

NEW GARSIDE STRUCTURES AND APPLICATIONS TO ARTIN GROUPS

THOMAS HAETTEL AND JINGYIN HUANG

ABSTRACT. Garside groups are combinatorial generalizations of braid groups which enjoy many nice algebraic, geometric, and algorithmic properties. In this article we propose a method for turning the direct product of a group G by \mathbb{Z} into a Garside group, under simple assumptions on G . This method gives many new examples of Garside groups, including groups satisfying certain small cancellation condition (including surface groups) and groups with a systolic presentation.

Our method also works for an infinite class of Artin groups, leading to many new group theoretic, geometric and topological consequences for them. In particular, we prove new cases of $K(\pi, 1)$ -conjecture for some hyperbolic type Artin groups.

1. INTRODUCTION

The notion of Garside group originated in Garside's work on word problems and conjugacy problems for braid groups [Gar69]. It turns out the key structure needed in Garside's argument also appears in more general groups later, notably in spherical Artin groups [BS72] and fundamental groups of complexified central simplicial arrangement complements [Del72]. An axiomatic setup was provided in [DP99, Deh02], to study groups that share a similar structure as a class, called *Garside groups*. Since then, other important classes of groups were proven to be Garside groups, including but not limited to some semi-direct products [CP05], some complex braid groups [Bes15, CP11, CLL15], structure groups of non-degenerate, involutive and braided set-theoretical solutions of the quantum Yang-Baxter equation [Cho10], affine Artin groups of \tilde{A}_n and \tilde{C}_n [Dig06, Dig12], crystallographic braid groups [MS17], rank three Artin groups [DPS22] etc. Garside groups

Date: October 16, 2024.

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Keywords : Garside groups, Artin groups, $K(\pi, 1)$ -conjecture, hyperplane arrangements, Coxeter groups, dual Garside structure, combinatorial Garside structure, interval groups, CUB spaces, Helly graphs, Helly groups, bicombing, Nonpositive curvature **AMS codes** : 20E42, 20F36, 20F55, 05B35, 06A12, 20F65, 05C25

are also known to be closed under certain kinds of amalgamation products and HNN extensions [Pic22], as well as Zappa–Szép products [GT16]. There are also several variations and generalizations of Garside groups, applying to more natural examples - we refer to the book [Deh15] for a comprehensive review.

Garside group in this article always means Garside group of finite type, i.e. the Garside element has finitely many divisors. If it has infinitely many, then we will call it a *quasi-Garside* group. Garside groups are known to enjoy a long list of nice geometric, group theoretic and topological properties - they are biautomatic [Cha92a, Deh02], hence have solvable word problems and conjugacy problems, they are torsion-free, and admit finite $K(\pi, 1)$ spaces [CMW04, DL03], they act geometrically on Helly graphs and injective metric spaces [HO21b, Hae24], hence satisfy the Farrell-Jones conjecture and coarse Baum-Connes conjecture [CCG⁺20] etc. Also, Garside groups of finite or infinite type play a central role in the proof of the $K(\pi, 1)$ -conjecture for different classes of complex hyperplane complements, see [Del72, Bes15, MS17, PS21].

1.1. New Garside groups. While Garside groups enjoy nice properties, they have a very strong algebraic constraint: since a power of the Garside element is central, they have infinite center. This explains why the list of known examples of Garside groups is somewhat limited. In this article, we propose a simple approach to circumvent this obstruction and use Garside theory to study some groups with possibly trivial center. Namely, given a group G , we will consider the direct product of G with \mathbb{Z} to artificially create a center, which will serve as the Garside axis. Then we work backward to find necessary conditions on G to make sure $G \times \mathbb{Z}$ is a Garside group, leading to the following simple criterion.

Theorem A. (=Theorem 3.9) *Let U be a finite set, endowed with a positive partial multiplication (see Definition 3.1), and associated prefix order \leq_L and suffix order \leq_R . Assume that the following requirements hold:*

- (U, \leq_L) and (U, \leq_R) are meet-semilattices.
- For any $a, u, v, w \in U$ such that $a \cdot u, a \cdot v \in U$ and w is the join for \leq_L of u and v , then $a \cdot w \in U$.
- For any $a, u, v, w \in U$ such that $u \cdot a, v \cdot a \in U$ and w is the join for \leq_R of u and v , then $w \cdot a \in U$.
- For any $a, b, u, v \in U$ such that $a \cdot u, a \cdot v, b \cdot u, b \cdot v \in U$, either a, b have a join for \leq_R , or u, v have a join for \leq_L .

Consider the group G_U given by the following presentation:

$$G_U = \langle U \mid \forall u, v, w \in U \text{ such that } u \cdot v = w, \text{ we have } uw = w \rangle.$$

Then the group $G_U \times \mathbb{Z}$ is a Garside group, with Garside element $(e, 1)$.

First note that if a group G is such that $G \times \mathbb{Z}$ is a Garside group, then we can deduce an impressive list of consequences for G , see Theorem D below.

This method can be applied to several classes of groups that we discuss in this article. For instance, it applies to some groups given by a $T(5)$ positive presentation, see Theorem 3.10 for a precise statement. In particular, we deduce the following nice consequence.

Corollary B. (=Corollary 3.11) *For any surface S of finite type (possibly non-orientable), except the projective plane, $\pi_1(S) \times \mathbb{Z}$ is a Garside group.*

Another interesting family of examples comes from groups given by a presentation such that the associated flag Cayley complex is systolic, called *systolic restricted presentation* by Soergel in [Soe21], where they are defined and studied. We refer to Definition 3.12. Examples include some amalgams of Garside groups and some 2-dimensional Artin groups. For these groups, we prove the following.

Corollary C. (=Corollary 3.13) *Let G denote a group with a systolic restricted presentation. Then $G \times \mathbb{Z}$ is a Garside group.*

Theorem A also applies to a class of groups with positive square presentations in the sense of Definition 3.14, where a criterion for such groups times \mathbb{Z} to be Garside is provided in Theorem 3.15. This applies to a subclass of groups arising from word labeled oriented graphs in the sense of [HR15], as well as some of the mock right-angled Artin groups defined in [Sco08].

In order to motivate the study of groups G for which $G \times \mathbb{Z}$ is a Garside group, we record here a list of direct consequences. We recall the definition of Garside groups in Section 2.4, and we recall various nonpositive curvature notions in Section 2.5.

Theorem D. *Assume that G is a group such that $G \times \mathbb{Z}$ is Garside. Then the following hold:*

- (1) *The group $G \times \mathbb{Z}$ is Helly.*
- (2) *The group G is torsion-free.*
- (3) *The group G is CUB, more precisely it acts geometrically on a finite-dimensional metric space with a unique convex geodesic bicombing. Moreover, this metric space is a simplicial complex such that each simplex is equipped with a polyhedral norm.*
- (4) *The group G acts geometrically on a weakly modular graph.*
- (5) *The group G is biautomatic, and in particular:*
 - *The centralizer of a finite set of elements of G is biautomatic.*
 - *G has solvable word and conjugacy problems.*
 - *Any polycyclic subgroup of G is virtually abelian, finitely generated and undistorted.*
 - *G has quadratic Dehn function, as well as Euclidean higher dimensional Dehn function.*
- (6) *Any element of G has rational translation length, with a uniformly bounded denominator.*
- (7) *The group G has contractible asymptotic cones.*

- (8) The group G satisfies the Farrell-Jones conjecture with finite wreath products.
- (9) The group G satisfies the coarse Baum-Connes conjecture.
- (10) The group ring $\mathbb{K}[G]$ satisfies Kaplansky's idempotent conjecture, if \mathbb{K} is a field with characteristic zero.

We defer the references for this theorem to Section 2.1.

1.2. Applications to Artin groups. One of the main motivation for our work comes from Artin groups, see Section 2.2 for basic definitions. To each Coxeter group, there is an associated Artin group, in the same fashion that the n -strand braid group is associated to the symmetric group of an n -element set. Here is the precise definition.

For every finite simple graph Γ with vertex set S and with edges labeled by some integer in $\{2, 3, \dots\}$, one associates the Coxeter group W_Γ with the following presentation:

$$W_\Gamma = \langle S \mid \forall \{s, t\} \in \Gamma^{(1)}, \forall s \in S, s^2 = 1, [s, t]_m = [t, s]_m \text{ if the edge } \{s, t\} \text{ is labeled } m \rangle,$$

where $[s, t]_m$ denotes the word $ststs\dots$ of length m . Such a graph Γ may be called a *Coxeter presentation graph*, emphasizing the fact that edges correspond to relations. We will also use W_S to denote W_Γ .

We will also be using a graph closely related to Γ , the *Dynkin diagram* Γ_D : it has the same vertex set S , with some edges labeled in $\{4, 5, \dots, \infty\}$, with the following edges between vertices $s, t \in S$:

- If there is an edge labeled 2 between s and t in Γ , there is no edge between s and t in Γ_D .
- If there is an edge labeled 3 between s and t in Γ , there is an unlabeled edge between s and t in Γ_D .
- If there is an edge labeled by $m \geq 4$ between s and t in Γ , there is the same edge between s and t in Γ_D labeled m .
- If there is no edge between s and t in Γ , there is an edge between s and t in Γ_D labeled ∞ .

The associated Artin group A_Γ is defined by a similar presentation:

$$A_\Gamma = \langle S \mid \forall \{s, t\} \in \Gamma^{(1)}, [s, t]_m = [t, s]_m \text{ if the edge } \{s, t\} \text{ is labeled } m \rangle.$$

The groups A_Γ are also called Artin-Tits groups, since they have been defined by Tits in [Tit66]. We will also use A_S to denote A_Γ .

Remark (Remark on convention). When the Coxeter presentation graph Γ of an Artin group is not a complete graph, we tend to work with Coxeter presentation graph. When the Coxeter presentation graph Γ of an Artin group is a complete graph, we tend to work with the associated Dynkin diagram Γ_D , which is often more informative. For example, for the braid group, its Coxeter presentation graph is a complete graph, but its Dynkin diagram is a linear graph. Also, note that the Coxeter presentation graph of

an Artin group is complete if and only if its Dynkin diagram does not have edges labeled by ∞ .

General Artin groups are largely mysterious, and even basic questions such as the following are still wide open (see [GP12], [Cha], [McC17]).

- (1) Are Artin groups torsion-free?
- (2) What is the center of Artin groups?
- (3) Do Artin groups have solvable word problem?
- (4) Is the natural hyperplane complement a classifying space for Artin groups (the $K(\pi, 1)$ conjecture, see Section 2.3)?

Note that a positive answer to the $K(\pi, 1)$ conjecture implies that the corresponding Artin group is torsion-free, and also that its center is known (see [JS23]).

For Artin groups of spherical type, i.e. when the associated Coxeter group is finite, all these questions have a precise answer, which all rely on the existence of Garside structures. In fact, Artin groups of spherical type enjoy two different Garside structures: the standard one, associated with the longest element in the associated finite Coxeter group, and the dual one, associated with a Coxeter element. For an Artin group of non-spherical type, only the dual structure could be studied. In this case, the dual interval is always infinite, so one can only hope for a quasi-Garside structure, which has much fewer consequences. Nevertheless, it is known that for an Artin group of affine type \widetilde{A}_n , \widetilde{C}_n or \widetilde{G}_2 ([Dig06, Dig12, McC15]), or for an Artin group of rank 3 [DPS22], this dual structure turns the Artin group into a quasi-Garside group. In fact, for every Artin group of affine type, McCammond and Sulway manage to provide a natural embedding of the Artin group into a quasi-Garside crystallographic braid group, which is central in the proof of the $K(\pi, 1)$ conjecture by Paolini and Salvetti ([PS21]).

However, even though a quasi-Garside structure might be sufficient to find classifying spaces, we already mentioned that a Garside structure on the direct product with \mathbb{Z} is also useful, see Theorem D.

We say an Artin group is of *cyclic type* if its Dynkin diagram is a cycle without ∞ -labeled edges, and any proper parabolic subgroup is spherical. Note that cyclic type Artin groups all have complete Coxeter presentation graphs. We refer to Table 1 for a complete list of cyclic type Artin groups, in terms of their Dynkin diagrams. In particular, it contains some Artin groups that are associated with certain Coxeter groups acting on the hyperbolic spaces \mathbb{H}^3 or \mathbb{H}^4 - all of the four basic questions are open for these Artin groups.

Theorem E. (=Proposition 5.6) *Suppose A_Γ is of cyclic type. Then $A_\Gamma \times \mathbb{Z}$ is a Garside group.*

As we will see later (Corollary H), Theorem E gives rise to new examples of Artin groups satisfying the $K(\pi, 1)$ -conjecture. We emphasize that an

advantage of the method here is that it not only gives the $K(\pi, 1)$ -conjecture, but also it implies a long list of highly nontrivial algorithmic, geometric and topological consequences as in Theorem D.

We can also treat a more general class of Artin groups which are obtained by gluing cyclic Artin groups and spherical Artin groups in the following way. To state the result, we will use the Coxeter presentation graphs.

Given a 4-cycle $\omega \subset \Gamma$ with consecutive vertices $\{x_i\}_{i=1}^4$, a pair of antipodal vertices in ω means either the pair $\{x_1, x_3\}$, or the pair $\{x_2, x_4\}$. A 4-cycle in Γ has a *diagonal* means it has a pair of antipodal vertices of ω which are connected by an edge in Γ . We say an induced subgraph of Γ is of *cyclic type* or *spherical type* if the Artin group defined on this subgraph is of cyclic type or spherical type. An edge of Γ is *large* if it has label ≥ 3 . For an induced subgraph Λ of Γ , let Λ^\perp be the induced subgraph of Γ spanned by vertices of $\Gamma \setminus \Lambda$ that commute with each vertex of Λ . We say the Coxeter presentation graph Γ is a *join* of two Coxeter presentation subgraphs Γ_1 and Γ_2 if Γ is a join of Γ_1 and Γ_2 as graphs (i.e. Γ is obtained from a disjoint union of Γ_1 and Γ_2 by adding a single edge between each vertex of Γ_1 and each vertex of Γ_2), and each vertex of Γ_1 commute with every vertex of Γ_2 . We say a Coxeter presentation graph Γ is spherical (resp. of cyclic type) if A_Γ is a spherical Artin group (resp. an Artin of cyclic type).

Theorem F. (=Theorem 6.3) *Let Γ be a Coxeter presentation graph such that*

- *each complete subgraph of Γ is a join of a cyclic type graph and a spherical type graph (we allow one of the join factors to be empty);*
- *for any cyclic type induced subgraph $\Lambda \subset \Gamma$, Λ^\perp is spherical.*

We assume in addition that there exists an orientation of all large edges of Γ such that

- (1) *the orientation restricted to each cyclic type subgraph of Γ gives a consistent orientation on the associated circle;*
- (2) *if ω is a 4-cycle in Γ with a pair of antipodal points x_1 and x_2 such that each edge of ω containing $x_i \in \{x_1, x_2\}$ is either not large or oriented towards x_i , then the cycle has a diagonal.*

Then $A_\Gamma \times \mathbb{Z}$ is a Garside group.

Below we include two simple examples of Coxeter presentation graph Γ where the Theorem F applies, see Figure 1. The first is an amalgamation of two Artin groups of type \widehat{A}_4 along a spherical parabolic subgroup of type A_3 . The second example is a bit more complicated, made of a few cyclic type Artin groups glued together in a cyclic way. Note that the edges without labels are assumed to be labeled by 2.

In particular, all consequences listed in Theorem D hold for this class of Artin groups. All of these consequences are new for this class, including the solvability of word problem. As a more precise comparison to previous

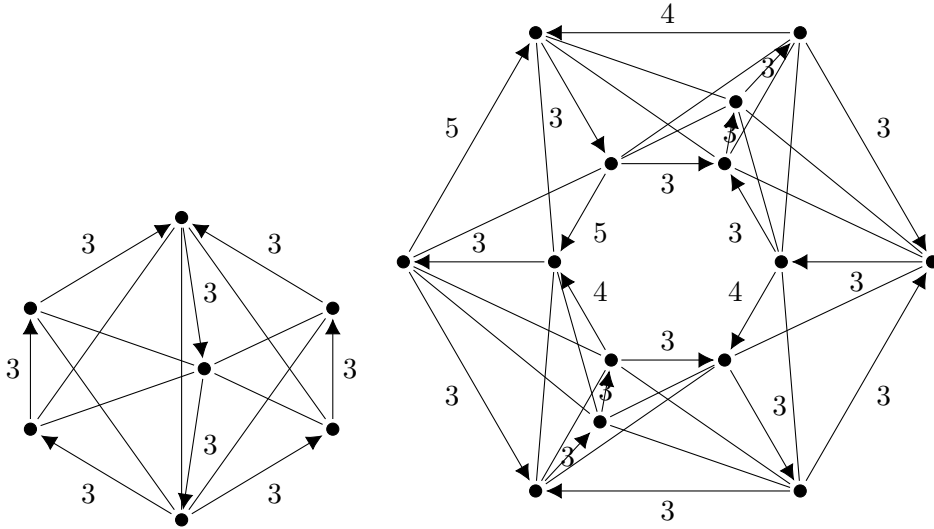


FIGURE 1. Examples of Coxeter presentation graphs of Artin groups to which Theorem F applies.

results, we view the class of Artin groups in the above theorem as a combination of basic building blocks made of cyclic type Artin groups and spherical Artin groups. Then

- (1) All consequences listed in Theorem D were known before for spherical Artin groups [Cha92b, HO21b, HH24], hence also known for the Artin group of type \tilde{A}_n , as the direct product of this Artin group with \mathbb{Z} has finite index in a spherical Artin groups [KIP02];
- (2) All consequences of Theorem D except the first one (acting geometrically on a Helly graph) were known before for cyclic Artin groups with at most three generators - as these groups act geometrically on CAT(0) complexes made of equilateral triangles [BM00];
- (3) All consequences of Theorem D are new for the remaining cyclic type Artin groups.
- (4) To the best of our knowledge, for each of the properties in the list of consequences of Theorem D, there does not exist a combination theorem which is powerful enough to cover the pattern of combination of cyclic type and spherical type Artin groups in Theorem F, thus all consequences are new for the class of Artin groups in Theorem F. For example, the most recent combination theorem for Farrell-Jones conjecture [Kno19] requires an acylindrical action of the group on a tree, which is not satisfied in our situation.

By a combination theorem, we mean a theorem of form that if a graph of groups G has each of its vertex groups and edge groups satisfying property X , then G also satisfies property X , under possibly some extra assumptions on G .

- (5) Artin groups in Theorem F are in general not of type FC, so consequences of Theorem D for this class do not follow from [HO21b].

These conditions are the most general that we can deal with. In particular, we isolate simple families of Artin groups to which this result applies.

Corollary G. *Assume that A_Γ is one of the following Artin groups:*

- A_Γ is of large type (i.e. no commuting relations in its standard presentation), and has rank at most 3.
- A_Γ is right-angled (i.e. only commuting relations in its standard presentation), without any induced squares.

Then $A_\Gamma \times \mathbb{Z}$ is Garside.

We emphasize that even for one of the simplest and most extensively studied classes of Artin groups, namely the class of right-angled Artin groups, not much is known about the connection to Garside groups. For the free group \mathbb{F}_r or rank r , Bessis has defined a quasi-Garside structure on \mathbb{F}_r ([Bes06]). In comparison, it is interesting that we are able to endow the direct product $\mathbb{F}_r \times \mathbb{Z}$ with an actual Garside structure, and not a mere quasi-Garside structure.

