The Cauchy Problem for defocusing Nonlinear Schrödinger Equations with non-vanishing initial data at infinity.

Clément Gallo
UMR de Mathématiques, Bat. 425
Université Paris-Sud
91405 Orsay, France.

Abstract. For rather general nonlinearities, we prove that defocusing nonlinear Schrödinger equations in $\mathbb{R}^n$ $(n \leqslant 4)$, with non-vanishing initial data at infinity u_0, are globally well-posed in $u_0 + H^1$. The same result holds in an exterior domain in $\mathbb{R}^n$, $n = 2, 3$.

1 Introduction

This paper is devoted to the study of the Cauchy problem for defocusing nonlinear Schrödinger equations in dimensions $n \leqslant 4$:

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \end{cases}, \tag{1}$$

where $\Omega = \mathbb{R}^n$. The initial data u_0 has the boundary condition

$$|u_0(x)|^2 \to \rho_0 \text{ as } x \to \infty , \tag{2}$$

where $\rho_0 > 0$ denotes the light intensity of the background. The real-valued function f is assumed to be defocusing. Namely, f satisfies the following assumption:

$$f(\rho_0) = 0 \text{ and } f'(\rho_0) < 0. \tag{H_f}$$

Under the same condition (H_f) on f, we also study the Cauchy Problem (1) where Ω is an exterior domain in $\mathbb{R}^n$, $n = 2, 3$, with a data u_0 which satisfies the same condition at infinity (2).

Equation (1) with $\Omega = \mathbb{R}^n$ admits many particular solutions with the boundary condition (2). These solutions may be gathered under the label "dark solitons". For general nonlinearities, let us mention for instance the stationary and the travelling bubbles. A. de Bouard [dB] gave a necessary and sufficient condition on the nonlinearity ensuring the existence of a stationary bubble, in any dimension. She also proved that the stationary bubbles are all unstable (see also [BGMP], [BP]). Z. Lin [L] studied the travelling bubbles in dimension one. He gave a criterion on the variation of the momentum with respect to the speed which determinates if these bubbles are stable or not. The Gross-Pitaevskii equation, which is (1) with $f(r) = 1 - r$ (here, $\rho_0 = 1$), has been the object of a deeper study. F. Bethuel and J.C. Saut [BS] proved the existence of travelling waves for the Gross-Pitaevskii equation for small non-zero speeds, in dimension two. Similar results have been obtained by D. Chiron [C] in dimension three and more.

The existence of all these dark solitons makes relevant the study of the Cauchy problem (1) with condition (2) at infinity. For example, it is a preliminary to the study of their stability when it is not known whether these solitons are stable or not[1]. For unstable dark solitons, the study of the Cauchy Problem (1) gives informations on the way this unstability occurs: for instance, a global well posedness result prohibits blowing-up.

A first step in the study of this Cauchy Problem has been done in [BS], where it was shown that the Gross-Pitaevskii equation is globally well-posed in $1 + H^1(\mathbb{R}^n)$ for $n = 2, 3$. However, this point of vue is not relevant in all cases. Indeed, in dimension one, most of the travelling bubbles have different limits at $+\infty$ and $-\infty$. Moreover, it was shown by P. Gravejat [Gr] that the two dimensional travelling waves for the Gross-Pitaevskii equation do not belong to the space $1 + H^1$ (they do not even belong to $1 + L^2$), in spite of the fact that they tend to 1 at infinity (up to the multiplication by a constant of modulus 1).

As a consequence, we need to find a more appropriate framework to study the Cauchy Problem. In [Ga] (see also the works of P.E. Zhidkov [Z0], [Z1], [Z2], [Z3]), we worked in the Zhidkov spaces

$$X^k(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n), \nabla u \in H^{k-1}(\mathbb{R}^n)\}.$$

We proved some global well-posedness results for (1) in dimension one, with

[1] As far as we know, the only dark solitons for which a stability result has been established are the stationary bubbles (see [dB]) which are known to be unstable, the travelling bubbles in dimension 1 (see [L]) and the black solitons in dimension 1, which are stationary solutions to (1)-(2) vanishing at one point, in the contrary to the bubbles (see [DMG], [G]).

condition (2) at infinity. However, we assumed the potential

$$V(r) = \int_r^{\rho_0} f(s)ds$$

to be positive, a condition which is not satisfied for all the nonlinearities for which there exists a stationary bubble (see [dB]). In the Gross-Pitaevskii case, as for the existence of travelling waves, the Cauchy problem has been the object of deeper investigations. Using a Brezis-Gallouët method, O. Goubet [Go] proved the global well-posedness for the Gross-Pitaevskii equation in $X^2(\mathbb{R}^2)$, if the initial data has finite energy. More recently, P. Gérard [Ge] obtained a global well-posedness result for the Gross-Pitaevskii equation in dimension two and three in the energy space

$$\{u \in H^1_{\rm loc}, \nabla u \in L^2, 1-|u|^2 \in L^2\}.$$

In this paper, we generalize this results to a larger class of nonlinearities. In particular, the potential is not assumed to be positive[2]. Our main result is as follows.

Theorem 1.1 *Let $n = 1, 2, 3$ or 4, $\rho_0 > 0$, and $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ ($k = 1$ if $n = 1$, $k = 2$ if $n = 2, 3$, $k = 3$ if $n = 4$) which satisfies (H_f). We assume moreover that there exists $\alpha_1 \geqslant 1$, with the supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ (where $\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$) , and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$ such that*

$$\exists C_0 > 0,\ A > \rho_0,\ \left\{ \begin{array}{l} \forall r \geqslant 1,\ \left\{ \begin{array}{ll} |f''(r)| \leqslant C_0 r^{\alpha_1 - 3} & \text{if } n = 1,2,3 \\ |f'''(r)| \leqslant C_0 r^{\alpha_1 - 4} & \text{if } n = 4 \end{array} \right. \\ \left\{ \begin{array}{l} \text{if } \alpha_1 \leqslant 3/2,\ V \text{ is bounded from below} \\ \text{if } \alpha_1 > 3/2,\ \forall r \geqslant A,\ r^{\alpha_2} \leqslant C_0 V(r) \end{array} \right. \end{array} \right. .(H_{\alpha_1,\alpha_2})^3$$

Then for any regular function of finite energy ϕ, which means

$$\phi \in \mathcal{C}_b^{k+1}(\mathbb{R}^n),\ \nabla\phi \in H^{k+1}(\mathbb{R}^n)^n,\ |\phi|^2 - \rho_0 \in L^2(\mathbb{R}^n), \qquad (H_\phi)$$

equation (1) is globally well-posed in $\phi + H^1(\mathbb{R}^n)$. Namely, for every $w_0 \in H^1(\mathbb{R}^n)$, there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\phi + w$ solves (1),

[2] In the usual "0 at infinity" case, the boundedness of the H^1 norm may be deduced from the conservation of the energy, the conservation of the charge and a Gagliardo-Nirenberg inequality. In our case, the analoguous to the charge is the quantity $\int(|u|^2 - \rho_0)$, the conservation of which is not so clear.

[3] Remark that the first condition in ($\mathrm{H}_{\alpha_1,\alpha_2}$) implies $|V(r)| \lesssim r^{\alpha_1}$ for $r \geqslant 1$. Thus, in the case $\alpha_1 > 3/2$, ($\mathrm{H}_{\alpha_1,\alpha_2}$) may be satisfied only if $\alpha_2 \leqslant \alpha_1$, so that $\alpha_2 \in [\alpha_1 - \frac{1}{2}, \alpha_1]$. Remark also that in the case $\alpha_1 \leqslant 3/2$, α_2 plays no role.

with the initial data $w(0) = w_0$.
For any $T > 0$, the flow map $w_0 \mapsto w$, $H^1 \mapsto \mathcal{C}([0,T], H^1)$ is Lipshitz continuous on the bounded sets of H^1.
The energy

$$\mathcal{E}(w) = \int_{\mathbb{R}^n} |\nabla(\phi + w)|^2 dx + \int_{\mathbb{R}^n} V(|\phi + w|^2) dx$$

is conserved by the flow.

For a very large class of defocusing nonlinearities, assumption $(\mathrm{H}_{\alpha_1,\alpha_2})$ is satisfied for some α_1, α_2 as required in Theorem 1.1. We give here some examples.

Examples.
1. The pure powers: $f(r) = (\rho_0^p - r^p)$ where p is a positive integer if $n = 1$ or 2, $p = 1$ if $n = 3$. In that case, $V(r) \geqslant 0$ on $\mathbb{R}_+$, $V(r) \sim \frac{1}{p+1} r^{p+1}$ as $r \to \infty$ and $f''(r) = -p(p-1)r^{p-2}$. Thus $(\mathrm{H}_{\alpha_1,\alpha_2})$ is satisfied for $\alpha_1 = \alpha_2 = p + 1$.
2. Saturated nonlinearity: $f(r) = \frac{1}{(1+ar)^2} - \frac{1}{(1+a)^2}$. In this case, $f''(r) = \frac{6a^2}{(1+ar)^4}$ and $V(r) = \frac{a(1-r)^2}{(1+ar)(1+a)^2} \geqslant 0$. In particular, $(\mathrm{H}_{1,\alpha_2})$ is satisfied for any α_2.
3. The cubic-quintic case: $f(r) = (r - \rho_0)(2a + \rho_0 - 3r)$, where $0 < a < \rho_0$. Then $V(r) = (r - \rho_0)^2(r - a)$ and $f''(r) = -6$. In the contrary to the two previous examples, V is not positive on $\mathbb{R}_+$, but $(\mathrm{H}_{\alpha_1,\alpha_2})$ is satisfied for $\alpha_1 = \alpha_2 = 3$. Thus Theorem 1.1 applies in dimensions one and two (in dimension three, Theorem 1.3 below applies).

The global well-posedness for an initial data in the energy space

$$E = \{u \in H^1_{\mathrm{loc}}, \nabla u \in L^2, \rho_0 - |u|^2 \in L^2\}.$$

is a consequence of Theorem 1.1 and of the following proposition, which directly follows from the results of P. Gérard in [Ge].

Proposition 1.1 *Let $u \in E$. Then there exists $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^n) \cap E$ such that $\nabla\phi \in H^\infty(\mathbb{R}^n)^n$ and $w \in H^1(\mathbb{R}^n)$ such that $u = \phi + w$.*

From this Proposition we deduce:

Theorem 1.2 *Under the same assumptions that in Theorem 1.1, for any $u_0 \in E$, there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $u := u_0 + w$ solves (1).*

Proof. Given $u_0 \in E$, let $u_0 = \phi + w_0$ be a decomposition as in Proposition 1.1. Thanks to Theorem 1.1, there exists an unique $\tilde{w} \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\phi + \tilde{w}$ solves (1). Therefore $w = \tilde{w} - w_0$ is the unique element of $\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $u := u_0 + w$ solves (1). □

In particular, the solution $u = \phi + w$ given by Theorem 1.1 does not depend on the choice of the decomposition of $u_0 \in E$ into $\phi + w_0$.

In the critical case $\alpha_1 = \alpha_1^*$, we obtain the following local result.

Theorem 1.3 *Under the same assumptions on f and ϕ that in Theorem 1.1, if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$, there exists $R > 0$ and $T > 0$ such that, for $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, there exists a unique $w \in \mathcal{C}([0,T], H^1)$ such that $\phi + w$ solves (1).*
For that T, the flow $w_0 \mapsto w$ is locally Lipshitz continuous from the ball of radius R in H^1 into $\mathcal{C}([0,T], H^1)$.
The energy is conserved on $[0,T]$.

Similarly to the sub-critical case, we deduce from Theorem 1.3 and Proposition 1.1 the following result.

Theorem 1.4 *Under the same assumptions on f that in Theorem 1.3, if $u_0 \in E$ satisfies (H_ϕ), there exists $T(u_0) > 0$ and a unique $w \in \mathcal{C}([0,T], H^1)$ such that $u_0 + w$ solves (1).*

It was shown in [Ge] that in dimensions two and three, the Gross-Pitaevskii equation is globally well-posed in the energy space E endowed with a structure of complete metric space by the distance

$$d_E(u,v) = \|u - v\|_{X^1 + H^1} + \||u|^2 - |v|^2\|_{L^2}.$$

It is quite clear that for any $T > 0$ and $u_0 \in E$, $u_0 + \mathcal{C}([0,T], H^1)$ is strictly included in $\mathcal{C}([0,T], E)$. In particular, P. Gérard obtained in [Ge] the uniqueness of the solutions to (1) in a bigger space than in Theorem 1.2. Using some of the arguments developed in [Ge] we get the uniqueness in the energy space for other non-linearities than Gross-Pitaevskii. More precisely, we have the following result.

Theorem 1.5 *Let $n = 2, 3, 4$. Under the assumptions of Theorem 1.1, let $T > 0$, $u_0 \in E$ and $u \in \mathcal{C}([0,T], E)$ be a solution of (1) with $u(0) = u_0$. Then $u - u_0 \in \mathcal{C}([0,T], H^1)$, and therefore u is the solution of (1) given by Theorem 1.2.*

Remarks.

1. In Theorem 1.3, $R = R(\phi)$ depends on ϕ. For general $u_0 \in E$, it is not clear whether we can find a function ϕ which satisfies (H_ϕ) and such that $w_0 = u_0 - \phi \in H^1$ has H^1-norm less than $R(\phi)$. That is why we need to assume that u_0 satisfies (H_ϕ) in Theorem 1.4.

2. In dimension 4, the Gross-Pitaevskii equation is critical (that is $\alpha_1 = \alpha_1^*$). In [Ge], P. Gérard proved that the four-dimensional Gross-Pitaevskii equation is globally well-posed in the energy space E, provided the initial condition u_0 has small energy. In Theorem 1.3, we prove that for critical non-linearities (and in particular for Gross-Pitaevskii in dimension 4), (1) is locally well-posed in $u_0 + H^1$, for any regular initial condition u_0 in the energy space, without the smallness assumption on the energy. However, we do not obtain any global well-posedness result, and we only prove the local Lipshitz continuity of the flow on small intervals of time. The missing argument is a persistency result. The reason of this missing is that for general non-linearities (in particular when the potential V is non-positive), the conservation of the energy does not imply the conservation of the smallness of w in H^1.

Our proof of Theorem 1.1 consists in looking for a solution of (1) under the form $\phi + w$. Thus the equation satisfied by w writes

$$\begin{cases} i\frac{\partial w}{\partial t} + \Delta w = F(w(t)) \\ w(0) = w_0 \end{cases}, \tag{3}$$

where

$$F(w) = -\Delta\phi - f(|\phi + w|^2)(\phi + w). \tag{4}$$

We prove that (3) is locally well-posed in $H^k(\mathbb{R}^n)$, $k = 1$ for $n = 1$, $k = 2$ for $n = 2, 3$ and $k = 3$ for $n = 4$, and we give estimations for the H^1-norm of w on the interval of existence in H^k. Next, for $n = 2, 3$ or 4, using Strichartz inequalities, we prove that (3) is locally well-posed in H^1, and globally in H^k. Finally, we approximate our (local) H^1 solution by the (global) H^k solution, and we deduce that its H^1-norm may not blow up on bounded intervals of time.

Using the Strichartz inequalities obtained by N. Burq, P. Gérard and N. Tzvetkov in [BGT], the same method gives similar results, in dimensions two and three, when $\mathbb{R}^n$ is replaced by an exterior domain $\Omega = \mathbb{R}^n \backslash K$, with either Dirichlet or Neumann boundary conditions. More precisely, we consider the initial value problem

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta_D u + f(|u|^2)u = 0, \ (t, x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \in E_D \end{cases}, \tag{5}$$

where the initial condition u_0 belongs to the energy space with Dirichlet boundary conditions

$$E_D := \{u \in H^1_{\text{loc}}(\Omega), \nabla u \in L^2(\Omega), \rho_0 - |u|^2 \in L^2(\Omega), \chi u \in H^1_0(\Omega)\}.$$

Here, $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\chi \equiv 1$ in a neighborhood V of the obstacle K. We also consider

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta_N u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \Omega \\ u(0) = u_0 \in E_N \end{cases}, \qquad (6)$$

where the initial condition u_0 belongs to the energy space with Neumann boundary conditions

$$E_N := \{u \in H^1_{\text{loc}}(\Omega), \nabla u \in L^2(\Omega), \rho_0 - |u|^2 \in L^2(\Omega), \chi u \in H^1_N(\Omega)\}.$$

The result we prove is as follows.

Theorem 1.6 *Let $n = 2$ or 3, and $\Omega \subset \mathbb{R}^n$ be the exterior domain of a smooth, compact, non-trapping, non-empty obstacle K, and $f \in \mathcal{C}^3(\mathbb{R}_+)$ which satisfies (H_f). We assume moreover that there exists $\alpha_1 \geqslant 1$, $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$ such that (H_{α_1,α_2}) is true. If $n = 3$, we assume moreover $\alpha_1 < 2$.*
Then, for every $u_0 \in E_D$ (resp. E_N), there exists an unique $w \in \mathcal{C}(\mathbb{R}, H^1_0(\Omega))$ (resp. $\mathcal{C}(\mathbb{R}, H^1_N(\Omega))$) such that $u_0 + w$ solves (5) (resp. (6)).
Given $\psi \in E_D$ (resp. E_N), for any $T > 0$, the flow map $w_0 \mapsto w$, $H^1_0 \mapsto \mathcal{C}([0,T], H^1_0)$ (resp $H^1_N \mapsto \mathcal{C}([0,T], H^1_N)$), where $w(0) = w_0$ and $\psi + w$ solves (5) (resp. (6)), is Lipshitz continuous on the bounded sets of H^1_0 (resp H^1_N).
The energy is conserved by the flow.

In the critical case $n = 3$, $\alpha_1 = 2$, we obtain:

Theorem 1.7 *Let $\Omega \subset \mathbb{R}^3$ be the exterior domain of a smooth, compact, non-trapping, non-empty obstacle K, and $f \in \mathcal{C}^3(\mathbb{R}_+)$ which satisfies (H_f). We assume moreover that there exists $\alpha_2 \in [3/2, 2]$ such that (H_{2,α_2}) is true.*
Then, for every ϕ satisfying

$$\phi \in \mathcal{C}^\infty_b(\Omega), \ \nabla\phi \in H^\infty(\Omega), \ Supp\phi \subset \Omega\backslash(V \cap \Omega), \ |\phi|^2 - \rho_0 \in L^2(\Omega)$$

(in particular, $\phi \in E_D \cap E_N$), there exists $R > 0$ and $T > 0$ such that for every $w_0 \in H^1_0$ (resp. H^1_N) with $\|w_0\|_{H^1(\Omega)} \leqslant R$, there exists an unique $w \in \mathcal{C}([0,T], H^1_0(\Omega))$ (resp. $\mathcal{C}([0,T], H^1_N(\Omega))$) such that $w(0) = w_0$ and $\phi + w$ solves (5) (resp. (6)).
For that T, the flow $w_0 \mapsto w$ is locally Lipshitz continuous from the ball of radius R in $H^1_0(\Omega)$ (resp. $H^1_N(\Omega)$) into $\mathcal{C}([0,T], H^1_0(\Omega))$ (resp. $\mathcal{C}([0,T], H^1_N(\Omega))$).
The energy is conserved by the flow.

Remark. In the case of an exterior domain, we only obtain uniqueness results in spaces like $u_0+\mathcal{C}([0,T],H_0^1)$, and not in $\mathcal{C}([0,T],E_D)$ (the continuity in E_D should be understood in the sense of the analogous to the distance d_E for an exterior domain). Indeed, even for the linear Schrödinger equation, the well-posedness in the energy space is not that clear.

Notations. If $m \in [0,\infty]$, $\mathcal{C}_b^m(\mathbb{R}^n)$ denotes the space of bounded functions of class $\mathcal{C}^m$ on $\mathbb{R}^n$.
We denote $H^\infty(\mathbb{R}^n) = \bigcap\limits_{s\geqslant 0} H^s(\mathbb{R}^n)$.
The notation $A \lesssim B$ means that there exists a harmless constant $C>0$ such that $A \leqslant CB$.
If $T>0$, $p,q \geqslant 1$, $L_T^pL^q$ denotes the Banach space $L^p([0,T],L^q)$ equipped with its natural norm.
If $p \in [1,\infty]$, we denote by $p' = \frac{p}{p-1}$ its conjugate exponent.

The structure of this paper is as follows. In section 2, we prove that (3) is locally well-posed in a space $H^k(\mathbb{R}^n)$ with k large. In section 3, we give an estimation on the H^1 norm of this solution on its maximal interval of existence in H^k. In section 4, thanks to a fixed point argument in $\mathcal{C}([0,T],H^1)$ and Strichartz estimates, we prove that (3) is locally well-posed in H^1. In section 5, we prove a persistence result and obtain the global well-posedness of equation (3) in H^1, in the sub-critical case. Section 6 is devoted to the proofs of Proposition 1.1 and Theorem 1.5. In that section, most of the arguments are due to P. Gérard (see [Ge]). In section 7, we adapt the method to the case of an exterior domain in $\mathbb{R}^n$, $n=2,3$. Section 8 is devoted to the proof of some technical lemmas concerning the L^p+L^q spaces, stated and used in section 6.

