FINITE TIME EXTINCTION BY NONLINEAR DAMPING FOR THE SCHRÖDINGER EQUATION

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Abstract. We consider the Schrödinger equation on a compact manifold, in the presence of a nonlinear damping term, which is homogeneous and sublinear. For initial data in the energy space, we construct a weak solution, defined for all positive time, which is shown to be unique. In the one-dimensional case, we show that it becomes zero in finite time. In the two and three-dimensional cases, we prove the same result under the assumption of extra regularity on the initial datum.

1. Introduction

We consider the Schrödinger equation with a homogeneous damping term,

$$i\frac{\partial u}{\partial t}+\Delta u=-i\gamma\frac{u}{|u|^\alpha},\quad t\in\mathbf{R}_+,\ x\in M\quad;\quad u_{|t=0}=u_0, \tag{1.1}$$

where $\gamma>0$, $0<\alpha\leqslant 1$, (M,g) is a smooth compact Riemannian manifold of dimension d, and u is complex-valued. If $d\leqslant 3$, we prove that if the initial datum u_0 is sufficiently regular (in $H^1(M)$ if $d=1$, in $H^2(M)$ if $d=2,3$), then every weak solution to (1.1) becomes zero in finite time. The reason why the space variable belongs to a compact manifold and not to the whole Euclidean space is most likely purely technical. It seems sensible to believe that the extinction phenomenon that we prove remains true on $\mathbf{R}^d$. Typically, only on a compact manifold does $u/|u|$ belong to L^p_x for finite p, so the nonlinear term is harder to control in the $\mathbf{R}^d$ case.

This phenomenon is to be compared with the case of the linear damping,

$$i\frac{\partial u}{\partial t}+\Delta u=-i\gamma u.$$

This case is particularly simple, since after the change of unknown function $v(t,x)=e^{\gamma t}u(t,x)$, v solves a free Schrödinger equation: its $L^2(M)$ norm does not depend on time, so the L^2-norm of u decays exponentially in time. Such a damping term is also used in some physical models involving an extra interaction nonlinearity, such as a cubic term; see e.g. [11] and references therein. Localized linear damping (replace γu with $a(x)u$) has been considered for control problems; see e.g. [15, 2] and references therein. Stabilization is obtained with an exponential rate in time. More recently, nonlinear damping terms have been considered, but with some homogeneity different from ours; see e.g. [18] and references therein. In [4], the authors consider

$$i\frac{\partial u}{\partial t}+\Delta u=\lambda|u|^2u-i\gamma|u|^4u,\quad x\in\mathbf{R}^3.$$

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See [4] also for references to situations where such a model is involved. The nonlinear damping is shown to stabilize the solution, in the sense that finite time blow-up (which may occur if $\lambda < 0$ and $\gamma = 0$) is prevented by the damping term ($\gamma > 0$). It can be inferred that the L^2-norm of u goes to zero as time goes to infinity, but probably more slowly than in the case of a linear damping. Roughly speaking, the damping is strong only where u is large, so it is less and less strong as u goes to zero.

The damping term present in (1.1) arises in Mechanics (a case where u is real-valued): in the case $\alpha = 1$, it is referred to as *Coulomb friction*. Its effects have been studied in [1] in the case of ordinary differential equations, and in [5] in the case of a wave equation. The intermediary case $0 < \alpha < 1$ has been studied in the ordinary differential equations case in [9, 10, 3], and the damping term is then called *strong friction*. As in the case of (1.1), the model one has in mind to understand the dynamics of the equation is the ordinary differential equation obtained by dropping the Laplacian in (1.1):

$$\frac{du}{dt} = -\gamma \frac{u}{|u|^\alpha}. \tag{1.2}$$

Multiplying the above equation by the conjugate of u, and setting $y(t) = |u(t)|^2$, (1.2) yields

$$\frac{dy}{dt} = -2\gamma y^{1-\alpha/2},$$

an equation which can be solved explicitly: so long as $y \geqslant 0$,

$$y(t) = \left(y(0)^{\alpha/2} - \alpha\gamma t\right)^{2/\alpha}.$$

Therefore, y (hence u) becomes zero at time

$$t_c = \frac{|u(0)|^\alpha}{\alpha\gamma}.$$

In this paper, we prove a similar phenomenon for weak solutions to (1.1). Before stating our main result, we have to specify the notion of weak solution, especially in the case $\alpha = 1$, where the right hand side in (1.1) does not make sense if $u(t, x) = 0$.

Definition 1.1 (Weak solution, case $0 < \alpha < 1$)**.** *Suppose* $0 < \alpha < 1$*. A (global) weak solution to* (1.1) *is a function* $u \in \mathcal{C}(\mathbf{R}_+; L^2(M)) \cap L^\infty(\mathbf{R}_+; H^1(M))$ *solving* (1.1) *in* $\mathcal{D}'(\mathbf{R}_+^* \times M)$*.*

Definition 1.2 (Weak solution, case $\alpha = 1$)**.** *Suppose* $\alpha = 1$*. A (global) weak solution to* (1.1) *is a function* $u \in \mathcal{C}(\mathbf{R}_+; L^2(M)) \cap L^\infty(\mathbf{R}_+; H^1(M))$ *solving*

$$i\frac{\partial u}{\partial t} + \Delta u = -i\gamma F$$

in $\mathcal{D}'(\mathbf{R}_+^* \times M)$*, where* F *is such that*

$$\|F\|_{L^\infty(\mathbf{R}_+ \times M)} \leqslant 1, \quad \text{and} \quad F = \frac{u}{|u|} \text{ if } u \neq 0.$$

Our main results are as follows.

Theorem 1.3. *Let* $d \geqslant 1$*,* $u_0 \in H^1(M)$*,* $\gamma > 0$ *and* $0 < \alpha \leqslant 1$*. Then* (1.1) *has a unique, global weak solution. In addition, it satisfies the* a priori *estimate:*

$$\|u\|_{L^\infty(\mathbf{R}_+, H^1(M))} \leqslant \|u_0\|_{H^1(M)}.$$

Remark 1.4 (Uniqueness). Since the nonlinearity in (1.1) is not Lipschitzean, uniqueness does not come from completely standard arguments. Note that since we consider complex-valued functions, the monotonicity arguments invoked in [5] (after [6]) cannot be used. Uniqueness relies in a crucial manner on the dissipation associated to the equation.

