Growth rate of the Schrödinger group on Zhidkov spaces

Clément Gallo
UMR de Mathématiques, Bat. 425
Université Paris-Sud
91405 Orsay, France.

Abstract. We give an upper bound on the growth rate of the Schrödinger group on Zhidkov spaces. In dimension 1, we prove that this bound is sharp.

Résumé. On donne une borne supérieure au taux de croissance du groupe de Schrödinger sur les espaces de Zhidkov. En dimension 1, on montre que cette borne est optimale.

Version française abrégée

Les équations de Schrödinger non linéaires défocalisantes, comme l'équation de Gross-Pitaevskii, admettent des ondes progressives non nulles à l'infini, appelées *dark solitons* ([1], [2], [3], [5], [7], [8]). Ces solutions interviennent dans de nombreux contextes physiques, en particulier en optique non linéaire (voir [6]) et dans l'étude de la superfluidité.

Afin de mieux comprendre le comportement asymptotique en temps des solutions de ces équations, notamment à proximité des *dark solitons*, il est intéressant d'étudier l'effet du propagateur de Schrödinger sur un espace qui contient ces *dark solitons*. On s'intéresse ici à l'équation de Schrödinger linéaire sur $\mathbb{R}^n$

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0) = u_0 \end{cases} \tag{1}$$

avec données non nulles à l'infini. Dans [4], on montre que (1) est bien posée sur les espaces de Zhidkov

$$X^k(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n), \nabla u \in H^{k-1}(\mathbb{R}^n)\},$$

sous la condition $k > n/2$. On améliore ici l'estimation montrée dans [4] sur le taux de croissance du groupe de Schrödinger sur $X^k(\mathbb{R}^n)$ (noté $S(t)$). Plus précisément, on montre qu'il existe une constante $C > 0$ telle que

$$\|S(t)\|_{\mathcal{L}(X^k(\mathbb{R}^n),X^k(\mathbb{R}^n))} \quad \leqslant \quad C(1 + |t|^\rho), \tag{2}$$

où

$$\rho = \begin{cases} \frac{1}{2}\frac{1}{1+1/n} & \text{si } n \text{ est pair} \\ \frac{1}{4}\frac{1}{1+1/2n} & \text{si } n \text{ est impair.} \end{cases} \tag{3}$$

De plus, en dimension $n = 1$, $\rho = 1/6$ est optimal.

1 Introduction

Many defocusing nonlinear Schrödinger equations, such as the Gross-Pitaevskii equation, admit travelling waves which do not vanish at infinity (see for instance [1], [2], [3], [5], [7], [8]). They are referred to as dark solitons. These solutions are important in many physical contexts, in particular in nonlinear optics (see [6]) and in the study of superfluidity.

To understand better the asymptotic behaviour of the solutions of these equations, which is particularly important in a neighbourhood of the dark solitons, it is interesting to study the effect of the free Schrödinger propagator on a space that contains the dark solitons.

In [4], we proved that the linear Schrödinger equation (1) is well-posed on the Zhidkov spaces $X^k(\mathbb{R}^n)$ under the condition $k > n/2$. Moreover, we gave a superior bound on the growth rate of the Schrödinger group on $X^k(\mathbb{R}^n)$, which will be denoted by $S(t)$. More precisely, we proved that there exists a constant $C > 0$ such that

$$\|S(t)\|_{\mathcal{L}(X^k(\mathbb{R}^n),X^k(\mathbb{R}^n))} \quad \leqslant \quad C(1+|t|^\rho), \tag{4}$$

where

$$\rho = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ 1/4 & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

The goal of this paper is to decrease as far as we can the upper bound on the growth rate of $S(t)$ given by (5), and to find the sharp exponent when it is possible. The main result we present here is as follows.

Theorem 1.1 *Let $k > n/2$. There exists a constant $C > 0$ such that for every $u_0 \in X^k(\mathbb{R}^n)$,*

$$\|S(t)u_0\|_{X^k(\mathbb{R}^n)} \leqslant C(1+|t|^\rho)\|u_0\|_{X^k(\mathbb{R}^n)}, \tag{6}$$

where

$$\rho = \begin{cases} \frac{1}{2}\frac{1}{1+\frac{1}{n}} & \text{if } n \text{ is even} \\ \frac{1}{4}\frac{1}{1+\frac{1}{2n}} & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Moreover, for $n = 1$, $\rho = 1/6$ is sharp.