2 Local theory for regular solutions

Lemma 2.1 *We assume* $(n,k)=(1,1),\ (2,2),\ (3,2)$ *or* $(4,3)$, $f \in \mathcal{C}^k(\mathbb{R}_+)$ *satisfies* (H_f), ϕ *satisfies* (H_ϕ). *Then* F *maps* $H^k(\mathbb{R}^n)$ *into itself.*

Proof. Let $w \in H^k(\mathbb{R}^n)$ and $\phi \in L^\infty$. Then $\phi + w \in L^\infty$ because of the Sobolev embedding $H^k(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. Since f and f' are continuous, it follows that $f(|\phi+w|^2)$, $f'(|\phi+w|^2) \in L^\infty$. Next, we write

$$\begin{aligned} f(|\phi+w|^2)(\phi+w) &= (|\phi|^2-\rho_0)\int_0^1 f'(\rho_0+s(|\phi|^2-\rho_0))ds(\phi+w) \\ &\quad +2\mathfrak{Re}\left[w\int_0^1 \overline{\phi+sw}f'(|\phi+sw|^2)ds\right](\phi+w)\ . \end{aligned} \tag{7}$$

Using (H_ϕ), it is easy to see that the right-hand side in (7) belongs to H^k. Thus $F(w) \in H^k$, because $\Delta\phi \in H^k$. □

Lemma 2.2 *We assume* $(n,k)=(1,1)$, $(2,2)$, $(3,2)$ *or* $(4,3)$, $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ *satisfies* (H_f), ϕ *satisfies* (H_ϕ). *Then* $F: H^k(\mathbb{R}^n) \mapsto H^k(\mathbb{R}^n)$ *is locally Lipshitz continuous.*

Proof. Let us take $R>0$ and $w_1, w_2 \in H^k$ such that $\|w_1\|_{H^k}, \|w_2\|_{H^k} \leqslant R$. Then

$$\begin{aligned} F(w_1)-F(w_2) &= \int_0^1 \Big[f(|\phi+w_1+s(w_2-w_1)|^2)(w_2-w_1) \\ &\quad +2\mathfrak{Re}\left[(w_2-w_1)\overline{\phi+w_1+s(w_2-w_1)}\right] \\ &\quad \times f'(|\phi+w_1+s(w_2-w_1)|^2)(\phi+w_1+s(w_2-w_1))\Big]ds \end{aligned} \tag{8}$$

Next, for all $x \in \mathbb{R}$,

$$|\phi(x)+w_1(x)+s(w_2(x)-w_1(x))| \leqslant \|\phi\|_{L^\infty}+2CR\ ,$$

where C is the norm of the continuous Sobolev embedding $H^k \subset L^\infty$. Thus there exists a constant $C(R)>0$ such that $\|f^{(\alpha)}(|\phi+w_1+s(w_2-w_1)|^2)\|_{L^\infty} \leqslant C(R)$, for $\alpha = 0,..,k+1$. Using again Sobolev embeddings $H^k(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, as well as $H^1(\mathbb{R}^n) \subset L^4(\mathbb{R}^n)$ for $n=2$ or 3, $H^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4)$ and $H^2(\mathbb{R}^4) \subset L^p(\mathbb{R}^4)$ for every $p \in [2,\infty)$, it follows from (8) and its differentiation that

$$\|F(w_1)-F(w_2)\|_{H^k} \leqslant \tilde{C}(R)\|w_1-w_2\|_{H^k},$$

where $\tilde{C}(R)$ only depends on R, and not on w_1, w_2. □

Once these two lemmas have been established, we can apply the classical results of the theory of nonlinear evolution equations (see for instance [P], Theorems 6.1.4 and 6.1.5, and [CH]). We deduce the following local well-posedness result:

Theorem 2.1 *$(n,k)=(1,1),\ (2,2),\ (3,2)$ or $(4,3)$, $f\in\mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), ϕ satisfies (H_ϕ). For every $w_0\in H^k(\mathbb{R}^n)$, there exists $T^*(w_0)>0$ such that (3) has a unique mild solution $w\in\mathcal{C}([0,T^*),H^k(\mathbb{R}^n))$, which means that $w(t)=e^{it\Delta}w_0-i\int_0^t e^{i(t-s)\Delta}F(w(s))ds$, where $e^{it\Delta}$ denotes the Schrödinger group. If $T^*<\infty$, then $\|w(t)\|_{H^k}\uparrow+\infty$ as $t\uparrow T^*$. Moreover, the map $T^*:H^k\mapsto\mathbb{R}_+^*$ is semi continuous from below, and if $w_0\in H^{k+2}(\mathbb{R}^n)$, w is a classical solution to (3), which means that $w\in\mathcal{C}([0,T^*),H^{k+2}(\mathbb{R}^n))\cap\mathcal{C}^1((0,T^*),H^k(\mathbb{R}^n))$.*

3 Estimate on the H^1 norm for regular solutions

We next prove under the supplementary assumption ($\mathrm{H}_{\alpha_1,\alpha_2}$) for some $\alpha_1\geqslant 1$, $\alpha_2\in[\alpha_1-1/2,\alpha_1]$ that the norm of $w(t)$ in $H^1(\mathbb{R}^n)$ (where w is the solution of (3) given by Theorem 2.1) can not blow up on $[0,T^*(w_0))$. In particular, in the one-dimensional case, this result and Theorem 2.1 imply that w is global. Namely, for every $w_0\in H^1(\mathbb{R})$, $T^*(w_0)=+\infty$, and Theorem 1.1 is proven in the case $n=1$. We first prove that the energy is conserved on $[0,T^*(w_0))$.

Lemma 3.1 *Let $(n,k)=(1,1),\ (2,2),\ (3,2)$ or $(4,3)$, $f\in\mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), ϕ satisfies (H_ϕ), $w_0\in H^k(\mathbb{R}^n)$. Then for every $t\in[0,T^*(w_0))$, the energy*

$$\mathcal{E}(t)\ :=\ \|\nabla\phi+\nabla w(t)\|_{L^2}^2+\int_{\mathbb{R}^n}V(|\phi(x)+w(t,x)|^2)dx \tag{9}$$

is conserved: $\mathcal{E}(t)\equiv\mathcal{E}(0)=:\mathcal{E}_0$, where $V(r):=\int_r^{\rho_0}f(s)ds$.

Proof. It suffices to prove Lemma 3.1 for $w_0\in H^{k+2}$. Indeed, once this is established, the lower semi-continuity of T^*, the continuity of the flow $w_0\mapsto w(t)$ from H^k into H^k for every $t<T^*$ (see [CH]), the density of H^{k+2} into H^k, and the continuity of the map $H^k\ni w\mapsto V(|\phi+w|^2)\in L^1$ imply the Lemma in all its generality. Let us first verify that

$$\begin{pmatrix} w & \mapsto & V(|\phi+w|^2)\\ H^k(\mathbb{R}^n) & \mapsto & L^1\end{pmatrix}$$

is continuous. We write

$$\begin{aligned}V(|\phi+w|^2) &= V(|\phi|^2)-\int_0^1 2\mathfrak{Re}\left[w\overline{\phi+sw}\right]\\ &\quad\times\left(f(|\phi|^2)+\int_0^1 2\mathfrak{Re}\left[sw\overline{\phi+s\tau w}\right]f'(|\phi+s\tau w|^2)d\tau\right)ds\end{aligned} \tag{10}$$

We have already seen in the proof of Lemma 2.1 that $f(|\phi|^2) \in L^2$. Thus, using the Cauchy Schwarz inequality and the Sobolev embedding $H^k \subset L^\infty$, it follows that the last term in the right-hand side of (10) is continuous from H^k into L^1. In order to prove that $V(|\phi|^2) \in L^1$, we just write

$$V(|\phi|^2) \quad = \quad (|\phi|^2 - \rho_0)^2 \int_0^1 s \int_0^1 -f'(\rho_0 + s\tau(|\phi|^2 - \rho_0)) d\tau ds,$$

and we use the assumption $|\phi|^2 - \rho_0 \in L^2$, as well as the boundedness of ϕ and the continuity of f'.

We next prove the Lemma for $w_0 \in H^{k+2}$, which will be assumed from now on. Let $t \in [0, T^*(w_0))$. Let us multiply (3) by $\overline{\partial_t w(t)}$, sum over $\mathbb{R}^n$ and take the real part. We get

$$-\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla(\phi + w(t))(x)|^2 dx - \int_{\mathbb{R}^n} \frac{d}{dt} V(|\phi(x) + w(t,x)|^2) dx = 0 \ .$$

Next, $w \mapsto V(|\phi + w|^2) \in \mathcal{C}^1(H^k, L^1)$, as is shown by the following expansion, which is obtained by the Taylor formula. For every $w, \delta w \in H^k$,

$$\begin{aligned} V(|\phi + w + \delta w|^2) &= V(|\phi + w|^2) - 2\mathfrak{Re}\left[\delta w \overline{\phi + w}\right] f(|\phi + w|^2) \\ &-4 \int_0^1 \int_0^1 \mathfrak{Re}\left[\delta w \overline{\phi + w}\right] \mathfrak{Re}\left[s\delta w \overline{\phi + w + s\tau\delta w}\right] f'(|\phi + w + s\tau\delta w|^2) d\tau ds \\ &+2 \int_0^1 s|\delta w|^2 f(|\phi + w + s\delta w|^2) ds. \end{aligned} \qquad (11)$$

Since moreover $w \in \mathcal{C}^1((0, T^*), H^k)$, the map $t \mapsto V(|\phi + w(t)|^2)$, $[0, T^*(w_0)) \mapsto L^1(\mathbb{R}^n)$ is of class $\mathcal{C}^1$. Thus

$$\int_{\mathbb{R}^n} \frac{d}{dt} V(|\phi(x) + w(t,x)|^2) dx = \frac{d}{dt} \int_{\mathbb{R}^n} V(|\phi(x) + w(t,x)|^2) dx \ ,$$

and the lemma is proved. □

Lemma 3.2 *$(n,k) = (1,1)$, $(2,2)$, $(3,2)$ or $(4,3)$, $f \in \mathcal{C}^{k+1}(\mathbb{R}_+)$ satisfies (H_f), ϕ satisfies (H_ϕ). Let us write V as $V = V_+ - V_-$, where $V_+, V_- \geqslant 0$ and V_- is assumed to be bounded. Then there exists a constant $C_1 > 0$ such that for every $w_0 \in H^k(\mathbb{R}^n)$, $t \in [0, T^*)$, we have*

$$\|\nabla\phi + \nabla w(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} V_+(|\phi(x) + w(t,x)|^2) dx \quad \leqslant \quad C_1(1 + \|w(t)\|_{L^2}^2) \ . \qquad (12)$$

Proof. Thanks to Lemma 3.1, it is clear that the left hand side in (12) equals

$$\mathcal{E}_0 + \int_{\mathbb{R}^n} V_-(|\phi(x) + w(t,x)|^2)dx \ .$$

Next, the definition of V and (H_f) imply $V(\rho_0) = 0$, $V'(\rho_0) = -f(\rho_0) = 0$ and $V''(\rho_0) = -f'(\rho_0) > 0$. It follows that there exists $C_2 > 0$ and $\delta > 0$ (for convenience, we assume $\delta < \rho_0$) such that $V(r) \geqslant C_2(\rho_0 - r)^2$ for every $r \in [\rho_0 - \delta, \rho_0 + \delta]$. In particular, $V_- \equiv 0$ on $[\rho_0 - \delta, \rho_0 + \delta]$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} V_-(|\phi(x) + w(t,x)|^2)dx \ &\leqslant \ \|V_-\|_{L^\infty} \left|\{x, |\phi + w(t)|^2 \leqslant \rho_0 - \delta\}\right| \\ &\quad + \|V_-\|_{L^\infty} \left|\{x, |\phi + w(t)|^2 \geqslant \rho_0 + \delta\}\right|. \quad (13)\end{aligned}$$

Next, using the triangle inequality, $\{x, |\phi + w|^2 \leqslant \rho_0 - \delta\}$ is a subset of $\{x, |w| \geqslant |\phi| - (\rho_0 - \delta)^{1/2}\}$, which is itself included in the union of $\{x, |w| \geqslant |\phi| - (\rho_0 - \delta)^{1/2} \geqslant \frac{\rho_0^{1/2} - (\rho_0 - \delta)^{1/2}}{2}\}$ and $\{x, |\phi| - (\rho_0 - \delta)^{1/2} \leqslant \frac{\rho_0^{1/2} - (\rho_0 - \delta)^{1/2}}{2}\}$. Similarly, $\{x, |\phi + w(t)|^2 \geqslant \rho_0 + \delta\} \subset \{x, |w(t)| \geqslant (\rho_0 + \delta)^{1/2} - |\phi| \geqslant \frac{(\rho_0 + \delta)^{1/2} - \rho_0^{1/2}}{2}\} \cup \{x, |\phi| \geqslant \frac{(\rho_0 + \delta)^{1/2} + \rho_0^{1/2}}{2}\}$. Thus

$$\begin{aligned}&\int_{\mathbb{R}^n} V_-(|\phi(x) + w(t,x)|^2)dx \\ &\quad \leqslant \ \|V_-\|_{L^\infty} \left(\int_{\mathbb{R}^n} \frac{4|w(t,x)|^2 dx}{(\rho_0^{1/2} - (\rho_0 - \delta)^{1/2})^2} + \left| \left\{ x, |\phi| \leqslant \frac{\rho_0^{1/2} + (\rho_0 - \delta)^{1/2}}{2} \right\} \right| \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \frac{4|w(t,x)|^2 dx}{((\rho_0 + \delta)^{1/2} - \rho_0^{1/2})^2} + \left| \left\{ x, |\phi| \geqslant \frac{\rho_0^{1/2} + (\rho_0 + \delta)^{1/2}}{2} \right\} \right| \right) \ .\end{aligned}$$

The result follows, since (H_ϕ) implies $|\phi(x)|^2 \to \rho_0$ as $x \to \infty$. □

We next use the assumption ($\mathrm{H}_{\alpha_1,\alpha_2}$) to control the L^2-norm of w. It will then follow that its H^1-norm remains bounded on bounded intervals, because of Lemma 3.2.

Lemma 3.3 *Let us assume that (H_f), (H_ϕ) and ($\mathrm{H}_{\alpha_1,\alpha_2}$) are satisfied, for some $\alpha_1 \geqslant 1$, $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$. Then there exists a constant $C_3 > 0$ such that for every $t \in [0, T^*)$, we have*

$$\|w(t)\|^2_{L^2(\mathbb{R}^n)} \ \leqslant \ (1 + \|w_0\|^2_{L^2(\mathbb{R}^n)}) e^{C_3 t}. \qquad (14)$$

Proof. Since $C_0 > 0$, in the case $\alpha_1 > 3/2$, $(\mathrm{H}_{\alpha_1,\alpha_2})$ implies $V(r) > 0$, and thus $V_-(r) \equiv 0$ for $r \geqslant A$. Therefore V_- is bounded, as it is required to apply Lemma 3.2. This is also true if $\alpha_1 \leqslant 3/2$. As in the proof of Lemma (3.1), the study may be reduced to the case $w_0 \in H^{k+2}$. Under this assumption, let us multiply (3) by $\overline{w(t)}$, sum over $\mathbb{R}^n$ and take the imaginary part. We get

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 \quad = \quad -2\mathfrak{Im}\int_{\mathbb{R}^n}\left[\Delta\phi + f(|\phi+w(t)|^2)\phi\right]\overline{w(t)}dx,$$

where

$$f(|\phi+w(t)|^2) = f(|\phi|^2) + \int_0^1 2\mathfrak{Re}[w(t)\overline{\phi+sw(t)}]f'(|\phi+sw(t)|^2)ds\ .$$

For any $\beta \geqslant 0$ we denote by $A_\beta \geqslant 1$ a constant such that for every $a, b > 0$, $(a+b)^\beta \leqslant A_\beta(a^\beta + b^\beta)$. From the first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$, we deduce the existence of $C_0' > 0$ such that for every $r \geqslant 0$, $r^{1/2}|f'(r)| \leqslant C_0'(1 + r^{\alpha_1 - 3/2})$ if $\alpha_1 > 3/2$ and $r^{1/2}|f'(r)| \leqslant C_0'$ if $\alpha_1 \leqslant 3/2$. Then, in the case $\alpha_1 > 3/2$,

$$\begin{aligned}
&\frac{d}{dt}\|w(t)\|_{L^2}^2 \\
&\leqslant \ 2(\|\Delta\phi\|_{L^2} + \|\phi\|_{L^\infty}\|f(|\phi|^2)\|_{L^2})\|w(t)\|_{L^2} \\
&\quad + \int_{\mathbb{R}^n}\int_0^1 4|w(t,x)|^2|\phi(x)|C_0'(1 + |\phi(x) + sw(t,x)|^{2(\alpha_1-3/2)})dsdx \\
&\leqslant \ 2(\|\Delta\phi\|_{L^2} + \|\phi\|_{L^\infty}\|f(|\phi|^2)\|_{L^2})\|w(t)\|_{L^2} \\
&\quad +4\|\phi\|_{L^\infty}C_0'A_{2\alpha_1-3}\left(\|w(t)\|_{L^2}^2(1 + \|\phi\|_{L^\infty}^{2\alpha_1-3}) + \|w(t)\|_{L^{2\alpha_1-1}}^{2\alpha_1-1}\right) \\
&\leqslant \ C_4(1 + \|w(t)\|_{L^2}^2 + \|w(t)\|_{L^{2\alpha_1-1}}^{2\alpha_1-1}),
\end{aligned} \tag{15}$$

where C_4 is a positive constant. When $\alpha_1 \leqslant 3/2$, we similarly obtain

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 \quad \leqslant \quad C_4(1 + \|w(t)\|_{L^2}^2).$$

If $\alpha_1 \leqslant 3/2$, the result follows by the Gronwall lemma with $C_3 = C_4$. If $\alpha_1 > 3/2$, it remains to control the $L^{2\alpha_1-1}$ norm of $w(t)$. In the sequel, $\alpha_1 > 3/2$ is assumed. Using the second assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$[4], the assumption $2\alpha_1 - 1 \leqslant 2\alpha_2$ and Lemma 3.2 we get

[4]Up to a change of A, we may assume in the sequel $A > \max(\rho_0, \|\phi\|_{L^\infty}^2, 1)$.

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |w(t,x)|^{2\alpha_1-1}dx \\
&= \int_{\{x,|\phi+w|\leqslant A^{1/2}\}} |w(t,x)|^2|w(t,x)|^{2\alpha_1-3}dx + \int_{\{x,|\phi+w|\geqslant A^{1/2}\}} |w(t,x)|^{2\alpha_1-1}dx \\
&\leqslant \int_{\{x,|w(t,x)|\leqslant A^{1/2}+\|\phi\|_{L^\infty}\}} |w(t,x)|^2|w(t,x)|^{2\alpha_1-3}dx \\
&\qquad +A_{2\alpha_1-1}\int_{\{x,|\phi+w|\geqslant A^{1/2}\}} (|\phi|^{2\alpha_1-1}+|\phi+w|^{2\alpha_1-1})dx \\
&\leqslant (A^{1/2}+\|\phi\|_{L^\infty})^{2\alpha_1-3}\int_{\mathbb{R}^n}|w(t,x)|^2dx + A_{2\alpha_1-1}\|\phi\|_{L^\infty}^{2\alpha_1-1}\int_{\{x,|\phi+w|\geqslant A^{1/2}\}} dx \\
&\qquad +A_{2\alpha_1-1}C_0\int_{\{x,|\phi+w|\geqslant A^{1/2}\}} V_+(|\phi+w|^2)dx \\
&\leqslant (A^{1/2}+\|\phi\|_{L^\infty})^{2\alpha_1-3}\|w(t)\|_{L^2}^2 + \frac{A_{2\alpha_1-1}\|\phi\|_{L^\infty}^{2\alpha_1-1}}{(A^{1/2}-\|\phi\|_{L^\infty})^2}\|w(t)\|_{L^2}^2 \\
&\qquad +A_{2\alpha_1-1}C_0C_1(1+\|w(t)\|_{L^2}^2)\ . \qquad (16)
\end{aligned}
$$

Next, concatenating the estimations (15) and (16), there exists a constant $C_3>0$ such that

$$
\frac{d}{dt}\|w(t)\|_{L^2}^2 \ \leqslant\ C_3(1+\|w(t)\|_{L^2}^2)\ .
$$

We conclude by the Gronwall lemma. □

In the one-dimensional case, the global well-posedness of (3) in H^1 is a straightforward consequence of Theorem 2.1, Lemma 3.2 and Lemma 3.3. More precisely, we have proven:

Theorem 3.1 *If $n=1$, under the conditions (H_f), (H_{α_1,α_2}), (H_ϕ), (3) is globally well posed in $H^1(\mathbb{R})$. Namely, for every $w_0\in H^1(\mathbb{R})$, $T^*(w_0)=+\infty$.*

Remarks.
1. In the one-dimensional case, the Lipshitz continuity of the flow from the bounded sets of H^1 into $\mathcal{C}([0,T],H^1)$ for any $T>0$ may be obtained by classical methods (see [P], [CH]), so that we will drop the proof.

