Injective metrics on buildings and symmetric spaces

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ABSTRACT. In this article, we show that the Goldman-Iwahori metric on the space of all norms on a fixed vector space satisfies the Helly property for balls.
On the non-Archimedean side, we deduce that most classical Bruhat-Tits buildings may be endowed with a natural piecewise ℓ^∞ metric which is injective. We also prove that most classical semisimple groups over non-Archimedean local fields act properly and cocompactly on Helly graphs. This gives another proof of biautomaticity for their uniform lattices.
On the Archimedean side, we deduce that most classical symmetric spaces of non-compact type may be endowed with a natural piecewise ℓ^∞ metric which is coarsely Helly. We also prove that most classical semisimple groups over Archimedean local fields act properly and cocompactly on injective metric spaces.
The only exception is the special linear group: if $n \geqslant 3$ and $\mathbb{K}$ is a local field, we show that $\mathrm{SL}(n, \mathbb{K})$ does not act properly and coboundedly on an injective metric space.

Introduction

In this article, we are interested in the relationship between symmetric spaces of non-compact type and Euclidean buildings, on one side, and injective metric spaces and Helly graphs, on the other side.

A geodesic metric space is called injective if the family of closed balls satisfies the Helly property, i.e. any family of pairwise intersecting balls has a non-empty global intersection. An injective metric space satisfies some properties of nonpositive curvature: it is contractible, any finite group action has a fixed point, and it has a conical geodesic bicombing. One key feature of injective metric spaces is that any metric space embeds isometrically in an essentially unique smallest injective metric space, called the injective hull. Injective metric spaces in geometric group theory have been notably popularized by Lang, who proved that any Gromov-hyperbolic group acts properly and cocompactly on an injective metric space, the injective hull of a Cayley graph (see [Lan13, Theorem 1.4]).

A geodesic metric space is called coarsely Helly if any family of pairwise intersecting balls has a non-empty global intersection, up to increasing the radii by a uniform amount. If a finitely generated group acts properly and cocompactly on a coarsely Helly metric space, we can deduce that it is semi-hyperbolic in the sense of Alonso-Bridson. This strategy has been used by Hoda, Petyt and the author to prove that any hierarchically hyperbolic group, including any mapping class group of a surface, is coarsely Helly and semi-hyperbolic.

Keywords : Injective metric, Helly graph, Bruhat-Tits building, symmetric space, biautomatic. **AMS codes** : 20E42, 53C35, 52A35, 22E46

The discrete analogue of injective metric spaces is the notion of Helly graphs: a connected graph is called Helly if the family of combinatorial balls satisfies the Helly property. The reader is referred to [CCG$^+$20] for the study of group actions on Helly graphs. One notable result is that a discrete group acting properly and cocompactly on a locally finite Helly graph is biautomatic (see [CCG$^+$20, Theorem 1.5]).

Symmetric spaces of non-compact type and Euclidean buildings already have a CAT(0) metric. Nevertheless, looking for injective metrics on those spaces may provide extra structure. For instance, deciding which CAT(0) groups are biautomatic is very subtle, as Leary and Minasyan recently provided the first counter-examples (see [LM21]). On the other hand, any Helly group is biautomatic.

Our work is based on a very simple remark that, given any set of norms on a vector space satisfying simple conditions, the Goldman-Iwahori metric satisfies the Helly property for closed balls (see [GI63]). The fact that the metric is geodesic will be verified in concrete examples.

Proposition A (Proposition 2.1)**.** *Let $\mathbb{K}$ denote a valued field, let V denote a $\mathbb{K}$-vector space, and let X denote a set of norms on V satisfying simple conditions (see Proposition 2.1). For any two elements η, η' in X, let us define the Goldman-Iwahori metric*

$$d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left| \log \frac{\eta(v)}{\eta'(v)} \right|.$$

The family of closed balls in the metric space (X, d) satisfies the Helly property.

Bruhat-Tits buildings

The first example to which Proposition A applies is the Goldman-Iwahori space of all ultrametrics norms (see [GI63]). It identifies with the Bruhat-Tits extended building of $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K}$ is a non-Archimedean valued field which is locally compact, or more generally spherically complete. Recall that the Bruhat-Tits building $\overline{X}$ of $\mathrm{SL}(n, \mathbb{K})$ can be described as the set of all homothety classes of ultrametric norms on $\mathbb{K}^n$ (see [Par99] for instance), and the Bruhat-Tits extended building X of $\mathrm{GL}(n, \mathbb{K})$ can be described as the set of all ultrametric norms on $\mathbb{K}^n$, also called the Goldman-Iwahori space. Each apartment in X naturally identifies with $\mathbb{R}^n$, and the Goldman-Iwahori metric from Proposition A is the length metric associated to the standard piecewise ℓ^∞ metric on each apartment. We therefore have the following.

