The black solitons of one-dimensional NLS equations

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Abstract. In this paper, we prove a criterion determining if a black soliton solution to a one-dimensional nonlinear Schrödinger equation is linearly stable or not. This criterion handles with the sign of the limit at 0 of the Vakhitov-Kolokolov function. For some nonlinearities, we compute numerically the black soliton and the Vakhitov-Kolokolov function. We then show that linearly unstable black solitons are also orbitally unstable. In the Gross-Pitaevskii case, we rigorously prove the linear stability of the black soliton. Finally, we numerically investigate the dynamical stability of these solutions solving both linearized and fully nonlinear equation with a finite differences algorithm.

1 Introduction

We consider here the Nonlinear Schrödinger equation (NLS) in the one-dimensional spatial case:

$$\begin{cases} i\partial_t u + \partial_x^2 u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R}^2, \\ u(0) = u_0, \end{cases} \tag{1}$$

where f is a real valued function and $|u_0(x)|^2 \to \rho_0$ as $x \to \infty$. In all that follows, we will assume that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. In nonlinear optics, this equation is a model for the evolution of the complex amplitude envelope of a electric field in optical fibers, or for the self-guided beams in planar waveguides. We refer to [KLD] for a list of physically relevant nonlinearities f. NLS equation also appears in the description of defectons or in the theory of one-dimensional ferromagnetic or molecular chains (see [BGMP], [BP], [B]).

Equation (1) has two invariants which are at least formally conserved by the flow: the charge

$$I(u) = \int_{\mathbb{R}} \left[|u|^2 - \rho_0\right] dx$$

and the energy

$$E(u) = \int_{\mathbb{R}} \left[|\nabla u|^2 + V(|u|^2)\right] dx,$$

where $V(r) = \int_r^{\rho_0} f(s)ds$ is the potential.

In this paper, we focus on stationary solutions $u(t,x) = \varphi(x)$ to (1). Thus we are faced with the second-order differential equation

$$\varphi'' + f(|\varphi|^2)\varphi = 0. \tag{2}$$

In [dB] (see also [BGMP], [BP]), A. de Bouard studies one special kind of these solutions: the stationary bubbles, which may exist under some conditions on the nonlinearity f. Namely, if V vanishes in the interval $(0, \rho_0)$ and if the largest zero of V in this interval is of multiplicity one, there does exist a stationary bubble solving (1), that is a real, strictly positive, even function, growing on $\mathbb{R}_+$, tending to $\rho_0^{1/2}$ at infinity.

Now, if V is positive on $[0, \rho_0)$, (1) admits an other kind of stationary solution: a black soliton, that is a real, odd function tending to $\pm\rho_0^{1/2}$ at $\pm\infty$. As an example, for the Gross-Pitaevskii equation, which is (1) with $f(r) = 1 - r$, $\rho_0 = 1$, $V(r) = (1-r)^2/2$, the black soliton is explicitly known as $\varphi(x) = \tanh(x/\sqrt{2})$. The term "black solitons" comes from optics, where they can be experimentally detected as black spots on bright backgrounds (see [KLD], [KK], [MS]).

In [dB], the stationary bubbles solving (1) are shown to be all unstable. The proof is based on a subtle analysis of the spectrum of the operator obtained by linearization of (1) near the bubble. On the contrary, it was first thought by physicists that the black solitons were all stable (see [KLD], [MS]). However, Y.S. Kivshar and W. Krolikowski were the first who observed numerically unstable black solitons ([KK]). In the same paper, they justify by variational arguments a criterion determining if a dark soliton (and in particular, a black soliton) is stable or not. We recall that a dark soliton solving (1) is a solution of the form $u(t,x) = u_v(x - vt)$, where $v \in \mathbb{R}$ is the speed of the soliton, and $|u_v(x)|^2 \to \rho_0$ as $x \to \infty$ (see [G1]). This criterion reduces to the analysis of the sign of the derivative dP_v/dv, where P_v is the momentum of the dark soliton of speed v, given by

$$P_v = \mathfrak{Im} \int_{-\infty}^{+\infty} \partial_x \overline{u_v} u_v \left(1 - \frac{\rho_0}{|u_v|^2}\right) dx.$$

Namely, u_v is stable if $dP_v/dv > 0$ and unstable if $dP_v/dv < 0$. This criterion has been rigorously proved by Z. Lin ([L]). However, the proof is based on the hydrodynamical form of Equation (1). A solution to (1) is written as $u = (\rho_0 - r)^{1/2} e^{i\theta}$, and Z. Lin makes the analysis on the system satisfied by (r, θ_x), which has a Hamiltonian structure. This analysis is valid for non-zero speeds (or for stationary bubbles), because the dark soliton is in that case a bubble, which in particular means that $|u_v(x)|>0$ for $x\in\mathbb{R}$. This approach breaks down for the black solitons, which vanish at one point.

The goal of this paper is to fill as far as we can this gap concerning the study of the stability of the black solitons. We make a spectral analysis as it was done by A. de Bouard on the stationary bubbles. Denoting by φ a black soliton solving (1), we look for solutions to (1) of the form $\varphi+u_1+iu_2$, where u_1 and u_2 are real valued. The linearization of (1) near $(u_1,u_2)=(0,0)$ writes

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3}$$

where

$$A=\begin{pmatrix} 0 & L_1 \\ -L_2 & 0 \end{pmatrix},$$

$$L_1=-\partial_x^2-f(\varphi^2)=-\partial_x^2+q_1,$$

$$L_2=-\partial_x^2-f(\varphi^2)-2\varphi^2 f'(\varphi^2)=-\partial_x^2+q_1+q_2.$$

We will see that $f(\varphi^2)$ and $\varphi^2 f'(\varphi^2)$ converge exponentially at infinity. It follows as in [dB] that the essential spectrum $\sigma_e(A)$ of A is contained in $i\mathbb{R}$.

Under this framework, we establish the following criterion of linear stability of the black soliton[1].

Theorem 1.1 *We assume that $f\in\mathcal{C}^3(\mathbb{R}_+)$[2]. L_1 is a self-adjoint operator on L^2 with domain H^2, its essential spectrum is $[0,+\infty)$, and L_1 has a unique negative eigenvalue λ_0 which is simple.*
The Vakhitov-Kolokolov function

$$g(\lambda) \quad := \quad <(L_1-\lambda)^{-1}\varphi',\varphi'>, \tag{4}$$

defined for $\lambda\in(\lambda_0,0)$, is of class $\mathcal{C}^1$, increases on $(\lambda_0,0)$, and $g(\lambda)\to-\infty$ as $\lambda\downarrow\lambda_0$. If the limit of $g(\lambda)$ as $\lambda\uparrow 0$ is strictly positive, then A has a positive eigenvalue, and φ is linearly unstable, while if it is non-positive, the spectrum of A lies in $i\mathbb{R}$, and φ is linearly stable.

As for the stationary bubbles studied in [dB], we prove that the black solitons which are linearly unstable are also orbitally unstable.

Theorem 1.2 *Let $f\in\mathcal{C}^3(\mathbb{R}_+)$ be such that there exists a black soliton φ solving (1). We assume that φ is linearly unstable. Then it is also orbitally unstable, in the sense that there exists $\varepsilon>0$ such that for every $\delta>0$, there exists $v_0\in H^1$ with $\|v_0\|_{H^1}\leqslant\delta$ such that,*

[1] The notion of linear stability that we consider has to be understood as spectral stability: φ is said to be linearly stable if the spectrum $\sigma(A)$ is a subset of $\{\lambda,\mathfrak{Re}\lambda\leqslant 0\}$, and linearly unstable in the other case.

[2] Only $f\in\mathcal{C}^1(\mathbb{R}_+)$ is needed for the sufficient condition of linear stability.

if $u(t) \in \varphi + \mathcal{C}((-T_*, T^*), H^1)$ *denotes the solution to (1)*[3] *with initial data* $\varphi + v_0$*, there exists* $t \in (0, T^*)$ *such that* $\inf_{s\in\mathbb{R}} \|u(t) - \tau_s\varphi\|_{L^\infty} \geqslant \varepsilon$*, where* $\tau_s f(x) = f(x - s)$*.*

Note that we used in Theorem 1.2 the notion of orbital instability associated to the L^∞ distance, and not H^1 distance as it was done in [dB]. The reason is that we observe numerically that the black solitons which are linearly stable, are also L^∞-orbitally stable, but H^1-orbitally unstable. More precisely, we observe that the solution of (1) we obtain by perturbation of a linearly stable black soliton differs from the black soliton from a small constant, on a spreading domain. A slightly modified version of our proof shows that such a kind of instability also holds for the stationary bubbles, at least in dimension one.

It has also to be mentioned that since φ does not vanish at infinity, the instability result above implies that

$$\exists \varepsilon > 0, \forall \delta > 0, \forall u_0 \in \varphi + H^1 \text{ with } \|u_0 - \varphi\|_{H^1} < \delta, \exists\, t, \quad \inf_{\theta\in\mathbb{R}} \inf_{s\in\mathbb{R}} \|u(t) - e^{i\theta}\tau_s\varphi\|_{L^\infty} \geqslant \frac{\varepsilon}{2}.$$

That is the reason why we call in the sequel *orbital instability* what seems to be at first glance only *instability modulo translations*. Indeed, if $\inf_{s\in\mathbb{R}} \|u(t) - \tau_s\varphi\|_{L^\infty} \geqslant \varepsilon$ and $\theta \in \mathbb{R}$, either $|e^{i\theta} - 1|\|\varphi\|_{L^\infty} \leqslant \varepsilon/2$, and thus for every $s \in \mathbb{R}$,

$$\|u(t) - e^{i\theta}\tau_s\varphi\|_{L^\infty} \geqslant \|u(t) - \tau_s\varphi\|_{L^\infty} - \|(1 - e^{i\theta})\tau_s\varphi\|_{L^\infty} \geqslant \varepsilon - \varepsilon/2 = \varepsilon/2,$$

or $|e^{i\theta} - 1|\|\varphi\|_{L^\infty} \geqslant \varepsilon/2$, which directly implies

$$\|u(t) - e^{i\theta}\tau_s\varphi\|_{L^\infty} \geqslant \lim_{x\to\infty} |u(t,x) - e^{i\theta}\varphi(x - s)| = \sqrt{\rho_0}|1 - e^{i\theta}| \geqslant \varepsilon/2,$$

since $u(t) \in \varphi + H^1$ and $\varphi(x) \to \sqrt{\rho_0} = \|\varphi\|_{L^\infty}$ as x tends to infinity.

As it will be seen, Theorem 1.1 is a numerically efficient criterion in order to check linear stability or linear instability. On the contrary, given a nonlinearity f such that there exists a black soliton solving (1), it is not easy to prove rigorously via this criterion the linear stability or the linear instability of the black soliton. We can however do it for the Gross-Pitaevskii equation, thanks to the explicit expression of the black soliton φ and the spectral properties of L_1.

Theorem 1.3 *The black soliton* $\varphi(x) = \tanh(x/\sqrt{2})$ *of the Gross-Pitaevskii equation is linearly stable.*

This paper is organized as follows. In section 2, we give a definition of the black solitons and a necessary and sufficient condition on the nonlinearity f ensuring that black solitons exist. Section 3 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in section

[3]It is not difficult to see that the Cauchy problem (1) is locally well-posed in $\varphi + H^1$, since $i\Delta$ generates a group on H^1 and $v \mapsto f(|\varphi + v|^2)(\varphi + v)$ is locally Lipschitz continuous on H^1. For a more detailed proof as well as conditions under which the problem is globally well-posed, see [G2]. $(-T_*, T^*)$ denotes here the maximal interval of existence of the solution in $\varphi + H^1$.

4. In section 5, we show that Theorem 1.1 is numerically efficient: we have checked for several nonlinearities if the condition in Theorem 1.1 is satisfied or not. Theorem 1.3 is proved in section 6. Finally, we numerically study in section 7 the dynamical linear and nonlinear stability of the black solitons, for the pure power defocusing equation as well as for the saturated equation. Each time, a finite differences scheme both in time and space is used. We prove in the appendix some technical lemmas which are used in section 4.

Notations. Throughout this paper, C denotes a harmless positive constant which can change from line to line.
If A is an unbounded operator on a Banach space, we denote by $\sigma(A)$ its spectrum, by $\sigma_e(A)$ its essential spectrum and by $\rho(A)$ its resolvent set.
For $s \in \mathbb{R}$, $p \geqslant 1$, we define the spaces $\mathbb{H}^s := H^s \times H^s$ and $\mathbb{L}^p := L^p \times L^p$, endowed with their natural norms. Finally, $< \cdot, \cdot >$ denotes the L^2 scalar product.

2 Existence of a black soliton

Definition 2.1 *Let $\rho_0 > 0$. A black soliton of (1) is a solution of (1) of the form*

$$u(t,x) = \varphi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

where $v \in \mathbb{R}$, $\varphi_v \in \mathcal{C}^1(\mathbb{R})$, $a_v(0) = 0$, $a_v(x) > 0$ for $x \in \mathbb{R}^\star$, θ_v is of class $\mathcal{C}^1$ on $\mathbb{R}^\star$, and piecewise $\mathcal{C}^1$ on $\mathbb{R}$, with $\theta_v(0_+) = \pi + \theta_v(0_-)$, and $a_v \to \rho_0^{1/2}$, $\partial_x a_v \to 0$, $\partial_x \theta_v \to 0$ as $x \to \pm\infty$.

Even if this definition does not a priori exclude the case where black solitons are travelling at a non-zero speed v, we will see thereafter that black solitons may only be stationary solutions (i.e. $v = 0$).

Next, we give a necessary and sufficient condition on the nonlinearity f which ensures the existence of a black soliton to (1).

Theorem 2.1 *Under the assumption $f \in \mathcal{C}(\mathbb{R}_+)$ and $\rho_0 > 0$, there exists a black soliton φ of (1), of speed $v \in \mathbb{R}$, with amplitude tending to $\rho_0^{1/2}$ at infinity if and only if the following conditions are satisfied:*
(i) $f(\rho_0) = 0$
(ii) $v = 0$
(iii) $V(r) := \int_r^{\rho_0} f(s)ds > 0$ *for* $r \in [0, \rho_0)$.
Moreover, if these conditions are satisfied, up to the multiplication by some $e^{i\theta}$, φ is unique, odd, and may be assumed to be real, positive for $x > 0$.
Under the further assumption $f \in \mathcal{C}^1(\mathbb{R}_+)$[4] and $f'(\rho_0) < 0$ ($f'(\rho_0) \leqslant 0$ is a consequence of (i) and (iii)),

$$\sqrt{\rho_0} - \varphi(x) = e^{-cx(1+o(1))},$$

[4]It is sufficient to assume only that f is differentiable at ρ_0.

$$\varphi'(x) \sim c(\sqrt{\rho_0} - \varphi(x)),$$
$$\varphi''(x) \sim -c^2(\sqrt{\rho_0} - \varphi(x))$$

as $x \to +\infty$, where $c := \sqrt{-2\rho_0 f'(\rho_0)}$ is the sound speed.

Proof. Let us assume the existence of a black soliton $\varphi_v = a_v e^{i\theta_v} = ae^{i\theta}$ of (1), with amplitude tending to $\rho_0^{1/2}$ at infinity.
$\varphi_v' = (a' + ia\theta')e^{i\theta}$ tends to 0 as $x \to \pm\infty$, and φ_v solves

$$-iv\varphi_v' + \varphi_v'' + f(|\varphi_v|^2)\varphi_v = 0, \tag{5}$$

thus $|\varphi_v''(x)| \to |f(\rho_0)|\rho_0^{1/2} =: l_0$ as $x \to \pm\infty$, and therefore $f(\rho_0) = 0$. Indeed, let us assume by contradiction that $l_0 > 0$. There exists $A > 0$ such that $x \geqslant A$ implies $3l_0/4 \leqslant |\varphi_v''(x)| \leqslant 5l_0/4$. φ_v solves (5), thus, since φ_v' and φ_v'' are bounded, φ_v and φ_v' are uniformly continuous, and so is φ_v''. It follows that there exists $\eta > 0$ such that $|x_1 - x_2| \leqslant \eta$ implies $|\varphi_v''(x_1) - \varphi_v''(x_2)| \leqslant l_0/4$. If $x_1, x_2 \geqslant A$ and $|x_1 - x_2| = \eta$,

$$\begin{aligned} &|\varphi_v'(x_1)| + |\varphi_v'(x_2)| \geqslant |\varphi_v'(x_1) - \varphi_v'(x_2)| \\ = \ &\left|\int_{x_1}^{x_2} \varphi_v''(x)dx\right| \geqslant \int_{x_1}^{x_2} (|\varphi_v''(x_1)| - l_0/4)dx \geqslant l_0\eta/2. \end{aligned} \tag{6}$$

Let $\varepsilon \in (0, l_0\eta/4)$ and $B \geqslant A$ such that $x \geqslant B$ implies $|\varphi_v'(x)| \leqslant \varepsilon$. For $x_1, x_2 \geqslant B$ and $|x_1 - x_2| = \eta$, (6) yields a contradiction.

For $x \in \mathbb{R}^\star$, $\varphi_v'(x) = (a'(x) + ia(x)\theta'(x))e^{i\theta(x)}$. Letting x tend to $0_\pm$ in this equality and using the regularity properties of φ and θ, it follows that $\varphi'(0_\pm) = a'(0_\pm)e^{i\theta(0_\pm)}$. Since $\theta(0_+) = \pi + \theta(0_-)$, we get $a'(0_-) = -a'(0_+)$.
Moreover, we deduce from (5) that $\varphi_v \in \mathcal{C}^2(\mathbb{R})$, and a straightforward computation yields for $x \in \mathbb{R}^\star$,

$$\theta_x(x) = \frac{1}{2}v\left(1 - \frac{\rho_0}{a(x)^2}\right) \tag{7}$$
$$-a_{xx}(x) = f_v(a(x)^2)a(x), \tag{8}$$

where $f_v(r) := f(r) + \frac{v^2}{4}(1 - \frac{\rho_0^2}{r^2})$. We also define $V_v(r) := \int_r^{\rho_0} f_v(s)ds = V(r) - \frac{v^2(r-\rho_0)^2}{4r}$.
Multiplying (8) by a_x and integrating between x_1 and x_2 ($x_1,\ x_2 \in \mathbb{R}_\pm^\star$) leads to

$$-a_x(x_1)^2 + a_x(x_2)^2 = -V_v(a(x_1)^2) + V_v(a(x_2)^2).$$

Letting x_2 tend to $\pm\infty$ (depending on the sign of x_1), we get

$$-a'(x_1)^2 = -V_v(a(x_1)^2),\ x_1 \in \mathbb{R}^\star.$$

Therefore $V_v(r) \geqslant 0$ for $r \in [0, \rho_0)$. In particular, $v = 0$.
Since $a \geqslant 0$ and $a(0) = 0$, we have $a'(0_+) \geqslant 0$. Now, if $a'(0_+) = 0$, $a \equiv 0$ and φ cannot

be a black soliton. Thus, in a neighborhood of 0 in $\mathbb{R}_+$, a is the solution of the Cauchy Problem

$$\begin{cases} a'(x) = V(a(x)^2)^{1/2}, \\ a(0) = 0. \end{cases} \tag{9}$$

Therefore a is strictly increasing as long as $V(a^2) > 0$. Let us assume by contradiction that there exists $r \in (0, \rho_0)$ such that $V(r) = 0$. Let $r_0 := \min\{r \in (0, \rho_0),\ V(r) = 0\} \in (0, \rho_0)$. Since $V \geqslant 0$ on $(0, \rho_0)$, we must have $f(r_0) = -V'(r_0) = 0$. The intermediate value theorem ensures that there exists $x_0 \in \mathbb{R}_+^\star$ such that $a(x_0) = r_0^{1/2}$, and then $a \equiv r_0^{1/2}$, which is a contradiction to the fact that φ is a black soliton.

