial^3 w$ one of these derivatives, $\partial^3 w$ solves an equation which may be written as

$$
i\partial_t \partial^3 w + \Delta \partial^3 w = f_0 + f_1 + f_2 + f_3,
$$

where

$$
|f_0(t)| \lesssim |\Delta\partial^3\varphi| + |\varphi|^2|f'(|\varphi|^2)|\partial^3(\varphi+w)| + |f(|\varphi|^2)||\partial^3(\varphi+w)|,
$$

$$
|f_1(t)| \lesssim |\partial(\varphi+w)|^3\left[|f'(|\varphi+w|^2)| + |\varphi+w|^2|f''(|\varphi+w|^2)| + |\varphi+w|^4|f'''(|\varphi+w|^2)|\right],
$$

$$
\begin{aligned}
|f_2(t)| &\lesssim |\partial^2(\varphi+w)||\partial(\varphi+w)|\left[|\varphi+w||f'(|\varphi+w|^2)| + |\varphi+w|^3|f''(|\varphi+w|^2)|\right] \\
&\lesssim |\partial^2(\varphi+w)||\partial(\varphi+w)|(1+|w|^{\max(0,2\alpha_1-3)})
\end{aligned}
\tag{6.69}
$$

and

$$
\begin{aligned}
|f_3(t)| &\lesssim |\partial^3(\varphi+w)|\int_0^1 \left[|\varphi+sw||f'(|\varphi+sw|^2)| + |\varphi+sw|^3|f''(|\varphi+sw|^2)|\right] ds|w| \\
&\lesssim |\partial^3(\varphi+w)||w|(1+|w|^{\max(0,2\alpha_1-3)}).
\end{aligned}
\tag{6.70}
$$

Thanks to the Strichartz estimates (6.19) and (6.20), for $T < T^*_3(w_0)$,

$$\|\partial^3 w\|_{L^\infty_T L^2} \lesssim \|\partial^3 w_0\|_{L^2} + \|f_0\|_{L^1_T L^2} + \|f_1\|_{L^2_T L^{4/3}} + \|f_2\|_{L^2_T L^{4/3}} + \|f_3\|_{L^2_T L^{4/3}}. \quad (6.71)$$

Next,

$$\|f_0\|_{L^1_T L^2} \lesssim T(1 + \|\partial^3 w\|_{L^\infty_T L^2}). \quad (6.72)$$

Since $\alpha_1 < 2$, $(\mathrm{H}_{\alpha_1,\alpha_2})$ implies that $r \mapsto r^2 f'''(r)$, $r \mapsto r f''(r)$ and f' are bounded. Thus

$$\|f_1\|_{L^2_T L^{4/3}} \lesssim \|\partial(\varphi + w)\|^3_{L^6_T L^4} \lesssim T^{1/2}\|\partial(\varphi + w)\|^3_{L^\infty_T H^1}. \quad (6.73)$$

Using (6.69), we get

$$\begin{aligned} \|f_2\|_{L^2_T L^{4/3}} \lesssim\ & T^{1/2}\|\partial^2(\varphi + w)\|_{L^\infty_T L^2}\|\partial(\varphi + w)\|_{L^\infty_T L^4} \\ & + \begin{cases} 0 & \text{if } \alpha_1 \leqslant 3/2 \\ \|\partial^2(\varphi + w)\partial(\varphi + w)|w|^{2\alpha_1 - 3}\|_{L^2_T L^{4/3}} & \text{if } \alpha_1 > 3/2 \end{cases}, \end{aligned} \quad (6.74)$$

where, by Hölder and Sobolev, if $\alpha_1 > 3/2$, choosing $\varepsilon = 2\alpha_1 - 3 \in (0, 1)$,

$$\begin{aligned} & \|\partial^2(\varphi + w)\partial(\varphi + w)|w|^{2\alpha_1 - 3}\|_{L^2_T L^{4/3}} \\ \lesssim\ & T^{1/2}\|\partial^2(\varphi + w)\|_{L^\infty_T L^2}\|\partial(\varphi + w)\|_{L^\infty_T L^{\frac{4}{1-\varepsilon}}}\||w|^{2\alpha_1 - 3}\|_{L^\infty_T L^{\frac{4}{\varepsilon}}} \\ \lesssim\ & T^{1/2}(1 + \|\partial^2 w\|_{L^\infty_T L^2})\|\partial(\varphi + w)\|_{L^\infty_T H^2}\|w\|^{2\alpha_1 - 3}_{L^\infty_T H^1}. \end{aligned} \quad (6.75)$$

Thanks to (6.70) and Sobolev,

$$\begin{aligned} \|f_3\|_{L^2_T L^{4/3}} \lesssim\ & T^{1/2}\|\partial^3(\varphi + w)\|_{L^\infty_T L^2}\left(\|w\|_{L^\infty_T L^4} + \|w\|^{\max(1, 2\alpha_1 - 2)}_{L^\infty_T L^{4\max(1, 2\alpha_1 - 2)}}\right) \\ \lesssim\ & T^{1/2}(1 + \|\partial^3 w\|_{L^\infty_T L^2})(\|w\|_{L^\infty_T H^1} + \|w\|^{\max(1, 2\alpha_1 - 2)}_{L^\infty_T H^2}) \end{aligned} \quad (6.76)$$

Concatenating (6.71), (6.72), (6.73),(6.74), (6.75), (6.76) and summing over the indices of length 3, we get

$$\begin{aligned} \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2} \lesssim & \sum_{|\alpha|=3}\|\partial^\alpha w_0\|_{L^2} + T(1 + \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2}) + T^{1/2}(1 + \|w\|_{L^\infty_{T_3^*(w_0)} H^2})^3 \\ & + T^{1/2}(1 + \|\partial^2 w\|_{L^\infty_{T_3^*} L^2})(1 + \|w\|_{L^\infty_{T_3^*(w_0)} H^2} + \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2})(1 + \|w\|_{L^\infty_{T_3^*(w_0)} H^2})^{\max(0, 2\alpha_1 - 3)} \\ & + T^{1/2}(1 + \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2})(\|w\|_{L^\infty_{T_3^*(w_0)} H^1} + \|w\|^{\max(1, 2\alpha_1 - 2)}_{L^\infty_{T_3^*(w_0)} H^2}). \end{aligned}$$

Therefore there exists $T_1 > 0$ sufficiently small and $C(\|w\|_{L^\infty_{T_3^*(w_0)} H^2}) > 0$, both depending only on $\|w\|_{L^\infty_{T_3^*(w_0)} H^2} < \infty$ such that

$$\sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2} \lesssim \sum_{|\alpha|=3}\|\partial^\alpha w_0\|_{L^2} + C(\|w\|_{L^\infty_{T_3^*(w_0)} H^2}).$$

We can recover $[0, T_3^*(w_0)]$ by a finite number of intervals of length T_1, and thus $\sum_{|\alpha|=3} \|\partial^\alpha w\|_{L_T^\infty L^2}$ remain bounded as $T \uparrow T_3^*(w_0)$. We have obtained a contradiction in the four dimensional case.

We are now ready to prove the global well-posedness of (6.3). Let $w_0 \in H^1(\mathbb{R}^n)$, $n = 2, 3$ or 4, and $T > 0$ be such that there exists a solution $w \in \mathcal{C}([0, T], H^1)$ to (6.3) with initial data w_0 (such a T exists thanks to Theorem 6.4.1). Let us take a sequence $(w_{0,n})_n \subset H^k$ ($k = 2$ if $n = 2, 3$, while $k = 3$ if $n = 4$) which converges to w_0 in H^1. By the lower semi-continuity of T_1^* (which is a byproduct of the local Lipshitz continuity of the flow map), $T_1^*(w_{0,n}) \geqslant T_1^*(w_0) \geqslant T$ for n large. We have just seen that $T_k^*(w_{0,n}) = T_1^*(w_{0,n})$. Therefore for n large, the energy is conserved for w_n on $[0, T]$. Namely, for all $t \in [0, T]$, $\mathcal{E}(w_n(t)) = \mathcal{E}(w_{0,n})$, where

$$\mathcal{E}(w) = \int_{\mathbb{R}^n} |\nabla(\varphi + w)|^2 dx + \int_{\mathbb{R}^n} V(|\varphi + w|^2) dx.$$

Moreover, $w_n \to w$ in X_T. The map $w \mapsto V(|\varphi + w|^2)$ is continuous from H^1 into L^1, as it can easily be deduced from (6.10), (6.11), Sobolev embeddings and the first condition in $(\mathrm{H}_{\alpha_1,\alpha_2})$. It follows that for every $t \in [0, T]$, $\mathcal{E}(w(t)) = \mathcal{E}(w_0)$. Moreover, for every $t \in [0, T]$, Lemmas 6.3.2 and 6.3.3 give the estimates

$$\|w_n(t)\|_{L^2}^2 \leqslant (1 + \|w_{0,n}\|_{L^2}^2) e^{C_3 t}$$

and

$$\|\nabla(\varphi + w_n(t))\|_{L^2}^2 \leqslant C_1(1 + (1 + \|w_{0,n}\|_{L^2}^2) e^{C_3 t}).$$

Passing to the limit $n \to \infty$, these inequalities remains true for w_n replaced by w, for every $t \in [0, T]$. It follows that the H^1 norm of w remains bounded on bounded intervals.

The proof of Theorem 6.1.1 will be complete if we show the Lipshitz continuity of the flow on bounded sets of H^1. That is what we next do.

Proof of the Lipshitz continuity of the flow. Let $T > 0$, $R > 0$ and $w_0, \tilde{w}_0 \in H^1$ with H^1 norm less than R. Let $w, \tilde{w} \in \mathcal{C}([0, T], H^1)$ be the associated solutions to (6.3). Then, as in the proof of the local Lipshitz continuity of the flow we gave in the previous section, slightly modified versions of Lemmas 6.4.3 and 6.4.4 yield for $\tilde{T} \leqslant T$,

$$\begin{aligned} \|w - \tilde{w}\|_{X_{\tilde{T}}} \;\leqslant\; & C\|w_0 - \tilde{w}_0\|_{H^1} \\ & + C\tilde{T}^\theta \|w - \tilde{w}\|_{X_{\tilde{T}}} (\|w\|_{X_T} + \|\tilde{w}\|_{X_T} + \|w\|_{X_T}^{\max(1, 2\alpha_1 - 2)} + \|\tilde{w}\|_{X_T}^{\max(1, 2\alpha_1 - 2)}) \end{aligned}$$

Thanks to our estimation on the H^1 norm of w, $\|w\|_{L_T^\infty H^1}$ and $\|\tilde{w}\|_{L_T^\infty H^1}$ are majored by a quantity which only depends on R and T. So are $\|w\|_{X_T}$ and $\|\tilde{w}\|_{X_T}$, because of the same arguments we used in the previous section to prove the local Lipshitz continuity of the flow. Thus there exists $h(R, T) > 0$ such that

$$\|w\|_{X_T} + \|\tilde{w}\|_{X_T} + \|w\|_{X_T}^{\max(1, 2\alpha_1 - 2)} + \|\tilde{w}\|_{X_T}^{\max(1, 2\alpha_1 - 2)} \leqslant h(R, T).$$

Choosing $\tilde{T} > 0$ small enough such that

$$C\tilde{T}^{\theta} h(R,T) \leqslant 1/2,$$

we obtain

$$\|w - \tilde{w}\|_{X_{\tilde{T}}} \leqslant 2C\|w_0 - \tilde{w}_0\|_{H^1}.$$

Next, recovering $[0,T]$ by small intervals of length $\tilde{T}$ and repeating this argument on each of these intervals, we get

$$\|w - \tilde{w}\|_{L^\infty_T H^1} \leqslant (2C)^{\lceil \frac{T}{\tilde{T}} \rceil}\|w_0 - \tilde{w}_0\|_{H^1},$$

which completes the proof. $\square$

6.6 Well-posedness in the energy space

This section is devoted to the well-posedness in the energy space. We prove Proposition 6.1.1 and Theorem 6.1.5, using arguments developed in [Ge].

6.6.1 Decomposition of a data in the energy space

Here, following P. Gérard, we give a proof of Proposition 6.1.1.

Let us take u in the energy space

$$E = \{u \in H^1_{\text{loc}}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), \rho_0 - |u|^2 \in L^2(\mathbb{R}^n)\}.$$

Let $\chi \in \mathcal{C}^\infty_c(\mathbb{C})$ be such that $0 \leqslant \chi \leqslant 1$, $\chi(z) \equiv \begin{cases} 1 & \text{if} \quad |z| \leqslant \sqrt{2\rho_0} \\ 0 & \text{if} \quad |z| \geqslant \sqrt{3\rho_0} \end{cases}$. We also choose $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho = 1$, $0 \leqslant \rho \leqslant 1$ and ρ is supported in the ball of radius 1.

We first decompose u as

$$u = (1-\chi)(u)u + \chi(u)u.$$

As it was mentioned by P. Gérard in [Ge], we have on the one side $(1-\chi)(u)u \in H^1(\mathbb{R}^n)$, and on the other side $\chi(u)u \in L^\infty(\mathbb{R}^n)$, $\nabla(\chi(u)u) \in L^2(\mathbb{R}^n)$, and thus $\chi(u)u \in H^1_{\text{loc}}(\mathbb{R}^n)$. Moreover, the choice of χ ensures

$$||\chi(u)u|^2 - \rho_0| = ||u|^2 - \rho_0| \text{ if } |u|^2 \leqslant 2\rho_0,$$

while if $|u|^2 \geqslant 2\rho_0$, one has

$$\begin{aligned} \big||\chi(u)u|^2 - \rho_0\big| &= \max(\chi(u)^2|u|^2 - \rho_0, \rho_0 - \chi(u)^2|u|^2) \\ &\leqslant \max(|u|^2 - \rho_0, \rho_0) \leqslant |u|^2 - \rho_0. \end{aligned}$$

In all these cases, we have $||\chi(u)u|^2-\rho_0|\leqslant ||u|^2-\rho_0|$. Therefore $\chi(u)u\in E$.

Next, we split $\chi(u)u$ as

$$\chi(u)u=\rho*(\chi(u)u)+(\chi(u)u-\rho*(\chi(u)u)).$$

Since $\chi(u)u\in X^1(\mathbb{R}^n)$, it is clear that $\psi:=\rho*(\chi(u)u)\in\mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\nabla\psi\in H^\infty(\mathbb{R}^n)$. In order to prove Proposition 6.1.1, it remains to verify that $|\psi|^2-\rho_0\in L^2(\mathbb{R}^n)$ and $\chi(u)u-\psi\in H^1(\mathbb{R}^n)$. Since $\chi(u)u\in E$, this is a consequence of the next two lemmas, which have been proved in [Ge].

Lemma 6.6.1 *If $v\in E$, $|\rho*v|^2-\rho_0\in L^2$*

Lemma 6.6.2 *If $v\in E$, $v-\rho*v\in H^1$.*

Proof of Lemma 6.6.1. This was proved in [Ge]. We recall the arguments.

$$\begin{aligned}|(\rho*v)(x)|^2-\rho_0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x-y)\rho(x-\tilde{y})(v(y)\overline{v}(\tilde{y})-\rho_0)dyd\tilde{y}\\ &= \underbrace{\rho*(|u|^2-\rho_0)}_{\in L^2}(x)+r(x),\end{aligned}$$

where

$$\begin{aligned}r(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x-y)\rho(x-\tilde{y})v(y)(\overline{v}(\tilde{y})-\overline{v}(y))dyd\tilde{y}\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x-y)\rho(x-\tilde{y})v(y)\int_0^1(\tilde{y}-y)\nabla\overline{v}(y+s(\tilde{y}-y))dsdyd\tilde{y}\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x-y)\rho(x-y-a)v(y)\int_0^1 a\nabla\overline{v}(y+sa)dsdyda\\ &= \int_0^1\int_{|a|\leqslant 2}a\left(\int_{\mathbb{R}^n}\rho_a(x-y)v(y)\nabla\overline{v}(y+sa)dy\right)dads,\end{aligned}$$

where $\rho_a(z)=\rho(z)\rho(z-a)$.

As it was mentioned in [Ge], for every positive integer n, $E\subset X^1(\mathbb{R}^n)+H^1(\mathbb{R}^n)$. In particular,

$$E\subset\begin{cases}L^\infty & \text{if } n=1\\ L^\infty+L^6 & \text{if } n=2\\ L^\infty+L^{\frac{2n}{n-2}} & \text{if } n\geqslant 3\end{cases}$$

since $\nabla v\in L^2$, the Hölder inequality yields

$$v\nabla\overline{v}(.+sa)\in\begin{cases}L^2 & \text{if } n=1\\ L^2+L^{3/2} & \text{if } n=2\\ L^2+L^{\frac{n}{n-1}} & \text{if } n\geqslant 3\end{cases},$$

with the norm of $v\nabla\overline{v}(.+sa)$ in the corresponding space uniformly bounded in s and a. Next, $\frac{1}{2}=\frac{1}{p}+\frac{1}{q}-1$, with respectively $q=2,\ 3/2,\ \frac{n}{n-1}$, yields respectively $p=1,\ 6/5,\ \frac{2n}{n+2}\in[1,\infty]$. ρ_a belongs to all these L^p spaces, with norm uniformly bounded in a. Therefore the Young inequality implies that the map

$$x\mapsto\int_{\mathbb{R}^n}\rho_a(x-y)u(y)\nabla\overline{u}(y+sa)dy$$

belongs to $L^2(\mathbb{R}^n)$, and the Lemma has been proved. □

Proof of Lemma 6.6.2. It is clear that $\nabla(v-\rho*v)\in L^2$. Let us verify that $v-\rho*v\in L^2$. Thanks to the properties of ρ,

$$\begin{aligned}\int_{\mathbb{R}^n}|\rho*v(x)-v(x)|^2dx &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\rho(y)(v(x-y)-v(x))dy\right|^2dx\\ &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\rho(y)\int_0^1-y\nabla v(x-sy)dsdy\right|^2dx\\ &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(y)\left|\int_0^1 y\nabla v(x-sy)ds\right|^2dydx\\ &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(y)\int_0^1|\nabla v(x-sy)|^2dsdydx\leqslant\|\nabla v\|_{L^2}^2.\end{aligned}$$

□

6.6.2 Uniqueness in the energy space.

Using arguments introduced by P. Gérard in [Ge], we prove here Theorem 6.1.5. In this section, χ denotes a cutoff function: $\chi\in\mathcal{C}_c^\infty(\mathbb{C})$, $0\leqslant\chi\leqslant1$, and

$$\chi(z)\equiv\begin{cases}1 \text{ for } |z|\leqslant1,\\ 0 \text{ for } |z|\geqslant2.\end{cases}$$

If $\mu>0$, we also denote $\chi_\mu(z)=\chi(z/\mu)$. We first state some elementary lemmas about the L^p+L^q spaces. The proofs are in the appendix.

Lemma 6.6.3 *Let $1\leqslant p<q\leqslant\infty$ and $f\in L^p+L^q$. Let $f_p\in L^p$ and $f_q\in L^q$ such that $f=f_p+f_q$. Then for every $\mu>0$ if $q<\infty$ (resp. $\mu>2\|f_q\|_{L^\infty}$ if $q=\infty$), $\chi_\mu(f)f\in L^q$ and $(1-\chi_\mu)(f)f\in L^p$. Moreover, we have the estimates*

$$\|\chi_\mu(f)f\|_{L^q}\leqslant\begin{cases}3\mu^{1-p/q}\|f_p\|_{L^p}^{p/q}+2\|f_q\|_{L^q} & \text{if } q<\infty\\ 2\mu & \text{if } q=\infty,\end{cases}$$

and

$$\|(1-\chi_\mu)(f)f\|_{L^p} \leqslant \begin{cases} \|f_p\|_{L^p} + \mu\left(\left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p + \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q\right)^{1/p} + \frac{\|f_q\|_{L^q}^{q/p}}{\mu^{\frac{q-p}{p}}} & \text{if } q<\infty \\ 3\|f_p\|_{L^p} & \text{if } q=\infty. \end{cases}$$

In particular, if $f_p \neq 0$, $f_q \neq 0$, defining μ_0 as follows,

$$\mu_0 = \begin{cases} \left(\frac{\|f_q\|_{L^q}^q}{\|f_p\|_{L^p}^p}\right)^{\frac{1}{q-p}} & \text{if } q<\infty \\ 3\|f_q\|_{L^\infty} & \text{if } q=\infty, \end{cases} \tag{6.77}$$

we have

$$\|\chi_{\mu_0}(f)f\|_{L^q} \quad \leqslant \quad 6\|f_q\|_{L^q} \tag{6.78}$$

and

$$\|(1-\chi_{\mu_0})(f)f\|_{L^p} \quad \leqslant \quad (2+(2^p+2^q)^{1/p})\|f_p\|_{L^p}. \tag{6.79}$$

Lemma 6.6.4 *If $|f| \leqslant |g|$ and $g \in L^p + L^q$ for some $p, q \in [1,\infty]$, then $f \in L^p + L^q$ with $\|f\|_{L^p+L^q} \leqslant C(p,q)\|g\|_{L^p+L^q}$, where $C(p,q)>0$.*

Lemma 6.6.5 *Let $1 \leqslant p_1 < p < p_2 \leqslant \infty$. Then $L^p \subset L^{p_1} + L^{p_2}$, with a continuous embedding.*

Lemma 6.6.6 *Let $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant q_1 < q_2 \leqslant \infty$, $f = f_1 + f_2$, where $f_j \in L^{p_j}$ and $g = g_1 + g_2$, where $g_j \in L^{q_j}$. Then $fg \in L^{\frac{p_1 q_1}{p_1+q_1}} + L^{\frac{p_2 q_2}{p_2+q_2}}$, and*

$$\|fg\|_{L^{\frac{p_1 q_1}{p_1+q_1}} + L^{\frac{p_2 q_2}{p_2+q_2}}} \leqslant C\|f\|_{L^{p_1}+L^{p_2}}\|q\|_{L^{q_1}+L^{q_2}},$$

where $C>0$ depends only on the p_j and the q_j.