2. In the case $V\geqslant 0$, Lemma 3.1 gives a better information than Lemma 3.2. Indeed, it says that for every $t\geqslant 0$, $\|\nabla w(t)\|_{L^2}\leqslant \mathcal{E}_0^{1/2}+\|\nabla\phi\|_{L^2}$. Coming back to the examples presented in the introduction, this implies that

$\|\nabla w(t)\|_{L^2}$ remains bounded on $\mathbb{R}$, for the pure powers and for the saturated nonlinearities.

3. In the one-dimensional case, if ϕ_v is one of the traveling bubbles studied by Zhiwu Lin or a black soliton, ϕ_v satisfies the assumption (H_ϕ). Thus (1) is globally well-posed in $\phi_v + H^1$. In the cases where ϕ_v is unstable, one can not expect to prove instability by blow-up and the mechanism of instability seems unknown so far.

4 Local theory for H^1 solutions

In this section, we prove that (3) is locally well-posed in $H^1(\mathbb{R}^n)$, for $n = 2, 3$ or 4, in both sub-critical and critical cases. Namely, we prove that for every $w_0 \in H^1(\mathbb{R}^n)$ (small in H^1 if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$), there exists $T > 0$ and a unique solution $w \in \mathcal{C}([0, T], H^1(\mathbb{R}^n))$ of

$$w \;=\; e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} F(w(s)) ds. \tag{17}$$

We employ a fix point argument for the map

$$\Phi(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} F(w(s)) ds. \tag{18}$$

in the space

$$X_T = L^\infty_T H^1 \cap L^{p_0}_T W^{1,q_0}$$

equipped with its natural norm $\|w\|_{X_T} = \|w\|_{L^\infty_T H^1} + \|w\|_{L^{p_0}_T W^{1,q_0}}$, where (p_0, q_0) is an admissible pair. A pair $(p, q) \in [2, \infty]$ is said to be admissible if

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},\ \ (p, q) \neq (2, \infty).$$

We fix $(p_0, q_0) = (2, 6)$ for $n = 3$, $(p_0, q_0) = (2, 4)$ for $n = 4$, while for $n = 2$, q_0 may be choosen large, but finite (and thus $p_0 > 2$ is close to 2). We will use the Strichartz estimates which we recall now (for a proof, we refer to [KeTa]).

Proposition 4.1 *For every admissible pairs (p, q) and $(\tilde{p}, \tilde{q})$, for every $v_0 \in L^2(\mathbb{R}^n)$ and $f \in L^{\tilde{p}'}(\mathbb{R}, L^{\tilde{q}'}(\mathbb{R}^n))$,*

$$\|e^{it\Delta} v_0\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \lesssim \|v_0\|_{L^2(\mathbb{R}^n)} \tag{19}$$

and

$$\| \int_{-\infty}^{t} e^{i(t-\tau)\Delta} f(\tau) d\tau \|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} \lesssim \|f\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))}. \tag{20}$$

The result we next prove is as follows.

Theorem 4.1 *We assume that f satisfies (H_{α_1,α_2}) for some $\alpha_1 \geqslant 1$, with the supplementary condition $\alpha_1 \leqslant \alpha_1^*$ if $n = 3, 4$ (where $\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$), and some $\alpha_2 \in [\alpha_1 - 1/2, \alpha_1]$.*

If $n = 2$, or $n = 3, 4$ and $\alpha_1 < \alpha_1^*$, then for every $R > 0$, there exists $T(R) > 0$ such that for every $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, there exists an unique $w \in X_{T(R)}$ solving (17). Moreover, $w \in \mathcal{C}([0, T(R)], H^1)$.
If for some $T > 0$, $\tilde{w} \in \mathcal{C}([0, T], H^1)$ solves (17) then $\tilde{w} \in X_T$, and $\tilde{w}$ is the unique solution to (17) in $\mathcal{C}([0, T], H^1)$.
The flow map is locally Lipshitz continuous.

For $n = 3, 4$ and $\alpha_1 = \alpha_1^*$, there exists $R > 0$ and $T > 0$ such that for every $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, there exists an unique $w \in X_T$ solving (17). It is the unique solution in $\mathcal{C}([0, T], H^1)$. The flow map $w_0 \mapsto w$, $H^1 \mapsto X_T$ (with the same small T) is locally Lipshitz continuous.

The next four lemmas give the estimates which will enable us to apply the fixed point argument.

Lemma 4.1 *Let $T > 0$ and $w \in X_T$. Then*

$$\begin{aligned} \|\Phi(w)\|_{L^\infty_T L^2} + \|\Phi(w)\|_{L^{p_0}_T L^{q_0}} \quad \leqslant \quad & \|w_0\|_{L^2} + CT(1 + \|w\|_{L^\infty_T L^2}) \\ & + CT^{1/p'}(\|w\|^2_{L^\infty_T H^1} + \|w\|^{\max(2,2\alpha_1-1)}_{L^\infty_T H^1}) \end{aligned} \tag{21}$$

where C is a positive constant.

Proof. We first decompose F in the following way:

$$\begin{aligned} F(w) &= -\Delta\phi - f(|\phi + w|^2)(\phi + w) \\ &= \left.-\Delta\phi - f(|\phi|^2)\phi - f(|\phi|^2)w - 2\mathfrak{Re}[w\overline{\phi}]f'(|\phi|^2)\phi \right\} =: F_1(w) \\ & \left. \begin{aligned} &-2\textstyle\int_0^1 \mathfrak{Re}[w\overline{\phi + sw}]f'(|\phi + sw|^2)dsw - 2|w|^2\phi \int_0^1 sf'(|\phi + sw|^2)ds \\ &-4\mathfrak{Re}[w\overline{\phi}]\phi \textstyle\int_0^1\int_0^1 \mathfrak{Re}[sw\overline{\phi + s\tau w}]f''(|\phi + s\tau w|^2)d\tau ds. \end{aligned} \right\} =: F_2(w) \end{aligned}$$

The Strichartz inequalities (19) and (20) yield

$$\|\Phi(w)\|_{L^\infty_T L^2} + \|\Phi(w)\|_{L^{p_0}_T L^{q_0}} \quad \leqslant \quad \|w_0\|_{L^2} + C\|F_1(w)\|_{L^1_T L^2} + C\|F_2(w)\|_{L^{p'}_T L^{q'}} \tag{22}$$

where (p,q) is any admissible pair and $C>0$. For the sequel, we fix

$$(p',q') \;=\; \begin{cases} (4/3,4/3) & \text{if} \quad n=2,\\ (2,6/5) & \text{if} \quad n=3,\\ (2,4/3) & \text{if} \quad n=4. \end{cases} \tag{23}$$

On the one hand,

$$\|F_1(w)\|_{L^1_TL^2} \;\lesssim\; T(1+\|w\|_{L^\infty_TL^2}), \tag{24}$$

while on the other hand, using the first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$,

$$|F_2(w)| \;\lesssim\; |w|^2(1+|w|)^{\max(0,2\alpha_1-3)}. \tag{25}$$

Thus,

$$\begin{aligned} \|F_2(w)\|_{L^{p'}_TL^{q'}} &\lesssim \|w\|^2_{L^{2p'}_TL^{2q'}} + \|w\|^{\max(2,2\alpha_1-1)}_{L_T^{p'\max(2,2\alpha_1-1)}L^{q'\max(2,2\alpha_1-1)}} \\ &\lesssim T^{1/p'}\|w\|^2_{L^\infty_TH^1} + T^{1/p'}\|w\|^{\max(2,2\alpha_1-1)}_{L^\infty_TH^1}, \end{aligned} \tag{26}$$

because of the Hölder inequality in time and Sobolev embeddings. Note that $q'\max(2,2\alpha_1-1)$ is finite if $n=2$, and is not larger than 6 if $n=3$, than 4 if $n=4$, thanks to $\alpha_1 \leqslant \alpha_1^*$.
Concatenating (22), (24) and (26), we obtain (21) (the constant C may have change). □

We next prove the same kind of estimation for $\nabla F(w)$.

Lemma 4.2 *There exists $\theta \geqslant 0$, with $\theta=0$ only if $n=3,4$ and $\alpha_1=\alpha_1^*$, such that for every $T>0$ and $w\in X_T$,*

$$\begin{aligned} \|\nabla\Phi(w)\|_{L^\infty_TL^2} + \|\nabla\Phi(w)\|_{L^{p_0}_TL^{q_0}} &\leqslant \|\nabla w_0\|_{L^2} + CT(1+\|\nabla w\|_{L^\infty_TL^2}) \\ &+C(1+\|\nabla w\|_{L^\infty_TL^2})(T^{1/p'}\|w\|_{L^\infty_TH^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{X_T}), \end{aligned} \tag{27}$$

where $C>0$.

Proof. Let us first write

$$\begin{aligned} \nabla F(w) &= -\nabla\Delta\phi - f(|\phi+w|^2)\nabla(\phi+w) - 2\mathfrak{Re}[\nabla(\phi+w)\overline{\phi+w}](\phi+w)f'(|\phi+w|^2) \\ &= \left.-\nabla\Delta\phi - f(|\phi|^2)\nabla(\phi+w) - 2\mathfrak{Re}[\nabla(\phi+w)\overline{\phi}]f'(|\phi|^2)\phi \quad\right\} \quad =: G_1(w) \\ &\quad \left.\begin{aligned} &-2\int_0^1 \mathfrak{Re}[w\overline{\phi+sw}]f'(|\phi+sw|^2)ds\nabla(\phi+w) \\ &-4\mathfrak{Re}[\nabla(\phi+w)\overline{\phi}]\phi\int_0^1 \mathfrak{Re}[w\overline{\phi+sw}]f''(|\phi+sw|^2)ds \\ &-2\mathfrak{Re}[\nabla(\phi+w)\overline{w}](\phi+w)f'(|\phi+w|^2) \\ &-2\mathfrak{Re}[\nabla(\phi+w)\overline{\phi}]wf'(|\phi+w|^2). \end{aligned}\right\} \quad =: G_2(w) \end{aligned}$$

Next, thanks to the Strichartz inequalities (19) and (20),

$$\|\nabla\Phi(w)\|_{L^\infty_T L^2} + \|\nabla\Phi(w)\|_{L^{p_0}_T L^{q_0}} \leqslant \|\nabla w_0\|_{L^2} + C\|G_1(w)\|_{L^1_T L^2} + C\|G_2(w)\|_{L^{p'}_T L^{q'}} \quad (28)$$

with the same choice of p', q' as in (23), and $C > 0$. Next,

$$\|G_1(w)\|_{L^1_T L^2} \lesssim T(1 + \|\nabla w\|_{L^\infty_T L^2}) \quad (29)$$

and

$$|G_2(w)| \lesssim |\nabla(\phi + w)||w|(1 + |w|)^{\max(0,2\alpha_1-3)}, \quad (30)$$

which implies, thanks to the Hölder inequality and Sobolev embeddings,

$$\begin{aligned}\|G_2(w)\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\phi+w)\|_{L^\infty_T L^2}\left(\|w\|_{L^{p'}_T L^\beta} + \|w\|^{\max(1,2\alpha_1-2)}_{L^{p'\max(1,2\alpha_1-2)}_T L^{\beta\max(1,2\alpha_1-2)}}\right)\\ &\lesssim (1 + \|\nabla w\|_{L^\infty_T L^2})\left(T^{1/p'}\|w\|_{L^\infty_T H^1} + T^\theta \|w\|^{\max(1,2\alpha_1-2)}_{L^s_T W^{1,r}}\right), \quad (31)\end{aligned}$$

where $1/q' = 1/2 + 1/\beta$ (our choice of q' gives $\beta = 4$ if $n = 2$ or 4, $\beta = 3$ if $n = 3$), $\theta = \frac{1}{p'} - \frac{\max(1,2\alpha_1-2)}{s}$, and the pair (s, r) is chosen such that:

- $(s, r) = (\infty, 2)$ if $\frac{1}{2} - \frac{1}{n} \leqslant \frac{1}{\beta\max(1,2\alpha_1-2)}$
- else, $r > 2$ and
  - (i) $\frac{2}{s} + \frac{n}{r} = \frac{n}{2}$ (which means that (s, r) is an admissible pair),
  - (ii) $\frac{1}{r} - \frac{1}{n} \leqslant \frac{1}{\beta\max(1,2\alpha_1-2)}$ (which gives the Sobolev embedding $W^{1,r} \subset L^{\beta\max(1,2\alpha_1-2)}$),
  - (iii) $\frac{1}{p'\max(1,2\alpha_1-2)} \geqslant \frac{1}{s}$ (that is $\theta \geqslant 0$).

Such a choice of s and r is possible if and only if

$$\frac{n}{2} - 1 \leqslant \left(\frac{n}{\beta} + \frac{2}{p'}\right)\frac{1}{\max(1, 2\alpha_1 - 2)}. \quad (32)$$

Indeed, if (32) is true, it suffices to choose $n/r \in [\frac{n}{2} - \frac{2}{p'\max(1,2\alpha_1-2)}, 1 + \frac{n}{\beta\max(1,2\alpha_1-2)}]$. Moreover, if (32) is a strict inequality, r and s may be chosen in such a way that $\theta > 0$. For $n = 2$, (32) is always satisfied and is strict, while for $n = 3$ or 4, (32) is equivalent to $\alpha_1 \leqslant \alpha_1^*$, and is strict if and only if $\alpha_1 < \alpha_1^*$.

Since $r \in [2, q_0]$ (taking q_0 large enough, this can always be assumed for $n = 2$, while for $n = 3, 4$, $n/r \geqslant n/2 - 1 = n/q_0$), and (s, r) is an admissible pair, we obtain by interpolation

$$\|w\|_{L^s_T W^{1,r}} \lesssim \|w\|^{\tilde{\theta}}_{L^\infty_T H^1} \|w\|^{1-\tilde{\theta}}_{L^{p_0}_T W^{1,q_0}} \lesssim \|w\|_{X_T}, \tag{33}$$

where $\tilde{\theta} \in [0, 1]$. We deduce (27) from (28), (29), (31) and (33). □

In the next two lemmas, we evaluate $\Phi(w_1) - \Phi(w_2)$ in X_T, provided $w_1, w_2 \in X_T$.

Lemma 4.3 *There exists $\theta_0 > 0$ and $\theta_1 \geqslant 0$ (with $\theta_1 = 0$ only if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$) such that for every $T > 0$ and $w_1, w_2 \in L^\infty_T H^1 \subset L^{p_0}_T L^{q_0}$,*

$$\begin{aligned} &\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2} + \|\Phi(w_1) - \Phi(w_2)\|_{L^{p_0}_T L^{q_0}} \\ &\lesssim \quad T\|w_1 - w_2\|_{L^\infty_T L^2} + (\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}}) \\ &\quad \times \big(T^{\theta_0}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + T^{\theta_1}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}\big) \end{aligned} \tag{34}$$

Proof. First,

$$\begin{aligned} &F(w_1) - F(w_2) \\ &= 2\mathfrak{Re}[(w_2 - w_1)\overline{\phi}]f'(|\phi|^2)\phi + f(|\phi|^2)(w_2 - w_1) \qquad \Big\}\delta_1(w_1, w_2) \\ &\left.\begin{aligned} &+4\mathfrak{Re}[(w_2 - w_1)\overline{\phi}] \int_0^1 \int_0^1 \mathfrak{Re}[(w_1 + s(w_2 - w_1))\overline{\phi + \tau(w_1 + s(w_2 - w_1))}] \\ &\qquad \times f''(|\phi + \tau(w_1 + s(w_2 - w_1))|^2) ds d\tau \phi \\ &+2\int_0^1 \mathfrak{Re}[(w_2 - w_1)\overline{\phi + w_1 + s(w_2 - w_1)}]f'(|\phi + w_1 + s(w_2 - w_1)|^2) ds w_2 \\ &+2\int_0^1 \mathfrak{Re}[(w_2 - w_1)\overline{w_1 + s(w_2 - w_1)}]f'(|\phi + w_1 + s(w_2 - w_1)|^2) ds \phi \\ &+2\int_0^1 \mathfrak{Re}[w_1\overline{\phi + sw_1}]f'(|\phi + sw_1|^2) ds (w_2 - w_1). \end{aligned}\right\}\delta_2(w_1, w_2) \end{aligned}$$

Next, with p', q' as in (23), we get by the non-homogeneous Strichartz estimate (20)

$$\begin{aligned} &\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2} + \|\Phi(w_1) - \Phi(w_2)\|_{L^{p_0}_T L^{q_0}} \\ &\quad \leqslant \quad \|\delta_1(w_1, w_2)\|_{L^1_T L^2} + \|\delta_2(w_1, w_2)\|_{L^{p'}_T L^{q'}}. \end{aligned} \tag{35}$$

It can easily be seen that

$$\|\delta_1(w_1, w_2)\|_{L^1_T L^2} \quad \lesssim \quad T\|w_1 - w_2\|_{L^\infty_T L^2} \ , \tag{36}$$

and

$$|\delta_2(w_1,w_2)| \quad \lesssim \quad |w_1-w_2|(|w_1|+|w_2|)(1+|w_1|+|w_2|)^{\max(0,2\alpha_1-3)} \ . \ (37)$$

Thus, by Hölder and Sobolev,

$$\begin{aligned}\|\delta_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} &\lesssim \|w_1-w_2\|_{L_T^{p'}L^{2q'}}(\|w_1\|_{L_T^\infty L^{2q'}}+\|w_2\|_{L_T^\infty L^{2q'}})\\ &+\|w_1-w_2\|_{L_T^{p'}L^{q_1}}\||w_1|+|w_2|\|^{\max(1,2\alpha_1-2)}_{L_T^\infty L^{q_2\max(1,2\alpha_1-2)}}, \qquad (38)\end{aligned}$$

where $1/q' = 1/q_1 + 1/q_2$, with $(q_1,q_2)=(2,\beta)$ if $n=2$ or $n=3$ and $\alpha_1 \leqslant 2$, whereas $q_2 = q_0/\max(1, 2\alpha_1-2)$ if n=3 and $\alpha_1 > 2$ or $n=4$. Thus, by Sobolev and Hölder in time,

$$\begin{aligned}\|\delta_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} &\lesssim T^{\frac{1}{p'}-\frac{1}{p_3}}\|w_1-w_2\|_{L_T^{p_3}L^{2q'}}(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})\\ &+T^{\frac{1}{p'}-\frac{1}{p_2}}\|w_1-w_2\|_{L_T^{p_2}L^{q_1}}\||w_1|+|w_2|\|^{\max(1,2\alpha_1-2)}_{L_T^\infty H^1}, \qquad (39)\end{aligned}$$

where $(p_3, 2q')$ and (p_2, q_1) are admissible pairs (q_1, q_2 have been chosen in sort that $2 \leqslant q_1 \leqslant q_0$). Moreover, $1/p' - 1/p_3 > 0$, and our choice of q_2 ensures that $1/p' - 1/p_2 > 0$ as soon as $n=2$ or $n=3,4$ and $\alpha_1 < \alpha_1^*$. An interpolation argument yields the announced result as in the proof of Lemma 4.2. □

We next estimate $\nabla(\Phi(w_1)-\Phi(w_2))$ in $L_T^\infty L_2 \cap L_T^{p_0}L^{q_0}$.