On the other side, if the assumptions of the statement are satisfied, we easily construct a black soliton $\varphi = ae^{i\theta}$ with amplitude tending to $\rho_0^{1/2}$ at infinity, in the following way : we show that the solution a to (9) is global. Indeed, let us denote by $[0, X^*)$ the maximal interval of existence of the solution to (9). If there exists $x_0 > 0$ such that $a(x_0) = \rho_0^{1/2}$, $a'(x_0) = 0$ and the uniqueness in the Cauchy-Lipschitz Theorem yields $a \equiv \rho_0^{1/2}$, which is a contradiction with $a(0) = 0$. Thus $0 \leqslant a(x) < \rho_0^{1/2}$ for $x \in [0, X^*)$, $X^* = +\infty$, and $a(x) \uparrow \rho_0^{1/2}$ as $x \to +\infty$. Next, we extend a to $\mathbb{R}$ in such a way that a is even, and we choose $\theta \equiv \theta_+ \in \mathbb{R}$ on $\mathbb{R}_+^\star$, $\theta \equiv \theta_- = \pi + \theta_+$ on $\mathbb{R}_-^\star$.

From now on, we assume $f \in \mathcal{C}^1(\mathbb{R}_+)$ and $f'(\rho_0) < 0$. It follows from (9) that

$$a'(x) \underset{x\to\infty}{\sim} c(\sqrt{\rho_0} - a(x)), \tag{10}$$

which may be integrated as

$$\sqrt{\rho_0} - a(x) \underset{x\to\infty}{=} e^{-cx(1+o(1))}. \tag{11}$$

The last asymptotics follows from the identity $a''(x) = -f(a(x)^2)a(x)$. □

3 A linear stability criterion

In this section, we prove Theorem 1.1. Let us denote by φ the soliton obtained under the conditions (i)-(ii)-(iii) of Theorem 2.1, with the extra assumption $f \in \mathcal{C}^3(\mathbb{R}_+)$ and $f'(\rho_0) < 0$. In our study of the linear stability of φ, we use the same notations as in [dB]. The only difference is that here, φ denotes a black soliton, whereas it denotes a stationary bubble in [dB]. As we mentioned it in the introduction, the linearization of (1) near the black soliton φ yields to the system (3).

We first focus on the spectra of L_1 and L_2 which are self-adjoint operators on L^2. As for the stationary bubbles studied in [dB], since $f(\varphi(x)^2) \to 0$ and $-2\varphi(x)^2 f'(\varphi(x)^2) \to c^2$ as $x \to \infty$, $\sigma_e(L_1) = [0, +\infty)$ and $\sigma_e(L_2) = [c^2, +\infty)$. The fact that φ solves (1) writes $L_1\varphi = 0$. The differentiation of this equality yields $L_2\varphi' = 0$. φ admits exactly one zero, whereas φ' does not vanish. Therefore Sturm's theory (see [BS]) ensures that 0 is the lowest eigenvalue of L_2 and that it is simple. It also suggests that L_1 has a unique strictly negative eigenvalue which is simple. That is what we prove in the next lemma[5].

Lemma 3.1 *L_1 has an unique strictly negative eigenvalue λ_0, which is simple. The associated eigenvector u_0 can be chosen positive.*

Proof of Lemma 3.1 If $v \in H^2$,

$$< L_1 v, v >= \int_{\mathbb{R}} \left[|\partial_x v|^2 - f(\varphi^2)|v|^2 \right] \, dx =: Q(v,v),$$

where Q is defined on H^1. Since $\sigma_e(L_1) = [0, +\infty)$, the existence of $v \in H^1$ such that $Q(v,v) < 0$ will imply the existence of a negative eigenvalue λ_0 of L_1. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ supported in $[-2,2]$, with $\chi(0) = 1$. For $\delta > 0$, we define the function φ_δ on $\mathbb{R}$ by

$$\varphi_\delta(x) := \begin{cases} |\varphi(x)| & \text{if } |x| \geqslant \delta, \\ \varphi(\delta) & \text{if } -\delta \leqslant x \leqslant \delta, \end{cases}$$

and for $n, \delta > 0$,

$$\varphi_{\delta,n}(x) := \varphi_\delta(x)\chi(x/n).$$

It is clear that $\varphi_{\delta,n} \in H^1$. Since $\varphi' \in L^1$, thanks to the asymptotics given in Theorem 2.1, so does φ'_δ. Thus

$$\begin{aligned} Q(\varphi_{\delta,n}, \varphi_{\delta,n}) &= \int_{\mathbb{R}} \Big[\varphi'_\delta(x)^2 \chi(x/n)^2 + \frac{2}{n} \varphi_\delta(x)\varphi'_\delta(x)\chi(x/n)\chi'(x/n) \\ &\qquad + \frac{1}{n^2}\varphi_\delta(x)^2\chi'(x/n)^2 - f(\varphi(x)^2)\varphi_\delta(x)^2\chi(x/n)^2 \Big] \, dx \\ &\underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \left[\varphi'_\delta(x)^2 - f(\varphi(x)^2)\varphi_\delta(x)^2 \right] dx. \end{aligned}$$

In spite of the fact that $\varphi_\delta \notin H^1$, we will denote this very last quantity by $Q(\varphi_\delta, \varphi_\delta)$.

$$\begin{aligned} Q(\varphi_\delta, \varphi_\delta) &= \int_{|x|\geqslant\delta} (\varphi'(x)^2 - f(\varphi(x)^2)\varphi(x)^2) dx + \int_{|x|\leqslant\delta} -f(\varphi(x)^2)\varphi(\delta)^2 dx \\ &= \underbrace{\int_{\mathbb{R}} (\varphi'(x)^2 - f(\varphi(x)^2)\varphi(x)^2) dx}_{=0} \\ &\qquad + \int_{-\delta}^{\delta} \left[-\varphi'(x)^2 + f(\varphi(x)^2)(\varphi(x)^2 - \varphi(\delta)^2) \right] dx \\ &\underset{\delta\to 0}{=} -2\delta\varphi'(0)^2 + o(\delta), \end{aligned}$$

[5]Note that in the case of the stationary bubbles studied in [dB], φ was positive while φ' vanished exactly once on $\mathbb{R}$. Thus L_1 was positive, whereas L_2 had an unique negative eigenvalue. This explains why in the sequel, the roles of L_1 and L_2 are exchanged in comparison with [dB].

where we have used the fact that $L_1\varphi = 0$. Thus we can choose $\delta > 0$ small enough such that $Q(\varphi_\delta, \varphi_\delta) < 0$, and then n large enough in such a way that $Q(\varphi_{\delta,n}, \varphi_{\delta,n}) < 0$. Therefore L_1 has a smallest eigenvalue $\lambda_0 < 0$. It is a classical result (see for instance [BS]) that λ_0 is simple, and that there exists a positive associated eigenvector u_0. Let us assume by contradiction that L_1 has another eigenvalue $\lambda_1 \in (\lambda_0, 0)$, and let u_1 be an associated eigenvector. Since $u_1 \perp u_0$ and $u_0 > 0$, u_1 must vanish at some $x_0 \in \mathbb{R}$ and the Sturm's theory implies that φ vanishes in both intervals $(-\infty, x_0)$ and $(x_0, +\infty)$, which is a contradiction, because φ vanishes only at 0. □

As we already mentioned in the introduction, the essential spectrum of A is purely imaginary, as it may be shown in a similar way that for the stationary bubbles (see [dB]). Moreover, if λ is an eigenvalue of A with associated eigenvector (u_1, u_2), $-\lambda$ is also an eigenvalue, with associated eigenvector $(-u_1, u_2)$. Thus the linear instability of φ is equivalent to the existence of an eigenvalue of A with non-zero real part. If λ $(\neq 0)$ is such an eigenvalue and if (u_1, u_2) is an associated eigenvector, we have

$$\begin{cases} L_1 u_2 = \lambda u_1, \\ -L_2 u_1 = \lambda u_2. \end{cases}$$

In particular, since L_2 is self-adjoint, $u_2 \in (\ker L_2)^\perp = (\varphi')^\perp$ and $u_1 \in -\lambda \tilde{L_2}^{-1} u_2 + \mathrm{Span}(\varphi')$, where we have denoted by $\tilde{L_2}$ the restriction of L_2 to $(\varphi')^\perp$ (it is clear that $\tilde{L_2}$ is invertible). Therefore, taking the scalar product with u_2 (which is orthogonal to φ'), and using the fact that $\tilde{L_2}$ is a self-adjoint coercive operator on $(\varphi')^\perp$, we obtain

$$-\lambda^2 = \frac{< L_1 u_2, u_2 >}{< \tilde{L_2}^{-1} u_2, u_2 >}.$$

We deduce that $-\lambda^2 \in \mathbb{R}$, thus $\lambda \in \mathbb{R} \cup i\mathbb{R}$. Therefore, if λ is an eigenvalue of A with non-zero real part, $\lambda \in \mathbb{R}^*$, and the quantity

$$\inf_{\substack{v \in (\varphi')^\perp \cap H^2 \\ v \neq 0}} \frac{< L_1 v, v >}{< \tilde{L_2}^{-1} v, v >} = \inf_{\substack{v \in (\varphi')^\perp \cap H^1 \\ v \neq 0}} \frac{Q(v,v)}{< \tilde{L_2}^{-1} v, v >}$$

is strictly negative. Since $\tilde{L_2}^{-1}$ is positive, we also have

$$\alpha := \inf_{\substack{v \in (\varphi')^\perp \cap H^1 \\ v \neq 0}} \frac{Q(v,v)}{\|v\|_{L^2}^2} < 0.$$

We next prove that if $\alpha < 0$, the infimum α is reached. Indeed, let $(v_n)_n$ be a sequence of $(\varphi')^\perp \cap H^1$ such that $\|v_n\|_{L^2} = 1$ and $Q(v_n, v_n) \to \alpha$.

$$\int_{\mathbb{R}} |\partial_x v_n|^2 dx = Q(v_n, v_n) + \int_{\mathbb{R}} f(\varphi^2)|v_n|^2 dx \leqslant \sup_n Q(v_n, v_n) + \|f\|_{L^\infty(0,\rho_0)},$$

thus $(v_n)_n$ is bounded in H^1. We can extract a subsequence (also denoted by v_n for convenience), such that $v_n \rightharpoonup v_*$ in H^1. We have $< v_*, \varphi' >= \lim_n < v_n, \varphi' >= 0$, and the fast decrease of $f(\varphi^2)$ at infinity ensures by compactness that $\int f(\varphi^2)|v_n|^2 \to \int f(\varphi^2)|v_*|^2$. On the other side, $\int |\partial_x v_*|^2 dx \leqslant \liminf \int |\partial_x v_n|^2 dx$, thus

$$Q(v_*, v_*) = \int_{\mathbb{R}} |\partial_x v_*|^2 dx - \int_{\mathbb{R}} f(\varphi^2)|v_*|^2 dx \leqslant \liminf \int_{\mathbb{R}} |\partial_x v_n|^2 dx - \lim \int_{\mathbb{R}} f(\varphi^2)|v_n|^2 dx = \alpha.$$

Moreover, $\|v_*\|_{L^2} \leqslant \liminf \|v_n\|_{L^2} = 1$. Assume by contradiction that $\|v_*\|_{L^2} < 1$. Then

$$\frac{Q(v_*, v_*)}{\|v_*\|_{L^2}^2} \leqslant \frac{\alpha}{\|v_*\|_{L^2}^2} < \alpha < 0,$$

which is a contradiction with the definition of α. Therefore $\|v_*\|_{L^2} = 1$, $\frac{Q(v_*,v_*)}{\|v_*\|_{L^2}^2} = \alpha$, and the infimum α is reached at v_*. Hence v_* satisfies in H^{-1} the Euler-Lagrange equation

$$L_1 v_* = \alpha v_* + \beta \varphi', \tag{12}$$

where $\beta \in \mathbb{R}$. We deduce that $v_* \in H^2 = D(L_1)$. It can be easily seen that $\alpha > \lambda_0$. Indeed, it is clear that $\alpha \geqslant \lambda_0$, and if $\alpha = \lambda_0$, (12) implies

$$0 =< v_*, (L_1 - \lambda_0)u_0 >=< (L_1 - \lambda_0)v_*, u_0 >= \beta < \varphi', u_0 >,$$

where u_0 denotes a positive eigenvector of L_1 associated with the eigenvalue λ_0. Since φ' and u_0 are both positive, it follows that $\beta = 0$. Eq. (12) with $\beta = 0$ implies that v_* is colinear to u_0, because λ_0 is the only negative eigenvalue of L_1 and is simple. This is a contradiction, because $u_0 \notin (\varphi')^\perp$.

Since $\sigma(L_1) \cap (\lambda_0, 0) = \emptyset$, we can define on the interval $(\lambda_0, 0)$ the Vakhitov-Kolokolov fonction g by

$$g(\lambda) \ := \ < (L_1 - \lambda)^{-1}\varphi', \varphi' > . \tag{13}$$

We have just seen that if φ is linearly unstable, the infimum $\alpha \in (\lambda_0, 0)$ of the ratio $Q(v,v)/\|v\|_{L^2}^2$ when v describes $(\varphi')^\perp \cap H^1$ is reached at some $v_* \in (\varphi')^\perp$ which satisfies the Euler-Lagrange equation (12). This last equation can be rewritten as $v_* = \beta(L_1 - \alpha)^{-1}\varphi'$, and thus $g(\alpha) = 0$.

Now, g is of class $\mathcal{C}^1$ on $(\lambda_0, 0)$ and $g'(\lambda) = \|(L_1 - \lambda)^{-1}\varphi'\|_{L^2}^2 > 0$, thus g is increasing on $(\lambda_0, 0)$. Moreover, if we denote by Π_0 the orthogonal projection on $u_0^\perp$,

$$(L_1 - \lambda)^{-1} = \frac{1}{\lambda_0 - \lambda} \frac{< u_0, . >}{\|u_0\|_{L^2}^2} u_0 + \Pi_0 (L_1 - \lambda)^{-1} \Pi_0,$$

and thus

$$g(\lambda) = \frac{1}{\lambda_0 - \lambda} \frac{< u_0, \varphi' >^2}{\|u_0\|_{L^2}^2} + < (L_1 - \lambda)^{-1} \Pi_0 \varphi', \Pi_0 \varphi' > . \tag{14}$$

Letting λ decrease to λ_0, the first term of the right-hand side tends to $-\infty$, whereas the second one remains bounded. It follows that $g(\lambda) \to -\infty$ as $\lambda \downarrow \lambda_0$. Therefore g vanishes on $(\lambda_0, 0)$ if and only if the limit of $g(\lambda)$ as $\lambda \uparrow 0$ is strictly positive.

At this stage, we have proved that if the limit of $g(\lambda)$ as $\lambda \uparrow 0$ is equal to or less than 0, then the black soliton is linearly stable.

Conversely, let us assume that $\lim_{\lambda \uparrow 0} g(\lambda) > 0$. Let $\alpha \in (\lambda_0, 0)$ be such that $g(\alpha) = 0$. Denoting $v := (L_1 - \alpha)^{-1}\varphi'$, $v \in H^5$ because $\varphi' \in H^4$ and $f(\varphi^2) \in \mathcal{C}_b^3$, and we have

$$0 = g(\alpha) = < v, \varphi' > = < v, (L_1 - \alpha)v > .$$

Thus $< L_1 v, v > = \alpha \|v\|_{L^2}^2 < 0$.

Since $\tilde{L_2}$ is a strictly positive self adjoint operator on $(\varphi')^\perp$, one can define the continuous operator $\tilde{L_2}^{-1/2} : L^2 \cap (\varphi')^\perp \mapsto H^1 \cap (\varphi')^\perp$. Moreover, $\tilde{L_2}^{-1/2}$ is continuous from $H^3 \cap (\varphi')^\perp$ into H^4, because $f \in \mathcal{C}^3(\mathbb{R}_+)$. Since $v \in (\varphi')^\perp \cap H^3$, $w_0 := \tilde{L_2}^{-1/2} v \in (\varphi')^\perp \cap H^4$. We have

$$< \tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} w_0, w_0 > = < L_1 v, v > < 0,$$

where $\tilde{L_1} := \Pi L_1 \Pi$ is the restriction of L_1 to $(\varphi')^\perp$ and Π denotes the orthogonal projection on $(\varphi')^\perp$. In particular,

$$\gamma := \inf_{\substack{w \in (\varphi')^\perp \cap H^4 \\ w \neq 0}} \frac{< \tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} w, w >}{\|w\|_{L^2}^2} < 0.$$

Next, we use the two following lemmas.

Lemma 3.2 *$\gamma > -\infty$.*

Lemma 3.3 *The essential spectrum of the self adjoint operator on $(\varphi')^\perp$, $\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2}$, with domain $H^4 \cap (\varphi')^\perp$, is included in $\mathbb{R}_+$.*

Let us temporarily admit these lemmas. Let $(w_n)_{n \in \mathbb{N}}$ be a sequence in $(\varphi')^\perp \cap H^4$ such that for every n, $\|w_n\|_{L^2} = 1$ and $< \tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} w_n, w_n > \to \gamma$. Thus

$$< (\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} - \gamma) w_n, w_n > \to 0$$

as n goes to infinity, which implies that 0 belongs to the spectrum of $\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} - \gamma I_d$. Next, thanks to Lemma 3.3, $\sigma_e(\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} - \gamma I_d) \subset [-\gamma, \infty)$. Therefore $0 < -\gamma$ is not in the essential spectrum of $\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} - \gamma$, which means that it is an eigenvalue. Let us choose an associated eigenvector $z_0 \in H^4 \cap (\varphi')^\perp$, and define $u_2 := \tilde{L_2}^{1/2} z_0 \in H^3 \cap (\varphi')^\perp$ as well as $u_1 := L_1 u_2 / \sqrt{-\gamma}$. We may easily see that (u_1, u_2) is an eigenvector of A associated to the eigenvalue $\sqrt{-\gamma} > 0$, which implies that φ is linearly unstable. □

In order to complete the proof of Theorem 1.1, it remains to prove Lemma 3.2 and Lemma 3.3.