Proof of Theorem 6.1.5. Let $T>0$, $u_0 \in E$, $u \in \mathcal{C}([0,T],E)$ be as in the statement of the theorem and $v \in u_0 + \mathcal{C}(\mathbb{R}, H^1) \subset \mathcal{C}(\mathbb{R}, E)$ the solution to (6.1) given by Theorem 6.1.2. Then, for every $t \in [0,T]$,

$$v(t) - u(t) = -i\int_0^t e^{i(t-s)\Delta}\left[G(v(s)) - G(u(s))\right] ds, \tag{6.80}$$

where $G(u) = -f(|u|^2)u$. Next,

$$\begin{aligned} G(v) - G(u) \quad &= \quad (|u|^2 - \rho_0)(u-v)\int_0^1 f'(\rho_0 + s(|u|^2-\rho_0))ds \\ &\qquad +v(|u|^2-|v|^2)\int_0^1 f'(|v|^2 + s(|u|^2-|v|^2))ds. \end{aligned} \tag{6.81}$$

The Sobolev embeddings ensure that $u, v \in L^\infty + H^1 \subset L^\infty + L^{q_0}$, where $q_0 > 2$ is as large as we want it to be if $n = 2$, $q_0 = 6$ if $n = 3$ and $q_0 = 4$ if $n = 4$. The first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$ ensures that for $s \in [0,1]$, $|f'(\rho_0 + s(|u|^2 - \rho_0))| \leqslant C(1 + |u|^2)^{\max(0,\alpha_1-2)}$, where $C > 0$. Thus, thanks to Lemma 6.6.4, $f'(\rho_0 + s(|u|^2 - \rho_0)) \in L^\infty + L^{\frac{q_0}{2\max(0,\alpha_1-2)}}$. In a similar way, $f'(|v|^2 + s(|u|^2 - |v|^2)) \in L^\infty + L^{\frac{q_0}{2\max(0,\alpha_1-2)}}$. It follows from Lemma 6.6.6 that

$$G(v) - G(u) \in L^2 + L^{q'},$$

where $\frac{1}{q'} = \frac{2\max(0,\alpha_1-2)}{q_0} + \frac{1}{2} + \frac{1}{q_0}$. For $n = 2$, q_0 is chosen large enough such that $q' > 1$. For $n = 3$, $q' = 6/\max(4, 2\alpha_1) \geqslant 6/5$ only if $\alpha_1 \leqslant 5/2$, while for $n = 4$, $q' = 4/3$. From (6.81), Lemma 6.6.6, Sobolev embeddings and Lemma 6.6.4 we deduce

$$\begin{aligned} \|G(v) - G(u)\|_{L^2+L^{q'}} \;\lesssim\; & (1 + \|u\|_{X^1+H^1})^{2\max(0,\alpha_1-2)} \|u - v\|_{X^1+H^1} d_E(u, \sqrt{\rho_0}) \\ & + \|v\|_{X^1+H^1} d_E(u, v)(1 + \|u\|_{X^1+H^1} + \|v\|_{X^1+H^1})^{2\max(0,\alpha_1-2)} \end{aligned} \tag{6.82}$$

Since $u, v \in \mathcal{C}([0,T], E)$, the right hand side in (6.82) is uniformly bounded on $[0,T]$. Therefore $G(v) - G(u) \in L^1_T L^2 + L^{p'}_T L^{q'}$, where (p, q) is an admissible pair. Thus, it follows from (6.80) and the non-homogeneous Strichartz estimate (6.20) that $u - v \in \mathcal{C}([0,T], L^2)$. Since $u, v \in \mathcal{C}([0,T], E)$, we already know that $\nabla(u - v) \in \mathcal{C}([0,T], L^2)$. Thus $u - v \in \mathcal{C}([0,T], H^1)$. The result follows for $n = 2$, $n = 4$, and $n = 3$ with the supplementary condition $\alpha_1 \leqslant 5/2$. Next, we prove the result for $n = 3$, $\alpha_1 \in (5/2, 3)$. In that case, (6.82) remains true, but with $q' \in (1, 6/5)$, in sort that the non-homogeneous Strichartz estimate may not be applied. However, we deduce from Lemma 6.6.3 and (6.82) that for every $\mu > 0$, $\chi_\mu(G(v) - G(u))(G(v) - G(u)) \in L^2$ (and $(1 - \chi_\mu)(G(v) - G(u))(G(v) - G(u)) \in L^{q'}$), with an L^2 norm uniformly bounded on $[0,T]$ by a quantity Q which only depends on μ, q', $\sup_{t\in[0,T]} \|u(t)\|_{X^1+H^1}$, $\sup_{t\in[0,T]} \|v(t)\|_{X^1+H^1}$, $\sup_{t\in[0,T]} d_E(u(t), v(t))$ and $\sup_{t\in[0,T]} d_E(u(t), \sqrt{\rho_0})$. Thus

$$\begin{aligned} & \|\chi_\mu(G(v) - G(u))(G(v) - G(u))\|_{L^1_T L^2} \\ & \leqslant\; TQ(\mu, q', \sup_{t\in[0,T]} \|u(t)\|_{X^1+H^1}, \sup_{t\in[0,T]} \|v(t)\|_{X^1+H^1}, \sup_{t\in[0,T]} d_E(u(t), v(t)), \sup_{t\in[0,T]} d_E(u(t), \sqrt{\rho_0})) \end{aligned} \tag{6.83}$$

Next, the first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$ ensures $|G(u)| \lesssim |u|(1 + |u|^2)^{\alpha_1 - 1}$. Since $u \in L^\infty + L^6$, Lemma 6.6.4 implies $G(u) \in L^\infty + L^{\frac{6}{2\alpha_1-1}}$. The same is true for $G(v)$, and we have

$$\|G(v) - G(u)\|_{L^\infty + L^{\frac{6}{2\alpha_1-1}}} \lesssim (1 + \|u\|_{X^1+H^1} + \|v\|_{X^1+H^1})^{2\alpha_1-1}.$$

The right hand side is uniformly bounded on $[0,T]$, thus, thanks to Lemma 6.6.3, $(1 - \chi_1)(G(v) - G(u))(G(v) - G(u)) \in L^{\tilde{p}'}_T L^{\tilde{q}'}$, where $(\tilde{p}, \tilde{q})$ is an admissible pair, and $\tilde{q}' = \frac{6}{2\alpha_1 - 1} \in (6/5, 3/2)$ if $\alpha_1 \in (5/2, 3)$. We have shown that $G(v) - G(u) \in L^1_T L^2 + L^{\tilde{p}'}_T L^{\tilde{q}'}$. As in the previous case, it follows from the non homogeneous Strichartz estimate that $u - v \in \mathcal{C}([0,T], H^1)$. $\square$

6.7 The case of an exterior domain

In this section, K denotes a smooth, compact, non-trapping, non-empty obstacle in $\mathbb{R}^n$, $n = 2, 3$, and $\Omega = \mathbb{R}^n \backslash K$. The Strichartz estimates we used in the previous sections on $\mathbb{R}^n$ fail when we are working on such an open set Ω. However, N. Burq, P. Gérard and N. Tzvetkov obtained in [BGT] the following Strichartz type estimates.

Proposition 6.7.1 *For every pair $(p, q) \in [2, \infty]$ such that*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2}, \tag{6.84}$$

for every $T > 0$, there exists $C(T) > 0$ such that for every $v_0 \in L^2(\Omega)$,

$$\|e^{it\Delta_D} v_0\|_{L^p_T L^q(\Omega)} \leqslant C(T) \|v_0\|_{L^2(\Omega)}, \tag{6.85}$$

and, if $(\tilde{p}, \tilde{q})$ satisfies (6.84) and moreover $p > 2$, $\tilde{p} > 2$, then for every $f \in L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)$,

$$\| \int_0^t e^{i(t-\tau)\Delta_D} f(\tau) d\tau \|_{L^p_T L^q(\Omega)} \leqslant C(T) \|f\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)}, \tag{6.86}$$

where Δ_D denotes the Dirichlet Laplacian. This is also true if Δ_D is replaced by the Neumann Laplacian Δ_N.

These Strichartz type inequalities are sufficient to prove the global well-posedness result for the initial value problem (6.5) stated in Theorem 6.1.6.

Proof of Theorem 6.1.6. The proof is very similar to that we made in the case of $\mathbb{R}^n$, so that we will only indicate the main changes. We only do it in the Dirichlet case, exactly the same arguments value for the Neumann case.

As in section 7, we remark that a data $u_0 \in E_D$ may be decomposed as $u_0 = \varphi + w_0$, where $w_0 \in H^1_0(\Omega)$ and

$$\varphi \in \mathcal{C}^\infty_b(\Omega), \ \nabla\varphi \in H^\infty(\Omega), \ \mathrm{Supp}\varphi \subset \Omega \backslash (V \cap \Omega), \ |\varphi|^2 - \rho_0 \in L^2(\Omega) \ .$$

Indeed, defining $\tilde{u}_0$ as the extension of u_0 to $\mathbb{R}^n$ by 0, the results of section 7 ensure that

$$\tilde{u}_0 = \tilde{\varphi} + \tilde{w}_0 = (1-\chi)\tilde{\varphi} + \chi\tilde{\varphi} + \tilde{w}_0 \ ,$$

where $\tilde{\varphi}$ satisfies (H_φ) and $\tilde{w}_0 \in H^1(\mathbb{R}^n)$. Then $\varphi := ((1-\chi)\tilde{\varphi})_{|\Omega}$ and $w_0 := (\chi\tilde{\varphi} + \tilde{w}_0)_{|\Omega}$ satisfy the required conditions.

As in the case $\Omega = \mathbb{R}^n$, we can look a solution to (6.5) as $u = \varphi + w$, where φ is as above and $w(t) \in H^1_0(\Omega)$. We are reduced to study the Cauchy problem

$$\left\{ \begin{array}{l} i\frac{\partial w}{\partial t} + \Delta_D w = F(w), \ (t,x) \in \mathbb{R} \times \Omega \\ w(0) = w_0 \end{array} \right. , \tag{6.87}$$

where $F(w) = -\Delta_D\varphi - f(|\varphi + w|^2)(\varphi + w)$. Revisiting the arguments developped in the proofs of Lemmas 6.2.1 and 6.2.2, it is easy to see that F maps $H_0^2(\Omega)$ into $H^2(\Omega)$ and that it is locally Lipshitz continuous. Moreover, if $w \in H_0^2(\Omega)$ and $(w_n)_n$ is a sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $w_n \to w$ in $H^2(\Omega)$, $F(w_n) \to F(w)$ in H^2 and $F(w_n) \in \mathcal{C}_c(\Omega)$. Thus $F(w) \in H_0^2(\Omega)$. Therefore F defines a locally Lipshitz continuous map from $H_0^2(\Omega)$ into itself.

Next, we remark that the operator A on $H_0^2(\Omega)$ defined by

$$\begin{cases} D(A) = \{w \in H_0^2(\Omega), \Delta_D w \in H_0^2(\Omega)\}, \\ Aw = i\Delta_D w \quad \text{for } w \in D(A) \end{cases}$$

generates a strongly continuous group e^{tA} which is the restriction of $e^{it\Delta_D}$ to $H_0^2(\Omega)$. Therefore the classical theory for nonlinear evolution equations implies that for every $w_0 \in H_0^2(\Omega)$, there exists a maximal time of existence $T^*(w_0)$ such that (6.87) has an unique mild solution $w \in \mathcal{C}([0, T^*), H_0^2(\Omega))$. If $w_0 \in H_0^4(\Omega)$, $w \in \mathcal{C}([0, T^*), D(A)) \cap \mathcal{C}^1((0, T^*), H_0^2(\Omega))$.

As in Lemma 6.3.1, we obtain the conservation of the energy first for $w_0 \in H_0^4(\Omega)$ and then for $w_0 \in H_0^2(\Omega)$ by density of $H_0^4(\Omega)$ into $H_0^2(\Omega)$. One may prove analogous results to Lemmas 6.3.2 and 6.3.3 with identical proofs.

As in the R^n case, the local well-posedness is obtained by a fixed point argument for the functional

$$\Phi(w) = e^{it\Delta_D} w - i\int_0^t e^{i(t-s)\Delta_D} F(w(s))ds$$

in a space $X_T = L_T^\infty H^1 \cap L_T^{p_0} W^{1,q_0}$, where (p_0, q_0) satisfies (6.84) and $p_0 > 2$ is close to 2. We begin by giving the analogous of the estimates established in Lemmas 6.4.1, 6.4.2, 6.4.3 and 6.4.4.

In the sequel, $\varepsilon_0 >$ and $\varepsilon' > 0$ satisfy the following conditions

$$\begin{cases} \varepsilon_0 < 1, \ \varepsilon' < 1 & \text{if } n = 2, \\ \begin{cases} \varepsilon' + \max(1, 2\alpha_1 - 2)\varepsilon_0 \leqslant 1 \\ \varepsilon' + \varepsilon_0 \leqslant 4 - 2\alpha_1 \end{cases} & \text{if } n = 3, \end{cases} \tag{6.88}$$

and $p_0,\ q_0,\ p',\ q'$ are defined by

	p_0	q_0	p'	q'
$n = 2$	$\frac{2}{1-\varepsilon_0} > 2$	$\frac{4}{1+\varepsilon_0} < 4$	$\frac{2}{1+\varepsilon'} < 2$	$\frac{4}{3-\varepsilon'} > 4/3$
$n = 3$	$\frac{2}{1-\varepsilon_0} > 2$	$\frac{6}{2+\varepsilon_0} < 3$	$\frac{2}{1+\varepsilon'} < 2$	$\frac{6}{4-\varepsilon'} > 3/2$

(6.89)

Lemma 6.7.1 *Let $n = 2, 3$, with $\alpha_1 < 2$ if $n = 3$, and let $T_0 > 0$, $w_0 \in H^1$. Then there exists $C > 0$ such that for every $w \in L_T^\infty H^1$, $T \leqslant T_0$,*

$$\begin{aligned} &\|\Phi(w)\|_{L_T^\infty L^2 \cap L_T^{p_0} L^{q_0}} \\ &\leqslant \ \|w_0\|_{L^2} + CT(1 + \|w\|_{L_T^\infty L^2}) + CT^{1/p'}(\|w\|^2_{L_T^\infty H^1} + \|w\|^{\max(2, 2\alpha_1 - 1)}_{L_T^\infty H^1}). \end{aligned} \tag{6.90}$$

Lemma 6.7.2 *Under the same assumptions, there exists $C > 0$ and $\theta > 0$ such that for every $w \in X_T$, $T \leqslant T_0$,*

$$\begin{aligned}\|\nabla\Phi(w)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} &\leqslant \|\nabla w_0\|_{L^2} + CT(1 + \|\nabla w\|_{L^\infty_T L^2}) \\ &+ CT^\theta(1 + \|w\|_{X_T})(\|w\|_{X_T} + \|w\|_{X_T}^{\max(1,2\alpha_1-2)}).\end{aligned} \tag{6.91}$$

Lemma 6.7.3 *Under the same assumptions that for Lemma 6.7.1, for every $w_1, w_2 \in L^\infty_T H^1$, $T \leqslant T_0$,*

$$\begin{aligned}&\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} \lesssim T\|w_1 - w_2\|_{L^\infty_T L^2} \\ &+ T^{1/p'}\|w_1 - w_2\|_{L^\infty_T H^1}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + \|w_1\|_{L^\infty_T H^1}^{\max(1,2\alpha_1-2)} + \|w_2\|_{L^\infty_T H^1}^{\max(1,2\alpha_1-2)})\end{aligned} \tag{6.92}$$

Lemma 6.7.4 *Under the same assumptions that for Lemma 6.7.1, there exists $\theta_1, \theta_2 > 0$ such that for every $w_1, w_2 \in X_T$, $T \leqslant T_0$,*

$$\begin{aligned}&\|\nabla\Phi(w_1) - \nabla\Phi(w_2)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} \lesssim T\|\nabla(w_1 - w_2)\|_{L^\infty_T L^2} \\ &\quad + T^{\theta_1}\|w_1 - w_2\|_{X_T}(\|w_1\|_{X_T} + \|w_2\|_{X_T}) \\ &\quad + T^{\theta_2}\|w_1 - w_2\|_{X_T}(1 + \|w_1\|_{X_T} + \|w_2\|_{X_T})(1 + \|w_1\|_{X_T}^{\max(0,2\alpha_1-3)} + \|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}).\end{aligned} \tag{6.93}$$

Proof of Lemma 6.7.1. Taking into account the new choice of parameters p_0, q_0, p', q' given by (6.89) and the new Strichartz estimates given in Proposition 6.7.1, the proof is similar to that of Lemma 6.4.1. In dimension 2, the Sobolev embedding $H^1 \subset L^{2q'}$ and $H^1 \subset L^{q'\max(2,2\alpha_1-1)}$ are true because $2q', q'\max(2, 2\alpha_1 - 1) \in [2, \infty)$. In dimension 3, they are true because $\varepsilon' \leqslant \min(2, 5 - 2\alpha_1)$, which implies $2q', q'\max(2, 2\alpha_1 - 1) \leqslant 6$. □

Proof of Lemma 6.7.2. The proof is rather similar to that of Lemma 6.4.2. ∇F may still be decomposed as $G_1 + G_2$. With the new parameters p_0, q_0, p', q', estimations (6.28), (6.29) and (6.30) remain true. In the two-dimensional case, using (6.30) as we did it to obtain (6.31), we easily get

$$\|G_2(w)\|_{L^{p'}_T L^{q'}} \lesssim T^{1/p'}(1 + \|w\|_{X_T})(\|w\|_{X_T} + \|w\|_{X_T}^{\max(1,2\alpha_1-2)}). \tag{6.94}$$

Let us give a little bit more details for the proof of a similar result in dimension three. Using (6.30), an estimation on the $L^{p'}_T L^{q'}$ norm of $G_2(w)$ may be reduced to estimations on both $\nabla(\varphi + w)w$ and $\nabla(\varphi + w)|w|^{\max(1,2\alpha_1-2)}$ for the same norm. Next, using the Hölder inequality and Sobolev embeddings and taking into account the value of parameters p' and q' given by (6.89), it follows that

$$\begin{aligned}\|\nabla(\varphi + w)w\|_{L_T^{\frac{2}{1+\varepsilon'}} L^{\frac{6}{4-\varepsilon'}}} &\lesssim \|\nabla(\varphi + w)\|_{L_T^{\frac{2}{1+\varepsilon'}} L^{\frac{6}{3-\varepsilon'}}}\|w\|_{L^\infty_T L^6} \\ &\lesssim T^{\frac{1+\varepsilon'}{2} - \frac{\varepsilon'}{2}}\|\nabla(\varphi + w)\|_{L_T^{\frac{2}{\varepsilon'}} L^{\frac{6}{3-\varepsilon'}}}\|w\|_{L^\infty_T H^1} \\ &\lesssim T^{1/2}(1 + \|w\|_{X_T})\|w\|_{X_T}.\end{aligned} \tag{6.95}$$