Remark 1 The sharpness result in Theorem 1.1 should be understood as follows: for every $C > 0$, every $\rho < 1/6$, there exists $\tau > 0$ and $u_{0\tau} \in X^k(\mathbb{R}^n)$ such that

$$\|S(\tau)u_{0\tau}\|_{X^k(\mathbb{R}^n)} \geqslant \|S(\tau)u_{0\tau}\|_{L^\infty(\mathbb{R}^n)} > C(1+\tau^\rho)\|u_{0\tau}\|_{X^k(\mathbb{R}^n)}.$$

In section 2, we retake and improve the proof of the well-posedness of (1) on $X^k(\mathbb{R}^n)$ we gave in [4], in order to verify (6)-(7). In section 3, we prove the sharpness of $\rho = 1/6$ in dimension 1.

Notations. If $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$ denotes the Fourier transform of f.
In all this note, C denotes a harmless positive constant which can change from line to line.

2 Proof of (6)-(7).

Given $u_0 \in X^k(\mathbb{R}^n)$, the proof of (4)-(5) we gave in [4] consists in giving a sense to the limit of

$$I_\varepsilon := \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x + 2\sqrt{t}z)dz$$

as $\varepsilon \to 0$ (for convenience, we will assume from now on $t > 0$). Up to the multiplication by a harmless constant, this limit is showed to be the solution $u(t)$ to (1) with initial data u_0.

Using a radial cut-off function χ which satisfies

$$\chi(x) \equiv \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2, \end{cases} \tag{8}$$

we write $\psi = 1 - \chi$, and we define, for $\beta > 0$, $\chi_\beta(z) = \chi(z/\beta)$, $\psi_\beta(z) = \psi(z/\beta)$. For convenience, we will also use the notation $\chi_\beta(|.|) = \chi_\beta(.)$ and $\psi_\beta(|.|) = \psi_\beta(.)$.

We split I_ε into two parts: $I_\varepsilon = A_\varepsilon + B_\varepsilon$, where

$$A_\varepsilon = \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_\beta(z) u_0(x + 2\sqrt{t}z) dz$$

and

$$B_\varepsilon = \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \psi_\beta(z) u_0(x + 2\sqrt{t}z) dz.$$

Using Lebesgue's convergence theorem, it is easy to see that A_ε has a limit as ε goes to 0, and that

$$\left| \lim_{\varepsilon \to 0} A_\varepsilon \right| \leqslant \|u_0\|_{L^\infty} (2\beta)^n |\mathcal{B}(0,1)|, \tag{9}$$

where $\mathcal{B}(0,1)$ denotes the unit ball in $\mathbb{R}^n$. Next, passing in polar coordinates, defining

$$g(r) = \int_{\mathbb{S}^{n-1}} u_0(x + 2\sqrt{t}rv) dv$$

and integrating by parts, we have (see [4] for more details)

$$\begin{aligned} B_\varepsilon &= \int_\beta^\infty e^{(i-\varepsilon)r^2} \psi_\beta(r) r^{n-1} g(r) dr \\ &= \left(\frac{-1}{2(i-\varepsilon)} \right)^k \sum_{j=0}^k \sum_{l=0}^j a_{k,j} \binom{j}{l} \int_\beta^\infty e^{(i-\varepsilon)r^2} \frac{1}{r^{2k-j}} g^{(l)}(r) \psi_\beta^{(j-l)}(r) r^{n-1} dr, \end{aligned}$$

where the $a_{k,j}$ depend only on integers k and j. We next apply Lebesgue's theorem to each term of this sum. For $l = 0$, $j \in \{0 \dots k\}$, we get

$$\left| e^{(i-\varepsilon)r^2} \frac{1}{r^{2k-j}} g(r) \psi_\beta^{(j)}(r) r^{n-1} \right| \leqslant \frac{|\mathbb{S}^{n-1}| \|u_0\|_{L^\infty}}{r^{2k-n+1}} 2^j \|\psi^{(j)}\|_{L^\infty},$$

and therefore the limit of these terms as ε tends to 0 exists, and

$$\left| \lim_{\varepsilon \to 0} \int_\beta^\infty e^{(i-\varepsilon)r^2} \frac{1}{r^{2k-j}} g(r) \psi_\beta^{(j)}(r) r^{n-1} dr \right| \leqslant \frac{|\mathbb{S}^{n-1}| \|u_0\|_{L^\infty} 2^j \|\psi^{(j)}\|_{L^\infty}}{(2k-n)\beta^{2k-n}}. \tag{10}$$

