THE COARSE HELLY PROPERTY, HIERARCHICAL HYPERBOLICITY, AND SEMIHYPERBOLICITY

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Abstract. We relate three classes of nonpositively curved metric spaces: hierarchically hyperbolic spaces, coarsely Helly spaces, and strongly shortcut spaces. We show that any hierarchically hyperbolic space admits a new metric that is coarsely Helly. The new metric is quasi-isometric to the original metric and is preserved under automorphisms of the hierarchically hyperbolic space. We show that any coarsely Helly metric space of uniformly bounded geometry is strongly shortcut. Consequently, hierarchically hyperbolic groups—including mapping class groups of surfaces—are coarsely Helly and coarsely Helly groups are strongly shortcut.

Using these results, we deduce several important properties of hierarchically hyperbolic groups, including that they are semihyperbolic, have solvable conjugacy problem, have finitely many conjugacy classes of finite subgroups, and that their finitely generated abelian subgroups are undistorted. Along the way we show that hierarchically quasiconvex subgroups of hierarchically hyperbolic groups have bounded packing.

1. Introduction

A principal theme of geometric group theory is the study of groups as metric spaces. This includes studying groups via the types of metric spaces they act on. In this vein, the study of groups acting on spaces satisfying various forms of nonpositive curvature conditions has been especially fruitful. In this article, we are concerned with three classes of spaces exhibiting nonpositive curvature: hierarchically hyperbolic spaces, coarsely Helly spaces, and strongly shortcut spaces.

1.1. The setting.

The first of our three classes is that of *hierarchically hyperbolic spaces*, which were introduced by Behrstock, Hagen, and Sisto in [BHS17b]. These spaces exhibit hyperbolic-like behaviour, and there is a growing body of interesting examples, including many quotients of mapping class groups [BHMS20] and all known cubical groups [HS20], amongst others. See Section 3.2 for more examples and discussion. The theory has had a number of successes, such as proving Farb's quasiflats conjecture for mapping class groups [BHS17c] and establishing uniform exponential growth for many cubical groups [ANS19]. We postpone describing the hierarchy structure until Section 3.1.

The next class we consider is that of *coarsely Helly* spaces. A metric space is said to be coarsely Helly, or to *have the coarse Helly property*, if there is a constant δ such

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that for any family $\{B(x_i, r_i) : i \in I\}$ of balls with $d(x_i, x_j) \leqslant r_i + r_j$ for all $i, j \in I$, the δ–neighbourhoods of those balls have nonempty total intersection. This property was first introduced by Chepoi and Estellon in [CE07], and the terminology comes from the classical Helly property for graphs.

This notion is closely related to that of *injective* metric spaces. A metric space is injective (also called *hyperconvex*) if for any family $\{B(x_i, r_i) : i \in I\}$ of balls with $d(x_i, x_j) \leqslant r_i + r_j$ for all $i, j \in I$, the balls have nonempty total intersection. In other words, it amounts to taking $\delta = 0$ in the coarse Helly property. (There are multiple equivalent ways to define injectivity of a metric space, by a theorem of Aronszajn and Panitchpakdi [AP56].) A construction of Isbell [Isb64], which was later rediscovered by Dress [Dre84] and by Chrobak and Larmore [CL94], shows that every metric space has an essentially unique *injective hull*. More precisely, the injective hull of a metric space X is an injective metric space $E(X)$, together with an isometric embedding $e : X \to E(X)$, such that no injective proper subspace of $E(X)$ contains $e(X)$. A nice description of the construction is given by Lang in [Lan13, §3].

The classes of coarsely Helly spaces and injective spaces are tied together by the useful fact that a metric space is coarsely Helly if and only if it is coarsely dense in its injective hull [CCG$^+$20, Proposition 3.12]. Moreover, if a group acts properly and coboundedly on a coarsely Helly space, then it acts properly and coboundedly on the injective hull of that space (see Lemma 3.10). Here and throughout the paper, a group G is said to act *properly* on a metric space X if $\{g \in G : gB \cap B \neq \varnothing\}$ is finite for every metric ball B of X. This is sometimes referred to as a *metrically proper* action.

Injective metric spaces satisfy a number of properties reminiscent of nonpositive curvature, and in particular of CAT(0) spaces. For instance, they admit a conical geodesic bicombing [Lan13], and proper injective spaces of finite combinatorial dimension have a canonical convex such bicombing [DL15]. Also, every bounded group action on an injective metric space has a fixed point, and the fixed point set is itself injective [Lan13]. These properties are what allow us to draw our conclusions for hierarchically hyperbolic groups. Although it will not be needed here, it is interesting to note that injective spaces are also complete [AP56] and contractible [Isb64].

The strong shortcut property was introduced by the second named author for graphs [Hod18] and then generalized to rough geodesic metric spaces [Hod20b]. A rough geodesic metric space X is *strongly shortcut* if for some $K > 1$, for every $C > 0$ there is a bound on the lengths of the (K, C)–quasi-isometric embeddings of Riemannian circles in X. A group is *strongly shortcut* if it acts properly and coboundedly on a strongly shortcut metric space. Many spaces and graphs of interest in geometric group theory and metric graph theory are strongly shortcut, including Gromov-hyperbolic spaces 1–skeletons of finite-dimensional CAT(0) cube complexes, Cayley graphs of Coxeter groups and asymptotically CAT(0) spaces. Despite being such a unifying notion, it remains possible to draw conclusions about strongly shortcut groups, including that they are finitely presented and have polynomial isoperimetric function, and so have decidable word problem.

1.2. **Comparison of the classes.**

Our main result is the definition of a new metric on hierarchically hyperbolic spaces, and more generally on coarse median spaces satisfying a nice approximation property of median intervals by CAT(0) cube complexes.

Our construction is directly inspired by work of Bowditch (see [Bow20]), in which he constructs an injective metric on any finite rank metric median space. Indeed, if one endows a finite-dimensional CAT(0) cube complex with the piecewise ℓ^∞ metric, it becomes an injective metric space. The new metric we construct is weakly roughly geodesic and has the property that balls are coarsely median-convex; see Theorem 2.10.

We then prove a hierarchical generalisation of a very nice result of Chepoi, Dragan, and Vaxès [CDV17] about pairwise close subsets of hyperbolic spaces. Combining this with work of Russell, Spriano, and Tran [RST18] enables us to deduce the coarse Helly property for balls.

Theorem A (Theorem 2.10, Corollary 3.6)**.** *Let $(X, \mathfrak{S})$ be a hierarchically hyperbolic space with metric d. There exists a metric σ on X such that (X, σ) is coarsely Helly and quasi-isometric to (X, d). Moreover, σ is invariant with respect to the automorphism group of $(X, \mathfrak{S})$.*

Our second result relates the class of coarsely Helly spaces to that of strongly shortcut spaces.

Theorem B (Theorem 4.2)**.** *Every coarsely Helly metric space of uniformly bounded geometry is strongly shortcut.*

Huang and Osajda proved that weak Garside groups of finite type and Artin groups of FC-type are Helly [HO19] and so we have the following corollary of Theorem B.

Corollary C. *Weak Garside groups of finite type and Artin groups of FC-type are strongly shortcut.*

Combining Theorem A with Theorem B, we deduce the following.

Corollary D. *Every hierarchically hyperbolic space admits a roughly geodesic metric in its quasi-isometry class that satisfies the strong shortcut property.*

In fact, in the case of hierarchically hyperbolic *groups*, the metric we construct is equivariant, by the "moreover" statement of Theorem A (also see Remark 3.9). In the setting of finitely generated groups, we therefore have that every hierarchically hyperbolic group acts properly cocompactly on a coarsely Helly space, and any group admitting such an action is a strongly shortcut group. Moreover, these three classes can be distinguished. Indeed, the $(3, 3, 3)$ Coxeter triangle group is strongly shortcut but not coarsely Helly [Hod20a]; and type-preserving uniform lattices in buildings of type C_n are coarsely Helly [CCG+20], but they cannot be hierarchically hyperbolic groups, because they do not admit any nonelementary actions on hyperbolic spaces [Hae20].

One can also ask how *Helly groups*, as defined in [CCG+20], fit into this framework. A *Helly graph* is a locally finite graph in which any set of pairwise intersecting balls in the vertex set have nonempty total intersection, and a group is Helly if it acts properly cocompactly on a Helly graph. Helly groups have some strong properties, including biautomaticity [CCG+20, Theorem 1.5].

1.3. **Metric consequences.** We now describe some of the consequences of Theorem A for hierarchically hyperbolic spaces. Recall that a quasigeodesic bicombing on a metric space (X,σ) is a map $\gamma : X \times X \times [0,1] \to X$ such that, for each distinct pair $a, b \in X$, the map

$$\begin{aligned}\big[0, \sigma(a,b)\big] &\to X\\ t &\mapsto \gamma_{a,b}\big(\tfrac{t}{\sigma(a,b)}\big)\end{aligned}$$

is a quasigeodesic from a to b with uniform constants. The bicombing is called bounded if it satisfies the following two-sided fellow-traveller property:

$$\exists C \geqslant 0, \forall a, b, a', b' \in X, \forall t \in [0,1], \sigma(\gamma_{a,b}(t), \gamma_{a',b'}(t)) \leqslant C \max(\sigma(a,a'), \sigma(b,b')) + C.$$

There are various ways to strengthen this fellow-travelling condition. We say that a bicombing is *roughly conical* if

$$\exists C \geqslant 0, \forall a, b, a', b' \in X, \forall t \in [0,1], \sigma(\gamma_{a,b}(t), \gamma_{a',b'}(t)) \leqslant (1-t)\sigma(a,a') + t\sigma(b,b') + C.$$

Note that if a bicombing is roughly conical, then it is bounded.

From the existence of conical, reversible, isometry-invariant geodesic bicombings in injective metric spaces [Lan13], we deduce the following.

Corollary E (Corollary 3.7). *Let $(X, \mathfrak{S})$ be a hierarchically hyperbolic space. Then (X,σ) admits a roughly conical, roughly reversible, quasigeodesic bicombing that is coarsely equivariant under the automorphism group of $(X, \mathfrak{S})$. More strongly, the combing lines are rough geodesics for the metric σ.*

In particular, this applies to Teichmüller space with either of the standard metrics, with equivariance under the action of the mapping class group. This particular application was unknown to us until comparing results with Durham, Minsky, and Sisto [DMS20].

Corollary E gives a positive answer to Question 8.1 of [EW17], as any roughly conical bicombing is *coherent and expanding*, in the terminology of [EW17]. Engel and Wulff proved that the existence of such a bicombing has a large number of K–theoretic consequences. This positive answer also allows one to apply work of Fukaya–Oguni (see [FO20]) to deduce the coarse Baum–Connes conjecture for hierarchically hyperbolic groups. The coarse Baum–Connes conjecture is also a consequence of finite asymptotic dimension, which is a known property of uniformly proper hierarchically hyperbolic spaces [BHS17a].

1.4. **Consequences for groups.**

We now turn to the case of hierarchically hyperbolic groups, which, as we have seen, act properly cocompactly on coarsely Helly spaces. Here we describe some of the consequences of such an action.

Following Alonso and Bridson, we say that a finitely generated group is semihyperbolic if it has a Cayley graph that admits an equivariant bounded quasigeodesic bicombing [AB95]. Among other results, Alonso and Bridson proved that this property implies the existence of a quadratic isoperimetric function, that the group has soluble word and conjugacy problems, and that an algebraic flat torus theorem holds [AB95]. For more discussion of the consequences of semihyperbolicity we refer the reader to [BH99]. Semihyperbolicity was introduced as a response to Gromov's call for a weaker form of hyperbolicity in his original essay on hyperbolic groups, and it fits into the framework of algorithmic properties developed in [ECH$^+$92]. For example, semihyperbolicity is implied by biautomaticity, but not by automaticity. A survey of this algorithmic framework can be found in [Bri19].

For hierarchically hyperbolic groups G, the freeness of the regular action of G on $(G\sigma)$ allows the bicombing of Corollary E to be pulled back to the Cayley graph of G [AB95].

Corollary F (Corollary 3.11)**.** *Every hierarchically hyperbolic group is semihyperbolic. In particular, the mapping class group of a surface of finite type is semihyperbolic.*

The mapping class group case is also a consequence of unpublished work of Hamenstädt [Ham09], and is related to Mosher's automaticity theorem [Mos95].

We should emphasise that the same result has been obtained by rather different methods, simultaneously and independently, by Durham, Minsky, and Sisto (see [DMS20]). This will be discussed more in Section 1.6.

It is well-known that mapping class groups have finitely many conjugacy classes of finite subgroups (see [Bri00]), a property that they share with hyperbolic groups. However, to the authors' knowledge, all existing proofs of this fact rely on difficult tools that do not generalise to other settings, such as Kerckhoff's celebrated solution of the Nielsen realisation problem [Ker83]. It is interesting to ask whether there is a proof that avoids such powerful machinery, and indeed a more general question about hierarchically hyperbolic groups was asked in [HP19]. The question of whether all hierarchically hyperbolic groups have finitely many conjugacy classes of finite subgroups has resisted a number of attempted resolutions.

The fact that hierarchically hyperbolic groups act properly cocompactly on coarsely Helly spaces makes the following a simple consequence of Lang's result about bounded actions on injective spaces [Lan13, Proposition 1.2].

Theorem G (Corollary 3.12)**.** *Hierarchically hyperbolic groups have finitely many conjugacy classes of finite subgroups.*

It is interesting to note that this applies in particular to many quotients of mapping class groups [BHS17a, BHMS20]. It is also a simple consequence that residually finite hierarchically hyperbolic groups are virtually torsionfree.

We now summarise the consequences for hierarchically hyperbolic groups of the results described above (also see Remark 3.9 for a comment on their generality).

Corollary H. *Every hierarchically hyperbolic group G has the following properties.*

- *G acts properly cocompactly on a coarsely Helly space.*
- *G has finitely many conjugacy classes of finite subgroups.*
- *G is semihyperbolic. In particular:*
  - *the conjugacy problem in G is soluble, and it can be solved in doubly exponential time;*
  - *any polycyclic subgroup of G is virtually abelian;*
  - *any finitely generated abelian subgroup of G is quasi-isometrically embedded;*
  - *the centraliser of any finite subset of G is finitely generated, quasi-isometrically embedded, and semihyperbolic.*

- *For any ring R, if the cohomological dimension $cd_R(G)$ is finite, then $cd_R(G) \leqslant asdim(G) + 1$.*
- *G is a strongly shortcut group.*

The result about polycyclic subgroups can also be deduced from the Tits alternative for hierarchically hyperbolic groups established in [DHS17]. The other consequences are new, however. The result about the conjugacy problem extends work of Abbott and Behrstock showing that it can be solved in exponential time for *Morse elements* of hierarchically hyperbolic groups [AB18], and generalises the fact that, in mapping class groups, it can always be solved in exponential time [MM00, Tao13, BD14]. In the case of cubical groups, a beautiful result of Niblo and Reeves states that every cubical group is biautomatic [NR98], and semihyperbolicity is a direct consequence of this. We emphasise, though, that the class of hierarchically hyperbolic groups is considerably larger than just cubical groups and mapping class groups; see Section 3.2.