For the very last inequality, we used that the pair $(\frac{2}{\varepsilon'}, \frac{6}{3-\varepsilon'})$ satisfies (6.84) and an interpolation argument as in Lemma 6.4.2. This can be done, provided $\frac{6}{3-\varepsilon'} \leqslant \frac{6}{2+\varepsilon_0}$, which is equivalent to $\varepsilon' + \varepsilon_0 \leqslant 1$. This is a consequence of (6.88). The same arguments yield

$$\begin{aligned}\|\nabla(\varphi+w)|w|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\varphi+w)\|_{L^{p'}_T L^{\beta_1}} \||w|^{\max(1,2\alpha_1-2)}\|_{L^\infty_T L^{\frac{6}{\max(1,2\alpha_1-2)}}} \\ &\lesssim T^{\frac{1}{p'}-\frac{1}{s_1}} \|\nabla(\varphi+w)\|_{L^{s_1}_T L^{\beta_1}} \|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1} \\ &\lesssim T^{\frac{1}{p'}-\frac{1}{s_1}} (1+\|w\|_{X_T})\|w\|^{\max(1,2\alpha_1-2)}_{X_T}, \qquad (6.96)\end{aligned}$$

where β_1 is given by $1/q' = 1/\beta_1 + \max(1, 2\alpha_1 - 2)/6$, and $1/s_1 + 3/\beta_1 = 3/2$. (6.88) ensures that $2 \leqslant \beta_1 \leqslant q_0$. Moreover, $1/p' - 1/s_1 = \min(1, 4-2\alpha_1)/2 > 0$. Thanks to (6.28), (6.29), (6.94), (6.95) and (6.96), the lemma easily follows as in Lemma 6.4.2. □

Proof of Lemma 6.7.3. We use the decomposition of $F(w_1) - F(w_2)$ given in the proof of Lemma 6.4.3. Inequalities (6.35), (6.36) and (6.37) are still valuable. Using the same arguments that in the proof of Lemma 6.7.1 (in particular, $q' \max(2, 2\alpha_1 - 1) \leqslant 6$ in dimension 3), we obtain, for $j = 1, 2$,

$$\||w_1 - w_2||w_j|\|_{L^{p'}_T L^{q'}} \lesssim T^{\frac{1}{p'}} \|w_1 - w_2\|_{L^\infty_T H^1} \|w_j\|_{L^\infty_T H^1}$$

and

$$\||w_1 - w_2||w_j|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} \lesssim T^{\frac{1}{p'}} \|w_1 - w_2\|_{L^\infty_T H^1} \|w_j\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1}.$$

The lemma follows. □

Proof of Lemma 6.7.4. We use the decomposition of $\nabla F(w_1) - \nabla F(w_2)$ into $\gamma_1 + \gamma_2 + \gamma_3$ written in the proof of Lemma 6.4.4. Using (6.41) and (6.42) (which remain true, with the new value of p', q', p_0, q_0), it suffices to control the $L^{p'}_T L^{q'}$ norm of γ_2 and γ_3. This will be done thanks to estimates (6.43) and (6.44) as follows. As in the proof of Lemma 6.7.2, for $j = 1, 2$, we get

$$\|\nabla(w_1 - w_2)|w_j|\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_1} \|w_1 - w_2\|_{X_T} \|w_j\|_{L^\infty_T H^1},$$

$$\|(|\nabla\varphi| + |\nabla w_1| + |\nabla w_2|)(w_1 - w_2)\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_1} \|w_1 - w_2\|_{X_T} (1 + \|w_1\|_{X_T} + \|w_2\|_{X_T}),$$

$$\|\nabla(w_1 - w_2)|w_j|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_2} \|w_1 - w_2\|_{X_T} \|w_j\|^{\max(1,2\alpha_1-2)}_{X_T}, \qquad (6.97)$$

as well as

$$\begin{aligned}&\|(|\nabla\varphi| + |\nabla w_1| + |\nabla w_2|)(w_1 - w_2)(|w_1| + |w_2|)^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}} \\ &\lesssim T^{\theta_2} \|w_1 - w_2\|_{X_T} (1 + \|w_1\|_{X_T} + \|w_2\|_{X_T})(\|w_1\|_{X_T} + \|w_2\|_{X_T})^{\max(0,2\alpha_1-3)}, \qquad (6.98)\end{aligned}$$

where $\theta_1 = \theta_2 = 1/p'$ for $n = 2$, $\theta_1 = 1/2$ and $\theta_2 = 1/p' - 1/s_1 = \min(1, 4 - 2\alpha_1)/2 > 0$ for $n = 3$. The Lemma easily follows. $\square$

The local well posedness in X_T may be deduced from Lemmas 6.7.1, 6.7.2, 6.7.3 and 6.7.4 as we did it in section 4 for the $\mathbb{R}^n$ subcritical case. The uniqueness of a solution to (6.87) in $\mathcal{C}([0,T], H^1)$ and the Lipshitz continuity of the flow may be proven as in section 4.

Next, we prove the global well-posedness result. The strategy is similar to that we employed in Section 5 for the $\mathbb{R}^n$ case. As in Section 5, if $w \in \mathcal{C}([0,T], H_0^2(\Omega))$ solves (6.87), then $\partial_{j,k}^2 w$ solves

$$i\partial_t \partial_{j,k}^2 w + \Delta \partial_{j,k}^2 w = g_0 + g_1 + g_2.$$

With the new choice of p', q' we made in (6.89), (6.53) and (6.54) are still valuable. It remains to estimate $\|g_j\|_{L_T^{p'}L^{q'}}$, for $j = 1, 2$. In dimension 2, (6.57) remains true (with $\theta = 1/p'$), while it follows from the Gagliardo-Nirenberg inequality, (6.56) and Hölder that

$$\begin{aligned} \|g_2(s)\|_{L^{q'}(\Omega)} &\lesssim \|\nabla(\varphi + w)\|_{L^2(\Omega)}^{\frac{3-\varepsilon'}{2}} \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^2(\Omega)}\Big)^{\frac{1+\varepsilon'}{2}} \\ &\quad + \|w\|_{L^{\frac{4\max(1,2\alpha_1-3)}{1-\varepsilon'}}}^{\max(1,2\alpha_1-3)} \|\nabla(\varphi + w)\|_{L^2(\Omega)} \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^2(\Omega)}\Big), \end{aligned} \tag{6.99}$$

which implies that

$$\begin{aligned} \|g_2\|_{L_T^{p'}L^{q'}} &\lesssim T^{1/p'} \big(1 + \|w\|_{L_T^\infty H^1}\big)^{\frac{3-\varepsilon'}{2}} \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}\Big)^{\frac{1+\varepsilon'}{2}} \\ &\quad + T^{1/p'} \|w\|_{L_T^\infty H^1}^{\max(1,2\alpha_1-3)} \big(1 + \|w\|_{L_T^\infty H^1}\big) \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}\Big). \end{aligned} \tag{6.100}$$

In dimension 3, for $j, k \in \{1, 2, 3\}$, using the Hölder inequality, Sobolev embeddings and an interpolation argument, we get

$$\begin{aligned} \|\partial_{j,k}^2(\varphi + w)\|_{L_T^{p'}L^{q'}} &\lesssim \|\partial_{j,k}^2(\varphi + w)\|_{L_T^\infty L^2} \|w\|_{L_T^{p'}L^\beta} \\ &\lesssim \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}\Big) \|w\|_{L_T^{p'}W^{1,r_1}} \\ &\lesssim T^{1/2} \Big(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}\Big) \underbrace{\|w\|_{L_T^{s_1}W^{1,r_1}}}_{\lesssim \|w\|_{X_T}}, \end{aligned} \tag{6.101}$$

where $1/\beta = 1/q' - 1/2 = (1-\varepsilon')/6$, $1/r_1 = 1/3 + 1/\beta = (3-\varepsilon')/6$ and $1/s_1 = 3/2 - 3/r_1 =$

$\varepsilon'/2$. In a similar way,

$$\begin{aligned}
&\|\partial^2_{j,k}(\varphi+w)|w|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}}\\
&\lesssim \|\partial^2_{j,k}(\varphi+w)\|_{L^\infty_T L^2}\|w\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}L^{\beta\max(1,2\alpha_1-2)}}\\
&\lesssim (1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}W^{1,r_2}}\\
&\lesssim T^{\frac{1}{p'}-\frac{\max(1,2\alpha_1-2)}{s_2}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\underbrace{\|w\|^{\max(1,2\alpha_1-2)}_{L^{s_2}_T W^{1,r_2}}}_{\lesssim\|w\|^{\max(1,2\alpha_1-2)}_{X_T}},
\end{aligned}\tag{6.102}$$

with $\frac{1}{r_2}=\frac{1}{3}+\frac{1}{\beta\max(1,2\alpha_1-2)}=\frac{2\max(1,2\alpha_1-2)+1-\varepsilon'}{6\max(1,2\alpha_1-2)}$, and $r_2\in[2,q_0]$ thanks to (6.88). Moreover, $\frac{1}{p'}-\frac{\max(1,2\alpha_1-2)}{s_2}=\min(1,4-2\alpha_1)/2>0$, because $\alpha_1<2$. We also have by Hölder and Gagliardo-Nirenberg

$$\begin{aligned}
\||\nabla(\varphi+w)|^2\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\varphi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\varphi+w)\|_{L^\infty_T L^{\check q}}\\
&\lesssim \|\nabla(\varphi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\varphi+w)\|^{\frac{1-\varepsilon_0-\varepsilon'}{2}}_{L^\infty_T L^2}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{\frac{1+\varepsilon_0+\varepsilon'}{2}}\\
&\lesssim T^{\frac{1}{p'}-\frac{1}{p_0}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{\frac{1+\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})^{\frac{3-\varepsilon_0-\varepsilon'}{2}},
\end{aligned}\tag{6.103}$$

where $1/q'=1/q_0+1/\check q$ and $1/p'-1/p_0=(\varepsilon'+\varepsilon_0)/2>0$. Next, we have to estimate $\||\nabla(\varphi+w)|^2|w|^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}}$. If $\alpha_1\leqslant 3/2$, this has just been done. Thus we assume $\alpha_1>3/2$. Provided $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'}\geqslant 2$, we have by Hölder, Gagliardo-Nirenberg and Sobolev

$$\begin{aligned}
\||\nabla(\varphi+w)|^2|w|^{2\alpha_1-3}\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\varphi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\varphi+w)\|_{L^\infty_T L^6}\||w|^{2\alpha_1-3}\|_{L^\infty_T L^{\frac{6}{1-\varepsilon_0-\varepsilon'}}}\\
&\lesssim T^{\frac{1}{p'}-\frac{1}{p_0}}\|\nabla(\varphi+w)\|_{L^{p_0}_T L^{q_0}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\||w|\|^{2\alpha_1-3}_{L^\infty_T L^{\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'}}}\\
&\lesssim T^{\frac{\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\|w\|^{2\alpha_1-3}_{L^\infty_T H^1}.
\end{aligned}\tag{6.104}$$

Note that (6.88) ensures $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'}\leqslant 6$. On the other side, if $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'}<2$, the same arguments imply

$$\begin{aligned}
&\||\nabla(\varphi+w)|^2|w|^{2\alpha_1-3}\|_{L^{p'}_T L^{q'}}\\
&\lesssim T^{\frac{\varepsilon_0+\varepsilon'}{2}}\|\nabla(\varphi+w)\|_{L^{p_0}_T L^{q_0}}\|\nabla(\varphi+w)\|_{L^\infty_T L^{\frac{6}{11-6\alpha_1-\varepsilon_0-\varepsilon'}}}\||w|^{2\alpha_1-3}\|_{L^\infty_T L^{\frac{2}{2\alpha_1-3}}}\\
&\lesssim T^{\frac{\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})(1+\|\nabla w\|^{1-\gamma}_{L^\infty_T L^2})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^\gamma\|w\|^{2\alpha_1-3}_{L^\infty_T H^1},
\end{aligned}\tag{6.105}$$

where $\gamma \in (0,1)$. Using the same arguments that in section 5, we can deduce from (6.36), (6.37) as well as (6.57) and (6.100) in dimension 2, (6.101), (6.102), (6.103), (6.104), (6.105) in dimension 3, that (6.87) is globally well-posed in X_T.

Proof Theorem 6.1.7. In the critical case $n = 3$, $\alpha_1 = 2$, a local well-posedness result may also be obtained. Indeed, the proofs of lemmas 6.7.1, 6.7.2, 6.7.3 and 6.7.4 above also work in that case, if we choose $p_0 = 2/(1-\varepsilon_0)$, $p' = 2/(1+\varepsilon')$, where $\varepsilon_0, \varepsilon' > 0$ satisfy $\varepsilon_0 + \varepsilon' \leqslant 1$ and $3\varepsilon_0 + \varepsilon' \leqslant 2$. The only difference is that we get $\theta = \theta_2 = 0$ in Lemmas 6.7.2 and 6.7.4. Instead of (6.96), we use the following estimate, which we prove as usual by Hölder, Sobolev and an interpolation argument.

$$\begin{aligned} \|\nabla(\varphi + w)|w|^2\|_{L_T^{p'}L^{q'}} &\lesssim \|\nabla(\varphi + w)\|_{L_T^{3p'}L^{\tilde{q}}}\|w\|^2_{L_T^{3p'}L^{\check{q}}} \\ &\lesssim \|\nabla(\varphi + w)\|_{L_T^{3p'}L^{\tilde{q}}}\|w\|^2_{L_T^{3p'}W^{1,\tilde{q}}} \\ &\lesssim (T^{\frac{1}{3p'}} + \|w\|_{X_T})\|w\|^2_{X_T}, \end{aligned} \tag{6.106}$$

where $\tilde{q}$ and $\check{q}$ must satisfy $1/\tilde{q} + 2/\check{q} = 1/q'$, $1/3p' + 3/\check{q} = 3/2$ and $1/\tilde{q} - 1/3 = 1/\check{q}$. Thanks to our choice of p_0, q_0, p', q' and to the condition we imposed on $\varepsilon_0, \varepsilon' > 0$, we obtain $\tilde{q} = 18/(8-\varepsilon') \in [2, q_0]$, which is the condition under which the above mentioned interpolation argument is valid. We similarly prove

$$\begin{aligned} &\|(|\nabla\varphi| + |\nabla w_1| + |\nabla w_2|)(w_1 - w_2)(|w_1| + |w_2|)\|_{L_T^{p'}L^{q'}} \\ &\lesssim \|w_1 - w_2\|_{X_T}(T^{\frac{1}{3p'}} + \|w_1\|_{X_T} + \|w_2\|_{X_T})(\|w_1\|_{X_T} + \|w_2\|_{X_T}), \end{aligned} \tag{6.107}$$

which we use instead of (6.98), and for $j = 1, 2$,

$$\|\nabla(w_1 - w_2)|w_j|^2\|_{L_T^{p'}L^{q'}} \lesssim \|w_1 - w_2\|_{X_T}\|w_j\|^2_{X_T}, \tag{6.108}$$

which will be used instead of (6.97). Thus, in the critical case, estimations (6.91) and (6.93) may be replaced respectively by

$$\begin{aligned} \|\nabla\Phi(w)\|_{L_T^\infty L^2 \cap L_T^{p_0}L^{q_0}} &\leqslant \|\nabla w_0\|_{L^2} + CT(1 + \|\nabla w\|_{L_T^\infty L^2}) \\ &+ CT^{\frac{1}{2}}(1 + \|w\|_{X_T})\|w\|_{X_T} + C(T^{\frac{1}{3p'}} + \|w\|_{X_T})(\|w\|_{X_T} + \|w\|^2_{X_T}) \end{aligned} \tag{6.109}$$

and

$$\begin{aligned} \|\nabla\Phi(w_1) - \nabla\Phi(w_2)\|_{L_T^\infty L^2 \cap L_T^{p_0}L^{q_0}} &\lesssim T\|\nabla(w_1 - w_2)\|_{L_T^\infty L^2} \\ &+ T^{\frac{1}{2}}\|w_1 - w_2\|_{X_T}(\|w_1\|_{X_T} + \|w_2\|_{X_T}) \\ &+ \|w_1 - w_2\|_{X_T}(T^{\frac{1}{3p'}} + \|w_1\|_{X_T} + \|w_2\|_{X_T})(\|w_1\|_{X_T} + \|w_2\|_{X_T}). \end{aligned} \tag{6.110}$$

From Lemma 6.7.1 (which remains true for $\alpha_1 = 2$) and (6.109), we deduce as in section 4 that for T and R small enough, if $\|w_0\|_{H^1} \leqslant R$, Φ maps the ball of radius $2R$ of H^1 into itself. Taking T and R even smaller if necessary, Lemma 6.7.3 and (6.110) ensures that this map is a contraction. The rest of the proof is identical to that of Theorem 6.1.4 which was done in section 4. $\square$

6.8 Appendix

Proof of Lemma 6.6.3. If μ, f_p and f_q are chosen as in the statement,

$$\begin{aligned}|\{x, |f(x)| > \mu\}| &= |\{x, \mu < |f(x)| \leqslant |f_p(x)| + |f_q(x)|\}| \\ &\leqslant |\{x, \mu/2 < |f_p(x)|\}| + |\{x, \mu/2 < |f_q(x)|\}|.\end{aligned}$$

Next, since $p < \infty$,

$$|\{x, \mu/2 < |f_p(x)|\}| \leqslant \int \left(\frac{2}{\mu}|f_p(x)|\right)^p dx = \left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p. \tag{6.111}$$

Similarly, if $q < \infty$,

$$|\{x, \mu/2 < |f_q(x)|\}| \leqslant \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q. \tag{6.112}$$

Thus

$$|\{x, |f(x)| > \mu\}| \leqslant \left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p + \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q. \tag{6.113}$$

If $q = \infty$, if $\mu > 2\|f_q\|_{L^\infty}$, one has $|\{x, \mu/2 < |f_q(x)|\}| = 0$, and

$$|\{x, |f(x)| > \mu\}| \leqslant \left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p. \tag{6.114}$$

If $q < \infty$, using (6.111) with μ replaced by 2μ, we have for $\mu > 0$,

$$\begin{aligned}
&\left(\int |\chi_\mu(f)f|^q\right)^{1/q} \\
&\leqslant \left(\int_{\{x,|f(x)|\leqslant 2\mu\}} |f(x)|^q dx\right)^{1/q} \\
&\leqslant \left(\int_{\{x,|f(x)|\leqslant 2\mu\}} |f_p(x)|^q dx\right)^{1/q} + \left(\int_{\{x,|f(x)|\leqslant 2\mu\}} |f_q(x)|^q dx\right)^{1/q} \\
&\leqslant \left(\int_{\left\{x,\ \begin{array}{l}|f(x)| \leqslant 2\mu, \\ |f_p(x)| \leqslant \mu\end{array}\right\}} |f_p(x)|^p |f_p(x)|^{q-p} dx\right)^{1/q} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu\end{array}\right\}} |f_p(x)|^q dx\right)^{1/q} + \|f_q\|_{L^q} \\
&\leqslant \mu^{\frac{q-p}{q}} \|f_p\|_{L^p}^{p/q} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu\end{array}\right\}} |f(x)|^q dx\right)^{1/q} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu\end{array}\right\}} |f_q(x)|^q dx\right)^{1/q} + \|f_q\|_{L^q} \\
&\leqslant \mu^{\frac{q-p}{q}} \|f_p\|_{L^p}^{p/q} + 2\mu|\{x, |f_p(x)| \geqslant \mu\}|^{1/q} + 2\|f_q\|_{L^q} \\
&\leqslant 3\mu^{1-p/q}\|f_p\|_{L^p}^{p/q} + 2\|f_q\|_{L^q},
\end{aligned} \tag{6.115}$$

while for $q=\infty$, $\|\chi_\mu(f)f\|_{L^\infty} \leqslant 2\mu$. On the other side, using (6.113), if $q<\infty$,

$$\left(\int |(1-\chi_\mu)(f)f|^p\right)^{1/p}$$
$$\leqslant \left(\int_{\{x,|f(x)|>\mu\}} |f_p(x)|^p dx\right)^{1/p} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)|>\mu,\\ |f_q(x)|<\mu\end{array}\right\}} |f_q(x)|^p dx\right)^{1/p} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)|>\mu,\\ |f_q(x)|\geqslant\mu\end{array}\right\}} |f_q(x)|^p dx\right)^{1/p}$$
$$\leqslant \|f_p\|_{L^p} + \mu|\{x,|f(x)|>\mu\}|^{1/p} + \left(\int_{\left\{x,\ \begin{array}{l}|f(x)|>\mu,\\ |f_q(x)|\geqslant\mu\end{array}\right\}} \frac{|f_q(x)|^q}{\mu^{q-p}} dx\right)^{1/p}$$
$$\leqslant \|f_p\|_{L^p} + \mu\left(\left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p + \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q\right)^{1/p} + \frac{\|f_q\|_{L^q}^{q/p}}{\mu^{\frac{q-p}{p}}}, \tag{6.116}$$

while if $q=\infty$, we similarly obtain thanks to (6.114), and because $\mu > 2\|f_q\|_{L^\infty}$

$$\|(1-\chi_\mu)(f)f\|_{L^p} \leqslant 3\|f_p\|_{L^p}.$$

Replacing μ by 1 or by μ_0, the estimates announced in the statement easily follow from (6.115), (6.116) and their analogous in the case $q=\infty$. □