In the multidimensional setting, our argument to prove finite time stabilization requires some extra regularity:

Theorem 1.5. *Let $d \leqslant 3$, $u_0 \in H^2(M)$, $\gamma > 0$ and $0 < \alpha \leqslant 1$. Then the solution of* (1.1) *belongs to $L^\infty(\mathbf{R}_+, H^2(M))$. In addition, there exists C, depending only on $\|u_0\|_{H^2(M)}$, M and γ, such that:*

$$\|u\|_{L^\infty(\mathbf{R}_+,H^2(M))} \leqslant C.$$

We will see that for such weak solutions, a complete dissipation occurs in finite time:

Theorem 1.6. *Let $d \leqslant 3$, $u_0 \in H^1(M)$, $\gamma > 0$ and $0 < \alpha \leqslant 1$. If $d = 2, 3$, suppose in addition that $u_0 \in H^2(M)$. Then there exists $T > 0$ such that the (unique) weak solution to* (1.1) *satisfies*

$$\text{for every} \quad t \geqslant T, \quad u(t,x) = 0, \quad \text{for almost every } x \in M.$$

Remark 1.7. With the results of [15, 3] in mind, it would seem interesting to consider an equation of the form

$$i\frac{\partial u}{\partial t} + \Delta u = -ia(x)\frac{u}{|u|^\alpha}$$

with $a \geqslant 0$, to stabilize u in finite time on the set $\{a > 0\}$ (one may think of a as an indicator function).

As a corollary to our approach, we can prove the same phenomenon for the "usual" nonlinear Schrödinger equation perturbed by the damping term that we consider in this paper, provided that $d = 1$ and that no finite time blow-up occurs without damping:

Corollary 1.8. *Let $d = 1$, $u_0 \in H^1(M)$, $\gamma, \sigma > 0$, $0 < \alpha \leqslant 1$ and $\lambda \in \mathbf{R}$. Assume in addition $\sigma < 2$ if $\lambda < 0$. Then there exists $T > 0$ such that the (unique) weak solution to*

$$i\frac{\partial u}{\partial t} + \Delta u = \lambda|u|^{2\sigma}u - i\gamma\frac{u}{|u|^\alpha} \quad ; \quad u_{|t=0} = u_0 \tag{1.3}$$

satisfies: for every $t \geqslant T$, $u(t,x) = 0$ for almost every $x \in M$.

Remark 1.9. The notion of weak solution for (1.3) is easily adapted, as well as the proof of uniqueness, since for $d = 1$, $H^1(M) \hookrightarrow L^\infty(M)$. Proving the analogue of Corollary 1.8 in a multi-dimensional framework, or in cases where finite time blow-up occurs when $\gamma = 0$ (e.g., $d = 1$, $\lambda < 0$ and $\sigma \geqslant 2$), seems to be an interesting open question. Note however that the results in [16] and the fact that the present damping is sublinear suggest that there is no universal conclusion in that case: in the competition between finite time blow-up and dissipation, either of the two effects may win.

2. Existence results

To prove the existence part of Theorem 1.3, we first regularize the nonlinearity to construct a mild solution, and then pass to the limit. This procedure allows us to prove Theorem 1.5 too. Uniqueness is established in Section 3.

2.1. **Construction of an approximating sequence.** In order to construct a solution of (1.1) as in Theorem 1.3, we solve first, for $\delta > 0$, the equation

$$i\frac{\partial u^\delta}{\partial t} + \Delta u^\delta = f_\delta(u^\delta) := -i\gamma \frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}. \tag{2.1}$$

Proposition 2.1. *Let $\delta > 0$, $u_0 \in L^2(M)$. There exists a unique $u^\delta \in \mathcal{C}(\mathbf{R}_+, L^2(M))$ such that*

$$u^\delta(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta} f_\delta(u^\delta(\tau))d\tau, \quad t \in \mathbf{R}_+.$$

If moreover $u_0 \in H^s(M)$ for some $s \geqslant 0$, then $u \in \mathcal{C}(\mathbf{R}_+, H^s(M)) \cap \mathcal{C}^1(\mathbf{R}_+, H^{s-2}(M))$. The flow map

$$\begin{array}{rcl} H^s(M) & \to & \mathcal{C}(\mathbf{R}_+, H^s(M)) \\ u_0 & \mapsto & u^\delta \end{array}$$

is continuous. If $u_0 \in H^1(M)$, for every $t \geqslant 0$, we have

$$\|u^\delta(t)\|^2_{L^2(M)} + 2\gamma \int_0^t \int_M \frac{|u^\delta(\tau)|^2}{(|u^\delta(\tau)|^2+\delta)^{\alpha/2}} dx d\tau = \|u_0\|^2_{L^2(M)}, \tag{2.2}$$

$$\begin{aligned} &\|\nabla u^\delta(t)\|^2_{L^2(M)} - \|\nabla u_0\|^2_{L^2(M)} = \\ &- 2\gamma \int_0^t \int_M \frac{\delta|\nabla u^\delta|^2 + (1-\alpha)|\mathrm{Re}(\overline{u^\delta}\nabla u^\delta)|^2 + |\mathrm{Im}(\overline{u^\delta}\nabla u^\delta)|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}(\tau,x) dx d\tau. \end{aligned} \tag{2.3}$$