Assumptions of Theorem F have a close connection to an existing result for a class of 2-dimensional Artin groups by [BM00]. More precisely, [BM00] studied the class of *large type* Artin groups, i.e. each edge in the Coxeter presentation graph has label ≥ 3 . A dihedral subgroup of A_Γ is a subgroup generated by two vertices in an edge of Γ . Interestingly, if we restrict Theorem F within the class of large type Artin groups, the left forbidden configuration in [BM00, Figure 5] corresponds exactly to Assumption 1 in Theorem F, and the right forbidden configuration in [BM00, Figure 5] corresponds exactly to Assumption 2 in Theorem F. There is an interesting geometric phenomenon behind this.

The strategy in [BM00] is to consider a dual Garside structure of each dihedral subgroup (choosing a dual Garside structure amounts to choosing an orientation of the associated edge), metrize each triangle in the presentation complex with respect to the dual Garside structure as flat equilateral triangles, and gluing these presentation complexes for dihedral subgroup in a natural way to obtain a complex with fundamental group A_Γ . Then the result [BM00, Theorem 7] implies that as long as the presentation graph Γ avoids two configurations in [BM00, Figure 5], then the resulting complex is locally CAT(0).

Theorem F has a geometric counterpart (cf. Corollary 6.4). More precisely, given an Artin group A_Γ , we can choose a dual Garside structure on each standard spherical parabolic subgroup in a consistent way (again such information can be encoded as an appropriate orientation of all large edges of Γ). The dual Garside structure on each spherical parabolic subgroup H gives an associated Garside complex (Definition 2.7) with fundamental group H . By gluing these Garside complexes in a natural way (as in [PS21, Section 5]),

we obtain a complex X_Γ with fundamental group A_Γ . Here we metrize each simplex in X_Γ by a polyhedral norm which is related to the \tilde{A}_n -geometry, see [Hae22] (the norm here is not Euclidean), which echoes the work of [BM00] where they metrize triangles with Euclidean \tilde{A}_2 shape. The assumptions in Theorem F will ensure that the universal cover of X_Γ with such metric is a space with convex geodesic bicombing (see Definition 2.13), which can be viewed as a form of non-positive curvature, and echoes the CAT(0) metric in [BM00].

It is natural to ask, if we metrize each simplex in X_Γ by Euclidean simplices with \tilde{A}_n -shape, whether the complex we obtain is locally CAT(0). However, it is notoriously difficult to verify local CAT(0)-ness in high dimension. Here this issue is bypassed by metrizing the simplices with different kinds of norms rather than the Euclidean norm. While the resulting metric is not locally CAT(0), it is almost as good as CAT(0) in the sense that it implies most of the consequences of CAT(0) groups. We refer to [Hae22, Hae24], as well as [DL15, DL16] for more discussion in this direction.

Interestingly, for every Artin group as in Theorem E, we have an answer to all four questions stated above for general Artin groups. In particular, we can deduce new cases of $K(\pi, 1)$ -conjecture from Theorem E.

Corollary H. (=Corollary 5.7) *Assume that A_Γ is of hyperbolic cyclic type. Then A_Γ satisfies the $K(\pi, 1)$ conjecture and has a trivial center.*

More precisely, the $K(\pi, 1)$ -conjecture is new for 6 examples of Artin groups whose Coxeter groups act cocompactly on \mathbb{H}^3 or \mathbb{H}^4 . These examples seem to be challenging from the viewpoint of other approaches to $K(\pi, 1)$ -conjecture. Though the $K(\pi, 1)$ -conjecture when A_Γ is 2-dimensional hyperbolic cyclic type follows from previous work [CD95], and there is also a more recent proof in [DPS22] using dual quasi-Garside structures.

To put Corollary H in another context, note that the $K(\pi, 1)$ -conjecture is proved by Artin groups associated with reflection groups acting on \mathbb{S}^n by Deligne [Del72], and Artin groups associated with reflection groups acting on \mathbb{E}^n by Paolini and Salvetti [PS21]. The next step is to look at Artin groups associated with reflection groups acting on \mathbb{H}^n (we call them hyperbolic type Artin groups), whose $K(\pi, 1)$ -conjecture is wide open. A fundamental subclass of hyperbolic type Artin groups is those associated with hyperbolic reflection groups whose fundamental domain is a compact simplex. This subclass is classified by Lanner [Lan50], which consists of infinitely many members in dimension 2 (whose $K(\pi, 1)$ -conjecture is already understood [CD95, DPS22]), and in 14 remaining cases in higher dimension. From this perspective, Corollary H treats 6 out of these 14 remaining cases.

Corollary H also follows from another article of the second named author [Hua23, Theorem 1.4], via an alternative approach to the $K(\pi, 1)$ -conjecture. However, the method here establishes all the properties in Theorem D for hyperbolic cyclic type Artin groups, which are not consequences of [Hua23].

1.3. Structure of the article. In Section 2, we collect some background, notably on Garside groups, Artin groups and nonpositively curved spaces. In Section 3, we discuss the general criterion of making $G \times \mathbb{Z}$ a Garside group and prove Theorem A. Then we discuss examples of $T(5)$ and systolic restricted presentation groups. In Section 4, we adapt Theorem A to the special situation of Artin groups, and produce a criterion for when an Artin group times \mathbb{Z} is Garside, see Proposition 4.2 and Corollary 4.4. In Section 5, we verify the criterion in Proposition 4.2 and Corollary 4.4 for cyclic type Artin groups. In Section 6 we treat more general Artin groups and prove Theorem F.

1.4. Acknowledgement. The authors would like to thank Anthony Genevois, Jon McCammond, Alex Martin, Damian Osajda and Dani Wise for interesting discussions. The authors thank the Centre de recherches mathématiques de Montréal for hospitality. The authors would like to thank the anonymous referees for their careful comments which greatly helped to improve this article.

Thomas Haettel was partially supported by French project ANR-22-CE40-0004 GOFR. Jingyin Huang was partially supported by a Sloan fellowship.

2. BACKGROUND

We start by giving references for Theorem D in the introduction, then we collect background definitions and results concerning Artin groups, Garside groups, and nonpositively curved spaces.

2.1. Proof of Theorem D. We now give precise references for the various items of Theorem D from the introduction, listing consequences for a group G such that $G \times \mathbb{Z}$ is Garside.

- Proof.**
- (1) This is a consequence of [HO21b], see also [Hae24].
 - (2) This is a consequence of [Deh15, Proposition 3.25].
 - (3) This is a consequence of [Hae22, Corollary 9.8].
 - (4) This is a consequence of [HH24].
 - (5) This is a consequence of [Mos97]. For consequences of biautomaticity, see for instance [BH99, Wen03, BD19].
 - (6) This is a consequence of [CCG⁺20, Proposition 7.10].
 - (7) This is a consequence of [LL07], see also [HO21a].
 - (8) Since G acts geometrically on a metric space with a convex geodesic bicombing, according to [KR17, Theorem 6.1], it satisfies the Farrell-Jones conjecture with finite wreath products.
 - (9) Since G acts geometrically on a metric space with a convex geodesic bicombing, according to [FO20], it satisfies the coarse Baum-Connes conjecture.
 - (10) This is a consequence of the Farrell-Jones conjecture and [BLR08, Theorem 0.12].

□

2.2. Coxeter groups and Artin groups. We recall the definitions of Coxeter groups and Artin groups.

For every finite simple graph Γ with vertex set S and with edges labeled by some integer in $\{2, 3, \dots\}$, one associates the Coxeter group W_Γ with the following presentation:

$$W_\Gamma = \langle S \mid \forall \{s, t\} \in \Gamma^{(1)}, \forall s \in S, s^2 = 1, [s, t]_m = [t, s]_m \text{ if the edge } \{s, t\} \text{ is labeled } m \rangle,$$

where $[s, t]_m$ denotes the word $ststs\dots$ of length m . Such a graph Γ may be called a *Coxeter presentation graph*, emphasizing the fact that edges correspond to relations. We will also use W_S to denote W_Γ .

We will also be using a graph closely related to Γ , the *Dynkin diagram* Γ_D : it has the same vertex set S , with some edges labeled in $\{4, 5, \dots, \infty\}$, with the following edges between vertices $s, t \in S$:

- If there is an edge labeled 2 between s and t in Γ , there is no edge between s and t in Γ_D .
- If there is an edge labeled 3 between s and t in Γ , there is an unlabeled edge between s and t in Γ_D .
- If there is an edge labeled by $m \geq 4$ between s and t in Γ , there is the same edge between s and t in Γ_D labeled m .
- If there is no edge between s and t in Γ , there is an edge between s and t in Γ_D labeled ∞ .

The associated Artin group A_Γ is defined by a similar presentation:

$$A_\Gamma = \langle S \mid \forall \{s, t\} \in \Gamma^{(1)}, [s, t]_m = [t, s]_m \text{ if the edge } \{s, t\} \text{ is labeled } m \rangle.$$

The groups A_Γ are also called Artin-Tits groups since they have been defined by Tits in [Tit66]. We will also use A_S to denote A_Γ .

Note that only the relations $s^2 = 1$ have been removed, so that there is a natural surjective morphism from A_Γ to W_Γ . Also note that when $m = 2$, then s and t commute, and when $m = 3$, then s and t satisfy the classical braid relation $sts = tst$.

For a subset S' of the generating S , the subgroup of A_Γ or W_Γ generated by S' is called a *standard parabolic subgroup*. A standard parabolic subgroup of an Artin group is itself an Artin group [VdL83]. A similar statement is true for Coxeter groups [Bou02]. A *parabolic subgroup* is a conjugate of a standard parabolic subgroup.

Most results about Artin-Tits groups concern particular classes. The Artin group A_Γ is called:

- of *spherical type* if its associated Coxeter group W_Γ is finite, i.e. may be realized as a reflection group of a sphere.
- of *Euclidean type* if its associated Coxeter group W_Γ may be realized as a reflection group of an Euclidean space.
- of *hyperbolic type* if its associated Coxeter group W_Γ may be realized as a reflection group of a real hyperbolic space.

Name	\widetilde{A}_n , for $n \geq 3$	Triangle	3 - 3 - 3 - 4	3 - 3 - 3 - 5
Dynkin diagram				
Name	3 - 4 - 3 - 4	3 - 4 - 3 - 5	3 - 5 - 3 - 5	3 - 3 - 3 - 3 - 4
Dynkin diagram				

TABLE 1. Dynkin diagrams of cyclic type

We say a Coxeter group W_S is of *cyclic type* if the associated Dynkin diagram is a cycle, and the parabolic subgroup generated by $S \setminus \{s\}$ is spherical for any vertex $s \in \Gamma$. We list in Table 1 the Dynkin diagrams of cyclic type. Note that we use in this table the convention of Dynkin diagrams: vertices that are not joined by an edge commute, and we drop the label 3 from edges. Note that cyclic type Coxeter groups are either of Euclidean type or of hyperbolic type.

For an element g in Coxeter group W_S , we can represent g as a word in the free monoid on S . Such representation is *reduced* if its length is the shortest possible among words in the free monoid that represent g . It is known that any two reduced words representing the same element in W_S differ by a finite sequence of moves applying the relation in W_S , see e.g. [Hum90]. Thus each element in W_S has a well-defined *support*, which is the collection of elements in S which appears in a reduced word representing this element.

A subset $S' \subset S$ is *irreducible* if it spans a connected subgraph of the Dynkin diagram, otherwise S' is *reducible*.

A conjugate of an element of the standard generating set S is called a *reflection* of W_S .

Lemma 2.1. *The support of each reflection is irreducible.*

Proof. Let $r = wsw^{-1}$ be a reflection in W_S with $s \in S$ and $w \in W_S$. If $\text{Supp}(r)$ is reducible, then $\text{Supp}(r) = I_1 \sqcup I_2$ with elements in I_1 commuting with elements in I_2 . As $\langle r \rangle$ is a parabolic subgroup of W_S which is contained in the standard parabolic subgroup $W_{I_1 \cup I_2}$, by the remark after [Dav15, Lemma 5.3.6], there exists $w' \in W_{I_1 \cup I_2}$ and $s' \in I_1 \cup I_2$ such that $r = w's'(w')^{-1}$. We assume without loss of generality that $s' \in I_1$. Write $w' = w'_1 w'_2$ with $w'_i \in W_{I_i}$ for $i = 1, 2$. Then $r = w'_1 s' (w'_1)^{-1}$ and $\text{Supp}(r) \subset I_1$, which is a contradiction. Thus the lemma is proved. \square

2.3. The $K(\pi, 1)$ -conjecture. Artin groups are closely related to hyperplane complements, which can be presented in a simple way in spherical, Euclidean, and hyperbolic types. Fix a Coxeter group $W = W_\Gamma$ of spherical type, Euclidean or hyperbolic type acting by isometries on a sphere \mathbb{S}^{n-1} , Euclidean space \mathbb{R}^{n-1} or a real hyperbolic space \mathbb{H}^{n-1} , where the standard generators act by reflections.

In the case of \mathbb{S}^{n-1} , we will consider W as a subgroup of $O(n)$ acting by linear transformations on $\Omega = \mathbb{R}^n$. In the case of \mathbb{R}^{n-1} , we will consider W as a subgroup of $GL(n)$ acting by linear transformations on \mathbb{R}^n , preserving the hyperplane $\{x_n = 1\}$ and acting by isometries on it. The group W preserves the open cone $\Omega = \{x_n > 0\}$ of \mathbb{R}^n . In the case of \mathbb{H}^{n-1} , we will consider W as a subgroup of $O(n-1, 1)$ acting linearly on \mathbb{R}^n , and preserving the open cone $\Omega = \mathbb{H}^{n-1}$. Let \mathcal{R} denote the set of reflections of W . Consider the family of linear hyperplanes of \mathbb{R}^n

$$\mathcal{H} = \{H_r \mid r \in \mathcal{R}\},$$

where $H_r \subset \mathbb{R}^n$ denotes the fixed point set of the reflection r .

The analogue of the complement of the complexified hyperplane arrangement is

$$M(\Gamma) = (\Omega \times \Omega) \setminus \bigcup_{r \in \mathcal{R}} (H_r \times H_r),$$

see [Par14] for more details. Note that W acts naturally on M , and we have the following (see [VdL83]):

$$\pi_1(W_\Gamma \backslash M(\Gamma)) \simeq A_\Gamma.$$

So the Artin group A_Γ appears as the fundamental group of (a quotient of) the complement of a complexified hyperplane arrangement. One very natural question is to decide whether it is a classifying space. This is the statement of the following conjecture.

Conjecture ($K(\pi, 1)$ conjecture). *The space $M(\Gamma)$ is aspherical.*

This conjecture has been proved for spherical type Artin groups by Deligne in [Del72], for 2-dimensional and type FC Artin groups by Charney and Davis in [CD95], and for Euclidean type Artin groups by Paolini and Salvetti in [PS21] very recently.

2.4. Interval groups and Garside groups. We will follow McCammond's article [McC05] for the description of interval groups.

Definition 2.2 (Posets). A poset P is called *bounded* if it has a minimum, denoted 0, and a maximum, denoted 1.

For $x \leq y$ in a poset P , the *interval* between x and y is the restriction of the poset to those elements z with $x \leq z \leq y$. We denote this interval by $[x, y]$. A poset P is called *graded* if for any $x \leq y$ in P , any chain in $[x, y]$ belongs to a maximal chain and all maximal chains have the same finite length.

A poset P is called *weakly graded* if there is a poset map $r : P \rightarrow \mathbb{Z}$, i.e. such that for every $x < y$ in P , we have $r(x) < r(y)$: the map r is called a *rank map*. A poset P is called *weakly boundedly graded* if there is a rank map $r : P \rightarrow \mathbb{Z}$ with finite image. Note that a finite weakly graded poset is always weakly boundedly graded.

An *upper bound* for a pair of elements $a, b \in P$ is an element $c \in P$ such that $a \leq c, b \leq c$. A *minimal upper bound* for a, b is an upper bound c such that there does not exist upper bound c' of a, b such that $c' < c$. The *join* of two elements a, b in P is an upper bound c of them such that for any other upper bound c' of a, b , we have $c \leq c'$. We define *lower bound*, *maximal lower bound*, and *meet* similarly. In general, the meet or join of two elements in P might not exist. A poset P is a *lattice* if any pair of elements have a meet and a join.

A poset P is a *meet-semilattice* (resp. *join-semilattice*) if any pair of elements have a meet (resp. a join).

Definition 2.3. We say that a poset P contains a *bowtie* if there exist pairwise distinct elements a, b, c and d such that $a, b < c, d$, and there exists no $x \in P$ such that $a, b \leq x \leq c, d$.

It turns out that bowties are the only obstruction to being a lattice, for a weakly graded poset. This is proved in [BM10, Proposition 1.5] for bounded graded lattices. This also holds for weakly graded lattices, so we give a proof here for the convenience of the reader.

Proposition 2.4. *Let L denote a weakly graded poset. Then $L \cup \{0, 1\}$ is a lattice if and only if L has no bowtie.*

Proof. Assume that $L \cup \{0, 1\}$ is a lattice, and consider $a, b < c, d$ in L . Then the meet x of c, d is such that $a, b \leq x \leq c, d$. So L has no bowties.

Conversely, assume that L has no bowtie. Note that $L \cup \{0, 1\}$ has no bowtie either. Fix $a, b \in L$, and let M denote the set of upper bounds of a and b in $L \cup \{0, 1\}$: we have $1 \in M$, so M is not empty. Let us consider a sequence $(x_n)_{n \in \mathbb{N}}$ in M such that for each $n \in \mathbb{N}$, we have $x_n \geq x_{n+1}$. Let $r : P \rightarrow \mathbb{Z}$ denote a weak grading on P . Then the sequence $(r(x_n))_{n \in \mathbb{N}}$ in \mathbb{Z} is non-increasing and bounded below by $r(a)$, so it is eventually constant. This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ itself is eventually constant.

We may therefore consider a minimal element x of M . We will prove that x is unique: by contradiction, assume that $y \in M$ is a minimal element distinct from x . Then $a, b < x, y$ form a bowtie. Hence x is the unique minimal element of M , and it is the join of a and b in $L \cup \{0, 1\}$.

Similarly, any two elements of L has a meet in $L \cup \{0, 1\}$. So $L \cup \{0, 1\}$ is a lattice. \square

Here is one definition of Garside groups. We refer the reader to [Deh15] and [McC05] for more background on Garside groups. We also refer the reader to [HH24] to equivalent definitions of Garside groups, which are more geometric in flavour.

Definition 2.5 (Garside group). Let G denote a group, $S \subset G$ a finite subset and $\Delta \in G$. The triple (G, S, Δ) is called a *Garside structure* if the following conditions hold. Let G^+ denote the submonoid of G generated by S .