Proof of Lemma 6.6.4. Let $g = g_p + g_q$ be a decomposition of g in $L^p + L^q$. If $g_p = 0$ (resp. $g_q = 0$), then $g \in L^q$ (resp. $g \in L^p$), and the result is clear. Let $\mu_0 > 0$ associated to this decomposition by (6.77) (where f_p, f_q are replaced by g_p, g_q). Thanks to Lemma 6.6.3, $\chi_{\mu_0}(g)g \in L^q$ and $(1-\chi_{\mu_0})(g)g \in L^p$, and the estimates (6.78) and (6.79) hold for g. Writing $f = \chi_{\mu_0}(g)f + (1-\chi_{\mu_0})(g)f$, we deduce that $f \in L^p + L^q$, with $\|f\|_{L^p+L^q} \leqslant C(p,q)\|g\|_{L^p+L^q}$, where $C(p,q) = \max(6, 2+(2^p+2^q)^{1/p})$. □

Proof of Lemma 6.6.5. Let $f \in L^p$, $f \neq 0$, and $\mu = \|f\|_{L^p}$. Then

$$\left(\int |\chi_\mu(f)f|^{p_2}\right)^{1/p_2} \leqslant \left(\int |\chi_\mu(f)f|^{p_2-p}|f|^p\right)^{1/p_2} \leqslant (2\mu)^{\frac{p_2-p}{p_2}} \|f\|_{L^p}^{p/p_2} = 2^{\frac{p_2-p}{p_2}}\|f\|_{L^p},$$

and

$$\left(\int |(1-\chi_\mu)(f)f|^{p_1}\right)^{1/p_1} \leqslant \left(\int |(1-\chi_\mu)(f)f|^{p_1}\left(\frac{|f|}{\mu}\right)^{p-p_1}\right)^{1/p_1} \leqslant \mu^{1-p/p_1}\|f\|_{L^p}^{p/p_1} = \|f\|_{L^p}.$$

Therefore $f \in L^{p_1} + L^{p_2}$ and

$$\|f\|_{L^{p_1}+L^{p_2}} \leqslant (1 + 2^{\frac{p_2-p}{p_2}})\|f\|_{L^p}.$$

□

Proof of Lemma 6.6.6. We write $fg = f_1g_1 + f_1g_2 + f_2g_1 + f_2g_2$. Thanks to Hölder, it is clear that for $i, j = 1, 2$, $f_ig_j \in L^{\frac{p_iq_j}{p_i+q_j}}$, and $\|f_ig_j\|_{L^{\frac{p_iq_j}{p_i+q_j}}} \leqslant \|f_i\|_{L^{p_i}}\|g_j\|_{L^{q_j}}$. For $i \neq j$, we have $\frac{p_iq_j}{p_i+q_j} \in (\frac{p_1q_1}{p_1+q_1}, \frac{p_2q_2}{p_2+q_2})$. The lemma follows, thanks to Lemma 6.6.5. $\square$

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Chapitre 7

Korteweg-de Vries and Benjamin-Ono equations on Zhidkov spaces

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Abstract. Motivated by the study of the Cauchy problem with bore-like initial data, we show the "well-posedness" for Korteweg-de Vries and Benjamin-Ono equations with initial data in Zhidkov spaces X^s, with respectively $s > 1$ and $s > 5/4$. Here, "well-posedness" includes local (global in some cases) existence, uniqueness under a supplementary assumption and continuity with respect to the initial data.

7.1 Introduction

In [ILS], the authors considered the Cauchy problems for Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations with bore-like initial data, namely

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(x,0) = g(x) \end{cases} \tag{7.1}$$

and

$$\begin{cases} \partial_t u + H\partial_x^2 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(x,0) = g(x) \end{cases} \tag{7.2}$$

where H denotes the Hilbert transform, and g satisfies

$$\begin{cases} \ \ \ i) \ \ g(x) \to C_\pm \text{ as } x \to \pm\infty \ . \\ \ \ ii) \ \ g' \in H^{s-1} \text{ where } s \geqslant 1 \ . \\ iii) \ \ g - C_+ \in L^2([0,\infty)), \ g - C_- \in L^2((-\infty,0]) \ . \end{cases} \tag{7.3}$$

They showed the local well-posedness of (7.1) and (7.2) with initial data g satisfying (7.3), under the assumption $s > 3/2$. Global well-posedness was obtained for $s \geqslant 2$.

Our aim here is to improve this result by weakening the assumptions on the initial data g: we replace (7.3) by

$$g \in X^s \tag{7.4}$$

where X^s denotes the Zhidkov space we introduced in [Ga1] (see also [Z1]) for integer values of s:

$$X^s := \{f \in \mathcal{D}'(\mathbb{R}),\ f \in L^\infty,\ f' \in H^{s-1}(\mathbb{R})\}. \tag{7.5}$$

Moreover (and that is probably the most interesting improvement), we assume only that $s > 1$ in the KdV case, and $s > 5/4$ in the BO case, instead of $s > 3/2$.

We first consider the KdV equation (7.1). Our strategy is as follows. Thanks to Lemma 7.2.1 below, for $g \in X^s$, there exists a function $\psi \in C^\infty(\mathbb{R})$ with $\psi' \in H^\infty$ such that $\varphi = g - \psi \in H^s$ (remark that it implies that ψ is bounded, since $s \geqslant 1$). Similarly to [ILS], we write a solution u of (7.1) as $u = v + \psi$, and we study the Cauchy problem associated with v, namely

$$\begin{cases} \partial_t v + \partial_x^3 v + v\partial_x v + \partial_x(v\psi) - (\partial_x^3\psi + \psi\psi')\ ,\ x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = \varphi(x) = g(x) - \psi(x) \end{cases} \tag{7.6}$$

Our main result is as follows:

Theorem 7.1.1 *Let $\psi \in \mathcal{C}_b^\infty(\mathbb{R})$ such that $\psi' \in H^\infty$, $s > 1$ and $\varphi \in H^s(\mathbb{R})$.*
Then there exists $T = T(\psi, s, \|\varphi\|_{H^s}) > 0$ and a unique $v \in \mathcal{C}([-T,T], H^s) \cap \mathcal{C}^1([-T,T], H^{s-3})$ solving (7.6), and such that $v_x \in L^1([-T,T], L^\infty)$.
Moreover, for any $R > 0$, the map $\varphi \to v$ is continuous from the ball of radius R in $H^s(\mathbb{R})$ to $\mathcal{C}([-T(R), T(R)], H^s)$.

From this local well-posedness result for (7.6) we deduce a local well-posedness result for (7.1):

Theorem 7.1.2 *Let $s > 1$, $g \in X^s$.*
Then there exists $\tilde{T} = \tilde{T}(s, \|g\|_{X^s}) > 0$ and a unique solution u of (7.1) such that $u \in \mathcal{C}([-\tilde{T}, \tilde{T}], X^s)$, $u - g \in \mathcal{C}([-\tilde{T}, \tilde{T}], H^s)$ and $u_x \in L^1([-\tilde{T}, \tilde{T}], L^\infty)$.
Moreover, for any $R > 0$, the map $g \to u$ is continuous from the ball of radius R in X^s to $\mathcal{C}([-\tilde{T}(R), \tilde{T}(R)], X^s)$.

Equation (7.6) is just KdV equation perturbed by some terms. In the case of KdV, Kenig, Ponce and Vega ([KPV4]) showed the local well-posedness in H^s, with $s > -3/4$, by the contraction principle. As it was mentionned in [ILS], this method fails here. Indeed, it does not seem to be possible to get an appropriate estimate on the H^s norm of the term $\psi\partial_x v$, because ψ does not vanish at infinity. Bourgain's method (see [Gi]) fails for the same reason, as well as the method used by Kenig and Koenig in [KeK]. Iorio, Linares and Scialom ([ILS]) used a parabolic regularization and the Bona-Smith approximation. To improve their result, we use here the method that was employed by Koch and Tzetvkov in [KoT] to show the local well-posedness of BO in H^s, $s > 5/4$. Namely, we show Strichartz estimates for a linearized version of (7.6). Next we derive the crucial non-linear estimate, using the Littlewood-Paley decomposition. To get an a priori estimate on the H^s norm of a solution of (7.6), a commutator lemma due to Kato ([Ka1]) was used in [ILS]. This lemma fails for $s \leqslant 3/2$. We use and prove here a new commutator lemma (Lemma 7.2.4 below), which is a variant of a lemma due to Kato and Ponce (see Lemma 7.2.3 below or [KP]).

Our method gives similar results for the BO equation:

Theorem 7.1.3 *Let $\psi \in C_b^\infty(\mathbb{R})$ such that $\psi' \in H^\infty$, $s > 5/4$ and $\varphi \in H^s(\mathbb{R})$. Then there exists $T = T(\psi, s, \|\varphi\|_{H^s})$ and an unique solution v of*

$$\left\{ \begin{array}{l} \partial_t v + H\partial_x^2 v + v\partial_x v + \partial_x(v\psi) = -(H\partial_x^2\psi + \psi\psi') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(x,0) = \varphi(x) = g(x) - \psi(x) \end{array} \right. , \tag{7.7}$$

such that $v \in C([-T,T], H^s) \cap C^1([-T,T], H^{s-2})$ and $v_x \in L^1([-T,T], L^\infty)$. Moreover, for any $R > 0$, the map $\varphi \to v$ is continuous from the ball of radius R in $H^s(\mathbb{R})$ to $C([-T(R), T(R)], H^s)$.

Theorem 7.1.4 *Let $s > 5/4$, $g \in X^s$. Then there exists $\tilde{T} = \tilde{T}(s, \|g\|_{X^s}) > 0$ and a unique solution u of (7.2) such that $u \in C([-\tilde{T}, \tilde{T}], X^s)$, $u - g \in C([-\tilde{T}, \tilde{T}], H^s)$ and $u_x \in L^1([-\tilde{T}, \tilde{T}], L^\infty)$. Moreover, for any $R > 0$, the map $g \to u$ is continuous from the ball of radius R in X^s to $C([-\tilde{T}(R), \tilde{T}(R)], X^s)$.*

In fact, it also works for all the dispersions between BO and KdV. Namely, for $\alpha \in [1,2]$, the problem

$$\left\{ \begin{array}{l} \partial_t v - D^\alpha \partial_x v + v\partial_x v + \partial_x(v\psi) = D^\alpha \partial_x \psi - \psi\psi' \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(0) = \varphi \in H^s \end{array} \right. , \tag{7.8}$$

where $D = (-\partial_x^2)^{1/2}$ is locally well posed in H^s for $s > 3/2 - \alpha/4$.

In [ILS], a global well-posedness result was obtained for (7.7) in H^s for $s \geqslant 2$. Since our local well-posedness result goes underneath 3/2, using the invariant of the Benjamin-Ono equation associated with the $H^{3/2}$ norm, we improve this result:

Theorem 7.1.5 *Let $\varphi \in H^s$, $s \geqslant 3/2$. Then the solution of (7.7) obtained in Theorem 7.1.3 can be extended to $\mathbb{R}$.*

Corollary 7.1.1 *Let $g \in X^s$, $s \geqslant 3/2$. Then the solution of (7.2) obtained in Theorem 7.1.3 can be extended to $\mathbb{R}$.*

Note that if Theorem 7.1.1 was true for $s = 1$, we would have a global well-posedness result. We failed to show this for general $\psi \in C_b^\infty$ with $\psi' \in H^\infty$. However, if $\psi \equiv a$ is a constant, a change of variables shows that v solves (7.6) if and only if $w(x,t) := v(x+at, t)$

solves the classical KdV equation, which is known to be globally well-posed in H^1. For other dispersions, if we assume that ψ is constant, the results of Kenig and Koenig similarly ensure that (7.8) is locally well-posed in H^s, $s > 3/2 - 3\alpha/8$.

The proof of Theorem 7.1.3 (resp. 7.1.4) is similar and simpler to that of Theorem 7.1.1 (resp. 7.1.2), so that we will omit it.

This paper is organized as follows. In section 2, we state some preliminary results, including the crucial commutator lemmas. In section 3, we prove Strichartz estimates for a linearized version of (7.6). In section 4, we prove a non-linear estimate. The proof is based on the Littlewood-Paley theory. In the last two sections, the main ideas are these of Koch and Tzvetkov explained in [KoT]. In section 5 and 6, we prove Theorem 7.1.1. In section 7, we derive Theorem 7.1.2. In section 8, we prove Theorem 7.1.5. Finally, we prove Lemma 7.2.4 in the appendix. The proof of this commutator lemma is inspired by that of Lemma 7.2.3 given in [KP].

Notations. Throughout this paper, the notation $A \lesssim B$ means that there exists an harmless constant $c > 0$ such that $A \leqslant cB$.
We denote by H^∞ the space $H^\infty = \bigcap\limits_{s\geqslant 0} H^s$, C_b^∞ the space of C^∞ bounded functions and $\mathcal{S}$ the space of Schwartz functions.
If X is a Banach space, T a positive number and $I \subset \mathbb{R}$ an interval, we define $L_T^p X := L^p([-T,T], X)$ and $L_I^p X := L^p(I, X)$ equipped with their natural norms.
We denote $L_T^\infty H^\infty := \bigcap\limits_{s\geqslant 0} L_T^\infty H^s$.
For $\sigma \geqslant 0$, we denote $J^\sigma := (1-\partial_x^2)^{\sigma/2}$, $D^\sigma := (-\partial_x^2)^{\sigma/2}$.
The letters λ and μ will design dyadic integers. The notation $\sum_\lambda f(\lambda)$ should be understood as $\sum_{k=0}^\infty f(2^k)$.
We call $(q,p) \in \mathbb{R}^2$ an admissible pair if

$$(q,p) = \left(\frac{6}{\theta(\beta+1)}, \frac{2}{1-\theta}\right), \ (\theta,\beta) \in [0,1] \times [0,1/2] \ .$$

Note that to prove theorems 7.1.3 and 7.1.4 dealing with the BO equation, we should replace this definition by:

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2}, \ q \in (4,\infty), \ p \in (2,\infty) \ .$$

We recall that the solution of the initial value problem

$$\begin{cases} \partial_t v + \partial_x^3 v = 0 \ , \ t,x \in \mathbb{R} \\ v(x,0) = v_0(x) \end{cases} \tag{7.9}$$

is given by the unitary Airy group which will be denoted by $\{W(t)\}_{t\in\mathbb{R}}$, i.e. $v(.,t) = W(t)v_0 = S_t \star v_0$ where for $t > 0$

$$S_t = t^{-1/3} K(t^{-1/3}.)$$

and

$$K(x) = c\int_{-\infty}^{\infty} e^{i(\xi^3+x\xi)}d\xi,\ x\in\mathbb{R}\ .$$

7.2 Preliminary results

We now state a result that gives a decomposition of the initial data g, which is used to reduce the study of (7.1) (resp. (7.2)) to that of (7.6) (resp. (7.7)).

Lemma 7.2.1 *Let $g\in X^s,\ s\geqslant 1$. Then there exists $\psi\in C^\infty(\mathbb{R}),\ \varphi\in H^s(\mathbb{R})$ such that*

$$\begin{aligned} &i)\quad \psi'\in H^\infty(\mathbb{R}).\\ &ii)\quad g=\psi+\varphi. \end{aligned}$$

Moreover, the maps $g\to\psi$ and $g\to\varphi$ can be defined as linear maps such that for every $s_1\geqslant 1$, $g\to\psi$ is continuous from X^s into X^{s_1} and $g\to\varphi$ is continuous from X^s into H^s.

Proof. Let $k(x)=(4\pi)^{-1/2}e^{-x^2/4}$, $\psi := k\star g$.
Then $\psi\in C_b^\infty(\mathbb{R})$, $\psi'=k\star g'\in H^\infty$, therefore $g-\psi\in L^\infty\subset\mathcal{S}'$ and

$$\widehat{g-\psi}(\xi) = (\hat{g}-\hat{\psi})(\xi) = (1-\hat{k})\hat{g}(\xi) = \underbrace{\frac{1-e^{-\xi^2}}{\xi}}_{\in L^\infty}\underbrace{\xi\hat{g}(\xi)}_{\in L^2}\ .$$

Hence $g-\psi\in L^2$, and $g-\psi\in H^s$.

Moreover, $\|\psi\|_{L^\infty}\leqslant\|g\|_{L^\infty}$ and

$$\|\psi'\|^2_{H^{s_1-1}} = \int(1+\xi^2)^{s_1-1}e^{-2\xi^2}|\xi\hat{g}(\xi)|^2d\xi \leqslant \sup_{\xi\in\mathbb{R}}\left((1+\xi^2)^{s_1-1}e^{-2\xi^2}\right)\|g'\|^2_{L^2}\ ,$$

which shows the continuity of $g\to\psi$ from X^s into X^{s_1}. Similarly,

$$\|\varphi\|^2_{H^s}\leqslant\sup_{\xi\in\mathbb{R}}\left((1+\xi^2)\left(\frac{1-e^{-\xi^2}}{\xi}\right)^2\right)\|g'\|^2_{H^{s-1}}$$

gives the continuity of $g\to\varphi$ from X^s into H^s. □

Lemma 7.2.2 *Let $(a_\lambda)_\lambda$, $(d_\lambda)_\lambda$ two sequences indexed on dyadic integers $\lambda=2^j,\ j\in\mathbb{N}$. Let $s>0$. Then*

$$\sum_\lambda\lambda^s\sum_{\mu\geqslant\lambda/8}a_\mu d_\lambda\lesssim\left(\sum_\lambda\lambda^{2s}a_\lambda^2\right)^{1/2}\left(\sum_\lambda d_\lambda^2\right)^{1/2},$$

and hence by duality

$$\sum_\lambda\lambda^{2s}\left(\sum_{\mu\geqslant\lambda/8}a_\mu\right)^2\lesssim\sum_\lambda\lambda^{2s}a_\lambda^2\ .$$

Proof. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_{\lambda}\lambda^s\sum_{\mu\geqslant\lambda/8}a_\mu d_\lambda &= \sum_{\lambda}\lambda^s\sum_{\substack{k=-3\\2^k\lambda\geqslant 1}}^{\infty}a_{2^k\lambda}d_\lambda=\sum_{k=-3}^{\infty}2^{-ks}\sum_{\lambda\geqslant 2^{-k}}(2^k\lambda)^s a_{2^k\lambda}d_\lambda\\ &\leqslant \sum_{k=-3}^{\infty}2^{-ks}\left(\sum_{\lambda\geqslant 2^{-k}}\left((2^k\lambda)^s a_{2^k\lambda}\right)^2\right)^{1/2}\left(\sum_{\lambda\geqslant 2^{-k}}d_\lambda^2\right)^{1/2}\\ &\leqslant \frac{2^{3s}}{1-2^{-s}}\left(\sum_{\lambda}\lambda^{2s}a_\lambda^2\right)^{1/2}\left(\sum_{\lambda}d_\lambda^2\right)^{1/2}.\end{aligned}$$

□

Commutator and Bilinear Estimates

We will use in the sequel two commutator lemmas. The first one is due to Kato and Ponce and is proved in [KP].

Lemma 7.2.3 *If $s>0$,*

$$\|\,[J^s,f]\,g\|_{L^2}\lesssim\|\partial_x f\|_{L^\infty}\|J^{s-1}g\|_{L^2}+\|J^s f\|_{L^2}\|g\|_{L^\infty}\ . \tag{7.10}$$

The second commutator lemma is proved in the appendix.

Lemma 7.2.4 *Let $s>0$, let $s_0\geqslant\max(0,3-s)$. Then*

$$\|\,[J^s,f]\,g\|_{L^2}\lesssim\|\partial_x f\|_{L^\infty}\|J^{s-1}g\|_{L^2}+\|J^{s_0+s-1}\partial_x f\|_{L^2}\|g\|_{L^\infty}\ . \tag{7.11}$$

Next lemmas (see [T]) will be used in the proof of Theorem 7.1.5.

Lemma 7.2.5 *Let $a,b,c\in\mathbb{R}$ such that $a\geqslant c$, $b\geqslant c$, $a+b\geqslant 0$ and $a+b-c>n/2$. Then the map $(f,g)\longrightarrow fg$ is a continuous bilinear form from $H^a(\mathbb{R}^n)\times H^b(\mathbb{R}^n)$ into $H^c(\mathbb{R}^n)$.*

Lemma 7.2.6 *If $s\geqslant 1$, then there is a constant C such that for all $f\in\mathcal{S}(\mathbb{R}^n)$, $g\in H^{s-1}(\mathbb{R}^n)$,*

$$\|\,[J^s,f]\,g\|_{L^2}\leqslant C\|f'\|_{H^s}\|g\|_{H^{s-1}}\ .$$

We will need a generalized version of Lemma 7.2.6.