Proof. Since $f_\delta \in \mathcal{C}^\infty(\mathbf{C},\mathbf{C})$ is globally Lipschitzean, the global well-posedness in H^s and the continuity of the flow map are well known, and follow from the standard fixed point argument and Gronwall lemma (see e.g. [8]). The identity (2.2) is first obtained for $u_0 \in H^s$ with s large, multiplying (2.1) by the conjugate of u^δ, taking the imaginary part and integrating in space and time. The identity (2.2) is then obtained for $u_0 \in L^2(M)$ thanks to the continuity of the flow map and the density of H^s in L^2. Similarly, (2.3) is first obtained for $u_0 \in H^s$ with s large, multiplying (2.1) by $\partial_t \overline{u^\delta}$, taking the imaginary part, and integrating. Alternatively, (2.3) can be obtained formally by applying the operator ∇ to (2.1), multiplying the result by $\nabla\overline{u^\delta}$, taking the imaginary part, and integrating. □

To prove Theorem 1.5, we will also use the following result:

Proposition 2.2. *Let $\delta > 0$, $u_0 \in H^2(M)$, and $u^\delta \in \mathcal{C}(\mathbf{R}_+, H^2(M)) \cap \mathcal{C}^1(\mathbf{R}_+, L^2(M))$ be as in Proposition 2.1. Then for every $t \geqslant 0$, we have*

$$\begin{aligned} &\|\partial_t u^\delta(t)\|^2_{L^2(M)} - \|\partial_t u^\delta(0)\|^2_{L^2(M)} = \\ &- 2\gamma \int_0^t \int_M \frac{\delta|\partial_t u^\delta|^2 + (1-\alpha)|\,\mathrm{Re}(\overline{u^\delta}\partial_t u^\delta)|^2 + |\,\mathrm{Im}(\overline{u^\delta}\partial_t u^\delta)|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}(\tau,x) dx d\tau. \end{aligned} \tag{2.4}$$

Proof. Since $u^\delta \in \mathcal{C}(\mathbf{R}_+, H^2(M))$ from Proposition 2.1, and

$$\left|\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}\right| \leqslant \frac{|u^\delta|}{\delta^{\alpha/2}},$$

Equation (2.1) implies $\partial_t u^\delta \in \mathcal{C}(\mathbf{R}_+, L^2(M))$, and $\partial_t u^\delta$ solves

$$\left(i\frac{\partial}{\partial t}+\Delta\right)\frac{\partial u^\delta}{\partial t} = -i\gamma\frac{\partial_t u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}} + i\gamma\frac{\alpha}{2}\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2+1}}\partial_t|u^\delta|^2.$$

On a formal level, we infer

$$\begin{aligned}
\frac{d}{dt}\|\partial_t u^\delta\|^2_{L^2(M)} &= 2\,\mathrm{Re}\int_M \frac{\partial\overline{u^\delta}}{\partial t}\frac{\partial^2 u^\delta}{\partial t^2}\\
&= -2\gamma\int_M \frac{|\partial_t u^\delta|^2}{(|u^\delta|^2+\delta)^{\alpha/2}} + 2\alpha\gamma\int_M \frac{\left(\mathrm{Re}(\overline{u^\delta}\partial_t u^\delta)\right)^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}\\
&= -2\gamma\int_M (|u^\delta|^2+\delta)\frac{|\partial_t u^\delta|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}} + 2\alpha\gamma\int_M \frac{\left(\mathrm{Re}(\overline{u^\delta}\partial_t u^\delta)\right)^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}.
\end{aligned}$$

The identity (2.4) then follows by decomposing

$$|u^\delta|^2|\partial_t u^\delta|^2 = \left(\mathrm{Re}(\overline{u^\delta}\partial_t u^\delta)\right)^2 + \left(\mathrm{Im}(\overline{u^\delta}\partial_t u^\delta)\right)^2.$$

The result follows from the same arguments as in the proof of Proposition 2.1. □

Remark 2.3. Finite time stabilization must not be expected to occur in (2.1) (for $\delta > 0$). The corresponding toy model is the ordinary differential equation

$$\frac{du^\delta}{dt} = -\gamma\frac{u^\delta}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}} \quad ; \quad u^\delta(0) = u_0.$$

Setting again $y_\delta = |u^\delta|^2$, it now solves

$$\frac{dy_\delta}{dt} = -2\gamma\frac{y_\delta}{\left(y_\delta+\delta\right)^{\alpha/2}} \quad ; \quad y_\delta(0) = |u_0|^2.$$

Now since

$$\int_\tau^1 \frac{\left(y+\delta\right)^{\alpha/2}}{y}dy$$

diverges logarithmically as $\tau \to 0^+$, y_δ decays exponentially in time. A change of time variable shows that there exists C independent of $\delta \in]0,1]$ such that

$$y_\delta(t) \leqslant Ce^{-Ct/\delta^{\alpha/2}}, \quad \forall t \geqslant 0.$$

The exponential decay is stronger and stronger as δ goes to zero. The example discussed in the introduction shows that in the limit $\delta \to 0$, this exponential decay becomes a finite time arrest. Proving Theorem 1.6 somehow amounts to showing the same phenomenon in a PDE setting.

2.2. **Convergence of the approximation.** The fact that the approximating sequence $(u^\delta)_\delta$ converges to a weak solution of (1.1) follows essentially from the same arguments as in [12].

A straightforward consequence from (2.2) and (2.3) is that for $u_0 \in H^1(M)$ fixed, the sequence $(u^\delta)_{0<\delta\leqslant 1}$ is uniformly bounded in $L^\infty(\mathbf{R}_+, H^1(M)) \cap L^{2-\alpha}(\mathbf{R}_+ \times M)$. Since $L^\infty(\mathbf{R}_+, H^1(M))$ is the dual of $L^1(\mathbf{R}_+, H^{-1}(M))$, we deduce the existence of $u \in L^\infty(\mathbf{R}_+, H^1(M))$ and of a subsequence u^{δ_n} such that

$$u^{\delta_n} \rightharpoonup u, \quad \text{in } w*\ L^\infty(\mathbf{R}_+, H^1(M)), \tag{2.5}$$

with, in view of (2.2) and (2.3),

$$\|u\|_{L^\infty(\mathbf{R}_+,H^1(M))} \leqslant \|u_0\|_{H^1(M)}.$$

Moreover, $\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}$ is uniformly bounded in $L^\infty(\mathbf{R}_+, L^{\frac{2}{1-\alpha}}(M))$ (with $2/(1-\alpha) = \infty$ if $\alpha = 1$), such that up to the extraction of an other subsequence, there is $F \in L^\infty(\mathbf{R}_+, L^{\frac{2}{1-\alpha}}(M))$ such that