Theorem B (Theorem 3.2)**.** *Let $\mathbb{K}$ denote any non-Archimedean valued field $\mathbb{K}$ which is spherically complete, and consider the extended Bruhat-Tits building X of $\mathrm{GL}(n, \mathbb{K})$. Endow X with the Goldman-Iwahori metric, i.e. the length metric associated to the standard piecewise ℓ^∞ metric on each apartment. Then (X, d) is injective.*

Note that a particular case of this result, when the valuation is discrete and the building is simplicial, was already known, combining works of Hirai and Chalopin et al.

Theorem ([Hir20],[CCHO21])**.** *Let X denote any extended Euclidean building of type $\widetilde{A}_{n-1}$. Endow X with the length metric associated to the standard piecewise ℓ^∞ metric on each apartment. Then (X, d) is injective.*

Our work has the advantage of being valid for a possibly non-discrete valuation if the field $\mathbb{K}$ is spherically complete, and furthermore our proof is extremely simple.

We can also wonder whether we can apply it to find a Helly graph related to Euclidean buildings. This is indeed the case.

Theorem C (Theorem 3.3). *Let $\mathbb{K}$ denote any non-Archimedean discretely valued field $\mathbb{K}$, and consider the extended Bruhat-Tits building X of* $\mathrm{GL}(n, \mathbb{K})$*. Then the thickening of the vertex set $X^{(0)}$ of X is a Helly graph. In particular,* $\mathrm{GL}(n, \mathbb{K})$ *acts properly and cocompactly by automorphisms on a Helly graph.*

The thickening of $X^{(0)}$ is the graph with vertex set $X^{(0)}$, and with an edge between two vertices if they are at ℓ^∞ distance 1 in some apartment.

For other classical groups, we can in fact deduce similar results using an embedding in $\mathrm{GL}(n, \mathbb{K})$.

Corollary D (Theorems 3.4 and 3.5). *Let $\mathbb{K}$ denote a local field of characteristic different from 2, and let G denote a classical connected semisimple group over $\mathbb{K}$, realized as the identity component of the fixed point set of an involution in the general linear group* $\mathrm{GL}(n, \mathbb{K})$*. Then the Bruhat-Tits building of G, endowed with the length metric induced from the ℓ^∞ metric on the extended Bruhat-Tits building of* $\mathrm{GL}(n, \mathbb{K})$*, is injective. Furthermore, the group G acts properly and cocompactly by automorphisms on a locally finite Helly graph.*

Note that Chalopin et al. proved that any cocompact lattice in a Euclidean building of type $\widetilde{C}_n$ acts properly and cocompactly on a Helly graph (see [CCG+20, Corollary 6.2]).

We also easily deduce a result for all classical semisimple Lie groups and their cocompact lattices.

Corollary E (Corollary 3.6). *Let G denote a classical reductive Lie group over a non-Archimedean local field of characteristic different from 2, and let $a \geqslant 0$ denote the number of almost simple factors of type A. Then $G \times \mathbb{Z}^a$ acts properly and cocompactly by automorphisms on a locally finite Helly graph.*

For any cocompact lattice Γ in G, the group $\Gamma \times \mathbb{Z}^a$ acts properly and cocompactly by automorphisms on a locally finite Helly graph, and the group Γ is biautomatic.

Note that Swiatkowski proved that any group acting properly and cocompactly on any Euclidean building is biautomatic (see [Ś06, Theorem 6.1]). Nevertheless, this provides another perspective on this result.

Symmetric spaces

The second example to which Proposition A applies is the symmetric space X of $\mathrm{GL}(n, \mathbb{R})/\mathbb{O}(n)$ of $\mathrm{GL}(n, \mathbb{R})$, which may be described as the space of all Euclidean norms on $\mathbb{R}^n$. However, it does not apply directly, since the supremum of two Euclidean norms is no longer Euclidean. So we apply Proposition A to the space $\hat{X}$ of all norms on $\mathbb{R}^n$, and use the John-Löwner ellipsoid to show that X is cobounded in $\hat{X}$.