Lemma 7.2.7 *Let $n=1$. In Lemma 7.2.6 above, the assumption $f\in\mathcal{S}$ can be replaced by $f\in X^\infty:=\bigcap_{s\geqslant 1}X^s$.*

Proof. Let $f \in X^\infty$. For $\varepsilon > 0$, $f_\varepsilon(x) := e^{-\varepsilon x^2} f(x)$ belongs to $\mathcal{S}$, therefore we can apply Lemma 7.2.6 to f_ε. Passing to the limit, we obtain Lemma 7.2.7. $\square$

In sections 3 to 6 below, we are interested in solving (7.6) where the functions ψ and φ are fixed.

7.3 The linear estimate

We first recall the Strichartz estimate with smoothing for the Airy group (see [KPV3]).

Lemma 7.3.1 *For any admissible pair $(q,p) = (6/(\theta(\beta+1)), 2/(1-\theta))$ with parameters $(\theta,\beta) \in [0,1] \times [0,1/2]$ and $u_0 \in L^2(\mathbb{R})$, we have*

$$\|D^{\frac{\theta\beta}{2}} W(t) u_0\|_{L^q_t(\mathbb{R},L^p)} \lesssim \|u_0\|_{L^2(\mathbb{R})} \ .$$

As in [KoT], we deduce next from Lemma 7.3.1 a Strichartz inequality for a linearized version of (7.6).

Lemma 7.3.2 *Let $\lambda \geqslant 1,\ T > 0,\ \sigma > 1/2$. Let $u : [-T,T] \times \mathbb{R} \mapsto \mathbb{R}$ be a regular solution of*

$$\partial_t u + \partial_x^3 u + V_1 \partial_x u + V_2 \partial_x u + V_3 u = f \tag{7.12}$$

where $V_1 \in L^\infty_T H^\sigma$, $V_2 \in L^\infty_T L^\infty$, $V_3 \in L^\infty_T H^\infty$ and $f \in L^1_T L^2$.
We assume moreover that there exists a constant $C > 0$ such that

$$Supp(\hat{u}(.,t)) \subset C[-\lambda,\lambda],\ \ t \in [-T,T].$$

Then for every admissible pair (q,p) with parameter $\beta = 1/2$ fixed, for any interval $I \subset [-T,T]$ such that $|I| \lesssim \lambda^{-1}$,

$$\|D^{\theta/4} u\|_{L^q_I L^p} \ \lesssim \ \Big(1 + \|J^\sigma V_1\|_{L^\infty_T L^2} + \|V_2\|_{L^\infty_T L^\infty} + \|V_3\|_{L^1_T L^\infty}\Big)\Big(\|u\|_{L^\infty_I L^2} + \|f\|_{L^1_I L^2}\Big)\,. \tag{7.13}$$

Moreover,

$$\begin{aligned}\|D^{\theta/4} u\|_{L^q_T L^p} \lesssim (1+T)^{1/q} \lambda^{1/q} &\Big(1 + \|J^\sigma V_1\|_{L^\infty_T L^2} + \|V_2\|_{L^\infty_T L^\infty} + \|V_3\|_{L^1_T L^\infty}\Big) \\ &\times \Big(\|u\|_{L^\infty_T L^2} + \|f\|_{L^1_T L^2}\Big).\end{aligned} \tag{7.14}$$

Proof. The solution v of the Cauchy problem

$$\begin{cases} \partial_t v + \partial_x^3 v = h \in L^1 L^2 \\ v(0) = v_0 \end{cases}$$

is given by

$$v(t) = W(t)v_0 + \int_0^t W(t-s)h(s)ds.$$

Applying $D^{\theta/4}$ to this equation, we obtain that for any $t \in [-T,T]$,

$$\|D^{\theta/4}v(t)\|_{L^p_x} \leqslant \|D^{\theta/4}W(t)v_0\|_{L^p_x} + \int_{-T}^{T} \|D^{\theta/4}W(t)W(-s)h(s)\|_{L^p_x}ds,$$

hence Lemma 7.3.1 yields

$$\|D^{\theta/4}v\|_{L^q_T L^p} \lesssim \|v_0\|_{L^2} + \|h\|_{L^1_T L^2}.$$

We apply this to u:

$$\begin{aligned}\|D^{\theta/4}u\|_{L^q_I L^p} \quad &\leqslant \quad \|u\|_{L^\infty_I L^2} + (\|V_1\|_{L^\infty_I L^\infty} + \|V_2\|_{L^\infty_I L^\infty})\|\partial_x u\|_{L^1_I L^2} \\ &\quad\quad +\|V_3\|_{L^1_I L^\infty}\|u\|_{L^\infty_I L^2} + \|f\|_{L^1_I L^2}.\end{aligned}$$

Now, as in [KT], using the Sobolev embedding, we have

$$\|V_1\|_{L^\infty_I L^\infty} \lesssim \|J^\sigma V_1\|_{L^\infty_I L^2},$$

Next using the assumptions on the support of $\hat{u}(.,t)$ and on the length of I and Plancherel's theorem, we get

$$\|\partial_x u\|_{L^1_I L^2} = \|\xi\hat{u}\|_{L^1_I L^2} \lesssim \lambda|I|\|\hat{u}\|_{L^\infty_I L^2} \lesssim \|u\|_{L^\infty_I L^2},$$

which completes the proof of (7.13). The proof of (7.14) is the same that in [KoT]: we write $[-T,T] = \overset{n}{\underset{k=1}{\cup}} I_k$ where $|I_k| \leqslant \lambda^{-1}$. We may assume that $n < 1 + 2\lambda T \leqslant 2\lambda(1+T)$, and hence using (7.13) applied to I_k and summing over k, we get (7.14). □

7.4 The nonlinear estimate

In the sequel, we use the same notations that in [KoT] about the Littlewood-Paley decomposition. Namely,

$$u = \sum_\lambda u_\lambda$$

where $u_\lambda := \Delta_\lambda u$, and the Fourier multiplier Δ_λ is defined by

$$\widehat{\Delta_\lambda u}(\xi) := \begin{cases} \varphi(\xi/\lambda)\hat{u}(\xi) & \lambda = 2^k, k \geqslant 1 \\ \chi(\xi)\hat{u}(\xi) & \lambda = 1 \end{cases},$$

where χ and φ are nonnegative, C_c^∞ functions on $\mathbb{R}$ satisfying

$$\chi(\xi) + \sum_{\lambda>1} \varphi(\xi/\lambda) = 1$$

and

$$\varphi(\xi) = \begin{cases} 0 & \text{if } |\xi| < 5/8 \text{ or } |\xi| > 2 \\ 1 & \text{if } 1 < |\xi| < 5/4 \end{cases} .$$

For a dyadic integer λ, we also define

$$\tilde{\Delta}_\lambda := \begin{cases} \Delta_{\lambda/2} + \Delta_\lambda + \Delta_{2\lambda} & \text{if } \lambda > 1 \\ \Delta_1 + \Delta_2 & \text{if } \lambda = 1 \end{cases} .$$

Let u be a regular solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u + \psi\partial_x u + \psi' u = -(\partial_x^3\psi + \psi\psi') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(0) = u_0 \in H^\infty \end{cases} . \tag{7.15}$$

By "regular solution" we mean that $u \in \bigcap_{s\geqslant 0} C(\mathbb{R}, H^s)$. Theorem 1.2 in [ILS] ensures that such a solution does exist (throughout [ILS], the assumption (7.3) can be replaced by "$g \in X^s$" for free).

Our aim in this section is to prove the following estimate on u.

Theorem 7.4.1 *Let $\sigma > 1/2$, $0 < T \leqslant 1$, (q,p) an admissible pair with parameters $\theta \in [0,1)$, $\beta = 1/2$, $s = \sigma + 1/q$ and u a regular solution of (7.15). Then*

$$\|D^{\theta/4} J^\sigma u\|_{L^q_T L^p} \lesssim (1 + \|J^\sigma u\|_{L^\infty_T L^2})(1 + \|u_x\|_{L^1_T L^\infty})^{3/2}(1 + \|J^s u\|^2_{L^\infty_T L^2})^{1/2} . \tag{7.16}$$

We split the proof in several lemmas.

Lemma 7.4.1 *Let (q,p) be an admissible pair with parameter $\theta \in [0,1)$ (i.e. $p < \infty$), let $\sigma > 1/2$. Then*

$$\|D^{\theta/4} J^\sigma u\|_{L^q_T L^p} \lesssim \left(\sum_\lambda \lambda^{2\sigma} \|D^{\theta/4} u_\lambda\|^2_{L^q_T L^p} \right)^{1/2} . \tag{7.17}$$

Proof. A similar lemma was stated in [KoT]. We recall the proof. We define $v := D^{\theta/4}u$. We have

$$
\begin{aligned}
\|J^\sigma v\|_{L^q_T L^p} &= \left(\int_{-T}^{T} \|\sum_\lambda J^\sigma v_\lambda(t)\|^q_{L^p_x} dt\right)^{1/q} \\
&\leqslant \left(\int_{-T}^{T} \|\left(\sum_\lambda |J^\sigma v_\lambda(t)|^2\right)^{1/2} \|^q_{L^p_x} dt\right)^{1/q} && (7.18) \\
&= \left(\int_{-T}^{T} \left(\int_x \left(\sum_\lambda |J^\sigma v_\lambda(t)|^2\right)^{p/2} dx\right)^{q/p} dt\right)^{1/q} \\
&\leqslant \left(\int_{-T}^{T} \left(\sum_\lambda \left(\int_x |J^\sigma v_\lambda(t)|^{2.p/2} dx\right)^{2/p}\right)^{q/2} dt\right)^{1/q} && (7.19) \\
&\leqslant \left(\int_{-T}^{T} \left(\sum_\lambda \|J^\sigma v_\lambda(t)\|^2_{L^p_x}\right)^{q/2} dt\right)^{1/q} \\
&\leqslant \left(\sum_\lambda \left(\int_{-T}^{T} \|J^\sigma v_\lambda(t)\|^{2.q/2}_{L^p_x} dt\right)^{2/q}\right)^{1/2} && (7.20) \\
&= \left(\sum_\lambda \|J^\sigma v_\lambda\|^2_{L^q_T L^p_x}\right)^{1/2} . && (7.21)
\end{aligned}
$$

Here we have used the square functions theorem for the Littlewood-Paley decomposition (see [S]) to obtain (7.18) ($1 < p < \infty$), and the Minkowski inequality (see [LL]) to get (7.19) and (7.20) (it works because $p/2 \geqslant 1$ and $q/2 \geqslant 1$, respectively). Next, using the Mikhlin-Hörmander theorem (or more precisely Lemma 6.2.1 in [BL]), we get, for all t,

$$\|J^\sigma v_\lambda(t)\|_{L^p} \lesssim \lambda^\sigma \|v_\lambda(t)\|_{L^p} \ . \tag{7.22}$$

(7.21) and (7.22) complete the proof of Lemma 7.4.1. □

Lemma 7.4.2 *There exists a constant $C > 0$ such that for all $w \in L^2$ and $v \in C_b^\infty$ such that $v_x \in H^\infty$,*

$$\| [\Delta_\lambda, v\partial_x] \, w\|_{L^2} \leqslant C\|v_x\|_{L^\infty}\|w\|_{L^2}.$$

Proof. By density of $\mathcal{S}$ in L^2, it suffices to show it for $w \in \mathcal{S}$. As in the proof of Lemma 2 in [KoT], we write for $\lambda \geqslant 2$

$$[\Delta_\lambda, v\partial_x] \, w(x) = \int_{-\infty}^{\infty} K(x,y)w(y)dy \ ,$$

where

$$K(x,y) = c\int_{-\infty}^{\infty} e^{i\lambda(x-y)\eta}\varphi(\eta)\left[i\lambda^2\eta(v(y)-v(x)) - \lambda v_x(y)\right]d\eta\ .$$

Therefore, using the mean value theorem,

$$|K(x,y)| \leqslant c\lambda\|v_x\|_{L^\infty} g(\lambda(y-x))$$

where g is in L^1. Hence

$$\sup_y \int_{-\infty}^{\infty} |K(x,y)|dx + \sup_x \int_{-\infty}^{\infty} |K(x,y)|dy \lesssim \|v_x\|_{L^\infty},$$

and the Schur lemma completes the proof of the lemma in the case $\lambda \geqslant 2$. The proof is similar in the case $\lambda = 1$. □

Lemma 7.4.3 *There exists a constant $C > 0$ such that for all $v \in C_b^\infty$ and $w \in L^2$,*

$$\|\left[\Delta_\lambda, v\right] w\|_{L^2} \leqslant C\|v\|_{L^\infty}\|w\|_{L^2}.$$

Proof. The proof is similar and easier to that of Lemma 7.4.2:

$$\begin{aligned}
\left[\Delta_\lambda, v\right] w(x) &= c\lambda\int_{-\infty}^{\infty}\hat{\varphi}(\lambda(y-x))(v(y)-v(x))w(y)dy \\
&\leqslant 2c\|v\|_{L^\infty}\int_{-\infty}^{\infty}\lambda|\hat{\varphi}(\lambda(y-x))||w(y)|dy,
\end{aligned}$$

and the Schur lemma completes the proof similarly to Lemma 7.4.2. □

Lemma 7.4.4 *Let $\sigma > 1/2$, (q,p) an admissible pair, $T > 0$ and u a regular solution of (7.15). Then*

$$\begin{aligned}
\sum_\lambda \lambda^{2\sigma}\|D^{\theta/4}u_\lambda\|^2_{L^q_T L^p} &\lesssim (1+T)^{2/q}\left(1+\|J^\sigma u\|_{L^\infty_T L^2} + \|\psi\|_{L^\infty} + T\|\psi'\|_{L^\infty}\right)^2 \\
\times\Bigg[\Big(1+\|u_x\|^2_{L^1_T L^\infty} + T^2\|\psi'\|^2_{L^\infty}\Big)&\sum_\lambda \lambda^{2\sigma+2/q}\|u_\lambda\|^2_{L^\infty_T L^2} + \|\psi'\|^2_{H^{\sigma+1/q}}\|u_x\|^2_{L^1_T L^\infty} \\
&+T^2\|\psi\psi' + \partial_x^3\psi\|^2_{H^{\sigma+1/q}} + T^2\|J^\sigma u\|^2_{L^\infty_T L^2}\|\psi'\|^2_{H^{\sigma+1/q}}\Bigg].
\end{aligned}$$

Proof. We apply Δ_λ to (7.15):

$$\begin{aligned}
&\partial_t u_\lambda + \partial_x^3 u_\lambda + (u+\psi)\partial_x u_\lambda + \psi' u_\lambda \\
&\quad = -\Delta_\lambda(\partial_x^3\psi + \psi\psi') - \left[\Delta_\lambda, (u+\psi)\partial_x\right] u - \left[\Delta_\lambda, \psi'\right] u. \qquad (7.23)
\end{aligned}$$

Therefore we can apply Lemma 7.3.2 to u_λ, with $V_1 = u$, $V_2 = \psi$, $V_3 = \psi'$, $f = -\Delta_\lambda(\partial_x^3\psi + \psi\psi') - [\Delta_\lambda, (u+\psi)\partial_x]\, u - [\Delta_\lambda, \psi']\, u$. Hence

$$\sum_\lambda \lambda^{2\sigma} \|D^{\theta/4} u_\lambda\|^2_{L^q_T L^p} \lesssim (1+T)^{2/q} \left(1 + \|J^\sigma u\|_{L^\infty_T L^2} + \|\psi\|_{L^\infty} + T\|\psi'\|_{L^\infty}\right)^2$$
$$\times \sum_\lambda \lambda^{2\sigma+2/q} \Big(\|u_\lambda\|_{L^\infty_T L^2} + T\|\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\|_{L^2} + \|\,[\Delta_\lambda, u\partial_x]\, u\|_{L^1_T L^2}$$
$$+ \|\,[\Delta_\lambda, \psi\partial_x]\, u\|_{L^1_T L^2} + \|\,[\Delta_\lambda, \psi']\, u\|_{L^1_T L^2}\Big)^2 \ . \tag{7.24}$$

We give now a bound for each term in the right hand side parenthesis of (7.24).

Like in the Lemma 3 in [KoT], we write

$$[\Delta_\lambda, v\partial_x] = [\Delta_\lambda, v\partial_x]\, \widetilde{\Delta_\lambda} + \Delta_\lambda v \partial_x (1 - \widetilde{\Delta_\lambda})$$

where $v = u$ or ψ. Therefore, thanks to Lemma 7.4.2, we get

$$\|\,[\Delta_\lambda, u\partial_x]\, u\|_{L^1_T L^2} \lesssim \|u_x\|_{L^1_T L^\infty} \|\widetilde{\Delta_\lambda} u\|_{L^\infty_T L^2} + \|\Delta_\lambda u \partial_x (1 - \widetilde{\Delta_\lambda}) u\|_{L^1_T L^2} \tag{7.25}$$

and

$$\|\,[\Delta_\lambda, \psi\partial_x]\, u\|_{L^1_T L^2} \lesssim T\|\psi'\|_{L^\infty} \|\widetilde{\Delta_\lambda} u\|_{L^\infty_T L^2} + \|\Delta_\lambda \psi \partial_x (1 - \widetilde{\Delta_\lambda}) u\|_{L^1_T L^2} \ . \tag{7.26}$$

For $v = u$ or ψ and $\lambda \geqslant 4$,

$$\mathcal{F}\left\{\Delta_\lambda \left(v\partial_x(1-\widetilde{\Delta_\lambda})u\right)\right\} \;=\; \underbrace{\varphi(\xi/\lambda)}_{\text{Supp } \subset\{\xi, \frac{5\lambda}{8} \leqslant |\xi| \leqslant 2\lambda\}} \quad \hat{v} \star \underbrace{\left(i\xi(1 - (\varphi(\frac{2\xi}{\lambda}) + \varphi(\frac{\xi}{\lambda}) + \varphi(\frac{\xi}{2\lambda})))\hat{u}\right)}_{\text{Supp } \subset \mathbb{R}\backslash\{\xi, \frac{\lambda}{2} \leqslant |\xi| \leqslant \frac{5\lambda}{2}\}},$$

where $\mathcal{F}$ denotes the Fourier transform. If $\mu \leqslant \lambda/16$, Supp $\hat{v}_\mu \subset [-\lambda/8, \lambda/8]$, and

$$\mathbb{R}\backslash\{\xi, \frac{\lambda}{2} \leqslant |\xi| \leqslant \frac{5\lambda}{2}\} + [-\lambda/8, \lambda/8] \subset \mathbb{R}\backslash\{\xi, \frac{5\lambda}{8} \leqslant |\xi| \leqslant 2\lambda\}.$$

Therefore

$$\mathcal{F}\left\{\Delta_\lambda \left(v\partial_x(1-\widetilde{\Delta_\lambda})u\right)\right\} = \varphi(\xi/\lambda) \sum_{\mu \geqslant \lambda/8} \widehat{v_\mu} \star \left(i\xi(1 - (\varphi(\frac{2\xi}{\lambda}) + \varphi(\frac{\xi}{\lambda}) + \varphi(\frac{\xi}{2\lambda})))\hat{u}\right)$$

and

$$\Delta_\lambda \left(v\partial_x(1-\widetilde{\Delta_\lambda})u\right) = \sum_{\mu \geqslant \lambda/8} \Delta_\lambda (v_\mu \partial_x (1-\widetilde{\Delta_\lambda})u) \ . \tag{7.27}$$

This equality trivially remains true in the cases $\lambda = 1$ or 2.