Proof of Lemma 3.2. A similar lemma is proved in [dB]. For every $w \in (\varphi')^{\perp} \cap H^4$, we have

$$\begin{aligned} < \tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} w, w > &= < \tilde{L_1} \tilde{L_2}^{1/2} w, \tilde{L_2}^{1/2} w > \\ &= < (L_2 - q_2) \tilde{L_2}^{1/2} w, \tilde{L_2}^{1/2} w > \\ &\geqslant \|L_2 w\|_{L^2}^2 - \|q_2\|_{L^\infty} < \tilde{L_2} w, w > \\ &\geqslant \|L_2 w\|_{L^2}^2 - \|q_2\|_{L^\infty} \|L_2 w\|_{L^2} \|w\|_{L^2} \geqslant -\frac{\|q_2\|_{L^\infty}^2}{4} \|w\|_{L^2}^2. \end{aligned}$$

We have used a Young inequality for the last estimate. It follows that $\gamma \geqslant -\|q_2\|_{L^\infty}^2/4 > -\infty$. □

Proof of Lemma 3.3. It is quite clear that $\tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2}$, with domain $H^4 \cap (\varphi')^{\perp}$ defines a self-adjoint operator on $L^2 \cap (\varphi')^{\perp}$. Next, we write

$$\begin{aligned} \tilde{L_2}^{1/2} \tilde{L_1} \tilde{L_2}^{1/2} &= \tilde{L_2}^{1/2} \Pi (L_2 - q_2) \Pi \tilde{L_2}^{1/2} \\ &= \tilde{L_2}^{1/2} (\tilde{L_2} - c^2) \tilde{L_2}^{1/2} + \tilde{L_2}^{1/2} \Pi (c^2 - q_2) \Pi \tilde{L_2}^{1/2} \\ &= \tilde{L_2}^2 - c^2 \tilde{L_2} + \tilde{L_2}^{1/2} \Pi (c^2 - q_2) \Pi \tilde{L_2}^{1/2}, \end{aligned}$$

and we will show that $\tilde{L_2}^{1/2} \Pi (c^2 - q_2) \Pi \tilde{L_2}^{1/2}$ is a relatively compact perturbation of $\tilde{L_2}^2 - c^2 \tilde{L_2}$. Indeed, it suffices to show that the image by $\tilde{L_2}^{1/2} \Pi (c^2 - q_2) \Pi \tilde{L_2}^{1/2}$ of the ball

$$B := \{f \in D(\tilde{L_2}^2 - c^2 \tilde{L_2}),\ \|f\|_{L^2}^2 \leqslant 1,\ \|(\tilde{L_2}^2 - c^2 \tilde{L_2}) f\|_{L^2}^2 \leqslant 1\}$$

is a compact set in L^2. If $f \in B$, then

$$\|\tilde{L_2} f\|_{L^2}^2 = < \tilde{L_2}^2 f, f > \leqslant \|\tilde{L_2}^2 f\|_{L^2} \|f\|_{L^2} \leqslant \|(\tilde{L_2}^2 - c^2 \tilde{L_2}) f\|_{L^2} \|f\|_{L^2} + c^2 \|\tilde{L_2} f\|_{L^2} \|f\|_{L^2}.$$

Thus

$$\|\tilde{L_2} f\|_{L^2}^2 \leqslant 1 + c^2 \|\tilde{L_2} f\|_{L_2}$$

and there exists a constant $C_0 > 0$ such that for every $f \in B$,

$$\|\tilde{L_2} f\|_{L^2} \leqslant C_0.$$

Thus, for every $f \in B$,

$$\|\tilde{L_2}^2 f\|_{L^2} \leqslant \|(\tilde{L_2}^2 - c^2 \tilde{L_2}) f\|_{L^2} + c^2 \|\tilde{L_2} f\| \leqslant 1 + c^2 C_0 =: C_1,$$

which means that B is a bounded set in $D(\tilde{L_2}^2) = H^4 \cap (\varphi')^{\perp}$. Next, $\tilde{L_2}^{1/2}$ is continuous from $H^4 \cap (\varphi')^{\perp}$ into $H^3 \cap (\varphi')^{\perp}$. Moreover, since $\partial^\alpha (q_2(x) - c^2) \to 0$ for $\alpha \leqslant 2$ as x goes to infinity, the multiplication by $c^2 - q_2$ is compact from $H^3 \cap (\varphi')^{\perp}$ into H^2. Π is continuous from H^2 into itself because $\varphi' \in H^2$. Finally, $\tilde{L_2}^{1/2}$ continuously maps $H^2 \cap (\varphi')^{\perp}$ into

$L^2 \cap (\varphi')^\perp$. Therefore $\tilde{L_2}^{1/2}\Pi(c^2 - q_2)\Pi\tilde{L_2}^{1/2}(B)$ is compact in L^2. We then deduce from the Weyl's criterion that

$$\sigma_e(\tilde{L_2}^{1/2}\tilde{L_1}\tilde{L_2}^{1/2}) = \sigma_e(\tilde{L_2}^2 - c^2\tilde{L_2}).$$

Next, since $\tilde{L_2}^2 - c^2\tilde{L_2}$ is the restriction of $L_2^2 - c^2 L_2$ to the orthogonal $(\varphi')^\perp$ of the kernel of L_2,

$$\sigma_e(\tilde{L_2}^2 - c^2\tilde{L_2}) = \sigma_e(L_2^2 - c^2 L_2).$$

The operator $L_2^2 - c^2 L_2$ is a differential operator which may be explicitly computed:

$$L_2^2 - c^2 L_2 = T + S,$$

where

$$T = \partial_x^4 - c^2\partial_x^2,$$
$$S = 2(c^2 - r(x))\partial_x^2 - 2r'(x)\partial_x - r''(x) + r(x)(r(x) - c^2)$$

and $r(x) = q_1(x) + q_2(x)$. It is clear that $D(T) = H^4 \subset D(S)$. Turning into Fourier variables, we easily compute $\sigma_e(T) = \sigma(T) = \mathbb{R}_+$. Let $U = \mathbb{C}\backslash\mathbb{R}_+$. Since the operator $S(-1-T)^{-1}$ is compact on L^2, Theorem A1 in [H] ensures that one of the two following properties occurs: either $U \subset \mathbb{C}\backslash\sigma_e(T+S)$, or $U \subset \sigma(T+S)$. The second possibility is excluded since $T+S$ is self adjoint, and thus $\sigma(T+S) \subset \mathbb{R}$. Therefore $\sigma_e(T+S) \subset \mathbb{R}_+$, and the lemma follows. □

4 Linear instability implies orbital instability

As it is mentioned by A. de Bouard in [dB], the linear instability of the stationary bubbles implies their orbital instability. Our aim here is to prove Theorem 1.2, which states that if a black soliton is linearly unstable, it is also orbitally unstable. As in the previous section, we assume that $f \in \mathcal{C}^3(\mathbb{R}_+)$ and $f'(\rho_0) < 0$.

Throughout this section, we denote as before by A the operator on $L^2 \times L^2$, defined by

$$\begin{cases} \forall (u_1, u_2) \in D(A) = \mathbb{H}^2, \\ A(u_1, u_2) = (L_1 u_2, -L_2 u_1), \end{cases}$$

and we assume that $\sigma(A) \not\subset i\mathbb{R}$. The analysis developed in section 3 ensures then that $\sigma(A)$ is the union of $i\mathbb{R}$ and of some pairs of opposite real eigenvalues with finite multiplicity. We denote by $\lambda_m > 0$ the greatest eigenvalue of A. We also define an operator A_1 on $\mathbb{H}^1$ by

$$\begin{cases} \forall (u_1, u_2) \in D(A_1) = \mathbb{H}^3, \\ A_1(u_1, u_2) = A(u_1, u_2). \end{cases}$$

Our proof of Theorem 1.2 follows the lines of that of Theorem 4.1 in [dB] (see also Theorem 2 in [HPW]). However, in order to obtain a result of L^∞ orbital instability (and not only H^1 orbital instability), more accuracy is required for the description of the spectrum of the group generated by A_1. This will be done thanks to the Spectral Mapping Theorem of J. Prüss [P], using a method introduced by F. Gesztesy, C.K.R.T. Jones, Y. Latushkin and M. Stanislavova in [GJLS] for localized solitary waves of Nonlinear Schrödinger Equations. Once we have proved that $\sigma(e^{A_1 t}) = e^{\sigma(A)t}$, orbital instability is obtained by perturbation of the black soliton in the direction of the eigenfunction associated to the maximal real eigenvalue of A. Since the proof may be adapted to the case of a one dimensional stationary bubble, it answers in this case to the question asked in [dB], Remark 4.1.

First, let us remark that $\sigma(A_1) = \sigma(A)$. Indeed, similarly to A, A_1 is a relatively compact perturbation of a differential operator with constant coefficients, which has $i\mathbb{R}$ as essential spectrum. Thus $\sigma_e(A_1) = i\mathbb{R}$. Next, if $(u_1, u_2) \in \mathbb{H}^2$ is an eigenvector of A associated to the eigenvalue λ, $\partial_x^2 u_1, \partial_x^2 u_2 \in H^1$ and thus $(u_1, u_2) \in \mathbb{H}^3 = D(A_1)$. Therefore $\lambda \in \sigma(A_1)$.

Next, if $(u_1, u_2) \in D(A_1)$, we have

$$\begin{aligned}
&\mathfrak{Re}\,\langle A_1(u_1,u_2),(u_1,u_2)\rangle_{\mathbb{H}^1}\\
&\quad = \mathfrak{Re}\left[\left\langle -\partial_x^2 u_2 + q_1 u_2, u_1\right\rangle_{H^1} - \left\langle -\partial_x^2 u_1 + q_1 u_1 + q_2 u_1, u_2\right\rangle_{H^1}\right]\\
&\quad = \mathfrak{Re}\left[\langle q_1 u_2, u_1\rangle_{H^1} - \langle (q_1+q_2)u_1, u_2\rangle_{H^1}\right].
\end{aligned}$$

In particular,

$$\begin{aligned}
|\mathfrak{Re}\,\langle A_1(u_1,u_2),(u_1,u_2)\rangle_{\mathbb{H}^1}| &\leqslant (2\|q_1\|_{H^1} + \|q_2\|_{H^1})\|u_1\|_{H^1}\|u_2\|_{H^1}\\
&\leqslant C\|(u_1,u_2)\|_{\mathbb{H}^1}^2.
\end{aligned}$$

It follows that A_1 is the generator of a C_0-group on $\mathbb{H}^1$, which will be denoted by $e^{A_1 t}$.

We are going to prove that the spectrum of $e^{A_1 t}$ equals the set $e^{\sigma(A)t}$. This will be done using the following Spectral Mapping Theorem, which is due to J. Prüss.

Theorem 4.1 *Let X be a Hilbert space, A an unbounded operator on X and $S(t)$ a C_0 semigroup on X, with generator A. Let $\mu \neq 0$. Then $\mu \in \rho(S(t))$ if and only if $E := \{\lambda \in \mathbb{C},\ e^{\lambda t} = \mu\} \subset \rho(A)$ and $\sup\{\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)}, \lambda \in E\} < \infty$.*

In order to apply Theorem 4.1 to the operator A_1, we will first prove uniform estimates on the resolvent of A for $\lambda \in \Gamma_{a,\tau_0} := \{a + i\tau, |\tau| \geqslant \tau_0\}$, where $a \in \mathbb{R}^*$ and $\tau_0 > 0$ is large. We will then easily deduce uniform estimates on the resolvent of A_1, for $\lambda \in \Gamma_{a,\tau_0}$.

Let us now prove the uniform estimates on the resolvent of A. Our method is similar to the one used by F. Gesztesy, C.K.R.T. Jones, Y. Latushkin and M. Stanislavova in [GJLS] for localized solitary waves. For $\lambda = a + i\tau$, where $a, \tau \in \mathbb{R}^*$, we have

$$\lambda - A = B_\lambda + M_Q,$$

where

$$B_\lambda = \begin{pmatrix} \lambda & \partial_x^2 \\ -\partial_x^2 + c^2 & \lambda \end{pmatrix}$$

and

$$M_Q = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(\varphi^2) \\ -f(\varphi^2) - 2\varphi^2 f'(\varphi^2) - c^2 & 0 \end{pmatrix}.$$

Since the spectrum of $\begin{pmatrix} 0 & -\partial_x^2 \\ -(-\partial_x^2 + c^2) & 0 \end{pmatrix}$ is included in $i\mathbb{R}$ and $\lambda \notin i\mathbb{R}$, B_λ is invertible, and $\lambda - A$ may be rewritten as

$$\lambda - A = B_\lambda \left[Id + B_\lambda^{-1} M_Q \right].$$

Next, we have the following lemma, which is proved in the appendix.

Lemma 4.1 *Let $a \neq 0$. Then there exists $C_a > 0$ such that for every $\lambda = a + i\tau$, $|\tau| \geqslant 1$,*

$$\|B_\lambda^{-1}\|_{\mathcal{L}(\mathbb{L}^2)} \leqslant C_a.$$

Let T_λ be the continuous operator on $\mathbb{L}^2$ defined by

$$\begin{aligned} T_\lambda &:= B_\lambda^{-1} M_Q \\ &= \begin{pmatrix} -\partial_x^2(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} Q_2 & \lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} Q_1 \\ \lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} Q_2 & -(-\partial_x^2 + c^2)(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} Q_1 \end{pmatrix}. \end{aligned} \quad (15)$$

We will next prove that each of the four block operators on L^2 in the right hand side of (15) tends to zero in $\mathcal{L}(L^2)$. This will imply that for τ_0 large enough, $\|T_\lambda\|_{\mathcal{L}(\mathbb{L}^2)} \leqslant 1/2$ as soon as $\lambda \in \Gamma_{a,\tau_0}$. We deduce that $Id + T_\lambda$ is invertible in this case and thanks to Lemma 4.1,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathbb{L}^2)} \leqslant 2C_a.$$

Let us define now

$$K_\lambda^1 = Q(x)(-\partial_x^2)(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1},$$

$$K_\lambda^2 = Q(x)\lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1}$$

and

$$K_\lambda^3 = Q(x)(-\partial_x^2 + c^2)(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1},$$

where $Q = Q_1$ or Q_2, in such a way that $K_\lambda^j \in \mathcal{L}(L^2)$, $j = 1, 2, 3$ are the adjoints of the block operators in the right-hand side of (15). Since $\|K_\lambda^j\|_{\mathcal{L}(L^2)} = \|{K_\lambda^j}^*\|_{\mathcal{L}(L^2)}$, we are reduced to prove that $\|K_\lambda^j\|_{\mathcal{L}(L^2)}$ tends to zero as τ goes to infinity. As in [GJLS], this will be done as follows.

We first remark that for $j = 1, 2, 3$, K_λ^j writes $Qg_\lambda^j(i\partial_x)$, where

$$g_\lambda^1(x) = \frac{x^2}{\lambda^2 + x^2(x^2 + c^2)},$$

$$g_\lambda^2(x) = \frac{\lambda}{\lambda^2 + x^2(x^2 + c^2)}$$

and

$$g_\lambda^3(x) = \frac{x^2 + c^2}{\lambda^2 + x^2(x^2 + c^2)}.$$

Thus K_λ^j is an integral operator with kernel $(x, y) \mapsto Q(x)\check{g}_\lambda^j(x - y)$, where $\check{g}$ denotes the inverse Fourier transform of g.

Note that Q converges exponentially to zero, therefore it clearly belongs to L^2. Thus, since $|g_\lambda^1| \leqslant |g_\lambda^3|$, it suffices to prove that for $j = 2, 3$, $g_\lambda^j \in L^2(\mathbb{R})$ and $\|g_\lambda^j\|_{L^2} \to 0$ as τ tends to infinity, which will imply that the K_λ^j are Hilbert-Schmidt operators which tend to 0 in $\mathcal{L}(L^2)$ as τ goes to infinity. More generally, we prove the following lemma.

Lemma 4.2 *Let $\lambda = a + i\tau$ where $a \neq 0$. For $j = 2, 3$, for every $q > 1$, g_λ^j belongs to $L^q(\mathbb{R})$ and $\|g_\lambda^j\|_{L^q} \to 0$ as τ tends to infinity.*

Next, we show that the uniform estimate on $\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathbb{L}^2)}$ for $\lambda \in \Gamma_{a,\tau_0}$ that we just obtained imply a similar uniform estimate for $\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathbb{H}^1)}$. Indeed, let $a \in \mathbb{R}^*$ and $\tau_0 > 0$, $C > 0$ such that for every $\lambda = a + i\tau \in \Gamma_{a,\tau_0}$,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathbb{L}^2)} \leqslant C.$$

Then if $\lambda \in \Gamma_{a,\tau_0}$, $\mathcal{F} \in \mathbb{H}^1$ and $\mathcal{V} = (\lambda - A)^{-1}\mathcal{F}$, then

$$\|\mathcal{V}\|_{\mathbb{L}^2} \leqslant C\|\mathcal{F}\|_{\mathbb{L}^2}. \tag{16}$$

We also get that $\mathcal{V} \in \mathbb{H}^3$ and

$$(\lambda - A)\partial_x\mathcal{V} = \partial_x\mathcal{F} - M_{Q'}\mathcal{V},$$

where $M_{Q'} = \begin{pmatrix} 0 & Q_1' \\ Q_2' & 0 \end{pmatrix}$. Thus

$$\begin{aligned} \|\partial_x\mathcal{V}\|_{\mathbb{L}^2} &\leqslant C(\|\partial_x\mathcal{F}\|_{\mathbb{L}^2} + (\|q_1'\|_{L^\infty} + \|q_2'\|_{L^\infty})\|\mathcal{V}\|_{\mathbb{L}^2}) \\ &\leqslant C\|\mathcal{F}\|_{\mathbb{H}^1}. \end{aligned} \tag{17}$$

It follows from (16) and (17) that $(\lambda - A_1)^{-1} : \mathbb{H}^1 \mapsto \mathbb{H}^1$ is continuous, with a norm uniformly bounded as λ stays in Γ_{a,τ_0}.

At this stage, we conclude by Theorem 4.1 that

$$\sigma(e^{A_1 t}) = e^{\sigma(A_1)t} = e^{\sigma(A)t}. \tag{18}$$

In particular, the spectral radius of $e^{A_1 t}$ is $e^{\lambda_m t}$. Once (18) has been established, the proof of Theorem 1.2 follows the broad lines of Theorem 2 in [HPW] and Theorem 4.1 in [dB].