1.5. **Bounded packing.**

The bounded packing property for subgroups of finitely generated groups was introduced as a metric abstraction of tools used by several authors to prove intersection properties of subgroups of hyperbolic groups [GMRS98, RS99], and in turn as a stepping stone towards ensuring cocompactness of the cube complex associated to a finite collection of quasiconvex codimension–1 subgroups [Sag97, NR03, HW14]. We recall the definition in Section 3.4; see [HW09, HW14] for more motivation and background. The prototypical example is that of a quasiconvex subgroup of a hyperbolic group. That such subgroups have bounded packing was first established by Gitik, Mitra, Rips, and Sageev, using compactness of the boundary [GMRS98], and another proof was given by Hruska–Wise, using induction on the *height* of the subgroups [HW09].

More general examples have been provided by Antolín, Mj, Sisto, and Taylor, who use induction on height to show that finite collections of *stable subgroups* in any finitely generated group have bounded packing [AMST19], again by using induction on height. Stability is a strong form of convexity that was introduced by Durham and Taylor [DT15], and stable subgroups are always hyperbolic. More generally, the notion of *Morse subgroups* was introduced independently by Tran [Tra19] and Genevois [Gen20], and the notion is implicit in earlier work of Sisto [Sis16]. Notably, Tran proved that any finite collection of Morse subgroups has bounded packing [Tra19, Theorem 1.2], again by using induction on height. However, being Morse is still quite restrictive for a subgroup.

Theorem I (Corollary 3.13). *Every finite collection of hierarchically quasiconvex subgroups of a group that is a hierarchically hyperbolic space (in particular, of any hierarchically hyperbolic group) has bounded packing.*

For example, this applies to subsurface stabilisers in the mapping class group, which are not Morse or stable. See Section 3.2 for the definition of hierarchical quasiconvexity and a more extensive list of examples. Our proof of this result is purely geometric. It relies on a very strong result for quasiconvex subsets of hyperbolic spaces that was proved by Chepoi, Dragan, and Vaxès; we state it as Theorem 3.4. Their theorem does not seem to have garnered the notice it deserves in geometric group theory. For instance, it yields what appears to be the simplest and most natural proof of bounded packing for quasiconvex subgroups of hyperbolic groups. One case of our hierarchical generalisation of their result can be stated as follows.

Theorem J (Theorem 3.5)**.** *Let X be a hierarchically hyperbolic space, and let $\mathcal{Q}$ be a finite collection of hierarchically quasiconvex subsets of X. If every pair of elements of $\mathcal{Q}$ is r–close, then there is a point of X that is R–close to every element of $\mathcal{Q}$, where R does not depend on the cardinality of $\mathcal{Q}$.*

1.6. **Comparison to the work of Durham, Minsky, and Sisto** [DMS20]**.**

Let us now say a few words about the difference between the present article and the work of Durham, Minsky, and Sisto [DMS20]. As noted, both articles independently prove that mapping class groups are semihyperbolic, but the approaches differ greatly. In both cases, this fact is deduced from a stronger statement in a more general setting, and those two statements are very different in flavour. Their results hold for hierarchically hyperbolic spaces with the extra assumption of *colorability*, and they deduce interesting corollaries about bicombings on the Teichmüller space with the Teichmüller metric, and the existence of barycentres. These results are also consequences of Theorem A and Corollary E.

Our construction is built on the fact that intervals in hierarchically hyperbolic spaces can be approximated by finite CAT(0) cube complexes (proved in [BHS17c]). The main result of Durham, Minsky, and Sisto is that these approximations are furthermore *stable*, meaning that a small change in the endpoints of the interval induces a small change in the approximating CAT(0) cube complex. This stability result may prove extremely useful for other purposes.

If we want to compare the bicombing we obtain to the one from [DMS20] in the simplest case of a CAT(0) cube complex, our bicombing looks like the geodesic CAT(0) bicombing, whereas their bicombing is more similar to (but not the same as) Niblo–Reeves normal cube paths [NR98]. One notable difference is that our bicombing is roughly conical and their bicombing is merely bounded, which is not enough to deduce the consequences of Section 1.3. On the other hand, their bicombing paths are known to be hierarchy paths, whilst ours are not.

1.7. **Structure of the article.**

In Section 2, we recall basic definitions of coarse median spaces, and we explain the extra property we need, a stronger approximation of median intervals by CAT(0) cube complexes. We then define a new distance, and we prove that it is quasi-isometric to the original one, is weakly roughly geodesic, and its balls are coarsely median-convex.

In Section 3, we recall basic definitions of hierarchically hyperbolic spaces, and we prove that hierarchically quasiconvex subsets satisfy a coarse Helly property. We use this to show that the new distance makes hierarchically hyperbolic spaces coarsely Helly, and deduce semihyperbolicity of hierarchically hyperbolic groups. We also show that hierarchically quasiconvex subgroups have bounded packing.

In Section 4, we recall the definition of a strongly shortcut group, and prove that the coarse Helly property implies the strong shortcut property.

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2. Coarse median spaces with quasicubical intervals

2.1. **Background on coarse median spaces.**

Coarse median spaces, defined by Bowditch in [Bow13], are a generalisation of CAT(0) cube complexes and Gromov-hyperbolic spaces, and the class is rich enough to encompass mapping class groups of finite type surfaces. The general idea is to associate to every triple of points in the space a point that satisfies the axioms of a usual median up to controlled error. This point will be called the coarse median.

Let us recall here that a *median* $\mu : X^3 \to X$ on a set X is a map satisfying (where we write equivalently $\mu(x,y,z)$ or $\mu_{x,y,z}$ to increase readability):

- $\mu(x,y,z)$ is symmetric in x,y,z,
- $\forall x,y \in X, \mu(x,x,y) = x$ and
- $\forall a,b,x,y,z \in X, \mu(a,b,\mu_{x,y,z}) = \mu(\mu_{a,b,x},\mu_{a,b,y},z)$.

The pair (X,μ) is called a *median algebra*. The *rank* of (X,μ) is the supremum of all $\nu \in \mathbb{N}$ such that there exists an injective median homomorphism from the ν–cube $\{0,1\}^\nu$ into X.

Let (X,d) be a metric space. For any $x,y \in X$, let $I_d(x,y) = \{z \in X \mid d(x,z)+d(z,y) = d(x,y)\}$ denote the interval between x and y. The metric space (X,d) is called *metric median* if $\forall x,y,z \in X, I_d(x,y) \cap I_d(y,z) \cap I_d(x,z)$ is a singleton, say $\{\mu(x,y,z)\}$. In this case, μ defines a median on X. Examples of median metric spaces include 1–skeletons of CAT(0) cube complexes with the combinatorial distance, trees, and L^1 spaces.

In a Gromov-hyperbolic space X, the three intervals joining three points may not intersect precisely in a singleton, but by definition they do coarsely intersect with uniformly bounded diameter. This suggests defining a map $X^3 \to X$ that satisfies the axioms of a median up to bounded error. This is made precise by the following definition due to Bowditch [Bow13].

Definition 2.1 (Coarse median space)**.** Let (X,d) be a metric space. A map $\mu : X^3 \to X$ is called a *coarse median* if there exist $k \in [0,+\infty)$ and $h : \mathbb{N} \to [0,+\infty)$ such that

- For all $a,b,c,a',b',c' \in X$, we have $d(\mu(a,b,c),\mu(a',b',c')) \leqslant k(d(a,a')+d(b,b')+d(c,c')) + h(0)$.
- For each finite non-empty set $A \subset X$, with $|A| \leqslant n$, there exists a finite median algebra (Q,μ_Q) and maps $\pi : A \to Q$, $\lambda : Q \to X$ such that for every $\alpha,\beta,\gamma \in Q$, we have $d(\lambda\mu_Q(\alpha,\beta,\gamma),\mu(\lambda\alpha,\lambda\beta,\lambda\gamma)) \leqslant h(n)$, and for every $a \in A$, we have $d(a,\lambda(\pi(a))) \leqslant h(n)$.

We say that the triple (X,μ,d) is a *coarse median space*. If Q can always be chosen to have rank at most ν, we say that μ has rank at most ν. Remark that a finite median algebra can be seen as the 0-skeleton of a CAT(0) cube complex (see [Che00, Rol98]).

In the case of mapping class groups, the coarse median operation is the *centroid* defined in [BM11].

We now recall the definition of intervals and coarse convexity in coarse median spaces.

Definition 2.2 (Median interval)**.** For a pair of points $a, b \in X$, the median interval between a and b is defined as

$$[a, b] = \{\mu(a, b, x) \mid x \in X\}.$$

Definition 2.3 (Coarse median-convexity)**.** For a constant $M \geqslant 0$, a subset Y of X is said to be M–coarsely median-convex if

$$d(Y, \mu(x, y, y')) \leqslant M \text{ for all } y, y' \in Y,\ x \in X.$$

2.2. **Construction of a new metric.**

Let (X, μ, d) be a coarse median space. Following Bowditch's construction of an injective metric on a median metric space in [Bow20], we shall define a new metric σ on X.

Definition 2.4 (Contraction)**.** For a constant $K \geqslant 0$, a map $\Phi : X \to \mathbb{R}$ is called a K–contraction if:

- Φ is $(1, K)$–coarsely Lipschitz, i.e. $\forall a, b \in X, |\Phi(x) - \Phi(y)| \leqslant d(x, y) + K$.
- Φ is a K–quasi-median homomorphism, i.e.
$$\forall a, b, c \in X, |\Phi(\mu(a, b, c)) - \mu_{\mathbb{R}}(\Phi(a), \Phi(b), \Phi(c))| \leqslant K,$$
where $\mu_{\mathbb{R}}$ denotes the standard median on $\mathbb{R}$.

Definition 2.5 (New metric)**.** For $K > 0$, we define a new metric σ on X as follows. Given $a, b \in X$, let $\sigma(a, b)$ denote the supremum of all $r \geqslant 0$ such that there exists a K–contraction $\Phi : X \to \mathbb{R}$ such that $\Phi(a) = 0$ and $\Phi(b) = r$.

The assumption that K is nonzero is notably needed to ensure that σ separates points.

Lemma 2.6. *The function σ is a metric on X.*

Proof. Let $a, b \in X$ be distinct. Then consider the map $\Phi : X \to \{0, K\}$ that sends b to K and everything else to 0. It is a K–contraction, and so $\sigma(a, b) \geqslant K > 0$.

The proof of the triangle inequality is identical to [Bow20, Lemma 3.1]. For the reader's convenience, we repeat it here. Let $a, b, c \in X$, let $r < \sigma(a, b)$ and consider a K–contraction $\Phi : X \to \mathbb{R}$ such that $\Phi(a) = 0$ and $\Phi(b) = r$. Then $\sigma(a, c) + \sigma(c, b) \geqslant \Phi(c) - \Phi(a) + \Phi(b) - \Phi(c) = \Phi(b) - \Phi(a) = r$. Hence $\sigma(a, c) + \sigma(c, b) \geqslant \sigma(a, b)$. ☐

We record the following simple consequence of the definition of σ.

Lemma 2.7. *If a group G is acting on a coarse median space (X, μ, d) by median isometries, in the sense that $g\mu(x, y, z) = \mu(gx, gy, gz)$ for all $g \in G$, $x, y, z \in X$, then the induced action of G on (X, μ, σ) is isometric.*

Proof. For any $g \in G$ and $x, y \in X$, if Φ is a K–contraction with $\Phi(x) = 0$ and $\Phi(y) = r$, then $\Phi' = \Phi g^{-1}$ is a K–contraction with $\Phi'(gx) = 0$ and $\Phi'(gy) = r$. ☐

In order to help understand the metric σ, we shall work with coarse median spaces that have the following property, which is a strengthening of the second axiom of coarse median spaces for sets $A = \{a, b\}$ with cardinality 2. We require an approximation of the entire interval $[a, b]$ with uniform constants, and also that the comparison map is a quasi-isometry and not just coarsely invertible.

Definition 2.8 (Quasicubical intervals)**.** Let (X, μ, d) be a coarse median space. We say that it has *quasicubical intervals* if it has finite rank ν and there exists $\kappa \geqslant 1$ such that the following hold. For every $a, b \in X$, there exists a finite CAT(0) cube complex Q of dimension at most ν, endowed with the ℓ^1 metric d_Q and the median μ_Q, such that there exists a map $\lambda : Q \to [a, b]$ satisfying:

- λ is a (κ, κ)–quasi-isometry, i.e. λ is κ–quasi-surjective and
$$\forall \alpha, \beta \in Q, \frac{1}{\kappa} d_Q(\alpha, \beta) - \kappa \leqslant d(\lambda(\alpha), \lambda(\beta)) \leqslant \kappa d_Q(\alpha, \beta) + \kappa;$$
- λ is a κ–quasi-median homomorphism, i.e.
$$\forall \alpha, \beta, \gamma \in Q, d(\lambda(\mu_Q(\alpha, \beta, \gamma)), \mu(\lambda(\alpha), \lambda(\beta), \lambda(\gamma)) \leqslant \kappa.$$

Obviously this is satisfied by finite dimensional CAT(0) cube complexes, or indeed by any space with a global quasi-median quasi-isometry to a CAT(0) cube complex. However, it actually holds in a much larger family of examples.

Proposition 2.9. *Hierarchically hyperbolic spaces have quasicubical intervals, as do coarse median spaces satisfying the axioms (B1)-(B10) in* [Bow19a].

Proof. In hierarchically hyperbolic spaces, the notion of coarse median intervals used here coincides coarsely with the notion of hierarchical convex hull of a pair of points defined in [BHS19] by [RST18, Corollary 5.12] and [Bow19b, Lemma 8.1]. The first statement is thus a special case of [BHS17c, Theorem 2.1]. The second statement is exactly [Bow19a, Theorem 1.3]. □

As noted by Bowditch, every hierarchically hyperbolic space satisfies the axioms (B1)-(B10) in [Bow19a]. It is not known whether all cocompact cube complexes can be given a structure that satisfies these axioms.

We can now state the main result of this section. It sums up Proposition 2.13, Proposition 2.18, and Lemma 2.20, and the proof is split over the next three subsections. We first introduce some terminology.