In the case $v = u$, using the fact that Δ_λ defines an operator on L^∞, with a norm uniformly bounded in λ (see Lemma II.1.1.2 in [AG]), we get

$$\begin{aligned} \|\Delta_\lambda \left(u\partial_x(1-\widetilde{\Delta_\lambda})u\right)\|_{L^1_T L^2} &\leqslant \sum_{\mu \geqslant \lambda/8} \|u_\mu\|_{L^\infty_T L^2} \|(1-\widetilde{\Delta_\lambda})u_x\|_{L^1_T L^\infty} \\ &\lesssim \|u_x\|_{L^1_T L^\infty} \sum_{\mu\geqslant \lambda/8} \|u_\mu\|_{L^\infty_T L^2}. \end{aligned} \tag{7.28}$$

The case $v = \psi$ is a bit more difficult, because $\psi_1 \notin L^2$. Nevertheless, if $\mu \geqslant 2$, hence

$$\widehat{\psi_\mu}(\xi) = \varphi(\xi/\mu)\hat{\psi}(\xi) = \frac{\varphi(\xi/\mu)}{\xi}\xi\hat{\psi}(\xi) \in L^2$$

because $\varphi \equiv 0$ near 0 and $\psi' \in L^2$. Therefore $\psi_\mu \in L^2$ for $\mu \geqslant 2$, and if $\lambda \geqslant 16$, we get

$$\|\Delta_\lambda \left(\psi\partial_x(1-\widetilde{\Delta_\lambda})u\right)\|_{L^1_T L^2} \lesssim \|u_x\|_{L^1_T L^\infty} \sum_{\mu\geqslant\lambda/8} \|\psi_\mu\|_{L^2}. \tag{7.29}$$

If $\lambda \in \{1, 2, 4, 8\}$, we write

$$\psi_1 = T\psi_1 + (1-T)\psi_1,$$

where $\widehat{Tf}(\xi) = \tilde{\chi}(\xi)\hat{f}(\xi)$, $0 \leqslant \tilde{\chi} \leqslant 1$, $\tilde{\chi} \equiv 1$ on $[-1/4, 1/4]$ and $\mathrm{Supp}\tilde{\chi} \subset [-1/2, 1/2]$. Proceeding the same way as to obtain (7.27), we have

$$\Delta_\lambda \left(T\psi_1\partial_x(1-\widetilde{\Delta_\lambda})u\right) = 0,$$

and $(1-T)\psi_1 \in L^2$, with $\|(1-T)\psi_1\|_{L^2} \leqslant 4\|\psi'\|_{L^2}$. Therefore, like for (7.29), we have

$$\|\Delta_\lambda \left(\psi\partial_x(1-\widetilde{\Delta_\lambda})u\right)\|_{L^1_T L^2} \lesssim \|u_x\|_{L^1_T L^\infty} \left(\|\psi'\|_{L^2} + \sum_{\mu\geqslant 2}\|\psi_\mu\|_{L^2}\right). \tag{7.30}$$

Similarly, we write

$$[\Delta_\lambda, \psi'] = [\Delta_\lambda, \psi']\,\widetilde{\Delta_\lambda} + \Delta_\lambda\psi'(1-\widetilde{\Delta_\lambda})$$

and the Lemma 7.4.3 yields

$$\|\,[\Delta_\lambda, \psi']\,u\|_{L^1_T L^2} \lesssim \|\psi'\|_{L^\infty}\|\widetilde{\Delta_\lambda}u\|_{L^1_T L^2} + \|\Delta_\lambda\psi'(1-\widetilde{\Delta_\lambda})u\|_{L^1_T L^2}.$$

Using the same arguments that for stating (7.28) and Sobolev embedding ($\sigma > 1/2$), we get

$$\begin{aligned} \|\Delta_\lambda\psi'(1-\widetilde{\Delta_\lambda})u\|_{L^1_T L^2} &\lesssim T\|u\|_{L^\infty_T L^\infty} \sum_{\mu\geqslant\lambda/8} \|(\psi')_\mu\|_{L^2} \\ &\lesssim T\|J^\sigma u\|_{L^\infty_T L^2} \sum_{\mu\geqslant\lambda/8} \|(\psi')_\mu\|_{L^2}. \end{aligned} \tag{7.31}$$

Now, using (7.24), (7.25), (7.26), (7.28), (7.29), (7.30), (7.31), we obtain

$$
\begin{aligned}
&\sum_\lambda \lambda^{2\sigma}\|D^{\theta/4}u_\lambda\|^2_{L^q_TL^p} \lesssim (1+T)^{2/q}\left(1+\|J^\sigma u\|_{L^\infty_TL^2}+\|\psi\|_{L^\infty}+T\|\psi'\|_{L^\infty}\right)^2 \\
&\quad\times \sum_\lambda \lambda^{2\sigma+2/q}\Bigg[\|u_\lambda\|^2_{L^\infty_TL^2}+T^2\|(\psi\psi'+\partial_x^3\psi)_\lambda\|^2_{L^2} \\
&\quad+\|u_x\|^2_{L^1_TL^\infty}\left(\|\widetilde{\Delta_\lambda}u\|^2_{L^\infty_TL^2}+\left(\sum_{\mu\geqslant\lambda/8}\|u_\mu\|_{L^\infty_TL^2}\right)^2\right)+T^2\|\psi'\|^2_{L^\infty}\|\widetilde{\Delta_\lambda}u\|^2_{L^\infty_TL^2} \\
&\quad+\|u_x\|^2_{L^1_TL^\infty}\left(\left(\sum_{\mu\geqslant\lambda/8}\|\psi_\mu\|_{L^\infty_TL^2}\right)^2\mathbf{1}_{\{\lambda\geqslant16\}}+\left(\|\psi'\|_{L^2}+\sum_{\mu\geqslant2}\|\psi_\mu\|_{L^2}\right)^2\mathbf{1}_{\{1\leqslant\lambda<16\}}\right) \\
&\qquad+T^2\|J^\sigma u\|^2_{L^\infty_TL^2}\left(\sum_{\mu\geqslant\lambda/8}\|(\psi')_\mu\|_{L^\infty_TL^2}\right)^2\Bigg]\ . \qquad (7.32)
\end{aligned}
$$

We will now majorize each term of the sum in the right hand side of (7.32). We first clearly have

$$
\sum_\lambda \lambda^{2\sigma+2/q}\|\widetilde{\Delta_\lambda}u\|^2_{L^\infty_TL^2} \;\;\lesssim\;\; \sum_\lambda \lambda^{2\sigma+2/q}\|u_\lambda\|^2_{L^\infty_TL^2}. \qquad (7.33)
$$

Using the fact that φ vanishes near 0, if $\lambda\geqslant 2$,

$$
\|\psi_\lambda\|^2_{L^2}=\int\frac{1}{\xi^2}\varphi(\xi/\lambda)^2\xi^2|\hat\psi(\xi)|^2d\xi\lesssim\lambda^{-2}\|(\psi')_\lambda\|^2_{L^2}\lesssim\|(\psi')_\lambda\|^2_{L^2}\ . \qquad (7.34)
$$

We also use the properties of the support of φ to obtain that if $v\in H^s$, $s\in\mathbb{R}$,

$$
\sum_\lambda \lambda^{2s}\|v_\lambda\|^2_{L^2} \;\;\lesssim\;\; \|v\|^2_{H^s}\ . \qquad (7.35)
$$

The lemma easily follows from (7.32), (7.33), (7.34), (7.35) and Lemma 7.2.2. □

To complete the proof of Theorem 7.4.1, it remains to control the quantity

$$
\sum_\lambda \lambda^{2\sigma+2/q}\|u_\lambda\|^2_{L^\infty_TL^2}\ .
$$

That is what we do in the following lemma.

Lemma 7.4.5 *Let u be a regular solution of (7.15), let $s>1/2$. Then, noting $(u_\lambda)_\lambda$ the Littlewood-Paley decomposition of u,*

$$
\begin{aligned}
\sum_\lambda \lambda^{2s}\|u_\lambda\|^2_{L^\infty_TL^2}&\lesssim\|J^su\|^2_{L^\infty_TL^2}\left(1+\|u_x\|_{L^1_TL^\infty}+T\|\psi'\|_{H^s}\right) \\
&+\|J^su\|_{L^\infty_TL^2}(T(1+\|\psi\psi'+\partial_x^3\psi\|^2_{H^s})+\|\psi'\|_{H^s}\|u_x\|_{L^1_TL^\infty})\ . \qquad (7.36)
\end{aligned}
$$

Proof. During the proof of Lemma 7.4.4, we saw that u_λ solves (7.23). We multiply now (7.23) by $\overline{u_\lambda}$, we take the real part and we sum in x and t variables, taking into account the fact that u and ψ are real-valued:

$$\|u_\lambda(t)\|_{L^2}^2 = \|u_\lambda(0)\|_{L^2}^2 - \int_0^t \int_{-\infty}^{\infty} \Big\{ (u+\psi)(\partial_x u_\lambda \overline{u_\lambda} + \partial_x \overline{u_\lambda} u_\lambda) + 2\psi' |u_\lambda(s)|^2$$
$$+2\mathfrak{Re}\left(\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\overline{u_\lambda}(s) + [\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda} + [\Delta_\lambda, \partial_x\psi]u\overline{u_\lambda}) \right) \Big\}\, dxds. \quad (7.37)$$

Therefore, integrating by parts, multipliying by λ^{2s}, taking the supremum in the t variable and summing over λ, we get

$$\begin{aligned}
\sum_\lambda \lambda^{2s}\|u_\lambda\|_{L_T^\infty L^2}^2 \;\lesssim\; & \sum_\lambda \lambda^{2s}\|u_\lambda(0)\|_{L^2}^2 \\
& + \int_{-T}^{T} (\|u_x(t)\|_{L^\infty} + \|\psi'\|_{L^\infty}) \sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 dt \\
& + \int_{-T}^{T} \sum_\lambda \lambda^{2s}\|\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\|_{L^2}\|u_\lambda(t)\|_{L^2} dt \\
& + \int_{-T}^{T} \sum_\lambda \lambda^{2s}\left| \int_{-\infty}^{\infty} [\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda} dx \right| dt \\
& + \int_{-T}^{T} \sum_\lambda \lambda^{2s}\left| \int_{-\infty}^{\infty} [\Delta_\lambda, \psi']u(t)\overline{u_\lambda}(t) dx \right| dt\ . \qquad (7.38)
\end{aligned}$$

We will now control each term in the right hand side of (7.38). We first use (7.35) to get the following inequalities:

$$\begin{aligned}
& \int_{-T}^{T} \sum_\lambda \lambda^{2s}\|\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\|_{L^2}\|u_\lambda(t)\|_{L^2} dt \\
\leqslant\; & \frac{1}{2}\int_{-T}^{T} \sum_\lambda \lambda^{2s}\left(\|J^s u\|_{L_T^\infty L^2}\|\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\|_{L^2}^2 + \frac{\|u_\lambda(t)\|_{L^2}^2}{\|J^s u\|_{L_T^\infty L^2}} \right) dt \\
\lesssim\; & \int_{-T}^{T} \left(\|J^s u\|_{L_T^\infty L^2}\|\psi\psi' + \partial_x^3\psi\|_{H^s}^2 + \frac{\|u(t)\|_{H^s}^2}{\|J^s u\|_{L_T^\infty L^2}} \right) dt \\
\lesssim\; & T\|J^s u\|_{L_T^\infty L^2}(1 + \|\psi\psi' + \partial_x^3\psi\|_{H^s}^2)\ , \qquad (7.39)
\end{aligned}$$

$$\sum_\lambda \lambda^{2s}\|u_\lambda(0)\|_{L^2}^2 \;\lesssim\; \|J^s u(0)\|_{L^2}^2 \;\leqslant\; \|J^s u\|_{L_T^\infty L^2}^2 \qquad (7.40)$$

and

$$\begin{aligned}
& \int_{-T}^{T} (\|u_x(t)\|_{L^\infty} + \|\psi'\|_{L^\infty}) \sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 dt \\
& \qquad \lesssim \left(\|u_x\|_{L_T^1 L^\infty} + T\|\psi'\|_{L^\infty} \right) \|J^s u\|_{L_T^\infty L^2}^2\ . \qquad (7.41)
\end{aligned}$$

Like in the proof of Lemma 7.4.4, we write

$$[\Delta_\lambda, (u+\psi)\partial_x]u(t) = [\Delta_\lambda, (u+\psi)\partial_x]\widetilde{\Delta_\lambda}u(t) + \Delta_\lambda\left((u+\psi)\partial_x(1-\widetilde{\Delta_\lambda})u(t)\right).$$

Then, Lemma 7.4.2 and a simplified version of (7.28) yield

$$\begin{aligned}\|[\Delta_\lambda, (u+\psi)\partial_x]u(t)\|_{L^2} \lesssim\ & (\|u_x(t)\|_{L^\infty} + \|\psi'\|_{L^\infty})\|\widetilde{\Delta_\lambda}u(t)\|_{L^2} \\ & +\|u_x(t)\|_{L^\infty}\sum_{\mu\geqslant\lambda/8}\|u_\mu(t)\|_{L^2} + \|\Delta_\lambda(\psi\partial_x(1-\widetilde{\Delta_\lambda})u(t))\|_{L^2}.\end{aligned}$$

Next, we bound up the last term in the right hand side of the above inequality like in (7.29) and (7.30)

$$\begin{aligned}&\|\Delta_\lambda(\psi\partial_x(1-\widetilde{\Delta_\lambda})u(t))\|_{L^2} \\ &\lesssim\ \|u_x\|_{L^\infty}\left(\sum_{\mu\geqslant\lambda/8}\|\psi_\mu\|_{L^2}\mathbf{1}_{\lambda\geqslant 16} + \left(\|\psi'\|_{L^2} + \sum_{\mu\geqslant 2}\|\psi_\mu\|_{L^2}\right)\mathbf{1}_{\lambda\leqslant 8}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{-T}^{T}\sum_\lambda \lambda^{2s}\left|\int_{-\infty}^{\infty}[\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda}dx\right|dt \\ &\lesssim\ \int_{-T}^{T}\sum_\lambda\lambda^{2s}\Bigg[(\|u_x(t)\|_{L^\infty} + \|\psi'\|_{L^\infty})\|\widetilde{\Delta_\lambda}u(t)\|_{L^2} \\ &\quad +\|u_x(t)\|_{L^\infty}\Bigg(\sum_{\mu\geqslant\lambda/8}\|u_\mu(t)\|_{L^2} + \sum_{\mu\geqslant\lambda/8}\|\psi_\mu\|_{L^2}\mathbf{1}_{\lambda\geqslant 16} \\ &\quad + \left(\|\psi'\|_{L^2} + \sum_{\mu\geqslant 2}\|\psi_\mu\|_{L^2}\right)\mathbf{1}_{\lambda\leqslant 8}\Bigg)\Bigg]\|u_\lambda(t)\|_{L^2}dt\ .\end{aligned} \tag{7.42}$$

Using again (7.35), it is easy to see that

$$\sum_\lambda\lambda^{2s}\|\widetilde{\Delta_\lambda}u(t)\|_{L^2}\|u_\lambda(t)\|_{L^2}\ \lesssim\ \|u(t)\|_{H^s}^2\ . \tag{7.43}$$

Lemma 7.2.2 with $a_\mu = \|u_\mu(t)\|_{L^2}$, $d_\lambda = \lambda^s\|u_\lambda(t)\|_{L^2}$ and (7.35) yield

$$\sum_\lambda\lambda^{2s}\sum_{\mu\geqslant\lambda/8}\|u_\mu(t)\|_{L^2}\|u_\lambda(t)\|_{L^2}\ \lesssim\ \|u(t)\|_{H^s}^2\ . \tag{7.44}$$

Using once again Lemma 7.2.2, (7.35) and (7.34),

$$\begin{aligned}
&\sum_{\lambda} \lambda^{2s} \left[\sum_{\mu \geqslant \lambda/8} \|\psi_\mu\|_{L^2} \mathbf{1}_{\lambda \geqslant 16} + (\|\psi'\|_{L^2} + \sum_{\mu \geqslant 2} \|\psi_\mu\|_{L^2}) \mathbf{1}_{\lambda \leqslant 8} \right] \|u_\lambda(t)\|_{L^2} \\
&\quad \lesssim \|u(t)\|_{H^s} \left(\sum_{\lambda \geqslant 2} \lambda^{2s} \|\psi_\lambda\|_{L^2}^2 \right)^{1/2} + \left(\|\psi'\|_{L^2} + \sum_{\mu \geqslant 2} \|\psi_\mu\|_{L^2} \right) \|u(t)\|_{L^2} \\
&\quad \lesssim \|u(t)\|_{H^s} \|\psi'\|_{H^s} \qquad (7.45)
\end{aligned}$$

Thanks to (7.42), (7.43), (7.44), (7.45), we have

$$\begin{aligned}
&\int_{-T}^{T} \sum_{\lambda} \lambda^{2s} \left| \int_{-\infty}^{\infty} [\Delta_\lambda, (u+\psi)\partial_x] u \overline{u_\lambda} dx \right| dt \\
&\lesssim \quad \left(\|u_x\|_{L^1_T L^\infty} + T\|\psi'\|_{L^\infty} \right) \|J^s u\|_{L^\infty_T L^2}^2 + \|\psi'\|_{H^s} \|u_x\|_{L^1_T L^\infty} \|J^s u\|_{L^\infty_T L^2} \qquad (7.46)
\end{aligned}$$

As in the proof of Lemma 7.4.4,

$$\| [\Delta_\lambda, \psi'] \, u(t)\|_{L^2} \quad \lesssim \quad \|\psi'\|_{L^\infty} \|\widetilde{\Delta_\lambda} u(t)\|_{L^2} + \|u(t)\|_{L^\infty} \sum_{\mu \geqslant \lambda/8} \|(\psi')_\mu\|_{L^2}$$

and therefore, using also the Sobolev embedding ($s > 1/2$),

$$\begin{aligned}
&\int_{-T}^{T} \sum_{\lambda} \lambda^{2s} \left| \int_{-\infty}^{\infty} [\Delta_\lambda, \psi'] u(t) \overline{u_\lambda}(t) dx \right| dt \\
&\lesssim \quad T\|\psi'\|_{L^\infty} \|J^s u\|_{L^\infty_T L^2}^2 + \|\psi'\|_{H^s} \|u\|_{L^1_T L^\infty} \|J^s u\|_{L^\infty_T L^2} \\
&\lesssim \quad T\|\psi'\|_{H^s} \|J^s u\|_{L^\infty_T L^2}^2 \ . \qquad (7.47)
\end{aligned}$$

Concatenating (7.38), (7.39), (7.40), (7.41), (7.46) and (7.47), we finally obtain the announced inequality. □

Theorem 7.4.1 directly follows from Lemmas 7.4.1, 7.4.4, 7.4.5.

7.5 The proof of Theorem 7.1.1 (existence and uniqueness)

We begin the proof of Theorem 7.1.1 with a lemma that gives an a priori estimate on the H^s norm of a regular solution of (7.6).

Lemma 7.5.1 *Let $u_0 \in H^\infty$, $s > 1/2$ and $u \in C([-T,T], H^\infty)$ a regular solution of (7.15). Then there exists $C = C(\psi) > 0$ such that*

$$\|u(t)\|_{H^s}^2 \leqslant \left(\|u_0\|_{H^s}^2 + 1 \right) \exp\left(C(T + \|u_x\|_{L^1_T L^\infty}) \right)$$

Proof. We apply J^s to (7.15), we multiply the obtained equation by $J^s u$ and we sum over $\mathbb{R}$:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|J^s u\|_{L^2}^2 + \int \frac{(u+\psi)}{2}\partial_x (J^s u)^2 + \int \psi'(J^s u)^2 \\ = \ -\int J^s(\psi\psi' + \partial_x^3\psi)J^s u - \int [J^s,(u+\psi)]\partial_x u J^s u - \int [J^s,\psi']uJ^s u\end{aligned}$$

After an integration by parts, we get

$$\begin{aligned}\frac{d}{dt}\|J^s u\|_{L^2}^2 &= \int \partial_x(u-\psi)(J^s u)^2 - 2\int J^s(\psi\psi' + \partial_x^3\psi)J^s u \\ &\quad -2\int [J^s,(u+\psi)]\partial_x u J^s u - 2\int [J^s,\psi']uJ^s u\end{aligned}$$

We use Lemma 7.2.3 to estimate the L^2 norms of $[J^s,u]\partial_x u$ and $[J^s,\psi']u$. We get

$$\|[J^s,u]\partial_x u\|_{L^2} \lesssim \|J^s u\|_{L^2}\|u_x\|_{L^\infty}$$

and

$$\|[J^s,\psi']u\|_{L^2} \lesssim \|\psi''\|_{L^\infty}\|J^{s-1}u\|_{L^2} + \|J^s\psi'\|_{L^2}\|u\|_{L^\infty} \ .$$

Since ψ does not belong to L^2, it does not work to estimate the L^2 norms of $[J^s,\psi]\partial_x u$. Lemma 7.2.4 (note that Lemma 7.2.7 could also have been used) yields

$$\|[J^s,\psi]\partial_x u\|_{L^2} \lesssim \|\psi'\|_{L^\infty}\|u\|_{H^s} + \|\psi'\|_{H^{s_0+s-1}}\|u_x\|_{L^\infty} \ .$$

Using the Cauchy Schwarz inequality and Sobolev embedding ($s>1/2$), we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^s}^2 &\lesssim \|u\|_{H^s}^2\left(\|u_x\|_{L^\infty} + \|\psi'\|_{L^\infty} + 1 + \|\psi'\|_{H^s} + \|\psi''\|_{L^\infty}\right) \\ &\quad + \|\psi\psi' + \partial_x^3\psi\|_{H^s}^2 + \|u_x\|_{L^\infty}\|\psi'\|_{H^{s_0+s-1}}^2 \ ,\end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}(1+\|u(t)\|_{H^s}^2) \leqslant C(\psi)(1+\|u_x(t)\|_{L^\infty})(1+\|u(t)\|_{H^s}^2),\ t\in(-T,T) \ . \tag{7.48}$$