It can easily be seen (see for instance [G2]) that there exists a neighborhood U of 0 in $\mathbb{H}^1$ such that for every $(v_1, v_2) \in U$, there is a unique solution $u \in \varphi + \mathcal{C}([0,1], H^1 + iH^1)$ to (1), such that $u(0) = \varphi + v_1 + iv_2$. For $t \in [0,1]$ and $(v_1, v_2) \in U$, we define the nonlinear flow

$$T(t)(v_1, v_2) := (\mathfrak{Re} u(t) - \varphi, \mathfrak{Im} u(t)).$$

Using these notations, Equation (1) equivalently expresses as

$$\mathcal{U}(t) = T(t)(v_1, v_2) = e^{A_1 t}(v_1, v_2) + \int_0^t e^{A_1(t-\tau)} g(T(\tau)(v_1, v_2)) d\tau, \tag{19}$$

where $g : \mathbb{H}^1 \mapsto \mathbb{H}^1$ is defined by

$$\begin{aligned} g(v_1, v_2) \quad = \quad & \big(\left[f(\varphi^2) - f((\varphi + v_1)^2 + v_2^2) \right] v_2, \\ & f((\varphi + v_1)^2 + v_2^2)(\varphi + v_1) - f(\varphi^2)(\varphi + v_1) - 2\varphi^2 f'(\varphi^2) v_1 \big) . \end{aligned}$$

It is quite clear that there exists $a_0, c_0 > 0$ such that U contains the ball of radius a_0 in $\mathbb{H}^1$ and $\|g(v_1, v_2)\|_{\mathbb{H}^1} \leqslant c_0 \|(v_1, v_2)\|_{\mathbb{H}^1}^2$ as soon as $\|(v_1, v_2)\|_{\mathbb{H}^1} \leqslant a_0$. If $\|(v_1, v_2)\|_{\mathbb{H}^1} < a_0$, let $t^* = \sup\{t \in [0,1], \|\mathcal{U}(t)\|_{\mathbb{H}^1} < a_0\} > 0$. Since the spectral radius of $e^{A_1 t}$ is $e^{\lambda_m t}$, for every $\eta > 0$, there exists $M > 0$ such that $\|e^{A_1 t}\|_{\mathcal{L}(\mathbb{H}^1)} \leqslant M e^{(\lambda_m + \eta)t}$. Thus for every $t \in [0, t^*]$,

$$\|\mathcal{U}(t)\|_{\mathbb{H}^1} \leqslant M e^{(\lambda_m + \eta)t} \|\mathcal{U}_0\|_{\mathbb{H}^1} + M a_0 \int_0^t e^{(\lambda_m + \eta)(t-\tau)} c_0 \|\mathcal{U}(\tau)\|_{\mathbb{H}^1} d\tau,$$

where $\mathcal{U}_0 = \mathcal{U}(0) = (v_1, v_2)$. Multipliying by $e^{-(\lambda_m + \eta)t}$ and using the Gronwall lemma, we get

$$\|\mathcal{U}(t)\|_{\mathbb{H}^1} \quad \leqslant \quad M \|\mathcal{U}_0\|_{\mathbb{H}^1} e^{(\lambda_m + \eta + M c_0 a_0)t}, \ t \in [0, t^*]. \tag{20}$$

Thus, one can choose $a_1 > 0$ small enough (for instance $a_1 = e^{-(\lambda_m + \eta + M c_0 a_0)} a_0 / M$), in such a way that $\|\mathcal{U}_0\|_{\mathbb{H}^1} < a_1$ implies $\|\mathcal{U}(t)\|_{\mathbb{H}^1} \leqslant a_0$ for every $t \in [0,1]$. For such a choice of $\mathcal{U}_0$, for every $t \in [0,1]$, it follows from (19) and (20) that

$$\begin{aligned}\|\mathcal{U}(t) - e^{A_1 t}\mathcal{U}_0\|_{\mathbb{H}^1} &\leqslant c_0 M^3 e^{(\lambda_m+\eta)t}\|\mathcal{U}_0\|_{\mathbb{H}^1}^2 \int_0^t e^{(2Mc_0a_0+\lambda_m+\eta)\tau}d\tau \\ &\leqslant b\|\mathcal{U}_0\|_{\mathbb{H}^1}^2, \end{aligned} \tag{21}$$

where $b = c_0 M^3 e^{(\lambda_m+\eta)}(e^{(2Mc_0a_0+\lambda_m+\eta)} - 1)/(2Mc_0a_0 + \lambda_m + \eta)$. From now on, we fix $\eta \in (0, \lambda_m)$. Let C_s be the norm of the continuous Sobolev embedding $\mathbb{H}^1 \subset \mathbb{L}^\infty$. Let $\mathcal{U}_m$ be an eigenvector of A_1 associated to the eigenvalue λ_m, normalized in such a way that $\|\mathcal{U}_m\|_{\mathbb{H}^1} = 1$. We fix $a_2 \in (0, a_1)$ and $\Delta = Mba_2e^{-(\lambda_m+\eta)}/(e^{\lambda_m-\eta} - 1)$, where a_2 is small enough, such that $\Delta < 1/2$ and $\Delta C_s/(1-\Delta) < \|\mathcal{U}_m\|_{\mathbb{L}^\infty}/2$. Let $R = 1/(1-\Delta) > 1$, let $N \in \mathbb{N}$, and $\varepsilon = a_2 e^{-\lambda_m N}/R$. Let us finally choose $\mathcal{U}_0 = \varepsilon\mathcal{U}_m$.

We first show by induction on n that, denoting by $T^*(\mathcal{U}_0)$ the maximal time of existence in $\mathbb{H}^1$ of the solution to (19) with initial data $\mathcal{U}_0$, we have for every $n \in \{0, \dots, N\}$, $T^*(\mathcal{U}_0) > n$ and

$$\|\mathcal{U}(n)\|_{\mathbb{H}^1} \leqslant \varepsilon R e^{\lambda_m n}. \tag{22}$$

For $n = 0$, $\|\mathcal{U}(0)\|_{\mathbb{H}^1} = \varepsilon < \varepsilon R$ by choice of $\mathcal{U}(0)$ and R. Let us now take $n \in \{1, \dots, N\}$, and assume that (22) holds at orders $k \in \{0, \dots, n-1\}$. Then

$$\|\mathcal{U}(n-1)\|_{\mathbb{H}^1} \leqslant \varepsilon R e^{\lambda_m(n-1)} \leqslant \varepsilon R e^{\lambda_m N} = a_2 < a_1,$$

therefore the time of existence of $\mathcal{U}$ is at least n, and we have

$$\begin{aligned}\|\mathcal{U}(n)\|_{\mathbb{H}^1} &\leqslant \|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{H}^1} + \|e^{nA_1}\mathcal{U}_0\|_{\mathbb{H}^1} \\ &\leqslant \left\|\sum_{k=0}^{n-1} e^{(n-1-k)A_1}\left[\mathcal{U}(k+1) - e^{A_1}\mathcal{U}(k)\right]\right\|_{\mathbb{H}^1} + \|e^{nA_1}\mathcal{U}_0\|_{\mathbb{H}^1} \\ &\leqslant \sum_{k=0}^{n-1} Me^{(\lambda_m+\eta)(n-1-k)} b\|\mathcal{U}(k)\|_{\mathbb{H}^1}^2 + e^{\lambda_m n}\|\mathcal{U}_0\|_{\mathbb{H}^1},\end{aligned}$$

because of (21), and since $\mathcal{U}_0$ is an eigenvector of e^{nA_1} associated to the eigenvalue $e^{\lambda_m n}$. The induction assumption then ensures that

$$\begin{aligned}\|\mathcal{U}(n)\|_{\mathbb{H}^1} &\leqslant \varepsilon e^{\lambda_m n}\left[Mbe^{-\lambda_m}e^{\eta(n-1)}\varepsilon R^2 \frac{e^{(\lambda_m-\eta)n} - 1}{e^{\lambda_m-\eta} - 1} + 1\right] \\ &\leqslant \varepsilon e^{\lambda_m n}(R\Delta + 1) = \varepsilon e^{\lambda_m n} R,\end{aligned}$$

which is the announced result. Note that the computation we just made also proves that for every $n \in \{0, \dots, N\}$,

$$\|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{H}^1} \leqslant \varepsilon e^{\lambda_m n} R\Delta. \tag{23}$$

Thus, for $n \in \{0, \ldots, N\}$,

$$\begin{aligned}
\|\mathcal{U}(n)\|_{\mathbb{L}^\infty} &\geqslant \|e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} - \|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} \\
&\geqslant \varepsilon e^{\lambda_m n}(\|\mathcal{U}_m\|_{\mathbb{L}^\infty} - C_s\Delta R) \geqslant \frac{\varepsilon e^{\lambda_m n}}{2}\|\mathcal{U}_m\|_{\mathbb{L}^\infty}.
\end{aligned} \tag{24}$$

For $n = N$, we get

$$\begin{aligned}
\|\mathcal{U}(N)\|_{\mathbb{L}^\infty} &\geqslant \frac{\varepsilon e^{\lambda_m N}}{2}\|\mathcal{U}_m\|_{\mathbb{L}^\infty} = \frac{a_2}{2R}\|\mathcal{U}_m\|_{\mathbb{L}^\infty} \\
&= \frac{a_2}{2}(1-\Delta)\|\mathcal{U}_m\|_{\mathbb{L}^\infty} \geqslant \frac{a_2}{4}\|\mathcal{U}_m\|_{\mathbb{L}^\infty}.
\end{aligned} \tag{25}$$

Finally, we show that $\mathcal{U}(N)$ stays far from the curve $\Gamma = \{v = (\varphi(\cdot - h) - \varphi, 0), h \in \mathbb{R}\}$ in $\mathbb{L}^\infty$, which will complete the proof of Theorem 1.2.

Let us first define, for $\rho > 0$, the cones

$$\Sigma_\rho = \left\{w \in \mathbb{L}^\infty,\ \left\|w - \|w\|_{\mathbb{L}^\infty}\widetilde{\mathcal{U}_m}\right\|_{\mathbb{L}^\infty} \leqslant \rho\|w\|_{\mathbb{L}^\infty}\right\}$$

and

$$\widetilde{\Sigma_\rho}^{\pm} = \left\{w \in \mathbb{L}^\infty,\ \left\|w \mp \|w\|_{\mathbb{L}^\infty}\widetilde{\varphi}'\right\|_{\mathbb{L}^\infty} \leqslant \rho\|w\|_{\mathbb{L}^\infty}\right\},$$

where $\widetilde{\mathcal{U}_m} = \mathcal{U}_m/\|\mathcal{U}_m\|_{\mathbb{L}^\infty}$ and $\widetilde{\varphi}' = (\varphi', 0)/\|(\varphi', 0)\|_{\mathbb{L}^\infty}$.

We claim that for $n \in \{0, \ldots, N\}$, we have $\mathcal{U}(n) \in \Sigma_{\rho_1}$, where $\rho_1 = 4\Delta R/\|\mathcal{U}_m\|_{\mathbb{L}^\infty}$. Indeed, since $e^{nA_1}\mathcal{U}_0 = e^{\lambda_m n}\mathcal{U}_0 = \varepsilon e^{\lambda_m n}\mathcal{U}_m$ and thanks to (23) and (24),

$$\begin{aligned}
\left\|\mathcal{U}(n) - \|\mathcal{U}(n)\|_{\mathbb{L}^\infty}\widetilde{\mathcal{U}_m}\right\|_{\mathbb{L}^\infty} &\leqslant \|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} + \left\|\varepsilon e^{\lambda_m n}\mathcal{U}_m - \frac{\|\mathcal{U}(n)\|_{\mathbb{L}^\infty}}{\|\mathcal{U}_m\|_{\mathbb{L}^\infty}}\mathcal{U}_m\right\|_{\mathbb{L}^\infty} \\
&\leqslant \|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} + \left|\varepsilon e^{\lambda_m n} - \frac{\|\mathcal{U}(n)\|_{\mathbb{L}^\infty}}{\|\mathcal{U}_m\|_{\mathbb{L}^\infty}}\right| \|\mathcal{U}_m\|_{\mathbb{L}^\infty} \\
&\leqslant \|\mathcal{U}(n) - e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} + \left|\|e^{nA_1}\mathcal{U}_0\|_{\mathbb{L}^\infty} - \|\mathcal{U}(n)\|_{\mathbb{L}^\infty}\right| \\
&\leqslant 2\Delta R\varepsilon e^{n\lambda_m} \leqslant \frac{4\Delta R}{\|\mathcal{U}_m\|_{\mathbb{L}^\infty}}\|\mathcal{U}(n)\|_{\mathbb{L}^\infty}.
\end{aligned}$$

Next, we show that if $v \in \Gamma$ is close to 0, then $v \in \widetilde{\Sigma_{\rho_2}}^{\pm}$ for some $\rho_2 > 0$. Let $v \in \Gamma$ with $\|v\|_{\mathbb{L}^\infty} < \sqrt{\rho_0}/2$. There exists $h \in \mathbb{R}$ such that $v = (v_1, 0)$ with $v_1 = \varphi(\cdot + h) - \varphi$. We assume for instance $h \geqslant 0$. Since φ is a diffeomorphism from $\big(-\varphi^{-1}(\sqrt{\rho_0}/2), \varphi^{-1}(\sqrt{\rho_0}/2)\big)$ onto $(-\sqrt{\rho_0}/2, \sqrt{\rho_0}/2)$, and since $v_1(0) = \varphi(h) \in (0, \sqrt{\rho_0}/2)$, the Mean Value Theorem induces

$$h < \frac{\sqrt{\rho_0}}{2} \sup_{|s| \leqslant \sqrt{\rho_0}/2} \big|(\varphi^{-1})'(s)\big| \leqslant \frac{\sqrt{\rho_0}}{2 \inf_{|s| \leqslant \varphi^{-1}(\sqrt{\rho_0}/2)} |\varphi'(s)|} =: \alpha.$$

It follows from the triangle inequality and a Taylor formula that

$$\begin{aligned}
\Big\| v - \|v\|_{\mathbb{L}^\infty} \widetilde{\varphi'} \Big\|_{\mathbb{L}^\infty} &= h \left\| \frac{v_1}{h} - \Big\| \frac{v_1}{h} \Big\|_{L^\infty} \frac{\varphi'}{\|\varphi'\|_{L^\infty}} \right\|_{L^\infty} \\
&\leqslant h \left[\Big\| \frac{v_1}{h} - \varphi' \Big\|_{L^\infty} + \|\varphi'\|_{L^\infty} \left| 1 - \frac{\|v_1/h\|_{L^\infty}}{\|\varphi'\|_{L^\infty}} \right| \right] \\
&\leqslant 2h \Big\| \frac{v_1}{h} - \varphi' \Big\|_{L^\infty} \leqslant 2h^2 \|\varphi''\|_{L^\infty}. \qquad (26)
\end{aligned}$$

Moreover, for $h \leqslant \alpha$, since φ' is decreasing on $\mathbb{R}_+$, the Taylor expansion at order one at $x = 0$ yields

$$h = \frac{\varphi(h)}{\int_0^1 \varphi'(sh)ds} \leqslant \frac{\|v_1\|_{L^\infty}}{\varphi'(\alpha)} = \frac{\|v\|_{\mathbb{L}^\infty}}{\varphi'(\alpha)}. \qquad (27)$$

We deduce from (26) and (27) that $v \in \widetilde{\Sigma_{\rho_2}}^+$ with $\rho_2 = 2\|v\|_{\mathbb{L}^\infty} \|\varphi''\|_{L^\infty} / \varphi'(\alpha)^2$. If $h \leqslant 0$, similar computations induce $v \in \widetilde{\Sigma_{\rho_2}}^-$.

Finally, if $v \in \widetilde{\Sigma_{\rho_2}}^\pm$, the triangle inequality gives

$$\begin{aligned}
\|\mathcal{U}(N) - v\|_{\mathbb{L}^\infty} \geqslant & \Big\| \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \widetilde{\mathcal{U}_m} \mp \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \widetilde{\varphi'} \Big\|_{\mathbb{L}^\infty} \\
& - \Big\| \mathcal{U}(N) - \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \widetilde{\mathcal{U}_m} \Big\|_{\mathbb{L}^\infty} - \Big\| (\|\mathcal{U}(N)\|_{\mathbb{L}^\infty} - \|v\|_{\mathbb{L}^\infty}) \widetilde{\varphi'} \Big\|_{\mathbb{L}^\infty} - \Big\| \|v\|_{\mathbb{L}^\infty} \widetilde{\varphi'} \mp v \Big\|_{\mathbb{L}^\infty},
\end{aligned}$$

thus, since $\mathcal{U}(N) \in \Sigma_{\rho_1}$,

$$(1 + \|\widetilde{\varphi'}\|_{\mathbb{L}^\infty}) \|\mathcal{U}(N) - v\|_{\mathbb{L}^\infty} \geqslant \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \Big\| \widetilde{\mathcal{U}_m} \mp \widetilde{\varphi'} \Big\|_{\mathbb{L}^\infty} - \rho_1 \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} - \rho_2 \|v\|_{\mathbb{L}^\infty}. \qquad (28)$$

Up to a change of a_2, one may assume that $\rho_1 = 4\Delta R / \|\mathcal{U}_m\|_{\mathbb{L}^\infty} < \|\widetilde{\mathcal{U}_m} \mp \widetilde{\varphi'}\|_{\mathbb{L}^\infty}/2$, $2C_s a_2 \leqslant \sqrt{\rho_0}/2$ and $8C_s a_2 \|\varphi''\|_{L^\infty}/\varphi'(\alpha)^2 < \|\widetilde{\mathcal{U}_m} \mp \widetilde{\varphi'}\|_{\mathbb{L}^\infty}/4$. Under these conditions, if $v \in \Gamma$, two cases may occur: either $\|v\|_{\mathbb{L}^\infty} > 2\|\mathcal{U}(N)\|_{\mathbb{L}^\infty}$, which implies by (25)

$$\|\mathcal{U}(N) - v\|_{\mathbb{L}^\infty} \geqslant \|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \geqslant \frac{a_2}{4} \|\mathcal{U}_m\|_{\mathbb{L}^\infty}, \qquad (29)$$

or $\|v\|_{\mathbb{L}^\infty} \leqslant 2\|\mathcal{U}(N)\|_{\mathbb{L}^\infty}$. In this last case, it follows from (22) that

$$\|v\|_{\mathbb{L}^\infty} \leqslant 2C_s a_2 \leqslant \sqrt{\rho_0}/2.$$

Thus $v = (\varphi(\cdot + h) - \varphi, 0)$ with $h \leqslant \alpha$, and (27) holds, as well as (28). Moreover, using once again (22)

$$\begin{aligned} \rho_2\|v\|_{\mathbb{L}^\infty} &\leqslant \frac{8\|\mathcal{U}(N)\|^2_{\mathbb{L}^\infty}\|\varphi''\|_{L^\infty}}{\varphi'(\alpha)^2} \\ &\leqslant \frac{8C_s\varepsilon e^{\lambda_m N} R\|\varphi''\|_{L^\infty}}{\varphi'(\alpha)^2}\|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \\ &= \frac{8C_s a_2\|\varphi''\|_{L^\infty}}{\varphi'(\alpha)^2}\|\mathcal{U}(N)\|_{\mathbb{L}^\infty} \leqslant \frac{1}{4}\left\|\widetilde{\mathcal{U}_m} \mp \widetilde{\varphi'}\right\|_{\mathbb{L}^\infty}\|\mathcal{U}(N)\|_{\mathbb{L}^\infty}. \end{aligned} \tag{30}$$

Using (28), (29) and (30), we finally obtain, for every $v \in \Gamma$,

$$\|\mathcal{U}(N) - v\|_{\mathbb{L}^\infty} \geqslant \min\left(\frac{a_2}{4}\|\mathcal{U}_m\|_{\mathbb{L}^\infty}, \frac{a_2}{16(1 + \|\widetilde{\varphi'}\|_{\mathbb{L}^\infty})}\left\|\widetilde{\mathcal{U}_m} \mp \widetilde{\varphi'}\right\|_{\mathbb{L}^\infty}\|\mathcal{U}_m\|_{\mathbb{L}^\infty}\right),$$

and the proof of Theorem 1.2 is now complete.