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2 + \delta_n)^{\alpha/2}} \rightharpoonup F, \quad \text{in } w*\ L^\infty(\mathbf{R}_+, L^{\frac{2}{1-\alpha}}(M)). \tag{2.6}$$

Moreover, $\|F\|_{L^\infty(\mathbf{R}_+,L^{\frac{2}{1-\alpha}}(M))} \leqslant \|u_0\|^{1-\alpha}_{L^2(M)}$. Let $\theta \in \mathcal{C}_c^\infty(\mathbf{R}_+^* \times M)$. Then

$$\begin{aligned}\left\langle -i\gamma \frac{u^{\delta_n}}{(|u^{\delta_n}|^2 + \delta_n)^{\alpha/2}}, \theta \right\rangle &= \left\langle i\frac{\partial u^{\delta_n}}{\partial t} + \Delta u^{\delta_n}, \theta \right\rangle = \left\langle u^{\delta_n}, -i\frac{\partial \theta}{\partial t} + \Delta\theta \right\rangle \\ &\underset{n\to\infty}{\longrightarrow} \left\langle u, -i\frac{\partial\theta}{\partial t} + \Delta\theta \right\rangle = \left\langle i\frac{\partial u}{\partial t} + \Delta u, \theta\right\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the distribution bracket on $\mathbf{R}_+^* \times M$. Thus, we deduce

$$i\frac{\partial u}{\partial t} + \Delta u = -i\gamma F, \quad \text{in } \mathcal{D}'(\mathbf{R}_+^* \times M).$$

We next show that $F = u/|u|^\alpha$ where the right hand side is well defined, that is if $\alpha < 1$, or $\alpha = 1$ and $u \neq 0$. We first suppose that $u_0 \in H^s(M)$ with s large. Let us fix $t' \in \mathbf{R}_+$ and $\delta > 0$. Thanks to (2.2), we infer, for any $t \in \mathbf{R}_+$,

$$\begin{aligned}\frac{d}{dt}\|u^\delta(t) - u^\delta(t')\|^2_{L^2} &\leqslant \frac{d}{dt}\left(-2\operatorname{Re}\left(u^\delta(t)|u^\delta(t')\right)\right) \\ &= -2\operatorname{Re}\left(i\Delta u^\delta(t) - \frac{\gamma u^\delta(t)}{(|u^\delta(t)|^2+\delta)^{\alpha/2}}\middle| u^\delta(t')\right),\end{aligned} \tag{2.7}$$

where $(\cdot|\cdot)$ denotes the scalar product in $L^2(M)$. By integration, we deduce

$$\begin{aligned}\|u^\delta(t) - u^\delta(t')\|^2_{L^2(M)} \leqslant 2|t-t'|\Big(&\|\Delta u^\delta\|_{L^\infty(\mathbf{R}_+,H^{-1}(M))}\|u^\delta\|_{L^\infty(\mathbf{R}_+,H^1(M))} \\ &+ \gamma\|u^\delta\|^{2-\alpha}_{L^\infty(\mathbf{R}_+,L^{2-\alpha}(M))}\Big).\end{aligned} \tag{2.8}$$

From the continuity of the flow map $H^1 \ni u_0 \mapsto u^\delta \in \mathcal{C}(\mathbf{R}_+, H^1)$ in Proposition 2.1, we deduce that (2.8) also holds if we only have $u_0 \in H^1(M)$. Next, since $(u^\delta)_{0<\delta\leqslant 1}$ is uniformly bounded in $L^\infty(\mathbf{R}_+, H^1(M))$ and M is compact, (2.8) gives the existence of a positive constant C such that for every $t, t' \in \mathbf{R}_+$,

$$\|u^\delta(t) - u^\delta(t')\|_{L^2(M)} \leqslant C|t-t'|^{1/2}.$$

In particular, for any $T>0$, $(u^\delta)_{0<\delta\leqslant 1}$ is a bounded sequence in $\mathcal{C}([0,T],L^2(M))$ which is uniformly equicontinuous from $[0,T]$ to $L^2(M)$. Moreover, the compactness of the embedding $H^1(M)\subset L^2(M)$ ensures that for every $t\in[0,T]$, the set $\{u^\delta(t)|\delta\in(0,1]\}$ is relatively compact in $L^2(M)$. As a result, Arzelà–Ascoli Theorem ensures that $(u^{\delta_n})_n$ is relatively compact in $\mathcal{C}([0,T],L^2(M))$. On the other hand, we already know from (2.5) that

$$u^{\delta_n} \rightharpoonup u \quad \text{in } w*\ L^\infty(\mathbf{R}_+,L^2(M)).$$

Therefore, we infer that u is the unique accumulation point of the sequence $(u^{\delta_n})_n$ in $\mathcal{C}([0,T],L^2(M))$. Thus

$$u^{\delta_n}\to u \quad \text{in } \mathcal{C}([0,T],L^2(M)),$$

which implies in particular $u\in\mathcal{C}([0,T],L^2(M))$ as well as $u(0)=u^{\delta_n}(0)=u_0$. This is true for any $T>0$, therefore

$$u\in\mathcal{C}(\mathbf{R}_+,L^2(M)).$$

Finally, up to the extraction of an other subsequence, $u^{\delta_n}(t,x)\to u(t,x)$ for almost every $(t,x)\in\mathbf{R}_+\times M$. Therefore, for almost every $(t,x)\in\mathbf{R}_+\times M$ such that $u(t,x)\neq 0$, we have

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}}(t,x)\to\frac{u}{|u|^\alpha}(t,x).$$

By comparison with (2.6), we deduce that up to a change of F on a set with zero measure,

$$F(t,x)=\frac{u}{|u|^\alpha}(t,x) \quad (\text{only if } u(t,x)\neq 0 \text{ in the case } \alpha=1),$$

which completes the proof of the existence part of Theorem 1.3.