- (1) The group G is generated by S .
- (2) For any element $g \in G^+$, there is a bound on the length of expressions $g = s_1 \dots s_n$, where $s_1, \dots, s_n \in S \setminus \{1\}$.
- (3) We define the partial orders \leq_L, \leq_R on G^+ by $a \leq_L b$ if and only if $b = ac$ for some $c \in G^+$ and $a \leq_R b$ if and only if $b = ca$ for some $c \in G^+$. The left \leq_L and right \leq_R orders on G^+ are lattices.
- (4) The element $\Delta \in G^+$, and the set S is a balanced interval between 1 and Δ , i.e.

$$S = \{g \in G^+ \mid 1 \leq_L g \leq_L \Delta\} = \{g \in G^+ \mid 1 \leq_R g \leq_R \Delta\}.$$

A group is called *Garside* if it admits such a Garside structure, and Δ is called the *Garside element*. If the set S is allowed to be infinite, but with the additional assumption that (S, \leq_L) and (S, \leq_R) are weakly boundedly graded, we may say that (G, S, Δ) is a *quasi-Garside structure*.

Definition 2.6 (Labeled posets). If P is a poset, the *set of intervals* is $I(P) = \{(x, y) \in P^2 \mid x \leq y\}$.

Let P denote a bounded poset, and let S denote a labeling set.

An *interval-labeling* of P is a map $\lambda : I(P) \rightarrow S$.

An interval-labeling λ is *group-like* if, for any two chains $x \leq y \leq z$ and $x' \leq y' \leq z'$ having two pairs of corresponding labels in common, the third pair of labels are equal.

An interval-labeling λ is *balanced* if

$$\{\lambda(0, x) \mid x \in P\} = \{\lambda(x, 1) \mid x \in P\} = \{\lambda(x, y) \mid (x, y) \in I(P)\}.$$

Note that McCammond's definition of balanced interval labeling ([McC05, Definition 1.11]) only requires the first equality to hold. However, McCammond states that the second inequality is a consequence of being balanced and group-like, which does not seem obvious. We therefore chose to strengthen the definition of a balanced labeling, to ensure that all consequences of a combinatorial Garside structure hold.

Definition 2.7 (Interval complex and interval group). Let P denote a poset with a group-like interval-labeling λ .

Let us consider the quotient K_P of the geometric realization $|P|$ of P , where the k -simplices corresponding to two k -chains $(x_0 < x_1 < \dots < x_k)$ and $(x'_0 < x'_1 < \dots < x'_k)$ are identified if and only if $\lambda(x_0, x_1) = \lambda(x'_0, x'_1), \dots, \lambda(x_{k-1}, x_k) = \lambda(x'_{k-1}, x'_k)$. It is called the *interval complex* of P .

The fundamental group G_P of K_P is called the *interval group* of P , it is naturally a quotient of the free group over the labeling set S .

Example. Let us consider the Boolean lattice $P = \mathcal{P}(S)$ consisting of all subsets of a finite set S . The geometric realization $|P|$ of P is isomorphic to a simplicial subdivision of the cube $[0, 1]^S$.

For each $x \subset y \subset S$, let us consider the labeling $\lambda(x, y) = y \setminus x \in P$. The corresponding quotient K_P is isomorphic to a simplicial subdivision of the torus $(\mathbb{S}^1)^S$. The interval group G_P is isomorphic to the free abelian group \mathbb{Z}^S , with the following presentation:

$$G_P = \langle P \mid \forall x \subset y \subset z \subset S, (y \setminus x) \cdot (z \setminus y) = (z \setminus x) \rangle \simeq \mathbb{Z}^S.$$

Definition 2.8 (Combinatorial Garside structure). A *combinatorial Garside structure* is a poset P with an interval-labeling $\lambda : I(P) \rightarrow S$ such that:

- P is a bounded, weakly graded lattice.
- λ is group-like and balanced.

We say the combinatorial Garside structure is *finite* if P consists of finitely many elements.

Combinatorial Garside structures are just an explicit combinatorial way to describe arbitrary Garside groups, as explained by McCammond.

Theorem 2.9. [McC05, Theorem 1.17] *A group G is a quasi-Garside group if and only if G is isomorphic to the interval group of a combinatorial Garside structure. A group G is a Garside group if and only if G is isomorphic to the interval group of a finite combinatorial Garside structure.*

Sketch of proof. As the proof of this theorem is not found in [McC05], we provide a sketch of how this can be deduced.

If G is a quasi-Garside group, then we can consider the interval $P = ([1, \Delta], \leq_L)$. If $a \leq_L b$ in P , then the interval $[a, b]$ is labeled by $a^{-1}b$. This makes P a combinatorial Garside structure, and it is clear that the relations in the presentation of the interval group of P hold in G . Conversely, let us show that these relations are sufficient to define G . Let us consider s_1, \dots, s_n in $P \cup P^{-1}$ such that $s_1 s_2 \dots s_n = 1$. Using Δ and the fact that P is balanced, we may assume that s_1, \dots, s_n are in P and satisfy $s_1 s_2 \dots s_n = \Delta^k$, for some $k \geq 0$. Without loss of generality, we may assume that $s_1, s_n \neq 1, \Delta$. Therefore the word $s_1 s_2 \dots s_n$ is not the left Garside normal form of this element, which is Δ^k . Hence there exists $1 \leq i \leq n - 1$ such that $\Delta \wedge (s_i s_{i+1}) = s'_i \neq s_i$. Let us write $s'_i = s_i u$, with $u \in P$, and $s_{i+1} = u s'_{i+1}$, with $s'_{i+1} \in P$. Both relations hold in the interval group of P , hence we can replace $s_i s_{i+1}$ by $s'_i s'_{i+1}$ in the word $s_1 s_2 \dots s_n$. By iterating this process, we arrive at the left Garside normal form Δ^k . We conclude that the equality $s_1 s_2 \dots s_n = \Delta^k$ holds in the interval group of P . Hence G is isomorphic to the interval group of P .

Conversely, given P being a combinatorial Garside structure, then the collection of interval labels of P form an associative Noetherian germ in the sense of [Dig12, Section 2.1]. This germ has a special element, Δ , which

is the interval label associated to the interval between the minimum and maximum of P . This germ defines a monoid $M(P)$, as explained in [Dig12, Section 2.1], and this germ satisfies the assumptions of [Dig12, Proposition 2.4]. Hence [Dig12, Proposition 2.4] implies that the group of fraction of $M(P)$ is a Garside group. This Garside group is isomorphic to the interval group of this combinatorial Garside structure. \square

2.5. Nonpositive curvature: Helly graphs and CUB spaces. We will present briefly several notions of metric spaces and graphs of nonpositive curvature which are relevant to Garside groups.

Let us start with Helly graphs: we refer the reader to [CCG⁺20] for more details. Let Γ be a simplicial graph. We endow Γ with a path metric d such that each edge has length 1. A *combinatorial ball* in Γ is the collection of all vertices in some metric ball $B(x, r)$ centered at a vertex of Γ . Recall that the action of a group G on a metric space X is *geometric*, if the action is isometric, proper, and cocompact (i.e. X/G is compact).

Definition 2.10 (Helly graph, Helly group). Let Γ be a simplicial graph, which is connected. Then Γ is called *Helly* if any family of pairwise intersecting combinatorial balls have a non-empty total intersection.

A group is called *Helly* if it acts geometrically by automorphisms on a Helly graph.

Helly graphs are combinatorial analogues of injective metric spaces, defined as follows.

Definition 2.11 (Injective metric spaces). A geodesic metric space X is *injective* if any family of pairwise intersecting closed metric balls in X have a non-empty total intersection.

Helly groups enjoy many properties which are typical of nonpositive curvature, see for instance [CCG⁺20], [Lan13] and [HO21a] and also Theorem D.

A much weaker, but way broader notion is that of weakly modular graphs, see [CCHO21]. These graphs encompass many "nonpositive curvature type" graphs, such as Helly graphs, (weakly) systolic graphs, median and quasi-median graphs, modular graphs.

Definition 2.12 (Weakly modular graph). A connected graph Γ is called *weakly modular* if it satisfies the triangle condition (TC) and the quadrangle condition (QC):

- (TC) For any $x, y, z \in \Gamma^{(0)}$ such that $d(y, z) = 1$ and $d(x, y) = d(x, z) = n \geq 2$, there exists $t \in \Gamma^{(0)}$ such that $d(t, y) = d(t, z) = 1$ and $d(x, t) = n - 1$.
- (QC) For any $x, y, z, u \in \Gamma^{(0)}$ such that $d(y, u) = d(z, u) = 1$, $d(y, z) = 2$, $d(x, u) = n \geq 3$ and $d(x, y) = d(x, z) = n - 1$, there exists $t \in \Gamma^{(0)}$ such that $d(t, y) = d(t, z) = 1$ and $d(x, t) = n - 2$.

Many of the consequences for Helly groups rely simply on the existence of a convex geodesic bicombing, whose definition we recall here. We also recall the definition of CUB spaces and groups, defined in [Hae22].

Definition 2.13 (Bicombing, CUB). A *convex geodesic bicombing* on a metric space X is a continuous map $\sigma : X \times X \times [0, 1] \rightarrow X$ such that:

- For each $x, y \in X$, the map $t \in [0, 1] \mapsto \sigma(x, y, t)$ is a constant speed reparametrized geodesic from x to y .
- For each $x, x', y, y' \in X$, the map $t \in [0, 1] \mapsto d(\sigma(x, y, t), \sigma(x', y', t))$ is convex.

A metric space is called *CUB*, for Convexly Uniquely Bicombable, if it admits a unique convex geodesic bicombing. A group is called *CUB* if it acts geometrically by isometries on a CUB space.

Groups acting on spaces with convex bicomblings enjoy many properties, see for instance [DL16] and [DL15]. Furthermore, CUB groups satisfy some extra properties presented in [Hae22], see also Theorem D.

One major incarnation of the nonpositive curvature properties of Garside groups is the following.

Theorem 2.14 ([HO21b], see also [Hae24]). *Any Garside group acts geometrically by automorphisms on a Helly graph.*

The quotient of a Garside group by the cyclic subgroup generated by the Garside element also has nonpositive curvature in the following sense.

Theorem 2.15 ([Hae22],[HH24]). *Let G denote a Garside group, with Garside element Δ . Then the group $G/\langle \Delta \rangle$ acts geometrically by isometries on a CUB space, and it acts geometrically by automorphisms on a weakly modular graph.*

2.6. Dual Garside structures on spherical type Artin groups. Dual Garside structure on spherical type Artin groups have been studied notably by Birman-Ko-Lee ([BKL98]) and Bessis ([Bes03]), see also [Pao21] for an overview of dual Garside structures on general Artin groups. We also refer the reader to [McC05] for the point of view of interval groups that we are presenting here.

Let Γ denote a Coxeter presentation graph, with vertex set S . Given any linear ordering $S = \{s_1, \dots, s_n\}$ of S , we have an associated *Coxeter element* $\delta = s_1 s_2 \dots s_n$ in the Coxeter group $W = W_\Gamma$.

Let R denote the set of *reflections* of W , i.e. the set of all conjugates of elements of S . Since R generates W , we may consider its associated word norm $\|\cdot\|_R$. In the Cayley graph of W with respect to R , let us consider the interval P between e and δ : more precisely

$$P = \{u \in W \mid \|u\|_R + \|u^{-1}\delta\|_R = \|\delta\|_R = n\}.$$

The set P has a natural partial (prefix) order \leq_L : if $u, v \in P$, we say that $u \leq_L v$ if $\|u\|_R + \|u^{-1}v\|_R = \|v\|_R$. Equivalently, u is a prefix of a minimal expression of v as a product of reflections. Also equivalently, u lies on a geodesic in the Cayley graph between e and v .

The poset P is easily seen to be bounded and graded. Let us define an interval-labeling $\lambda : I(P) \rightarrow W$ by $\lambda(u, v) = u^{-1}v \in W$: this labeling is group-like and balanced. The poset P is finite if and only if W is finite, i.e. if Γ is of spherical type.

Definition 2.16 (Dual Artin group). The *dual Artin group* associated to Γ and δ is the interval group $A_\delta(\Gamma)$ of the poset P .

Theorem 2.17 (Birman-Ko-Lee [BKL98], Bessis [Bes03]). *If Γ is of spherical type, for any Coxeter element δ , the dual Artin group $A_\delta(\Gamma)$ is isomorphic to the standard Artin group A_Γ . Moreover, the poset P is a lattice: in particular, the Artin group A_Γ is a Garside group.*

2.7. Complexes associated with Garside groups. Consider a Garside group G , with positive monoid G^+ , Garside element Δ and Garside generating set S as in Definition 2.5. Let \leq_L and \leq_R be the orders as in Definition 2.5, which also extend to orders in G . More precisely, for $a, b \in G$, $a \leq_L b$ if $b = ac$ for some $c \in G^+$, and $a \leq_R b$ if $b = ca$ for some $c \in G^+$.

The *Garside complex* of G is the simplicial complex \widehat{X}_G with vertex set G , and with simplices corresponding to chains $g_1 <_L g_2 <_L \cdots <_L g_n$ such that $g_n \leq_L g_1 \Delta$. Note that G acts properly and cocompactly by simplicial automorphisms on its Garside complex. Alternatively, from the Garside group G , we can define an associated combinatorial Garside structure with the underlying poset P being the set $\{e\} \cup S$ equipped with the partial order \leq_L , and $\lambda(x, y) = x^{-1}y$ for $x, y \in P$. Then the universal cover of the interval complex associated with this combinatorial Garside structure is the Garside complex.

The *Bestvina complex* of G is the simplicial complex X_G whose vertices correspond to left cosets of $\langle \Delta \rangle$ in G ([Bes99]). There is an edge between two vertices $a\langle \Delta \rangle$ and $b\langle \Delta \rangle$ if we can find $a' \in a\langle \Delta \rangle$ and $b' \in b\langle \Delta \rangle$ such that $b' = a'x$ for some $x \in S \setminus \{\Delta\}$. The Bestvina complex is the flag complex induced by its 1-skeleton. Note that $\bar{G} = G/\langle \Delta \rangle$ acts properly and cocompactly by simplicial automorphisms on the Bestvina complex. Topologically \widehat{X}_G is homeomorphic to $X_G \times \mathbb{R}$.

Theorem 2.18. ([Hae24, Theorem E]) *For a Garside group G , if we metrize each simplex in the Garside complex \widehat{X}_G as orthoschemes with ℓ^∞ -metric as in [Hae24], then \widehat{X}_G is an injective metric space. In particular, it is CUB. Moreover, the injective metric on \widehat{X}_G descends to a CUB metric on the Bestvina complex X_G , whose simplices are equipped with special polyhedral norms in the sense of [Hae22].*

3. GARSIDE STRUCTURE ON $G \times \mathbb{Z}$

3.1. General construction. We will now present a general construction of a Garside structure on the direct product $G \times \mathbb{Z}$, where G is a group given by a specific presentation with generating set denoted U . We will consider U as an abstract set endowed with a partial multiplication as defined below.

Definition 3.1 (Positive partial multiplication). Let U denote a set. A map \cdot defined on a subset of $U \times U$ with range U is called a *positive partial multiplication* if the following hold:

- **Left associativity** For any $u, v, w \in U$ such that $u \cdot v$ and $(u \cdot v) \cdot w$ are defined, we require that $v \cdot w$ and $u \cdot (v \cdot w)$ are defined, and that we have the equality $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.
- **Right associativity** For any $u, v, w \in U$ such that $v \cdot w$ and $u \cdot (v \cdot w)$ are defined, we require that $u \cdot v$ and $(u \cdot v) \cdot w$ are defined, and that we have the equality $u \cdot (v \cdot w) = (u \cdot v) \cdot w$.
- **Identity** There exists a distinguished element $e \in U$ such that, for every $u \in U$, we have $e \cdot u = u \cdot e = u$.
- **Positivity** For any $u, v \in U$ such that $u \cdot v = e$, we have $u = v = e$.
- **Left cancellability** For any $u, v, w \in U$ such that $u \cdot v = u \cdot w$, we have $v = w$.
- **Right cancellability** For any $u, v, w \in U$ such that $v \cdot u = w \cdot u$, we have $v = w$.

Let us define relations \leq_L, \leq_R on U by:

$$\begin{aligned} u \leq_L v & \quad \text{if there exists } w \in U \text{ such that } u \cdot w = v \\ u \leq_R v & \quad \text{if there exists } w \in U \text{ such that } w \cdot u = v. \end{aligned}$$

Remark. Given $u, v \in U$, we will often write in the sequel " $u \cdot v \in U$ " in place of " $u \cdot v$ is defined".

Lemma 3.2. *The relations \leq_L, \leq_R are partial orders on U .*

Proof. By the existence of $e \in U$, we know that both relations are reflexive.

By the associativity assumption, we know that both relations are transitive.

We will now prove that \leq_L is antisymmetric, the proof for \leq_R is similar. Let us assume that $u, v \in U$ are such that $u \leq_L v$ and $v \leq_L u$. There exists $w, w' \in U$ such that $v = u \cdot w$ and $u = v \cdot w'$, hence $u = (u \cdot w) \cdot w' = u \cdot (w \cdot w')$ by associativity. Since U is cancellable, we deduce that $w \cdot w' = e$. Since U is positive, we conclude that $w = w' = e$, hence $u = v$. \square

If $u, v \in (U, \leq_L)$ has a join (resp. meet), then we denote the join (resp. meet) by $u \vee_L v$ (resp. $u \wedge_L v$). Similarly, we define $u \vee_R v$ and $u \wedge_R v$.

Note that the poset (U, \leq_L) admits an interval-labeling with labels in U , i.e. for $u, v \in U$, the label of the interval between u and $u \cdot v$ is $v \in U$. One readily verifies that this interval-labeling is group-like, so it makes

sense to define the interval group G_U . In particular, G_U has the following presentation:

$$G_U = \langle U \mid \forall u, v, w \in U \text{ such that } u \cdot v = w, \text{ we have } uv = w \rangle.$$

We will now describe the construction of a bounded poset E consisting of one copy of U and another “inverted” copy of U as follows, which will be such that G_E is isomorphic to $G_U \times \mathbb{Z}$.

Let \bar{U} be another copy of U , and we denote $\bar{u} \in \bar{U}$ to be the element associated with $u \in U$. We will think \bar{u} as a formal inverse of u .

Consider the set $E = (U, 0) \sqcup (\bar{U}, 1)$, with the following relation \preceq :

- $(u, 0) \preceq (v, 0)$ if and only if $u \leq_L v$.
- $(u, 0) \preceq (\bar{v}, 1)$ if and only if $v \cdot u \in U$.
- $(\bar{u}, 1) \preceq (\bar{v}, 1)$ if and only if $v \leq_R u$.

Lemma 3.3. *The relation \prec is a partial order on E , with minimum $(e, 0)$ and maximum $(\bar{e}, 1)$.*

Proof. The reflexivity is clear. For transitivity, if $(u, 0) \prec (\bar{v}, 1)$ and $(\bar{v}, 1) \prec (\bar{w}, 1)$, then $v \cdot u \in U$ and $w \leq_R v$. Thus $v = w' \cdot w$ for some $w' \in U$. Thus $(w' \cdot w) \cdot u \in U$. By right associativity of the partial multiplication, we know $w \cdot u \in U$. Thus $(u, 0) \prec (\bar{w}, 1)$. Other cases of transitivity are similar. The antisymmetry of \prec follows from the antisymmetry of \leq_L and \leq_R as in Lemma 3.2. \square

The poset E is interval-labeled, with labels in E :

- For $u, v \in U$, the label of the interval between $(u, 0)$ and $(u \cdot v, 0)$ is $(v, 0) \in E$.
- For $u, v, v \cdot u \in U$, the label of the interval between $(u, 0)$ and $(\bar{v}, 1)$ is $(\overline{v \cdot u}, 1) \in E$.
- For $u, v, v \cdot u \in U$, the label of the interval between $(\overline{v \cdot u}, 1)$ and $(\bar{u}, 1)$ is $(v, 0) \in E$.