Lemma 4.4 *There exists $\theta_2, \theta_3 > 0$, with $\theta_2 = 0$ and $\theta_3 = 0$ only if $n \neq 2$ and $\alpha_1 = \alpha_1^*$, such that for every $T > 0$, $w_1, w_2 \in X_T$,*

$$\begin{aligned}\|\nabla\Phi(w_1)-\nabla\Phi(w_2)\|_{L_T^\infty L^2}+\|\nabla\Phi(w_1)-\nabla\Phi(w_2)\|_{L_T^{p_0}L^{q_0}} &\lesssim \quad T\|\nabla(w_1-w_2)\|_{L_T^\infty L^2}\\ +T^{1/p'}(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})^{\max(1,2\alpha_1-2)}\|w_1-w_2\|_{L_T^\infty H^1}&\\ +T^{\theta_2}\|w_1-w_2\|_{X_T}(\|w_1\|_{X_T}^{\max(1,2\alpha_1-2)}+\|w_2\|_{X_T}^{\max(1,2\alpha_1-2)})& \qquad (40)\\ +T^{\theta_3}\|w_1-w_2\|_{X_T}(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})(\|w_1\|_{X_T}^{\max(0,2\alpha_1-3)}+\|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}).&\end{aligned}$$

Proof. We first write

$$
\begin{aligned}
&\nabla F(w_1) - \nabla F(w_2)\\
&= 2\mathfrak{Re}[\nabla(w_2 - w_1)\overline{\phi}]f'(|\phi|^2)\phi + f(|\phi|^2)\nabla(w_2 - w_1) \qquad \} \ \gamma_1(w_1, w_2)\\
&\left.\begin{aligned}
&+2\mathfrak{Re}[\nabla(w_2 - w_1)\overline{\phi}] \int_0^1 2\mathfrak{Re}[w_2\overline{\phi + sw_2}]f''(|\phi + sw_2|^2)ds\phi\\
&+2\mathfrak{Re}[\nabla(w_2 - w_1)\overline{\phi + w_2}]f'(|\phi + w_2|^2)w_2\\
&+2\mathfrak{Re}[\nabla(w_2 - w_1)\overline{w_2}]f'(|\phi + w_2|^2)\phi\\
&+2\int_0^1 \mathfrak{Re}[w_1\overline{\phi + sw_1}]f'(|\phi + sw_1|^2)ds\nabla(w_2 - w_1)
\end{aligned}\right\} \gamma_2(w_1, w_2)\\
&\left.\begin{aligned}
&+2\int_0^1 \mathfrak{Re}[\nabla(\phi + w_1)\overline{w_2 - w_1}]f'(|\phi + w_1 + s(w_2 - w_1)|^2)(\phi + w_1 + s(w_2 - w_1))ds\\
&+4\int_0^1 \mathfrak{Re}[\nabla(\phi + w_1)\overline{\phi + w_1 + s(w_2 - w_1)}]\mathfrak{Re}[(w_2 - w_1)\overline{\phi + w_1 + s(w_2 - w_1)}]\\
&\qquad\qquad \times f''(|\phi + w_1 + s(w_2 - w_1)|^2)(\phi + w_1 + s(w_2 - w_1))ds\\
&+2\int_0^1 \mathfrak{Re}[\nabla(\phi + w_1)\overline{\phi + w_1 + s(w_2 - w_1)}]f'(|\phi + w_1 + s(w_2 - w_1)|^2)(w_2 - w_1)ds\\
&+2\int_0^1 \mathfrak{Re}[(w_2 - w_1)\overline{\phi + w_1 + s(w_2 - w_1)}]f'(|\phi + w_1 + s(w_2 - w_1)|^2)ds\nabla(\phi + w_2).
\end{aligned}\right\} \gamma_3(w_1, w_2)
\end{aligned}
$$

The non-homogeneous Strichartz inequality (20) implies

$$
\begin{aligned}
&\|\nabla(\Phi(w_1) - \Phi(w_2))\|_{L_T^\infty L^2} + \|\nabla(\Phi(w_1) - \Phi(w_2))\|_{L_T^{p_0} L^{q_0}}\\
&\quad \leqslant \ \|\gamma_1(w_1, w_2)\|_{L_T^1 L^2} + \|\gamma_2(w_1, w_2)\|_{L_T^{p'} L^{q'}} + \|\gamma_3(w_1, w_2)\|_{L_T^{p'} L^{q'}},
\end{aligned} \tag{41}
$$

where (p', q') is given by (23). Next,

$$
\|\gamma_1(w_1, w_2)\|_{L_T^1 L^2} \ \lesssim \ T\|\nabla(w_1 - w_2)\|_{L_T^\infty L^2}, \tag{42}
$$

while

$$
|\gamma_2(w_1, w_2)| \ \lesssim \ |\nabla(w_2 - w_1)|(|w_1| + |w_2|)(1 + |w_1| + |w_2|)^{\max(0, 2\alpha_1 - 3)} \tag{43}
$$

and

$$
|\gamma_3(w_1, w_2)| \ \lesssim \ (|\nabla\phi| + |\nabla w_1| + |\nabla w_2|)|w_1 - w_2|(1 + |w_1| + |w_2|)^{\max(0, 2\alpha_1 - 3)} \tag{44}
$$

Using the Hölder inequality, we get

$$
\begin{aligned}
\|\gamma_2(w_1, w_2)\|_{L_T^{p'} L^{q'}} \ &\lesssim \ \|\nabla(w_1 - w_2)\|_{L_T^\infty L^2}\\
&\times \left(\|w_1\|_{L_T^{p'} L^\beta} + \|w_2\|_{L_T^{p'} L^\beta} + \||w_1| + |w_2|\|^{\max(1, 2\alpha_1 - 2)}_{L_T^{p' \max(1, 2\alpha_1 - 2)} L^{\beta \max(1, 2\alpha_1 - 2)}}\right)
\end{aligned} \tag{45}
$$

and

$$\|\gamma_3(w_1,w_2)\|_{L_T^{p'}L^{q'}} \lesssim (\|\nabla\phi\|_{L^2}+\|\nabla w_1\|_{L_T^\infty L^2}+\|\nabla w_2\|_{L_T^\infty L^2}) \times\Big(\|w_1-w_2\|_{L_T^{p'}L^\beta}+\|(w_1-w_2)|w_1|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta}+\|(w_1-w_2)|w_2|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta}\Big) \quad (46)$$

Next, the Hölder inequality in time and the Sobolev embeddings ensure that for $j=1,2$,

$$\|w_j\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_j\|_{L_T^\infty H^1}$$

and

$$\|w_j\|^{\max(1,2\alpha_1-2)}_{L_T^{p'\max(1,2\alpha_1-2)}L^{\beta\max(1,2\alpha_1-2)}} \lesssim T^{\theta_2}\|w_j\|^{\max(1,2\alpha_1-2)}_{L_T^sW^{1,r}},$$

with the same choice of s, r and $\theta_2=\theta$ we did to get (31). It follows then from (45) as in the proof of Lemma 4.2 that

$$\|\gamma_2(w_1,w_2)\|_{L_T^{p'}L^{q'}} \lesssim \|\nabla(w_1-w_2)\|_{L_T^\infty L^2} \times\Big(T^{1/p'}(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1})+T^{\theta_2}(\|w_1\|^{\max(1,2\alpha_1-2)}_{X_T}+\|w_2\|^{\max(1,2\alpha_1-2)}_{X_T})\Big) \quad (47)$$

Using the same arguments, we also have

$$\|w_1-w_2\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_1-w_2\|_{L_T^\infty H^1} \quad (48)$$

and, if $\beta\max(1,2\alpha_1-2)\leqslant q_0$, for $j=1,2$,

$$\|(w_1-w_2)|w_j|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\begin{cases}\|w_1-w_2\|_{L_T^\infty H^1} & \text{if } 2\alpha_1-3\leqslant 0\\ \|w_1-w_2\|_{L_T^\infty L^{\frac{\beta}{1-\varepsilon}}}\|w_j\|^{2\alpha_1-3}_{L_T^\infty L^{\frac{\beta(2\alpha_1-3)}{\varepsilon}}} & \text{if } 2\alpha_1-3>0,\end{cases}$$

where $\varepsilon=\beta(2\alpha_1-3)/q_0$ (note that $2\alpha_1-3>0$ and $\beta(2\alpha_1-2)\leqslant q_0$ imply $\varepsilon\in(0,1)$). In that case, it follows from the Sobolev embeddings that

$$\|(w_1-w_2)|w_j|^{\max(0,2\alpha_1-3)}\|_{L_T^{p'}L^\beta} \lesssim T^{1/p'}\|w_1-w_2\|_{L_T^\infty H^1}\|w_j\|^{\max(0,2\alpha_1-3)}_{L_T^\infty H^1}. \quad (49)$$

Next, if $\beta\max(1,2\alpha_1-2)>q_0$, since $\beta\leqslant q_0$, we have $2\alpha_1-3>0$ and the Hölder inequality yields

$$\|(w_1-w_2)|w_j|^{2\alpha_1-3}\|_{L_T^{p'}L^\beta} \lesssim \|w_1-w_2\|_{L_T^{p_1}L^{q_1}}\|w_j\|^{2\alpha_1-3}_{L_T^{p_2(2\alpha_1-3)}L^{q_2(2\alpha_1-3)}},$$

where $(p_1, q_1) = (2\alpha_1 - 2)(p', \beta)$ and $(p_2, q_2) = \frac{2\alpha_1-2}{2\alpha_1-3}(p', \beta)$. Then,

$$\|(w_1 - w_2)|w_j|^{2\alpha_1-3}\|_{L^{p'}_T L^\beta} \lesssim \begin{cases} T^{\frac{1}{p'}-\frac{2\alpha_1-2}{s}}\|w_1 - w_2\|_{L^s_T W^{1,r}}\|w_j\|^{2\alpha_1-3}_{L^s_T W^{1,r}} & \text{if } q_1 > q_0 \\ T^{1/p'}\|w_1 - w_2\|_{L^\infty_T H^1}\|w_j\|^{2\alpha_1-3}_{L^\infty_T H^1} & \text{if } q_1 \leqslant q_0. \end{cases} \quad (50)$$

If $n = 2$, it is possible to choose q_0 large enough, such that $q_1 \leqslant q_0$. If $n = 3, 4$ and $q_1 > q_0$, r and s are chosen such that

(i) $\frac{2}{s} + \frac{n}{r} = \frac{n}{2}$,

(ii) $\frac{1}{r} - \frac{1}{n} \leqslant \frac{1}{q_1} < \frac{1}{q_0} = \frac{1}{2} - \frac{1}{n}$ (thus $r > 2$),

(iii) $\frac{1}{p_1} - \frac{1}{s} \geqslant 0$.

These conditions may be satisfied if and only if $\frac{n}{2} - \frac{2}{p_1} \leqslant 1 + \frac{n}{q_1}$, which is true if $\alpha_1 \leqslant 3$ for $n = 3$, $\alpha_1 \leqslant 2$ for $n = 4$. Moreover, as in the proof of Lemma 4.2, if $n = 2$ or $n = 3, 4$ and $\alpha_1 < \alpha_1^*$, s and r may be chosen in such a way that $\theta_3 := \frac{1}{p'} - \frac{2\alpha_1-2}{s} > 0$.

As in the proof of Lemma 4.2, it follows from an interpolation argument and from (46), (48), (49) and (50) that

$$\begin{aligned}\|\gamma_3(w_1, w_2)\|_{L^{p'}_T L^{q'}} \quad &\lesssim \quad T^{1/p'}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}\|w_1 - w_2\|_{L^\infty_T H^1} \\ &+T^{\theta_3}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})\|w_1 - w_2\|_{X_T}(\|w_1\|^{\max(0,2\alpha_1-3)}_{X_T} + \|w_2\|^{\max(0,2\alpha_1-3)}_{X_T}). \quad (51)\end{aligned}$$

for some $\theta_3 \geqslant 0$, with $\theta_3 = 0$ only if $n = 3, 4$ and $\alpha_1 = \alpha_1^*$. Concatenating (41), (42), (47) and (51), we obtain the announced result. □

The fixed point argument in the sub-critical case. The last four lemmas enable us to apply a fix-point argument to Φ in X_T. We first consider the cases $n = 2$ and $n = 3, 4$ with $\alpha_1 < \alpha_1^*$. Let us take $R > 0$ and $w_0 \in H^1$, $\|w_0\|_{H^1} \leqslant R$. Let B_{R+1} be the ball of radius $R + 1$ in X_T, for some $T > 0$. Thanks to Lemmas 4.1 and 4.2, since $\theta > 0$ in (27), Φ maps B_{R+1} into itself as soon as T is chosen small enough. Since $\theta_0, \theta_1, \theta_2, \theta_3 > 0$, Lemmas 4.3 and 4.4 then ensure that Φ defines a contraction on B_{R+1}, taking if necessary T even smaller. Then, existence and uniqueness of a solution to (17) in X_T follows from a fixed point argument. Retaking all the arguments above with $X_T = L^\infty_T H^1 \cap L^{p_0}_T W^{1,q_0}$ replaced by $\mathcal{C}([0,T], H^1) \cap L^{p_0}_T W^{1,q_0}$, we deduce that this solution belongs to $\mathcal{C}([0,T], H^1)$.

The fixed point argument in the critical case. Let us now take care of the case $n = 3$ or 4, $\alpha_1 = \alpha_1^*$. Since $\theta = \theta_1 = \theta_2 = \theta_3 = 0$ in Lemmas 4.2, 4.3 and 4.4, the argument we used for $\alpha_1 < \alpha_1^*$ breaks down. However, since $\max(1, 2\alpha_1 - 2) > 1$, using Lemmas 4.1 and 4.2, Φ maps B_{2R} into itself for every $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, provided R and T are small enough. In a similar way, taking R and T even smaller if necessary, Φ defines a contraction on B_{2R}, thanks to Lemmas 4.3 and 4.4, and because $\max(0, 2\alpha_1 - 3) > 0$.

In order to complete the proof of Theorem 4.1, it remains to show the uniqueness in $\mathcal{C}([0,T], H^1)$, as well as the Lipshitz-continuity of the flow. We first prove this in the case $n = 2$ or $n = 3, 4$ and $\alpha_1 < \alpha_1^*$.

Proof of Theorem 1.1: the uniqueness. Let $T > 0$ and $w_1, w_2 \in \mathcal{C}([0,T], H^1)$ be two solutions to (17) with initial data $w_1(0) = w_2(0) = w_0 \in H^1$. Then by Lemma 4.3, for $\tilde{T} \leqslant \min(T, 1)$, defining $\tilde{\theta} := \min(1, \theta_0, \theta_1) > 0$,

$$\begin{aligned}
&\|w_1 - w_2\|_{L^\infty_{\tilde{T}} L^2} + \|w_1 - w_2\|_{L^{p_0}_{\tilde{T}} L^{q_0}} \\
&\quad \leqslant C\tilde{T}^{\tilde{\theta}} \left(\|w_1 - w_2\|_{L^\infty_{\tilde{T}} L^2} + \|w_1 - w_2\|_{L^{p_0}_{\tilde{T}} L^{q_0}} \right) \\
&\quad \times \left(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + (\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1, 2\alpha_1 - 2)}\right).
\end{aligned}$$

Since $\tilde{\theta} > 0$, we can choose $\tilde{T}$ small enough, in such a way that

$$C\tilde{T}^{\tilde{\theta}} \left(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + (\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1, 2\alpha_1 - 2)}\right) < 1,$$

which implies that $w_1 \equiv w_2$ on $[0, \tilde{T}]$. Since $\tilde{T}$ only depends on $\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}$, we can reiterate this argument on small intervals of length $\tilde{T}$ until the whole interval $[0, T]$ is recovered. This proves the uniqueness of a solution to (17) in $\mathcal{C}([0,T], H^1)$.

We next prove that if for some $T > 0$, $w \in \mathcal{C}([0,T], H^1)$ solves (17), then $w \in X_T$. Let $T > 0$ and $w \in \mathcal{C}([0,T], H^1)$ be a solution to (17). Let us define $R_0 := \|w\|_{L^\infty_T H^1}$. From the contraction argument developped above, we deduce that there exists $T(R_0) > 0$ such that for every data in the ball of radius R_0 in H^1, there exists an unique solution in $X_{T(R_0)}$ with this initial condition. It is the unique solution in $\mathcal{C}([0, T(R_0)], H^1)$ with that initial data. Thanks to this argument, for every $k \in \mathbb{N}$ such that $I_k := [kT(R_0), (k+1)T(R_0)] \cap [0, T] \neq \emptyset$, there exists $w_k \in \mathcal{C}(I_k, H^1) \cap L^{p_0}(I_k, W^{1,q_0})$ which solves (17), with $w_k(kT(R_0)) = w(kT(R_0))$. The uniqueness of the solution in $\mathcal{C}(I_k, H^1)$ implies that w coincides with w_k on I_k. In particular, $w_{|I_k} \in L^{p_0}(I_k, W^{1,q_0})$, and thus $w \in X_T$.

Next, we prove the local Lipshitz continuity of the flow, in the sub-critical case.

Proof of the local Lipschitz continuity of the flow in the sub-critical case. We first define $R = \|w\|_{L^\infty_T H^1} + 1$. The contraction argument we employed above ensures that there exists $T(R) \in (0,1)$ such that for every data $\tilde{w_0}$ in the ball of radius R in H^1, there exists an unique solution to (17) (with w_0 replaced by $\tilde{w_0}$) in $X_{T(R)}$. This solution has been obtained by a contraction argument in the ball of radius $R+1$ in $X_{T(R)}$. In particular, its $X_{T(R)}$-norm is less than $R+1$. Thus, if $w_{0,k} \in H^1$ satisfies $\|w_{0,k} - w(kT(R))\|_{H^1} \leqslant 1$ for some $k \in \mathbb{N}$ such that $kT(R) \leqslant T$, there exists $w_k \in X_{T(R)}$ solving (17) (with w_0 replaced by $w_{0,k}$). Defining $\theta_4 = \min(\theta_0, \theta_1, \theta_2, \theta_3) > 0$, slightly modified versions of Lemmas 4.3 and 4.4 yield

$$\begin{aligned}&\|w_k - w(kT(R) + .)\|_{X_{T(R)}}\\ &\quad \leqslant C\|w_{0,k} - w(kT(R))\|_{H^1} + C\|w_k - w(kT(R) + .)\|_{X_{T(R)}} T(R)^{\theta_4}(1 + R^{\max(1,2\alpha_1-1)}),\end{aligned}$$

where $C > 0$. Up to a change of $T(R)$, one may assume that

$$CT(R)^{\theta_4}(1 + R^{\max(1,2\alpha_1-1)}) \leqslant 1/2,$$

in such a way that

$$\|w_k - w(kT(R) + .)\|_{X_{T(R)}} \leqslant 2C\|w_{0,k} - w(kT(R))\|_{H^1}.$$

If $\tilde{w}_0$ satisfies $\|\tilde{w}_0 - w_0\|_{H^1} \leqslant (1/\max(1, 2C))^{\left\lceil \frac{T}{T(R)} \right\rceil}$, a solution $\tilde{w}$ to (17) with w_0 replaced by $\tilde{w_0}$ may be constructed step by step by this argument, recovering $[0,T]$ by intervals of length $T(R)$. We deduce that $T^*(\tilde{w}_0) \geqslant T^*(w_0) \geqslant T$. Moreover,

$$\|w - \tilde{w}\|_{X_T} \lesssim \|\tilde{w}_0 - w_0\|_{H^1},$$

which completes the proof of the local Lipshitz continuity of the flow if $n = 2$ or $n = 3, 4$ and $\alpha_1 < \alpha_1^*$.

Proof of Theorem 1.3: the uniqueness. In the critical case $n = 3, 4$ and $\alpha_1 = \alpha_1^*$, let as before be T and R small enough such that Φ defines a contraction on B_{2R}. Let $w_1, w_2 \in \mathcal{C}([0,T], H^1)$ be two solutions to (17) with the same initial condition $w_0 \in H^1$, which satisfies $\|w_0\|_{H^1} \leqslant R$. Defining $\tilde{\theta} = \min(1, \theta_0)$, Lemma 4.3 provides, replacing T by $\min(T, 1)$,

$$\begin{aligned}\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}} \quad &\leqslant \quad C(\|w_1 - w_2\|_{L^\infty_T L^2} + \|w_1 - w_2\|_{L^{p_0}_T L^{q_0}})\\ &\quad \times (T^{\tilde{\theta}}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + (4R)^{2\alpha_1^*-2}).\end{aligned}$$

Taking T and R even smaller if necessary, one may assume that

$$C(T^{\tilde{\theta}}(1+\|w_1\|_{L^\infty_T H^1}+\|w_2\|_{L^\infty_T H^1})+(4R)^{2\alpha_1^*-2})<1,$$

which implies that $w_1 \equiv w_2$ on $[0,T]$.