For $l \geqslant 1$, $j \in \{l \dots k\}$, using

$$\|g^{(j)}\|_{L^2(\beta,\infty,r^{n-1}dr)} \quad \leqslant \quad (2\sqrt{t})^{j-n/2} |\mathbb{S}^{n-1}|^{1/2} \|u_0\|_{X^k(\mathbb{R}^n)}$$

(which has been established in [4]), we similarly obtain

$$\left| \lim_{\varepsilon \to 0} \int_\beta^\infty e^{(i-\varepsilon)r^2} \frac{1}{r^{2k-j}} g^{(l)}(r) \psi_\beta^{(j-l)}(r) r^{n-1} dr \right| \quad \leqslant \quad \frac{|\mathbb{S}^{n-1}|^{1/2} \|\psi^{(j-l)}\|_{L^\infty}}{(4k-2j-n)^{1/2}} \|u_0\|_{X^k} \frac{(2\sqrt{t})^{l-\frac{n}{2}}}{\beta^{2k-l-\frac{n}{2}}}. \tag{11}$$

Thanks to (9), (10) and (11), I_ε has a limit as ε tends to 0, and

$$\left| \lim_{\varepsilon \to 0} I_\varepsilon \right| \leqslant C \|u_0\|_{X^k} \left(\beta^n + \frac{1}{\beta^{2k-n}} + \frac{\sqrt{t}^{-n/2}}{\beta^{2k-n/2}} (\beta\sqrt{t} + (\beta\sqrt{t})^k) \right).$$

We next choose $\beta = t^\gamma$ (and not $\beta = 1$ as we did in [4]), where

$$\gamma = \frac{1}{2}\frac{k-\frac{n}{2}}{k+\frac{n}{2}} > 0.$$

Taking into account this choice of γ, it follows that

$$\begin{aligned} \left|\lim_{\varepsilon\to 0} I_\varepsilon\right| &\leqslant C\|u_0\|_{X^k}\left(t^{\gamma n} + t^{-\gamma(2k-n)} + t^{\gamma(1-2k+n/2)+1/2-n/4} + t^{\gamma(-k+n/2)+k/2-n/4}\right) \\ &\leqslant C\|u_0\|_{X^k}(1+t^\rho), \end{aligned} \tag{12}$$

where $\rho = \gamma n$. When $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$, we obtain the ρ given in (7). The rest of the proof is similar to that of Theorem 3.1 in [4]. In particular, we deduce that for every integer $k > n/2$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|u_0\|_{X^k(\mathbb{R}^n)}(1+t^\rho).$$

□

2.1 Proof of the sharpness of (7) in dimension 1.

Let $k \geqslant 1$. For $\tau \geqslant 1$, let us define

$$u_{0\tau}(x) = e^{-\frac{i|x|^2}{4\tau}}\psi\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^{2(k+1)/3}}{|x|^{k+1}}.$$

Then $u_{0\tau} \in L^\infty$, with $\|u_{0\tau}\|_{L^\infty} = \sup_{r\geqslant 0}\frac{\psi(r)}{r^{k+1}}$. Next, the derivatives from order 1 to k of $u_{0\tau}$ are in L^2, with L^2 norm uniformly bounded with respect to τ. Indeed, it is easy to see by induction on $m \in \{1\dots k\}$ that $u_{0\tau}^{(m)}$ is a linear combination of terms which look like

$$Q(x) = se^{-\frac{i|x|^2}{4\tau}}\psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^\alpha}{x^q},$$

where $s = (-1)^{(k+1)\frac{1-sign(x)}{2}}$ and $q \geqslant k+1-m$ (the only term with $q = k+1-m$ is obtained by differentiating m times the exponential factor in $u_{0\tau}$). $q \geqslant 1$ because $m \leqslant k$, and since moreover $x \mapsto \psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)$ is uniformly bounded on $\mathbb{R}$ and supported in $[\tau^{2/3}, \infty)$, it follows that $u_{0\tau}^{(m)} \in L^2$, for $m \in \{1\dots k\}$. Next, we easily compute