Lemma 3.4. *The interval-labeled poset E is group-like.*

Proof. Consider a chain with 3 elements $a \prec b \prec c$ in E . Among the three labels $\lambda(a, b)$, $\lambda(a, c)$ and $\lambda(b, c)$, we will show that two of them determine the third one uniquely.

If $\lambda(a, b)$ and $\lambda(b, c)$ are known, there are three possibilities.

- Assume that $\lambda(a, b) = (u, 0)$ and $\lambda(b, c) = (v, 0)$. Then $\lambda(a, c) = (u \cdot v, 0) \in E$.
- Assume that $\lambda(a, b) = (u, 0)$ and $\lambda(b, c) = (\bar{v}, 1)$. Then $a = (w, 0)$ for $w \in U$, $b = (w \cdot u, 0)$ and $c = (\bar{x}, 1)$, where $x \cdot w \cdot u = v \in U$. Then $\lambda(a, c) = (\overline{x \cdot w}, 1) \in E$, and the value of $x \cdot w$ is determined by u and v through the equality $(x \cdot w) \cdot u = v$.

- Assume that $\lambda(a, b) = (\bar{u}, 1)$ and $\lambda(b, c) = (v, 0)$. Then $a = (w, 0)$ for some $w \in U$, $b = (\bar{x}, 1)$ and $c = (\bar{y}, 1)$ such that $u = x \cdot w \in U$ and $x = v \cdot y \in U$. Thus $u = v \cdot y \cdot w$. Since $a \prec c$, we know that $y \cdot w \in U$, so $\lambda(a, c) = (\overline{y \cdot w}, 1)$, and the value of $y \cdot w$ is determined by u and v through the equality $v \cdot (y \cdot w) = u$.

If $\lambda(a, b)$ and $\lambda(a, c)$ are known, there are three possibilities.

- Assume that $\lambda(a, b) = (u, 0)$ and $\lambda(a, c) = (v, 0)$. Since $b \prec c$, there exists $w \in U$ such that $u \cdot w = v$. Such w is unique by cancellability. Hence $\lambda(b, c) = (w, 0) \in E$.
- Assume that $\lambda(a, b) = (u, 0)$ and $\lambda(a, c) = (\bar{v}, 1)$. Then $\lambda(b, c) = (\overline{v \cdot u}, 1) \in E$.
- Assume that $\lambda(a, b) = (\bar{u}, 1)$ and $\lambda(a, c) = (\bar{v}, 1)$. Then $a = (w, 0)$ for some $w \in U$, $b = (\bar{x}, 1)$ and $c = (\bar{y}, 1)$, with $x, y \in U$ such that $x \cdot w = u$ and $y \cdot w = v$. Since $b \prec c$, there exists $z \in U$ such that $z \cdot y = x$. Hence $z \cdot y \cdot w = x \cdot w$, so $z \cdot v = u$. By cancellability, z is uniquely determined by u, v . Then $\lambda(b, c) = (z, 0)$.

By symmetry, the remaining case is similar. \square

Lemma 3.5. *The interval-labeled poset E is balanced.*

Proof. The interval between $(u, 0)$ and $(u \cdot v, 0)$ has label $(v, 0) \in E$, which is also the label of the interval between $(e, 0)$ and $(v, 0)$, and also between $(\bar{v}, 1)$ and $(\bar{e}, 1)$.

The interval between $(\overline{v \cdot u}, 1)$ and $(\bar{u}, 1)$ has label $(v, 0) \in E$, which is also the label of the interval between $(e, 0)$ and $(v, 0)$, and also between $(\bar{v}, 1)$ and $(\bar{e}, 1)$.

The interval between $(u, 0)$ and $(\bar{v}, 1)$ has label $(\overline{v \cdot u}, 1) \in E$, which is also the label of the interval between $(e, 0)$ and $(\overline{v \cdot u}, 1)$, and also between $(v \cdot u, 0)$ and $(\bar{e}, 1)$. \square

Proposition 3.6. *Let us consider the interval groups G_U, G_E associated with the interval-labeled posets U, E . Then the natural map*

$$\begin{aligned} E &\mapsto G_U \times \mathbb{Z} \\ (u, 0) \in U \times \{0\} \subset E &\mapsto (u, 0) \in G_U \times \mathbb{Z} \\ (\bar{u}, 1) \in \bar{U} \times \{1\} \subset E &\mapsto (u^{-1}, 1) \in G_U \times \mathbb{Z} \end{aligned}$$

extends to an isomorphism of groups between G_E and $G_U \times \mathbb{Z}$.

Proof. Note that $G_U \times \mathbb{Z}$ has generating set $(U \times \{0\}) \cup \{(e, 1)\}$, and there are two types of relations:

- (1) $(e, 1)(u, 0) = (u, 0)(e, 1)$ for each $u \in U$;
- (2) $(u, 0)(v, 0) = (w, 0)$ for any $u, v, w \in U$ with $u \cdot v = w$.

On the other hand, the group G_E has generating set E , and the relations are:

- (1) $(u, 0)(v, 0) = (w, 0)$ for any $u, v, w \in U$ with $u \cdot v = w$;
- (2) $(u, 0)(\bar{v}, 1) = (\bar{w}, 1)$ for any $u, v, w \in U$ with $w \cdot u = v$;
- (3) $(\bar{u}, 1)(v, 0) = (\bar{w}, 1)$ for $u, v, w \in U$ with $v \cdot w = u$.

One readily checks that the map defined in the proposition extends to a group homomorphism $G_E \rightarrow G_U \times \mathbb{Z}$ as it is compatible with the relations.

We now define the inverse of this map on the standard generators of $G_U \times \mathbb{Z}$:

$$\begin{aligned} (U \times \{0\}) \cup \{(e, 1)\} \subset G_U \times \mathbb{Z} &\mapsto G_E \\ (u, 0) &\mapsto (u, 0) \in G_E \\ (e, 1) &\mapsto (\bar{e}, 1) \in G_E. \end{aligned}$$

This map is also compatible with the relations of G_U and G_E , note that we see relations of $G_U \times \mathbb{Z}$ of type (1) (as in the beginning of the proof) in G_E as follows:

$$(u, 0)(\bar{e}, 1) = (u, 0)((\bar{u}, 1)(u, 0)) = ((u, 0)(\bar{u}, 1))(u, 0) = (\bar{e}, 1)(u, 0).$$

Note that the composition of these two maps are identity on the generators, thus they are inverses of each other. Then we are done. \square

Given $u, v \in U$, a *left upper common bound* for u, v is a upper bound for \leq_L . A *left join* of u and v is an element $w \in U$ with $u \leq_L w$ and $v \leq_L w$, such that $w \leq_L w'$ for any other left upper common bound w' of u, v . A left join, if exists, must be unique. Similarly, we define right upper common bound and right join for u, v .

Proposition 3.7. *Assume that U satisfies the following:*

- (1) (U, \leq_L) and (U, \leq_R) are weakly boundedly graded posets.
- (2) (U, \leq_L) and (U, \leq_R) are meet-semilattices.
- (3) For any $a, u, v, w \in U$ such that $a \cdot u, a \cdot v \in U$ and w is the join for \leq_L of u and v , then $a \cdot w \in U$.
- (4) For any $a, u, v, w \in U$ such that $u \cdot a, v \cdot a \in U$ and w is the join for \leq_R of u and v , then $w \cdot a \in U$.
- (5) For any $a, b, u, v \in U$ such that $a \cdot u, a \cdot v, b \cdot u, b \cdot v \in U$, either a, b have a join for \leq_R , or u, v have a join for \leq_L .

Then E is a lattice.

Proof. Assumption 1 implies E is a weakly boundedly graded poset. More precisely, we define the rank function on $(U, 0)$ using a rank function f_1 on (U, \leq_L) , and the rank function on (U, \leq_R) using the an (inverted) rank function f_2 on (U, \leq_R) . As f_1 and f_2 has finite image, we can assume any image of f_1 is strictly smaller than each element in the image of f_2 . So the new rank function on E gives a weak grading, with finite image. By Proposition 2.4, it is sufficient to prove that E contains no bowtie.

Assume that $(u, 0), (v, 0) \prec (w, 0), (x, 0)$ is a bowtie in E , where $u, v, w, x \in U$: hence $u, v \leq_L w, x$ is a bowtie in U , which contradicts that (U, \leq_L) is a meet-semilattice by Proposition 2.4.

Assume that $(\bar{a}, 1), (\bar{b}, 1) \leq (\bar{c}, 1), (\bar{d}, 1)$ is a bowtie in E , where $a, b, c, d \in U$: hence $a \in U \cdot c$, and $c \leq_R a$. So $c, d \leq_R a, b$ is a bowtie in U , which contradicts that (U, \leq_R) is a meet-semilattice.

Assume that $(u, 0), (v, 0) \prec (w, 0), (\bar{a}, 1)$ is a bowtie in E , where $u, v, w, a \in U$: hence $u, v \leq_L w$. Then u and v has a left join, denoted $u \vee_L v$, in (U, \leq_L) . Indeed, $u \vee_L v$ is the element with the lowest rank in (U, \leq_L) which is an upped bound of $\{u, v\}$, and this element is unique since (I, \leq_L) is a meet-semilattice. Since we are assumed to have a bowtie, $(w, 0)$ is a minimal upper bound of $(u, 0)$ and $(v, 0)$ with respect to \prec , so w is a minimal upper bound of u and v with respect to \leq_L . Thus $w = u \vee_L v$. As $(u, 0), (v, 0) \prec (\bar{a}, 1)$, we know $a \cdot u, a \cdot v \in U$. By assumption (3), this implies that $a \cdot w \in U$, so $(w, 0) \prec (\bar{a}, 1)$, which contradicts that $(u, 0), (v, 0) \prec (w, 0), (\bar{a}, 1)$ is a bowtie in E .

Assume that $(a, 0), (\bar{w}, 1) \prec (\bar{u}, 1), (\bar{v}, 1)$ is a bowtie in E , where $u, v, w, a \in U$: hence $u, v \leq_R w$. Similarly to the previous paragraph, u and v has a right join, denoted $u \vee_R v$, in (U, \leq_R) ; and $w = u \vee_R v$. As $(a, 0) \prec (\bar{u}, 1), (\bar{v}, 1)$, we have $u \cdot a, v \cdot a \in U$. By assumption, this implies that $w \cdot a \in U$, so $(\bar{w}, 1) \prec (a, 0)$, which contradicts that $(a, 0), (\bar{w}, 1) \prec (\bar{u}, 1), (\bar{v}, 1)$ is a bowtie in E .

Assume that $(u, 0), (v, 0) \prec (\bar{a}, 1), (\bar{b}, 1)$ is a bowtie in E , where $u, v, a, b \in U$: hence $a \cdot u, a \cdot v, b \cdot u, b \cdot v \in U$. By assumption, this implies that either a, b have a right join c for \leq_R or u, v have a left join w for \leq_L . In the former case, we deduce from assumption (4) of the proposition that $c \cdot u \in U$ and $c \cdot v \in U$. Hence $(\bar{c}, 1)$ is in the middle of the bowtie, i.e. $\{(u, 0), (v, 0)\} \prec (\bar{c}, 1) \prec (\bar{a}, 1), (\bar{b}, 1)$, which is a contradiction. Similarly, in the latter case, we deduce from assumption (3) of the proposition that $(w, 0)$ is in the middle of the bowtie, which is a contradiction. \square

The following is combination of Definition 2.8, Theorem 2.9, Theorem 2.18, Lemma 3.4, Lemma 3.5 and Proposition 3.7 (we refer to Definition 2.11 and Definition 2.13 for definitions of injective metric spaces and CUB spaces).

Theorem 3.8. *Under the assumption of Proposition 3.7, the group $G_E \simeq G_U \times \mathbb{Z}$ is quasi-Garside with Garside element $(e, 1)$ and set of simple elements E . If U is finite, then E is finite and G_E is Garside.*

Moreover, suppose U is finite. Let K_E denote the interval complex of the labeled poset E , whose universal cover is isomorphic to the Garside complex \widehat{X}_{G_E} of the Garside group G_E (as discussed in Section 2.7). If we metrize each simplex in \widehat{X}_G as orthoschemes with ℓ^∞ -metric as in [Hae24], then \widehat{X}_G is an injective metric space. In particular, it is CUB.

In particular, we can deduce the following result stated in the introduction.

Theorem 3.9. *Let U be a finite set, endowed with a positive partial multiplication, and let G_U denote the associated interval group. Assume that the following hold:*

- (U, \leq_L) and (U, \leq_R) are meet-semilattices.

- For any $a, u, v, w \in U$ such that $a \cdot u, a \cdot v \in U$ and w is the join for \leq_L of u and v , then $a \cdot w \in U$.
- For any $a, u, v, w \in U$ such that $u \cdot a, v \cdot a \in U$ and w is the join for \leq_R of u and v , then $w \cdot a \in U$.
- For any $a, b, u, v \in U$ such that $a \cdot u, a \cdot v, b \cdot u, b \cdot v \in U$, either a, b have a join for \leq_R , or u, v have a join for \leq_L .

Then the group $G_U \times \mathbb{Z}$ is a Garside group, with Garside element $(e, 1)$.

Remark. We may remark that there are very simple situations where we can apply Theorem 3.9. For instance, let us consider the free group F over a finite set S , and let $U = S \cup \{e\}$. Then U satisfies the assumptions of Proposition 3.7, and in particular the group $F \times \mathbb{Z}$ is Garside. This particular case can also be deduced from [Pic22] via different methods. We will however see, in the rest of the article, more interesting applications of this result.

3.2. Some examples where Theorem 3.9 applies. Recall that for a group G with a given presentation, there is a *presentation complex*, which is a 2-dimensional cell complex with the 0-skeleton being a single vertex, the 1-skeleton being a wedge of oriented circles in 1-1 correspondence with elements in the generating set (each circle is labeled by a generator), and for each relator we attached a 2-cell such that the boundary of this 2-cell goes to the loop in the 1-skeleton tracing out the word of this relator. The presentation complex has fundamental group being G .

For a (not necessarily simplicial) graph, its *girth* is defined to the minimal possible number of edges in an embedded \mathbb{S}^1 in the graph.

Theorem 3.10. *Let us consider a group G given by a finite presentation $\langle S \mid r_1 = r'_1, \dots, r_n = r'_n \rangle$. Assume that the following hold:*

- For each $1 \leq i \leq n$, the words r_i, r'_i are positive words in S , without common prefix or suffix.
- For each $1 \leq i \leq n$, the word r_i (and r'_i) does not appear as a subword of some of the other $2n - 1$ words.
- For each distinct $s, t \in S$, there exist at most one $1 \leq i \leq n$ such the the first letters of $\{r_i, r'_i\}$ are $\{s, t\}$.
- For each distinct $s, t \in S$, there exist at most one $1 \leq i \leq n$ such the the last letters of $\{r_i, r'_i\}$ are $\{s, t\}$.
- The presentation is $T(5)$, i.e. the link of the vertex in the presentation complex has girth at least 5.

Then $G \times \mathbb{Z}$ is Garside.

Proof. Let $\mathbb{F}^+(S)$ be the positive free monoid on the generating set S . Let U denote the quotient of the set of subwords of the words $R = \{r_1, r'_1, \dots, r_n, r'_n\}$ in $\mathbb{F}^+(S)$, under the equivalence relation defined by $r_i \sim r'_i$, for each $1 \leq i \leq n$. Given two positive words u_1, u_2 of $\mathbb{F}^+(S)$, we will write $u_1 = u_2$ if they are the same word in $\mathbb{F}^+(S)$, and $u_1 \equiv u_2$ if they give the same element in U . Note that if u is a proper non-trivial subword of r_i and u' is a proper

non-trivial subword of r'_i , then $u \neq u'$ - this follows from the second bullet point of the assumption of the theorem.

Let us endow U with the partial multiplication induced by the free monoid $\mathbb{F}^+(S)$ on S . For any $u \in U$, let us define $u \cdot e \equiv e \cdot u \equiv u$. For any $u, v \in U \setminus \{e\}$, then $u \cdot v$ exists and is equal to $uv \in \mathbb{F}^+(S)$ if and only if there exists $r \in R$ such that uv is a subword of r .

We will show that this defines a positive partial multiplication on U .

Assume that $u, v, w \in U$ are such that $u \cdot v \in U$ and $(u \cdot v) \cdot w \in U$. We will consider u, v, w as representatives inside $\mathbb{F}^+(S)$. Then there exists $r_1 \in R$ such that uv is a subword of r_1 . Moreover, $uv, w \notin R$ and uvw is a subword of $r \in R$. Hence vw is a subword of r . Note that $v \notin R$, otherwise we will contradict the second assumption of the theorem. Thus by the definition of product, $v \cdot w \equiv vw \in U$. So U is left associative, and similarly we can show it is right associative.

The identity element $e \in \mathbb{F}^+(S)$ is an identity element for (U, \cdot) . Now we check positivity. If $u \cdot v \in U$ and $u \cdot v \equiv e$, then $u, v \notin R$ and uv is a subword of $r \in R$. This forces $u = v = e$ as u and v are positive words. For the cancellability, if $u \cdot v \equiv u \cdot w$, then either $uv = uw$ in $\mathbb{F}^+(S)$ and $v = w$ follows from the cancellability in $\mathbb{F}^+(S)$, or there exists i such that $uv = r_i$ and $uw = r'_i$, which implies $u = e$ as we assume that for each $1 \leq i \leq n$, the words r_i, r'_i have no common prefix or suffix.

So U satisfies Definition 3.1. In particular, we may consider the left and right orders on U .

Let us prove that (U, \leq_L) is weakly boundedly graded. It is clear that $e \in U$ is the minimum of U . For each $u \in U$, let $r(u) \in \mathbb{N}$ denote the maximal length of a representative for u in $\mathbb{F}^+(S)$ (note that the second bullet point in the statement of the theorem implies that we can not keep replacing subwords in u by longer subwords, so there is a bound on the length of all representatives). The map $r : U \rightarrow \mathbb{N}$ is a rank map with respect to \leq_L and \leq_R . Since R is finite, r has finite image. So (U, \leq_L) and (U, \leq_R) are weakly boundedly graded.

Let us prove that (U, \leq_L) is a meet-semilattice. According to Proposition 2.4, it is sufficient to prove that (U, \leq_L) does not contain a bowtie. By contradiction, assume that $u, v \leq_L x, y$ is a bowtie in U , with $r(x) + r(u) + r(v) + r(y)$ minimal, where $r : U \rightarrow \mathbb{N}$ is a weak grading. Let $s, t \in S$ denote the first letters of u, v respectively. If $s = t$, then s is a common lower bound (with respect to \leq_L) of $\{u, v, x, y\}$, which enables us to find a bowtie with a smaller value of $r(x) + r(u) + r(v) + r(y)$. Thus $s \neq t$. We have $s, t \leq_L x, y$. Thus x can be represented by two different positive words x_s and x_t in the free monoid, with x_s starting with s and x_t starting with t . If $x_s \notin R$, then x_s is a proper non-trivial subword of some element r in R (by the second bullet point of the theorem). However, we can transform x_s to x_t by using relators finitely many times, this means a subword of x_s (hence a subword

of r) belongs to R , which contradicts the second bullet point of the theorem. Hence $x_s \in R$, and similarly y represents a word in R . According to the third bullet point of the theorem, we deduce that there exists $1 \leq i \leq n$ such that x, y both represent $r_i \equiv r'_i$. Hence $x = y$ in U , so $u, v \leq_L x, y$ is not a bowtie: this is a contradiction. Hence (U, \leq_L) is a meet-semilattice.