5 Numerical check of linear stability

In this section, we numerically investigate the linear stability of the black solitons for different kinds of nonlinearities, using the criterion given in Theorem 1.1. First, it is possible to find an approximate value Φ of the black soliton φ using a differential solver, since $\varphi(0) = 0$ and the value $\varphi'(0)$ is explicitly known. In fact, if we multiply (2) by φ' and integrate between 0 and infinity, it can be shown that

$$\varphi'(0) = \sqrt{V(0)}.$$

For instance, for the pure power defocusing equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + (1 - |u|^{2\sigma})u = 0, \tag{31}$$

which corresponds to $f(r) = 1 - r^\sigma$, we have

$$\varphi'(0) = \sqrt{\frac{\sigma}{\sigma + 1}}.$$

It is sufficient to compute the approximate solution of (2) on a sufficiently large domain in such a way that the derivative almost vanishes at the boundary. For some prescribed λ, the Vakhitov-Kolokolov function is given by

$$g(\lambda) = < (L_1 - \lambda I_d)^{-1}\varphi', \varphi' > := < \psi, \varphi' >,$$

where $\psi \in L^2(\mathbb{R})$ satisfies the differential equation $(L_1 - \lambda I_d)\psi = \varphi'$. This equation can be solved with use of finite differences, seeking the approximate values ψ_j of the solution at gridpoints $x_j = jh$ where $j = -J, \cdots, J$. In this case, the discretized linear system can be written as $M\Psi = \Phi'$, where $\Psi = (\psi_{-J}, \psi_{-J+1}, \cdots, \psi_{J-1}, \psi_J)^T \in \mathbb{R}^{2J+1}$, $\Phi' \in \mathbb{R}^{2J+1}$ stands for the approximate value of φ' computed from $\Phi = (\varphi_{-J}, \varphi_{-J+1}, \cdots, \varphi_{J-1}, \varphi_J)^T$ and the matrix M is given by

$$M = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} + \operatorname{diag}\Big(q_1(x_j) - \lambda, -J \leq j \leq J\Big).$$

The approximation G of the Vakhitov-Kolokolov function g is then given by $G(\lambda) = < \Psi, \Phi >$. In Figures 1 and 2 are respectively plotted the profiles of black solitons obtained for (31) considered with different values of σ ($\sigma = 0.5$, 1, 2 and 3) and for each case the Vakhitov-Kolokolov function with respect to log $(-\lambda)$.

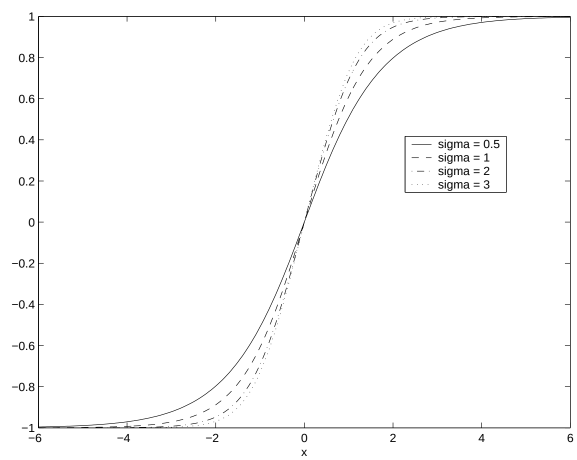

Figure 1: Black solitons for different values of σ.

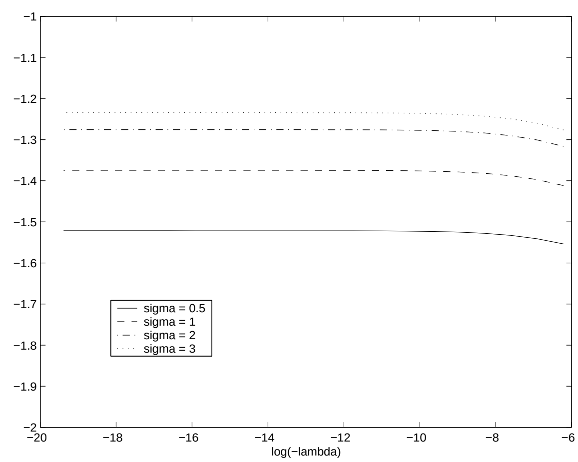

Figure 2: G with respect to $\log(-\lambda)$ for different values of σ.

It can be observed that the function G is always strictly negative when λ tends to zero and admits at each time a strictly negative limit at zero. This numerically checks the linear stability in this case. It can be also observed that this limit seems to be an increasing function of σ. Nevertheless, when σ becomes large, it has been found that the limit value $l(\sigma)$ when λ tends to zero is always strictly negative. As indicated in Figure 3,

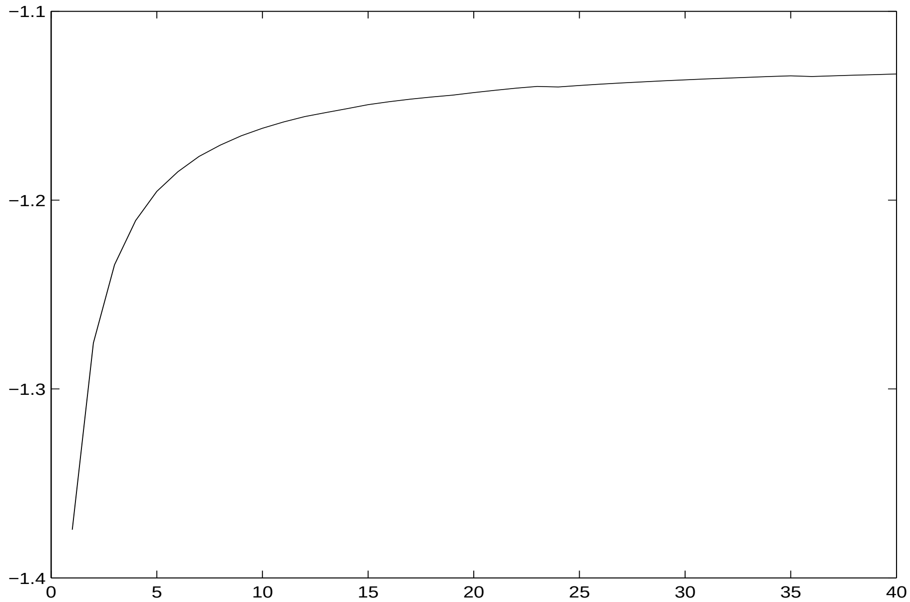

Figure 3: Plot of $l(\sigma)$ versus σ.

it seems that $l := \lim_{\sigma \to \infty} l(\sigma)$ exists and remains strictly negative (we have $l \simeq -1.13$). This result suggests the linear stability of black solitons for any nonnegative value of σ.

We then focus on the case of the saturated equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \Big(\frac{1}{(1+a|u|^2)^2} - \frac{1}{(1+a)^2}\Big)u = 0, \tag{32}$$

where a is a prescribed positive parameter. Here, the nonlinear term still vanishes for $|u|^2 = 1$ and remains positive between the two values 0 and 1. The value at the origin of the black soliton derivative is given by

$$\varphi'(0) = \frac{\sqrt{a}}{1+a}.$$

It is still possible here to compute the soliton and the Vakhitov-Kolokolov function. In Figures 4 are shown the profiles of black solitons for different values of a. It can be noticed that contrary to the first equation investigated, the black soliton derivative at zero is no more a monotonous function of the parameter (in fact, the maximum value obtained is reached for $a_* = 1$). In figure 5 is plotted the value $G(\lambda_0)$ for $\lambda_0 = -10^{-5}$ for different values of a. We observe that for sufficiently large values of a (say, $a > a_{VK}$), the Vakhitov-Kolokolov function becomes positive if λ is close to zero, which ensures that the black soliton is linearly stable when $a < a_{VK} \simeq 7.47$ and linearly unstable when $a > a_{VK}$. This kind of instability threshold has been already observed in the literature (see [KK]).

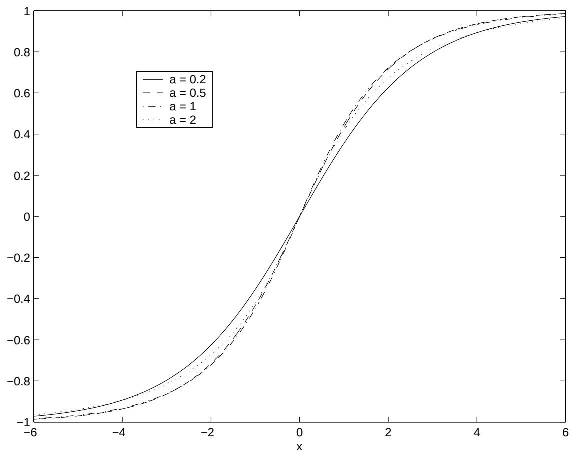

Figure 4: Black solitons for different a.

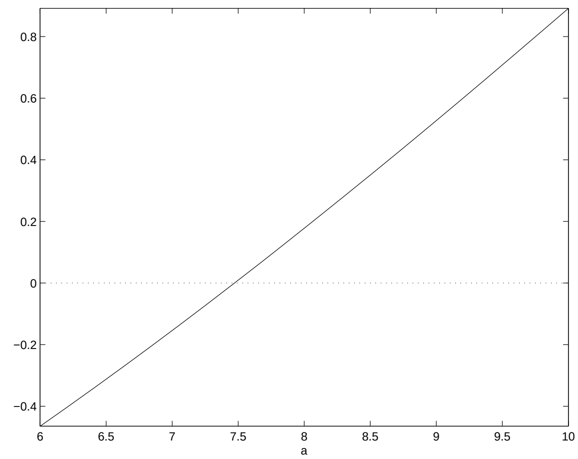

Figure 5: $G(\lambda_0)$ versus a.

6 The linear stability of the black soliton of the 1D Gross-Pitaevskii equation

In the cubic defocusing case $f(r) = 1 - r$, the black soliton φ is explicitly known:

$$\varphi(x) = \tanh \frac{x}{\sqrt{2}}.$$

Thus

$$\varphi'(x) = \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}},$$

$$f(\varphi(x)^2) = \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}},$$

and

$$L := L_1 = -\frac{d^2}{dx^2} - \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} \ .$$

We also know that the lowest eigenvalue of L is $\lambda_0 = -1/2$, and an associated eigenvector is

$$u_0(x) = \frac{1}{\cosh \frac{x}{\sqrt{2}}}.$$

The explicit form of this data enables us to prove the linear stability of the black soliton of the Gross-Pitaevskii equation stated in Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda \in (-1/2, 0)$, and

$$v := (L - \lambda)^{-1}\varphi' \in H^2 \subset L^\infty.$$

Then it is clear that v is even, $v(x) \to 0$ as $x \to \infty$, and v solves

$$v''(x) \quad = \quad -\frac{v(x)}{\cosh^2 \frac{x}{\sqrt{2}}} - \frac{1}{\sqrt{2}\cosh^2 \frac{x}{\sqrt{2}}} - \lambda v(x). \tag{33}$$

For every integer $n \geqslant 1$, let us define

$$u_n := \int_{-\infty}^{+\infty} \frac{v(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx$$

and

$$a_n := \int_{-\infty}^{+\infty} \frac{1}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx.$$

Note that $u_1 = \sqrt{2}g(\lambda)$. The strategy of the proof is as follows. First, multiplying (33) by $1/\cosh^{2n} \frac{x}{\sqrt{2}}$ and integrating over $\mathbb{R}$, we obtain an induction relation between u_n and u_{n+1}. Summing this relation, we express the first term in the asymptotic expansion of u_n in function of u_1. Next, using the definition of u_n, we express the same quantity in function of $v(0)$. The comparison of these two expressions yields a link between u_1 and $v(0)$. In particular, this relation implies that $v(0) > 0$ as soon as $u_1 \geqslant 0$. Finally, we show that the condition $v(0) > 0$ is inconsistent with the fact that $v(x) \to 0$ as $x \to \infty$.

Let us as announced multiply (33) by $1/\cosh^{2n} \frac{x}{\sqrt{2}}$ and sum the obtained equality over $\mathbb{R}$. We get

$$\int_{-\infty}^{+\infty} \frac{v''(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx \quad = \quad -u_{n+1} - \frac{1}{\sqrt{2}} a_{n+1} - \lambda u_n. \tag{34}$$

Using integration by parts, the left hand side of (34) can be expressed as

$$\int_{-\infty}^{+\infty} \frac{v''(x)}{\cosh^{2n} \frac{x}{\sqrt{2}}} dx \quad = \quad 2n^2 u_n - (2n+1) n u_{n+1}. \tag{35}$$

From (34) and (35) we infer the following induction relation:

$$u_{n+1} \quad = \quad \frac{1 + \frac{\lambda}{2n^2}}{1 + \frac{1}{2n} - \frac{1}{2n^2}} u_n + \frac{1}{\sqrt{2}} \frac{a_{n+1}}{2n^2 + n - 1}. \tag{36}$$

(36) yields a relation between u_{n+1} and u_1:

$$u_{n+1} = \prod_{l=1}^{n} \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) u_1 + \frac{1}{\sqrt{2}} \sum_{k=1}^{n} \prod_{l=k+1}^{n} \left(\frac{1 + \frac{\lambda}{2l^2}}{1 + \frac{1}{2l} - \frac{1}{2l^2}} \right) \frac{a_{k+1}}{2k^2 + k - 1}. \tag{37}$$

We will next compute asymptotic expansions of each term in (37). We begin with the term containing u_1. First remark that the product $P_\lambda := \prod_{l=1}^{\infty}\left(1+\frac{\lambda}{2l^2}\right)$ converges. Thus

$$\prod_{l=1}^{n}\left(1+\frac{\lambda}{2l^2}\right) = P_\lambda + o(1),$$

with $P_\lambda \in (0,1)$ (because $\lambda \in (-1/2, 0)$). On the other side, there exists a constant $P > 0$ such that

$$\prod_{l=1}^{n}\left(1+\frac{1}{2l}-\frac{1}{2l^2}\right) = P\sqrt{n}(1+o(1)).$$

Therefore

$$\prod_{l=1}^{n}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right) \;=\; \frac{P_\lambda}{P\sqrt{n}}(1+o(1)). \tag{38}$$

We begin the computation of the asymptotic expansions of the two others terms in (37) with the following lemma:

Lemma 6.1 *Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$, bounded, even, with $\varphi(0) \neq 0$. Then*

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x}dx \underset{n\to\infty}{=} \frac{\varphi(0)}{\sqrt{n}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}}dy(1+o(1)).$$

Proof. We write

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x}dx = 2\int_0^\infty \varphi(x)e^{-2n\ln\cosh x}dx,$$

and we make the change of variables $y = 2n\ln\cosh x$, such that

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\cosh^{2n} x}dx &= 2\int_0^\infty \frac{\varphi(\mathrm{arccosh} e^{\frac{y}{2n}})}{\sqrt{e^{\frac{y}{n}}-1}}e^{-y}e^{\frac{y}{2n}}\frac{dy}{2n} \\ &= \frac{1}{\sqrt{n}}\int_0^\infty \tilde{\varphi}(\frac{y}{n})\frac{e^{-y}}{\sqrt{y}}dy,\end{aligned}$$

where for $y > 0$,

$$\tilde{\varphi}(y) := \varphi(\mathrm{arccosh} e^{\frac{y}{2}})\frac{e^{\frac{y}{2}}\sqrt{y}}{\sqrt{e^y-1}}$$

and $\tilde{\varphi}(0) = \varphi(0)$. $\tilde{\varphi}$ is continuous on $\mathbb{R}_+$, and the dominated convergence theorem yields

$$\int_0^\infty \tilde{\varphi}(\frac{y}{n})\frac{e^{-y}}{\sqrt{y}}dy \underset{n\to\infty}{\longrightarrow} \int_0^\infty \varphi(0)\frac{e^{-y}}{\sqrt{y}}dy,$$

and the lemma follows. $\square$

We apply this lemma on the one side with $\varphi(x) = v(\sqrt{2}x)$, on the other side with $\varphi(x) = 1$. We obtain respectively

$$u_n = \sqrt{2}\int_{-\infty}^{\infty} \frac{v(\sqrt{2}x)}{\cosh^{2n} x}dx \quad = \quad \frac{\sqrt{2}v(0)}{\sqrt{n}}\int_0^{\infty} \frac{e^{-y}}{\sqrt{y}}dy(1+o(1)) \tag{39}$$

and

$$a_n = \sqrt{2}\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n} x} = \frac{\sqrt{2}}{\sqrt{n}}\int_0^{\infty} \frac{e^{-y}}{\sqrt{y}}dy(1+o(1)). \tag{40}$$

We next give an asymptotic expansion of the last term in (37). We rewrite it under the form

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\sum_{k=1}^{n}\prod_{l=k+1}^{n}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)\frac{a_{k+1}}{2k^2+k-1} \\
&= \frac{1}{\sqrt{2}}\left(\prod_{l=1}^{n}\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)\sum_{k=1}^{n}\frac{a_{k+1}}{\prod_{l=1}^{k}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)(2k^2+k-1)}.
\end{aligned}$$

The series in the right hand side converges and we will denotes its sum by S_λ. Indeed, it follows from (38) and (40) that

$$\frac{a_{k+1}}{\prod_{l=1}^{k}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)(2k^2+k-1)} = \frac{\sqrt{2}P\sqrt{k}}{2\sqrt{k}P_\lambda}\int_0^{\infty}\frac{e^{-y}}{\sqrt{y}}dy\frac{1}{k^2}(1+o(1)) = O(\frac{1}{k^2}).$$

Therefore, using once again (38), we get

$$\frac{1}{\sqrt{2}}\sum_{k=1}^{n}\prod_{l=k+1}^{n}\left(\frac{1+\frac{\lambda}{2l^2}}{1+\frac{1}{2l}-\frac{1}{2l^2}}\right)\frac{a_{k+1}}{2k^2+k-1} \quad = \quad \frac{P_\lambda S_\lambda}{P\sqrt{2n}}(1+o(1)). \tag{41}$$

We inject (39), (38) and (41) into (37), and we obtain the relation

$$\sqrt{2}v(0)\int_0^{\infty}\frac{e^{-y}}{\sqrt{y}} = \frac{P_\lambda}{P}(u_1+\frac{S_\lambda}{\sqrt{2}}). \tag{42}$$

For $\lambda \in (-1/2, 0)$, it is easy to see that $S_\lambda > S_0 \in (0, +\infty)$. Therefore, if we assume by contradiction that $u_1 > -S_0/2\sqrt{2} > -S_\lambda/2\sqrt{2}$, (42) implies $v(0) > 0$.

In the sequel, $\lambda \in (-1/2, 0)$ is fixed, and we assume by contradiction that $u_1 > -S_0/2\sqrt{2}$, and thus that $v(0) > 0$. We will show that this implies that $v(x) \to -\infty$ as $x \to +\infty$, which is a contradiction with $v \in L^2$.