Next, the Gronwall's lemma concludes the proof of the lemma. □

We are now ready to begin the proof of Theorem 7.1.1 itself. Let $s>1$, $\theta\in(0,1)$ such that $(1-\theta)/2 < s-1$, $p := 2/(1-\theta)$ and $\sigma := s-\theta/4 > 3/4$. We also define $q := 4/\theta > 4$ and $\alpha := (1-1/q)/2 < 2$. For $T \geqslant 0$, and u a regular solution of (7.15), we define

$$F(T) := \|u_x\|_{L^1_T L^\infty} + T^\alpha \|J^\sigma u\|_{L^\infty_T L^2}.$$

Our choice of p and Sobolev embedding ensure that $W^{\sigma-(1-\theta/4),p} \subset L^\infty$ with continuous embedding, because $\sigma - (1-\theta/4) = s-1 > 1/p$. Hence, using also the continuity of the Hilbert transform on L^p and Hölder inequality in the t variable,

$$\begin{aligned} \|u_x\|_{L^1_T L^\infty} &\lesssim \|D^{1-\theta/4} H D^{\theta/4} u\|_{L^1_T W^{\sigma-(1-\theta/4),p}} \lesssim \|H D^{\theta/4} J^\sigma u\|_{L^1_T L^p} \\ &\lesssim \|D^{\theta/4} J^\sigma u\|_{L^1_T L^p} \lesssim T^{1-1/q} \|D^{\theta/4} J^\sigma u\|_{L^q_T L^p} \ . \end{aligned}$$

Thanks to Theorem 7.4.1, for $T \leqslant 1$,

$$\|u_x\|_{L^1_T L^\infty} \ \lesssim \ (T^{1-1/q} + T^{1-1/q-\alpha} F(T))(1+F(T))^{3/2}(1+\|u\|_{L^\infty_T H^s}) \ .$$

Therefore, since $\sigma < s$,

$$F(T) \ \lesssim \ T^\alpha (1+F(T))^{5/2}(1+\|u\|_{L^\infty_T H^s}) \ .$$

We control the quantity $\|u\|_{L^\infty_T H^s}$ by Lemma 7.5.1:

$$\|u\|_{L^\infty_T H^s} \ \lesssim \ (1+\|u_0\|_{H^s}) \exp(CF(T)) \ .$$

Finally, we obtain that there exists a constant $C' > 0$ (which only depends on ψ) such that

$$F(T) \ \leqslant \ C' T^\alpha (1+F(T))^{5/2}(1+\|u_0\|_{H^s}) \exp(CF(T)), \ T \in [0,1] \ . \tag{7.49}$$

We define $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(x) := \frac{x}{(1+x)^{5/2} \exp(Cx)}, \ x \in \mathbb{R} \ .$$

Let $T(\|u_0\|_{H^s}) \in [0,1]$ small enough such that

$$C' T(\|u_0\|_{H^s})^\alpha (1+\|u_0\|_{H^s}) < \|g\|_\infty \ .$$

Then by continuity of $g(F(T))$, since $g(F(0)) = g(0) = 0$, there exists some $A > 0$, which does not depend on u_0, such that

$$F(t) \leqslant A, \ t \in [0, T(\|u_0\|_{H^s})] \ .$$

In particular, $\|u_x\|_{L^1_{T(\|u_0\|_{H^s})} L^\infty} \leqslant A$. Therefore, thanks to Lemma 7.5.1,

$$\|u\|_{L^\infty_{T(\|u_0\|_{H^s})} H^s} \leqslant \tilde{A}(\|u_0\|_{H^s}) < \infty \ . \tag{7.50}$$

Let now $u_0 \in H^s$, and $u_{0,n} \to u_0$ in H^s, where $u_{0,n}$ is regular ($u_{0,n} \in H^\infty$). We denote by u_n the solution of (7.15) with initial data $u_n(0) = u_{0,n}$. The results in [ILS] ensure that u_n is global. Thanks to (7.50), we can extract a subsequence (that we also denote by u_n) such that u_n converges for the weak $\star$ topology of $L^\infty_{T(\|u_0\|_{H^s}+1)} H^s$.

Now, if v and w are two solutions of (7.15), then

$$\frac{1}{2}\frac{d}{dt}\|(v-w)(t)\|_{L^2}^2 + \int (v-w)^2\partial_x v - \frac{1}{2}\int (v-w)^2\partial_x w + \frac{1}{2}\int (v-w)^2\psi' = 0 \ .$$

Therefore

$$\frac{d}{dt}\|(v-w)(t)\|_{L^2}^2 \lesssim \|(v-w)(t)\|_{L^2}^2 \left(\|v_x(t)\|_{L^\infty} + \|w_x(t)\|_{L^\infty} + \|\psi'\|_{L^\infty}\right) \ ,$$

and the Gronwall lemma yields

$$\|(v-w)(t)\|_{L^2}^2 \quad \lesssim \quad \|(v-w)(0)\|_{L^2}^2 \exp\Big(C(\|v_x(t)\|_{L^1_T L^\infty} + \|w_x(t)\|_{L^1_T L^\infty} + \|\psi'\|_{L^\infty})\Big) \ . \tag{7.51}$$

Applying this to $v = u_n$, $w = u_m$, we get that for n and m large enough, $|t| \leqslant T(\|u_0\|_{H^s}+1)$,

$$\|u_n(t) - u_m(t)\|_{L^2}^2 \lesssim \|u_{0,n} - u_{0,m}\|_{L^2}^2 \exp\left(C(2F(T(\|u_0\|_{H^s}+1)) + \|\psi'\|_{L^\infty})\right) \ ,$$

which shows that $(u_n)_n$ is a Cauchy sequence, and hence converges to u strongly in the Banach space $L^\infty_T L^2$, because $u_{0,n} \to u$ in L^2. We deduce that the map $t \to u(t)$ is weakly continuous from $[-T,T]$ into H^s. Indeed, if we choose some $\varepsilon > 0$, if $\varphi \in H^s$, let $\tilde\varphi \in H^{2s}$ such that $\|\varphi - \tilde\varphi\|_{H^s} < \varepsilon/(2\|u\|_{L^\infty_T H^s})$. For $t, \tau \in [-T,T]$,

$$\begin{aligned} |(u(t)-u(\tau),\varphi)_{H^s}| &\leqslant \varepsilon + \left|\left(u(t)-u(\tau), J^{2s}\tilde\varphi\right)_{L^2}\right| \\ &\leqslant \varepsilon + 2\|u_n - u\|_{L^\infty_T L^2}\|\tilde\varphi\|_{H^{2s}} + \|u_n(t)-u_n(\tau)\|_{L^2}\|\tilde\varphi\|_{H^{2s}}. \end{aligned}$$

Choosing n large enough, and $t-\tau$ small enough, we have

$$|(u(t)-u(\tau),\varphi)_{H^s}| \leqslant 2\varepsilon.$$

In order to show that u is continuous with value in H^s, we define the norm

$$|||v||| := \left(\sum_\lambda \lambda^{2s}\|v_\lambda\|_{L^2}^2\right)^{1/2} ,$$

which is equivalent to the H^s norm (indeed, (7.35) gives one of the inequalities, the proof of the other one is similar). It suffices to show that $|||v(t)||| \underset{t\to\tau}{\longrightarrow} |||v(\tau)|||$. That is what we do in the following lemma.

Lemma 7.5.2 *Let u be as above the weak $\star$ limit in $L^\infty_T H^s$ of u_n and $(u_\lambda)_\lambda$ its Littlewood-Paley decomposition, $t,\tau \in [-T,T]$. Then*

$$\sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 \leqslant \exp\Big(C\|u_x\|_{L^1_I L^\infty}\Big)\sum_\lambda \lambda^{2s}\|u_\lambda(\tau)\|_{L^2}^2 + g(t,\tau) \ ,$$

where $I = [\tau, t]$ or $[t,\tau]$, depending on the sign of $t-\tau$, and $g(t,\tau) \underset{t-\tau\to 0}{\to} 0$.

Proof. We first remark that u solves (7.6), since u_n solves (7.15). We assume for instance that $\tau \leqslant t$. Like in the proof of Lemma 7.4.5, we show

$$\begin{aligned}\sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 &\leqslant \sum_\lambda \lambda^{2s}\|u_\lambda(\tau)\|_{L^2}^2 + C(t-\tau) + C\|u_x\|_{L^1_I L^\infty} \\ &\quad + C\int_\tau^t (1+\|u_x(\sigma)\|_{L^\infty})\sum_\lambda \lambda^{2s}\|u_\lambda(\sigma)\|_{L^2}^2 d\sigma \ .\end{aligned}$$

We conclude by Gronwall lemma that

$$\begin{aligned}\sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 &\leqslant \left(\sum_\lambda \lambda^{2s}\|u_\lambda(\tau)\|_{L^2}^2 + C(t-\tau)\right) \\ &\quad \times \left[1 + C\left(\|u_x\|_{L^1_I L^\infty} + t - \tau\right)\exp\left(C(\|u_x\|_{L^1_I L^\infty} + t-\tau)\right)\right] \\ &= \exp\left(C\|u_x\|_{L^1_I L^\infty}\right)\sum_\lambda \lambda^{2s}\|u_\lambda(\tau)\|_{L^2}^2 + \underset{t-\tau\to 0}{o}(1) \ .\end{aligned}$$

□

Coming back to te proof of Theorem 7.1.1, since $\|u_x\|_{L^1_I L^\infty} \underset{t-\tau\to 0}{\longrightarrow} 0$, using Lemma 7.5.2 and exchanging t and τ, we get that

$$\sum_\lambda \lambda^{2s}\|u_\lambda(t)\|_{L^2}^2 = \sum_\lambda \lambda^{2s}\|u_\lambda(\tau)\|_{L^2}^2 + \underset{t-\tau\to 0}{o}(1) \ .$$

To complete the first part of the proof of Theorem 7.1.1, it remains to show the uniqueness result. It is a straightforward consequence of (7.51). □

7.6 The proof of Theorem 7.1.1 (continuous dependence on the initial data).

We want now to prove the part of Theorem 7.1.1 about the continuous dependence on the initial data of a solution of (7.6). We use the method presented in [KoT]. We begin with two lemmas.

Lemma 7.6.1 *Let u be a solution of (7.6) in $\mathcal{D}'$. Let $1 < \delta \leqslant \kappa$, and $(\omega_\lambda)_\lambda$ a sequence of positive numbers such that for every dyadic integer λ, $\delta\omega_\lambda \leqslant \omega_{2\lambda} \leqslant \kappa\omega_\lambda$. Then for all $t, \tau \in [-T, T]$,*

$$\begin{aligned}\sum_\lambda \omega_\lambda^2\|u_\lambda(t)\|_{L^2}^2 &\leqslant \left(1 + C(\|u_x\|_{L^1_I L^\infty} + |t-\tau|)\right)\exp\left(C(\|u_x\|_{L^1_I L^\infty} + |t-\tau|)\right) \\ &\quad \times \left(\sum_\lambda \omega_\lambda^2\|u_\lambda(\tau)\|_{L^2}^2 + C|t-\tau|\right),\end{aligned}$$

where $I = [\tau, t]$ or $[t, \tau]$, depending on the sign of $t - \tau$.

Proof. The proof is identical to this of Lemma 7.5.2. The slight difference is that we use a variant version of Lemma 7.2.2 which is obtained by replacing the sequence (λ^s) by (ω_λ). This works because $\sum_\lambda \omega_\lambda^{-1}$ is finite. □

The second lemma is proved in [KoT].

Lemma 7.6.2 *Assume that $v_n \to v$ in H^s. Then there exists a sequence (ω_λ) of positive numbers which satisfies*

$$2^s\omega_\lambda \leqslant \omega_{2\lambda} \leqslant 2^{s+1}\omega_\lambda \ ,\forall\lambda$$

and

$$\omega_\lambda/\lambda^s \to \infty$$

such that

$$\sup_n \sum_\lambda \omega_\lambda^2 \|(v_n)_\lambda\|_{L^2}^2 < \infty \ .$$

Let u_{0n} be a sequence such that $u_{0n} \to u_0$ in H^s. Let $u_n, u \in C([0,T], H^s)$ be the associated solutions of (7.6). Then

$$u_n \to u \text{ in } C([-T(R), T(R)], L^2) \ , \tag{7.52}$$

because of (7.51) and $\|(u_n)_x\|_{L^1_{T(R)}L^\infty} \leqslant A$, where R is such that $R \geqslant \sup_n \|u_{0n}\|_{H^s}$.
We define a sequence $(\omega_\lambda)_\lambda$ like in Lemma 7.6.2, with $v_n = u_{0n}$. Lemma 7.6.1 with $\tau = 0$ yields

$$\sup_{n,t} \sum_\lambda \omega_\lambda^2 (\|(u_n)_\lambda(t)\|_{L^2}^2 + \|u_\lambda(t)\|_{L^2}^2) < \infty \tag{7.53}$$

Let

$$u_\Lambda := \sum_{\lambda \leqslant \Lambda} u_\lambda.$$

We first have

$$\|u_n - u\|_{L^\infty_T H^s} \leqslant \|u_n - (u_n)_\Lambda\|_{L^\infty_T H^s} + \|(u_n)_\Lambda - u_\Lambda\|_{L^\infty_T H^s} + \|u_\Lambda - u\|_{L^\infty_T H^s}.$$

Let $\varepsilon > 0$.

$$\begin{aligned} &\max\left(\sup_{n,t} \|(u_n)_\Lambda(t) - u_n(t)\|_{H^s}, \sup_t \|u_\Lambda(t) - u(t)\|_{H^s} \right) \\ &\qquad \leqslant \ \sup_{n,t} \underbrace{\left(\sup_{\lambda > \Lambda} \left(\frac{\lambda^s}{\omega_\lambda} \right) \right)}_{\underset{\Lambda\to\infty}{\longrightarrow} 0} \left(\sum_{\lambda > \Lambda} \omega_\lambda^2 (\|(u_n)_\lambda(t)\|_{L^2}^2 + \|u_\lambda(t)\|_{L^2}^2) \right)^{1/2} , \end{aligned}$$

hence, because of (7.53), we can choose Λ large enough such that

$$\max\left(\sup_{n,t} \|(u_n)_\Lambda(t) - u_n(t)\|_{H^s}, \sup_t \|u_\Lambda(t) - u(t)\|_{H^s} \right) \leqslant \varepsilon/4 \ .$$

Next, thanks to (7.52), we can choose n_0 large enough such that for every $n \geqslant n_0$ and $t \in [-T(R), T(R)]$,

$$\|(u_n)_\Lambda(t) - u_\Lambda(t)\|_{H^s} \leqslant C\Lambda^s \|(u_n)_\Lambda(t) - u_\Lambda(t)\|_{L^2} \leqslant \varepsilon/2 \ .$$

This completes the proof of Theorem 7.1.1. $\square$

7.7 The proof of Theorem 7.1.2

Existence. We first remark that in Theorem 7.1.1 above,

$$T(\psi, s, \|\varphi\|_{H^s}) = T(\|\psi\|_{X^{s_1}}, s, \|\varphi\|_{H^s}) \ ,$$

where $s_1 > 1$ is large, and T can be assumed to be non-increasing with respect to $\|\psi\|_{X^{s_1}}$ and $\|\varphi\|_{H^s}$. Hence, if we define

$$\tilde{T}(\|g\|_{X^s}) := T(C_{s_1}\|g\|_{X^s}, s, \tilde{C}\|g\|_{X^s})$$

where C_{s_1} (resp. $\tilde{C}$) is the norm of the bounded linear map $g \to \psi$ (resp. $g \to \varphi$) from X^s into X^{s_1} (resp H^s) defined in Lemma 7.2.1, we have

$$\tilde{T}(\|g\|_{X^s}) \leqslant T(\|\psi\|_{X^{s_1}}, s, \|\varphi\|_{H^s}) = T(\psi, s, \|\varphi\|_{H^s}) \ .$$

Let $v \in C([-T,T], H^s)$ be the solution of (7.6) given in Theorem 7.1.1. Then $u := v + \psi \in C([-\tilde{T}, \tilde{T}], X^s)$ solves (7.1). Moreover, since $\psi - g \in H^s$, $u - g \in C([-\tilde{T}, \tilde{T}], H^s)$. $u_x \in L^1_{\tilde{T}} L^\infty$ because v_x does.

Uniqueness. We choose ψ and $\tilde{T}$ as above. Let u be a solution of (7.1) as required. Let $\tilde{v} = u - \psi$. It is easy to see that $\tilde{v}$ solves (7.6), and the uniqueness in Theorem 7.1.1 yields $\tilde{v} = v$, where v is the solution of (7.6) we obtained in Theorem 7.1.1.

Continuity with respect to the initial data. Let $R > 0$ and $g \in X^s$ with $\|g\|_{X^s} < R$. Let $(g_n)_n$ be a sequence in X^s such that $g_n \to g$ in X^s. We assume that for all n, $\|g_n\|_{X^s} \leqslant R$.
As in Lemma 7.2.1, we define $k(x) = (4\pi)^{-1/2} e^{-x^2/4}$, $\psi = k \star g$, $\psi_n = k \star g_n$, $\varphi = g - \psi$, $\varphi_n = g - \psi_n$.
Let $v \in C([-\tilde{T}, \tilde{T}], H^s)$ be the solution of (7.6), and $v_n \in C([-\tilde{T}, \tilde{T}], H^s)$ solving

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(w\psi_n) = -(\partial_x^3 \psi_n + \psi_n \psi_n') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ w(x,0) = \varphi_n(x) \end{cases}$$

Lemma 7.6.2 applied to the sequence $\varphi_n \to \varphi$ in H^s and Lemma 7.6.1 yield

$$\sup_{n,t} \sum_\lambda \omega_\lambda^2 \left(\|(v_n)_\lambda(t)\|_{L^2}^2 + \|v_\lambda(t)\|_{L^2}^2 \right) < \infty \ ,$$

and we conclude as in section 7.6 that $w_n := v_n - v \to 0$ in $C([-\tilde{T},\tilde{T}],H^s)$. Since $s>1$ and $\psi_n \to \psi$ in X^s, we deduce that

$$u_n - u = (v_n+\psi_n) - (v+\psi) \to 0 \text{ in } X^s.$$

□

7.8 Global well-posedness of (7.7) in H^s, $s \geqslant 3/2$.

The Benjamin-Ono equation possesses infinitely many invariants (for instance, see [ABFS]). There is an invariant associated with each H^s norm, $s \in \mathbb{N}/2$. Here are the first ones.

$$I_0(u) := \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx \ ,$$

$$I_1(u) := -\int_{-\infty}^{\infty}\left[\frac{u^3}{3} + uH(\partial_x u)\right] dx \ ,$$

$$I_2(u) := \int_{-\infty}^{\infty}\left[\frac{u^4}{4} + \frac{3}{2}u^2 H(\partial_x u) + 2(\partial_x u)^2\right] dx \ ,$$

$$\begin{aligned} I_3(u) := \int_{-\infty}^{\infty}\Big[-\frac{u^5}{5} - \frac{4}{3}u^3 H(\partial_x u) - u^2 H(u\partial_x u) - 2uH(\partial_x u)^2 \\ -6u(\partial_x u)^2 + 4uH\partial_x^3 u\Big] dx \ . \end{aligned}$$

The first two following lemmas are proved in [ILS].

Lemma 7.8.1 *Let $\varphi \in H^{s_0}$ where s_0 is large. Let $u \in C(\mathbb{R}, H^{s_0})$ be the solution of (7.7) given in [ILS]. Then for all $t \in \mathbb{R}$,*

$$\|u(t)\|_{L^2}^2 \ \leqslant \ h(\psi, t, \|\varphi\|_{L^2}). \tag{7.54}$$

Lemma 7.8.2 *Under the same assumptions that in Lemma 7.8.1, for all $t \in \mathbb{R}$,*

$$\|u(t)\|_{H^1}^2 \ \leqslant \ k(\psi, t, \|\varphi\|_{H^1}). \tag{7.55}$$

Here, h and k are valued in $\mathbb{R}_+$. They are even, non-decreasing functions of t on $\mathbb{R}_+$, and continuous with respect to t and $\|\varphi\|_{L^2}$ or $\|\varphi\|_{H^1}$.

Our aim is now to prove a similar lemma, where the L^2 and H^1 norm are replaced by the $H^{3/2}$ norm.