2.3. **Proof of Theorem 1.5.** To prove Theorem 1.5, we resume the idea due to T. Kato [14] (see also [7]), based on the general idea for Schrödinger equation, that two space derivative cost the same as one time derivative.

The time derivative of u^δ at time $t=0$ is given by the equation: from (2.1),

$$\frac{\partial u^\delta}{\partial t}(0)=i\Delta u^\delta-if_\delta\left(u^\delta\right)\Big|_{t=0}=i\Delta u_0-\gamma\frac{u_0}{(|u_0|^2+\delta)^{\alpha/2}}.$$

We infer, for $u_0\in H^2(M)$,

$$\begin{aligned}\left\|\frac{\partial u^\delta}{\partial t}(0)\right\|_{L^2(M)}&\leqslant\|\Delta u_0\|_{L^2(M)}+\gamma\left\||u_0|^{1-\alpha}\right\|_{L^2(M)}\\&\leqslant\|\Delta u_0\|_{L^2(M)}+C(\alpha,M)\|u_0\|^{1-\alpha}_{L^2(M)},\end{aligned}$$

since M is compact. By Proposition 2.2, the L^2-norm of $\partial_t u^\delta$ is a non-increasing function of time, and there exists C such that

$$\left\|\frac{\partial u^\delta}{\partial t}(t)\right\|_{L^2(M)}\leqslant C,\quad \forall t\in\mathbf{R}_+,\ \forall\delta\in]0,1].$$

Using (2.1) again, we infer

$$\|\Delta u^\delta(t)\|_{L^2(M)}\leqslant C+\gamma\left\||u(t)|^{1-\alpha}\right\|_{L^2(M)}\leqslant\widetilde{C},$$

where $\widetilde{C}$ is independent of $t \in \mathbf{R}_+$ and $\delta \in]0,1]$, since M is compact and since (2.2) implies

$$\left\|u^\delta(t)\right\|_{L^2(M)} \leqslant \|u_0\|_{L^2(M)}\,, \quad \forall t \in \mathbf{R}_+,\ \forall \delta \in]0,1].$$

Therefore, there exists C depending only on $\|u_0\|_{H^2(M)}$, M and γ such that

$$\|\Delta u^\delta\|_{L^\infty(\mathbf{R}_+,L^2(M))} \leqslant C.$$

By Fatou's Lemma, we conclude that if $u_0 \in H^2(M)$, then the weak solution u satisfies $u \in L^\infty(\mathbf{R}_+, H^2(M))$.

3. Uniqueness

We start the proof of uniqueness in Theorem 1.3 with the following lemma.

Lemma 3.1. *Let $\alpha \in]0,1]$. For all $z_1, z_2 \in \mathbf{C}$,*

$$\operatorname{Re}\left(\left(\frac{z_1}{|z_1|^\alpha} - \frac{z_2}{|z_2|^\alpha}\right)\left(\overline{z_1 - z_2}\right)\right) \geqslant 0.$$

Proof. Pick $\rho_1, \rho_2 \geqslant 0$ and $\theta_1, \theta_2 \in [0, 2\pi[$ such that

$$z_j = \rho_j e^{i\theta_j}, \quad j = 1,2.$$

We write

$$\begin{aligned}\operatorname{Re}\left(\left(\frac{z_1}{|z_1|^\alpha} - \frac{z_2}{|z_2|^\alpha}\right)\left(\overline{z_1 - z_2}\right)\right) &= \rho_1^{2-\alpha} + \rho_2^{2-\alpha} - \rho_1^{1-\alpha}\rho_2 \cos(\theta_1 - \theta_2) \\ &\quad - \rho_1\rho_2^{1-\alpha}\cos(\theta_1 - \theta_2) \\ &\geqslant \rho_1^{2-\alpha} + \rho_2^{2-\alpha} - \rho_1^{1-\alpha}\rho_2 - \rho_1\rho_2^{1-\alpha}.\end{aligned}$$

For $\alpha = 1$, the conclusion is then obvious. For $\alpha \in]0,1[$, we use Young's inequality:

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \forall a,b \geqslant 0,\ \forall p \in]1,\infty[.$$

With $p = \frac{2-\alpha}{1-\alpha}$ and $p = 2-\alpha$, respectively, we infer

$$\rho_1^{1-\alpha}\rho_2 \leqslant \left(\frac{1-\alpha}{2-\alpha}\right)\rho_1^{2-\alpha} + \frac{\rho_2^{2-\alpha}}{2-\alpha} \quad ; \quad \rho_1\rho_2^{1-\alpha} \leqslant \frac{\rho_1^{2-\alpha}}{2-\alpha} + \left(\frac{1-\alpha}{2-\alpha}\right)\rho_2^{2-\alpha}.$$

The lemma follows. □

Next, we prove the following energy estimate, which is shown to hold for any solution to (1.1).

Proposition 3.2. *Let $d \geqslant 1$. Let $u_0, v_0 \in H^1(M)$ and $u, v \in \mathcal{C}(\mathbf{R}_+, L^2(M)) \cap L^\infty(\mathbf{R}_+, H^1(M))$ be two solutions of (1.1) with initial data $u(0) = u_0$ and $v(0) = v_0$ respectively. Then the map $m_{u,v} : t \mapsto \|(u-v)(t)\|^2_{L^2(M)}$ is differentiable everywhere on $\mathbf{R}_+$, $m'_{u,v} \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ and for every $t \in \mathbf{R}_+$,*

$$\frac{d}{dt}\|(u-v)(t)\|^2_{L^2(M)} + 2\gamma \int_M \operatorname{Re}\left(\left(\frac{u(t)}{|u(t)|^\alpha} - \frac{v(t)}{|v(t)|^\alpha}\right)\overline{u(t) - v(t)}\right) dx = 0 \tag{3.1}$$