A metric space (X, d) (or, more briefly, the metric d) is called *roughly geodesic* if there exists a constant $C_d \geqslant 0$ such that, for any $a, b \in X$, there exists a $(1, C_d)$–quasi-isometric embedding of the interval $f : [0, d(a, b)] \to X$ such that $f(0) = a$ and $f(d(a, b)) = b$. A metric space (X, d) is called *weakly roughly geodesic* if there exists a constant $C'_d \geqslant 0$ such that, for any $a, b \in X$ and any nonnegative $r \leqslant d(a, b)$, there is a point $c \in X$ with $|d(a, c) - r| \leqslant C'_d$ and $d(a, c) + d(c, b) \leqslant d(a, b) + C'_d$. Note that if a metric space is roughly geodesic, then it is weakly roughly geodesic.

Theorem 2.10. *Assume that the coarse median space (X, μ, d) has quasicubical intervals and is roughly geodesic. The distances σ and d are quasi-isometric, the distance σ is weakly roughly geodesic, and balls for the distance σ are uniformly coarsely median-convex. Moreover, σ is invariant under the group of median isometries of (X, μ, d).*

2.3. The metrics d and σ are quasi-isometric.

Here we shall prove that the new distance σ is quasi-isometric to the original distance d. We need the following technical result for coarse median spaces, which is a special case of Lemmas 2.18 and 2.19 of [NWZ19].

Lemma 2.11. *In any coarse median space (X, d, μ), there exists a constant $H_5 \geqslant 0$ such that the following inequalities hold for any $a, b, c, x, z \in X$.*

$$d(\mu(x, z, \mu_{a,b,c}), \mu(\mu_{x,z,a}, \mu_{x,z,b}, c)) \leqslant H_5$$
$$d(\mu(x, z, \mu_{a,b,c}), \mu(\mu_{x,z,a}, \mu_{x,z,b}, \mu_{x,z,c})) \leqslant H_5.$$

We will now prove that, up to multiplicative and additive constants, one can restrict to contractions defined on the interval between two points for the definition of σ.

Lemma 2.12. *For each $a, b \in X$, let $\sigma'(a, b)$ denote the supremum of all $r \geqslant 0$ such that there exists a K–contraction $\Phi' : [a, b] \to \mathbb{R}$ for which $\Phi'(a) = 0$ and $\Phi'(b) = r$. There exists $L \geqslant 1$ such that, for each $a, b \in X$, we have $\sigma(a, b) \leqslant \sigma'(a, b) \leqslant L\sigma(a, b)$.*

Proof. It is immediate that $\sigma(a, b) \leqslant \sigma'(a, b)$. Consider $r \geqslant 0$ and a K–contraction $\Phi' : [a, b] \to \mathbb{R}$ such that $\Phi'(a) = 0$ and $\Phi'(b) = r$. Define $\Phi : X \to \mathbb{R}$ by $c \mapsto \Phi'(\mu(a, b, c))$. Since the map $c \mapsto \mu(a, b, c)$ is $(k, h(0))$–coarsely Lipschitz and Φ' is $(1, K)$–coarsely Lipschitz, we deduce that Φ is $(k, h(0) + K)$–coarsely Lipschitz.

Now let $x, y, z \in X$. According to Lemma 2.11 we have

$$d(\mu(a, b, \mu_{x,y,z}), \mu(\mu_{a,b,x}, \mu_{a,b,y}, \mu_{a,b,z})) \leqslant H_5.$$

Hence, since Φ' is $(1, K)$–coarsely Lipschitz,

$$|\Phi'(\mu(a, b, \mu_{x,y,z})) - \Phi'(\mu(\mu_{a,b,x}, \mu_{a,b,y}, \mu_{a,b,z}))| \leqslant H_5 + K.$$

But Φ' is also a K–quasi-median homomorphism, and so

$$|\Phi'(\mu(\mu_{a,b,x}, \mu_{a,b,y}, \mu_{a,b,z})) - \mu_{\mathbb{R}}(\Phi'(\mu_{a,b,x}), \Phi'(\mu_{a,b,y}), \Phi'(\mu_{a,b,z}))| \leqslant K.$$

Combining these and recalling the definition of Φ, we conclude that $|\Phi(\mu(x, y, z)) - \mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z))| \leqslant H_5 + 2K$. Thus, if we set $L = \max\{k, 1 + \frac{h(0)}{K}, 2 + \frac{H_5}{K}\}$, then we have that $\frac{1}{L}\Phi$ is a K–contraction, and so $\sigma'(a, b) \leqslant L\sigma(a, b)$. $\square$

We can now deduce that the metric σ is quasi-isometric to d. We resume the assumptions of Theorem 2.10.

Proposition 2.13. *The metrics d and σ are quasi-isometric.*

Proof. Fix $a, b \in X$. First of all, since any K–contraction is $(1, K)$–coarsely Lipschitz, we have $\sigma(a, b) \leqslant d(a, b) + K$.

According to the quasicubicality of intervals, there exists a finite CAT(0) cube complex Q of dimension at most ν, and a map $\lambda : (Q, d_Q) \to [a, b]$ that is a (κ, κ)–quasi-isometry and a κ–quasi-median homomorphism. Then λ has a quasi-inverse $\pi : [a, b] \to (Q, d_Q)$ that is a (κ', κ')–quasi-isometry and a κ'–quasi-median homomorphism, where κ' is a constant depending only on κ and $h(0)$.

Note that we shall in fact use Q to denote the vertex set, d_Q to denote the combinatorial (piecewise ℓ^1) distance on Q, and μ_Q to denote the median on Q. Let us denote by σ_Q the piecewise ℓ^∞ distance on Q: we have $\sigma_Q \leqslant d_Q \leqslant \nu\sigma_Q$.

Since Q is a CAT(0) cube complex, there exists a 0–contraction $\Phi_Q : (Q, d_Q) \to \mathbb{R}$ such that $\Phi_Q(\pi(a)) = 0$ and $\Phi_Q(\pi(b)) = \sigma_Q(\pi(a), \pi(b))$ (see [Bow20, §7]). Let us consider $\Phi' = \frac{\min\{1,K\}}{\kappa'}\Phi_Q \circ \pi : [a, b] \to \mathbb{R}$. Since $\pi : [a, b] \to (Q, d_Q)$ is a (κ', κ')–quasi-isometry and $\Phi_Q : (Q, d_Q) \to \mathbb{R}$ is 1–Lipschitz, we deduce that Φ' is $(1, K)$–coarsely Lipschitz. Furthermore, for every $x, y, z \in [a, b]$, we have:

$$\begin{aligned} & |\Phi_Q \circ \pi(\mu(x,y,z)) - \mu_{\mathbb{R}}(\Phi_Q \circ \pi(x), \Phi_Q \circ \pi(y), \Phi_Q \circ \pi(z))| \\ \leqslant\ & |\Phi_Q \circ \pi(\mu(x,y,z)) - \Phi_Q(\mu_Q(\pi(x), \pi(y), \pi(z)))| \\ +\ & |\Phi_Q(\mu_Q(\pi(x), \pi(y), \pi(z))) - \mu_{\mathbb{R}}(\Phi_Q \circ \pi(x), \Phi_Q \circ \pi(y), \Phi_Q \circ \pi(z))| \\ \leqslant\ & d_Q(\pi(\mu(x,y,z)), \mu_Q(\pi(x), \pi(y), \pi(z))) \quad \leqslant \quad \kappa', \end{aligned}$$

so Φ' is K–quasi-median.

The map Φ' is therefore a K–contraction on $[a, b]$, and $\Phi'(a) = 0$ and $\Phi'(b) = \frac{\min\{1,K\}}{\kappa'} \sigma_Q(\pi(a), \pi(b)) \geqslant \frac{\min\{1,K\}}{\nu\kappa'} d_Q(\pi(a), \pi(b))$. Using Lemma 2.12, we deduce that $d_Q(\pi(a), \pi(b)) \leqslant \frac{\nu\kappa' L}{\min\{1,K\}} \sigma(a, b)$. But π is a (κ', κ')–quasi-isometry, so we also have $d_Q(\pi(a), \pi(b)) \geqslant \frac{1}{\kappa'} d(a, b) - \kappa'$.

In conclusion, we have

$$\frac{\min\{1,K\}}{\nu\kappa'^2 L} d(a,b) - \frac{\min\{1,K\}}{\nu L} \quad \leqslant \quad \sigma(a,b) \quad \leqslant \quad d(a,b) + K$$

for all $a, b \in X$. □

2.4. **The metric σ is weakly roughly geodesic.**

Recall that (X, μ, d) is a coarse median space with quasicubical intervals, and that the metric d is C_d–roughly geodesic: for any $a, b \in X$, there exists a $(1, C_d)$–quasi-isometric embedding of the interval $f : [0, d(a, b)] \to X$ such that $f(0) = a$ and $f(d(a, b)) = b$.

We shall prove that the new metric σ is weakly roughly geodesic, i.e. there exists a constant $C'_\sigma \geqslant 0$ such that, for any $a, b \in X$ and $0 \leqslant r \leqslant \sigma(a, b)$, there exists $c \in X$ such that $|\sigma(a, c) - r| \leqslant C'_\sigma$ and $\sigma(a, c) + \sigma(c, b) \leqslant \sigma(a, b) + C'_\sigma$.

This will be the most difficult part of the proof of Theorem 2.10.

Let $a, b \in X$, and consider K–contractions $\Phi_1 : X \to [0, r]$ and $\Phi_2 : X \to [r, r + s]$ (for some $r, s \geqslant 0$) such that $\Phi_1(a) = 0$ and $\Phi_2(b) = r + s$. We want to find a criterion to ensure that we can combine Φ_1 and Φ_2 into a contraction Φ such that $\Phi(a) = 0$ and $(r + s) - \Phi(b)$ is bounded above by some constant.

Lemma 2.14. *There exists a constant $D \geqslant 0$ such that the following holds. Assume that $a, b, \Phi_1, \Phi_2, r, s$ are as above. If $t \geqslant 0$ is such that the sets*

$$\{x \in [a, b] \,|\, \Phi_1(x) \leqslant r - t\} \text{ and } \{x \in [a, b] \,|\, \Phi_2(x) \geqslant r + t\}$$

are disjoint, then $\sigma(a, b) \geqslant r + s - 2t - 2D$.

Proof. Let us denote $Z_1 = \{x \in [a, b] \,|\, \Phi_1(x) \leqslant r - t\}$ and $Z_2 = \{x \in [a, b] \,|\, \Phi_2(x) \geqslant r + t\}$.

Let $D_1 = k(3K + 4C_d) + K + h(0)$, and let us write $Y_1 = \{x \in [a, b] \,|\, \Phi_1(x) \leqslant r - t - D_1\}$ and $Y_2 = \{x \in [a, b] \,|\, \Phi_2(x) \geqslant r + t + D_1\}$.

Claim 1: $d(Y_1, Y_2) \geqslant D_1 - K$

Proof of Claim 1: Let $y_1 \in Y_1$ and $y_2 \in Y_2$. Since $Y_2 \subset Z_2$, we have $y_2 \notin Z_1$, so $\Phi_1(y_2) > r - t$. We also have $\Phi_1(y_1) \leqslant r - t - D_1$, so $|\Phi_1(y_1) - \Phi_1(y_2)| \geqslant D_1$. As Φ_1 is $(1, K)$–coarsely Lipschitz, we have $|\Phi_1(y_1) - \Phi_1(y_2)| \leqslant d(y_1, y_2) + K$, and hence $d(y_1, y_2) \geqslant D_1 - K$. ♢

Let us write $X_1' = \{x \in X \mid \Phi_1(x) \leqslant r - t - D_1 - K\}$ and $X_2' = \{x \in X \mid \Phi_2(x) \geqslant r + t + D_1 + K\}$.

Claim 2: $d(X_1', X_2') \geqslant \frac{D_1 - K - h(0)}{k}$.

Proof of Claim 2: Let $x_1 \in X_1'$ and $x_2 \in X_2'$, and set $y_1 = \mu(a, b, x_1) \in [a, b]$ and $y_2 = \mu(a, b, x_2) \in [a, b]$. We shall first prove that $y_1 \in Y_1$ and $y_2 \in Y_2$.

We know that $\Phi_1(y_1) \leqslant \mu_{\mathbb{R}}(\Phi_1(a), \Phi_1(b), \Phi_1(x_1)) + K$. We have $\Phi_1(a) = 0$ and $\Phi_1(x_1) \leqslant r - t - D_1 - K$. Hence $\mu_{\mathbb{R}}(\Phi_1(a), \Phi_1(b), \Phi_1(x_1)) \leqslant r - t - D_1 - K$ and $\Phi_1(y_1) \leqslant r - t - D_1 - K + K = r - t - D_1$. As a consequence $y_1 \in Y_1$, and similarly $y_2 \in Y_2$.

We have proved in Claim 1 that $d(Y_1, Y_2) \geqslant D_1 - K$, so $d(y_1, y_2) \geqslant D_1 - K$. Since μ is $(k, h(0))$–coarsely Lipschitz with respect to each variable, we have $d(y_1, y_2) \leqslant kd(x_1, x_2) + h(0)$, so $d(x_1, x_2) \geqslant \frac{d(y_1, y_2) - h(0)}{k} \geqslant \frac{D_1 - K - h(0)}{k}$. ♢

Let $D = D_1 + 2K$, and let us denote $X_1 = \{x \in X \mid \Phi_1(x) \leqslant r - t - D\} \subset X_1'$ and $X_2 = \{x \in X \mid \Phi_2(x) \geqslant r + t + D\} \subset X_2'$.

Claim 3: The coarse-median convex hull $\mathrm{Hull}(X_1) = \{\mu(x, y, z) \mid x, y \in X_1, z \in X\}$ is disjoint from X_2, and $\mathrm{Hull}(X_2)$ is disjoint from X_1.

Proof of Claim 3: Fix $x, y \in X_1$ and $z \in X$. Since $\Phi_1(x) \leqslant r - t - D$ and $\Phi_1(y) \leqslant r - t - D$, we deduce that $\mu_{\mathbb{R}}(\Phi_1(x), \Phi_1(y), \Phi_1(z)) \leqslant r - t - D$, and it follows that $\Phi_1(\mu(x, y, z)) \leqslant r - t - D + K = r - t - D_1 - K$, so $\mu(x, y, z) \in X_1'$. As $d(X_1', X_2') \geqslant \frac{D_1 - K - h(0)}{k} > 0$, we know that X_1' and X_2' are disjoint, so $\mu(x, y, z) \notin X_2'$, and, in particular, $\mu(x, y, z) \notin X_2$. The other case is similar. ♢

Let us define $X_0 = X \smallsetminus (X_1 \cup X_2)$. Consider the map $\Phi : X \to [0, r + s - 2t - 2D]$ defined by:

$$\begin{array}{rll} \text{If } x \in X_1 & \text{then} & \Phi(x) = \Phi_1(x). \\ \text{If } x \in X_2 & \text{then} & \Phi(x) = \Phi_2(x) - 2t - 2D. \\ \text{If } x \notin X_1 \cup X_2 & \text{then} & \Phi(x) = r - t - D. \end{array}$$

We have $\Phi(a) = 0$ and $\Phi(b) = r + s - 2t - 2D$, so if we prove that Φ is a K–contraction, then we may deduce that $\sigma(a, b) \geqslant r + s - 2t - 2D$, the desired conclusion.