Let $a, u, v \in U$ such that $a \cdot u, a \cdot v \in U$ and $u \vee_L v = w \in U$. We want to prove that $a \cdot w \in U$. We will actually prove that $u \leq_L v$ or $v \leq_L u$: if not, this means that there exists $1 \leq i \leq n$ such that u, v are prefixes of r_i, r'_i respectively (up to switching u and v). Then the words $au, av, u^{-1}v$ give rise to a triangle in the link of the vertex in the presentation complex, which contradicts the $T(5)$ assumption. So $u \leq_L v$ or $v \leq_L u$, and hence $a \cdot w \in \{a \cdot u, a \cdot v\} \subset U$.

Let us assume that $a, b, u, v \in U$ are pairwise distinct such that $a \cdot u, a \cdot v, b \cdot u, b \cdot v \in U$. We will prove that either a, b are comparable for \leq_R , or u, v are comparable for \leq_L . If not, then the words au, av, bv, bu give rise to a 4-cycle in the link of the vertex in the presentation complex, which contradicts the $T(5)$ assumption. So for instance a, b are comparable for \leq_R , in which case a and b have a join for \leq_R .

According to Proposition 3.7, we deduce that $G \times \mathbb{Z}$ is a Garside group. \square

Corollary 3.11. *For any surface S of finite type (possibly non-orientable), except the projective plane, $\pi_1(S) \times \mathbb{Z}$ is a Garside group.*

Proof. If S is a surface with boundary, its fundamental group is a free group.

If S is the torus, then $\pi_1(S) \simeq \mathbb{Z}^2$, which is a Garside group, so $\pi_1(S) \times \mathbb{Z} \simeq \mathbb{Z}^3$ is a Garside group.

If S is the closed orientable surface with genus $g \geq 2$, consider the standard presentation, see e.g. [Hat02, Section 1.2],

$$G_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle.$$

This presentation is not positive, so we will modify it as follows, where \bar{a}_1 and \bar{b}_1 play the role of a_1^{-1} and b_1^{-1} :

$$G_g = \langle \bar{a}_1, \bar{b}_1, a_2, b_2, \dots, a_g, b_g, h_2, h_3, \dots, h_{g-1} \mid \bar{a}_1 \bar{b}_1 h_2 h_3 \dots h_{g-1} a_g b_g = \bar{b}_1 \bar{a}_1 b_g a_g, \\ a_2 b_2 = h_2 b_2 a_2, a_3 b_3 = h_3 b_3 a_3, \dots, a_{g-1} b_{g-1} = h_{g-1} b_{g-1} a_{g-1} \rangle.$$

Note that these two presentations give the same group - \bar{a}_1 and \bar{b}_1 go to a_1^{-1} and b_1^{-1} , and h_i goes to the commutator of a_i and b_i for $2 \leq i \leq g-1$. Then this new presentation satisfies the assumptions of Theorem 3.10.

If S is the projective plane, then $\pi_1(S) \simeq \mathbb{Z}/2\mathbb{Z}$, so $\pi_1(S) \times \mathbb{Z}$ has torsion, hence it is not a Garside group.

If S is the closed non-orientable surface with genus 2, i.e. the Klein bottle, then its fundamental group has the following presentation

$$\pi_1(S) = \langle a, b \mid a^2 = b^2 \rangle,$$

which is a Garside presentation with Garside element $\Delta = a^2 = b^2$. Hence $\pi_1(S) \times \mathbb{Z}$ is a Garside group.

If S is the closed non-orientable surface with genus $g \geq 3$, consider the (almost) standard presentation (see e.g. [Hat02, Section 1.2])

$$G = \langle a_1, \dots, a_g \mid a_1^2 \dots a_{g-1}^2 = a_g^2 \rangle,$$

then one readily checks that it satisfies the assumptions of Theorem 3.10. Here we explain the verification of the $T(5)$ condition. Note that the link of the unique vertex in the presentation complex has $2g$ vertices - for each generator a_i , it gives two vertices in the link, denoted by a_i^+ and a_i^- , corresponding to the outgoing and incoming direction of the oriented loop corresponding to a_i . The each i , there is an edge joining a_i^+ and a_i^- . For $1 \leq i \leq g-2$, there is an edge joining a_i^- and a_{i+1}^+ . There is an edge joining a_{g-1}^- and a_g^- , and an edge joining a_g^+ and a_1^+ . These are all the edges. Thus the link has girth $2g$. Since $g \geq 3$, we know the presentation is $T(5)$. \square

Another easy class of groups for which we can apply Theorem 3.8 is the following class of groups with a systolic presentation. They have been defined and studied by Soergel in [Soe21].

Recall that a flag simplicial complex is *systolic* if it is simply connected and all of its vertex links are 6-large, i.e. all cycles of length 4 or 5 have diagonals, see [JS06].

Definition 3.12 (Soergel [Soe21]). A finite presentation $\langle S \mid R \rangle$ of a group G is called a *systolic restricted presentation* if the following hold:

- Each relation $r \in R$ is of the form $r = abc^{-1} \in \mathbb{F}(S)$, where $a, b, c \in S$ and $\mathbb{F}(S)$ is the free group on S .
- The flag completion of the Cayley graph of G with respect to S is simplicial and systolic.

Note that asking that the Cayley graph of G with respect to S is simplicial is equivalent to asking that any $s \in S$ has image in G different from e , and also for any distinct $s, t \in S$, their image in G are neither equal nor inverse. Soergel gives a complete characterization of such systolic restricted presentations in [Soe21, Theorem 1].

Among Garside presentations, Soergel gives a characterization of those which are systolic, see [Soe21, Theorem 2]. There are essentially amalgams of the following Garside groups $G_{n,m}$, for $n, m \geq 1$, defined by the following systolic restricted presentation:

$$G_{n,m} = \langle x_1, \dots, x_n \mid x_1 x_2 \dots x_m = x_2 x_3 \dots x_{m+1} = x_n x_1 x_2 \dots x_{m-1} \rangle.$$

While the presentation of $G_{n,m}$ is not systolic restricted on the nose, [Soe21, Theorem 2] proves that these groups, as well as some amalgamations of them, actually have systolic restricted presentations.

Among 2-dimensional Artin groups, Soergel gives a sufficient criterion in terms of orientations of the edges of the Coxeter presentation graph, see [Soe21, Theorem 3]. As a very restricted example, if the Coxeter presentation graph Γ has no triangles and no squares, then A_Γ admits a systolic restricted presentation.

Theorem 3.13. *Let G denote a group with a systolic restricted presentation. Then $G \times \mathbb{Z}$ is a Garside group.*

Proof. Let us denote by $U = S \cup \{e\}$ in G , and let us consider the induced partial multiplication from G . Since the Cayley graph of G with respect to S is simplicial, we deduce that U embeds in G . We will denote the flag completion of the Cayley graph of G by X .

The only non-trivial assumption to check for this partial multiplication is the positivity: if there exist $s, t \in S$ such that $st = e$ in G , this contradicts the fact that the Cayley graph of G with respect to S is simplicial.

Since U is finite, it is weakly boundedly graded, and it has minimum e . Let us show that (U, \leq_L) is a meet-semilattice by contradiction: let us assume that we have a bowtie $a, b <_L u, v$, with $a, b, u, v \in S$. Then this corresponds to a loop of length 4 in the link of the vertex e in X . By systolicity, we deduce that there exists a diagonal: either a, b are comparable, or u, v are comparable. Hence $a, b <_L u, v$ is not a bowtie. Similarly, (U, \leq_R) is a meet-semilattice.

Let us now consider $a, u, v \in U$ such that $au, av \in U$ and u, v have a join $w \in U$ for \leq_L . Then a^{-1}, u, w, v form a loop of length 4 in the link of the vertex e in X . By systolicity, we deduce that there exists a diagonal: either u, v are comparable, in which case $w \in \{u, v\}$ and $aw \in \{au, av\} \subset U$, or there is an edge between a^{-1} and w , in which case $aw \in U$.

Let us now consider $a, b, u, v \in U$ such that $au, av, bu, bv \in U$. Then a^{-1}, u, b^{-1}, v form a loop of length 4 in the link of the vertex e in X . By systolicity, we deduce that there exists a diagonal. If there is an edge between a^{-1} and b^{-1} , this means that a and b are \leq_R -comparable, so they have a right join. If there is an edge between u and v , this means that u and v are \leq_L -comparable, so they have a left join.

According to Theorem 3.8, we conclude that $G \times \mathbb{Z}$ is a Garside group. \square

Definition 3.14. A finite presentation $\langle S \mid R \rangle$ is a *positive square presentation* if each relator r is of form $ab = cd$ where a, b, c, d are (not necessarily distinct) elements in S .

Some natural examples of groups of square presentation include right-angled Artin groups, mock right-angled Artin groups in the sense of [Sco08] and groups arising from word labeled oriented graphs in the sense of [HR15]. We give a criterion showing some of these groups are Garside groups after taking the product with \mathbb{Z} .

Given a finite square presentation, let X be the associated presentation complex. Each edge loop of X is oriented and labeled by an element in S . Let Θ be the link of the unique vertex of X . A vertex of Θ is of type o or i if it corresponds to outgoing or incoming edge at the base vertex.

Theorem 3.15. *Let G denote a group with a positive square presentation such that the link Θ of its presentation complex satisfies the following conditions:*

- (1) *there does not exist embedded 2-cycles in Θ of type (o, o) (means a 2-cycle with two vertices of type o) or type (i, i) ;*
- (2) *there does not exist embedded 3-cycles in Θ of type (o, o, i) or (i, i, o) ;*
- (3) *there does not exist embedded 4-cycles in Θ of type (o, i, o, i) .*

Then $G \times \mathbb{Z}$ is a Garside group.

Proof. Suppose the collection of relators are of form $\{a_i b_i = a'_i b'_i\}_{i=1}^k$ where $a_i, b_i, a'_i, b'_i \in S$ for each $1 \leq i \leq k$. Let U be the set of equivalence classes of words in $\{e\} \cup S \cup \{a_i b_i\}_{i=1}^k \cup \{a'_i b'_i\}_{i=1}^k$, under the equivalence relation generated by $a_i b_i \sim a'_i b'_i$ for $1 \leq i \leq k$. We endow U with the partial multiplication as in the proof of Theorem 3.10. Now we verify the assumptions of Theorem 3.9: (U, \leq_L) is a meet-semilattice follows from the lack of 2-cycles of type (o, o) in Θ . Indeed, a bowtie in (U, \leq_L) consists of two elements $a, b \in S$, such that they have two different upper bounds in U . Thus in a small neighborhood of the base vertex, we see a small outgoing a segment, and a small outgoing b segment, span the corners of two different squares. This gives rise a 2-cycle of type (o, o) in the link of the base vertex, which contradicts assumption (1) of the theorem. Similarly, we know that (U, \leq_R) is a meet-semilattice follows from the lack of 2-cycles of type (i, i) in Θ . Now take $a, u, v \in S$ with $u \neq v$ such that $au, av \in U$ and u and v have a left join, then this gives a 3-cycle in Θ made of vertices of type incoming a , outgoing u , outgoing v , which is excluded by the lack of 3-cycles of type (o, o, i) . Similarly, the third item of Theorem 3.9 follows from the lack of 3-cycles of (i, i, o) . For the last item of Theorem 3.9, let $a, b, u, v \in S$ with $u \neq v$ and $a \neq b$. If $au, av, bu, bv \in U$, this gives a 4-cycle in Θ with consecutive vertices of type a incoming, u outgoing, b incoming and v outgoing, which is ruled out by the lack of 4-cycles of type (o, i, o, i) . Thus we are done by Theorem 3.9. \square

4. APPLICATION TO ARTIN GROUPS

We will now explain how we can apply Theorem 3.9 for some Artin groups.

Let (W, S) denote a Coxeter group, and let A denote the associated Artin group. Let R denote the set of all reflections (i.e. conjugates of elements in S) of W . A *reflection decomposition* of an element $w \in W$ is a way of written w as a product of elements in S . A reflection decomposition of w is *minimal* if the length of the product is as small as possible. Let $|w|$ denote the reflection length of w , i.e. the length of a minimal reflection decomposition of w . We define two relations \leq_L and \leq_R in W as follows. We say $u \leq_L v$ if u appears as a prefix of a minimal reflection decomposition of v ; and $u \leq_R v$ if u appears as a suffix of a minimal reflection decomposition of v . One readily sees that \leq_R and \leq_L are partial orders on W .

We will define the set U inside W , and we will look for conditions on U and W ensuring that the assumptions from Theorem 3.9 are satisfied.

Given a finite subset $U \subset W$, we consider the following partial multiplication \cdot on U : if $u, v \in U$ are such that their product uv in the Coxeter group W lies in U and furthermore $|uv| = |u| + |v|$, we define $u \cdot v = uv \in U$. Let $R_U = R \cap U$, i.e. the collection of reflections that are in U . Similarly, if $u_1, u_2, \dots, u_n \in U$, we will write $w = u_1 \cdot u_2 \cdot \dots \cdot u_n \in U$ if $u_1 u_2 \cdot \dots \cdot u_n \in U$ and $|w| = |u_1| + |u_2| + \dots + |u_n|$.

Lemma 4.1. *Suppose U satisfies the following conditions:*

- (1) *For every $u \in U$, there exist $r_1, \dots, r_n \in R_U$ such that $u = r_1 \cdot r_2 \cdot \dots \cdot r_n$.*
- (2) *For every $r_1, \dots, r_n \in R_U$ such that $r_1 \cdot r_2 \cdot \dots \cdot r_n \in U$, we have $r_1 \cdot r_2 \cdot \dots \cdot r_{n-1} \in U$ and $r_2 \cdot r_3 \cdot \dots \cdot r_n \in U$.*

Then the set (U, \cdot) satisfies Definition 3.1.

Proof. It suffices to verify that for $u, v, w \in U$ such that $u \cdot v \cdot w \in U$, we have $u \cdot v \in U$ and $v \cdot w \in U$. Indeed, by Assumption 1, let us write reflection factorizations in R_U : $u = r_1 \cdot r_2 \cdot \dots \cdot r_n$, $v = r'_1 \cdot r'_2 \cdot \dots \cdot r'_{n'}$ and $w = r''_1 \cdot r''_2 \cdot \dots \cdot r''_{n''}$. We then have $u \cdot v \cdot w \in U$, so by assumption 2, we have both $u \cdot v \in U$ and $v \cdot w \in U$. \square

Note that the relations \leq_L and \leq_R on U defined right after Definition 3.1 is the same as the restriction of the relations (W, \leq_L) and (W, \leq_R) (as defined in the beginning of this section) to U . So there is no ambiguity in our notation.

Let E be the poset constructed from U as in Section 3.1. We have a criterion for E to be a lattice.

Proposition 4.2. *Suppose $U \subset W$ is finite. Assume that all the following requirements are satisfied.*

- (1) *For every $u \in U$, there exist $r_1, \dots, r_n \in R_U$ such that $u = r_1 \cdot r_2 \cdot \dots \cdot r_n$.*
- (2) *For every $r_1, \dots, r_n \in R_U$ such that $r_1 \cdot r_2 \cdot \dots \cdot r_n \in U$, we have $r_1 \cdot r_2 \cdot \dots \cdot r_{n-1} \in U$ and $r_2 \cdot r_3 \cdot \dots \cdot r_n \in U$.*

- (3) Now we consider the partial orders \leq_L and \leq_R restricted on U . We require that these partial order satisfy that, for every $r_1 \in R_U$ and $r_2 \in R_U$ with a common left upper bound in (U, \leq_L) , they have a left join in (U, \leq_L) ; similarly, if r_1 and r_2 have a common right upper bound in (U, \leq_R) , then they have a right join in (U, \leq_R) .
- (4) For every $a \in U$, for any $u, v \in R_U$ such that u, v have a left join $w \in (U, \leq_L)$ and $a \cdot u, a \cdot v \in U$, we have $a \cdot w \in U$.
- (5) For every $a \in U$, for any $u, v \in R_U$ such that u, v have a right join $w \in (U, \leq_R)$ and $u \cdot a, v \cdot a \in U$, we have $w \cdot a \in U$.
- (6) For every $a, b, u, v \in R_U$ and any $x \in U$ such that $a \cdot x \cdot u, a \cdot x \cdot v, b \cdot x \cdot u, b \cdot x \cdot v \in U$, we have that either u, v have a left join, or a, b have a right join.

Then U satisfies all the conditions in Proposition 3.7. In particular E is a lattice.

Proof. By Lemma 3.2, it remains to verify that U satisfies the assumptions of Proposition 3.7.

Assumption 1 of Proposition 3.7 follows by considering the reflection length on U (the bounded assumption follows from finiteness of U).

We will now prove that (U, \leq_L) is a meet-semilattice. We artificially add a largest element $\hat{1}$ to U , so $P = (U \cup \{\hat{1}\}, \leq_L)$ is a bounded poset of finite length (i.e. there is a finite upper bound on the lengths of its chains). Recall that an element $p_1 \in P$ covers $p_2 \in P$ if $p_1 > p_2$ and there does not exist $p \in P$ with $p_1 > p > p_2$. We claim that if u_1, u_2 are two distinct elements in P that covers v , then u_1 and u_2 has a join. By assumption 2, we can write $u_1 = v \cdot r_1$ and $u_2 = v \cdot r_2$ with $r_1, r_2 \in R_U$. If $\hat{1}$ is the only common left upper bound of u_1 and u_2 , then clearly they have a join. If u_1 and u_2 have a common left upper bound u' other than $\hat{1}$, then we can write $u' = u_i \cdot w_i$ with $w_i \in U$ for $i = 1, 2$. Thus r_1 and r_2 has a common left upper bound, which is $w' = r_1 \cdot w_1 = r_2 \cdot w_2 \in U$. Let r be the left join of r_1 and r_2 . Then $r \leq_L w'$, which implies that $v \cdot r \leq_L v \cdot w' = u'$. Thus $v \cdot r$ is the join for u_1 and u_2 . Now it follows from [BEZ90, Lemma 2.1] that P is a lattice. Thus (U, \leq_L) is a meet-semilattice. Similarly we can prove (U, \leq_R) is a meet-semilattice.

We now prove Assumption 3 of Proposition 3.7, i.e. for all $a, u, v \in U$ such that u, v have a join $w \in U$ for \leq_L , and $a \cdot u, a \cdot v \in U$, then $a \cdot w \in U$.