Proof of Theorem 1.3: the local Lipschitz continuity of the flow. Let $w \in X_T$ be a solution to (17), with $\|w_0\|_{H^1} \leqslant R/2$. Let $\tilde{w_0} \in H^1$ be such that $\|w_0 - \tilde{w_0}\|_{H^1} \leqslant R/2$. Previous results ensure that there exists $\tilde{w} \in X_T$ which is a solution to (17) with w_0 replaced by w_0. Then, taking $\check{\theta} = \min(1, 1/p', \theta_0)$, modified versions of Lemmas 4.3 and 4.4 yield

$$\begin{aligned}\|w-\tilde{w}\|_{X_T} \leqslant\ & C\|w_0-\tilde{w_0}\|_{H^1}+CT^{\check{\theta}}\|w-\tilde{w}\|_{X_T}(1+\|w\|_{L^\infty_T H^1}+\|\tilde{w}\|_{L^\infty_T H^1})^{2\alpha_1^*-2}\\ & +C\|w-\tilde{w}\|_{X_T}(\|w\|_{X_T}+\|\tilde{w}\|_{X_T})^{2\alpha_1^*-2}\\ & +C\|w-\tilde{w}\|_{X_T}(1+\|w\|_{L^\infty_T H^1}+\|\tilde{w}\|_{L^\infty_T H^1})(\|w\|_{X_T}+\|\tilde{w}\|_{X_T})^{2\alpha_1^*-3}\\ \leqslant\ & C\|w_0-\tilde{w_0}\|_{H^1} \qquad (52)\\ & +C\|w-\tilde{w}\|_{X_T}(T^{\check{\theta}}(1+4R)^{2\alpha_1^*-2}+(4R)^{2\alpha_1^*-2}+(1+4R)(4R)^{2\alpha_1^*-3}).\end{aligned}$$

Choosing T and R even smaller if necessary, one may assume that

$$C(T^{\check{\theta}}(1+4R)^{2\alpha_1^*-2}+(4R)^{2\alpha_1^*-2}+(1+4R)(4R)^{2\alpha_1^*-3})<1/2.$$

Under this condition, (52) induces the Lipshitz continuity of the flow on small intervals of time.

5 The global well-posedness

As it was remarked at the end of section 3, Theorem 1.1 has already been proved in the one-dimensional case. This section is devoted to the proof of a persistency result which, once combined with the results of the previous sections, will give the global well-posedness of (17) in H^1, for dimensions $n = 2, 3, 4$, in the sub-critical case. In the following, $(n, m) = (2, 2), (3, 2)$ or $(4, 3)$.

Let $w_0 \in H^m$, and $T_m^*(w_0) > 0$ be the maximal time of existence of a solution w to (3) in $H^m(\mathbb{R}^n)$, given by Theorem 2.1, in such a way that $\|w(t)\|_{H^m} \underset{t\uparrow T_m^*(w_0)}{\longrightarrow} \infty$ if $T_m^*(w_0)$ is finite. We also define

$$T_1^*(w_0) = \sup\{T > 0, \text{there exists a solution to (17) in } X_T\}.$$

Since $\mathcal{C}([0,T], H^m) \subset X_T$, it is clear that $T_m^*(w_0) \leqslant T_1^*(w_0)$. Let us assume by contradiction that $T_m^*(w_0) < T_1^*(w_0)$. In particular, $T_m^*(w_0) < \infty$.

The uniqueness result in Theorem 4.1 ensures that w is the restriction to $[0, T_m^*(w_0))$ of a function which lives in $X_{T_1^*(w_0)-\varepsilon}$, for every $\varepsilon \in (0, T_1^*(w_0) - T_m^*(w_0))$. The results of section 3 ensure that $\|w(t)\|_{H^1}$ remains bounded on $[0, T_m^*(w_0))$. Therefore $\sum_{2\leqslant|\alpha|\leqslant m} \|\partial^\alpha w(t)\|_{L^2} \to \infty$ as $t \uparrow T_m^*(w_0)$.

Let us differentiate (3) twice, in directions x_j and x_k. Using also the Taylor formula, it follows that $\partial_{j,k}^2 w$ solves

$$
\begin{aligned}
& i\partial_t\partial_{j,k}^2 w + \Delta\partial_{j,k}^2 w \\
= \;& -\Delta\partial_{j,k}^2\phi - f(|\phi|^2)\partial_{j,k}^2(\phi+w) - 2\mathfrak{Re}\left[\partial_{j,k}^2(\phi+w)\overline{\phi}\right] f'(|\phi|^2)\phi \qquad \} \, g_0(t) \\
& \left.\begin{aligned}
& -2\int_0^1 \mathfrak{Re}\left[w\overline{\phi+sw}\right] f'(|\phi+sw|^2)ds\,\partial_{j,k}^2(\phi+w) \\
& -2\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\phi+w)\overline{w}\right] f'(|\phi+sw|^2)(\phi+sw)ds \\
& -4\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\phi+w)\overline{\phi+sw}\right] \mathfrak{Re}\left[w\overline{\phi+sw}\right] f''(|\phi+sw|^2)(\phi+sw)ds \\
& -2\int_0^1 \mathfrak{Re}\left[\partial_{j,k}^2(\phi+w)\overline{\phi+sw}\right] f'(|\phi+sw|^2)w\,ds
\end{aligned}\right\} g_1(t) \\
& \left.\begin{aligned}
& -2\mathfrak{Re}\left[\partial_k(\phi+w)\overline{\phi+w}\right]\partial_j(\phi+w)f'(|\phi+w|^2) \\
& -2\mathfrak{Re}\left[\partial_j(\phi+w)\overline{\phi+w}\right]\partial_k(\phi+w)f'(|\phi+w|^2) \\
& -2\mathfrak{Re}\left[\partial_j(\phi+w)\overline{\partial_k(\phi+w)}\right](\phi+w)f'(|\phi+w|^2) \\
& -4\mathfrak{Re}\left[\partial_j(\phi+w)\overline{\phi+w}\right]\mathfrak{Re}\left[\partial_k(\phi+w)\overline{\phi+w}\right] f''(|\phi+w|^2)(\phi+w).
\end{aligned}\right\} g_2(t)
\end{aligned}
$$

It follows from the Strichartz estimates (19) and (20) that for $T < T_m^*(w_0)$,

$$
\|\partial_{j,k}^2 w\|_{L_T^\infty L^2} \leqslant \|\partial_{j,k}^2 w_0\|_{L^2} + \|g_0\|_{L_T^1 L^2} + \|g_1\|_{L_T^{p'} L^{q'}} + \|g_2\|_{L_T^{p'} L^{q'}}, \tag{53}
$$

where p', q' are given by (23). First,

$$
\|g_0\|_{L_T^1 L^2} \;\lesssim\; T(1 + \|\partial_{j,k}^2 w\|_{L_T^\infty L^2})\ . \tag{54}
$$

Next,

$$
|g_1(s)| \lesssim |w|(1 + |w|^{\max(0,2\alpha_1-3)})|\partial_{j,k}^2(\phi+w)|,
$$

while

$$
\begin{aligned}
|g_2(s)| \;&\lesssim\; (1 + |w|^{\max(0,2\alpha_1-3)})|\partial_j(\phi+w)||\partial_k(\phi+w)| & (55)\\
&\lesssim\; (1 + |w|^{\max(1,2\alpha_1-3)})|\partial_j(\phi+w)||\partial_k(\phi+w)|. & (56)
\end{aligned}
$$

Then, using arguments developped in the proof of Lemma 4.2 to control $\|G_2(w)\|_{L_T^{p'} L^{q'}}$, we obtain

$$\begin{aligned}\|g_1\|_{L^{p'}_T L^{q'}} &\lesssim \|\partial^2_{j,k}(\phi+w)\|_{L^\infty_T L^2}(\|w\|_{L^{p'}_T L^\beta} + \|w\|^{\max(1,2\alpha_1-2)}_{L^{p'\max(1,2\alpha_1-2)}_T L^{\beta\max(1,2\alpha_1-2)}}) \\ &\lesssim \|\partial^2_{j,k}(\phi+w)\|_{L^\infty_T L^2}(T^{1/p'}\|w\|_{L^\infty_T H^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{X_T}), \quad (57)\end{aligned}$$

where β and θ are the same as in Lemma 4.2. Using (56), we also have

$$\begin{aligned}\|g_2\|_{L^{p'}_T L^{q'}} \lesssim T^{1/p'}\|\partial_j(\phi+w)\|_{L^\infty_T L^{2q'}}\|\partial_k(\phi+w)\|_{L^\infty_T L^{2q'}} \\ +\|\partial_j(\phi+w)\partial_k(\phi+w)|w|^{\max(1,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}} \quad (58)\end{aligned}$$

From now on, we distinguish the cases $n=2$, $n=3$ and $n=4$. For $n=2$, thanks to Hölder, Sobolev and Gagliardo-Nirenberg inequalities, (58) yields

$$\begin{aligned}\|g_2\|_{L^{4/3}_T L^{4/3}} \lesssim T^{3/4}(1+\|w\|_{L^\infty_T H^1})^{3/2}(1+\|\Delta w\|_{L^\infty_T L^2})^{1/2} \\ +T^{3/4}(1+\|w\|_{L^\infty_T H^1})(1+\|\Delta w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-3)}_{L^\infty_T H^1}. \quad (59)\end{aligned}$$

For $n=3$, the same arguments yield

$$\begin{aligned}\|g_2\|_{L^2_T L^{6/5}} \lesssim T^{1/2}(1+\|w\|_{L^\infty_T H^1})^{3/2}(1+\|\Delta w\|_{L^\infty_T L^2})^{1/2} \\ +T^{\frac{1}{2}-\frac{\max(1,2\alpha_1-3)}{s}}(1+\|w\|_{L^\infty_T H^1})(1+\|\Delta w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-3)}_{L^s_T W^{1,r}}, \quad (60)\end{aligned}$$

where s and r are chosen in such a way that

(i) $\frac{2}{s}+\frac{3}{r}=\frac{3}{2}$,

(ii) $\frac{1}{r}-\frac{1}{3}\leqslant\frac{1}{6\max(1,2\alpha_1-3)}$,

(iii) $\frac{1}{2}-\frac{\max(1,2\alpha_1-3)}{s}>0$.

These conditions may be satisfied, provided $\alpha_1<3$.

For $n=4$, we deduce from (55) and the Gagliardo-Nirenberg inequality that if $\alpha_1\leqslant 3/2$,

$$\begin{aligned}\|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}\|\partial_j(\phi+w)\|_{L^\infty_T L^{8/3}}\|\partial_k(\phi+w)\|_{L^\infty_T L^{8/3}} \\ &\lesssim T^{1/2}(1+\|w\|_{L^\infty_T H^1})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}), \quad (61)\end{aligned}$$

while if $\alpha_1 > 3/2$ (and $\alpha_1 < 2$),

$$\begin{aligned}\|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}(1+\|w\|_{L^\infty_T H^1})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad +\|\partial_j(\phi+w)\partial_k(\phi+w)|w|^{2\alpha_1-3}\|_{L^2_T L^{4/3}}. \qquad (62)\end{aligned}$$

Let us fix $\varepsilon := \frac{1}{2\alpha_1-3} - 1 > 0$. Then by Hölder,

$$\begin{aligned}&\|\partial_j(\phi+w)\partial_k(\phi+w)|w|^{2\alpha_1-3}\|_{L^2_T L^{4/3}} \\ &\leqslant \||\nabla(\phi+w)|^{\frac{1}{1+\varepsilon}}\|_{L^2_T L^{4(1+\varepsilon)}}\||\nabla(\phi+w)|^{\frac{1+2\varepsilon}{1+\varepsilon}}\|_{L^\infty_T L^{4\frac{1+\varepsilon}{1+3\varepsilon}}}\||w|^{2\alpha_1-3}\|_{L^\infty_T L^{4(1+\varepsilon)}} \\ &= \|\nabla(\phi+w)\|^{\frac{1}{1+\varepsilon}}_{L_T^{\frac{2}{1+\varepsilon}}L^4}\|\nabla(\phi+w)\|^{\frac{1+2\varepsilon}{1+\varepsilon}}_{L^\infty_T L^{4\frac{1+2\varepsilon}{1+3\varepsilon}}}\|w\|^{2\alpha_1-3}_{L^\infty_T L^4} \qquad (63)\end{aligned}$$

Next, the Hölder inequality in time yields

$$\|\nabla(\phi+w)\|^{\frac{1}{1+\varepsilon}}_{L_T^{\frac{2}{1+\varepsilon}}L^4} \lesssim T^{\frac{1}{1+\varepsilon}(\frac{1+\varepsilon}{2}-\frac{1}{2})}\|\nabla(\phi+w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4} = T^{\frac{\varepsilon}{2(1+\varepsilon)}}\|\nabla(\phi+w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4} \qquad (64)$$

It follows from the Gagliardo-Nirenberg inequality that

$$\|\nabla(\phi+w)\|^{\frac{1+2\varepsilon}{1+\varepsilon}}_{L^\infty_T L^{4\frac{1+2\varepsilon}{1+3\varepsilon}}} \lesssim \|\nabla(\phi+w)\|^{\frac{\varepsilon}{1+\varepsilon}}_{L^\infty_T L^2}\|\Delta(\phi+w)\|_{L^\infty_T L^2}, \qquad (65)$$

and by Sobolev

$$\|w\|^{2\alpha_1-3}_{L^\infty_T L^4} \lesssim \|w\|^{2\alpha_1-3}_{L^\infty_T H^1}. \qquad (66)$$

We deduce from (62), (63), (64), (65) and (66) that if $\alpha_1 > 3/2$,

$$\begin{aligned}\|g_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}(1+\|w\|_{L^\infty_T H^1})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad +T^{\frac{\varepsilon}{2(1+\varepsilon)}}\|\nabla(\phi+w)\|^{\frac{1}{1+\varepsilon}}_{L^2_T L^4}\|\nabla(\phi+w)\|^{\frac{\varepsilon}{1+\varepsilon}}_{L^\infty_T L^2}\|\Delta(\phi+w)\|_{L^\infty_T L^2}\|w\|^{2\alpha_1-3}_{L^\infty_T H^1}. \quad (67)\end{aligned}$$

Concatenating (53), (54), (57), as well as (59), (60), (61) or (67), and summing over the indices of length 2, we get

$$\begin{aligned}\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2} &\lesssim \sum_{|\alpha|=2}\|\partial^\alpha w_0\|_{L^2} + T(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad +(T^{1/p'}\|w\|_{L^\infty_{T^*_k(w_0)}H^1} + T^\theta\|w\|^{\max(1,2\alpha_1-2)}_{X_{T^*_k(w_0)}})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) \\ &\quad + \begin{cases} T^{1/p'}(1+\|w\|_{L^\infty_{T^*_k(w_0)}H^1})^{3/2}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{1/2} & \\ +T^{\tilde\theta}(1+\|w\|_{L^\infty_{T^*_k(w_0)}H^1})\|w\|^{\max(1,2\alpha_1-3)}_{X_{T^*_k(w_0)}} & \text{if } n=2,3 \\ +T^{\check\theta}(1+\|w\|_{X_{T^*_k(w_0)}})\|w\|^{\max(0,2\alpha_1-3)}_{X_{T^*_k(w_0)}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2}) & \text{if } n=4, \end{cases} \qquad (68)\end{aligned}$$

where $\tilde{\theta}, \check{\theta} > 0$. Thus there exists a small $T_0 > 0$ depending only on $\|w\|_{X_{T_k^*(w_0)}} < \infty$ such that

$$\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_{T_0} L^2} \leqslant C(\|w\|_{X_{T_k^*(w_0)}}) + C \sum_{|\alpha|=2} \|\partial^\alpha w_0\|_{L^2},$$

where $C > 0$ and $C(\|w\|_{X_{T_k^*(w_0)}})$ only depends on $\|w\|_{X_{T_k^*(w_0)}}$. Recovering $[0, T_k^*(w_0)]$ by a finite number of intervals of length T_0, it follows that $\sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^\infty_T L^2}$ remains bounded as $T \uparrow T_k^*(w_0)$. This is a contradiction in dimensions $n = 2$ and $n = 3$.

For $n = 4$, we need to control the derivatives of order 3 of w. Denoting by $\partial^3 w$ one of these derivatives, $\partial^3 w$ solves an equation which may be written as

$$i\partial_t \partial^3 w + \Delta \partial^3 w = f_0 + f_1 + f_2 + f_3,$$

where

$$|f_0(t)| \lesssim |\Delta \partial^3 \phi| + |\phi|^2 |f'(|\phi|^2)| \partial^3(\phi + w)| + |f(|\phi|^2)||\partial^3(\phi + w)|,$$

$$|f_1(t)| \lesssim |\partial(\phi+w)|^3 \left[|f'(|\phi + w|^2)| + |\phi + w|^2 |f''(|\phi + w|^2)| + |\phi + w|^4 |f'''(|\phi + w|^2)| \right],$$

$$\begin{aligned} |f_2(t)| &\lesssim |\partial^2(\phi + w)||\partial(\phi + w)| \left[|\phi + w||f'(|\phi + w|^2)| + |\phi + w|^3 |f''(|\phi + w|^2)| \right] \\ &\lesssim |\partial^2(\phi + w)||\partial(\phi + w)|(1 + |w|^{\max(0, 2\alpha_1 - 3)}) \end{aligned} \quad (69)$$

and

$$\begin{aligned} |f_3(t)| &\lesssim |\partial^3(\phi + w)| \int_0^1 \left[|\phi + sw||f'(|\phi + sw|^2)| + |\phi + sw|^3 |f''(|\phi + sw|^2)| \right] ds |w| \\ &\lesssim |\partial^3(\phi + w)||w|(1 + |w|^{\max(0, 2\alpha_1 - 3)}). \end{aligned} \quad (70)$$

Thanks to the Strichartz estimates (19) and (20), for $T < T_3^*(w_0)$,

$$\|\partial^3 w\|_{L^\infty_T L^2} \lesssim \|\partial^3 w_0\|_{L^2} + \|f_0\|_{L^1_T L^2} + \|f_1\|_{L^2_T L^{4/3}} + \|f_2\|_{L^2_T L^{4/3}} + \|f_3\|_{L^2_T L^{4/3}} \quad (71)$$

Next,

$$\|f_0\|_{L^1_T L^2} \lesssim T(1 + \|\partial^3 w\|_{L^\infty_T L^2}). \quad (72)$$

Since $\alpha_1 < 2$, $(\mathrm{H}_{\alpha_1,\alpha_2})$ implies that $r \mapsto r^2 f'''(r)$, $r \mapsto r f''(r)$ and f' are bounded. Thus

$$\|f_1\|_{L^2_T L^{4/3}} \lesssim \|\partial(\phi+w)\|^3_{L^6_T L^4} \lesssim T^{1/2}\|\partial(\phi+w)\|^3_{L^\infty_T H^1}. \qquad (73)$$

Using (69), we get

$$\begin{aligned} \|f_2\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}\|\partial^2(\phi+w)\|_{L^\infty_T L^2}\|\partial(\phi+w)\|_{L^\infty_T L^4} \\ &\quad + \begin{cases} 0 & \text{if } \alpha_1 \leqslant 3/2 \\ \|\partial^2(\phi+w)\partial(\phi+w)|w|^{2\alpha_1-3}\|_{L^2_T L^{4/3}} & \text{if } \alpha_1 > 3/2 \end{cases} \end{aligned} \qquad (74)$$

where, by Hölder and Sobolev, if $\alpha_1 > 3/2$, choosing $\varepsilon = 2\alpha_1 - 3 \in (0,1)$,

$$\begin{aligned} &\|\partial^2(\phi+w)\partial(\phi+w)|w|^{2\alpha_1-3}\|_{L^2_T L^{4/3}} \\ &\lesssim T^{1/2}\|\partial^2(\phi+w)\|_{L^\infty_T L^2}\|\partial(\phi+w)\|_{L^\infty_T L^{\frac{4}{1-\varepsilon}}}\||w|^{2\alpha_1-3}\|_{L^\infty_T L^{\frac{4}{\varepsilon}}} \\ &\lesssim T^{1/2}(1+\|\partial^2 w\|_{L^\infty_T L^2})\|\partial(\phi+w)\|_{L^\infty_T H^2}\|w\|^{2\alpha_1-3}_{L^\infty_T H^1}. \end{aligned} \qquad (75)$$

Thanks to (70) and Sobolev,

$$\begin{aligned} \|f_3\|_{L^2_T L^{4/3}} &\lesssim T^{1/2}\|\partial^3(\phi+w)\|_{L^\infty_T L^2}\left(\|w\|_{L^\infty_T L^4} + \|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T L^{4\max(1,2\alpha_1-2)}}\right) \\ &\lesssim T^{1/2}(1+\|\partial^3 w\|_{L^\infty_T L^2})(\|w\|_{L^\infty_T H^1} + \|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^2}) \end{aligned} \qquad (76)$$

Concatenating (71), (72), (73),(74), (75), (76) and summing over the indices of length 3, we get

$$\begin{aligned} \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2} &\lesssim \sum_{|\alpha|=3}\|\partial^\alpha w_0\|_{L^2} + T(1+\sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2}) + T^{1/2}(1+\|w\|_{L^\infty_{T_3^*(w_0)}H^2})^3 \\ &+T^{1/2}(1+\|\partial^2 w\|_{L^\infty_{T_3^*}L^2})(1+\|w\|_{L^\infty_{T_3^*(w_0)}H^2} + \sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2})(1+\|w\|_{L^\infty_{T_3^*(w_0)}H^2})^{\max(0,2\alpha_1-3)} \\ &+T^{1/2}(1+\sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2})(\|w\|_{L^\infty_{T_3^*(w_0)}H^1} + \|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_{T_3^*(w_0)}H^2}). \end{aligned}$$

Therefore there exists $T_1 > 0$ sufficiently small and $C(\|w\|_{L^\infty_{T_3^*(w_0)}H^2}) > 0$, both depending only on $\|w\|_{L^\infty_{T_3^*(w_0)}H^2} < \infty$ such that

$$\sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2} \lesssim \sum_{|\alpha|=3}\|\partial^\alpha w_0\|_{L^2} + C(\|w\|_{L^\infty_{T_3^*(w_0)}H^2}).$$

We can recover $[0, T_3^*(w_0)]$ by a finite number of intervals of length T_1, and thus $\sum_{|\alpha|=3}\|\partial^\alpha w\|_{L^\infty_T L^2}$ remain bounded as $T \uparrow T_3^*(w_0)$. We have obtained a contradiction in the four dimensional case.