Claim 4: Φ is $(1, K)$–coarsely Lipschitz.

Proof of Claim 4: Notice that Φ coincides on $X_1 \cup X_0$ with the composition of $\Phi_1 : X \to [0, r]$ with the map $m_t = \min(\cdot, r - t - D) : [0, r] \to [0, r - t - D]$, which is 1–Lipschitz. Hence, if $x, y \in X_1 \cup X_0$, then $|\Phi(x) - \Phi(y)| \leqslant |\Phi_1(x) - \Phi_1(y)| \leqslant d(x, y) + K$, and similarly if $x, y \in X_2 \cup X_0$.

If $x \in X_1$ and $y \in X_2$, then since d is C_d–roughly geodesic, one may consider $z_1 \in X_1$ such that $d(x, z_1) + d(z_1, y) \leqslant d(x, y) + C_d$ and such that $r - t - D - C_d - K \leqslant \Phi_1(z_1) \leqslant r - t - D$. Similarly consider $z_2 \in X_2$ such that $r + t + D \leqslant \Phi_2(z_2) \leqslant r + t + D + C_d + K$

and $d(z_1, z_2) + d(z_2, y) \leqslant d(z_1, y) + C_d$. We then have

$$\begin{array}{rcl} |\Phi(x) - \Phi(y)| & \leqslant & |\Phi(x) - \Phi(z_1)| + |\Phi(z_2) - \Phi(y)| + 2C_d + 2K \\ & = & |\Phi_1(x) - \Phi_1(z_1)| + |\Phi_2(z_2) - \Phi_2(y)| + 2C_d + 2K \\ & \leqslant & d(x, z_1) + K + d(z_2, y) + K + 2C_d + 2K \\ & \leqslant & d(x, y) - d(z_1, z_2) + 4K + 4C_d \\ & \leqslant & d(x, y) + K, \end{array}$$

since $d(z_1, z_2) \geqslant d(X_1, X_2) \geqslant d(X_1', X_2') \geqslant \frac{D_1 - K - h(0)}{k} = 3K + 4C_d$. ♢

Claim 5: Φ is K–quasi-median.

Proof of Claim 5: As noted in the proof of Claim 4, on $X_1 \cup X_0$ we have $\Phi = m_t \circ \Phi_1$. Also, m_t is a median homomorphism with respect to $\mu_{\mathbb{R}}$. Hence, if $x, y, z \in X_1 \cup X_0$, we have

$$\begin{array}{rl} & |\Phi(\mu(x, y, z)) - \mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z))| \\ = & |m_t(\Phi_1(\mu(x, y, z))) - \mu_{\mathbb{R}}(m_t(\Phi_1(x)), m_t(\Phi_1(y)), m_t(\Phi_1(z)))| \\ \leqslant & |\Phi_1(\mu(x, y, z)) - \mu_{\mathbb{R}}(\Phi_1(x), \Phi_1(y), \Phi_1(z))| \\ \leqslant & K, \end{array}$$

and similarly if $x, y, z \in X_2 \cup X_0$.

Assume now that $x, y \in X_1$ and $z \in X_2$. Since $\Phi(x) = \Phi_1(x) \leqslant r - t - D$, $\Phi(y) = \Phi_1(y) \leqslant r - t - D$ and $\Phi(z) = \Phi_2(z) - 2t - 2D \geqslant r - t - D$, we have $\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) = \mu_{\mathbb{R}}(\Phi_1(x), \Phi_1(y), r - t - D)$. Furthermore, since $z \notin X_1$, we have $\Phi_1(z) > r - t - D$, so $\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) = \mu_{\mathbb{R}}(\Phi_1(x), \Phi_1(y), \Phi_1(z))$. As Φ_1 is K–quasi-median, we deduce that $|\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) - \Phi_1(\mu(x, y, z))| \leqslant K$. According to Claim 3, we know that $\mu(x, y, z) \notin X_2$, and so $\Phi(\mu(x, y, z)) = m_t \circ \Phi_1(\mu(x, y, z))$. But $\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) \leqslant r - t - D$, so we conclude that $|\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) - \Phi(\mu(x, y, z))| \leqslant K$. A similar argument applies when $x, y \in X_2$ and $z \in X_1$.

Assume finally that $x \in X_1$, $y \in X_0$, and $z \in X_2$. Since $\Phi(x) = \Phi_1(x) \leqslant r - t - D$, $\Phi(y) = r - t - D$ and $\Phi(z) = \Phi_2(z) - 2t - 2D \geqslant r - t - D$, we have $\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) = r - t - D$. Let $m = \mu(x, y, z)$. If $m \in X_0$, then $\Phi(m) = r - t - D = \mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z))$. If $m \in X_1$, then $\Phi(m) = \Phi_1(m) = \Phi_1(\mu(x, y, z)) \geqslant \mu_{\mathbb{R}}(\Phi_1(x), \Phi_1(y), \Phi_1(z)) - K \geqslant r - t - D - K$. Hence $|\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) - \Phi(m)| \leqslant K$, and similarly if $m \in X_2$. Thus, in each case we have $|\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z)) - \Phi(m)| \leqslant K$. ♢

We have proved that Φ is a K–contraction. Hence $\sigma(a, b) \geqslant |\Phi(a) - \Phi(b)| = r + s - 2t - 2D$. □

We need the following iterative description of the convex hull in a CAT(0) cube complex. Note that the constant ν is probably far from optimal.

Lemma 2.15. *Let Q be a CAT(0) cube complex of dimension at most ν, let $A \subset Q^{(0)}$ be a non-empty subset, and let* Hull(A) *denote its convex hull. Let $\mu_Q : {Q^{(0)}}^3 \to Q^{(0)}$ denote the median. Let $A_0 = A$, and for each $i \in \mathbb{N}$, let*

$$A_{i+1} = \mu_Q(Q^{(0)}, A_i, A_i) = \{\mu_Q(x, a, b) \,|\, a \in A_i, b \in A_i, x \in Q^{(0)}\}.$$

Then $A_\nu = \mathrm{Hull}(A)$.

Proof. Assume that A is not convex, otherwise the result is clear.

Remark that A_1 is connected, so up to replacing A with A_1, we will assume that A is connected and we will prove that $A_{\nu-1} = \mathrm{Hull}(A)$.

Fix $x \in \mathrm{Hull}(A) \setminus A$, and note that every hyperplane of $\mathrm{Hull}(A)$ separates x from some point of A. Consider the set $\mathcal{H}$ of all hyperplanes of $\mathrm{Hull}(A)$ adjacent to x, and for each $H \in \mathcal{H}$, let $Q^{(0)} = H^+ \sqcup H^-$ denote the partition defined by H, where $x \in H^+$.

Let $H_1, \ldots, H_n$ be a maximal pairwise intersecting family in $\mathcal{H}$. We have $n \leqslant \nu$. Furthermore, we cannot have $n = 1$, for then x would be a cut-point (or leaf) of $\mathrm{Hull}(A)$, and hence would be in A by connectedness (or convexity). Thus $n \geqslant 2$. For each $1 \leqslant i \leqslant n$, let $\mathcal{H}_i$ be the subset of $\mathcal{H}$ consisting of those hyperplanes that are disjoint from H_i in $\mathrm{Hull}(A)$, together with H_i, and set

$$J_i = \bigcap_{H \in \mathcal{H}_i} H^+.$$

As an intersection of halfspaces, J_i is convex. For each i, fix a point $z_i \in A \cap J_i$, which is nonempty by connectedness of A and disjointness of the elements of $\mathcal{H}_i$. Let us define

$$y_n = \mu(x, z_n, \mu(x, z_{n-1}, \mu(\ldots \mu(x, z_2, z_1) \ldots))).$$

More formally, let $y_1 = z_1$ and, for every $i \geqslant 2$, let $y_i = \mu(x, z_i, y_{i-1})$. We have $y_n \in A_{n-1}$, so $y_n \in A_{\nu-1}$. We shall prove that $y_n = x$.

Let us prove by induction on $1 \leqslant i \leqslant n$ that for each $1 \leqslant j \leqslant i$, we have $y_i \in J_j$. When $i = 1$, we have $y_1 = z_1 \in J_1$ by definition.

Assume that for some $2 \leqslant i \leqslant n$, we have $y_{i-1} \in J_j$ for all $1 \leqslant j \leqslant i-1$. Fix $1 \leqslant j \leqslant i-1$: since $x \in J_j$ and $y_{i-1} \in J_j$, we deduce that $y_i = \mu(x, z_i, y_{i-1}) \in J_j$ by convexity. Since $x \in J_i$ and $z_i \in J_i$, we deduce that $y_i = \mu(x, z_i, y_{i-1}) \in J_i$. This concludes the induction.

So we know that for every $1 \leqslant i \leqslant n$, we have $y_n \in J_i$.

Assume for a contradiction that $x \neq y_n$, so there exists a hyperplane H in Q that is adjacent to x and separates x from y_n. Since $x, y_n \in \mathrm{Hull}(A)$, we know that the hyperplane H intersects A, so $H \in \mathcal{H}$. Since $y_n \in J_i$ for each i, we deduce that H is not contained in any $\mathcal{H}_i$. In particular, H is not disjoint from, or equal to, any H_i, so H crosses every H_i. This contradicts the maximality of the family $H_1, \ldots, H_n$. We deduce that $x = y_n$. □

In order to apply Lemma 2.14, we shall focus on contractions on CAT(0) cube complexes.

Lemma 2.16. *Let Q be a CAT(0) cube complex of dimension at most ν, and let $\Phi : Q^{(0)} \to \mathbb{R}$ be a K'–quasi-median, (K', K')–coarsely Lipschitz map (for the ℓ^1 metric) with bounded image. There exists an interval $[u, v]$ of $\mathbb{Z}$ and a family $(H_n)_{u \leqslant n \leqslant v}$ of pairwise disjoint hyperplanes of Q satisfying the following:*

- *if $u \leqslant n < m < p \leqslant v$, then H_m separates H_n and H_p,*
- *for each vertex x in Q, there exists a unique $n = \Psi(x) \in [u-1, v]$ such that:*
  - ⋆ *either $u \leqslant n \leqslant v-1$ and x is between H_n and H_{n+1},*

- ⋆ *or* $n = u - 1$ *and* H_u *separates* x *from* H_{u+1}*,*
  - ⋆ *or* $n = v$ *and* H_v *separates* x *from* H_{v-1}*, and*
- *for each vertex* x *in* Q*, we have* $|\Phi(x) - 4K'\nu\Psi(x)| \leqslant 4K'\nu$*.*

Proof. Fix $n \in \mathbb{Z}$, and consider $K_n = \Phi^{-1}((2An-A, 2An])$, where $A = 2K'\nu$. Since $Q^{(0)}$ is 1–connected, we know that $\Phi(Q)$ is $2K'$–connected. In particular, the set of integers $n \in \mathbb{Z}$ such that $K_n \neq \varnothing$ is an interval $[u-1, v]$. Furthermore, for each $u \leqslant n \leqslant v-1$, we know that K_n disconnects Q.

Let $\mathrm{Hull}(K_n) \subset Q^{(0)}$ denote the (median) convex hull of K_n. According to Lemma 2.15, for each $x \in \mathrm{Hull}(K_n)$, we have $|\Phi(x) - \Phi(K_n)| \leqslant K'\nu$. In particular, we deduce that if $n \neq m$, then the convex subcomplexes $\mathrm{Hull}(K_n)$ and $\mathrm{Hull}(K_m)$ are disjoint. Then, for each $u \leqslant n \leqslant v$, there exists a hyperplane H_n of Q that separates $\mathrm{Hull}(K_{n-1})$ from $\mathrm{Hull}(K_n)$ [Che94, Corollary 1].

For each vertex $x \in Q^{(0)}$, let $u-1 \leqslant n \leqslant v$ such that $\Phi(x) \in (2An-2A, 2An]$. Then $\Psi(x)$ is either equal to $n-1$ or to n. So $|\Phi(x) - 2A\Psi(x)| \leqslant 2A = 4K'\nu$. □

Before stating the next lemma, we remark that, given any collection $\mathcal{H}$ of disjoint hyperplanes in a finite CAT(0) cube complex Q, there is an associated map $Q^{(0)} \to \mathbb{Z}$: the cube complex dual to $\mathcal{H}$ is a finite interval of $\mathbb{Z}$, and each vertex of Q determines a consistent orientation of the hyperplanes in $\mathcal{H}$. This is a special case of the *restriction quotient* described in [CS11], and it is clearly a median map. Conversely, any 0–contraction on Q can be realised as restriction quotient in this manner. Moreover, after a translation of $\mathbb{Z}$, we may assume that the codomain is contained in $\mathbb{N}$ if it is bounded.

Lemma 2.17. *Let* Q *be a finite CAT(0) cube complex of dimension at most* ν*. Let* $\mathcal{C}$ *be a family of* 0*–contractions on* Q*, i.e. each* $\Psi \in \mathcal{C}$ *is a map* $Q^{(0)} \to \mathbb{N}$ *given by a family* $H_{\Psi,1}, \ldots, H_{\Psi,n_\Psi}$ *of ordered disjoint hyperplanes of* Q*. Let* $\sigma_{\mathcal{C}}$ *denote the pseudometric on* $Q^{(0)}$ *defined by*

$$\forall x, y \in Q^{(0)}, \sigma_{\mathcal{C}}(x,y) = \sup_{\Psi \in \mathcal{C}} |\Psi(x) - \Psi(y)|.$$

Then for each $a, b \in Q^{(0)}$ *and for each integer* $0 \leqslant r \leqslant \sigma_{\mathcal{C}}(a,b)$*, there is a vertex* $c \in [a,b]$ *and contractions* $\Psi_1, \Psi_2 \in \mathcal{C}$ *such that the following hold.*

(1) $\sigma_{\mathcal{C}}(a,c) = r$*,*
(2) $\sigma_{\mathcal{C}}(a,c) = |\Psi_1(a) - \Psi_1(c)|$ *and* $\sigma_{\mathcal{C}}(c,b) = |\Psi_2(c) - \Psi_2(b)|$*,*
(3) for each hyperplane H_1 *defining* Ψ_1 *that separates* a *and* c*, and for each hyperplane* H_2 *defining* Ψ_2 *that separates* c *and* b*, the hyperplanes* H_1 *and* H_2 *are disjoint.*

Proof. Fix $a, b \in Q^{(0)}$ and an integer $0 < r < \sigma_{\mathcal{C}}(a,b)$. Since $\sigma_{\mathcal{C}}$ is 1–Lipschitz with respect to the combinatorial distance on $Q^{(0)}$, we know that there exists $c \in [a,b]$ such that $\sigma_{\mathcal{C}}(a,c) = r$. Among all possible choices, choose such c as far away from a as possible, in the sense that:

$$\text{if } c' \in [a,b] \text{ has } \sigma_{\mathcal{C}}(a,c') = r \text{ and } c \in [a,c'], \text{ then } c' = c.$$

Let $\Psi_2 \in \mathcal{C}$ such that $\sigma_{\mathcal{C}}(c,b) = |\Psi_2(c) - \Psi_2(b)|$. Let $H_{2,1}, \ldots, H_{2,n_2}$ be the ordered disjoint hyperplanes defining Ψ_2 separating c and b, numbered from c to b.