We will prove it by decreasing induction on $|a|$, and for a fixed value of $|a|$ by increasing induction on $|u| + |v|$. Since U is finite, if $|a|$ is maximal, then $u, v = e$, so the property is true. Now consider $a, u, v \in U$, and assume that the property holds for any larger value of $|a|$. If $u, v \in R_U$, the property holds by assumption. So let us assume that the property holds for smaller values of $|u| + |v|$. Let us assume that $u \notin R_U$, and write $u = u_1 \cdot r$, with $u_1 \in U, r \in R_U$ and $|u_1| = |u| - 1$. According to Properties 1 and 2, we know that $a \cdot u_1 \in U$. Since u_1, v have an upper bound w , and since U is a meet-semilattice, they have a left join $w_1 = u_1 \cdot w'$, with $w' \in U$. Since $|u_1| + |v| < |u| + |v|$, we

deduce by induction that $a \cdot w_1 = a \cdot u_1 \cdot w' \in U$. Now $u_1 \cdot w'$ and $u = u_1 \cdot r$ have a left upper bound $w \in U$, so we deduce by Lemma 4.1, Properties 1 and 2 that w' and r have a left upper bound in U , hence they also have a left join: let us write $w' \vee_L r = w'' \in U$. Since $|au_1| > |a|$, we deduce by induction that $au_1 \cdot w'' \in U$, in particular $|au_1 w''| = |au_1| + |w''| = |a| + |u_1| + |w''|$. Note that $u = u_1 r \leq_L u_1 \cdot w''$ and $v \leq_L w_1 = u_1 w' \leq_L u_1 w''$, we know $w \leq_L u_1 \cdot w''$. On the other hand, $u_1^{-1} w$ is a left common upper bound for r and w' . Hence $w'' \leq_L u_1^{-1} w$ and $u_1 \cdot w'' \leq_L w$. Then $w = u_1 \cdot w''$. In particular $|aw| = |au_1 w''| = |a| + |u_1| + |w''| = |a| + |w|$. As $aw = wu_1 w'' \in U$, we know $a \cdot w \in U$.

Assumption 4 of Proposition 3.7 can be proved in a similar way.

We will now prove Assumption 5 of Proposition 3.7, i.e. for every $a, b, u, v, x \in U$ such that $a \cdot x \cdot u, a \cdot x \cdot v, b \cdot x \cdot u, b \cdot x \cdot v \in U$, we have that either u, v have a left join, or a, b have a right join.

We will prove it by decreasing induction on $|x|$, and for a fixed value of $|x|$ by increasing induction on $|a| + |b| + |u| + |v|$. Since U is finite, if $|x|$ is maximal, then a, b, u, v are all equal to e , so the property is true. Now consider $a, b, u, v, x \in U$, and assume that the property holds for any larger value of $|x|$. If $a, b, u, v \in R_U$, then the property holds by assumption. Without loss of generality, assume that $a \in U \setminus R_U$, and write $a = r \cdot a'$, for some $r \in R_U$ and $a' \in U \setminus \{e\}$ so that $|a'| = |a| - 1$. So $a' \cdot x \cdot u, a' \cdot x \cdot v, b \cdot x \cdot u, b \cdot x \cdot v \in U$. Since $|a'| < |a|$, we deduce by induction that either a', b have a right join or u, v have a left join, and in the latter case we have the desired conclusion. Let us then assume that a', b have a right join $c \in U$. Let us write $c = c' \cdot a'$, where $c' \in U$.

Since $a' \cdot xu, b \cdot xu \in U$ and a', b have a right join $c \in U$, according to Property 5, we deduce that $c \cdot xu \in U$, and similarly $c \cdot xv \in U$. We now consider the four elements

$$c' \cdot a' x \cdot u = cxu, c' \cdot a' x \cdot v = cxv, r \cdot a' x \cdot u, r \cdot a' x \cdot v$$

in U . Since $a' \cdot x \in U$ and $|a' x| > |x|$, we deduce by induction that either c', r have a right join or u, v have a left join, and in the latter case we have the desired conclusion. Let us then assume that c', r have a right join $d \in U$. Since $c' \cdot a' = c, r \cdot a' = a \in U$ and c', r have a right join $d \in U$, according to Property 5, we deduce that $d \cdot a' \in U$. Now remark that $a = ra' \leq_R da'$ and $b \leq_R c = c'a' \leq_R da'$, so a, b have a common right upper bound for \leq_R . Since (U, \leq_R) is a meet-semilattice, we conclude that a, b have a right join (which is the meet of all common right upper bounds of $\{a, b\}$). \square

Let us denote by K_E the interval complex of the poset E as in Definition 2.7 (the interval labeling on E is defined after Lemma 3.3), and let G_E denote the corresponding interval group. We will find a simple criterion ensuring that the interval group G_E is isomorphic to $A \times \mathbb{Z}$, where A is the Artin group associated to W .

Theorem 4.3. *Assume that, for each spherical $T \subset S$, there is a choice of Coxeter element $w_T \in W_T$ such that, for every spherical $T' \subset T$, we know that $w_{T'} \leq_L w_T$ in (W_S, \leq_L) . Assume that*

$$U = \bigcup_{T \subset S \text{ spherical}} [e, w_T].$$

Here $[e, w_T]$ denotes the interval between e and w_T in (W_S, \leq_L) . Then K_E has the homotopy type of the Salvetti complex of the Artin group $A \times \mathbb{Z}$. In particular, the interval group G_E is isomorphic to $A \times \mathbb{Z}$.

Proof. For each spherical $T \subset S$, let us denote $U_T = [e, w_T] \subset U$. Consider the subposet $E_T = (U_T \times \{0\}) \sqcup (\overline{U_T} \times \{1\}) \subset E$, and denote by $K_{E_T} \subset K_E$ the subcomplex corresponding to the quotient of the geometric realization of E_T .

We claim that K_{E_T} has the homotopy type of the Salvetti complex X_T of the Artin group $A_T \times \mathbb{Z}$. Indeed, let us denote by $s_0 \in A_T \times \mathbb{Z}$ a generator of \mathbb{Z} , so that the Artin group $A_T \times \mathbb{Z}$ has standard generating system $T' = T \cup \{s_0\}$. Now $w'_T = w_T s_0$ is a Coxeter element for the spherical Coxeter group $W'_T = W_T \times \mathbb{Z}/2\mathbb{Z}$, and K_{E_T} coincides with the dual Salvetti complex for w'_T as described in [PS21, Section 5]. According to [PS21, Remark 5.4], we deduce that K_{E_T} has the same homotopy type as the standard Salvetti complex X_T for the spherical Artin group $A_T \times \mathbb{Z}$.

By assumption on U , it is clear that K is equal to the union of all K_{E_T} , for $T \subset S$ spherical. Also remark that the standard Salvetti complex X for the Artin group $A \times \mathbb{Z}$ is equal to the union of all X_T , for $T \subset S$ spherical.

According to the proof of [PS21, Theorem 5.5], we deduce that K_E has the homotopy type of X . In particular, the interval group G_E of E , which is the fundamental group of K_E , is naturally isomorphic to the Artin group $A \times \mathbb{Z}$. \square

Corollary 4.4. *Assume that W is a Coxeter group, with a subset $U \subset W$ satisfying the conditions of Proposition 4.2 and of Theorem 4.3. Let A be the Artin group associated with W . Then $A \times \mathbb{Z}$ is Garside, with Garside element $(e, 1)$. Moreover, the $K(\pi, 1)$ conjecture holds for A .*

Proof. This follows from a combination of Proposition 4.2, Proposition 3.7, Theorem 3.8 and Theorem 4.3. Moreover, the standard Salvetti complex X of $A \times \mathbb{Z}$ is aspherical, so in particular the standard Salvetti complex of A itself is aspherical: we deduce that the $K(\pi, 1)$ conjecture holds for A . \square

Remark. There are some Artin groups for which it is not possible to find a subset $U \subset A$ satisfying the conditions of Proposition 4.2 and of Theorem 4.3. Here are two simple examples.

- (1) Consider the right-angled Artin group $A \simeq \mathbb{F}_2 \times \mathbb{F}_2$ with defining graph a square with vertices a, u, b, v in this cyclic order (see Figure 2), and assume that the conditions of Theorem 4.3 hold. Then

we have $au, av, bu, bv \in U$, but neither a, b nor u, v have a join for \leq_L . Then the conditions of Proposition 4.2 do not hold.

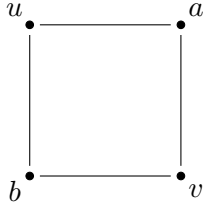


FIGURE 2. The right-angled Artin group A over a square.

- (2) Consider the A with defining graph a complete graph over 7 vertices, whose Dynkin diagram is a line with vertices s_1, s_2, \dots, s_7 , with all edge labels equal to 4 (see Figure 3). Assume that the conditions of Theorem 4.3 hold. Consider the four elements of U : $a = s_1s_2$ or s_2s_1 (depending on the ordering on S), $b = s_2s_3$ or s_3s_2 , $u = s_5s_6$ or s_6s_5 and $v = s_6s_7$ or s_7s_6 . Then $au, av, bu, bv \in U$, but neither a, b nor u, v have a join for \leq_L . Then the conditions of Proposition 4.2 do not hold.

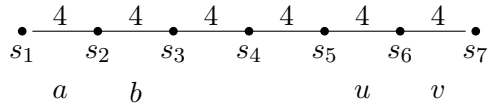


FIGURE 3. The Dynkin diagram of an Artin group for which Corollary 4.4 does not apply.

Corollary 4.5. *Assume that, for each spherical $T \subset S$, there is a choice of Coxeter element $w_T \in W_T$ such that, for every spherical $T' \subset T$, we have $w_{T'} \leq_L w_T$. Assume that*

$$U = \bigcup_{T \subset S \text{ spherical}} [e, w_T].$$

Let \widehat{U} be the lift of U from W_S to A_S via the (compatible) isomorphism between the dual Artin group associated with A_T for each $T \subset S$ spherical and A_T (cf. Theorem 2.17, and more precisely [Bes03, Theorem 2.2.5]).

Assume that U satisfying the conditions of Proposition 4.2. Let X_S be the flag complex of the Cayley graph of A_S with generating set \widehat{U} . Then X_S admits an A_S -equivariant CUB metric such that each simplex of X_S is equipped with a polyhedral norm as in [Hae22].

Proof. By Corollary 4.4, $A_S \times \mathbb{Z}$ is a Garside group with the choice of fundamental interval E as in Section 3. Note that Bestvina complex (cf. Section 2.7) for the Garside group $A_S \times \mathbb{Z}$ is isomorphic to flag complex

of the Cayley graph of A_S with generating set \widehat{U} . Thus we are done by Theorem 2.18. \square

5. CYCLIC-TYPE ARTIN GROUPS

In this section and the next, we will describe families of Artin groups for which we can find a set U satisfying the conditions of Proposition 4.2 and of Theorem 4.3. This section will focus on cyclic type Artin groups. The main goal of this section will be Proposition 5.6.

5.1. Spherical Artin group with linear Dynkin diagram. The goal of this subsection is to prove Corollary 5.5, which is a main ingredient towards Proposition 5.6. We will first establish a special case of Corollary 5.5, which is Lemma 5.4, then use Lemma 5.4 to deduce the more general Corollary 5.5.

Lemma 5.1. *Let W_S be an arbitrary Coxeter group (not necessarily spherical). Let w denote a word in S representing the trivial element of W_S . Then each letter of w appears at least twice.*

Proof. By contradiction, assume that we can write $w = usv$, where $s \in S$ and u, v are words in $S \setminus \{s\}$. Then, in the Coxeter groups W_S , the words s and $u^{-1}v^{-1}$ represent the same element. Since s is in the support of s and not of $u^{-1}v^{-1}$, this is a contradiction. \square

Lemma 5.2. *Let W_S be an arbitrary Coxeter group (not necessarily spherical). Let $s \in S$, and let w denote a reduced word in $S \setminus \{s\}$ representing an element commuting with s . Then every letter of w commutes with s .*

Proof. Since w and s have disjoint supports, the words sw and ws are reduced. We can pass from the reduced word sw to the reduced word ws by applying only standard relations (see for instance [Dav15, Theorem 3.4.2]). This implies that s commutes with every letter of w . \square

Lemma 5.3. *Let W_S be an arbitrary Coxeter group (not necessarily spherical). Let $\{s_1, s_2, \dots, s_n\} \subset S$ such that, for each $1 \leq i \leq n-1$, there exists $i < j \leq n$ such that s_i and s_j do not commute. Then the word*

$$s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$$

is reduced.

Proof. We induct on n . Then case $n = 1$ is trivial. For the general case, by contradiction, assume that the word $w = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$ is not reduced. According to the deletion condition (see for instance [Hum90, Corollary 5.8]), w can also be represented by a word w' obtained from w by deleting two letters.

Since w represents a reflection of W_S , w' also represents a reflection. According to the strong exchange condition (see [Hum90, Theorem 5.8]), if we remove one letter from w' we may obtain the trivial element. According to Lemma 5.1, there are only two possibilities:

- (1) there exists $1 \leq i \leq n-1$ such that w' is obtained from w by removing the two occurrences of s_i ;
- (2) there exists $1 \leq i \leq n-1$ such that w' is obtained from w by removing s_n and one of the s_i .

First we rule out possibility (1). In this case

$$w' = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_n \dots s_{i+1} s_{i-1} \dots s_1.$$

By conjugating by $s_1 s_2 \dots s_{i-1}$, we deduce that the words

$$s_i \dots s_{n-1} s_n s_{n-1} \dots s_i = s_{i+1} \dots s_{n-1} s_n s_{n-1} \dots s_{i+1}.$$

In particular, the element $u = s_{i+1} \dots s_{n-1} s_n s_{n-1} \dots s_{i+1}$ commutes with s_i . As $u = s_{i+1} \dots s_{n-1} s_n s_{n-1} \dots s_{i+1}$ is reduced by induction assumption, according to Lemma 5.2, we deduce that s_i commutes with every letter s_{i+1}, \dots, s_n , which contradicts the assumption.

Now we rule out possibility (2). In this case, $w' = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1$. By conjugation by $s_1 s_2 \dots s_{i-1}$, we deduce that $s_i = s_i \dots s_{n-1} s_n s_{n-1} \dots s_i$. Thus $s_{i+1} \dots s_n s_{n-1} \dots s_i$ represents the trivial element, contradicting Lemma 5.1. \square

Lemma 5.4. *Suppose S is spherical with linear Dynkin diagram. We label elements of S as $\{s_1, \dots, s_n\}$ using one of the two linear orders induced by the Dynkin diagram. Let U be the dual Garside interval with respect to the dual Garside element $\delta = s_1 s_2 \dots s_n$ and let R_U be the set of reflections in U . Assume that $u, v \in R_U$ are such that $u \cdot v \in U$. Let $I = \text{Supp}(u)$ and $J = \text{Supp}(v)$. Define $\min(I)$ be the smallest element in I with respect to the chosen linear order.*

If $\min(I) - 1 \in J$, then $I \subset J$.

Here we slightly abuse notation, and will use $s_i - 1$ to denote s_{i-1} , and $s_i + 1$ to denote s_{i+1} .

Proof. Suppose we fix an order of $\{s_1, s_2, \dots, s_n\}$ from left to right of the Dynkin diagram. Then we have the following cases to consider.

- (1) Type I_m with $m \geq 3$, and $n = 2$.
- (2) Type A_n .
- (3) Type B_n , with Dynkin diagram $3 - 3 - \dots - 3 - 4$.
- (4) Type B_n , with Dynkin diagram $4 - 3 - \dots - 3 - 3$.
- (5) Type H_3 , with Dynkin diagram $3 - 5$, and $n = 3$.
- (6) Type H_3 , with Dynkin diagram $5 - 3$, and $n = 3$.
- (7) Type H_4 , with Dynkin diagram $3 - 3 - 5$, and $n = 4$.
- (8) Type H_4 , with Dynkin diagram $5 - 3 - 3$, and $n = 4$.
- (9) Type F_4 , with Dynkin diagram $3 - 4 - 3$, and $n = 4$.

A priori, two different linear orders on the Dynkin diagrams gives two different cases. However, we claim that if the lemma holds for a given linear order on the Dynkin diagram, then it also holds with respect to the reserved linear order. Indeed, suppose $\{s_1, \dots, s_n\}$ are named according a given linear

order on the Dynkin diagram. Let u, v, I, J and δ be as in the lemma. For each element $g \in W_S$ with its reduced expression $g = s_{i_1} s_{i_2} \cdots s_{i_k}$, we define $\bar{g} = s_{i_k} \cdots s_{i_2} s_{i_1}$. Note that $\bar{g} \in W_S$ does not depend on the choice of reduced expression of g . Then $uv \leq_L \delta$ if and only if $\bar{v}\bar{u} \leq_R \bar{\delta}$ if and only if $\bar{v}\bar{u} \leq_L \bar{\delta}$ (by Theorem 2.17 and Definition 2.5 (4)). As $\min(\text{Supp}(u)) - 1 \in \text{Supp}(v)$ with respect to the given order, and I and J are irreducible (Lemma 2.1), we deduce that either $\min(\text{Supp}(\bar{v})) - 1 \in \text{Supp}(\bar{u})$ with respect to the reserved order, or $I \subset J$. In the latter case we are already done. In the former case, if we know the lemma holds with respect to the original order, then we deduce that $\text{Supp}(\bar{v}) \subset \text{Supp}(\bar{u})$, hence $J \subset I$, which contradicts that $\min(I) - 1 \in J$ with respect to the original order, so the former case cannot happen. This proves the claim.

Assume first that $|S| = 2$, i.e. W_S is of type I_m with $m \geq 3$. Assume that $I, J \neq S = \{s_1, s_2\}$, so that $u, v \in \{s_1, s_2\}$. We only have to prove that $s_2 s_1 \not\leq_L \delta$.

By contradiction, assume that $s_2 s_1 \leq_L \delta$. Since $\delta = s_1 s_2$ has reflection length 2, we deduce that $s_2 s_1 = \delta$, so $s_1 s_2 = s_2 s_1$. This contradicts $m \geq 3$.

Suppose $|S| > 2$. We argue by contradiction and assume I is not a subset of J . As J contains $\min(I) - 1$, and $S = I \cup J$, by Lemma 2.1, we assume that $I = \{s_i, \dots, s_n\}$ and $J = \{s_1, \dots, s_j\}$. Note that $j \leq n - 1$, otherwise we already have $I \subset J$.

Assume that W_S is not of type F_4 with $i = 2$ and $j = 3$. Then either J is of type A_j or I is of type A_{n-i+1} . By the order-reversing argument in the above claim, we can assume without loss of generality that J is of type A_j . Then we know that $v = s_1 s_2 \cdots s_{j-1} s_j s_{j-1} \cdots s_2 s_1$. Indeed, this is the unique reflection in A_j with support being J , and we can see v must be of this form by treating A_j as the permutation group of $j + 1$ elements $\{\alpha_1, \dots, \alpha_{j+1}\}$, with s_i being the transposition permuting α_i and α_{i+1} . Then v is the transposition permuting α_1 and α_{j+1} . We also have

$$\begin{aligned} \delta &= s_1 \cdots s_j s_{j+1} \cdots s_n = v s_1 \cdots s_{j-1} s_{j+1} \cdots s_n \\ &= (v s_1 v^{-1}) \cdots (v s_{j-1} v^{-1}) (v s_{j+1} \cdots s_n v^{-1}) v. \end{aligned}$$

By the Garside property, there exists $w \in [1, \delta]$ with reflection length $n - 1$ such that $w \cdot v = \delta$. Thus

$$(v s_1 v^{-1}) \cdots (v s_{j-1} v^{-1}) (v s_{j+1} v^{-1}) \cdots (v s_n v^{-1})$$

is a minimal reflection decomposition of w . By [McC15, Lemma 3.7], $v s_k v^{-1} \leq_L w$ for $k \neq j$. Since $u \cdot v \leq_L \delta$, we know that $u \cdot v \leq_R \delta$ by Theorem 2.17. Thus $u \leq_R w$.