$$\|Q\|_{L^2} = \tau^{\alpha - 2q/3 + 1/3}\left\|\frac{\psi^{(p)}(x)}{x^q}\right\|_{L^2}, \tag{13}$$

and

$$sQ'(x) = -\frac{ix}{2\tau}Q(x) + e^{-\frac{i|x|^2}{4\tau}}\psi^{(p+1)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^{\alpha-2/3}}{x^q} - qe^{-\frac{i|x|^2}{4\tau}}\psi^{(p)}\left(\frac{x}{\tau^{2/3}}\right)\frac{\tau^\alpha}{x^{q+1}}. \tag{14}$$

The L^2 norms of the three terms in the right hand side of (14) are respectively

$$\frac{1}{2}\tau^{\alpha-1-2(q-1)/3+1/3}\left\|\frac{\psi^{(p)}(x)}{x^{q-1}}\right\|_{L^2},$$

$$\tau^{\alpha-2/3-2q/3+1/3}\left\|\frac{\psi^{(p+1)}(x)}{x^q}\right\|_{L^2}$$

and

$$q\tau^{\alpha-2(q+1)/3+1/3}\left\|\frac{\psi^{(p)}(x)}{x^{q+1}}\right\|_{L^2}.$$

Therefore there exists a constant $C > 0$ such that $\|Q'\|_{L^2} \leqslant C\tau^{-1/3}\|Q\|_{L^2}$. Thus, for $m \in \{1 \dots k\}$, $\|u_{0\tau}^{(m)}\|_{L^2} \leqslant C\tau^{-m/3}\|u_{0\tau}\|_{L^2}$. Since $u_{0\tau} \in L^2$ and $\|u_{0\tau}\|_{L^2} = C\tau^{1/3}$, we get the announced result: $u_{0\tau} \in X^k$ and $\|u_{0\tau}\|_{X^k} \leqslant C$, where C does not depend on τ.

On the other side, since $z \mapsto \chi(z)$ and $z \mapsto \psi(2z/\tau^{1/6})$ have disjoint supports for τ large enough,

$$\int_{-\infty}^{\infty} e^{(i-\varepsilon)z^2}\chi(z)u_{0\tau}(2\sqrt{\tau}z)dz = \int_{-\infty}^{\infty} e^{(i-\varepsilon)z^2}\chi(z)e^{-iz^2}\psi\left(\frac{2z}{\tau^{1/6}}\right)\frac{\tau^{\frac{k+1}{6}}}{2^{k+1}|z|^{k+1}}dz = 0, \tag{15}$$

while

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{(i-\varepsilon)z^2}\psi(z)u_{0\tau}(2\sqrt{\tau}z)dz &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{(i-\varepsilon)z^2}\psi(z)e^{-iz^2}\psi\left(\frac{2z}{\tau^{1/6}}\right)\frac{\tau^{\frac{k+1}{6}}}{2^{k+1}|z|^{k+1}}dz \\ &= \tau^{1/6}\int_{-\infty}^{\infty}\frac{\psi(2r)}{2^{k+1}|r|^{k+1}}dr. \end{aligned} \tag{16}$$

As a conclusion, for τ large enough, denoting by u_τ the solution to (1) with initial data $u_{0\tau}$,

$$\|u_\tau(\tau)\|_{X^k} \geqslant \|u_\tau(\tau)\|_{L^\infty} \geqslant |u_\tau(\tau,0)| = C\tau^{1/6} \geqslant C\tau^{1/6}\|u_{0\tau}\|_{X^k}.$$

This completes the proof of the sharpness of the exponent 1/6 in (7). □

Remark 2 We can also prove this sharpness result by using the initial conditions

$$u_{0\tau}(x) = e^{-\frac{ix^2}{4\tau}\chi\left(\frac{x}{\tau^{2/3+\delta}}\right)}$$

or

$$u_{0\tau}(x) = e^{-\frac{ix^2}{4\tau}}\chi\left(\frac{x}{\tau^{2/3-\delta}}\right),$$

and letting the parameter $\delta > 0$ tend to 0. In these two cases, taking the notations of section 3.1, the predominant part in I_ε is A_ε (on the contrary, in the proof we developed here, $B_\varepsilon(\tau) \gg A_\varepsilon(\tau)$ as τ is large).

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