In particular, if v is taken to be the trivial solution $v \equiv 0$, we have for any solution of (1.1):

$$\frac{d}{dt}\|u(t)\|^2_{L^2(M)} + 2\gamma\|u(t)\|^{2-\alpha}_{L^{2-\alpha}(M)} = 0. \tag{3.2}$$

Proof. First, notice that if $u \in \mathcal{C}(\mathbf{R}_+, L^2(M)) \cap L^\infty(\mathbf{R}_+, H^1(M))$, then for every $t \in \mathbf{R}_+$, $u(t) \in H^1(M)$. This is so because u is weakly continuous in time, with values in $H^1(M)$: $u \in \mathcal{C}_w(\mathbf{R}_+, H^1(M))$. Indeed, if $t \geqslant 0$ is fixed, since $u \in L^\infty(\mathbf{R}_+, H^1(M))$, there exists a sequence $t_n \to t$ such that for every n, $\|u(t_n)\|_{H^1(M)} \leqslant \|u\|_{L^\infty(\mathbf{R}_+, H^1(M))}$. Then, for every $\phi \in H^1(M)$ and $j \in \{1, \cdots, d\}$,

$$\begin{aligned}\left\langle \partial_j u(t), \phi \right\rangle_{H^{-1},H^1} &= -\left\langle u(t), \partial_j \phi \right\rangle_{L^2,L^2} = -\lim_{n\to\infty} \left\langle u(t_n), \partial_j \phi \right\rangle_{L^2,L^2} \\ &= \lim_{n\to\infty} \left\langle \partial_j u(t_n), \phi \right\rangle_{L^2,L^2},\end{aligned}$$

thus

$$\left| \left\langle \partial_j u(t), \phi \right\rangle_{H^{-1},H^1} \right| \leqslant \|u\|_{L^\infty(\mathbf{R}_+, H^1(M))} \|\phi\|_{L^2(M)},$$

which implies that $\nabla u(t) \in L^2(M)^d$. As a result, for every $t \in \mathbf{R}_+$,

$$\frac{\partial u}{\partial t}(t) = i\Delta u(t) - \gamma \frac{u(t)}{|u(t)|^\alpha} \in H^{-1}(M).$$

Then, if u, v are as in the statement of Proposition 3.2, $m_{u,v}$ is differentiable everywhere on $\mathbf{R}_+$, and

$$\begin{aligned} m'_{u,v}(t) &= 2\,\mathrm{Re} \left\langle \frac{\partial (u-v)}{\partial t}(t), \overline{(u-v)(t)} \right\rangle_{H^{-1}(M),H^1(M)} \\ &= 2\,\mathrm{Re} \left\langle i\Delta(u-v)(t) - \gamma \left(\frac{u(t)}{|u(t)|^\alpha} - \frac{v(t)}{|v(t)|^\alpha} \right), \overline{(u-v)(t)} \right\rangle_{H^{-1}(M),H^1(M)} \\ &= -2\gamma \int_M \mathrm{Re} \left(\left(\frac{u(t)}{|u(t)|^\alpha} - \frac{v(t)}{|v(t)|^\alpha} \right) \overline{(u-v)(t)} \right) dx. \end{aligned} \tag{3.3}$$

Since $u, v \in L^\infty(\mathbf{R}_+, L^2(M))$ and $L^2(M) \subset L^{2-\alpha}(M)$ by compactness of M, we deduce from the Cauchy-Schwarz inequality that $m'_{u,v} \in L^\infty(\mathbf{R}_+) \subset L^1_{\mathrm{loc}}(\mathbf{R}_+)$. □

It follows from Proposition 3.2, Lemma 3.1 and the Fundamental Theorem of Calculus, that if u and v are chosen as in Proposition 3.2, $m_{u,v}(t) = \|(u-v)(t)\|^2_{L^2(M)}$ is non-increasing on $\mathbf{R}_+$. The uniqueness part of Theorem 1.3 follows, choosing two solutions u and v of (1.1) with the same initial datum $u(0) = v(0)$.

4. Finite time stabilization: proof of Theorem 1.6

We next show that under the assumptions of Theorem 1.6, u vanishes in finite time. The proof relies on a Nash type inequality:

Lemma 4.1. *Let (M, g) be a smooth compact Riemannian manifold, of dimension d, and $\alpha \in]0, 1]$. There exists $C > 0$ such that*

$$\|f\|^{\alpha d + 4 - 2\alpha}_{L^2(M)} \leqslant C \left(\|f\|^{2-\alpha}_{L^{2-\alpha}(M)} \right)^2 \|f\|^{\alpha d}_{H^1(M)}, \quad \forall f \in H^1(M). \tag{4.1}$$

$$\|f\|^{\alpha d + 8 - 4\alpha}_{L^2(M)} \leqslant C \left(\|f\|^{2-\alpha}_{L^{2-\alpha}(M)} \right)^4 \|f\|^{\alpha d}_{H^2(M)}, \quad \forall f \in H^2(M). \tag{4.2}$$

Proof. As it is standard in geometry, inequalities valid on $\mathbf{R}^d$ are easily transported to the case of compact manifolds (see e.g. [13]). Since M is compact, M can be covered by a finite number of charts

$$(\Omega_n, \varphi_n)_{1 \leqslant n \leqslant N}$$

such that for any n, the components g_{ij}^n of g in (Ω_n,φ_n) satisfy

$$\frac{1}{2}\delta_{ij} \leqslant g_{ij}^n \leqslant 2\delta_{ij}$$

as bilinear forms. Let $(\eta_n)_{1\leqslant n\leqslant N}$ be a smooth partition of unity subordinate to the covering $(\Omega_n)_{1\leqslant n\leqslant N}$. For any $f\in C^\infty(M)$ and any n, we have

$$\int_M |\eta_n f|^p \leqslant 2^{d/2}\int_{\mathbf{R}^d}\left|(\eta_n f)\circ\varphi_n^{-1}(x)\right|^p dx,\quad 1\leqslant p\leqslant 2,$$
$$\int_M \left|\nabla\left(\eta_n f\right)\right|^2 \geqslant 2^{-d/2}\int_{\mathbf{R}^d}\left|\nabla\left((\eta_n f)\circ\varphi_n^{-1}\right)(x)\right|^2 dx,$$
$$\int_M \left|\Delta\left(\eta_n f\right)\right|^2 \geqslant 2^{-d/2}\int_{\mathbf{R}^d}\left|\Delta\left((\eta_n f)\circ\varphi_n^{-1}\right)(x)\right|^2 dx.$$