Let H be a hyperplane of Q adjacent to c and either equal to $H_{2,1}$ or separating c from $H_{2,1}$, and let $c' \in [a,b]$ be the vertex adjacent to c such that H crosses the edge

$[c,c']$. First note that, since $H_{2,1}$ separates c and b, we deduce that H separates c and b. Thus H does not separate a and c, because $c \in [a,b]$. In particular, $c \in [a,c']$. Since c is chosen as far from a as possible among points at $\sigma_{\mathcal{C}}$–distance equal to r, and every hyperplane separating a and c separates a and c', we deduce that $\sigma_{\mathcal{C}}(a,c') > \sigma_{\mathcal{C}}(a,c) = r$, so $\sigma_{\mathcal{C}}(a,c') = \sigma_{\mathcal{C}}(a,c)+1$.

Let $\Psi_1 \in \mathcal{C}$ such that $\sigma_{\mathcal{C}}(a,c') = |\Psi_1(a) - \Psi_1(c')|$. Let $H_{1,1},\dots,H_{1,n_1}$ be the ordered disjoint hyperplanes defining Ψ_1 separating a and c', numbered from a to c'. Since $\sigma_{\mathcal{C}}(a,c') = \sigma_{\mathcal{C}}(a,c)+1$, we know that $H = H_{1,n_1}$ and that $\sigma_{\mathcal{C}}(a,c) = |\Psi_1(a) - \Psi_1(c)|$.

In particular, H is disjoint from $H_{1,1},\dots,H_{1,n_1-1}$. So we deduce that H separates $H_{1,1},\dots,H_{1,n_1-1}$ from $H_{2,1},\dots,H_{2,n_2}$. We deduce the desired conclusion: for any hyperplane $H_1 = H_{1,i}$ (for some $1 \leqslant i \leqslant n_1-1$) defining Ψ_1 separating a and c and for each hyperplane $H_2 = H_{2,j}$ (for some $1 \leqslant j \leqslant n_2$) defining Ψ_2 separating c and b, the hyperplanes H_1 and H_2 are disjoint. $\square$

We can now use these lemmas to prove that, in the setting of Theorem 2.10, the metric σ is weakly roughly geodesic.

Proposition 2.18. *The metric σ is weakly roughly geodesic. More precisely, there exists a constant C'_σ such that, for any $a,b \in X$ and $0 \leqslant r \leqslant \sigma(a,b)$, there is some $c \in [a,b]$ such that $|\sigma(a,c) - r| \leqslant C'_\sigma$ and $\sigma(a,c) + \sigma(c,b) \leqslant \sigma(a,b) + C'_\sigma$.*

Proof. According to Proposition 2.13, there exists a constant $K_\sigma \geqslant 1$ such that d and σ are (K_σ, K_σ)–quasi-isometric.

Since X has quasicubical intervals, there exists a finite CAT(0) cube complex Q (with the ℓ^1 metric) of dimension at most ν, and a map $\lambda : Q \to [a,b]$ that is a (κ,κ)–quasi-isometric embedding and a κ–quasi-median homomorphism.

For each K–contraction $\Phi : X \to \mathbb{R}$, the composition $\Phi \circ \lambda : Q \to \mathbb{R}$ is a K'–quasi-median, (K',K')–coarsely Lipschitz map, where $K' = K + \kappa$. According to Lemma 2.16, there exists a 0–contraction $\Psi : Q \to \mathbb{Z}$ such that $|\Phi \circ \lambda(x) - 4K'\nu\Psi(x)| \leqslant 4K'\nu$ for all $x \in Q^{(0)}$. Let $\mathcal{C}$ denote the set of all 0–contractions $\Psi : Q \to \mathbb{Z}$ such that there is some K–contraction $\Phi : X \to \mathbb{Z}$ with $|\Phi \circ \lambda(x) - 4K'\nu\Psi(x)| \leqslant 4K'\nu$ for all $x \in Q^{(0)}$.

We shall prove that σ is weakly roughly geodesic, with constant

$$C'_\sigma = 48K'\nu + 2K_\sigma(\kappa+1) + 2\kappa + 2K + 2D,$$

where D is the constant from Lemma 2.14.

Let $\alpha,\beta \in Q$ such that $d(\lambda(\alpha),a) \leqslant \kappa$ and $d(\lambda(\beta),b) \leqslant \kappa$. Then $\sigma(\lambda(\alpha),a) \leqslant K_\sigma(\kappa+1)$ and $\sigma(\lambda(\beta),b) \leqslant K_\sigma(\kappa+1)$. Lemma 2.17 applied to α, β, the family $\mathcal{C}$, and $r' = \lfloor \frac{r}{4K'\nu} \rfloor$ provides a vertex $\gamma \in [\alpha,\beta]$ and 0–contractions $\Psi_1,\Psi_2 \in \mathcal{C}$. Let $c = \lambda(\gamma) \in [a,b]$.

Let us start by computing $\sigma(a,c)$. By definition of the set $\mathcal{C}$, we have $|\sigma(\lambda(\alpha),\lambda(\gamma)) - 4K'\nu\sigma_{\mathcal{C}}(\alpha,\gamma)| \leqslant 4K'\nu$. By the choice of γ, we have $\sigma_{\mathcal{C}}(\alpha,\gamma) = r'$. Thus $|\sigma(a,c) - r| \leqslant K_\sigma(\kappa+1) + 8K'\nu \leqslant C'_\sigma$.

The aim for the rest of the proof is to confirm the second restriction on c. The first step is to apply Lemma 2.14.

Recall that $\Psi_1,\Psi_2 \in \mathcal{C}$ are the 0–contractions provided by Lemma 2.17: they are such that $\sigma_{\mathcal{C}}(\alpha,\gamma) = |\Psi_1(\alpha) - \Psi_1(\gamma)| = r'$ and $\sigma_{\mathcal{C}}(\gamma,\beta) = |\Psi_2(\gamma) - \Psi_2(\beta)| = s'$. After translations of $\mathbb{Z}$, we may also assume that $\Psi_1(\alpha) = 0$, $\Psi_1(\gamma) = \Psi_2(\gamma) = r'$, and $\Psi_2(\beta) = r' + s'$. By definition of $\mathcal{C}$, there exist K–contractions Φ_1 and Φ_2 on X such that $|\Phi_1 \circ \lambda(x) - 4K'\nu\Psi_1(x)| \leqslant 4K'\nu$ and $|\Phi_2 \circ \lambda(x) - 4K'\nu\Psi_2(x)| \leqslant 4K'\nu$ for all $x \in Q^{(0)}$. In particular, $\Phi_1(a) \leqslant 4K'\nu$ and $\Phi_2(b) \geqslant r + 4K'\nu s' - 8K'\nu$.

Let $t = \kappa + K + 12K'\nu$. We shall prove that the subspaces $Z_1 = \{x \in [a, b] \mid \Phi_1(x) \leqslant r - t\}$ and $Z_2 = \{x \in [a, b] \mid \Phi_2(x) \geqslant r + t\}$ are disjoint. Fix $x \in [a, b]$, and pick any $\xi \in Q^{(0)}$ such that $d(\lambda(\xi), x) \leqslant \kappa$.

Suppose that $\Phi_1(x) \leqslant r - t$. Then $\Phi_1(\lambda(\xi)) \leqslant r - t + \kappa + K$, so $\Psi_1(\xi) \leqslant \frac{r-t+\kappa+K+4K'\nu}{4K'\nu} \leqslant \frac{r-8K'\nu}{4K'\nu} \leqslant r' - 1$. Similarly, if $\Phi_2(x) \geqslant r + t$, then $\Psi_2(\xi) \geqslant r' + 1$.

According to Property (3) of Lemma 2.17, if $\xi \in Q^{(0)}$ then we cannot simultaneously have both $\Psi_1(\xi) \leqslant r' - 1$ and $\Psi_2(\xi) \geqslant r' + 1$. As a consequence, we cannot have both $x \in Z_1$ and $x \in Z_2$. This implies that $Z_1 \cap Z_2 = \varnothing$.

According to Lemma 2.14, we deduce that $\sigma(a, b) \geqslant (r - 4K'\nu) + (4K'\nu s' - 8K'\nu) - 2t - 2D$. Moreover, recall that $s' = \sigma_{\mathcal{C}}(\gamma, \beta)$, that $c = \lambda(\gamma)$, and that $\sigma(b, \lambda(\beta)) \leqslant K_\sigma(\kappa + 1)$. Thus, by definition of $\mathcal{C}$, we have

$$\begin{aligned} |\sigma(c, b) - 4K'\nu s'| \leqslant& |\sigma(\lambda(\gamma), \lambda(\beta)) - 4K'\nu\sigma_{\mathcal{C}}(\gamma, \beta)| + K_\sigma(\kappa + 1) \\ \leqslant& 4K'\nu + K_\sigma(\kappa + 1). \end{aligned}$$

To sum up, $\sigma(a, b)$ is bounded below by

$$\begin{aligned} &\geqslant r + 4K'\nu s' - 12K'\nu - 2t - 2D \\ &\geqslant \big(\sigma(a, c) - K_\sigma(\kappa + 1) - 8K'\nu\big) + \big(\sigma(c, b) - 4K'\nu - K_\sigma(\kappa + 1)\big) - 12K'\nu - 2t - 2D \\ &= \sigma(a, c) + \sigma(c, b) - 24K'\nu - 2K_\sigma(\kappa + 1) - 2(\kappa + K + 12K'\nu) - 2D \\ &= \sigma(a, c) + \sigma(c, b) - C'_\sigma. \end{aligned}$$

We have proved that σ is C'_σ–weakly roughly geodesic. □

2.5. **Coarse convexity of balls.**

To complete the proof of Theorem 2.10, it remains to show that balls in (X, σ) are uniformly coarsely median-convex.

Lemma 2.19. *There is a constant $\epsilon \geqslant 0$ such that for any $x, y, z \in X$ with $x \in [y, z]$, we have $d(x, \mu(x, y, z)) \leqslant \epsilon$.*

Proof. According to [Bow19b, Lemma 8.1], there are constants r_0 and r'_0 such that x lies at distance at most r'_0 from a point x' with $d(x', \mu(x', y, z)) \leqslant r_0$. The lemma follows from the fact that the median operation is coarsely Lipschitz. □

Lemma 2.20. *There is a constant M such that each ball in (X, σ) is M–coarsely median-convex.*

Proof. Fix $w \in X$ and $R \geqslant 0$. Let $y, z \in B_\sigma(w, R)$. Given any $a \in X$, we want to bound the distance from $x = \mu(a, y, z)$ to $B_\sigma(w, R)$.

Let $r < \sigma(w, x)$, and let $\Phi : X \to [0, r]$ be a K–contraction such that $\Phi(w) = 0$ and $\Phi(x) = r$. Since Φ is a K–quasi-median homomorphism, we have $|\Phi(\mu(x, y, z)) -$

$\mu_{\mathbb{R}}(\Phi(x), \Phi(y), \Phi(z))| \leqslant K$. Lemma 2.19 tells us that $d(\mu(x,y,z),x) \leqslant \epsilon$, and so

$$\begin{aligned} |\Phi(\mu(x,y,z)) - r| &= |\Phi(\mu(x,y,z)) - \Phi(x)| \\ &\leqslant d(\mu(x,y,z),x) + K \\ &\leqslant \epsilon + K. \end{aligned}$$

This means that one of $\Phi(y)$ and $\Phi(z)$ must be at least $r - \epsilon - 2K$, and so $\sigma(w,x) \leqslant \max\{\sigma(w,y), \sigma(w,z)\} + \epsilon + 2K$.

This proves that $x \in B_\sigma(w, R + \epsilon + 2K)$. Since σ is C'_σ-weakly roughly geodesic, it follows that $x = \mu(a,y,z)$ is uniformly close to some point in $B_\sigma(w,R)$, so balls in (X,σ) are M–coarsely median-convex, where $M = \varepsilon + 2K + 3C'_\sigma$. □

3. Quasiconvexity and the coarse Helly property in HHSs

The goal of this section is to prove that *hierarchically quasiconvex* subsets of hierarchically hyperbolic spaces satisfy the coarse Helly property. Since coarsely median-convex subsets of a hierarchically hyperbolic space are hierarchically quasiconvex [RST18, Proposition 5.11], this applies in particular to balls for the metric σ constructed in Section 2, by Theorem 2.10, allowing us to deduce Theorem A.

The coarse Helly property for hierarchically quasiconvex subsets is interesting in its own right; we give more discussion in Section 3.4, where we also deduce the bounded packing property for hierarchically quasiconvex subgroups of groups that are HHSs.

3.1. **Background on hierarchical hyperbolicity.**

Here we give a description of hierarchically hyperbolic spaces (HHSs) and hierarchically hyperbolic groups (HHGs). For full definitions, see [BHS19, Def. 1.1, 1.21]. Briefly, an HHS consists of a quasigeodesic space (X,d), a constant E, and a set $\mathfrak{S}$, elements of which are called *domains*. Each domain U has an associated E–hyperbolic space $\mathcal{C}U$, and the various axioms give structure for extracting information about X from these hyperbolic spaces. This includes:

- Each domain U has an associated E–coarsely onto, (E,E)–coarsely Lipschitz *projection* map $\pi_U : X \to \mathcal{C}U$.
- $\mathfrak{S}$ has a partial order $\sqsubset$, called nesting, and a symmetric relation $\perp$, called orthogonality. If $U \sqsubset V$ and $V \perp W$, then $U \perp W$. The relations $\subsetneq$, $\perp$, and $=$ are mutually exclusive, and their complement, denoted $\pitchfork$, is called transversality.
- There is a bound on the size of $\subsetneq$–chains and pairwise orthogonal sets.
- If $U \subsetneq V$ or $U \pitchfork V$ then there is a set $\rho^U_V \subset \mathcal{C}V$ of diameter at most E.
- If $U \subsetneq V$ then there is also a map $\rho^V_U : \mathcal{C}V \to \mathcal{C}U$. If $\gamma \subset \mathcal{C}V$ is a geodesic and $d_{\mathcal{C}V}(\gamma, \rho^U_V) > E$, then $\operatorname{diam} \rho^V_U(\gamma) \leqslant E$.