Let us first define, for $x \in \mathbb{R}$,

$$w(x) := v(x) \cosh \frac{x}{\sqrt{2}}. \tag{43}$$

Then for $x \in \mathbb{R}$, we have

$$v(x) = \frac{w(x)}{\cosh \frac{x}{\sqrt{2}}},$$

$$v'(x) = \frac{w'(x)}{\cosh \frac{x}{\sqrt{2}}} - \frac{1}{\sqrt{2}} \frac{\tanh \frac{x}{\sqrt{2}}}{\cosh \frac{x}{\sqrt{2}}} w(x),$$

$$v''(x) = \frac{w''(x)}{\cosh \frac{x}{\sqrt{2}}} - \frac{2}{\sqrt{2}} \frac{\tanh \frac{x}{\sqrt{2}}}{\cosh \frac{x}{\sqrt{2}}} w'(x) - \frac{w(x)}{\cosh^3 \frac{x}{\sqrt{2}}} + \frac{1}{2} \frac{w(x)}{\cosh \frac{x}{\sqrt{2}}}.$$

Since v satisfies (33), it follows that w solves

$$w''(x) = \sqrt{2} \tanh \frac{x}{\sqrt{2}} w'(x) - \frac{1}{\sqrt{2} \cosh \frac{x}{\sqrt{2}}} - (\frac{1}{2} + \lambda) w(x). \tag{44}$$

Remark that $w(0) = v(0) > 0$, w is even and $w'(0) = 0$. Evaluating (44) at $x = 0$ and using the fact that $\lambda > -1/2$, we thus have

$$w''(0) = -\frac{1}{\sqrt{2}} - (\frac{1}{2} + \lambda) w(0) < 0.$$

Therefore there exists $\eta_0 > 0$ such that if $x \in [0, \eta_0]$, $w''(x) < 0$ and $w(x) > 0$. Thus we can define

$$x_1 := \sup\{x > 0, w(y) > 0 \text{ and } w''(y) < 0 \text{ for every } y \in [0, x]\}.$$

If $x_1 = +\infty$, we have $w''(y) < 0$ on $[0, \infty)$ which implies that w' strictly decreases on $\mathbb{R}_+$; thus for $x \geqslant 1$, $w'(x) \leqslant w'(1) < w'(0) = 0$. Therefore for $x \geqslant 1$, $w(x) \leqslant w'(1)(x-1) + w(1)$. In particular, $w(x) < 0$ for x large enough, which is a contradiction with the assumption $x_1 = +\infty$. Thus $x_1 < \infty$.

Next, we show that $w(x_1) = 0$ or $w''(x_1) = 0$. Indeed, if it was not the case, thanks to the continuity of w and w'', there would exist a neighbourhood of x_1 in $\mathbb{R}_+$ on which $w > 0$ and $w'' < 0$, which would yield a contradiction with the definition of x_1. We next prove that $w''(x_1) < 0$, and thus $w(x_1) = 0$. Indeed,

$$\begin{aligned} w''(x_1) &= \sqrt{2} \tanh \frac{x_1}{\sqrt{2}} w'(x_1) - \frac{1}{\sqrt{2} \cosh \frac{x_1}{\sqrt{2}}} - (\frac{1}{2} + \lambda) w(x_1) \\ &\leqslant -\frac{1}{\sqrt{2} \cosh \frac{x_1}{\sqrt{2}}} < 0, \end{aligned}$$

because w' is strictly decreasing on $[0, x_1]$ (thus $w'(x_1) < w'(0) = 0$), $1/2 + \lambda > 0$ and $w(x_1) \geqslant 0$. Therefore $v(x_1) = w(x_1) = 0$, and $v'(x_1) = \frac{w'(x_1)}{\cosh \frac{x_1}{\sqrt{2}}} < 0$. It follows that there exists $\eta_1 > 0$ such that $-1/\sqrt{2} < v(x) < 0$ for $x \in (x_1, x_1 + \eta_1]$. Thus we can define

$$x_2 := \sup\{x > x_1, -1/\sqrt{2} < v(y) \leqslant 0 \text{ for } y \in [x_1, x)\} \geqslant x_1 + \eta_1 > x_1.$$

For $x \in [x_1, x_2)$, since $-\lambda > 0$, $v(x) \leqslant 0$ and $1/\sqrt{2} + v(x) > 0$, we have

$$v''(x) \;=\; -\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} \left(\frac{1}{\sqrt{2}} + v(x) \right) - \lambda v(x) < 0.$$

Therefore v' is decreasing on $[x_1, x_2)$, and $v(x) \leqslant v'(x_1)(x - x_1)$ if $x \in [x_1, x_2)$. Now, since $v'(x_1) < 0$, we have $x_2 < \infty$, $v(x_2) = -1/\sqrt{2}$ and $v'(x_2) < v'(x_1) < 0$.

From now on, we distinguish the two cases:

1. $x_2 \geqslant x_\lambda := \sqrt{2}\text{arccosh}\frac{1}{\sqrt{-\lambda}}$,
2. $x_2 < x_\lambda$.

In the first case, for $x \geqslant x_2$, $\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda \leqslant 0$. $v'(x_2) < 0$, therefore there exists $\eta_2 > 0$, $v'(x) < 0$ for $x \in [x_2, x_2 + \eta_2]$, and the definition of $x_3 > x_2$ as

$$x_3 := \sup\{x > x_2,\ v'(y) < 0 \text{ for } y \in (x_2, x)\} > x_2$$

makes sense. v is decreasing on $[x_2, x_3)$, thus $v(x) \leqslant -1/\sqrt{2}$ for every $x \in [x_2, x_3)$. It follows that for $x \in [x_2, x_3)$, $-\left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda \right) v(x) \leqslant 0$, and

$$v''(x) = -\frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} - \left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda \right) v(x) \leqslant -\frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} < 0.$$

Therefore v' is decreasing on $[x_2, x_3)$. This implies that $x_3 = +\infty$, and for $x \geqslant x_2$, $v(x) \leqslant v(x_2) + v'(x_2)(x - x_2)$, thus $v(x) \to -\infty$ as $x \to +\infty$.

We are now concerned with the second case $x_2 < x_\lambda$. We introduce

$$x_4 := \sup\{x \in (x_2, x_\lambda), v'(y) < 0 \text{ for } y \in [x_2, x)\}$$

and the open set $\Omega := \{x \in (x_2, x_4), v''(x) > 0\}$. Ω is the disjointed union of its connex parts $\omega_i = (a_i, b_i)$ for $i \in I$, where I is a finite or countable set. For $x \in \omega_i$, $v''(x) > 0$, and thus integrating twice, for $x \in \omega_i$,

$$v'(x) \geqslant v'(a_i),$$

$$v(x) - v(a_i) \geqslant v'(a_i)(x - a_i).$$

Using (33) and the fact that $x < x_\lambda$, it follows that

$$v''(x) \leqslant \left(\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} + \lambda \right) (-v(a_i) - v'(a_i)(x - a_i)) - \frac{1}{\sqrt{2}} \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}},$$

which may be integrated between a_i and x. After an integration by parts, we obtain

$$\begin{aligned} v'(x) &\leqslant v'(a_i) - (\sqrt{2}v(a_i) + 1) \left(\tanh \frac{x}{\sqrt{2}} - \tanh \frac{a_i}{\sqrt{2}} \right) - \lambda v(a_i)(x - a_i) \\ &-\lambda \frac{v'(a_i)}{2}(x - a_i)^2 - v'(a_i)\sqrt{2} \tanh \frac{x}{\sqrt{2}}(x - a_i) + v'(a_i) \int_{a_i}^{x} \sqrt{2} \tanh \frac{y}{\sqrt{2}} dy. \qquad (45) \end{aligned}$$

For $x \in (x_2, x_4)$, $v'(x) < 0$, thus $v(x) < v(x_2) = -1/\sqrt{2}$. In particular, $v'(a_i) < 0$ and $v(a_i) + 1/\sqrt{2} < 0$ (indeed, $a_i > x_2$ because $v''(x_2) = \lambda/\sqrt{2} < 0$ while $v''(a_i) = 0$). For $x \in \omega_i$, it follows from (45), the mean value theorem and the inequality $\tanh \frac{x}{\sqrt{2}} \leqslant 1$ that

$$\begin{aligned} v'(x) &\leqslant v'(a_i) - \underbrace{\left[(v(a_i) + \frac{1}{\sqrt{2}}) \frac{1}{\cosh^2 \frac{a_i}{\sqrt{2}}} + \lambda v(a_i) \right]}_{=-v''(a_i)=0.} (x - a_i) \\ &\quad -\lambda \frac{v'(a_i)}{2}(x - a_i)^2 - v'(a_i)\sqrt{2} \int_{a_i}^{x} (1 - \tanh \frac{y}{\sqrt{2}}) dy \\ &= v'(a_i) \left[1 - \sqrt{2} \int_{a_i}^{x} (1 - \tanh \frac{y}{\sqrt{2}}) dy - \frac{\lambda (x - a_i)^2}{2} \right] \\ &\leqslant v'(a_i) \left[1 - \sqrt{2} \int_{x_2}^{\infty} (1 - \tanh \frac{y}{\sqrt{2}}) dy \right] = v'(a_i)(1 - 2\ln(1 + e^{-\sqrt{2}x_2})). \qquad (46) \end{aligned}$$

We provisionally admit the fact that $(1 - 2\ln(1 + e^{-\sqrt{2}x_2})) > 0$. Let us assume by contradiction that $v'(x_4) = 0$.

If $x_4 \notin \overline{\Omega}$, there exists $\eta > 0$ such that $v''(x) \leqslant 0$ for $x \in [x_4 - \eta, x_4]$, and thus $0 = v'(x_4) \leqslant v'(x_4 - \eta) < 0$, which is a contradiction.

Thus $x_4 \in \overline{\Omega}$. Let $(c_n)_n$ be a sequence in Ω such that $c_n \to x_4$. For each n, $c_n \in \omega_{i(n)}$, and $c_n < b_{i(n)} \leqslant x_4$. Thus $b_{i(n)} \to x_4$. If $b_{i(n)}$ was constant (equal to $x_4 = b_{i_0}$) for $n \geqslant n_0$, we would have by (46)

$$0 = v'(x_4) = v'(b_{i_0}) \leqslant v'(a_{i_0})(1 - 2\ln(1 + e^{-\sqrt{2}x_2})) < 0,$$

which is a contradiction. Thus $b_{i(n)}$ takes infinitely many values. We next remark that if $y \in (0, x_\lambda)$ satisfies $v''(y) = 0$, it follows from (33) that

$$v(y) = -\frac{1}{\sqrt{2}(1 + \lambda \cosh^2 \frac{y}{\sqrt{2}})} =: f(y).$$

Here, we have $0 = v''(b_{i(n)}) \to v''(x_4)$, thus $v''(x_4) = 0$, and $v(x_4) = f(x_4)$. Next,

$$f'(x_4) = \lambda \frac{\sinh \frac{x_4}{\sqrt{2}} \cosh \frac{x_4}{\sqrt{2}}}{\left(1 + \lambda \cosh^2 \frac{x_4}{\sqrt{2}}\right)^2} < 0 = v'(x_4).$$

This implies that there exists $\eta_4 > 0$ such that $v(x) < f(x)$ for $x \in [x_4 - \eta_4, x_4)$. For n large enough, $b_{i(n)} \in [x_4 - \eta_4, x_4)$, $v''(b_{i(n)}) = 0$, thus $v(b_{i(n)}) = f(b_{i(n)})$. This is a contradiction.

If $x_4 < x_\lambda$, the definition of x_4 implies that we must have $v'(x_4) = 0$, which we have just seen to be impossible. Thus $x_4 = x_\lambda$ and $v'(x_4) < 0$. Then, we conclude as in the case $x_2 \geqslant x_\lambda$, with x_2 replaced by x_4.

To complete the proof, it remains to prove the estimate

$$2 \ln \left(1 + e^{-\sqrt{2} x_2}\right) \quad < \quad 1 \tag{47}$$

in the case $x_2 < x_\lambda$, which will be done by bounding x_2 from below. For $x \in [0, x_2]$, $\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} > 0$, and we have seen that v is decreasing on $[0, x_2]$, thus $v(x) \leqslant v_0 = v(0)$ for $x \in [0, x_2]$, and

$$\begin{aligned} v''(x) &= -\left(\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}}\right) v(x) - \frac{1}{\sqrt{2} \cosh^2 \frac{x}{\sqrt{2}}} \\ &\geqslant -\left(\lambda + \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}}\right) v_0 - \frac{1}{\sqrt{2} \cosh^2 \frac{x}{\sqrt{2}}} \\ &\geqslant -\left(v_0 + \frac{1}{\sqrt{2}}\right) \frac{1}{\cosh^2 \frac{x}{\sqrt{2}}}, \end{aligned}$$

because $-\lambda v_0 > 0$. Integrating this inequality between 0 and $x \in [0, x_2]$, we get

$$|v'(x)| = -v'(x) \leqslant (1 + v_0 \sqrt{2}) \tanh \frac{x}{\sqrt{2}} \leqslant (1 + v_0 \sqrt{2}).$$

Using the mean value Theorem, it follows that

$$\frac{1}{\sqrt{2}} + v_0 = |v(x_2) - v(0)| \leqslant (1 + v_0 \sqrt{2}) x_2.$$

Therefore $x_2 \geqslant 1/\sqrt{2}$, and

$$2 \ln(1 + e^{-\sqrt{2} x_2}) \leqslant 2 \ln(1 + e^{-1}) < 1.$$

This completes the proof. □

7 Dynamical stability

In this section, we numerically investigate the dynamical stability of the black solitons. For this purpose, we use finite differences in both time and space.

7.1 Description of the numerical method

Space and time steps h and δt being given, all the computations will be made in a rectangular grid at spatial points $x_j = jh$ and at discrete times $t_n = n\delta t$. Let us define u_j^n as the approximate value of $u(x_j, t_n)$. For any given smooth function w, we also define the operators L_x as

$$(L_x w)_j = \frac{1}{h^2}\Big(w_{j+1} - 2w_j + w_{j-1}\Big) = \Big(\frac{\partial^2 w}{\partial x^2}\Big)_j + \mathcal{O}(h^2)$$

that can be seen as the discretization of the differential operator $\partial^2/\partial x^2$ with use of Taylor formulas. We then consider the symmetric Crank-Nicolson discretization of (1): setting $u^{n+1/2} = (u^n + u^{n+1})/2$, the system writes

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + (L_x u^{n+1/2})_j + f(|u_j^{n+1/2}|^2)u_j^{n+1/2} = 0$$

for $n \geqslant 0$ and $|j| \leqslant J$, with J given. This is a nonlinear algebraic discrete system that is solved at each time step using a fixed point method. This way of discretizing the system enables us to have the conservation of the discrete first invariant on the whole space

$$I_n = \sum_{j\in\mathbb{Z}} \Big(|u_j^n|^2 - 1\Big) = I_0, \quad \forall\, n \geq 0.$$

7.2 The boundary problem

When dealing with asymptotic behaviour of time evolution problems, numerics may become delicate since for obvious CPU reasons, computations have to be made in a bounded interval of $\mathbb{R}$, often referred to as the computational domain. In this case, an inappropriate boundary condition can lead to artificial reflection and may cause a wrong approximation of the exact solution. In all the cases that will be investigated, the solutions under study tend to different limit values at $\pm\infty$. The most natural way to proceed is to set these values for the numerical approximation at each extremity of the computational domain. This is justified because given a black soliton φ; the Cauchy problem for (1) is well-posed in $\varphi + H^1$ (see [G2]). We show here that even in the linear case

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{48}$$

with boundary conditions -1 and 1 at $-\infty$ and ∞, this treatment can seriously affect the solution asymptotics. In order to avoid such problems, a numerical trick consists in adding a diffusion term in (48), that will come into play near the boundary. We thus replace (48) by

$$(i - \alpha(x))\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0, \tag{49}$$

where the function α vanishes at the interior of the numerical domain. This means that the original problem is turned into a diffusive-like equation where the diffusion term only plays a role close to the boundary. Consequently, waves that could reflect on the boundary are absorbed. The term α is referred to in the literature as a *sponge factor*. Such a function is plotted in Figure 6. Remark that since we deal with a modified equation, conservation of invariants for initial problem no more holds.

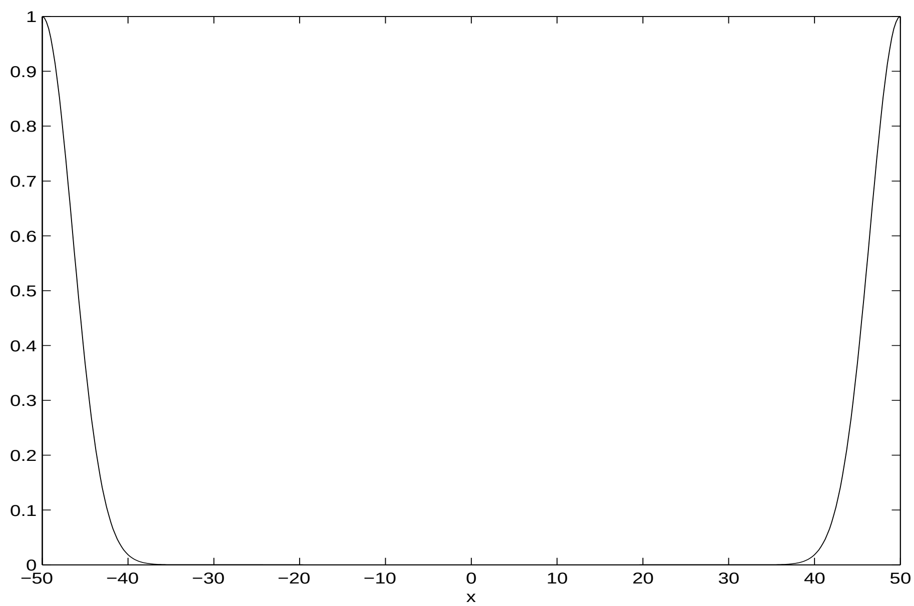

Figure 6: Plot of α.

We now show the results obtained for the numerical resolution of (48) considered with the initial data $u_0(x) = \tanh x$, for the following choice of grid parameters: $h = 0.1$, $J = 200$, $\delta t = 0.1$ and a sponge factor given by

$$\alpha(x) = \exp(-(|x| - L)^2/4),$$

where $L = Jh$. In Figures 7 and 8, we view the solution computed with the sponge factor compared to both solution computed with nonhomogeneous Dirichlet conditions $u(t, -L) = -1$, $u(t, L) = 1$ and solution considered in the larger domain $]-3L, 3L[$, for which the solution is not affected by the boundary on the small domain. It can be observed that the first solution is a much better approximation of the reference one than the solution obtained with Dirichlet conditions. In all what follows, such sponge terms will be used for our tests. Note that in this case, there is no need in using a numerical scheme which preserves the charge on the numerical domain since the modified equation is no more conservative, even if the lack of conservation is only relevant close to the boundary.

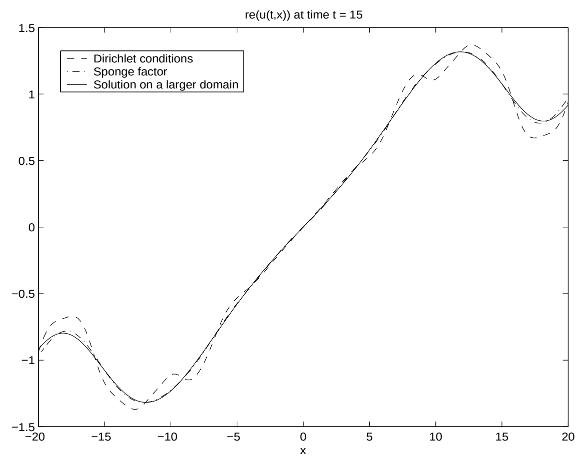

Figure 7: Profile of the real part of the solutions of (48) with Dirichlet boundary conditions, with sponge factor and computed on the larger domain $]-60,60[$, plot on the small domain $]-20,20[$, $T=15$.