The lemma follows from inequalities on $\mathbf{R}^d$, adapted from the Nash inequality [17]: for all $\alpha\in]0,1]$ and all $s>0$, there exists $C=C(\alpha,s)$ such that

$$\|g\|_{L^2(\mathbf{R}^d)}^{\alpha d+2s(2-\alpha)} \leqslant C\left(\|g\|_{L^{2-\alpha}(\mathbf{R}^d)}^{2-\alpha}\right)^{2s}\|g\|_{\dot H^s(\mathbf{R}^d)}^{\alpha d},\quad \forall g\in H^s(\mathbf{R}^d)\cap L^{2-\alpha}(\mathbf{R}^d), \tag{4.3}$$

where $\dot H^s(\mathbf{R}^d)$ denotes the homogeneous Sobolev space. Note that for $s=1$ and $s=2$, we recover the numerology of (4.1) and (4.2), respectively. To prove (4.3), use Plancherel formula and decompose the frequency space: for $R>0$, write

$$\|g\|_{L^2(\mathbf{R}^d)} \lesssim \|\widehat g\|_{L^2(|\xi|\leqslant R)} + \|\widehat g\|_{L^2(|\xi|>R)} \lesssim R^{d/q}\|\widehat g\|_{L^p(\mathbf{R}^d)} + R^{-s}\left\||\xi|^s\widehat g\right\|_{L^2(\mathbf{R}^d)},$$

where $1/2=1/q+1/p$. Choose p so that its Hölder conjugate exponent is $p'=2-\alpha\in[1,2[$. Hausdorff–Young inequality implies

$$\|g\|_{L^2(\mathbf{R}^d)} \lesssim R^{d/q}\|g\|_{L^{2-\alpha}(\mathbf{R}^d)} + R^{-s}\|g\|_{\dot H^s(\mathbf{R}^d)}.$$

We compute $q=2(2-\alpha)/\alpha$. Optimizing in R yields

$$R^{s+(\alpha d)/(2(2-\alpha))} = \frac{\|g\|_{\dot H^s(\mathbf{R}^d)}}{\|g\|_{L^{2-\alpha}(\mathbf{R}^d)}},$$

where we point out that getting the best possible constant is not our goal. This value of R yields (4.3). Lemma 4.1 then follows by using the chain rule, the fact that η_n is smooth on M, and summing over n. □

To prove finite time extinction, we treat separately the cases $d=1$ on the one hand, and $d=2,3$ on the other hand. In the one-dimensional case, the identity (3.2) and Nash inequality (4.1) yield

$$\frac{d}{dt}\|u(t)\|_{L^2(M)}^2 + \frac{C\gamma}{\|u(t)\|_{H^1(M)}^{\alpha/2}}\|u(t)\|_{L^2(M)}^{2-\alpha/2} \leqslant 0, \tag{4.4}$$

for some $C>0$ independent of t, γ and u. From Theorem 1.3, we infer

$$\frac{d}{dt}\|u(t)\|_{L^2(M)}^2 + \frac{C\gamma}{\|u_0\|_{H^1(M)}^{\alpha/2}}\|u(t)\|_{L^2(M)}^{2-\alpha/2} \leqslant 0.$$

By integration, we deduce, as long as $\|u(t)\|_{L^2(M)}$ is not equal to zero:

$$\|u(t)\|_{L^2(M)} \leqslant \left(\|u_0\|_{L^2(M)}^{\alpha/2} - \frac{C\gamma}{\|u_0\|_{H^1(M)}^{\alpha/2}}t\right)^{2/\alpha}.$$

We infer that $\|u(t)\|_{L^2(M)}$ vanishes in finite time, at a time

$$T_v := \sup\{t \in \mathbf{R}_+ | \|u(t)\|_{L^2(M)} \neq 0\},$$

which is bounded from above by

$$T_v \leqslant \frac{1}{C\gamma} \|u_0\|_{L^2(M)}^{\alpha/2} \|u_0\|_{H^1(M)}^{\alpha/2}.$$

Using (3.2) again, and the mere fact that the L^2-norm of u is a non-increasing function of time, we conclude that $\|u(t)\|_{L^2(M)} = 0$ for all $t > T_v$.

Remark 4.2. Without the information $u \in L^\infty(\mathbf{R}_+, H^1(M))$, we cannot conclude after (4.4), in general. For instance, if we have $\|u(t)\|_{H^1(M)} \leqslant Ce^{Ct}$, the integration of (4.4) does not necessarily yield finite time stabilization.

Remark 4.3. Similarly, it might be tempting to first integrate (3.2) with respect to time, and then use the fact that M is compact, to write

$$\|u(t)\|_{L^{2-\alpha}(M)}^2 + C\gamma \int_0^t \|u(\tau)\|_{L^{2-\alpha}(M)}^{2-\alpha} d\tau \lesssim \|u_0\|_{L^2(M)}^2.$$

However, this inequality does not rule out, e.g., an exponential decay in time.