This last point is referred to as *bounded geodesic image*. For $x, y \in X$, it is standard to write $d_U(x,y)$ in place of $d_{\mathcal{C}U}(\pi_U(x), \pi_U(y))$, and similarly for subsets of X. Moreover, we can always assume that X and the associated hyperbolic spaces are graphs (for example by [CdlH16, Lemma 3.B.6]). In particular, we can and shall assume that X and the $\mathcal{C}U$ are geodesic.

We say that X admits an HHS structure if there is an HHS whose underlying metric space is X, and we write $(X,\mathfrak{S})$ as shorthand for the entirety of a choice of HHS structure.

An HHG is a finitely generated group G whose Cayley graph admits an HHS structure $(G,\mathfrak{S})$ such that G acts cofinitely on $\mathfrak{S}$ and elements of G induce isometries $\mathcal{C}U \to \mathcal{C}gU$ for all $U \in \mathfrak{S}$. (There are a couple of other natural regulatory assumptions that we shall not concern ourselves with here.)

The idea behind two domains being orthogonal is that one can see a direct product of associated sub-HHSs inside X. This is made precise by the *partial realisation* axiom.

Axiom (Partial realisation)**.** *If $\{U_i\}$ is a set of pairwise orthogonal domains, then for any choice of points $p_i \in \mathcal{C}U_i$, there is some $x \in X$ with $d_{U_i}(x,p_i) \leqslant E$ for all i, and with $d_V(x,\rho_V^{U_i}) \leqslant E$ whenever $U_i \subsetneq V$ or $U_i \pitchfork V$.*

In fact, one of the main tools for dealing with HHSs is the *realisation theorem* [BHS19, Theorem 3.1], which extends the partial realisation axiom. Roughly, it says that any *consistent tuple* is well-approximated by the projections of some point in X. In other words, performing constructions in X can be reduced to performing constructions in the associated hyperbolic spaces and checking that the points produced by this process are consistent.

Definition 3.1 (Consistent tuple)**.** For a constant $\kappa \geqslant E$, a tuple $(b_U) \in \prod_{U\in\mathfrak{S}} \mathcal{C}U$ is said to be κ–consistent if

$$\min\left\{d_U(b_U,\rho_U^V), d_V(b_V,\rho_V^U)\right\} \leqslant \kappa \quad \text{whenever } U \pitchfork V, \text{ and}$$

$$\min\left\{d_V(b_V,\rho_V^U), \operatorname{diam}(b_U \cup \rho_U^V(b_V))\right\} \leqslant \kappa \quad \text{whenever } U \subsetneq V.$$

Axiom (Consistency)**.** *For any $x \in X$, the tuple $(\pi_U(x))_{U\in\mathfrak{S}}$ is E–consistent.*

It will be useful to be able to talk about consistency for subsets of $\mathfrak{S}$. Given $u \in \mathcal{C}U$ and $v \in \mathcal{C}V$, we say that u and v *satisfy the consistency inequalities* for U and V if

- $U \pitchfork V$ and $\min\left\{d_U(u,\rho_U^V), d_V(v,\rho_V^U)\right\} \leqslant E$, or
- (after relabelling) $U \subsetneq V$ and $\min\left\{d_V(v,\rho_V^U), \operatorname{diam}(\{u\} \cup \rho_U^V(v))\right\}$.

Let us now state the realisation theorem, which will be the mechanism for our proof of Theorem 3.5. We shall only need the existence part.

Theorem 3.2 (Realisation, [BHS19, Theorem 3.1])**.** *For each $\kappa \geqslant E$, there are numbers $\theta_e(\kappa)$ and $\theta_u(\kappa)$ such that, if $(b_U)_{U\in\mathfrak{S}}$ is a κ–consistent tuple, then there is some $x \in X$ with $d_U(x,b_U) \leqslant \theta_e(\kappa)$ for all domains U. Moreover, the set of such x has diameter at most $\theta_u(\kappa)$.*

A key application of the realisation theorem is for the construction of a coarse median operation for HHSs. Given three points x,y,z in an HHS $(X,\mathfrak{S})$, let $(m_U)_{U\in\mathfrak{S}}$ be the tuple whose U–entry is (a point in) the median of the triple $\pi_U(x),\pi_U(y),\pi_U(z)$ in the hyperbolic space $\mathcal{C}U$. This tuple is consistent [BHS19, Theorem 7.3], so we define the median to be a point obtained by applying the realisation theorem to the tuple (m_U). (One also needs a proposition of Bowditch [Bow13, Proposition 10.1] to conclude that (X,μ,d) is a coarse median space.) When X is an HHG, this can be arranged to be equivariant, because elements of X induce isometries $\mathcal{C}U \to \mathcal{C}xU$ that interact well with the HHS structure.

The action on the index set is what distinguishes HHGs from groups that are HHSs, and this turns out to be an important distinction. For example, the property of being an HHS is invariant under quasi-isometries, but there are groups that are virtually HHGs but not HHGs themselves. Indeed, the $(3,3,3)$ triangle group is virtually abelian, but, as mentioned in the introduction, it is not coarsely Helly [Hod20a], and it therefore cannot be an HHG by Corollary H. A more direct proof not relying on the results of this paper is given in [PS20]. Conversely, any group that is an HHS can be equipped with a coarse median [BHS19], but this is may fail to be equivariant if the structure is only an HHS structure.

A related notion is that of a group that acts on an HHS $(X, \mathfrak{S})$ by *HHS automorphisms.* In other words, it acts on X isometrically, and on $\mathfrak{S}$ with the regulatory assumptions alluded to above, but the action on $\mathfrak{S}$ need not be cofinite. The median is still equivariant for such actions. The class of groups acting on HHSs by HHS automorphisms strictly contains the class of HHGs.

3.2. **Quasiconvexity and examples.**

In the theory of hyperbolic spaces, an important class of subsets are the quasiconvex subsets, because they inherit the structure of the ambient space. There is a natural analogue in the setting of hierarchical hyperbolicity, namely the *hierarchically quasiconvex* subsets. Just as in the hyperbolic setting, these inherit the structure of the ambient space, which in general has rather more data attached to it than a hyperbolic space.

Definition 3.3 (Hierarchical quasiconvexity)**.** A subset Y of an HHS $(X, \mathfrak{S})$ is said to be hierarchically quasiconvex if there is a function k such that: every $\pi_U(Y)$ is $k(0)$–quasiconvex; and if $x \in X$ has $d_U(x, Y) \leqslant r$ for all $U \in \mathfrak{S}$, then $d_X(x, Y) \leqslant k(r)$.

We now give examples of hierarchically hyperbolic spaces and groups, and their hierarchically quasiconvex subsets.

All hyperbolic groups are hierarchically hyperbolic, as are mapping class groups of finite type surfaces [BHS17b]; Teichmüller space with either of the standard metrics [BHS17b]; many graphs associated to curves on surfaces, including the pants graph [Vok17]; quotients of the mapping class group by powers of pseudo-Anosovs [BHS17a] and Dehn-twist subgroups [BHMS20]; extensions of Veech groups [DDLS20]; the genus-two handlebody group [Mil20]; fundamental groups of closed 3–manifolds without Nil or Sol components [BHS19]; all right angled Artin groups [BHS17b]; and in fact all known cubical groups [HS20]. Aside from the extensions of Veech groups and some 3–manifold groups, the groups listed here are all known to be HHGs, not merely HHSs.

There are also various ways to combine HHSs and HHGs to produce new ones. For example, any direct product, or more generally graph product, of HHGs is an HHG [BR20]; many graphs of groups are HHGs [BHS19, BR18, RS20]; and both classes are closed under relative hyperbolicity [BHS19].

There are some standard constructions to produce hierarchically quasiconvex subsets, such as taking the median interval between a pair of points, or more generally the *hull* of a finite set, or taking the *standard product region* of a domain, but we shall not detail these here: see [BHS19]. Also, for many groups that are HHSs (including all HHGs), every *stable* subgroup is hierarchically quasiconvex [ABD17, RST18].

Let us now give some more specific examples. In general, if a space admits an HHS structure, then it admits many different ones. In the following list, when we say that Y is a hierarchically quasiconvex subset of a metric space X, we mean that there is an HHS $(X, \mathfrak{S})$ of which Y is a hierarchically quasiconvex subset.

- In the case of the mapping class group with the standard HHS structure, the hull of a finite set is exactly the Σ–hull introduced in [BKMM12]. These have been used to describe top-dimensional flats in the asymptotic cone [BKMM12], and to prove the rapid decay property [BM11].
- If S is a surface of finite type, then the stabiliser of an embedded multicurve is a hierarchically quasiconvex subgroup of $\mathrm{MCG}(S)$. These contain, at finite index, subsurface stabilisers, which are therefore hierarchically quasiconvex as well.
- Let X be the pants graph of a finite type surface S, and let S' be a subsurface. Fix a pants decomposition P of $S \smallsetminus S'$, and let Y be the subgraph of X consisting of all pants decompositions of S that restrict to P. Then Y is hierarchically quasiconvex in X.
- If X is a CAT(0) cube complex that is an HHS, then any convex subcomplex of X is hierarchically quasiconvex. In this case, Theorem 3.5 partially recovers and extends [HW09, Corollary 3.6].
- If M is a closed 3–manifold without Nil or Sol components in its prime decomposition, then $\pi_1(M)$ is an HHS, and the cut tori in its geometric decomposition give hierarchically quasiconvex subgroups.
- Any vertex group of a graph product of HHGs is hierarchically quasiconvex. More generally, so is any graphical subgroup of a graph product of groups that are HHSs.

3.3. **The coarse Helly property.**

Here we prove our result on hierarchically quasiconvex subsets of an HHS and deduce that HHSs are coarsely Helly when equipped with the metric σ from Section 2. We then deduce that every HHG acts properly cocompactly by isometries on a coarsely Helly space.

We shall make use of the following powerful result for hyperbolic spaces. The version stated here is a combination of [CDV17, Lemma 5.1] and the proof of [CDV17, Theorem 5.1]. Throughout this section, we say that subsets Z_1 and Z_2 of a metric space (X, d) are r–close if there exist $z_1 \in Z_1$ and $z_2 \in Z_2$ with $d(z_1, z_2) \leqslant r$.

Theorem 3.4 ([CDV17])**.** *Let Y be an E–hyperbolic graph and let y be any vertex of Y. Let $\mathcal{Q}$ be a collection of pairwise $2Er$–close k_0–quasiconvex subsets of Y^0 with the property that $\{d(y, Q) : Q \in \mathcal{Q}\}$ is bounded. By discreteness, we can fix $Q \in \mathcal{Q}$ with $d(y, Q)$ maximal. Let $z \in Q$ have $d(y, z) = d(y, Q)$, and let c be the point on a geodesic $[y, z]$ with $d(c, z) = \min\{Er, d(y, z)\}$. Then $d(c, Q') \leqslant r'$ for all $Q' \in \mathcal{Q}$, where $r' = \max\{2k_0 + 5E, Er + k_0 + 3E\}$.*

The strength of this theorem is twofold. Firstly, the constant r' is independent of the size of the set $\mathcal{Q}$—a statement with this independence does not seem to appear elsewhere in the geometric group theory literature. The second strength is in the construction of the point c: it has a lot of flexibility, as the restriction on the input point y is fairly weak, and it also is completely explicit.

Theorem 3.5 (Coarse Helly property). *Let $(X, \mathfrak{S})$ be an HHS with constant E, and let $\mathcal{Q}$ be a collection of k–hierarchically quasiconvex subsets of X such that either $\mathcal{Q}$ is finite or $\mathcal{Q}$ contains a bounded element. Suppose that there is a constant r such that any two elements of $\mathcal{Q}$ are r–close. Then there is a constant $R = R(E, k, r)$ such that there is a point $x \in X$ with $d_X(x, Q) \leqslant R$ for all $Q \in \mathcal{Q}$.*

Proof. Let us say that a domain U *begets* a domain V if either $U \pitchfork V$ or $U \subsetneq V$. If U begets V then there is a well-defined bounded set ρ^U_V.

Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a maximal collection of pairwise orthogonal, nest-minimal domains. For any domain $V \in \mathfrak{S} \smallsetminus \mathcal{U}$, there is some i such that U_i begets V. By [DHS17, Lemma 1.5], for any domain $V \in \mathfrak{S}$ we have $d_V(\rho^{U_i}_V, \rho^{U_j}_V) \leqslant 2E$ whenever U_i and U_j both beget V. Moreover, recall that $\operatorname{diam} \rho^{U_i}_V \leqslant E$. At the cost of increasing the hierarchical hyperbolicity constant to at most $10E$, we can therefore perturb the HHS structure to assume that every $\rho^{U_i}_V$ is a singleton, and that $\rho^{U_i}_V = \rho^{U_j}_V$ whenever both U_i and U_j beget V. We write $\rho^{\mathcal{U}}_V$ for the singleton

$$\rho^{\mathcal{U}}_V = \bigcup_{\{i \,:\, U_i \text{ begets } V\}} \rho^{U_i}_V .$$

As mentioned, the construction of $\mathcal{U}$ ensures that the point $\rho^{\mathcal{U}}_V$ exists for all $V \in \mathfrak{S} \smallsetminus \mathcal{U}$.

We are free to assume that if $r > 0$ then $r > 1$. Thus, by definition of hierarchical quasiconvexity and the fact that projection maps are (E, E)–coarsely Lipschitz, we have that, for any domain V, the sets $\pi_V(Q)$ are pairwise $2Er$–close and k_0–quasiconvex, where $k_0 = k(0)$. Let r' be as in the statement of Theorem 3.4. That theorem now allows us to choose, for each $U \in \mathcal{U}$, a point b_U in $\mathcal{C}U$ with $d_U(b_U, Q) \leqslant r'$ for all $Q \in \mathcal{Q}$. For any other domain V, let b_V be a point of $\mathcal{C}V$ obtained by applying Theorem 3.4 in the hyperbolic graph $\mathcal{C}V$, with quasiconvex subsets $\{\pi_V(Q) : Q \in \mathcal{Q}\}$ and starting vertex $\rho^{\mathcal{U}}_V$.

Claim: The tuple $(b_V)_{V \in \mathfrak{S}}$ is $(r' + 7E)$–consistent.