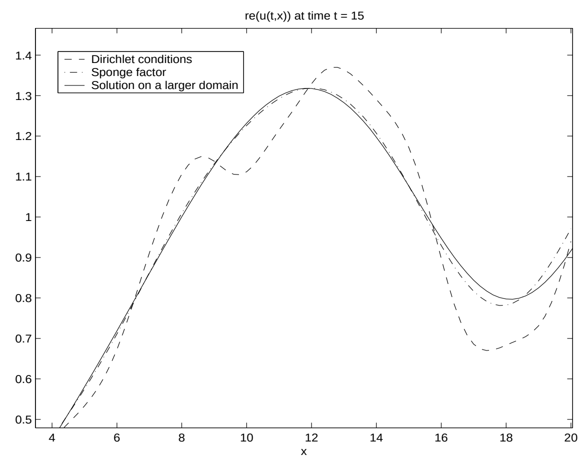

Figure 8: Profile of the real part of the solutions of (48) with Dirichlet boundary conditions, with sponge factor and computed on the larger domain $]-60,60[$, $T=15$, zoom near the right boundary.

7.3 The pure power defocusing equation

Since the stability condition stated in Theorem 1.1 deals with linear stability, we first perform computations for the linearized equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+f(\varphi^2)u+2\varphi^2 f'(\varphi^2)\mathfrak{Re}\,u=0, \tag{50}$$

with $f(r)=1-r^\sigma$ (which is an other way to write system (3)) in order to check if a stability threshold occurs. We start from the initial condition

$$\varepsilon(x)=e^{-0.01x^2}, \tag{51}$$

that can be seen as a perturbation of zero. Note that due to the linearity of (50), this perturbation is not needed to be small. We numerically solve equation (50) using the same semi-implicit discretization of the linear term as in the nonlinear case and plot the evolution of the maximal amplitude in order to look for the generation of an unstable mode. For the sake of clarity, we have preferred to view the profile of $k(t):=\log(1+\|u(t)\|_{L^\infty})/t$: if k

tends to zero as t is large, then no exponential growth is obtained and the black soliton can be considered as linearly stable. If k tends to some positive constant λ, then instability occurs with the corresponding growth rate given by λ.

Our first simulations showed successive drops of k, caused by periodic changes of the sign of the real part. It has to be noticed that this phenomenon still occurs if the sponge term is removed. Surprisingly, dealing with well-adapted grid parameters enables us to delay it. In most of the tests that will be discussed, we have decided to deal with large numerical domains considered with small space step h, for the following reasons: firstly, if we look for an unstable mode for which k tends to a positive limit value, the space domain has to be large enough in such a way that the eigenvector is well-localized. Furthermore, the black soliton φ computed using a differential solver (see Section 5) has to be accurately described around the origin where the variations of φ are most significant: this requires to work with reasonably small values of h.

We have computed the evolution of k for different values of σ ($\sigma = 1$, $\sigma = 2$ and $\sigma = 5$) with parameters $J = 50000$, $h = 0.002$, $\delta t = 0.2$ and $N = 10000$. Figure 9 indicates that k decreases to 0: there is no positive eigenvalue for the linearized operator, which illustrates linear stability for any value of σ.

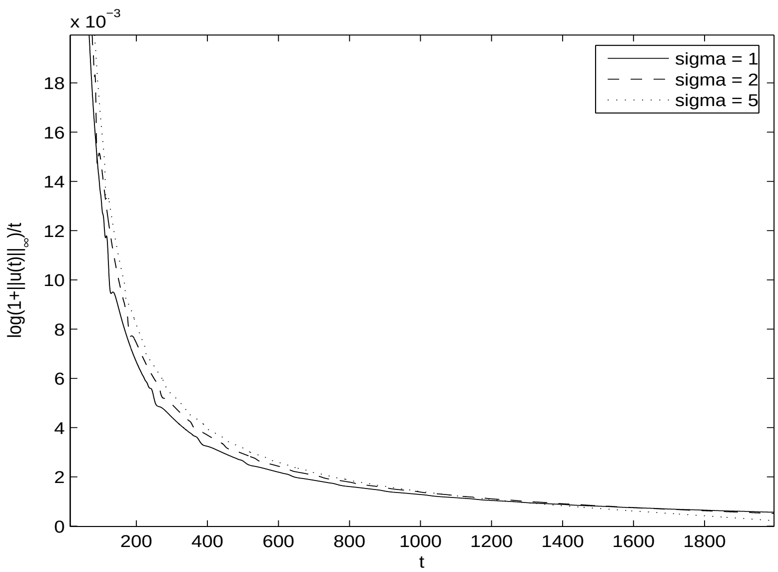

Figure 9: Plot of k versus time, $\sigma = 1$, $\sigma = 2$ and $\sigma = 5$.

We then numerically solved problem (31), injecting as initial data a small perturbation of the black soliton numerically calculated: we take $u_0 = \varphi + \varepsilon$, with

$$\varepsilon(x) = q(1+i)e^{-0.01x^2}\cos x. \tag{52}$$

We took the following parameters : $h = 0.01$, $J = 20000$, $\delta t = 1$, $N = 1000$ and $q = 3.10^{-4}$. For any value of σ, the numerical solution seems to travel with a velocity depending on σ. This may suggest that the black soliton is not stable. However, we translate the solution in such a way that the real part changes its sign at the origin and we study the L^∞ error with respect to the black soliton. We plot in Figure 10 the evolution of the L^∞ norm of the error between the black soliton and the translated numerical solution obtained for $\sigma = 1$, with respect to time. It can be observed that this difference is bounded through time, which is a grant of stability. Other tests performed for different values of σ have shown a similar behaviour. Note that it can be observed small oscillations of the error caused by the translation of the solution from an integer number of grid cells and the large value of the derivative of the black soliton at $x = 0$. Thus, the period observed in Figure 10 corresponds to the time the solution takes to go through a grid cell.

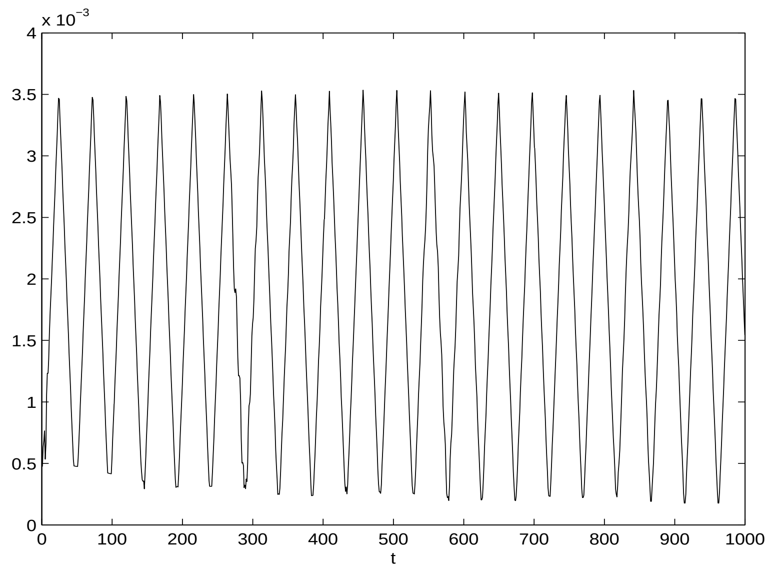

Figure 10: Plot of the L^∞ error between the translated solution and the exact black soliton versus time, $\sigma = 1$.

We then perturbed the initial data with an oscillatory function $\varepsilon(x) = 0.5e^{-x^2} \sin 5x\varphi(x)$. In our test, we set $J = 800$, $h = \delta t = 0.05$ and for simulations performed until final time $T = 5$. In Figures 11 is plotted the solution profile at successive times $t = 1$, $t = 3$ and $t = 5$. It can be observed a propagation to the right of the wavetrain, with a similar effect to the left side for negative x. Since its length seems quite constant through time, the global L^2 error between the numerical solution and the black soliton will remain bounded. Since it is worth investigating the behaviour at the left of the perturbed wave, we also define the *local* L^2 error, that is computed around the origin (in our simulations, we chose the domain $D = [-10, 10]$). In Figure 12 are plotted the two errors and it can be noticed that once the perturbation has left D, then the approximate solution mimics the soliton profile and the local L^2 error becomes small, while the global error stays bounded since the travelling perturbation keeps the same amplitude. This effect could be referred as *local* asymptotic stability, meaning that in compact domains of the real line, the black soliton attracts the solutions for large times.

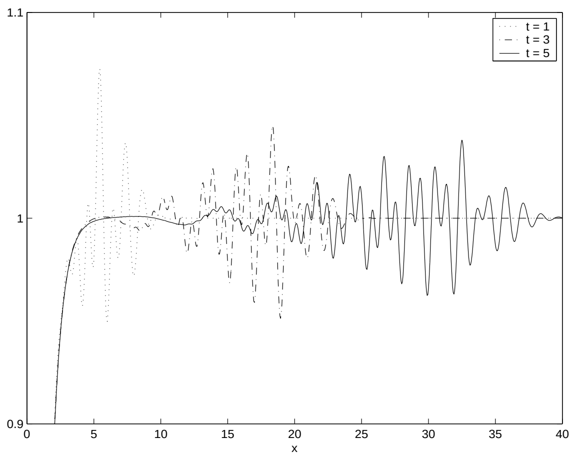

Figure 11: Snapshots of the perturbed solution at different times ($t = 1$, $t = 3$ and $t = 5$) around the black soliton profile, $\sigma = 1$.

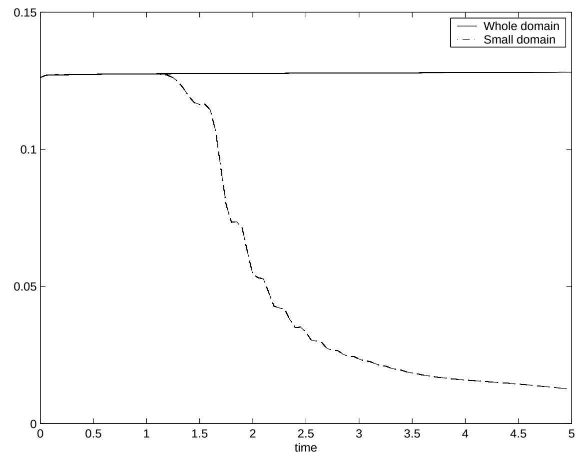

Figure 12: Evolution of global and local L^2 error between the perturbed solution and the black soliton versus time.

These two simulations have shown different behaviours for the solution of (1): the first perturbation leads us to a travelling wave that remains close to the black soliton in L^∞ norm, whereas the second perturbation generates local oscillations going away to infinity and a stationary global profile (that is with no translation effect). This could suggest that the perturbed solution mimics travelling solitons of (1) with velocity depending on the initial perturbation, the velocity being equal to zero in the last experiment. Such states are referred as travelling bubbles and are explictly given in the case $\sigma = 1$ by

$$\varphi_v(x - vt) = \sqrt{1 - \frac{v^2}{2}} \tanh\left(\sqrt{1 - \frac{v^2}{2}} \frac{x - vt}{\sqrt{2}}\right) + i\frac{v}{\sqrt{2}}, \tag{53}$$

see for instance [G0]. Note that for $v \neq 0$, the phase shift of the bubble differs from the black soliton one. In particular, the numerical boundary conditions $u = \pm 1$ at $x = \pm x_{max}$, ($x_{max} = Jh$) are not well-adapted to describe these bubbles. However, it is possible to compare the numerical solution with such a bubble on a subdomain $D \subset] - x_{max}, x_{max}[$: first, up to a multiplication by a constant phase term $\exp i\theta$, the imaginary part of $\exp i\theta\, u_j^n$ is observed to be constant, say equal to γ (that does not depend on time), on D. The numerical value of the velocity is then deduced from (53) by $v = \sqrt{2}\gamma$ and can be compared to the velocity of the solution deduced from Figure 10 by the relation $v = \Delta d/\Delta t$, where Δd stands for the distance covered by the bubble between two local extrema of the L^∞ profile (taking into account that the distance h is covered during a period) and Δt is the corresponding time. For instance, the computations made for perturbation (51) with $q = 3\ 10^{-4}$ give $\sqrt{2}\gamma \simeq 2.086\ 10^{-4}$ and show a good agreement with $\Delta d/\Delta t \simeq 2.083\ 10^{-4}$ (the values are chosen in such a way that $\Delta t \sim T$ in the plot). For $\sigma > 1$, the explicit form of travelling bubbles is not known but it is still possible to compute the velocity v of the bubble from the minimum intensity η_0 of the solution: indeed, setting

$$V_v(r) := V(r) - v^2 \frac{(r-1)^2}{4r},$$

η_0 is the unique zero of V_v on the interval $(0,1)$ (see [G1]); consequently, v expresses as

$$v = 2\frac{\sqrt{V(\eta_0)\eta_0}}{1-\eta_0}, \tag{54}$$

the value η_0 being determined from the numerical solution. Starting from the same perturbation as in the previous test for the resolution of (31) with $\sigma = 2$, (54) gives $v \simeq 2.77\ 10^{-4}$ which is again very close to the observed velocity that has been found close to $2.74\ 10^{-4}$. Other experiments have shown that this still holds for other values of σ and other initial perturbations. Furthermore, the velocity of the numerical travelling bubble essentially depends on the perturbation profile at the origin.

7.4 The saturated equation

As in the previous section, we first investigate the linearized equation (50), starting from (51) as initial data. We used the following parameters : $J = 3\ 10^5$, $h = 1/3\ 10^{-2}$, $\delta t = 10$ and $N = 1000$. The choice of such a δt is justified by the smallness of the nonlinearity which prescribes large characteristic time scales. We also plot in Figures 13, 14, 15 and 16 snapshots of the real and imaginary parts of the solution of (50) at different times in the two cases $a = 7.4$ and $a = 7.6$ (note that $7.4 < a_{VK} < 7.6$). It can be observed that in the case $a = 7.4$, the imaginary part spreads out in the numerical domain but keeps the same order of magnitude through time. The solution we obtain here closely looks like the derivative of the black soliton, which is not that surprising since $L_2\varphi' = 0$, meaning that φ' is a stationary solution to (50). On the contrary, the profile of the imaginary part obtained for $a = 7.6$ is amplified without any alteration of its shape, after a transient regime at moderate times. This suggests that in the latter case, an unstable mode has been captured from initial data (51). It points out the existence of a threshold value a_l for which we observe both dynamical linear instability if $a > a_l$ and stability if $a < a_l$. Note that $a_l \simeq a_{VK}$.

In order to study more precisely the amplification rate of the solution, k has been plotted in the cases $a = 7$, $a = 8$ and $a = 9$ (see Figure 17). It gives another illustration of linear instability of the black soliton above a_l. In figure 18, we have plotted $k_\infty := \lim_{t\to\infty} k(t)$ versus a for $a > a_l$. In each case, this limit has been estimated by extrapolation from computations until $t = 10^4$. It has to be pointed out that computing the value of the instability growth rate is delicate when a becomes too close to the critical threshold: the

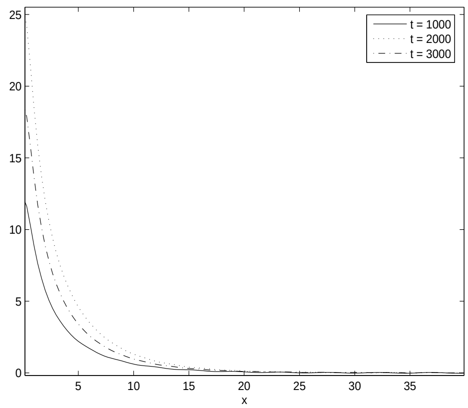

Figure 13: Plot of the real part at different times, $a = 7.4$.

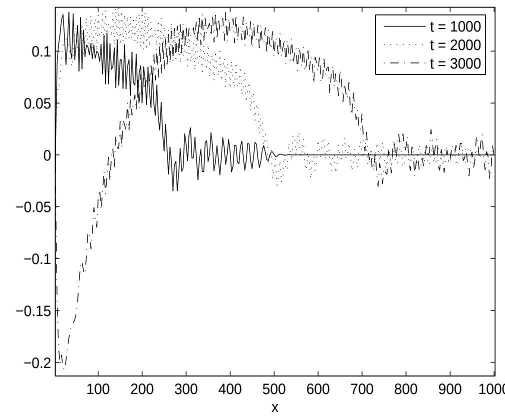

Figure 14: Plot of the imaginary part at different times, $a = 7.4$.

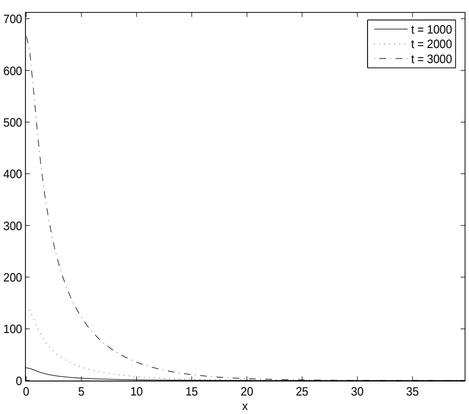

Figure 15: Plot of the real part at different times, $a = 7.6$.

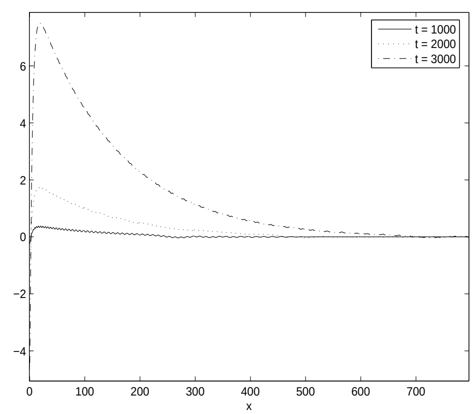

Figure 16: Plot of the imaginary part at different times, $a = 7.6$.

code only gives an approximate solution that shares the same properties of the exact one up to a numerical tolerance. Moreover, the nonlinearity is always very small for values of a close to a_l, which requires our experiments to be made for sufficiently large times in such a way that the nonlinear contribution can be observed.

We also solved Equation (32) starting from the Gaussian modulated initial perturbation (52). In our computations, we chose $\delta t = 1$, $J = 20000$, $h = 0.01$, $q = 3.10^{-4}$ in the cases $a = 5$ and $a = 9$ (see Figures 19 and 20). For $a = 5$, it can be noticed that the profile of the real part cannot be distinguished from the black soliton. However, the solution has been translated to the right: the real part now vanishes at $x = 0.12$. In the case $a = 9$, the situation completely changes: the translation to the right is now much faster and the profile of the translated real part differs from the black soliton.

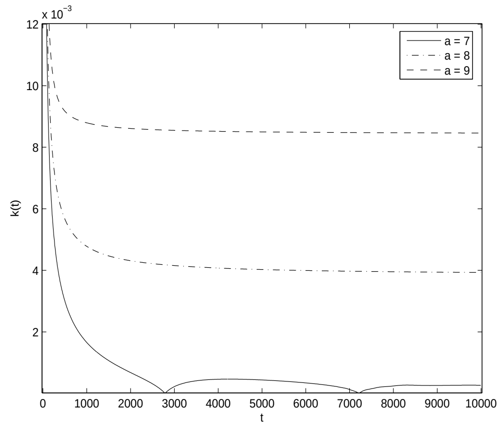

Figure 17: Plot of k versus time, $a = 7$, $a = 8$ and $a = 9$.