We see that the key in the above argument is that we have controlled the term $\|u(t)\|_{L^{2-\alpha}(M)}^{2-\alpha}$ by $\|u(t)\|_{L^2(M)}^\beta$ for some $\beta < 2$, in order to recover the ODE mechanism presented in the introduction. With the uniform H^1 estimate given in Theorem 1.3, (4.1) yields such a control provided that

$$\alpha d + 4 - 2\alpha < 4, \text{ that is, if } \alpha \left(\frac{d}{2} - 1\right) < 0.$$

Since $\alpha \in]0,1]$, this is possible if, and only if, $d = 1$. For $d = 2, 3$, we therefore use (4.2) and Theorem 1.5. We infer similarly

$$\frac{d}{dt}\|u(t)\|_{L^2(M)}^2 + C\gamma \|u(t)\|_{L^2(M)}^{2-(1-d/4)\alpha} \leqslant 0.$$

Again since $(1 - d/4)\alpha > 0$, we infer that $\|u(t)\|_{L^2(M)}$ vanishes in finite time T_v, with

$$T_v \leqslant \frac{1}{C\gamma} \|u_0\|_{L^2(M)}^{(1-d/4)\alpha}.$$

Note that unlike in the one-dimensional case, this constant C depends on u_0 (on $\|u_0\|_{H^2(M)}$ only), M and γ.

5. Finite time stabilization: proof of Corollary 1.8

Since Corollary 1.8 includes the assumption $d = 1$, we shall be rather brief for the analogue of Theorem 1.3. We can resume the strategy presented in Section 2, to construct an approximating sequence solution to

$$i\frac{\partial u^\delta}{\partial t} + \Delta u^\delta = \lambda |u^\delta|^{2\sigma} u^\delta - i\gamma \frac{u^\delta}{(|u^\delta|^2 + \delta)^{\alpha/2}}. \tag{5.1}$$

Since $d = 1$, $H^1(M) \hookrightarrow L^\infty(M)$, so the extra nonlinear term $|u|^{2\sigma}u$ is well controlled in view of a limiting procedure, provided that we have a uniform bound for u^δ in $L^\infty(\mathbf{R}_+, H^1(M))$. This is the most important step to infer Corollary 1.8 from the proof of Theorem 1.6.

Again because $H^1(M) \hookrightarrow L^\infty(M)$, the global well-posedness of (5.1) for $\delta > 0$ is straightforward. The analogues of (2.2) and (2.3) are, since $\lambda \in \mathbf{R}$:

$$
\begin{aligned}
&\|u^\delta(t)\|^2_{L^2(M)} + 2\gamma \int_0^t \int_M \frac{|u^\delta(\tau)|^2}{(|u^\delta(\tau)|^2+\delta)^{\alpha/2}} dx d\tau = \|u_0\|^2_{L^2(M)},\\
&\frac{d}{dt}\left(\|\nabla u^\delta(t)\|^2_{L^2(M)} + \frac{\lambda}{\sigma+1}\|u^\delta(t)\|^{2\sigma+2}_{L^{2\sigma+2}(M)}\right) =\\
&\quad -2\gamma \int_M \frac{\delta |\nabla u^\delta|^2 + (1-\alpha)|\mathrm{Re}(\overline{u^\delta}\nabla u^\delta)|^2 + |\mathrm{Im}(\overline{u^\delta}\nabla u^\delta)|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}(t,x)dx.
\end{aligned}
$$

They are obtained by the same procedure as in the proof of Proposition 2.1: formally, multiply (5.1) by $\overline{u^\delta}$, integrate over M, and take the imaginary part to get the first evolution law; multiply (5.1) by $\partial_t \overline{u^\delta}$, integrate over M, and take the real part to get the second evolution law.

The first law yields a global *a priori* estimate for $\|u^\delta(t)\|_{L^2(M)}$, uniformly with respect to $\delta \in]0,1]$. We infer a uniform H^1 control from the second law: in the same fashion as in the proof of Lemma 4.1, Gagliardo–Nirenberg inequalities on $\mathbf{R}$ yield, in particular,

$$
\|f\|_{L^\infty(M)} \leqslant C \|f\|^{1/2}_{L^2(M)} \|f\|^{1/2}_{H^1(M)}, \quad \forall f \in H^1(M).
$$

Denoting

$$
E^\delta(t) = \|\nabla u^\delta(t)\|^2_{L^2(M)} + \frac{\lambda}{\sigma+1}\|u^\delta(t)\|^{2\sigma+2}_{L^{2\sigma+2}(M)},
$$

we have of course $E^\delta(t) \leqslant E^\delta(0)$, a quantity which does not depend on δ. In the defocusing case $\lambda \geqslant 0$, this yields the required *a priori* H^1 estimate. In the focusing case $\lambda < 0$, write as on $\mathbf{R}$,

$$
\begin{aligned}
\|\nabla u^\delta(t)\|^2_{L^2(M)} &= E^\delta(t) + \frac{|\lambda|}{\sigma+1}\|u^\delta(t)\|^{2\sigma+2}_{L^{2\sigma+2}(M)}\\
&\leqslant E^\delta(0) + C\|u^\delta(t)\|^{2\sigma}_{L^\infty(M)}\|u^\delta(t)\|^2_{L^2(M)}\\
&\leqslant C\left(\|u_0\|_{H^1(M)}\right) + C\|u^\delta(t)\|^{\sigma+2}_{L^2(M)}\|u^\delta(t)\|^{\sigma}_{H^1(M)}.
\end{aligned}
$$

The L^2 *a priori* estimate and the assumption $\sigma < 2$ yield a uniform *a priori* H^1 estimate. The analogue of Theorem 1.3 follows as in Section 2 (see [12] for more details concerning the nonlinearity $|u|^{2\sigma}u$).

Uniqueness stems from the same arguments as in Section 3, and the fact that the nonlinearity $|u|^{2\sigma}u$ is uniformly Lipschitzean on balls of $H^1(M)$, since

$$
\left||u|^{2\sigma}u - |v|^{2\sigma}v\right| \lesssim \left(|u|^{2\sigma}+|v|^{2\sigma}\right)|u-v| \lesssim \left(\|u\|^{2\sigma}_{L^\infty(M)} + \|v\|^{2\sigma}_{L^\infty(M)}\right)|u-v|,
$$

and $H^1(M) \hookrightarrow L^\infty(M)$.

The end of the proof of Corollary 1.8 is exactly the same as the proof of Theorem 1.6 in the case $d=1$: since $\lambda \in \mathbf{R}$, (3.2) remains valid.

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