Lemma 7.8.3 *Under the same assumptions that in Lemma 7.8.1 and 7.8.2,*

$$\begin{aligned} -I_3(u(t)) \ &\leqslant \ -I_3(\varphi) + C(\psi)\int_0^{|t|}(1 + h(\psi,s,\|\varphi\|_{L^2}) + k(\psi,s,\|\varphi\|_{H^1}))^5 ds \\ &+C(\psi)\int_0^{|t|}(1 + h(\psi,s,\|\varphi\|_{L^2}) + k(\psi,s,\|\varphi\|_{H^1}))\|u(s)\|_{H^{3/2}}^2 ds \ . \end{aligned} \tag{7.56}$$

Proof. We will prove the lemma only for $t \geqslant 0$. We remark as in [ILS] that (7.7) can be rewritten as

$$\begin{cases} \partial_t u - \frac{1}{2}\partial_x I_1'(u) + \partial_x(\psi u) + (H\psi'' + \psi\psi') = 0 \\ u(0) = \varphi \end{cases} . \tag{7.57}$$

For convenience, we will denote in the sequel $\rho := \psi\psi' + H\psi''$. Using the properties of the invariants of the Benjamin-Ono equation (see [ABFS]), we compute

$$\begin{aligned} \frac{d}{dt}(-I_3(u)) &= \left(I_3'(u), -\frac{1}{2}\partial_x I_1'(u) + \partial_x(\psi u) + \rho \right) \\ &= (I_3'(u), \partial_x(\psi u)) + (I_3'(u), \rho) \ . \end{aligned} \tag{7.58}$$

We note that

$$\begin{aligned} I_3'(u) &= -u^4 - 4(u^2 H(\partial_x u) + uH(u\partial_x u) + H(u^2\partial_x u)) - 2H(\partial_x u)^2 \\ &\quad -4H\partial_x(uH(\partial_x u)) + 6(\partial_x u)^2 + 12u\partial_x^2 u + 8H\partial_x^3 u \end{aligned} \tag{7.59}$$

Using (7.59), the Cauchy-Schwarz inequality, Sobolev inequalities and integration by parts, we first show that

$$\begin{aligned} (I_3'(u), \rho) \lesssim\ & \|u\|_{H^1}^4 \|\rho\|_{L^\infty} + \|u\|_{H^1}^3 \|\rho\|_{L^2} + \|u\|_{H^{3/2}}^2 \|\rho\|_{L^2} + \|u\|_{H^1}^2 \|\rho'\|_{L^2} \\ & + \|u\|_{H^1}^2 \|\rho\|_{L^\infty} + \|u\|_{H^1}(\|\rho'\|_{L^\infty}\|u\|_{L^2} + \|\rho\|_{L^\infty}\|u\|_{H^1}) + \|u\|_{L^2}\|\rho'''\|_{L^2}. \end{aligned} \tag{7.60}$$

Next, we control the quantity $(I_3'(u), \partial_x(\psi u))$. We will majorize separately each term that we obtain as we substitute (7.59) into $(I_3'(u), \partial_x(\psi u))$. Our arguments are again the Cauchy-Schwarz inequality, Sobolev inequalities, integration by parts and the continuity of the Hilbert transform on L^p, $1 < p < \infty$.

$$\left(-u^4, \partial_x(\psi u)\right) = -\frac{4}{5}\left(u^5, \psi'\right) \lesssim \|\psi'\|_{L^\infty}\|u\|_{H^1}^5 \ , \tag{7.61}$$

$$\begin{aligned} -4 & \left(u^2 H(\partial_x u) + uH(u\partial_x u) + H(u^2\partial_x u), \partial_x(\psi u)\right) \\ & \lesssim \|u\|_{H^1}^3 (\|\psi'\|_{L^\infty}\|u\|_{L^2} + \|\psi\|_{L^\infty}\|u\|_{H^1}) \ , \end{aligned} \tag{7.62}$$

$$\left(-2H(\partial_x u)^2 + 6(\partial_x u)^2, \partial_x(\psi u)\right) \lesssim \|u\|_{H^{5/4}}^2 (\|\psi'\|_{L^\infty}\|u\|_{L^2} + \|\psi\|_{L^\infty}\|u\|_{H^1}) \tag{7.63}$$

and

$$\begin{aligned} \left(u\partial_x^2 u, \partial_x(\psi u)\right) &= -\left(\partial_x u, \partial_x(\psi' u^2)\right) - \frac{1}{2}\left((\partial_x u)^2, \partial_x(\psi u)\right) \\ &\lesssim \|u\|_{H^1}^3 (\|\psi''\|_{L^2} + \|\psi'\|_{L^\infty}) + \|u\|_{H^{5/4}}(\|\psi'\|_{L^\infty}\|u\|_{L^2} + \|\psi\|_{L^\infty}\|u\|_{H^1}) \end{aligned} \tag{7.64}$$

It is a bit more delicate to estimate the two remaining terms, $(H\partial_x(uH(\partial_x u)), \partial_x(\psi u))$ and $(H\partial_x^3 u, \partial_x(\psi u))$. We begin with the last one. Using the properties of the Hilbert transform, the fact that $D = H\partial_x$ and an integration by parts,

$$\begin{aligned}
\big(H\partial_x^3 u, \partial_x(\psi u)\big) &= \big(\partial_x D^{3/2} u, D^{3/2}(\psi u)\big) \\
&= -\big(D^{3/2}u, D^{3/2}(\psi' u)\big) + \frac{1}{2}\big((D^{3/2}u)^2, \psi'\big) - \big(D^{3/2}u, \left[D^{3/2}, \psi\right]\partial_x u\big) .
\end{aligned}$$

Using Lemma 7.2.5, we have

$$-\big(D^{3/2}u, D^{3/2}(\psi' u)\big) \lesssim \|u\|_{H^{3/2}}^2 \|\psi'\|_{H^{3/2}} , \tag{7.65}$$

and it is easy to see that

$$\frac{1}{2}\big((D^{3/2}u)^2, \psi'\big) \lesssim \|\psi'\|_{L^\infty}\|u\|_{H^{3/2}}^2 . \tag{7.66}$$

Next,

$$\left[D^{3/2}, \psi\right]\partial_x u = \left[J^{3/2}, \psi\right]\partial_x u - [R, \psi]\,\partial_x u$$

where R is defined by

$$\widehat{Rf}(\xi) = \left[(1+\xi^2)^{3/4} - |\xi|^{3/2}\right]\widehat{f}(\xi) = |\xi|^{3/2}\left((1+\frac{1}{\xi^2})^{3/4} - 1\right)\widehat{f}(\xi) .$$

In particular, R is bounded on L^2, and

$$\|\,[R, \psi]\,\partial_x u\|_{L^2} \lesssim \|\psi\|_{L^\infty}\|u\|_{H^1} .$$

Using finally Lemma 7.2.7, we obtain

$$\big(H\partial_x^3 u, \partial_x(\psi u)\big) \lesssim (\|\psi'\|_{L^\infty} + \|\psi'\|_{H^{3/2}})\|u\|_{H^{3/2}}^2 + \|\psi\|_{L^\infty}\|u\|_{H^1}\|u\|_{H^{3/2}} . \tag{7.67}$$

We now control the last term.

$$(H\partial_x(uH(\partial_x u)), \partial_x(\psi u)) = \big(D^{1/2}(uDu), D^{1/2}\partial_x(\psi u)\big) .$$

Lemma 7.2.5 yields

$$\|D^{1/2}(uDu)\|_{L^2} \lesssim \|u\|_{H^1}\|u\|_{H^{3/2}} , \tag{7.68}$$

and with Lemma 7.2.7, we obtain that

$$\begin{aligned}
\|D^{1/2}\partial_x(\psi u)\|_{L^2} &\lesssim \|J^{3/2}(\psi u)\|_{L^2} \\
&\lesssim \|\psi J^{3/2} u\|_{L^2} + \|\left[J^{3/2}, \psi\right] u\|_{L^2} \\
&\lesssim \|\psi\|_{L^\infty}\|u\|_{H^{3/2}} + \|\psi'\|_{H^{3/2}}\|u\|_{H^{1/2}} .
\end{aligned} \tag{7.69}$$

Bringing together (7.58) and the estimates (7.60-7.64), (7.67), (7.68) and (7.69), we get

$$\frac{d}{dt}(-I_3(u)) \;\lesssim\; C(\psi)\left[(1+h+k)^5 + (1+h+k)\|u\|^2_{H^{3/2}}\right] \; . \tag{7.70}$$

Integrating with respect to time, we obtain the announced result. □

Lemma 7.8.4 *Under the same assumptions that in Lemma 7.8.3,*

$$\|u(t)\|_{H^{3/2}} \leqslant m(\psi, t, \|\varphi\|_H^{3/2}) \; , \tag{7.71}$$

where m is an even positive function which growth with t on $\mathbb{R}_+$, and which is continuous with respect to t and $\|\varphi\|_H^{3/2}$. m can be expressed in function of h and k, but we will omit this expression, which is of low interest.

Proof. Using the conservation law I_3, Lemma 7.8.3 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|u(t)\|^2_{H^{3/2}} \;\lesssim\; & C(\|\varphi\|_H^{3/2} + (1+k)^5) + C(\psi)\int_0^t (1+h+k)^5 ds \\ & + C(\psi)\int_0^t (1+h+k)\|u(s)\|^2_{H^{3/2}} ds \; .\end{aligned}$$

The result follows by applying the Gronwall's lemma. □

Passing to the proof of Theorem 7.1.5, take now $\varphi \in H^{3/2}$. Since (7.7) has been shown to be locally well-posed in $H^{3/2}$, there exists $T^* > 0$ and a solution $u \in C([0, T^*[, H^{3/2})$ of (7.7). Moreover, either $T^* = \infty$ or T^* is finite and $\|u(t)\|_{H^{3/2}} \underset{t \uparrow T^*}{\longrightarrow} +\infty$. We assume by contradiction that T^* is finite. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence of H^{s_1} functions such that $\varphi_j \to \varphi$ in $H^{3/2}$, where s_1 is large. We denote by $u_j \in C(\mathbb{R}, H^{s_0})$ the solution of (7.7) with initial data φ_j, which is given by the results of [ILS]. Then, using Theorem 7.1.3, it is easy to see that for all $T < T^*$, $u_j \to u$ in $C([0,T], H^{3/2})$. In particular,

$$\|u\|_{L^\infty_T H^{3/2}} \leqslant \liminf \|u_j\|_{L^\infty_T H^{3/2}} \leqslant \lim m(\psi, t, \|\varphi_j\|_{H^{3/2}}) = m(\psi, t, \|\varphi\|_{H^{3/2}}) \; .$$

Letting $T \to T^*$, we obtain a contradiction with the fact that $\|u(t)\|_{H^{3/2}}$ blows up as $t \uparrow T^*$.

We have shown that (7.7) is globally well-posed in $H^{3/2}$. We next prove that it is globally well-posed in H^s, where $s \in (3/2, 2)$.

Inequality (7.48) was shown for a regular solution of (7.6), but it clearly remains true for a regular solution of (7.7).

Like in [Po], we use the following inequality due to H. Brezis and T. Gallouët (see [BG]): if $f \in H^a(\mathbb{R}^n)$ with $a > n/2$, then

$$\|f\|_{L^\infty} \leqslant C_{a,n}(1 + \|f\|_{H^{n/2}}\sqrt{\log(2 + \|f\|_{H^a})}) \; . \tag{7.72}$$

Since $s - 1 > 1/2$, it follows that

$$\|\partial_x u\|_{L^\infty} \lesssim 1 + \|u\|_{H^{3/2}} \sqrt{\log(2 + \|u\|_{H^s})} . \tag{7.73}$$

Thanks to (7.73) and (7.48), we obtain that there is $C > 0$ such that for all $u_0 \in H^\infty$, $T > 0$, $t \in [-T, T]$,

$$\frac{d}{dt}\|u(t)\|_{H^s}^2 \leqslant C(1 + \|u(t)\|_{H^{3/2}}) \log(3 + \|u(t)\|_{H^s}^2)(3 + \|u(t)\|_{H^s}^2) ,$$

using Lemma 7.8.4, we get

$$\|u(t)\|_{H^s}^2 \leqslant (3 + \|u_0\|_{H^s}^2) \exp\exp(C(1 + m(\psi, T, \|u_0\|_{H^{3/2}}))t) .$$

By continuity with respect to the initial data, this inequality remains true for $u_0 \in H^s$. This shows that (7.7) is globally well-posed in H^s, and the proof of Theorem 7.1.5 is complete.

7.9 Appendix

The proof of Lemma 7.2.4 is similar and simpler to that of Lemma 7.2.3 which was done in [KP]. It is based on the following result due to Coifman and Meyer (see [CM1], [CM2]).

Theorem 7.9.1 *Let $\sigma(\eta, \xi) \in C^\infty((\mathbb{R}^n)^k \times \mathbb{R}^n \backslash (0, 0))$ satisfy*

$$|\partial^\nu \sigma(\eta, \xi)| \leqslant C_\nu(|\xi| + |\eta|)^{-|\nu|} , \tag{7.74}$$

where $\nu = (\nu_0, ..., \nu_k)$, $\nu_j \in \mathbb{N}^n$, and

$$|\nu| = |\nu_0| + .. + |\nu_k| \leqslant N := n(k+1) + 1 .$$

Then

$$\|\sigma(D)(a_1...a_k, f)\|_{L^2} \lesssim \|a_1\|_{L^\infty}...\|a_k\|_{L^\infty}\|f\|_{L^2} ,$$

where $\sigma(D)(a, f) = \sigma(D)(a_1...a_k, f)$ is defined by

$$\sigma(D)(a, f) := \int\int e^{ix(\xi+\tilde{\eta})}\sigma(\eta, \xi)\hat{a}(\eta)\hat{f}(\xi)d\eta d\xi ,$$

$$a(x) = a_1(x_1)...a_k(x_k) , \ x_j \in \mathbb{R}^n ,$$

$$\tilde{\eta} = \eta_1 + ... + \eta_k , \ \eta = (\eta_1, ..., \eta_k) .$$

In our case, $n = k = 1$, hence $N = 3$.

Proof of Lemma 7.2.4. We begin as in the proof of Lemma 7.2.3 in [KP]. We write

$$\begin{aligned}[J^s, f]g(x) &= c\int\int e^{ix(\xi+\eta)}\left[(1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2}\right]\hat{f}(\xi)\hat{g}(\eta)d\eta d\xi \\ &= c\sum_{j=1}^{2}\sigma_j(D)(f,g)(x)\end{aligned}$$

where

$$\sigma_j(\xi,\eta) = \left[(1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2}\right]\varphi_j(\frac{\xi}{\eta})$$

and the φ_j are positive even functions on $\mathbb{R}$ such that $\varphi_1+\varphi_2 \equiv 1$ on $\mathbb{R}$ and $\mathrm{Supp}\varphi_1 \subset [-1/3, 1/3]$, $\mathrm{Supp}\varphi_2 \subset \mathbb{R}\backslash[-1/4, 1/4]$.

It was shown in [KP] that

$$\|\sigma_1(D)(f,g)\|_{L^2} \lesssim \|\partial_x f\|_{L^\infty}\|J^{s-1}g\|_{L^2}. \tag{7.75}$$

Next, we define

$$\begin{aligned}&\sigma_2(D)(f,g)(x) \\ &= \int\int e^{ix(\xi+\eta)}\frac{(1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2}}{\xi(1+\xi^2)^{\frac{s_0+s-1}{2}}}\varphi_2(\frac{\xi}{\eta})(1+\xi^2)^{\frac{s_0+s-1}{2}}\xi\hat{f}(\xi)\hat{g}(\eta)d\eta d\xi \\ &=: \tilde{\sigma}_2(D)(J^{s_0+s-1}\partial_x f, g)(x)\ .\end{aligned}$$

It suffices to prove that $\tilde{\sigma}_2$ satisfies the assumptions of Theorem 7.9.1 to obtain that

$$\|\sigma_2(D)(f,g)\|_{L^2} \lesssim \|J^{s_0+s-1}\partial_x f\|_{L^2}\|g\|_{L^\infty}\ , \tag{7.76}$$

which will complete the proof of Lemma 7.2.4, combined with (7.75).

It is clear that $\tilde{\sigma}_2 \in C^\infty(\mathbb{R}^2\backslash(0,0))$ (because it vanishes if $|\xi| \leqslant |\eta|/4$). In fact, $\tilde{\sigma}_2 \in C^\infty(\mathbb{R}^2)$. Indeed, if $|\xi| \leqslant 1/8$ and $\tilde{\sigma}_2(\xi,\eta) \neq 0$, then $|\eta| \leqslant 4|\xi| \leqslant 1/2$, and $|\xi+\eta| \leqslant 5/8$. Therefore, using integer series, for $|\xi| \leqslant 1/8$, we can write

$$\tilde{\sigma}_2(\xi,\eta) = \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}}\sum_{r=0}^{\infty} c_r\frac{(\xi+\eta)^{2r} - \eta^{2r}}{\xi}\varphi_2(\frac{\xi}{\eta})\ ,$$

which is clearly of class C^∞ on $\{(\xi,\eta),\ |\xi| \leqslant 1/8\}$, thanks to the mean value theorem and because the ray of convergence of $\sum c_r X^r$ is 1, and $1/8 + 1/2 < 1$.
Therefore $\tilde{\sigma}_2$ is bounded on the compact set $K := \{(\xi,\eta),\ |\xi| \leqslant 1,\ |\eta| \leqslant 4\}$, hence on the set $\{(\xi,\eta),\ |\xi| \leqslant 1\}$, because it vanishes on $\{(\xi,\eta),\ |\xi| \leqslant 1\}\backslash K$, and we have a similar result for the derivates of $\tilde{\sigma}_2$.

It remains to see that the derivates of $\tilde{\sigma}_2$ of order less or equal to 3 are bounded on $\{(\xi,\eta),\ |\xi| \geqslant 1\}$. That is what we will do now.

Let ν be a multiindex such that $|\nu| \leqslant 3$. Then by the Leibniz formula,

$$\partial^\nu \tilde{\sigma}_2(\xi,\eta) = \sum_{\alpha+\beta+\gamma+\delta=\nu} c_{\alpha\beta\gamma\delta} \partial^\alpha \left((1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2}\right)$$
$$\partial^\beta \frac{1}{\xi} \partial^\gamma \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}} \partial^\delta \varphi_2(\frac{\xi}{\eta}) \ . \tag{7.77}$$

We estimate now each derivate in the right hand side of (7.77).

$$\left|\partial^\alpha \left((1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2}\right)\right| \lesssim (1+(\xi+\eta)^2)^{\frac{s-|\alpha|}{2}} + (1+\eta^2)^{\frac{s-|\alpha|}{2}} \ ,$$

$$\left|\partial^\beta \frac{1}{\xi}\right| \lesssim \frac{1}{\xi^{1+|\beta|}}$$

and

$$\left|\partial^\gamma \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}}\right| \lesssim \frac{1}{(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \ .$$

If $\delta = 0$, $\partial^\delta \varphi_2(\xi/\eta) = \varphi_2(\xi/\eta)$ is bounded and vanishes on the set $\{(\xi,\eta),\ |\xi/\eta| \leqslant 1/4\}$. Else, $\partial^\delta \varphi_2(\xi/\eta)$ vanishes on $\{(\xi,\eta),\ |\xi/\eta| \leqslant 1/4 \text{ or } |\xi/\eta| \geqslant 1/3\}$, and

$$\left|\partial^\delta \varphi_2(\frac{\xi}{\eta})\right| \lesssim \frac{1}{\eta^{|\delta|}} \lesssim \frac{1}{\xi^{|\delta|}} \ .$$

Therefore, on the set $\{(\xi,\eta),\ |\xi| \geqslant 1\}$,

$$|\partial^\nu \tilde{\sigma}_2(\xi,\eta)| \, (|\xi|+|\eta|)^{|\nu|} \lesssim \operatorname*{Max}_{\alpha+\beta+\gamma+\delta=\nu} Q_{\alpha\beta\gamma\delta} \ , \tag{7.78}$$

where

$$Q_{\alpha\beta\gamma\delta} := \frac{(|\xi|+|\eta|)^{|\nu|} \left((1+(\xi+\eta)^2)^{\frac{s-|\alpha|}{2}} + (1+\eta^2)^{\frac{s-|\alpha|}{2}}\right)}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \ .$$

To control $Q_{\alpha\beta\gamma\delta}$, we distinguish two cases:

– if $s - |\alpha| \geqslant 0$,

$$\begin{aligned} Q_{\alpha\beta\gamma\delta} &\lesssim \frac{|\xi|^{|\nu|}(1+|\xi|^2)^{\frac{s-|\alpha|}{2}}}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \\ &\lesssim (1+\xi^2)^{-s_0/2} \lesssim 1 \end{aligned}$$

because $s_0 \geqslant 0$.

– if $s - |\alpha| < 0$,

$$\begin{aligned} Q_{\alpha\beta\gamma\delta} &\lesssim \frac{|\xi|^{|\nu|}}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \\ &\lesssim (1+\xi^2)^{(|\alpha|-s_0-s)/2} \lesssim 1 \end{aligned}$$

because $|\alpha| \leqslant 3$ and $3 - s_0 - s \leqslant 0$.

The proof of Lemma 7.2.4 is complete. $\square$

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