Proof of Claim: Suppose that W begets V, and that $d_V(\rho^W_V, \rho^{\mathcal{U}}_V) \leqslant 2E$. By the construction of b_V, if $d_V(b_V, \rho^W_V) > k_0 + 3E$ (in particular if it is greater than r'), then there is some $Q \in \mathcal{Q}$ such that $d_V(Q, \rho^W_V) > k_0 + 3E + Er - \operatorname{diam} \rho^W_V$, which is at least $Er + k_0 + E$. If $W \pitchfork V$, then $\pi_W(Q)$ is contained in the E–neighbourhood of ρ^V_W by consistency for elements of Q. In particular, $d_W(\rho^V_W, b_W) \leqslant r' + E$ as b_W is r'–close to $\pi_W(Q)$. If $W \subsetneq V$, then since $\pi_V(Q)$ is k_0–quasiconvex, bounded geodesic image and consistency show that the set $\rho^V_W \pi_V(Q)$ has diameter at most E, and its E–neighbourhood contains $\pi_W(Q)$. Moreover, its E–neighbourhood contains $\rho^V_W(b_V)$ by bounded geodesic image, as witnessed by the geodesic used to construct b_V. Thus

$$\begin{aligned}\operatorname{diam}(b_W \cup \rho^V_W(b_V)) &\leqslant d_W(b_W, Q) + \operatorname{diam} \pi_W(Q) + d_W(Q, \rho^V_W(b_V)) + \operatorname{diam} \rho^V_W(b_V) \\ &\leqslant r' + 3E + 3E + E = r' + 7E.\end{aligned}$$

The above paragraph will be referred to as $(*)$ for the rest of the proof of the claim. We split the checking of the consistency inequalities for pairs (V, W) of domains into three cases.

<u>Case 1</u>. $W \in \mathcal{U}$ begets V.

In this case, $\rho^W_V = \rho^{\mathcal{U}}_V$, so we are done by $(*)$.

Case 2. There is some $U \in \mathcal{U}$ that begets both V and W.

Proposition 1.8 of [BHS19] states that if W begets V then ρ^U_V and ρ^U_W satisfy the consistency inequalities for V and W. Consequently, by $(*)$, the only case we need to check here is when $U \pitchfork W$, $W \subsetneq V$, and $\operatorname{diam}(\rho^U_W \cup \rho^V_W(\rho^U_V)) \leqslant 2E$. Assuming that $d_V(\rho^W_V, b_V) > r' + E$, there are two possibilities, depending on the location of ρ^U_V.

If there is a geodesic $[\rho^U_V, b_V]$ that is disjoint from the E–neighbourhood of ρ^W_V, then $\operatorname{diam}(\rho^V_W(\rho^U_V) \cup \rho^V_W(b_V)) \leqslant E$, so $d_W(\rho^U_W, \rho^V_W(b_V)) \leqslant 3E$. Moreover, for each $Q \in \mathcal{Q}$ there is some $q \in Q$ such that any geodesic $[b_V, \pi_V(q)]$ is disjoint from the E–neighbourhood of ρ^W_V. In particular, $\rho^V_W(b_V)$ is $2E$–close to each $\pi_W(q)$, and hence ρ^U_W is $5E$–close to each $\pi_W(Q)$. Since b_W lies on a shortest geodesic between ρ^U_W and some $\pi_W(Q)$, we get that $d_W(b_W, \rho^U_W) \leqslant 5E$, and so b_W is $8E$–close to $\rho^V_W(b_V)$.

Otherwise, every geodesic $[\rho^U_V, b_V]$ meets the E–neighbourhood of ρ^W_V. By construction of b_V, there exists $Q \in \mathcal{Q}$ such that $d_V(\rho^W_V, Q) > r' + Er$. By the same argument as in $(*)$, we now get that $\rho^V_W(b_V)$ is $3E$–close to $\pi_W(Q)$, which has diameter at most $3E$. Hence $\operatorname{diam}(b_W \cup \rho^V_W(b_V)) \leqslant r' + 7E$.

Case 3. No U_i begets both V and W, and neither V nor W is in $\mathcal{U}$.

After relabelling we can assume that U_1 begets V and U_2 begets W. Since U_1 does not beget W we have $U_1 \perp W$, and similarly $U_2 \perp V$. In particular, the only case that needs checking is when $V \pitchfork W$. The partial realisation axiom provides a point $z \in X$ such that $d_V(z, \rho^{U_1}_V) \leqslant E$ and $d_W(z, \rho^{U_2}_W) \leqslant E$. By consistency for z, we have that either $d_V(\rho^W_V, \rho^{U_1}_V) \leqslant 2E$ or $d_W(\rho^V_W, \rho^{U_2}_W) \leqslant 2E$. We are done by $(*)$. $\diamondsuit$

In light of the claim, Theorem 3.2 provides a point $x \in X$ such that $d_V(x, b_V) \leqslant \theta_e(r' + 7E)$ for all $V \in \mathfrak{S}$. By construction of the points b_V, we have that $d_V(x, Q) \leqslant r' + \theta_e(r' + 7E)$ for all $Q \in \mathcal{Q}$. Hierarchical quasiconvexity of the Q now gives that $d_X(x, Q) \leqslant k(r' + \theta_e(r' + 7E))$ for all $Q \in \mathcal{Q}$. $\square$

It is worth noting that the proof of Theorem 3.5 gives flexibility of a similar kind to that in Theorem 3.4. Indeed, we are free in our choice of $\mathcal{U}$, and once this is chosen we apply the Chepoi–Dragan–Vaxès construction in each of the hyperbolic spaces associated to $\mathcal{U}$, without restriction on the choice of starting point therein. We shall not need to make use of this in the present paper.

Corollary 3.6. *If X is an HHS, then (X, σ) is coarsely Helly, hence roughly geodesic.*

Proof. By Proposition 2.9, the coarse median space (X, μ, d) has quasicubical intervals, so Theorem 2.10 tells us that the metric σ is weakly roughly geodesic on X, that it is quasi-isometric to d, and that σ–balls are coarsely median–convex. Let $\{B_\sigma(x_i, r_i) : i \in I\}$ be a family of balls in (X, σ) with the property that $\sigma(x_i, x_j) \leqslant r_i + r_j$ for all $i, j \in I$. Since σ is weakly roughly geodesic, there is a constant δ, independent of the family of balls, such that the balls $B_\sigma(x_i, r_i + \delta)$ intersect pairwise.

Let B_i be the image of the ball $B_\sigma(x_i, r_i + \delta)$ under the identity quasi-isometry $(X, \sigma) \to (X, d)$. The B_i are uniformly coarsely median-convex, and so they are uniformly hierarchically quasiconvex by [RST18, Proposition 5.11]. They also intersect pairwise, and each is bounded, so Theorem 3.5 produces a point at uniformly bounded

d–distance from each B_i. As d and σ are quasi-isometric, this point is at uniformly bounded σ–distance from each $B_\sigma(x_i, r_i + \delta)$. Thus (X, σ) is coarsely Helly.

Since any injective space is geodesic, we deduce that the coarsely Helly metric space (X, σ) is not merely weakly roughly geodesic, but actually roughly geodesic, as it is coarsely dense in its injective hull. □

Usually it really is necessary to change the metric: [CCG+20, Ex. 5.13] shows that $\mathbb{Z}^3$ with the standard metric is not coarsely Helly, though it is an HHS.

We now explain how to deduce the existence of a bicombing from work of Lang. We say that a bicombing is *roughly reversible* if it satisfies the following coarse version of symmetry.

$$\exists C \geqslant 0, \forall a, b \in X, \forall t \in [0,1], \sigma(\gamma_{a,b}(t), \gamma_{b,a}(1-t)) \leqslant C.$$

See Section 1.3 for the definition of a roughly conical bicombing.

Corollary 3.7. *If $(X, \mathfrak{S})$ is an HHS, then (X, σ) admits a roughly conical, roughly reversible, bicombing by rough geodesics that is coarsely equivariant under the automorphism group of $(X, \mathfrak{S})$.*

Proof. According to Corollary 3.6, the metric space (X, σ) is coarsely Helly, so it is D–coarsely dense in its injective hull for some δ. A construction of Lang shows that every injective metric space E admits a conical, reversible, geodesic, $\operatorname{Isom} E$–invariant bicombing γ' [Lan13]. Take $E = E((X, \sigma))$. For each $a, b \in X$ and $t \in [0,1]$, define $\gamma_{a,b}(t)$ as any point of X at distance at most D from $\gamma'_{a,b}(t)$. Since $\gamma(t)$ is at uniform distance D from $\gamma'(t)$, we deduce that γ is a bicombing on (X, σ) with the listed properties. □

Note that if the action of the automorphism group of $(X, \mathfrak{S})$ on X is free, then the bicombing may be chosen to be actually equivariant.

We now discuss the consequences of our construction for HHGs.

Corollary 3.8. *If G is an HHG, then G admits a proper, cocompact, isometric action on the coarsely Helly space (G, σ).*

Proof. (G, σ) is coarsely Helly by Corollary 3.6. Since the median is equivariant in an HHG, Lemma 2.7 tells us that the action is isometric. Properness and cocompactness follow from Proposition 2.13. □

Remark 3.9. In fact, we do not quite need to assume that we have a hierarchically hyperbolic group: we only need a proper cocompact action by median isometries on an HHS. In fact, cocompactness can be relaxed to coboundedness for the sake of the applications in this paper. For example, it would be sufficient to assume that G is a group acting properly coboundedly by HHS automorphisms on an HHS. The consequences for HHGs listed here and in the introduction therefore apply in this generality.

The next lemma is a modified version of [CCG+20, Proposition 6.7], in which the assumption that the hull is proper has been dropped.

Lemma 3.10. *If a group G acts properly coboundedly on a coarsely Helly space X, then G acts properly coboundedly on the injective hull $E(X)$. In particular, every HHG admits a proper, cobounded action on an injective space.*

Proof. The Hausdorff distance between $e(X)$ and $E(X)$ is bounded by some constant D, so the induced action of G on $E(X)$, provided by [Lan13, Proposition 3.7], is cobounded. For properness, let $Y \subset E(X)$ be bounded and let $Y' = \{x \in X : d(Y, e(x)) \leqslant D\} \neq \varnothing$. Since e is an isometric embedding, Y' is bounded. If $g \in G$ has $gY \cap Y \neq \varnothing$, then pick $y \in Y$ with $gy \in Y$ and let $x \in X$ have $d(y, e(x)) \leqslant D$. Then $d(gy, ge(x)) \leqslant D$, so $e(gx) = ge(x)$ is D–close to Y. That is, $gY' \cap Y' \neq \varnothing$, so since Y' is bounded and the action of G on X is proper, there are only finitely many such g. The final sentence follows from Corollary 3.8. □

We now strengthen Corollary 3.7 in the case of HHGs.

Corollary 3.11. *If G is an HHG, then G is semihyperbolic.*

Proof. By Lemma 3.10, G acts properly coboundedly on an injective space E. Every orbit map $G \to E$ is a G–equivariant quasi-isometry. By [Lan13, Proposition 3.8], E has a G–invariant, bounded, geodesic bicombing in the sense of [AB95]. As the action of G on itself is free, it is semihyperbolic by [AB95, Theorem 4.1]. □

Corollary 3.12. *If G is an HHG, then G has finitely many conjugacy classes of finite subgroups.*

Proof. By Lemma 3.10, G acts properly coboundedly on an injective space E. Let $x \in E$ and let r be a constant such that $G \cdot x$ is r–coarsely dense in E. Let F be a finite subgroup of G. By [Lan13, Proposition 1.2], there is a point $z \in E$ that is fixed by F, and hence F fixes the ball $B(z, r)$ in E, which contains a point of $G \cdot x$. It follows that a conjugate of F fixes a point in $B(x, r)$, and we are done by properness of the action. □

3.4. **Packing subgroups.**

Here we describe the application to bounded packing mentioned in the introduction. Following Hruska and Wise [HW09], we say that a finite collection $\mathcal{H}$ of subgroups of a discrete group G has *bounded packing in G* if for each N there is a constant r such that for any collection of N distinct cosets of elements of $\mathcal{H}$, at least two are separated by a distance of at least r (with respect to some left-invariant, proper distance). If $\mathcal{H}$ consists of a single subgroup H, then we say that H has bounded packing in G.

Corollary 3.13. *If $\mathcal{H}$ is a finite collection of hierarchically quasiconvex subgroups of a group G that is an HHS, then $\mathcal{H}$ has bounded packing in G.*

Proof. By Theorem 3.5, any finite collection of cosets of elements of $\mathcal{H}$ that are pairwise r–close must all come R–close to a single point $x \in G$. In other words, they all intersect the R–ball about x. Since distinct cosets of a given subgroup are disjoint and balls in G are finite, this bounds the size of the collection of cosets. □

In the case of quasiconvex subgroups of hyperbolic groups, one can use Theorem 3.4 in place of Theorem 3.5 in this argument to provide a new, simpler proof of bounded packing. This type of argument is also implicit in [HP19, Remark 4.4, Corollary 4.5], though the Helly property for quasiconvex subgroups of hyperbolic groups is established in a much less efficient way there.

Previous proofs of this result work by induction on the height of subgroups. However, this line of reasoning does not generalise outside the setting of strict negative curvature;

indeed, no subgroup of a flat can ever have finite height. Moreover, Theorem 3.5 is purely geometric: there is no group action involved. It therefore seems that the most natural way to establish bounded packing for quasiconvex subgroups of hyperbolic groups is via the Chepoi–Dragan–Vaxès theorem as described above.

If a group G has a codimension–1 subgroup H, then Sageev's construction yields an action of G on a CAT(0) cube complex, and if the conjugates of H satisfy the coarse Helly property, then it follows that the action of G on the CAT(0) cube complex is cocompact ([Sag97]). This raises the following question.

Question. *Does the mapping class group have property FW_∞, i.e. does any action of the mapping class group on a finite-dimensional CAT(0) cube complex have a fixed point?*

Note that property FW_∞ is intermediate between having no virtual surjection onto $\mathbb{Z}$ [Had20] and Kazhdan's property (T). There are known restrictions on what an action of the mapping class group on a CAT(0) cube complex could look like. Indeed, the mapping class group of a surface of genus at least three does not admit a properly discontinuous action by semisimple isometries on a complete CAT(0) space ([KL96, BH99, Bri10]), nor, more specifically, does it act properly on a CAT(0) cube complex (even an infinite dimensional one) [Gen19]. Moreover, if such a mapping class group acts essentially on a CAT(0) cube complex X, then, as a consequence of [Had20], X cannot have a $\mathbb{Z}$ factor in the canonical decomposition of [CS11].

More generally, in relationship with property (T) and the Haagerup property, the existence of non-trivial actions of the mapping class group on various generalisations of CAT(0) cube complexes remains mysterious; for example median spaces, Hilbert spaces, CAT(0) spaces, and L^p spaces. The coarse Helly property may prove useful in the study of such actions.

4. Strong Shortcut Property

In this section we will prove that coarsely Helly spaces of uniformly bounded geometry are strongly shortcut. A *Riemannian circle* S is S^1 endowed with a geodesic metric of some length $|S|$. Let (X, σ) be a roughly geodesic metric space. Then (X, σ) is *strongly shortcut* if there exists $K > 1$ such that for any $C > 0$ there is a bound on the lengths $|S|$ of (K, C)–quasi-isometric embeddings $S \to X$ of Riemannian circles S in (X, σ) [Hod20b]. A group G is *strongly shortcut* if it acts properly and coboundedly on a strongly shortcut metric space [Hod18, Hod20b].