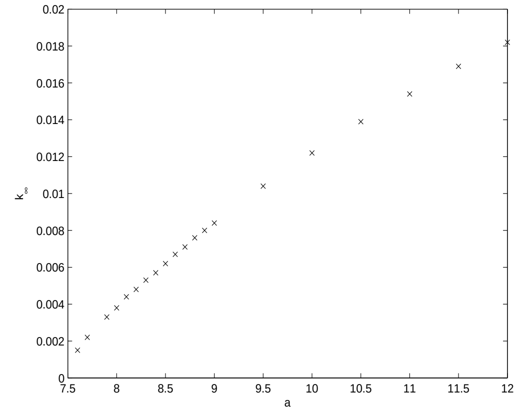

Figure 18: Plot of k_∞ as a function of a.

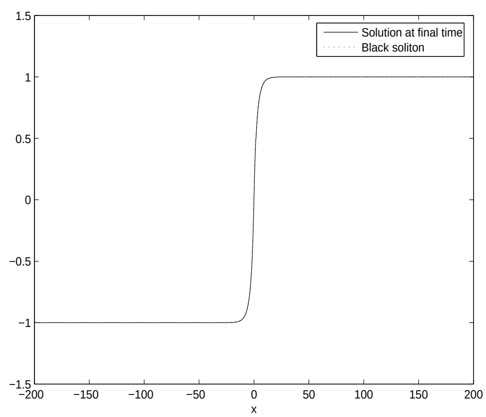

Figure 19: Profile of the real part of the solution at final time $T = 1000$ compared with the black soliton, $a = 5$.

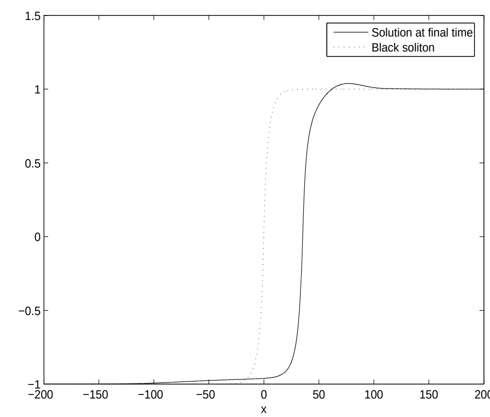

Figure 20: Profile of the real part of the solution at final time $T = 1000$ compared with the black soliton, $a = 9$.

In order to see more precisely the influence of parameter a, we view in Figure 21 on the same plot the errors between the black soliton and the translated solution of (32) obtained for different values of a: $a = 6$, $a = 7$, $a = 8$ and $a = 9$, with the same values of both grid parameters and initial perturbation. As in the test performed in the linearized case, this figure suggests the occurrence of a threshold value $a_{nl} \in (7, 8)$: taking $a < a_{nl}$ leads us to a bounded variation of the error, whereas for $a > a_{nl}$ the error seems to increase with time. Note that the value of a_{nl} is in agreement with the threshold values obtained on the one hand with the Vakhitov-Kolokolov function (a_{VK}) and on the other hand with the study of the linearized system (a_l). In the cases where the translated profiles do not differ much from the black soliton (that is for $a = 6$ and $a = 7$), the oscillations that have been already observed in the pure power case (see Figure 10) remain visible for large times.

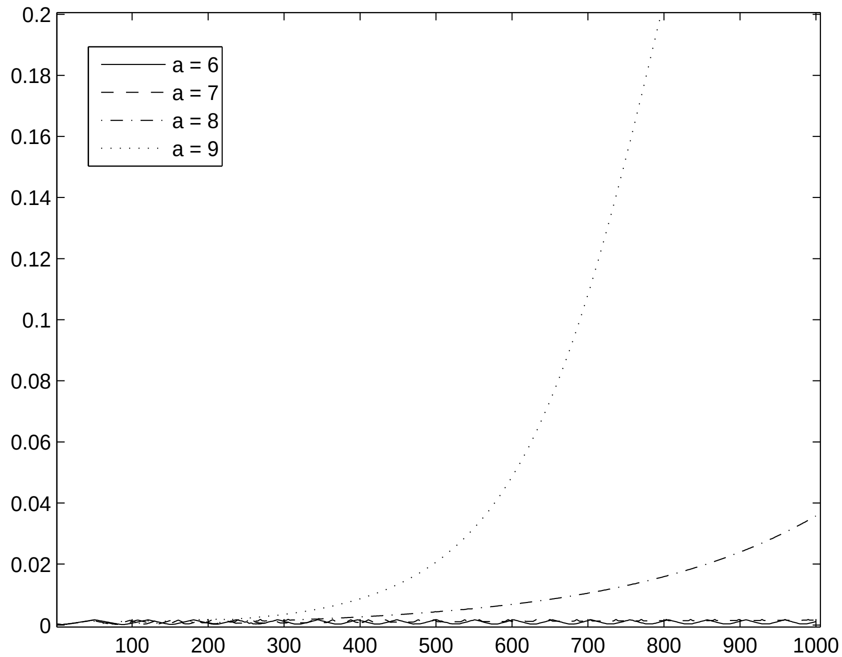

Figure 21: Plot of the error between the black soliton and the translated solution of (32) $a = 6$, $a = 7$, $a = 8$ and $a = 9$.

In the stable case, the moving soliton can be identified as a travelling bubble, as for the pure power nonlinearity. Following the same strategy as before, we have measured the velocity $v = 9.86\ 10^{-5}$ of the solution for $a = 2$, which gives a good agreement with respect to the velocity $v = 9.73\ 10^{-5}$ obtained by (54).

7.5 Conclusion

We have investigated the stability of black solitons for different nonlinearities using different points of view. First, the Vakhitov-Kolokolov function g has been computed numerically. Thanks to Theorem 1.1, linear stability holds if and only if $g(\lambda)$ has a negative limit when λ tends to zero. Secondly, the linearized equation has been solved starting from a perturbation of the black soliton. Linear instability manifests itself by the emergence of an exponentially increasing unstable mode. Finally, the fully nonlinear evolution has been computed and has shown that even in the linearly stable cases, the black soliton may be translated with a velocity depending on the initial perturbation and behaves like a travelling bubble on a spreading domain. The evaluation of the error between the computed solution and the initial data does not seem well-adapted for the stability analysis. The stability has to be studied modulo translations: this is exactly orbital stability.

Our numerical simulations bring to the fore that in the pure power case, the black soliton is linearly stable and is also orbitally stable. In the saturated equation, these three approaches tend to exhibit the same threshold value $a_{VK} \simeq a_l \simeq a_{nl}$ for the parameter a. Let us denote by a_c this threshold value. If $a < a_c$, the black soliton is stable in all the above mentioned meanings, whereas instability occurs when $a > a_c$. The need to investigate orbital stability instead of stability in usual sense may be surprising, taking into account previous results in the literature concerning travelling bubbles for the $\psi^3 - \psi^5$ nonlinear Schrödinger equation (see [BP], [dB]).

8 Appendix

We prove here some technical lemmas which have been used in section 4. All these proofs are slightly modified versions of proofs given in [GJLS].

Proof of Lemma 4.1. Note first that since $\lambda \notin i\mathbb{R}$, $-\lambda^2 \notin \mathbb{R}_+ = \sigma(-\partial_x^2(-\partial_x^2 + c^2))$. Thus the operator $\lambda^2 - \partial_x^2(-\partial_x^2 + c^2)$ is invertible, and B_λ^{-1} may be expressed as

$$B_\lambda^{-1} = \begin{pmatrix} \lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} & -\partial_x^2(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} \\ -(-\partial_x^2 + c^2)(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} & \lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1} \end{pmatrix}. \tag{55}$$

Thus the lemma reduces to prove uniform bounds on each four operators on L^2 in the right-hand side of (55). Passing into Fourier variables, we get

$$\|\lambda(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1}\|_{\mathcal{L}(L^2)} \leqslant \sup_{\xi\in\mathbb{R}} \frac{|\lambda|}{|\lambda^2 + \xi^2(\xi^2 + c^2)|},$$

$$\| - \partial_x^2(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1}\|_{\mathcal{L}(L^2)} \leqslant \sup_{\xi\in\mathbb{R}} \frac{\xi^2}{|\lambda^2 + \xi^2(\xi^2 + c^2)|}$$

and

$$\|(-\partial_x^2 + c^2)(\lambda^2 - \partial_x^2(-\partial_x^2 + c^2))^{-1}\|_{\mathcal{L}(L^2)} \leqslant \sup_{\xi\in\mathbb{R}} \frac{\xi^2 + c^2}{|\lambda^2 + \xi^2(\xi^2 + c^2)|}.$$

Next,

$$\sup_{\xi\in\mathbb{R}} \frac{|\lambda|}{|\lambda^2 + \xi^2(\xi^2 + c^2)|} \leqslant \frac{|\lambda|}{|\mathfrak{Im}(\lambda^2)|} = \frac{(a^2 + \tau^2)^{1/2}}{2|a||\tau|} \leqslant \frac{(a^2 + 1)^{1/2}}{2|a|}.$$

Similarly,

$$\frac{c^2}{|\lambda^2 + \xi^2(\xi^2 + c^2)|} \leqslant \frac{c^2}{2|a||\tau|} \leqslant \frac{c^2}{2|a|}.$$

Finally, if $\xi^2(\xi^2 + c^2) \geqslant 2\tau^2$, we also have $\xi^2(\xi^2 + c^2) - \tau^2 \geqslant \xi^2(\xi^2 + c^2)/2$, and

$$\frac{\xi^2}{((a^2+\xi^2(\xi^2+c^2)-\tau^2)^2+4a^2\tau^2)^{1/2}} \leqslant \frac{\xi^2}{a^2+\xi^2(\xi^2+c^2)/2} \leqslant C,$$

while if $\xi^2(\xi^2+c^2) \leqslant 2\tau^2$,

$$\frac{\xi^2}{((a^2+\xi^2(\xi^2+c^2)-\tau^2)^2+4a^2\tau^2)^{1/2}} \leqslant \frac{\xi^2}{2|a||\tau|} \leqslant \frac{1}{2|a|}\left(\frac{\xi^2(\xi^2+c^2)}{\tau^2}\right)^{1/2} \leqslant \frac{\sqrt{2}}{2|a|},$$

which completes the proof of the lemma. □

Proof of Lemma 4.2. We first prove the result for $j=3$. We write

$$\begin{aligned} \|g_\lambda^3\|_{L^q}^q &= 2\int_0^\infty \frac{(x^2+c^2)^q}{[(a^2+x^2(x^2+c^2)-\tau^2)^2+4a^2\tau^2]^{q/2}}dx \\ &= 2(I_1+I_2), \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_0^{c/\sqrt{2}} \frac{(x^2+c^2)^q}{[(a^2+x^2(x^2+c^2)-\tau^2)^2+4a^2\tau^2]^{q/2}} \\ &\leqslant \int_0^{c/\sqrt{2}} \frac{(3c^2/2)^q}{(2|a||\tau|)^q}dx \underset{|\tau|\to\infty}{\longrightarrow} 0, \end{aligned}$$

and

$$I_2 := \int_{c/\sqrt{2}}^\infty \frac{1}{(2|a||\tau|)^q} \frac{(x^2+c^2)^q}{\left[1+\left(\frac{a^2+x^2(x^2+c^2)-\tau^2}{2|a||\tau|}\right)^2\right]^{q/2}}dx.$$

The change of variable $y=\frac{x^2(x^2+c^2)+a^2-\tau^2}{2|a||\tau|}$ yields, setting $\tau'=\frac{\tau^2+c^4/4-a^2}{2|a||\tau|}$, if $\tau^2 \geqslant a^2-c^4/4+1$,

$$I_2 = \frac{1}{4}\int_{\frac{c^4}{2|a||\tau|}-\tau'}^\infty \frac{1}{(2|a||\tau|)^{q-1}} \frac{(\tau^2+c^4/4-a^2)^{q/2-3/4}\left((\frac{y}{\tau'}+1)^{1/2}+\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^q}{(1+y^2)^{q/2}\left((\frac{y}{\tau'}+1)^{1/2}-\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^{1/2}(\frac{y}{\tau'}+1)^{1/2}}dy.$$

Next, $y \geqslant \frac{c^4}{2|a||\tau|}-\tau'$ implies $(\frac{y}{\tau'}+1)^{1/2} \geqslant \frac{c^2}{(\tau^2+c^4/4-a^2)^{1/2}}$, thus for $q>1$,

$$\begin{aligned}
I_2 &\leqslant \frac{1}{4}\frac{(\tau^2+c^4/4-a^2)^{q/2-3/4}}{(2|a||\tau|)^{q-1}}\int_{-\tau'}^{\infty}\frac{\left(\frac{3}{2}\left(\frac{y}{\tau'}+1\right)^{1/2}\right)^q}{(1+y^2)^{q/2}\left(\frac{1}{2}\left(\frac{y}{\tau'}+1\right)^{1/2}\right)^{1/2}\left(\frac{y}{\tau'}+1\right)^{1/2}}dy\\
&\leqslant C|\tau|^{-1/2}\int_{-\tau'}^{\infty}\frac{\left(\frac{y}{\tau'}+1\right)^{q/2-3/4}}{(1+y^2)^{q/2}}dy\\
&\leqslant C|\tau|^{-1/2}\left(\int_{-\tau'}^{\tau'}\frac{dy}{(1+y^2)^{q/2}}+\int_{\tau'}^{\infty}\frac{y^{q/2-3/4}}{\tau'^{q/2-3/4}(1+y^2)^{q/2}}dy\right)\\
&\leqslant C|\tau|^{-1/2}\left(1+|\tau'|^{1-q}\int_1^{\infty}y^{-q/2-3/4}dy\right)\underset{|\tau|\to\infty}{\longrightarrow}0.
\end{aligned}$$

We now perform similar computations for $j=2$: if $\tau\geqslant\tau_0$ where $\tau_0\geqslant|a^2-c^4/4|^{1/2}$ is large enough, in such a way that for every $\tau\geqslant\tau_0$, $c^4/(8|a||\tau|)-\tau'<-\tau'/2<0$ and $1/\sqrt{2}>c^2/(\tau^2+c^4/4-a^2)^{1/2}$, we compute

$$\begin{aligned}
\|g_\lambda^2\|_{L^q}^q &= 2\int_0^{\infty}\frac{(a^2+\tau^2)^{q/2}}{[(a^2+x^2(x^2+c^2)-\tau^2)^2+4a^2\tau^2]^{q/2}}dx\\
&= 2\frac{(a^2+\tau^2)^{q/2}}{(2|a||\tau|)^{q-1}}\frac{1}{4}\int_{\frac{c^4}{8|a||\tau|}-\tau'}^{\infty}(\tau^2+c^4/4-a^2)^{-3/4}(1+y^2)^{-q/2}\\
&\qquad\times\left(\left(\frac{y}{\tau'}+1\right)^{1/2}-\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^{-1/2}\left(\frac{y}{\tau'}+1\right)^{-1/2}dy\\
&= \frac{(a^2+\tau^2)^{q/2}}{2(2|a||\tau|)^{q-1}(\tau^2+c^4/4-a^2)^{3/4}}(J_1+J_2),
\end{aligned}$$

where

$$\begin{aligned}
J_2 &:= \int_{-\tau'/2}^{\infty}\frac{dy}{(1+y^2)^{q/2}\left(\left(\frac{y}{\tau'}+1\right)^{1/2}-\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^{1/2}\left(\frac{y}{\tau'}+1\right)^{1/2}}\\
&\leqslant C\int_{-\infty}^{\infty}\frac{dy}{(1+y^2)^{q/2}}
\end{aligned}$$

and

$$\begin{aligned}
J_1 &:= \int_{\frac{c^4}{8|a||\tau|}-\tau'}^{-\tau'/2}\frac{dy}{(1+y^2)^{q/2}\left(\left(\frac{y}{\tau'}+1\right)^{1/2}-\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^{1/2}\left(\frac{y}{\tau'}+1\right)^{1/2}}\\
&\leqslant \int_{\frac{c^4}{8|a||\tau|}-\tau'}^{-\tau'/2}\frac{1}{\left(1+\left(\frac{\tau'}{2}\right)^2\right)^{q/2}}\frac{dy}{\left(\left(\frac{y}{\tau'}+1\right)^{1/2}-\frac{c^2}{2(\tau^2+c^4/4-a^2)^{1/2}}\right)^{1/2}\left(\frac{y}{\tau'}+1\right)^{1/2}}
\end{aligned}$$

We make the change of variables $z = \left(\frac{y}{\tau'} + 1\right)^{1/2} - \frac{c^2}{2(\tau^2 + c^4/4 - a^2)^{1/2}}$, in such a way that for $|\tau| \geqslant \tau_0$,

$$J_1 \leqslant C|\tau|^{-q} \int_0^{1/\sqrt{2}} \frac{|\tau'| dz}{z^{1/2}} \leqslant C|\tau|^{1-q},$$

which completes the proof of Lemma 4.2. □

References

[B] I.V. Barashenkov, *Stability criterion for dark solitons*, Phys. Rev. Lett., 77 (1996), 1193-1197.

[BGMP] I.V. Barashenkov, A.D. Gocheva, V.G. Makhankov, I.V. Puzynin, *Stability of the soliton-like "bubbles"*, Physica D, 34 (1989), 240-254.

[BP] I.V. Barashenkov, E.Y. Panova, *Stability and evolution of the quiescent and traveling solitonic bubbles*, Physica D, 69 (1993), 114-134.

[BS] F.A. Berezin, M.A. Shubin, *The Schrödinger equation*, Kluwer Academic Publishers (1991).

[dB] A. de Bouard, *Instability of stationnary bubbles*, SIAM J. Math. Anal. 26 (3) (1995) 566-582.

[G0] C. Gallo, *Propriétés qualitatives d'ondes solitaires solutions d'EDP non linéaires dispersives*, Ph. D. Thesis, Paris-Sud Orsay University (2005).

[G1] C. Gallo, *The dark solitons of the one-dimensional nonlinear Schrödinger equation*, submitted.

[G2] C. Gallo, *The Cauchy Problem for defocusing Nonlinear Schrödinger equations with non-vanishing initial data at infinity*, submitted.

[GJLS] F. Gesztesy, C.K.R.T. Jones, Y. Latushkin, M. Stanislavova, *A Spectral Mapping Theorem and Invariant Manifolds for Nonlinear Schrödinger Equations*, Indiana University Math. Journal 49 (2000) 221-243.

[H] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics 840, Springer, Berlin (1981), p. 136.

[HPW] D.B. Henry, J.F. Perez, W.F. Wreszinski, *Stability Theory for Solitary-Wave Solutions of Scalar Field Equations*, Commun. Math. Phys. 85 (1982), 351-361.

[KK] Y.S. Kivshar, W. Krolikowski, *Instabilities of dark solitons*, Opt. Lett. 20 (1995), 1527-1529.

[KLD] Y.S. Kivshar, B. Luther-Davies, *Dark solitons: physics and applications*, Physics Reports 298 (1998), 81-197.

[MS] D.J. Mitchell, A.W. Snyder, *Stability of nonlinear guided waves*, J. Opt. Soc. Am. B 10 (1993), 1572-1580.

[P] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. 284, 2 (1984), 847-857.

[L] Z. Lin, *Stability and instability of traveling solitonic bubbles*, Adv. Diff. Eq., 7 (2002), 897-918.