We are now ready to prove the global well-posedness of (3). Let $w_0 \in H^1(\mathbb{R}^n)$, $n = 2, 3$ or 4, and $T > 0$ be such that there exists a solution $w \in \mathcal{C}([0,T], H^1)$ to (3) with initial data w_0 (such a T exists thanks to Theorem 4.1). Let us take a sequence $(w_{0,n})_n \subset H^k$ ($k = 2$ if $n = 2, 3$, while $k = 3$ if $n = 4$) which converges to w_0 in H^1. By the lower semi-continuity of T_1^* (which is a byproduct of the local Lipshitz continuity of the flow map), $T_1^*(w_{0,n}) \geqslant T_1^*(w_0) \geqslant T$ for n large. We have just seen that $T_k^*(w_{0,n}) = T_1^*(w_{0,n})$. Therefore for n large, the energy is conserved for w_n on $[0,T]$. Namely, for all $t \in [0,T]$, $\mathcal{E}(w_n(t)) = \mathcal{E}(w_{0,n})$, where

$$\mathcal{E}(w) = \int_{\mathbb{R}^n} |\nabla(\phi + w)|^2 dx + \int_{\mathbb{R}^n} V(|\phi + w|^2) dx.$$

Moreover, $w_n \to w$ in X_T. The map $w \mapsto V(|\phi + w|^2)$ is continuous from H^1 into L^1, as it can easily be deduced from (10), (11), Sobolev embeddings and the first condition in ($\mathrm{H}_{\alpha_1,\alpha_2}$). It follows that for every $t \in [0,T]$, $\mathcal{E}(w(t)) = \mathcal{E}(w_0)$. Moreover, for every $t \in [0,T]$, Lemmas 3.2 and 3.3 give the estimates

$$||w_n(t)||_{L^2}^2 \leqslant (1 + ||w_{0,n}||_{L^2}^2) e^{C_3 t}$$

and

$$||\nabla(\phi + w_n(t))||_{L^2}^2 \leqslant C_1(1 + (1 + ||w_{0,n}||_{L^2}^2) e^{C_3 t}).$$

Passing to the limit $n \to \infty$, these inequalities remains true for w_n replaced by w, for every $t \in [0,T]$. It follows that the H^1 norm of w remains bounded on bounded intervals.

The proof of Theorem 1.1 will be complete if we show the Lipshitz continuity of the flow on bounded sets of H^1. That is what we next do.

Proof of the Lipshitz continuity of the flow. Let $T > 0$, $R > 0$ and $w_0, \tilde{w}_0 \in H^1$ with H^1 norm less than R. Let $w, \tilde{w} \in \mathcal{C}([0,T], H^1)$ be the associated solutions to (3). Then, as in the proof of the local Lipshitz continuity of the flow we gave in the previous section, slightly modified versions of Lemmas 4.3 and 4.4 yield for $\tilde{T} \leqslant T$,

$$\begin{aligned} \|w - \tilde{w}\|_{X_{\tilde{T}}} \;&\leqslant\; C\|w_0 - \tilde{w}_0\|_{H^1} \\ &+ C\tilde{T}^\theta \|w - \tilde{w}\|_{X_{\tilde{T}}} (\|w\|_{X_T} + \|\tilde{w}\|_{X_T} + \|w\|_{X_T}^{\max(1,2\alpha_1 - 2)} + \|\tilde{w}\|_{X_T}^{\max(1,2\alpha_1-2)}) \end{aligned}$$

Thanks to our estimation on the H^1 norm of w, $\|w\|_{L_T^\infty H^1}$ and $\|\tilde{w}\|_{L_T^\infty H^1}$ are majored by a quantity which only depends on R and T. So are $\|w\|_{X_T}$ and $\|\tilde{w}\|_{X_T}$, because of the same arguments we used in the previous section to prove the local Lipshitz continuity of the flow. Thus there exists $h(R,T) > 0$ such that

$$\|w\|_{X_T} + \|\tilde{w}\|_{X_T} + \|w\|_{X_T}^{\max(1,2\alpha_1-2)} + \|\tilde{w}\|_{X_T}^{\max(1,2\alpha_1-2)} \leqslant h(R,T).$$

Choosing $\tilde{T} > 0$ small enough such that

$$C\tilde{T}^{\theta} h(R,T) \leqslant 1/2,$$

we obtain

$$\|w - \tilde{w}\|_{X_{\tilde{T}}} \leqslant 2C\|w_0 - \tilde{w}_0\|_{H^1}.$$

Next, recovering $[0,T]$ by small intervals of length $\tilde{T}$ and repeating this argument on each of these intervals, we get

$$\|w - \tilde{w}\|_{L^\infty_T H^1} \leqslant (2C)^{\lceil \frac{T}{\tilde{T}} \rceil} \|w_0 - \tilde{w}_0\|_{H^1},$$

which completes the proof. □

6 Well-posedness in the energy space

This section is devoted to the well-posedness in the energy space. We prove Proposition 1.1 and Theorem 1.5, using arguments developed in [Ge].

6.1 Decomposition of a data in the energy space

Here, following P. Gérard, we give a proof of Proposition 1.1.

Let us take u in the energy space

$$E = \{u \in H^1_{\text{loc}}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), \rho_0 - |u|^2 \in L^2(\mathbb{R}^n)\}.$$

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{C})$ be such that $0 \leqslant \chi \leqslant 1$, $\chi(z) \equiv \begin{cases} 1 & \text{if} \quad |z| \leqslant \sqrt{2\rho_0} \\ 0 & \text{if} \quad |z| \geqslant \sqrt{3\rho_0} \end{cases}$. We also choose $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho = 1$, $0 \leqslant \rho \leqslant 1$ and ρ is supported in the ball of radius 1.

We first decompose u as

$$u = (1-\chi)(u)u + \chi(u)u.$$

As it was mentioned by P. Gérard in [Ge], we have on the one side $(1-\chi)(u)u \in H^1(\mathbb{R}^n)$, and on the other side $\chi(u)u \in L^\infty(\mathbb{R}^n)$, $\nabla(\chi(u)u) \in L^2(\mathbb{R}^n)$, and thus $\chi(u)u \in H^1_{\text{loc}}(\mathbb{R}^n)$. Moreover, the choice of χ ensures

$$||\chi(u)u|^2 - \rho_0| = ||u|^2 - \rho_0| \text{ if } |u|^2 \leqslant 2\rho_0,$$

while if $|u|^2 \geqslant 2\rho_0$, one has

$$\begin{aligned} \big||\chi(u)u|^2-\rho_0\big| &= \max(\chi(u)^2|u|^2-\rho_0, \rho_0-\chi(u)^2|u|^2) \\ &\leqslant \max(|u|^2-\rho_0, \rho_0) \leqslant |u|^2-\rho_0. \end{aligned}$$

In all these cases, we have $||\chi(u)u|^2-\rho_0| \leqslant ||u|^2-\rho_0|$. Therefore $\chi(u)u \in E$.

Next, we split $\chi(u)u$ as

$$\chi(u)u = \rho * (\chi(u)u) + (\chi(u)u - \rho * (\chi(u)u)).$$

Since $\chi(u)u \in X^1(\mathbb{R}^n)$, it is clear that $\psi := \rho * (\chi(u)u) \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\nabla\psi \in H^\infty(\mathbb{R}^n)$. In order to prove Proposition 1.1, it remains to verify that $|\psi|^2-\rho_0 \in L^2(\mathbb{R}^n)$ and $\chi(u)u-\psi \in H^1(\mathbb{R}^n)$. Since $\chi(u)u \in E$, this is a consequence of the next two lemmas, which have been proved in [Ge].

Lemma 6.1 *If $v \in E$, $|\rho * v|^2 - \rho_0 \in L^2$*

Lemma 6.2 *If $v \in E$, $v - \rho * v \in H^1$.*

Proof of Lemma 6.1. This was proved in [Ge]. We recall the arguments.

$$\begin{aligned} |(\rho * v)(x)|^2 - \rho_0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x-y)\rho(x-\tilde{y})(v(y)\overline{v}(\tilde{y})-\rho_0)dyd\tilde{y} \\ &= \underbrace{\rho * (|u|^2-\rho_0)}_{\in L^2}(x) + r(x), \end{aligned}$$

where

$$\begin{aligned} r(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x-y)\rho(x-\tilde{y})v(y)(\overline{v}(\tilde{y})-\overline{v}(y))dyd\tilde{y} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x-y)\rho(x-\tilde{y})v(y)\int_0^1 (\tilde{y}-y)\nabla\overline{v}(y+s(\tilde{y}-y))dsdyd\tilde{y} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x-y)\rho(x-y-a)v(y)\int_0^1 a\nabla\overline{v}(y+sa)dsdyda \\ &= \int_0^1\int_{|a|\leqslant 2} a\left(\int_{\mathbb{R}^n}\rho_a(x-y)v(y)\nabla\overline{v}(y+sa)dy\right)dads, \end{aligned}$$

where $\rho_a(z) = \rho(z)\rho(z-a)$.

As it was mentioned in [Ge], for every positive integer n, $E \subset X^1(\mathbb{R}^n) + H^1(\mathbb{R}^n)$. In particular,

$$E \subset \begin{cases} L^\infty & \text{if } n=1 \\ L^\infty + L^6 & \text{if } n=2 \\ L^\infty + L^{\frac{2n}{n-2}} & \text{if } n \geqslant 3 \end{cases}$$

since $\nabla v \in L^2$, the Hölder inequality yields

$$v\nabla\overline{v}(.+sa) \in \begin{cases} L^2 & \text{if } n=1 \\ L^2 + L^{3/2} & \text{if } n=2 \\ L^2 + L^{\frac{n}{n-1}} & \text{if } n \geqslant 3 \end{cases},$$

with the norm of $v\nabla\overline{v}(.+sa)$ in the corresponding space uniformly bounded in s and a. Next, $\frac{1}{2} = \frac{1}{p} + \frac{1}{q} - 1$, with respectively $q = 2,\ 3/2,\ \frac{n}{n-1}$, yields respectively $p = 1,\ 6/5,\ \frac{2n}{n+2} \in [1, \infty]$. ρ_a belongs to all these L^p spaces, with norm uniformly bounded in a. Therefore the Young inequality implies that the map

$$x \mapsto \int_{\mathbb{R}^n} \rho_a(x-y)u(y)\nabla\overline{u}(y+sa)dy$$

belongs to $L^2(\mathbb{R}^n)$, and the Lemma has been proved. □

Proof of Lemma 6.2. It is clear that $\nabla(v - \rho * v) \in L^2$. Let us verify that $v - \rho * v \in L^2$. Thanks to the properties of ρ,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\rho * v(x) - v(x)|^2 dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \rho(y)(v(x-y) - v(x))dy \right|^2 dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \rho(y) \int_0^1 -y\nabla v(x-sy)dsdy \right|^2 dx \\
&\leqslant \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(y) \left| \int_0^1 y\nabla v(x-sy)ds \right|^2 dydx \\
&\leqslant \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(y) \int_0^1 |\nabla v(x-sy)|^2 dsdydx \leqslant ||\nabla v||_{L^2}^2.
\end{aligned}$$

□

6.2 Uniqueness in the energy space.

Using arguments introduced by P. Gérard in [Ge], we prove here Theorem 1.5. In this section, χ denotes a cutoff function: $\chi \in \mathcal{C}_c^\infty(\mathbb{C})$, $0 \leqslant \chi \leqslant 1$, and

$$\chi(z) \equiv \begin{cases} 1 \text{ for } |z| \leqslant 1, \\ 0 \text{ for } |z| \geqslant 2. \end{cases}$$

If $\mu > 0$, we also denote $\chi_\mu(z) = \chi(z/\mu)$. We first state some elementary lemmas about the $L^p + L^q$ spaces. The proofs are in the appendix.

Lemma 6.3 *Let $1 \leqslant p < q \leqslant \infty$ and $f \in L^p + L^q$. Let $f_p \in L^p$ and $f_q \in L^q$ such that $f = f_p + f_q$. Then for every $\mu > 0$ if $q < \infty$ (resp. $\mu > 2\|f_q\|_{L^\infty}$ if $q = \infty$), $\chi_\mu(f)f \in L^q$ and $(1-\chi_\mu)(f)f \in L^p$. Moreover, we have the estimates*

$$\|\chi_\mu(f)f\|_{L^q} \leqslant \begin{cases} 3\mu^{1-p/q}\|f_p\|_{L^p}^{p/q} + 2\|f_q\|_{L^q} & \text{if } q < \infty \\ 2\mu & \text{if } q = \infty, \end{cases}$$

and

$$\|(1-\chi_\mu)(f)f\|_{L^p} \leqslant \begin{cases} \|f_p\|_{L^p} + \mu\left(\left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p + \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q\right)^{1/p} + \frac{\|f_q\|_{L^q}^{q/p}}{\mu^{\frac{q-p}{p}}} & \text{if } q < \infty \\ 3\|f_p\|_{L^p} & \text{if } q = \infty. \end{cases}$$

In particular, if $f_p \neq 0$, $f_q \neq 0$, defining μ_0 as follows,

$$\mu_0 = \begin{cases} \left(\frac{\|f_q\|_{L^q}^q}{\|f_p\|_{L^p}^p}\right)^{\frac{1}{q-p}} & \text{if } q < \infty \\ 3\|f_q\|_{L^\infty} & \text{if } q = \infty, \end{cases} \tag{77}$$

we have

$$\|\chi_{\mu_0}(f)f\|_{L^q} \leqslant 6\|f_q\|_{L^q} \tag{78}$$

and

$$\|(1-\chi_{\mu_0})(f)f\|_{L^p} \leqslant (2 + (2^p + 2^q)^{1/p})\|f_p\|_{L^p}. \tag{79}$$

Lemma 6.4 *If $|f| \leqslant |g|$ and $g \in L^p + L^q$ for some $p, q \in [1, \infty]$, then $f \in L^p + L^q$ with $\|f\|_{L^p+L^q} \leqslant C(p,q)\|g\|_{L^p+L^q}$, where $C(p,q) > 0$.*

Lemma 6.5 *Let $1 \leqslant p_1 < p < p_2 \leqslant \infty$. Then $L^p \subset L^{p_1} + L^{p_2}$, with a continuous embedding.*

Lemma 6.6 *Let $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant q_1 < q_2 \leqslant \infty$, $f = f_1 + f_2$, where $f_j \in L^{p_j}$ and $g = g_1 + g_2$, where $g_j \in L^{q_j}$. Then $fg \in L^{\frac{p_1 q_1}{p_1+q_1}} + L^{\frac{p_2 q_2}{p_2+q_2}}$, and*

$$\|fg\|_{L^{\frac{p_1 q_1}{p_1+q_1}} + L^{\frac{p_2 q_2}{p_2+q_2}}} \leqslant C\|f\|_{L^{p_1}+L^{p_2}}\|q\|_{L^{q_1}+L^{q_2}},$$

where $C > 0$ depends only on the p_j and the q_j.

Proof of Theorem 1.5. Let $T>0$, $u_0\in E$, $u\in\mathcal{C}([0,T],E)$ be as in the statement of the theorem and $v\in u_0+\mathcal{C}(\mathbb{R},H^1)\subset\mathcal{C}(\mathbb{R},E)$ the solution to (1) given by Theorem 1.2. Then, for every $t\in[0,T]$,

$$v(t)-u(t)=-i\int_0^t e^{i(t-s)\Delta}\left[G(v(s))-G(u(s))\right]ds, \tag{80}$$

where $G(u)=-f(|u|^2)u$. Next,

$$\begin{aligned} G(v)-G(u) &= (|u|^2-\rho_0)(u-v)\int_0^1 f'(\rho_0+s(|u|^2-\rho_0))ds \\ &\quad +v(|u|^2-|v|^2)\int_0^1 f'(|v|^2+s(|u|^2-|v|^2))ds. \end{aligned} \tag{81}$$

The Sobolev embeddings ensure that $u,v\in L^\infty+H^1\subset L^\infty+L^{q_0}$, where $q_0>2$ is as large as we want it to be if $n=2$, $q_0=6$ if $n=3$ and $q_0=4$ if $n=4$. The first assertion in $(\mathrm{H}_{\alpha_1,\alpha_2})$ ensures that for $s\in[0,1]$, $|f'(\rho_0+s(|u|^2-\rho_0))|\leqslant C(1+|u|^2)^{\max(0,\alpha_1-2)}$, where $C>0$. Thus, thanks to Lemma 6.4, $f'(\rho_0+s(|u|^2-\rho_0))\in L^\infty+L^{\frac{q_0}{2\max(0,\alpha_1-2)}}$. In a similar way, $f'(|v|^2+s(|u|^2-|v|^2))\in L^\infty+L^{\frac{q_0}{2\max(0,\alpha_1-2)}}$. It follows from Lemma 6.6 that

$$G(v)-G(u)\in L^2+L^{q'},$$

where $\frac{1}{q'}=\frac{2\max(0,\alpha_1-2)}{q_0}+\frac{1}{2}+\frac{1}{q_0}$. For $n=2$, q_0 is chosen large enough such that $q'>1$. For $n=3$, $q'=6/\max(4,2\alpha_1)\geqslant 6/5$ only if $\alpha_1\leqslant 5/2$, while for $n=4$, $q'=4/3$. From (81), Lemma 6.6, Sobolev embeddings and Lemma 6.4 we deduce

$$\begin{aligned} \|G(v)-G(u)\|_{L^2+L^{q'}} &\lesssim (1+\|u\|_{X^1+H^1})^{2\max(0,\alpha_1-2)}\|u-v\|_{X^1+H^1}d_E(u,\sqrt{\rho_0}) \\ &\quad +\|v\|_{X^1+H^1}d_E(u,v)(1+\|u\|_{X^1+H^1}+\|v\|_{X^1+H^1})^{2\max(0,\alpha_1-2)}. \end{aligned} \tag{82}$$

Since $u,v\in\mathcal{C}([0,T],E)$, the right hand side in (82) is uniformly bounded on $[0,T]$. Therefore $G(v)-G(u)\in L^1_TL^2+L^{p'}_TL^{q'}$, where (p,q) is an admissible pair. Thus, it follows from (80) and the non-homogeneous Strichartz estimate (20) that $u-v\in\mathcal{C}([0,T],L^2)$. Since $u,v\in\mathcal{C}([0,T],E)$, we already know that $\nabla(u-v)\in\mathcal{C}([0,T],L^2)$. Thus $u-v\in\mathcal{C}([0,T],H^1)$. The result follows for $n=2$, $n=4$, and $n=3$ with the supplementary condition $\alpha_1\leqslant 5/2$. Next, we prove the result for $n=3$, $\alpha_1\in(5/2,3)$. In that case, (82) remains true, but with $q'\in(1,6/5)$, in sort that the non-homogeneous Strichartz estimate may not be applied. However, we deduce from Lemma 6.3 and (82) that for every $\mu>0$, $\chi_\mu(G(v)-G(u))(G(v)-G(u))\in L^2$ (and $(1-\chi_\mu)(G(v)-G(u))(G(v)-$

$G(u)) \in L^{q'}$), with an L^2 norm uniformly bounded on $[0,T]$ by a quantity Q which only depends on μ, q', $\sup_{t\in[0,T]} \|u(t)\|_{X^1+H^1}$, $\sup_{t\in[0,T]} \|v(t)\|_{X^1+H^1}$, $\sup_{t\in[0,T]} d_E(u(t),v(t))$ and $\sup_{t\in[0,T]} d_E(u(t),\sqrt{\rho_0})$. Thus

$$\begin{aligned} &\|\chi_\mu(G(v)-G(u))(G(v)-G(u))\|_{L^1_T L^2} \\ &\leqslant\ TQ(\mu,q',\sup_{t\in[0,T]}\|u(t)\|_{X^1+H^1},\sup_{t\in[0,T]}\|v(t)\|_{X^1+H^1},\sup_{t\in[0,T]} d_E(u(t),v(t)),\sup_{t\in[0,T]} d_E(u(t),\sqrt{\rho_0})) \end{aligned} \tag{83}$$

Next, the first assertion in ($\mathrm{H}_{\alpha_1,\alpha_2}$) ensures $|G(u)| \lesssim |u|(1+|u|^2)^{\alpha_1-1}$. Since $u \in L^\infty + L^6$, Lemma 6.4 implies $G(u) \in L^\infty + L^{\frac{6}{2\alpha_1-1}}$. The same is true for $G(v)$, and we have

$$\|G(v)-G(u)\|_{L^\infty+L^{\frac{6}{2\alpha_1-1}}} \lesssim (1+\|u\|_{X^1+H^1}+\|v\|_{X^1+H^1})^{2\alpha_1-1}.$$

The right hand side is uniformly bounded on $[0,T]$, thus, thanks to Lemma 6.3, $(1-\chi_1)(G(v)-G(u))(G(v)-G(u)) \in L^{\tilde{p}'}_T L^{\tilde{q}'}$, where $(\tilde{p},\tilde{q})$ is an admissible pair, and $\tilde{q}' = \frac{6}{2\alpha_1-1} \in (6/5,3/2)$ if $\alpha_1 \in (5/2,3)$. We have shown that $G(v)-G(u) \in L^1_T L^2 + L^{\tilde{p}'}_T L^{\tilde{q}'}$. As in the previous case, it follows from the non homogeneous Strichartz estimate that $u-v \in \mathcal{C}([0,T],H^1)$. □

7 The case of an exterior domain

In this section, K denotes a smooth, compact, non-trapping, non-empty obstacle in $\mathbb{R}^n$, $n=2,3$, and $\Omega = \mathbb{R}^n \backslash K$. The Strichartz estimates we used in the previous sections on $\mathbb{R}^n$ fail when we are working on such an open set Ω. However, N. Burq, P. Gérard and N. Tzvetkov obtained in [BGT] the following Strichartz type estimates.