We will now give a brief description of the injective hull construction of Isbell [Isb64], which was later rediscovered by Dress [Dre84] and Chrobak and Larmore [CL94]. For a nice discussion on this construction, see Lang [Lan13]. Let (X, σ) be a metric space. A *radius function* on X is a function $f \colon X \to \mathbb{R}_{\geqslant 0}$ for which

$$\sigma(x, y) \leqslant f(x) + f(y)$$

for every $x, y \in X$. A radius function $f \colon A \to \mathbb{R}_{\geqslant 0}$ on any subspace of $A \subseteq X$ is called a *partial radius function* on X. If $f, g \colon X \to \mathbb{R}_{\geqslant 0}$ are two radius functions then f *dominates* g if $f(x) \geqslant g(x)$ for all $x \in X$. A radius function $f \colon X \to \mathbb{R}_{\geqslant 0}$ is minimal if the only radius function it dominates is itself.

If $f\colon A \to \mathbb{R}_{\geqslant 0}$ is a partial radius function on X then there exists a minimal radius function $g\colon X \to \mathbb{R}_{\geqslant 0}$ such that $g|_A$ is dominated by f. For any $x \in X$, the function $\sigma(\cdot, x)$ is a minimal radius function. If $f, g\colon X \to \mathbb{R}_{\geqslant 0}$ are two minimal radius functions then

$$|f - g|_\infty = \sup_{x \in X} \big| f(x) - g(x) \big|$$

is bounded. The set of minimal radius functions on X, with metric given by $d(f, g) = |f - g|_\infty$, is the *injective hull* $E(X)$ of X. The isometric embedding $e\colon X \hookrightarrow E(X)$ is given by $x \mapsto \sigma(\cdot, x)$. The metric space X is coarsely Helly with constant δ if and only if it is δ–coarsely dense in its injective hull [CCG+20, Proposition 3.12].

Lemma 4.1. *Let (X, σ) be a metric space. Let $g\colon X \to \mathbb{R}_{\geqslant 0}$ be a minimal radius function, let $\bar{f}\colon X \to \mathbb{R}_{\geqslant 0}$ be a radius function and let $f\colon X \to \mathbb{R}_{\geqslant 0}$ be any minimal radius function dominated by $\bar{f}$. Then $|g - f|_\infty \leqslant |g - \bar{f}|_\infty$.*

Proof. Let $y \in X$. Then $f(y) \leqslant \bar{f}(y) \leqslant g(y) + |g - \bar{f}|_\infty$ and so $f(y) - g(y) \leqslant |g - \bar{f}|_\infty$. It remains to prove that $g(y) - f(y) \leqslant |g - \bar{f}|_\infty$. By minimality of g, for any $\epsilon > 0$, there exists $z \in X$ for which $g(y) + g(z) < \sigma(y, z) + \epsilon$. Then, since f is a radius function dominated by $\bar{f}$, we have

$$\begin{aligned} f(y) &\geqslant \sigma(y, z) - f(z) \\ &\geqslant \sigma(y, z) - \bar{f}(z) \\ &\geqslant \sigma(y, z) - g(z) - |g - \bar{f}|_\infty \\ &> g(y) - \epsilon - |g - \bar{f}|_\infty \end{aligned}$$

and so $g(y) - f(y) < |g - \bar{f}|_\infty + \epsilon$ which completes the proof since we chose $\epsilon > 0$ arbitrarily. $\square$

Theorem 4.2. *Let (X, σ) be a coarsely Helly metric space. If (X, σ) has uniformly bounded geometry then (X, σ) is strongly shortcut.*

Proof. Let $\delta > 0$ be a coarse Helly constant for (X, σ). Let $X \to E(X)$ be the embedding of (X, σ) into its injective hull and view this embedding as an inclusion of a subspace. Then X is δ–coarsely dense in $E(X)$ so there is a retraction $r\colon E(X) \to X$ such that r is a $(1, 2\delta)$–quasi-isometry.

Let $\phi\colon S \to X$ be a (K, C)–quasi-isometric embedding of a Riemannian circle. Let $f''\colon \phi(S) \to \mathbb{R}_{\geqslant 0}$ be the constant function taking the value $K\frac{|S|}{4} + C$. Then f'' is a radius function on $\phi(S) \subset X$. Let $f'\colon \phi(S) \to \mathbb{R}_{\geqslant 0}$ be a minimal radius function on $\phi(S)$ dominated by f''. Then for each $x \in \phi(S)$ and each $\epsilon > 0$, there exists a $y \in \phi(S)$ for which $f'(x) + f'(y) < \sigma(x, y) + \epsilon$. Since f' is a partial radius function on X we can let $f\colon X \to \mathbb{R}_{\geqslant 0}$ be a minimal radius function on X dominated by f'. Then f corresponds to a vertex of $E(X)$ at distance at most $K\frac{|S|}{4} + C$ from every point in $\phi(S)$. Then, by consideration of pairs of antipodes of S and the fact that ϕ is a (K, C)–quasi-isometry, we see that f is bounded below by $\frac{2 - K^2}{4K}|S| - 2C$ on $\phi(S)$.

For $x, y \in X$ let $\ell_{x,y} = f(x) + f(y) - \sigma(x,y)$. Then, for each $x \in \phi(S)$ and each $\epsilon > 0$, there exists $y \in \phi(S)$ such that $\ell_{x,y} < \epsilon$. Moreover, for $a, b \in S$ we have

$$\begin{aligned} \frac{2-K^2}{2K}|S| - 4C &\leqslant f\big(\phi(a)\big) + f\big(\phi(b)\big) \\ &= \sigma\big(\phi(a), \phi(b)\big) + \ell_{\phi(a),\phi(b)} \\ &\leqslant K d_S(a,b) + C + \ell_{\phi(a),\phi(b)} \end{aligned}$$

and so $d_S(a,b) \geqslant \frac{2-K^2}{2K^2}|S| - \frac{\ell_{\phi(a),\phi(b)}+5C}{K}$.

Let $x \in \phi(S)$. There exists a sequence of minimal radius functions $(f_x^k : X \to \mathbb{R}_{\geqslant 0})_k$ where k ranges in $\{0, 1, \ldots, M_x\}$ such that $M_x = \left\lfloor \frac{f(x)}{\delta} \right\rfloor$ and the following properties hold for all k, k' and y.

(1) $f_x^0 = f$
(2) $f_x^{M_x}(x) < \delta$
(3) $d_{E(X)}(f_x^k, f_x^{k'}) = \delta|k - k'|$
(4) $f(y) + k\delta - \ell_{x,y} \leqslant f_x^k(y) \leqslant f(y) + \max\{0, k\delta - \ell_{x,y}\}$

We construct the $(f_x^k)_k$ by induction on k. By property (1), we must start with $f_x^0 = f$. Assuming we have f_x^{k-1}, we will begin by defining a radius function $\bar{f}_x^k$. Set $\bar{f}_x^k(x) = f_x^{k-1}(x) - \delta$. By minimality of f_x^{k-1}, there exists $y \in X$ for which $f_x^{k-1}(y) + f_x^{k-1}(x) - \delta < \sigma(x,y)$. For all such $y \in X$, set $\bar{f}_x^k(y) = \sigma(x,y) - f_x^{k-1}(x) + \delta$. For all other $y \in X$, set $\bar{f}_x^k(y) = f_x^{k-1}(y)$. Then $\bar{f}_x^k$ is a radius function. Define f_x^k as any minimal radius function that is dominated by $\bar{f}_x^k$.

Since $\bar{f}_x^k(y) = \sigma(x,y) - f_x^{k-1}(x) + \delta = \sigma(x,y) - \bar{f}_x^k(x)$ for some $y \in X$, we must have $f_x^k(x) = \bar{f}_x^k(x) = f_x^{k-1}(x) - \delta$. Thus $|f_x^{k-1} - f_x^k|_\infty \geqslant \delta$ and

$$d_{E(X)}(f_x^{M_x}, x) = f_x^{M_x}(x) = f(x) - M_x\delta = f(x) - \left\lfloor \frac{f(x)}{\delta} \right\rfloor \delta < \delta$$

which establishes (2). On the other hand, by Lemma 4.1, we have $|f_x^{k-1} - f_x^k|_\infty \leqslant |f_x^{k-1} - \bar{f}_x^k|_\infty \leqslant \delta$ and so $d_{E(X)}(f_x^{k-1}, f_x^k) = |f_x^{k-1} - f_x^k|_\infty = \delta$. Therefore,

$$\begin{aligned} d_{E(X)}(f, x) &= f(x) \\ &= M_x\delta + f(x) - M_x\delta \\ &= M_x\delta + d_{E(X)}(f_x^{M_x}, x) \\ &= \sum_{k=1}^{M_x} d_{E(X)}(f_x^{k-1}, f_x^k) + d_{E(X)}(f_x^{M_x}, x) \end{aligned}$$

where $f_x^0 = f$. Then, by the triangle inequality, property (3) is satisfied.

To verify property (4), let $y \in X$. We have

$$f(y) + k\delta - \ell_{x,y} = \sigma(x,y) + k\delta - f(x) = \sigma(x,y) - f_x^k(x) \leqslant f_x^k(y)$$

so the lower bound holds. The upper bound on $f_x^k(y)$ given by property (4) is $R_k = f(y) + \max\{0, k\delta - \ell_{x,y}\}$. Suppose property (4) doesn't hold and let k be the least integer for which $f_x^k(y) > R_k$. Then $k > 0$ and k must satisfy $f_x^k(y) - f_x^{k-1}(y) > R_k - R_{k-1} \geqslant 0$. By the construction of f_x^k, the fact that $f_x^k(y) > f_x^{k-1}(y)$ implies that

$f_x^{k-1}(y) + f_x^{k-1}(x) - \delta < \sigma(x,y)$ and that $\bar{f}_x^k(y) = \sigma(x,y) - f_x^{k-1}(x) + \delta = \sigma(x,y) - \bar{f}_x^k(x)$. Then we must have

$$\begin{aligned}
f_x^k(y) &= \bar{f}_x^k(y) \\
&= \sigma(x,y) - \bar{f}_x^k(x) \\
&= \sigma(x,y) - f_x^k(x) \\
&= f(y) + k\delta - \ell_{x,y} \\
&\leqslant R_k
\end{aligned}$$

which contradicts $f_x^k(y) > R_k$. Thus we have verified property (4).

We will now use the sequence $(f_x^k)_k$ of minimal radius functions to prove the theorem. Let $a, a' \in S$ satisfy $d_S(a,a') \geqslant \frac{2(K^2-1)}{K^2}|S| + \frac{4\delta+10C}{K}$ and take $b \in S$ for which $\ell_{\phi(a),\phi(b)} < \delta$. Then $d_S(a,a') + d_S(a',b) + d_S(b,a) \leqslant |S|$ so we have

$$\begin{aligned}
d_S(a',b) &\leqslant |S| - d_S(a,b) - d_S(a,a') \\
&\leqslant |S| - \frac{2-K^2}{2K^2}|S| + \frac{\ell_{\phi(a),\phi(b)} + 5C}{K} - d_S(a,a') \\
&< |S| - \frac{2-K^2}{2K^2}|S| + \frac{5C+\delta}{K} - d_S(a,a') \\
&\leqslant |S| - \frac{2-K^2}{2K^2}|S| + \frac{5C+\delta}{K} - \frac{2(K^2-1)}{K^2}|S| - \frac{4\delta+10C}{K} \\
&= \frac{2-K^2}{2K^2}|S| - \frac{3\delta+5C}{K}
\end{aligned}$$

and so

$$\frac{2-K^2}{2K^2}|S| - \frac{\ell_{\phi(a'),\phi(b)} + 5C}{K} \leqslant d_S(a',b) < \frac{2-K^2}{2K^2}|S| - \frac{3\delta+5C}{K}$$

which implies $\ell_{\phi(a'),\phi(b)} > 3\delta$. So we have

$$\begin{aligned}
f^3_{\phi(a')}\big(\phi(b)\big) &\leqslant f\big(\phi(b)\big) + \max\{0, 3\delta - \ell_{\phi(a'),\phi(b)}\} \\
&= f\big(\phi(b)\big) \\
&\leqslant f^3_{\phi(a)}\big(\phi(b)\big) - 3\delta + \ell_{\phi(a),\phi(b)} \\
&< f^3_{\phi(a)}\big(\phi(b)\big) - 2\delta
\end{aligned}$$

where the inequalities are applications of property (4). Thus

$$d_{E(X)}(f^3_{\phi(a')}, f^3_{\phi(a)}) > 2\delta$$

and so $r(f^3_{\phi(a')})$ and $r(f^3_{\phi(a)})$ are distinct elements of the metric ball $B\big(r(f), 5\delta\big)$ of radius 5δ centered at $r(f)$ in X. So, if $\{a_i\}_{i=1}^N$ are points of S that subdivide S into segments of length at least $\frac{2(K^2-1)}{K^2}|S| + \frac{4\delta+10C}{K}$ then $B\big(r(f), 5\delta\big)$ contains at least N points. Subdividing S evenly we can achieve $N = \left\lfloor \left(\frac{2(K^2-1)}{K^2} + \frac{4\delta+10C}{K|S|}\right)^{-1} \right\rfloor$. So we have shown that if X admits a (K,C)–quasi-isometric embedding of a Riemannian circle S then for some $x \in X$ we have $\big|B(x,5\delta)\big| \geqslant \left\lfloor \left(\frac{2(K^2-1)}{K^2} + \frac{4\delta+10C}{K|S|}\right)^{-1} \right\rfloor$.

To complete the proof, suppose X is not strongly shortcut. Then, for each $K > 1$, there exists $C_K > 0$ and a sequence $(\phi_n\colon S_n \to X)_n$ of (K, C_K)–quasi-isometric embeddings of circles where $|S_n| \to \infty$ as $n \to \infty$. The argument above shows that, for each $K > 1$ and $n \in \mathbb{N}$ there exists $x_{K,n} \in X$ satisfying $\left|B(x_{K,n}, 5\delta)\right| \geqslant \left\lfloor \left(\frac{2(K^2-1)}{K^2} + \frac{4\delta+10C_k}{K|S_n|}\right)^{-1} \right\rfloor$. The expression $\left(\frac{2(K^2-1)}{K^2} + \frac{4\delta+10C_K}{K|S_n|}\right)^{-1}$ tends to $\frac{K^2}{2(K^2-1)}$ as n tends to infinity so if $n_K \in \mathbb{N}$ is large enough then $\left|B(x_{K,n_K}, 5\delta)\right| \geqslant \frac{K^2}{2(K^2-1)} - 1$. But this contradicts the uniform bounded geometry assumption on X since $\frac{K^2}{2(K^2-1)}$ tends to infinity as K tends to 1. $\square$

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