According to [Bes03, Lemma 1.4.3], the element w is a Garside element for the parabolic subgroup P_w of P generated by the reflections which are \leq_L -smaller than w . Since w has reflection length $n - 1$, this subgroup P_w

equals

$$P_w = \langle vs_1v^{-1}, \dots, vs_{j-1}v^{-1}, vs_{j+1}v^{-1}, \dots, vs_nv^{-1} \rangle = v(\langle s_1, \dots, s_{j-1} \rangle \times \langle s_{j+1}, \dots, s_n \rangle)v^{-1}.$$

As $u \leq_R w$, we know that $u \in P_w$ by [BDSW14, Theorem 1.4]. Hence $v^{-1}uv \in W_{S \setminus \{s_j\}}$. By Lemma 2.1, $\text{Supp}(v^{-1}uv) \subset \{s_1, \dots, s_{j-1}\}$ or $\text{Supp}(v^{-1}uv) \subset \{s_{j+1}, \dots, s_n\}$. Thus $u \in v(\langle s_1, \dots, s_{j-1} \rangle)v^{-1}$ or $u \in v(\langle s_{j+1}, \dots, s_n \rangle)v^{-1}$. Also since

$$v(\langle s_1, \dots, s_{j-1} \rangle)v^{-1} \subset W_{\{s_1, \dots, s_j\}},$$

we rule out that $u \in v(\langle s_1, \dots, s_{j-1} \rangle)v^{-1}$, hence $u \in v(\langle s_{j+1}, \dots, s_n \rangle)v^{-1}$. In particular,

$$u \in v(\langle s_{j+1}, \dots, s_n \rangle)v^{-1} \cap W_I = \langle vs_{j+1}v^{-1}, s_{j+2}, \dots, s_n \rangle \cap \langle s_i, \dots, s_n \rangle.$$

As $s_{\min(I)-1} \in J$, we know $i \leq j+1$. Let $P = \langle vs_{j+1}v^{-1}, s_{j+2}, \dots, s_n \rangle \cap \langle s_i, \dots, s_n \rangle$. By [Qi07], P is a parabolic subgroup of W_S . Note that $P \supset W_{\{s_{j+2}, \dots, s_n\}}$. On the other hand, $vs_{j+1}v^{-1} \notin W_{\{s_i, \dots, s_n\}}$ as $vs_{j+1}v^{-1} = s_1s_2 \cdots s_js_{j+1}s_j \cdots s_1$ and the word $s_1s_2 \cdots s_js_{j+1}s_j \cdots s_1$ is reduced by Lemma 5.3. Hence $u \in P = W_{\{s_{j+2}, \dots, s_n\}}$. As $j+2 \geq i+1$, this contradicts that $\text{Supp}(u) = \{s_i, \dots, s_n\}$. We deduce this contradiction as we were assuming I is not a subset of J in the beginning. This proves that $I \subset J$.

The remaining case is in type F_4 with $i = 2$ and $j = 3$. Then v is a reflection inside the Coxeter group of B_3 generated by s_1, s_2 and s_3 , which has Dynkin diagram 3–4. Consider the canonical representation of Coxeter group of type B_3 acting on \mathbb{R}^3 . Then s_1 acts by the orthogonal reflection along $x_1 = x_2$, s_2 acts by the orthogonal reflection along $x_2 = x_3$, and s_3 acts by the orthogonal reflection along $x_3 = 0$. Note that there are nine reflection in W_{s_1, s_2, s_3} , whose reflection hyperplanes are $x_i = \pm x_j$ for $1 \leq i \neq j \leq 3$ and $x_i = 0$ for $1 \leq i \leq 3$. Note that reflections along $x_i = x_j$ for $1 \leq i \neq j \leq 3$ are supported on W_{s_1, s_2} ; reflections along $x_2 = \pm x_3$ or $x_i = 0$ for $i = 2, 3$ are supported on W_{s_2, s_3} . This gives 6 reflections in total. The remaining three reflections in W_{s_1, s_2, s_3} give all the possibilities of v . More precisely, reflection along $x_1 = 0$ gives $v = s_1s_2s_3s_2s_1$, reflection along $x_1 + x_3 = 0$ gives $v = s_1s_3s_2s_3s_1$, and reflection along $x_1 + x_2 = 0$ gives $v = s_2s_3s_2s_1s_2s_3s_2$.

The case $v = s_1s_2s_3s_2s_1$ is identical to before. Now we assume $v = s_1s_3s_2s_3s_1 = s_3s_1s_2s_1s_3 = s_3s_2s_1s_2s_3$. Then

$$\begin{aligned} \delta &= s_1s_2s_3s_4 = s_1s_2s_3s_4s_3s_2s_1s_2s_3v \\ &= (s_1s_2s_3s_4s_3s_2s_1)(s_2)(s_3)v. \end{aligned}$$

By the same argument as before, we know $u \in \langle s_1s_2s_3s_4s_3s_2s_1, s_2, s_3 \rangle$. Let

$$\begin{aligned} P &= \langle s_1s_2s_3s_4s_3s_2s_1, s_2, s_3 \rangle = s_1s_2s_3 \langle s_4, s_1, s_2s_3s_2 \rangle s_3s_2s_1 = s_1s_2s_3s_2 \langle s_4, s_2s_1s_2, s_3 \rangle s_2s_3s_2s_1 \\ &= s_1s_2s_3s_2 \langle s_4, s_1s_2s_1, s_3 \rangle s_2s_3s_2s_1 = s_1s_2s_3s_2s_1 \langle s_4, s_2, s_3 \rangle s_1s_2s_3s_2s_1. \end{aligned}$$

In particular, P is a parabolic subgroup. Note that $u \in P \cap W_{s_2, s_3, s_4}$.

By [Qi07], $P \cap W_{s_2, s_3, s_4}$ is a parabolic subgroup of W_S . Note that $\langle s_2, s_3 \rangle \subset P \cap W_{s_2, s_3, s_4}$. Moreover, $s_1 s_2 s_3 s_4 s_3 s_2 s_1 \in P \setminus W_{s_2, s_3, s_4}$ as $s_1 s_2 s_3 s_4 s_3 s_2 s_1$ is a reduced word by Lemma 5.3. Thus $P \cap W_{s_2, s_3, s_4} = \langle s_2, s_3 \rangle$. Thus $s_4 \notin \text{Supp}(u)$, which is a contradiction.

It remains to look at the case $v = s_2 s_3 s_2 s_1 s_2 s_3 s_2$. Then

$$\begin{aligned} \delta &= s_1 s_2 s_3 s_4 = s_1 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_2 v \\ &= (s_1 s_2 s_3 s_4 s_3 s_2 s_1)(s_1 s_2 s_3 s_2 s_3 s_2 s_1)(s_2 s_3 s_2)v. \end{aligned}$$

Note that $s_2 s_3 s_2 = s_1 s_2 s_3 (s_3 s_2 s_1 s_2 s_3 s_2 s_1 s_2 s_3) s_3 s_2 s_1 = s_1 s_2 s_3 (s_1 s_2 s_3 s_2 s_1) s_3 s_2 s_1$. Thus by repeating the previous discussion, we know

$$u \in s_1 s_2 s_3 \langle s_4, s_2, s_1 s_2 s_3 s_2 s_1 \rangle s_3 s_2 s_1 = s_1 s_2 s_3 s_1 s_2 \langle s_4, s_1, s_3 \rangle s_2 s_1 s_3 s_2 s_1.$$

As u belongs to a parabolic subgroup which splits as a product and $s_4 \in \text{Supp}(u)$, we argue as before to deduce that

$$u \in s_1 s_2 s_3 s_1 s_2 \langle s_4, s_3 \rangle s_2 s_1 s_3 s_2 s_1 := P.$$

Thus $u \in P \cap W_{s_2, s_3, s_4}$. Note that

$$\begin{aligned} s_1 s_2 s_3 s_1 s_2 (s_3) s_2 s_1 s_3 s_2 s_1 &= s_1 s_2 s_1 (s_3 s_2 s_3 s_2 s_3) s_1 s_2 s_1 \\ &= s_2 s_1 s_2 (s_2 s_3 s_2) s_2 s_1 s_2 = s_2 s_1 s_3 s_1 s_2 = s_2 s_3 s_2 \in P \cap W_{s_2, s_3, s_4}. \end{aligned}$$

Thus $P \cap W_{s_2, s_3, s_4}$ is a parabolic subgroup of rank ≥ 1 . On the other hand,

$$s_1 s_2 s_3 s_1 s_2 (s_4) s_2 s_1 s_3 s_2 s_1 = s_1 s_2 s_3 s_4 s_3 s_2 s_1.$$

As $s_1 s_2 s_3 s_4 s_3 s_2 s_1$ is a reduced word in W_S by Lemma 5.3, it can not be contained in W_{s_2, s_3, s_4} . Thus $P \cap W_{s_2, s_3, s_4} \subsetneq P$. It follows that $P \cap W_{s_2, s_3, s_4} = \langle s_2 s_3 s_2 \rangle$, hence $s_4 \notin \text{Supp}(u)$, which is a contradiction. \square

Corollary 5.5. *Suppose S is spherical with linear Dynkin diagram. We label elements of S as $\{s_1, \dots, s_n\}$ using one of the two linear orders induced by the Dynkin diagram. Let U be the dual Garside interval with respect to the dual Garside element $\delta = s_1 s_2 \cdots s_n$ and let R_U be the set of reflections in U . Assume that $u, v \in U$ are such that $\text{Supp}(u)$ and $\text{Supp}(v)$ are irreducible, and $u \cdot v \in U$. Let $I = \text{Supp}(u)$ and $J = \text{Supp}(v)$.*

If $\min(I) - 1 \in J$, then $I \subset J$.

Recall that we slightly abuse notation, use $s_i - 1$ to denote s_{i-1} , and $s_i + 1$ to denote s_{i+1} .

Proof. Let $v = r_1 r_2 \cdots r_k$ with $r_i \in R_U$ be a minimal reflection decomposition of v . As $\min(I) - 1 \in J$, there exists $1 \leq i \leq k$ such that $\min(I) - 1 \in \text{Supp}(r_i)$. Suppose $I \subset J$ is not true. Then $\max(\text{Supp}(r_i)) < \max(I)$, where $\max(I)$ is defined to be the biggest element in I . Hence $\max(\text{Supp}(r_i)) + 1 \in I$. We write a minimal reflection decomposition of u as $u = t_1 t_2 \cdots t_m$. Then there exists $1 \leq j \leq m$ such that $\max(\text{Supp}(r_i)) + 1 \in \text{Supp}(t_j)$. As $u \cdot v \in \delta$, we know $t_1 \cdots t_m r_1 \cdots r_k$ is a minimal reflection decomposition of uv . In particular $t_1 \cdots t_m r_1 \cdots r_k \leq_L \delta$. By [McC15, Lemma 3.7], $t_j \cdot r_i \leq_L \delta$. By construction we have $\min(\text{Supp}(t_j)) - 1 \in \text{Supp}(r_i)$, and $\text{Supp}(r_i)$ does

not contain $\text{Supp}(t_j)$ (as $\max(\text{Supp}(r_i)) + 1 \in \text{Supp}(t_j)$), which contradicts Lemma 5.4. Thus the corollary is proved. \square

5.2. Cyclic-type Artin groups. Let W_S be a cyclic type Coxeter group (cf. Table 1). We take a cyclic order on S coming from its Dynkin diagram, and denote elements in S as elements in $\mathbb{Z}/n\mathbb{Z}$. For each $i \in \mathbb{Z}/n\mathbb{Z}$, consider the dual Garside interval U_i in $P_{S \setminus i}$ with respect to the dual Garside element $\delta_i = s_{i+1}s_{i+2} \dots s_n s_1 \dots s_{i-1}$. Let $U = \cup_{i \in \mathbb{Z}/n\mathbb{Z}} U_i$. It is clear that this set U satisfies the assumptions of Theorem 4.3.

For each $i \in \mathbb{Z}/n\mathbb{Z}$, consider the set $R_{U_i} \subset U_i$ of reflections in the spherical parabolic subgroup $P_{S \setminus i}$, and let $R_U = \cup_{i \in \mathbb{Z}/n\mathbb{Z}} R_{U_i} \subset U$.

Proposition 5.6. *The sets R and U satisfy all assumptions from Proposition 4.2. In particular, if A_Γ is of cyclic type, then $A_\Gamma \times \mathbb{Z}$ is a Garside group.*

Proof. We verify each assumption of Proposition 4.2 as follows.

- (1) Any $u \in U_i$ can be written as a product of elements in R_{U_i} which is a minimal length reflection factorization.
- (2) Let $r_1, \dots, r_m \in U_R$ such that $u = r_1 \cdot r_2 \cdot \dots \cdot r_m \in U$. Let $i \in \mathbb{Z}/n\mathbb{Z}$ such that $u \in U_i$. Then $r_1, \dots, r_m \in P_{S \setminus i}$ by [BDSW14, Theorem 1.4], and $r_1 \cdot r_2 \cdot \dots \cdot r_m \leq_L \delta_i$. So both $r_1 \cdot r_2 \cdot \dots \cdot r_{m-1}$ and $r_2 \cdot r_3 \cdot \dots \cdot r_m$ belong to $P_{S \setminus i}$, and also they are both prefixes of δ_i . Hence $r_1 \cdot r_2 \cdot \dots \cdot r_{m-1}$ and $r_2 \cdot r_3 \cdot \dots \cdot r_m$ belong to U_i .
- (3) Let $r, r' \in U_R$ which admit a common left upper bound $u \in U$. Let $i \in \mathbb{Z}/n\mathbb{Z}$ such that $u \in U_i$. By [BDSW14, Theorem 1.4], $r, r' \in U_i$ and u is a common left upper bound for r, r' in (U_i, \leq_L) . In particular, $r, r' \leq_L \delta_i$, so r and r' admit a unique left join u_i in (U_i, \leq_L) . Now we show u_i is also the join of r and r' in (U, \leq_L) . Indeed, take an arbitrary left upper bound u' of r, r' in U . Suppose $u' \in U_j$. Then as before we know $r, r' \in U_j$ and u' is a common left upper bound of r, r' in (U_j, \leq_L) by Theorem 2.17. Let u_j (resp. u_{ij}) be the left join of r, r' in (U_j, \leq_L) (resp. $(U_i \cap U_j, \leq_L)$). One readily verifies that $u_{ij} = u_i$, $u_{ij} = u_j$ and $u_j \leq_L u'$. Thus $u_i \leq_L u'$, implying u_i is the left join of r and r' in U . The case of common right upper bound is similar.
- (4) Let a, u, v, w be as in Proposition 4.2 (4). Let $I, J, K \subset \mathbb{Z}/n\mathbb{Z}$ denote $\text{Supp}(a)$, $\text{Supp}(u)$ and $\text{Supp}(v)$ respectively. We first prove $I \cup J \cup K \subsetneq \mathbb{Z}/n\mathbb{Z}$. Suppose by contradiction that $I \cup J \cup K = \mathbb{Z}/n\mathbb{Z}$. Let $\{I_i\}_{i=1}^k$ be the irreducible components of I . By Lemma 2.1, $a = a_1 \cdot a_2 \cdot \dots \cdot a_k$ such that $\text{Supp}(a_i) = I_i$. By [McC15, Lemma 3.7], $a_i \cdot u \leq_L a \cdot u$, thus $a_i \cdot u \in U$. Similarly, $a_i \cdot v \in U$ for $1 \leq i \leq k$. As $I \cup J \cup K = \mathbb{Z}/n\mathbb{Z}$, for each a_i , we know either $\min(I_i) - 1 \in J$ or $\min(I_i) - 1 \in K$ (here I_i inherits a linear order from the cyclic order on $\mathbb{Z}/n\mathbb{Z}$, hence it make senses to talk about $\min(I_i)$ and $\min(I_i) - 1$). We first look at the case that $\min(I_j) - 1 \in J$. Consider $I_i \cup J$, which

is irreducible. As $a_i \cdot u \in U$, there exists i_0 such that $a_i \cdot u \in U_{i_0}$. Then $I_i \cup J \subset S \setminus \{i_0\}$ by [BDSW14, Theorem 1.4]. We endow $S \setminus \{i_0\}$ with the linear order induced from the cyclic order on S , then Corollary 5.5 implies $I_i \subset J$. This shows that $I_i \subset J \cup K$ for each i . Thus $I \cup J \cup K = J \cup K$. However, as u and v has a left join w in U , there exists i'_0 such that $w \in U_{i'_0}$, hence $u, v \in U_{i'_0}$ by [BDSW14, Theorem 1.4]. Thus $J \cup K \subsetneq S$, which is a contradiction. The case that $\min(I_j) - 1 \in K$ can be treated similarly.

Let $i \in S$ such that $I \cup J \cup K \subset S \setminus \{i\}$. As w is a left join of u and v in U , by the discussion in item 3, $\text{Supp}(w) \in J \cup K$ and w is the left join of u and v in (U_i, \leq_L) . Thus $\text{Supp}(a) \cup \text{Supp}(w) \subset U_i$. Clearly $aw \in U_i \subset U$. It remains to show $|aw| = |a| + |w|$. As (U_i, \leq_L) is a lattice, $a \cdot u$ and $a \cdot v$ has a left join in U_i , denoted by a' . As $a \leq_L a'$, we know $a' = a \cdot w'$ for some $w' \in U_i$. By cancellation property in U_i , we know $u \leq_L w'$ and $v \leq_L w'$. Thus $w \leq_L w'$. Then $w' = w \cdot w_0$ for $w_0 \in U_i$. Then $|a'| = |a| + |w'| = |a| + |w| + |w_0|$. As $a' = aww_0$, $|a'| \leq |aw| + |w_0|$. Thus $|aw| = |a| + |w|$.

- (5) This is similar to the previous item.
- (6) Let a, b, u, v, x be as in Proposition 4.2 (6). Let $I_a = \text{Supp}(a)$. Similarly we define I_b, I_u and I_v . We claim either $I_a \cup I_b \subsetneq S$ or $I_u \cup I_v \subsetneq S$. Assume by contradiction that $I_a \cup I_b = S$ and $I_u \cup I_v = S$. As $a \cdot x \cdot u \in U$, there exists $i \in S$ such that $a \cdot x \cdot u \leq_L \delta_i$. By [McC15, Lemma 3.7], $a \cdot u \leq_L a \cdot x \cdot u \leq_L \delta_i$, thus $a \cdot u \in U_i \subset U$. Similarly, $a \cdot v, b \cdot u, b \cdot v \in U$. As $I_a \cup I_b = S$, either $\min(I_u) - 1 \in I_a$ or $\min(I_u) - 1 \in I_b$. If $\min(I_u) - 1 \in I_a$, as $a \cdot u \in U_i$, we know from Lemma 5.4 that $I_u \subset I_a$. As $I_u \cup I_v = S$, $I_a \cup I_v = S$, which contradicts $a \cdot v \in U$. The case of $\min(I_u) - 1 \in I_b$ is similar.