Proposition 7.1 *For every pair $(p,q) \in [2,\infty]$ such that*

$$\frac{1}{p}+\frac{n}{q}=\frac{n}{2}, \tag{84}$$

for every $T>0$, there exists $C(T)>0$ such that for every $v_0 \in L^2(\Omega)$,

$$\|e^{it\Delta_D}v_0\|_{L^p_T L^q(\Omega)} \leqslant C(T)\|v_0\|_{L^2(\Omega)}, \tag{85}$$

and, if $(\tilde{p},\tilde{q})$ satisfies (84) and moreover $p>2$, $\tilde{p}>2$, then for every $f \in L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)$,

$$\|\int_0^t e^{i(t-\tau)\Delta_D} f(\tau)d\tau\|_{L^p_T L^q(\Omega)} \leqslant C(T)\|f\|_{L^{\tilde{p}'}_T L^{\tilde{q}'}(\Omega)}, \tag{86}$$

where Δ_D denotes the Dirichlet Laplacian. This is also true if Δ_D is replaced by the Neumann Laplacian Δ_N.

These Strichartz type inequalities are sufficient to prove the global well-posedness result for the initial value problem (5) stated in Theorem 1.6.

Proof of Theorem 1.6. The proof is very similar to that we made in the case of $\mathbb{R}^n$, so that we will only indicate the main changes. We only do it in the Dirichlet case, exactly the same arguments value for the Neumann case.

As in section 7, we remark that a data $u_0 \in E_D$ may be decomposed as $u_0 = \phi + w_0$, where $w_0 \in H_0^1(\Omega)$ and

$$\phi \in \mathcal{C}_b^\infty(\Omega),\ \nabla\phi \in H^\infty(\Omega),\ \mathrm{Supp}\phi \subset \Omega\backslash(V \cap \Omega),\ |\phi|^2 - \rho_0 \in L^2(\Omega)\ .$$

Indeed, defining $\tilde{u}_0$ as the extension of u_0 to $\mathbb{R}^n$ by 0, the results of section 7 ensure that

$$\tilde{u}_0 = \tilde{\phi} + \tilde{w}_0 = (1-\chi)\tilde{\phi} + \chi\tilde{\phi} + \tilde{w}_0\ ,$$

where $\tilde{\phi}$ satisfies (H_ϕ) and $\tilde{w}_0 \in H^1(\mathbb{R}^n)$. Then $\phi := ((1-\chi)\tilde{\phi})_{|\Omega}$ and $w_0 := (\chi\tilde{\phi} + \tilde{w}_0)_{|\Omega}$ satisfy the required conditions.

As in the case $\Omega = \mathbb{R}^n$, we can look a solution to (5) as $u = \phi + w$, where ϕ is as above and $w(t) \in H_0^1(\Omega)$. We are reduced to study the Cauchy problem

$$\begin{cases} i\frac{\partial w}{\partial t} + \Delta_D w = F(w),\ (t,x) \in \mathbb{R}\times\Omega \\ w(0) = w_0 \end{cases}, \tag{87}$$

where $F(w) = -\Delta_D\phi - f(|\phi + w|^2)(\phi + w)$. Retaking the arguments developped in the proofs of Lemmas 2.1 and 2.2, it is easy to see that F maps $H_0^2(\Omega)$ into $H^2(\Omega)$ and that it is locally Lipshitz continuous. Moreover, if $w \in H_0^2(\Omega)$ and $(w_n)_n$ is a sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $w_n \to w$ in $H^2(\Omega)$, $F(w_n) \to F(w)$ in H^2 and $F(w_n) \in \mathcal{C}_c(\Omega)$. Thus $F(w) \in H_0^2(\Omega)$. Therefore F defines a locally Lipshitz continuous map from $H_0^2(\Omega)$ into itself.

Next, we remark that the operator A on $H_0^2(\Omega)$ defined by

$$\begin{cases} D(A) = \{w \in H_0^2(\Omega), \Delta_D w \in H_0^2(\Omega)\}, \\ Aw = i\Delta_D w \ \text{ for } w \in D(A) \end{cases}$$

generates a strongly continuous group e^{tA} which is the restriction of $e^{it\Delta_D}$ to $H_0^2(\Omega)$. Therefore the classical theory for nonlinear evolution equations implies that for every $w_0 \in H_0^2(\Omega)$, there exists a maximal time of existence $T^*(w_0)$ such that (87) has an unique mild solution $w \in \mathcal{C}([0,T^*), H_0^2(\Omega))$. If $w_0 \in H_0^4(\Omega)$, $w \in \mathcal{C}([0,T^*), D(A)) \cap \mathcal{C}^1((0,T^*), H_0^2(\Omega))$.

As in Lemma 3.1, we obtain the conservation of the energy first for $w_0 \in H_0^4(\Omega)$ and then for $w_0 \in H_0^2(\Omega)$ by density of $H_0^4(\Omega)$ into $H_0^2(\Omega)$. One may prove analogous results to Lemmas 3.2 and 3.3 with identical proofs.

As in the R^n case, the local well-posedness is obtained by a fix point argument for the functional

$$\Phi(w) = e^{it\Delta_D} w - i \int_0^t e^{i(t-s)\Delta_D} F(w(s)) ds$$

in a space $X_T = L^\infty_T H^1 \cap L^{p_0}_T W^{1,q_0}$, where (p_0, q_0) satisfies (84) and $p_0 > 2$ is close to 2. We begin by giving the analogous of the estimates established in Lemmas 4.1, 4.2, 4.3 and 4.4.

In the sequel, $\varepsilon_0 >$ and $\varepsilon' > 0$ satisfy the following conditions

$$\begin{cases} \varepsilon_0 < 1,\ \varepsilon' < 1 & \text{if } n = 2, \\ \begin{cases} \varepsilon' + \max(1, 2\alpha_1 - 2)\varepsilon_0 \leqslant 1 \\ \varepsilon' + \varepsilon_0 \leqslant 4 - 2\alpha_1 \end{cases} & \text{if } n = 3, \end{cases} \tag{88}$$

and $p_0,\ q_0,\ p',\ q'$ are defined by

	p_0	q_0	p'	q'
$n = 2$	$\frac{2}{1-\varepsilon_0} > 2$	$\frac{4}{1+\varepsilon_0} < 4$	$\frac{2}{1+\varepsilon'} < 2$	$\frac{4}{3-\varepsilon'} > 4/3$
$n = 3$	$\frac{2}{1-\varepsilon_0} > 2$	$\frac{6}{2+\varepsilon_0} < 3$	$\frac{2}{1+\varepsilon'} < 2$	$\frac{6}{4-\varepsilon'} > 3/2$

(89)

Lemma 7.1 *Let $n = 2, 3$, with $\alpha_1 < 2$ if $n = 3$, and let $T_0 > 0$, $w_0 \in H^1$. Then there exists $C > 0$ such that for every $w \in L^\infty_T H^1$, $T \leqslant T_0$,*

$$\begin{aligned} &\|\Phi(w)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} \\ &\leqslant \|w_0\|_{L^2} + CT(1 + \|w\|_{L^\infty_T L^2}) + CT^{1/p'}(\|w\|^2_{L^\infty_T H^1} + \|w\|^{\max(2,2\alpha_1-1)}_{L^\infty_T H^1}) \end{aligned} \tag{90}$$

Lemma 7.2 *Under the same assumptions, there exists $C > 0$ and $\theta > 0$ such that for every $w \in X_T$, $T \leqslant T_0$,*

$$\begin{aligned} \|\nabla\Phi(w)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} &\leqslant \|\nabla w_0\|_{L^2} + CT(1 + \|\nabla w\|_{L^\infty_T L^2}) \\ &+ CT^\theta(1 + \|w\|_{X_T})(\|w\|_{X_T} + \|w\|^{\max(1,2\alpha_1-2)}_{X_T}). \end{aligned} \tag{91}$$

Lemma 7.3 *Under the same assumptions that for Lemma 7.1, for every $w_1, w_2 \in L^\infty_T H^1$, $T \leqslant T_0$,*

$$\begin{aligned} &\|\Phi(w_1) - \Phi(w_2)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} \lesssim T\|w_1 - w_2\|_{L^\infty_T L^2} \\ &+ T^{1/p'}\|w_1 - w_2\|_{L^\infty_T H^1}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1} + \|w_1\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1} + \|w_2\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1}). \end{aligned} \tag{92}$$

Lemma 7.4 *Under the same assumptions that for Lemma 7.1, there exists $\theta_1, \theta_2 > 0$ such that for every $w_1, w_2 \in X_T$, $T \leqslant T_0$,*

$$\begin{aligned} \|\nabla\Phi(w_1) - \nabla\Phi(w_2)\|_{L^\infty_T L^2 \cap L^{p_0}_T L^{q_0}} &\lesssim T\|\nabla(w_1 - w_2)\|_{L^\infty_T L^2} \\ &+T^{\theta_1}\|w_1 - w_2\|_{X_T}(\|w_1\|_{X_T} + \|w_2\|_{X_T}) \\ &+T^{\theta_2}\|w_1 - w_2\|_{X_T}(1 + \|w_1\|_{X_T} + \|w_2\|_{X_T})(1 + \|w_1\|_{X_T}^{\max(0,2\alpha_1-3)} + \|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}). \end{aligned} \tag{93}$$

Proof of Lemma 7.1. Taking into account the new choice of parameters p_0, q_0, p', q' given by (89) and the new Strichartz estimates given in Proposition 7.1, the proof is similar to that of Lemma 4.1. In dimension 2, the Sobolev embedding $H^1 \subset L^{2q'}$ and $H^1 \subset L^{q' \max(2,2\alpha_1-1)}$ are true because $2q', q'\max(2, 2\alpha_1 - 1) \in [2, \infty)$. In dimension 3, they are true because $\varepsilon' \leqslant \min(2, 5 - 2\alpha_1)$, which implies $2q', q'\max(2, 2\alpha_1 - 1) \leqslant 6$. □

Proof of Lemma 7.2. The proof is rather similar to that of Lemma 4.2. ∇F may still be decomposed as $G_1 + G_2$. With the new parameters p_0, q_0, p', q', estimations (28), (29) and (30) remain true. In the two-dimensional case, using (30) as we did it to obtain (31), we easily get

$$\|G_2(w)\|_{L^{p'}_T L^{q'}} \lesssim T^{1/p'}(1 + \|w\|_{X_T})(\|w\|_{X_T} + \|w\|_{X_T}^{\max(1,2\alpha_1-2)}). \tag{94}$$

Let us give a little bit more details for the proof of a similar result in dimension three. Using (30), an estimation on the $L^{p'}_T L^{q'}$ norm of $G_2(w)$ may be reduced to estimations on both $\nabla(\phi + w)w$ and $\nabla(\phi + w)|w|^{\max(1,2\alpha_1-2)}$ for the same norm. Next, using the Hölder inequality and Sobolev embeddings and taking into account the value of parameters p' and q' given by (89), it follows that

$$\begin{aligned} \|\nabla(\phi + w)w\|_{L_T^{\frac{2}{1+\varepsilon'}} L^{\frac{6}{4-\varepsilon'}}} &\lesssim \|\nabla(\phi + w)\|_{L_T^{\frac{2}{1+\varepsilon'}} L^{\frac{6}{3-\varepsilon'}}} \|w\|_{L^\infty_T L^6} \\ &\lesssim T^{\frac{1+\varepsilon'}{2} - \frac{\varepsilon'}{2}} \|\nabla(\phi + w)\|_{L_T^{\frac{2}{\varepsilon'}} L^{\frac{6}{3-\varepsilon'}}} \|w\|_{L^\infty_T H^1} \\ &\lesssim T^{1/2}(1 + \|w\|_{X_T})\|w\|_{X_T}. \end{aligned} \tag{95}$$

For the very last inequality, we used that the pair $(\frac{2}{\varepsilon'}, \frac{6}{3-\varepsilon'})$ satisfies (84) and an interpolation argument as in Lemma 4.2. This can be done, provided $\frac{6}{3-\varepsilon'} \leqslant \frac{6}{2+\varepsilon_0}$, which is equivalent to $\varepsilon' + \varepsilon_0 \leqslant 1$. This is a consequence of (88).

The same arguments yield

$$\begin{aligned}\|\nabla(\phi+w)|w|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\phi+w)\|_{L^{p'}_T L^{\beta_1}} \||w|^{\max(1,2\alpha_1-2)}\|_{L^\infty_T L^{\frac{6}{\max(1,2\alpha_1-2)}}} \\ &\lesssim T^{\frac{1}{p'}-\frac{1}{s_1}} \|\nabla(\phi+w)\|_{L^{s_1}_T L^{\beta_1}} \|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1} \\ &\lesssim T^{\frac{1}{p'}-\frac{1}{s_1}} (1+\|w\|_{X_T})\|w\|^{\max(1,2\alpha_1-2)}_{X_T}, \qquad (96)\end{aligned}$$

where β_1 is given by $1/q' = 1/\beta_1 + \max(1, 2\alpha_1 - 2)/6$, and $1/s_1 + 3/\beta_1 = 3/2$. (88) ensures that $2 \leqslant \beta_1 \leqslant q_0$. Moreover, $1/p' - 1/s_1 = \min(1, 4-2\alpha_1)/2 > 0$. Thanks to (28), (29), (94), (95) and (96), the lemma easily follows as in Lemma 4.2. □

Proof of Lemma 7.3. We use the decomposition of $F(w_1) - F(w_2)$ given in the proof of Lemma 4.3. Inequalities (35), (36) and (37) are still valuable. Using the same arguments that in the proof of Lemma 7.1 (in particular, $q' \max(2, 2\alpha_1 - 1) \leqslant 6$ in dimension 3), we obtain, for $j = 1, 2$,

$$\||w_1 - w_2||w_j|\|_{L^{p'}_T L^{q'}} \lesssim T^{\frac{1}{p'}} \|w_1 - w_2\|_{L^\infty_T H^1} \|w_j\|_{L^\infty_T H^1}$$

and

$$\||w_1 - w_2||w_j|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} \lesssim T^{\frac{1}{p'}} \|w_1 - w_2\|_{L^\infty_T H^1} \|w_j\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1}.$$

The lemma follows. □

Proof of Lemma 7.4. We use the decomposition of $\nabla F(w_1) - \nabla F(w_2)$ into $\gamma_1 + \gamma_2 + \gamma_3$ written in the proof of Lemma 4.4. Using (41) and (42) (which remain true, with the new value of p', q', p_0, q_0), it suffices to control the $L^{p'}_T L^{q'}$ norm of γ_2 and γ_3. This will be done thanks to estimates (43) and (44) as follows. As in the proof of Lemma 7.2, for $j = 1, 2$, we get

$$\|\nabla(w_1 - w_2)|w_j|\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_1} \|w_1 - w_2\|_{X_T} \|w_j\|_{L^\infty_T H^1},$$

$$\|(|\nabla\phi|+|\nabla w_1|+|\nabla w_2|)(w_1-w_2)\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_1} \|w_1-w_2\|_{X_T}(1+\|w_1\|_{X_T}+\|w_2\|_{X_T}),$$

$$\|\nabla(w_1 - w_2)|w_j|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} \lesssim T^{\theta_2} \|w_1 - w_2\|_{X_T} \|w_j\|^{\max(1,2\alpha_1-2)}_{X_T}, \quad (97)$$

as well as

$$\begin{aligned}&\|(|\nabla\phi| + |\nabla w_1| + |\nabla w_2|)(w_1 - w_2)(|w_1| + |w_2|)^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}} \\ &\lesssim T^{\theta_2} \|w_1 - w_2\|_{X_T} (1 + \|w_1\|_{X_T} + \|w_2\|_{X_T})(\|w_1\|_{X_T} + \|w_2\|_{X_T})^{\max(0,2\alpha_1-3)}, \quad (98)\end{aligned}$$

where $\theta_1 = \theta_2 = 1/p'$ for $n = 2$, $\theta_1 = 1/2$ and $\theta_2 = 1/p' - 1/s_1 = \min(1, 4 - 2\alpha_1)/2 > 0$ for $n = 3$. The Lemma easily follows. $\square$

The local well posedness in X_T may be deduced from Lemmas 7.1, 7.2, 7.3 and 7.4 as we did it in section 4 for the $\mathbb{R}^n$ subcritical case. The uniqueness of a solution to (87) in $\mathcal{C}([0,T], H^1)$ and the Lipshitz continuity of the flow may be proven as in section 4.

Next, we prove the global well-posedness result. The strategy is similar to that we employed in Section 5 for the $\mathbb{R}^n$ case. As in Section 5, if $w \in \mathcal{C}([0,T], H_0^2(\Omega))$ solves (87), then $\partial_{j,k}^2 w$ solves

$$i\partial_t \partial_{j,k}^2 w + \Delta \partial_{j,k}^2 w = g_0 + g_1 + g_2.$$

With the new choice of p', q' we made in (89), (53) and (54) are still valuable. It remains to estimate $\|g_j\|_{L_T^{p'} L^{q'}}$, for $j = 1, 2$. In dimension 2, (57) remains true (with $\theta = 1/p'$), while it follows from the Gagliardo-Nirenberg inequality, (56) and Hölder that

$$\begin{aligned} \|g_2(s)\|_{L^{q'}(\Omega)} \lesssim\ & \|\nabla(\phi + w)\|_{L^2(\Omega)}^{\frac{3-\varepsilon'}{2}} (1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^2(\Omega)})^{\frac{1+\varepsilon'}{2}} \\ & + \|w\|_{L^{\frac{4\max(1,2\alpha_1-3)}{1-\varepsilon'}}}^{\max(1,2\alpha_1-3)} \|\nabla(\phi+w)\|_{L^2(\Omega)} (1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L^2(\Omega)}), \end{aligned} \tag{99}$$

which implies that

$$\begin{aligned} \|g_2\|_{L_T^{p'} L^{q'}} \lesssim\ & T^{1/p'} (1 + \|w\|_{L_T^\infty H^1})^{\frac{3-\varepsilon'}{2}} (1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2})^{\frac{1+\varepsilon'}{2}} \\ & + T^{1/p'} \|w\|_{L_T^\infty H^1}^{\max(1,2\alpha_1-3)} (1 + \|w\|_{L_T^\infty H^1})(1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}) \end{aligned} \tag{100}$$

In dimension 3, for $j, k \in \{1, 2, 3\}$, using the Hölder inequality, Sobolev embeddings and an interpolation argument, we get

$$\begin{aligned} \|\partial_{j,k}^2(\phi + w)\|_{L_T^{p'} L^{q'}} & \lesssim \|\partial_{j,k}^2(\phi+w)\|_{L_T^\infty L^2} \|w\|_{L_T^{p'} L^\beta} \\ & \lesssim (1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}) \|w\|_{L_T^{p'} W^{1,r_1}} \\ & \lesssim T^{1/2} (1 + \sum_{|\alpha|=2} \|\partial^\alpha w\|_{L_T^\infty L^2}) \underbrace{\|w\|_{L_T^{s_1} W^{1,r_1}}}_{\lesssim \|w\|_{X_T}}, \end{aligned} \tag{101}$$

where $1/\beta = 1/q' - 1/2 = (1-\varepsilon')/6$, $1/r_1 = 1/3 + 1/\beta = (3-\varepsilon')/6$ and $1/s_1 = 3/2 - 3/r_1 = \varepsilon'/2$. In a similar way,

$$\begin{aligned}
&\|\partial^2_{j,k}(\phi+w)|w|^{\max(1,2\alpha_1-2)}\|_{L^{p'}_T L^{q'}} \\
&\lesssim \|\partial^2_{j,k}(\phi+w)\|_{L^\infty_T L^2}\|w\|^{\max(1,2\alpha_1-2)}_{L^{p'\max(1,2\alpha_1-2)}_T L^{\beta\max(1,2\alpha_1-2)}} \\
&\lesssim (1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\|w\|^{\max(1,2\alpha_1-2)}_{L^{p'\max(1,2\alpha_1-2)}_T W^{1,r_2}} \\
&\lesssim T^{\frac{1}{p'}-\frac{\max(1,2\alpha_1-2)}{s_2}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\underbrace{\|w\|^{\max(1,2\alpha_1-2)}_{L^{s_2}_T W^{1,r_2}}}_{\lesssim \|w\|^{\max(1,2\alpha_1-2)}_{X_T}}, \qquad (102)
\end{aligned}$$

with $\frac{1}{r_2} = \frac{1}{3} + \frac{1}{\beta\max(1,2\alpha_1-2)} = \frac{2\max(1,2\alpha_1-2)+1-\varepsilon'}{6\max(1,2\alpha_1-2)}$, and $r_2 \in [2, q_0]$ thanks to (88). Moreover, $\frac{1}{p'} - \frac{\max(1,2\alpha_1-2)}{s_2} = \min(1, 4-2\alpha_1)/2 > 0$, because $\alpha_1 < 2$. We also have by Hölder and Gagliardo-Nirenberg

$$\begin{aligned}
\||\nabla(\phi+w)|^2\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\phi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\phi+w)\|_{L^\infty_T L^{\check{q}}} \\
&\lesssim \|\nabla(\phi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\phi+w)\|^{\frac{1-\varepsilon_0-\varepsilon'}{2}}_{L^\infty_T L^2}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{\frac{1+\varepsilon_0+\varepsilon'}{2}} \\
&\lesssim T^{\frac{1}{p'}-\frac{1}{p_0}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})^{\frac{1+\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})^{\frac{3-\varepsilon_0-\varepsilon'}{2}}, \qquad (103)
\end{aligned}$$

where $1/q' = 1/q_0 + 1/\check{q}$ and $1/p' - 1/p_0 = (\varepsilon' + \varepsilon_0)/2 > 0$. Next, we have to estimate $\||\nabla(\phi+w)|^2|w|^{\max(0,2\alpha_1-3)}\|_{L^{p'}_T L^{q'}}$. If $\alpha_1 \leqslant 3/2$, this has just been done. Thus we assume $\alpha_1 > 3/2$. Provided $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'} \geqslant 2$, we have by Hölder, Gagliardo-Nirenberg and Sobolev

$$\begin{aligned}
\||\nabla(\phi+w)|^2|w|^{2\alpha_1-3}\|_{L^{p'}_T L^{q'}} &\lesssim \|\nabla(\phi+w)\|_{L^{p'}_T L^{q_0}}\|\nabla(\phi+w)\|_{L^\infty_T L^6}\||w|^{2\alpha_1-3}\|_{L^\infty_T L^{\frac{6}{1-\varepsilon_0-\varepsilon'}}} \\
&\lesssim T^{\frac{1}{p'}-\frac{1}{p_0}}\|\nabla(\phi+w)\|_{L^{p_0}_T L^{q_0}}(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\||w|\|^{2\alpha_1-3}_{L^\infty_T L^{\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'}}} \\
&\lesssim T^{\frac{\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L^\infty_T L^2})\|w\|^{2\alpha_1-3}_{L^\infty_T H^1}. \qquad (104)
\end{aligned}$$

Note that (88) ensures $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'} \leqslant 6$. On the other side, if $\frac{6(2\alpha_1-3)}{1-\varepsilon_0-\varepsilon'} < 2$, the same arguments imply

$$\begin{aligned}
&\||\nabla(\phi+w)|^2|w|^{2\alpha_1-3}\|_{L_T^{p'}L^{q'}}\\
&\lesssim \; T^{\frac{\varepsilon_0+\varepsilon'}{2}}\|\nabla(\phi+w)\|_{L_T^{p_0}L^{q_0}}\|\nabla(\phi+w)\|_{L_T^\infty L^{\frac{6}{11-6\alpha_1-\varepsilon_0-\varepsilon'}}}\||w|^{2\alpha_1-3}\|_{L_T^\infty L^{\frac{2}{2\alpha_1-3}}}\\
&\lesssim \; T^{\frac{\varepsilon_0+\varepsilon'}{2}}(1+\|w\|_{X_T})(1+\|\nabla w\|^{1-\gamma}_{L_T^\infty L^2})(1+\sum_{|\alpha|=2}\|\partial^\alpha w\|_{L_T^\infty L^2})^\gamma\|w\|^{2\alpha_1-3}_{L_T^\infty H^1},
\end{aligned} \quad (105)$$

where $\gamma\in(0,1)$. Using the same arguments that in section 5, we can deduce from (36), (37) as well as (57) and (100) in dimension 2, (101), (102), (103), (104), (105) in dimension 3, that (87) is globally well-posed in X_T.

Proof Theorem 1.7. In the critical case $n=3$, $\alpha_1=2$, a local well-posedness result may also be obtained. Indeed, the proofs of lemmas 7.1, 7.2, 7.3 and 7.4 above also work in that case, if we choose $p_0=2/(1-\varepsilon_0)$, $p'=2/(1+\varepsilon')$, where $\varepsilon_0,\varepsilon'>0$ satisfy $\varepsilon_0+\varepsilon'\leqslant 1$ and $3\varepsilon_0+\varepsilon'\leqslant 2$. The only difference is that we get $\theta=\theta_2=0$ in Lemmas 7.2 and 7.4. Instead of (96), we use the following estimate, which we prove as usual by Hölder, Sobolev and an interpolation argument.

$$\begin{aligned}
\|\nabla(\phi+w)|w|^2\|_{L_T^{p'}L^{q'}} &\lesssim \|\nabla(\phi+w)\|_{L_T^{3p'}L^{\tilde q}}\|w\|^2_{L_T^{3p'}L^{\check q}}\\
&\lesssim \|\nabla(\phi+w)\|_{L_T^{3p'}L^{\tilde q}}\|w\|^2_{L_T^{3p'}W^{1,\tilde q}}\\
&\lesssim (T^{\frac{1}{3p'}}+\|w\|_{X_T})\|w\|^2_{X_T},
\end{aligned} \quad (106)$$

where $\tilde q$ and $\check q$ must satisfy $1/\tilde q+2/\check q=1/q'$, $1/3p'+3/\check q=3/2$ and $1/\tilde q-1/3=1/\check q$. Thanks to our choice of p_0,q_0,p',q' and to the condition we imposed on $\varepsilon_0,\varepsilon'>0$, we obtain $\tilde q=18/(8-\varepsilon')\in[2,q_0]$, which is the condition under which the above mentioned interpolation argument is valid. We similarly prove

$$\begin{aligned}
&\|(|\nabla\phi|+|\nabla w_1|+|\nabla w_2|)(w_1-w_2)(|w_1|+|w_2|)\|_{L_T^{p'}L^{q'}}\\
&\lesssim \; \|w_1-w_2\|_{X_T}(T^{\frac{1}{3p'}}+\|w_1\|_{X_T}+\|w_2\|_{X_T})(\|w_1\|_{X_T}+\|w_2\|_{X_T}),
\end{aligned} \quad (107)$$

which we use instead of (98), and for $j=1,2$,

$$\|\nabla(w_1-w_2)|w_j|^2\|_{L_T^{p'}L^{q'}} \lesssim \|w_1-w_2\|_{X_T}\|w_j\|^2_{X_T}, \quad (108)$$

which will be used instead of (97). Thus, in the critical case, estimations (91) and (93) may be replaced respectively by

$$\begin{aligned}
\|\nabla\Phi(w)\|_{L_T^\infty L^2\cap L_T^{p_0}L^{q_0}} &\leqslant \|\nabla w_0\|_{L^2}+CT(1+\|\nabla w\|_{L_T^\infty L^2})\\
&+CT^{\frac12}(1+\|w\|_{X_T})\|w\|_{X_T}+C(T^{\frac{1}{3p'}}+\|w\|_{X_T})(\|w\|_{X_T}+\|w\|^2_{X_T})
\end{aligned} \quad (109)$$

and

$$\|\nabla\Phi(w_1)-\nabla\Phi(w_2)\|_{L^\infty_T L^2\cap L^{p_0}_T L^{q_0}} \lesssim T\|\nabla(w_1-w_2)\|_{L^\infty_T L^2}$$
$$+T^{\frac{1}{2}}\|w_1-w_2\|_{X_T}(\|w_1\|_{X_T}+\|w_2\|_{X_T})$$
$$+\|w_1-w_2\|_{X_T}(T^{\frac{1}{3p'}}+\|w_1\|_{X_T}+\|w_2\|_{X_T})(\|w_1\|_{X_T}+\|w_2\|_{X_T}). \tag{110}$$

From Lemma 7.1 (which remains true for $\alpha_1 = 2$) and (109), we deduce as in section 4 that for T and R small enough, if $\|w_0\|_{H^1} \leqslant R$, Φ maps the ball of radius $2R$ of H^1 into itself. Taking T and R even smaller if necessary, Lemma 7.3 and (110) ensures that this map is a contraction. The rest of the proof is identical to that of Theorem 1.4 which was done in section 4. □

8 Appendix

Proof of Lemma 6.3. If μ, f_p and f_q are chosen as in the statement,

$$\begin{aligned}|\{x,|f(x)|>\mu\}| &= |\{x,\mu<|f(x)|\leqslant|f_p(x)|+|f_q(x)|\}|\\ &\leqslant |\{x,\mu/2<|f_p(x)|\}|+|\{x,\mu/2<|f_q(x)|\}|.\end{aligned}$$

Next, since $p<\infty$,

$$|\{x,\mu/2<|f_p(x)|\}| \leqslant \int\left(\frac{2}{\mu}|f_p(x)|\right)^p dx = \left(\frac{2}{\mu}\right)^p \|f_p\|^p_{L^p}. \tag{111}$$

Similarly, if $q<\infty$,

$$|\{x,\mu/2<|f_q(x)|\}| \leqslant \left(\frac{2}{\mu}\right)^q \|f_q\|^q_{L^q}. \tag{112}$$

Thus

$$|\{x,|f(x)|>\mu\}| \leqslant \left(\frac{2}{\mu}\right)^p \|f_p\|^p_{L^p}+\left(\frac{2}{\mu}\right)^q \|f_q\|^q_{L^q}. \tag{113}$$

If $q=\infty$, if $\mu>2\|f_q\|_{L^\infty}$, one has $|\{x,\mu/2<|f_q(x)|\}|=0$, and

$$|\{x,|f(x)|>\mu\}| \leqslant \left(\frac{2}{\mu}\right)^p \|f_p\|^p_{L^p}. \tag{114}$$

If $q < \infty$, using (111) with μ replaced by 2μ, we have for $\mu > 0$,

$$
\begin{aligned}
&\left(\int |\chi_\mu(f) f|^q\right)^{1/q} \\
&\leqslant \left(\int_{\{x, |f(x)| \leqslant 2\mu\}} |f(x)|^q dx\right)^{1/q} \\
&\leqslant \left(\int_{\{x, |f(x)| \leqslant 2\mu\}} |f_p(x)|^q dx\right)^{1/q} + \left(\int_{\{x, |f(x)| \leqslant 2\mu\}} |f_q(x)|^q dx\right)^{1/q} \\
&\leqslant \left(\int_{\left\{x,\ \substack{|f(x)| \leqslant 2\mu, \\ |f_p(x)| \leqslant \mu}\right\}} |f_p(x)|^p |f_p(x)|^{q-p} dx\right)^{1/q} + \left(\int_{\left\{x,\ \substack{|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu}\right\}} |f_p(x)|^q dx\right)^{1/q} + \|f_q\|_{L^q} \\
&\leqslant \mu^{\frac{q-p}{q}} \|f_p\|_{L^p}^{p/q} + \left(\int_{\left\{x,\ \substack{|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu}\right\}} |f(x)|^q dx\right)^{1/q} + \left(\int_{\left\{x,\ \substack{|f(x)| \leqslant 2\mu, \\ |f_p(x)| \geqslant \mu}\right\}} |f_q(x)|^q dx\right)^{1/q} + \|f_q\|_{L^q} \\
&\leqslant \mu^{\frac{q-p}{q}} \|f_p\|_{L^p}^{p/q} + 2\mu |\{x, |f_p(x)| \geqslant \mu\}|^{1/q} + 2\|f_q\|_{L^q} \\
&\leqslant 3\mu^{1-p/q} \|f_p\|_{L^p}^{p/q} + 2\|f_q\|_{L^q}, \qquad (115)
\end{aligned}
$$

while for $q = \infty$, $\|\chi_\mu(f) f\|_{L^\infty} \leqslant 2\mu$. On the other side, using (113), if $q < \infty$,

$$
\begin{aligned}
&\left(\int |(1-\chi_\mu)(f) f|^p\right)^{1/p} \\
&\leqslant \left(\int_{\{x, |f(x)| > \mu\}} |f_p(x)|^p dx\right)^{1/p} + \left(\int_{\left\{x,\ \substack{|f(x)| > \mu, \\ |f_q(x)| < \mu}\right\}} |f_q(x)|^p dx\right)^{1/p} + \left(\int_{\left\{x,\ \substack{|f(x)| > \mu, \\ |f_q(x)| \geqslant \mu}\right\}} |f_q(x)|^p dx\right)^{1/p} \\
&\leqslant \|f_p\|_{L^p} + \mu |\{x, |f(x)| > \mu\}|^{1/p} + \left(\int_{\left\{x,\ \substack{|f(x)| > \mu, \\ |f_q(x)| \geqslant \mu}\right\}} \frac{|f_q(x)|^q}{\mu^{q-p}} dx\right)^{1/p} \\
&\leqslant \|f_p\|_{L^p} + \mu \left(\left(\frac{2}{\mu}\right)^p \|f_p\|_{L^p}^p + \left(\frac{2}{\mu}\right)^q \|f_q\|_{L^q}^q\right)^{1/p} + \frac{\|f_q\|_{L^q}^{q/p}}{\mu^{\frac{q-p}{p}}}, \qquad (116)
\end{aligned}
$$

while if $q = \infty$, we similarly obtain thanks to (114), and because $\mu > 2\|f_q\|_{L^\infty}$

$$\|(1-\chi_\mu)(f) f\|_{L^p} \leqslant 3\|f_p\|_{L^p}.$$

Replacing μ by 1 or by μ_0, the estimates announced in the statement easily follow from (115), (116) and their analogous in the case $q = \infty$. $\square$

Proof of Lemma 6.4. Let $g = g_p + g_q$ be a decomposition of g in $L^p + L^q$. If $g_p = 0$ (resp. $g_q = 0$), then $g \in L^q$ (resp. $g \in L^p$), and the result is clear. Let $\mu_0 > 0$ associated to this decomposition by (77) (where f_p, f_q are replaced by g_p, g_q). Thanks to Lemma 6.3, $\chi_{\mu_0}(g)g \in L^q$ and $(1-\chi_{\mu_0})(g)g \in L^p$, and the estimates (78) and (79) hold for g. Writing $f = \chi_{\mu_0}(g)f + (1-\chi_{\mu_0})(g)f$, we deduce that $f \in L^p + L^q$, with $\|f\|_{L^p+L^q} \leqslant C(p,q)\|g\|_{L^p+L^q}$, where $C(p,q) = \max(6, 2 + (2^p + 2^q)^{1/p})$. $\square$

Proof of Lemma 6.5. Let $f \in L^p$, $f \neq 0$, and $\mu = \|f\|_{L^p}$. Then

$$\left(\int |\chi_\mu(f)f|^{p_2}\right)^{1/p_2} \leqslant \left(\int |\chi_\mu(f)f|^{p_2-p}|f|^p\right)^{1/p_2} \leqslant (2\mu)^{\frac{p_2-p}{p_2}}\|f\|_{L^p}^{p/p_2} = 2^{\frac{p_2-p}{p_2}}\|f\|_{L^p},$$

and

$$\left(\int |(1-\chi_\mu)(f)f|^{p_1}\right)^{1/p_1} \leqslant \left(\int |(1-\chi_\mu)(f)f|^{p_1}\left(\frac{|f|}{\mu}\right)^{p-p_1}\right)^{1/p_1} \leqslant \mu^{1-p/p_1}\|f\|_{L^p}^{p/p_1} = \|f\|_{L^p}.$$

Therefore $f \in L^{p_1} + L^{p_2}$ and

$$\|f\|_{L^{p_1}+L^{p_2}} \leqslant (1 + 2^{\frac{p_2-p}{p_2}})\|f\|_{L^p}.$$

$\square$

Proof of Lemma 6.6. We write $fg = f_1g_1 + f_1g_2 + f_2g_1 + f_2g_2$. Thanks to Hölder, it is clear that for $i,j = 1,2$, $f_ig_j \in L^{\frac{p_iq_j}{p_i+q_j}}$, and $\|f_ig_j\|_{L^{\frac{p_iq_j}{p_i+q_j}}} \leqslant \|f_i\|_{L^{p_i}}\|g_j\|_{L^{q_j}}$. For $i \neq j$, we have $\frac{p_iq_j}{p_i+q_j} \in (\frac{p_1q_1}{p_1+q_1}, \frac{p_2q_2}{p_2+q_2})$. The lemma follows, thanks to Lemma 6.5. $\square$

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References

[BGMP] I.V. Barashenkov, A.D. Gocheva, V.G. Makhankov, I.V. Puzynin, *Stability of the soliton-like "bubbles"*, Physica D, 34 (1989), 240-254.

[BP] I.V. Barashenkov, E.Y. Panova, *Stability and evolution of the quiescent and traveling solitonic bubbles*, Physica D, 69 (1993), 114-134.

[BS] F. Bethuel, J.C. Saut, *Travelling waves for the Gross-Pitaevskii equation I*, Ann. Inst. Henri Poincaré physique théorique 70 (2) (1999), 147-238

[dB] A. de Bouard, *Instability of stationnary bubbles*, SIAM J. Math. Anal. 26 (3) (1995) 566-582.

[BGT] N. Burq, P. Gérard, N. Tzvetkov, *On nonlinear Schrödinger equations in exterior domains*, Ann. Inst. H. Poincaré, AN 21 (2004), 295-318.

[CH] T. Cazenave, A. Haraux, *Introduction aux problèmes d'évolution semi-linéaires*, Mathématiques et applications, Ellipses, (1990)

[C] D. Chiron, *Travelling waves for the Gross-Pitaevskii equation in dimension larger than two*, Nonlinear Analysis, 58 (2004), 175-204

[DMG] L. Di Menza, C. Gallo, *The black solitons of the NLS equation*, in preparation.

[G] C. Gallo, *The dark solitons of the one-dimensional nonlinear Schrödinger equation*, in preparation

[Ga] C. Gallo, *Schrödinger group on Zhidkov spaces*, Adv. Diff. Eq., 9 (2004), 509-538

[Ge] P. Gérard, *The Cauchy problem for the Gross-Pitaevskii equation*, to appear in Ann. IHP

[Go] O. Goubet, personal communication

[Gr] P. Gravejat, *Asymptotics for subsonic travelling waves in the Gross-Pitaevskii equation*, preprint (2003)

[K] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré, 46, 1 (1987), 113-129.

[KeTa] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., 120 (1998), 955-980.

[L] Z. Lin, *Stability and instability of traveling solitonic bubbles*, Adv. Diff. Eq., 7 (2002), 897-918

[P] A. Pazy, *Semigroups of linear operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, New-York (1983)

[Z0] P.E. Zhidkov, *The Cauchy problem for a nonlinear Schrödinger equation*, Soobshch. OIYaI, R5-87-373, Dubna (1987).

[Z1] P.E. Zhidkov, *Solvability of the Cauchy problems and stability of some solutions of the nonlinear Schrödinger equation*, preprint OIYaI, P5-89-322, Dubna (1989), Matem. Model., 1, no.10,155-160 (1989).

[Z2] P.E. Zhidkov, *Existence of solutions to the Cauchy problem and stability of kink-solutions of the nonlinear Schrödinger equation*, Siberian. Math. J., 33 (1992), 239-246.

[Z3] P.E. Zhidkov, *Korteweg-de-Vries and nonlinear Schrödinger equations: qualitative theory*, Lecture Notes in Mathematics 1756, Springer-Verlag (2001).