If $I_a \cup I_b \subsetneq S$, then there is $i \in S$ such that $a, b \in U_i$. As (U_i, \leq_R) is lattice by Theorem 2.17, a and b have a right join in (U_i, \leq_R) , which is also a right join of a and b in U by the argument in item 3. The case $I_u \cup I_v \subsetneq S$ is similar.

□

Now the following is a consequence of Proposition 5.6, Corollary 4.4 and the main result of [JS23].

Corollary 5.7. *Assume that A_Γ is of cyclic type. Then A_Γ satisfies the $K(\pi, 1)$ conjecture and has trivial center.*

6. COMBINATION OF CYCLIC TYPE AND SPHERICAL TYPE ARTIN GROUPS

In this section, we will describe another family of Artin groups for which we can find a set U satisfying the conditions of Proposition 4.2 and of Theorem 4.3. The main goal of this section is to prove Theorem 6.3.

An edge of a Coxeter presentation graph is *large* if its label is ≥ 3 . For each induced subgraph $\Lambda \subset \Gamma$, we define Λ^\perp to be the induced subgraph spanned by vertices of $\Gamma \setminus \Lambda$ which commute with every vertex in Λ .

We will be considering orientation of each large edges of Γ . For the moment suppose Γ is spherical and we orient each large edge. We say a Coxeter element of Γ is compatible with such orientation if whenever there is an oriented edge from $s_1 \in S$ to $s_2 \in S$, then s_1 appears before s_2 in the expression of the Coxeter element.

Lemma 6.1. *Given a spherical Coxeter presentation graph Γ with an orientation on each of its large edges, any two Coxeter elements that are compatible with the orientation are equal.*

Proof. We will prove it by induction on the rank of Γ .

Let us assume that $w = s_1 \dots s_n$ and $w' = s'_1 \dots s'_n$ are two Coxeter elements that are compatible with the orientation. Let $i \in \{1, \dots, n\}$ such that $s'_i = s_1$. Assume that, among all possible reduced expressions of w' that are compatible with the orientation, the position i of s_1 is minimal. We will prove that $i = 1$.

Assume by contradiction that $i > 1$. Since i is minimal, we deduce that the edge between s'_{i-1} and s'_i has label ≥ 3 . As w' is compatible with the orientation, we deduce that the edge between s'_{i-1} and s'_i is oriented from s'_{i-1} to s'_i . As w is also compatible with the orientation and $s'_i = s_1$, we deduce that this edge is oriented from s'_i to s'_{i-1} . This is a contradiction.

So $s'_1 = s_1$. By induction, we deduce that $s_2 \dots s_n = s'_2 \dots s'_n$, hence $w = w'$. \square

Lemma 6.2. *Given a spherical Coxeter presentation graph Γ with orientation on its large edges. Let δ be the Coxeter element which is compatible with the orientation. Let $[1, \delta]$ be the collection of elements in W_Γ that are prefixes of δ with respect to the reflection length on W_Γ .*

Give two reflections $r_1, r_2 \in [1, \delta]$ such that $r_1 r_2 \in [1, \delta]$. Suppose there exist s_1 and s_2 such that $s_1 \in \text{Supp}(r_1) \setminus \text{Supp}(r_2)$ and $s_2 \in \text{Supp}(r_2) \setminus \text{Supp}(r_1)$. Then either s_1 and s_2 commute, or there is an oriented edge from s_1 to s_2 .

Proof. We argue by contradiction and assume there is an oriented edge from s_2 to s_1 . Let Λ be the Dynkin diagram, which is a tree. Then we cut Λ along the midpoint of the edge $\overline{s_2 s_1}$ into two subtrees with $s_i \in \Lambda_i$ for $i = 1, 2$. By Lemma 2.1, $\text{Supp}(r_i) \subset \Lambda_i$ for $i = 1, 2$. The edge orientation on Λ induces edge orientation on Λ_i for $i = 1, 2$. Let δ_i be the Coxeter element in A_{Λ_i} which is compatible with the edge orientation on Λ_i . As vertices in $\Lambda_1 \setminus \{s_1\}$ commute with vertices in $\Lambda_2 \setminus \{s_2\}$, Lemma 6.1 implies that $\delta = \delta_2 \delta_1$. As r_i is a reflection in A_{Λ_i} , we know $r_1 \leq_L \delta_1$ and $r_2 \leq_R \delta_2$ by [Bes03, Lemma 1.3.3]. In particular δ_1 has a minimal reflection decomposition of form $\delta_1 = r_1 \cdot r'_1 \cdot r'_2 \cdots r'_k$, and δ_2 has a minimal reflection decomposition of form $\delta_2 = r''_1 \cdots r''_m \cdot r_2$. Thus

$$r''_1 \cdots r''_m \cdot r_2 \cdot r_1 \cdot r'_1 \cdot r'_2 \cdots r'_k = \delta.$$

By [McC15, Lemma 3.7], $r_2 \cdot r_1 \leq_L \delta$. Thus $r_2 \cdot r_1, r_1 \cdot r_2 \in [1, \delta]$, and these two elements are both common upper bound for r_1 and r_2 with respect to

\leq_L . Then $r_2r_1 = r_1r_2$ as $([1, \delta], \leq_L)$ is a lattice. We write r_i as an reducible word w_i in W_S . Then w_i only uses from letters from Λ_i , and w_1w_2, w_2w_1 are reduced words. Then by Tits's solution to the word problem of Coxeter group, we know that it is possible to apply the relators finitely many times to transform w_1w_2 into w_2w_1 . However, as s_2 is on the right side of s_1 in w_1w_2 and $m(s_1, s_2) \geq 3$, and the property of having at least one s_2 on the right side of s_1 is preserved under applying the relations, this leads to a contradiction. \square

Given a 4-cycle $\omega \subset \Gamma$ with consecutive vertices $\{x_i\}_{i=1}^4$, a pair of antipodal vertices in ω means either the pair $\{x_1, x_3\}$, or the pair $\{x_2, x_4\}$. A 4-cycle in Γ has a *diagonal* means it has a pair of antipodal vertices of ω which are connected by an edge in Γ . We say the Coxeter presentation graph Γ is a *join* of two Coxeter presentation subgraphs Γ_1 and Γ_2 if Γ is a join of Γ_1 and Γ_2 as graphs (i.e. Γ is obtained from a disjoint union of Γ_1 and Γ_2 by adding a single edge between each vertex of Γ_1 and each vertex of Γ_2), and each vertex of Γ_1 commute with every vertex of Γ_2 . We say a Coxeter presentation graph Γ is spherical (resp. of cyclic type) if A_Γ is a spherical Artin group (resp. an Artin of cyclic type).

Theorem 6.3. *Let Γ be a Coxeter presentation graph such that*

- *each complete subgraph of Γ is a join of a cyclic type graph and a spherical type graph (we allow one of the join factors to be empty);*
- *for any cyclic type induced subgraph $\Lambda \subset \Gamma$, Λ^\perp is spherical.*

We assume in addition that there exists an orientation of all large edges of Γ such that

- *the orientation restricted to each cyclic type subgraph of Γ gives a consistent orientation on the associated circle;*
- *if ω is a 4-cycle in Γ with a pair of antipodal points x_1 and x_2 such that each edge of ω containing $x_i \in \{x_1, x_2\}$ is either not large or oriented towards x_i , then the cycle has a diagonal.*

Then $A_\Gamma \times \mathbb{Z}$ is a Garside group.

Proof. Let S be the vertex set of Γ . Let $I \subset S$ be a spherical subset. We define δ_I be a product of all elements in I in an order which is compatible with the orientation of Γ in sense explained before Lemma 6.1. Then δ_I is well-defined by Lemma 6.1.

We also view δ_I as an element in the Coxeter group W_Γ . Let \mathcal{S} be the collection of all spherical subset of S . Define $U = \cup_{I \in \mathcal{S}} [1, \delta_I]$, where $[1, \delta_I]$ denotes the interval in (W_Γ, \leq_L) with \leq_L being the partial order induced considering prefix with respect to minimal reflection decompositions. It is clear that U satisfies the assumptions of Theorem 4.3.

We now verify that U satisfies all the requirements in Proposition 4.2.

By [BDSW14, Theorem 1.4], any minimal length reflection decomposition of an element $a \in [1, \delta_I]$ only involves reflections in W_I . On the other hand,

by [Bes03, Lemma 1.3.3], for any reflection $r \in W_I$, there exists a minimal length reflection decomposition of δ_I starting with r , thus $r \in U$. Now Assumptions 1 and 2 of Proposition 4.2 follow.

Recall that R_U denotes the collection of reflections in U . For Assumption 3 of Proposition 4.2, if $r_1, r_2 \in R_U$ has a left common upper bound $a \in U$, then there exists a spherical subset $I \in S$ such that $a \in [1, \delta_I]$. By [BDSW14, Theorem 1.4], $r_1, r_2 \in W_I$. As in the previous paragraph, we know $r_1, r_2 \in [1, \delta_I]$. As $([1, \delta_I], \leq_L)$ is a lattice (Theorem 2.17 and [BDSW14, Theorem 1.4]), we know r_1 and r_2 has left join a in $([1, \delta_I], \leq_L)$. By the same argument as in the verification of Assumption 3 in Proposition 4.2, we know a is also the left join of r_1 and r_2 in (U, \leq_L) . Similarly, we know if $r_1, r_2 \in R_U$ have a common right upper bound in (U, \leq_R) , then they have a right join in (U, \leq_R) .

Now we verify Assumption 4 of Proposition 4.2.

For any $w \in W_\Gamma$, let $I_w = \text{Supp}(w)$. We claim that if $a, b \in U$ and $a \cdot b \in U$ (recall that $a \cdot b$ means $|ab| = |a| + |b|$ with $|\cdot|$ denotes the reflection length), then $I_{ab} = I_a \cup I_b$. Note that $I_{ab} \subset I_a \cup I_b$ is clear. Now let $a = r_1 r_2 \cdots r_n$ and $b = r'_1 r'_2 \cdots r'_m$ be minimal length reflection decomposition of a and b . By [BDSW14, Theorem 1.4], $\text{Supp}(r_i) \subset I_a$ for each i , thus $I_a = \cup_{i=1}^n \text{Supp}(r_i)$. Similarly $I_b = \cup_{i=1}^m \text{Supp}(r'_i)$. As $a \cdot b \in U$, $r_1 \cdots r_n r'_1 \cdots r'_m$ is a minimal length reflection decomposition of a and b . As $ab \in A_{I_{ab}}$, we know from [BDSW14, Theorem 1.4] that $r_i, r'_i \in A_{I_{ab}}$. Similarly as before $I_{ab} = (\cup_{i=1}^n \text{Supp}(r_i)) \cup (\cup_{i=1}^m \text{Supp}(r'_i))$. Thus $I_a \subset I_{ab}$ and $I_b \subset I_{ab}$. Now the claim follows.

Let a, u, v, w be as in Assumption 4 of Proposition 4.2. Then $I_a \cup I_u = I_{au}$, which spans a complete subgraph of Γ . Similarly, $I_a \cup I_v$ spans a complete subgraph of Γ . By the previous paragraph, if $u \leq_L w$ and $v \leq_L w$, then $I_u \subset I_w$ and $I_v \subset I_w$. Hence $I_v \cup I_u$ spans a complete subgraph of Γ . Thus $I = I_v \cup I_u \cup I_a$ spans a complete subgraph of Γ . Then by the first bullet point of the theorem, $I = I_1 \cup I_2$ where I_1 is a cyclic type irreducible component of I and I_2 is the union of all irreducible spherical components of I . Note that $a = a_1 \cdot a_2$ for $a_i \in W_{I_i} \cap U$ for $i = 1, 2$ - to see this, we consider a minimal reflection decomposition of a , and by Lemma 2.1 and [BDSW14, Theorem 1.4], each reflection in this decomposition has support contained in either I_1 and I_2 . As elements supported in I_1 commute with elements supported in I_2 , we can group reflections supported in I_i to form a_i for $i = 1, 2$, and have $a = a_1 \cdot a_2$, as well as $a_i \in U$ for $i = 1, 2$. Lemma 2.1 also implies that u belongs to either W_{I_1} or W_{I_2} , and v belongs to either W_{I_1} or W_{I_2} . We consider the following cases.

- (1) If $u, v \in W_{I_1}$, then $a_1 u, a_1 v \in W_{I_1}$. As $a \cdot u = a_1 \cdot a_2 \cdot u \in U$, by [McC15, Lemma 3.7], $a_1 \cdot u \leq_L a \cdot u$. Thus $a_1 \cdot u \in U$. Similarly, $a_1 \cdot v \in U$. Thus $a_1 \cdot v, a_2 \cdot u \in U \cap W_{I_1}$. Proposition 5.6 implies that $a_1 \cdot w \in U \cap W_{I_1}$. Note that $\text{Supp}(a_2)$ and $\text{Supp}(a_1 \cdot w)$ commute, so

their union is spherical. As $a_2 \in U$ and $a_1 \cdot w \in U$, it follows from the definition of U that $a \cdot w = a_2 \cdot (a_1 \cdot w)$ also belongs to U .

- (2) If exactly one of $\{u, v\}$, say u , is in W_{I_1} , then $w = u \cdot v$ as $\text{Supp}(u)$ and $\text{Supp}(v)$ commute. As $\text{Supp}(a \cdot u)$ and $\text{Supp}(v)$ commute, their union is spherical. As $a \cdot u \in U$ and $v \in U$, it follows from the definition of U that $a \cdot w = (a \cdot u) \cdot v$ is contained in U .
- (3) Suppose each of u, v is in W_{I_2} . As $a \cdot u \in U$ and $a \cdot u = a_1 \cdot (a_2 \cdot u)$ with $\text{Supp}(a_1)$ and $\text{Supp}(a_2 \cdot u)$ commuting, by a similar argument as before using Lemma 2.1 and [BDSW14, Theorem 1.4], we know $a_2 \cdot u \in U \cap W_{I_2}$. Thus $a_2 \cdot u \in [1, \delta_{I_2}]$. Similarly, $a_2 \cdot v \in [1, \delta_{I_2}]$. As $([1, \delta_{I_2}], \leq_L)$ is a lattice (Theorem 2.17 and [BDSW14, Theorem 1.4]), $a_2 \cdot v$ and $a_2 \cdot u$ have a join in $([1, \delta_{I_2}], \leq_L)$, which must be $a_2 \cdot w$ as $w = u \vee_L v$. In particular, $a_2 \cdot w \in U$. As $\text{Supp}(a_2 \cdot w)$ and $\text{Supp}(a_1)$ commute, we know $a \cdot w = a_1 \cdot (a_2 \cdot w) \in U$.

Assumption 5 of Proposition 4.2 can be verified similarly.

Now we verify Assumption 6 of Proposition 4.2. Let a, b, u, v, x be as in Assumption 6. As $a \cdot x \cdot u \in U$, by previous discussion we know that $I_a \cup I_x \cup I_u = I_{axu}$. Thus $I_a \cup I_u$ spans a complete subgraph of Γ . Similarly, $I_a \cup I_v, I_b \cup I_u, I_b \cup I_v$ span complete graphs of Γ .

First we consider the case when $I_a \cup I_b$ spans a complete subgraph. By Lemma 2.1, I_a and I_b are irreducible. If I_a commute with I_b , then $I_a \cup I_b$ is spherical and $a \cdot b$ is the right join of a and b in (U, \leq_R) . It remains to consider the case when $I_a \cup I_b$ is irreducible. By the first bullet point of the theorem, either $I_a \cup I_b$ is spherical, or $I_a \cup I_b$ is contained in an induced subgraph of Γ of cyclic type. If $I_a \cup I_b$ is spherical, then a and b have a right join in $([1, \delta_{I_a \cup I_b}], \leq_L)$ by Theorem 2.17, hence in (U, \leq_L) . Now suppose $I_a \cup I_b$ is contained in a cyclic type subgraph Λ of Γ . It suffices to consider the case that $I_a \cup I_b = \Lambda$, otherwise $I_a \cup I_b$ is spherical and we finish as before. Note that $I_a \cup I_b \cup I_u$ spans a complete subgraph of Γ . As I_u is irreducible by Lemma 2.1, thus either $I_u \subset I_a \cup I_b$, or $I_u \subset (I_a \cup I_b)^\perp$ by our assumption on complete subgraphs of Γ . Similarly, either $I_v \subset I_a \cup I_b$, or $I_v \subset (I_a \cup I_b)^\perp$. If both $I_u \subset I_a \cup I_b$ and $I_v \subset I_a \cup I_b$ hold, then $a \cdot u, b \cdot u, a \cdot v, b \cdot v \in U \cap W_{I_a \cup I_b}$ by [BDSW14, Theorem 1.4] and we are reduced to Proposition 5.6. If at least one of the two statements $I_u \subset I_a \cup I_b$ and $I_v \subset I_a \cup I_b$ is false, then $I_u \cup I_v$ is spherical (this uses the second bullet point in the assumption of the theorem), which implies that u and v have a left join in U .

The case when $I_u \cup I_v$ spans a complete subgraph is similar. It remains to consider the case that $I_u \cup I_v$ does not span a complete subgraph of Γ , and $I_a \cup I_b$ does not span a complete subgraph of Γ . Now we will show this remaining case actually can not happen, hence finishes the proof.

Suppose $I_u \cup I_v$ is not complete. Take $s_u \in I_u$ and $s_v \in I_v$ such that they are not adjacent in Γ . We hope to show $I_a \cup I_b$ spans a complete subgraph of Γ . Take $s \in I_a$ and $t \in I_b$. If $s \in I_u$, then s and t are adjacent in Γ as $I_u \cup I_b$ spans a complete subgraph. Now we assume $s \notin I_u$. Note that $s_u \notin I_a$,

otherwise s_u and s_v are adjacent in Γ . As $a \cdot x \cdot u \in U$, we know $a \cdot u \in U$ by [McC15, Lemma 3.7]. Now by Lemma 6.2, either s and s_u commute, or there is an oriented edge from s to s_u . Similarly, we know this sentence is still true if we replace the ordered pair (s, s_u) in the statement by (t, s_u) , (s, s_v) and (t, s_v) . Thus by our assumption, the 4-cycle s, s_u, t, s_v in Γ must have a diagonal. The diagonal must connect s and t , as s_u and s_v are not adjacent. Thus $I_a \cup I_b$ spans a complete subgraph of Γ . \square

The following is a consequence of Theorem 6.3 and Corollary 4.5.

Corollary 6.4. *Let Γ be a Coxeter presentation graph with vertex set S satisfying all the assumptions in Theorem 6.3. For each spherical $T \subset S$, we choose a Coxeter element $w_T \in W_T$ compatible with the orientation of Γ . Let*

$$U = \bigcup_{T \subset S \text{ spherical}} [e, w_T].$$

Let \widehat{U} be the lift of U from W_S to A_S via the isomorphism between the dual Artin group associated with A_T for each $T \subset S$ spherical and A_T ([Bes03, Theorem 2.2.5]).

Let X_S be the flag complex of the Cayley graph of A_S with generating set \widehat{U} . Then X_S admits an A_S -equivariant CUB metric such that each simplex of X_S is equipped with a polyhedral norm as in [Hae22].

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