Theorem F (Theorem 4.4). *Let $X = \mathrm{GL}(n, \mathbb{R})/\mathbb{O}(n)$ denote the symmetric space of* $\mathrm{GL}(n, \mathbb{R})$*, and endow X with the Finsler length metric associated to the standard piecewise ℓ^∞ metric on each apartment. Then (X, d) is coarsely Helly, and its injective hull is locally compact. In particular,* $\mathrm{GL}(n, \mathbb{R})$ *acts properly and cocompactly by isometries on an injective metric space.*

For other classical groups, we can in fact deduce similar results using an embedding in $\mathrm{GL}(n, \mathbb{R})$.

Theorem G (Theorem 4.6). *Let G denote a classical almost simple non-compact real Lie group which is not of type A, and let X denote its symmetric space. Then X has a natural Finsler length metric d such that (X, d) is coarsely Helly, and its injective hull is locally compact. In particular, G acts properly and cocompactly by isometries on an injective metric space.*

We also easily deduce a result for all classical semisimple Lie groups and their cocompact lattices.

Corollary H (Corollary 4.7)**.** *Let G denote any reductive real Lie group, with classical non-compact almost simple factors. Let $a \geqslant 0$ denote the number of almost simple factors of type A. Then $G \times \mathbb{R}^a$ acts properly and cocompactly on an injective metric space. In particular, for any cocompact lattice Γ in G, the group $\Gamma \times \mathbb{Z}^a$ acts properly and cocompactly on an injective metric space.*

Recall that Chalopin et al. proved that any Helly group is biautomatic. This motivates the question whether the non-discrete analogue of this result holds:

Question. *Assume that a finitely generated group Γ acts properly and cocompactly on an injective metric space. Is Γ biautomatic ?*

The special linear group

We now turn to the special linear group. According to Theorems B and F, if $\mathbb{K}$ is a local field, we have seen that $\mathrm{GL}(n, \mathbb{K})$ acts properly and cocompactly on an injective metric space. It is natural to ask what happens for $\mathrm{SL}(n, \mathbb{K})$. Inspired by the work of Hoda on crystallographic Helly groups (see [Hod20]), we prove the following.

Theorem I (Theorem 5.1)**.** *Let $\mathbb{K}$ be a local field (with characteric different from 2 if $\mathbb{K}$ is non-Archimedean), and let $n \geqslant 3$. Then $\mathrm{SL}(n, \mathbb{K})$ is not coarsely Helly: $\mathrm{SL}(n, \mathbb{K})$ does not act properly and coboundedly on an injective metric space.*

This is also evidence that cocompact lattices in $\mathrm{SL}(n, \mathbb{K})$ are not expected to be coarsely Helly.

Structure of the article

In Section 1, we review the notions of injective metric spaces, Helly graphs and group actions. In Section 2, we present Proposition 2.1 stating that the Goldman-Iwahori metric on the space of all norms satisfies a Helly property for balls. In Section 3, we apply this construction to Bruhat-Tits buildings, and in Section 4, we apply it to symmetric spaces of non-compact type. In the final Section 5, we prove that the special linear group is not coarsely Helly.

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1 Injective metric spaces and Helly graphs

In this section, we recall some basic definitions about injective metric spaces and Helly graphs. We refer the reader to [Lan13] and [CCG⁺20] for more details.

A metric space (X, d) is called *injective* if, for any family $(x_i)_{i \in I}$ of points in X and $(r_i)_{i \in \mathbb{N}}$ of nonnegative real numbers satisfying

$$\forall i, j \in I, r_i + r_j \geqslant d(x_i, x_j),$$

the family of balls $(B(x_i, r_i))_{i \in \mathbb{N}}$ has a non-empty global intersection.

In case the metric space (X,d) is geodesic, it is injective of and only if the family of balls satisfy the *Helly property*: any family of pairwise intersecting closed balls has a non-empty global intersection.

Examples of geodesic injective metric spaces are normed vector spaces with the ℓ^∞ norm, and also finite-dimensional CAT(0) cube complexes with the piecewise ℓ^∞ metric (see [Bow20]).

One key feature of the theory is that any metric space X embeds isometrically in a unique minimal injective metric space, called the *injective hull* of X and denoted EX (see [Isb64]).

A metric space (X,d) is called *coarsely Helly* if there exists a constant $C \geqslant 0$ such that, for any family $(x_i)_{i\in I}$ of points in X and $(r_i)_{i\in\mathbb{N}}$ of nonnegative real numbers satisfying

$$\forall i,j \in I, r_i + r_j \geqslant d(x_i,x_j),$$

the family of balls $(B(x_i, r_i + C))_{i\in\mathbb{N}}$ has a non-empty global intersection.

There is also a discrete version of injective metric spaces concerning graphs: a connected graph is called a *Helly graph* if the family of combinatorial balls satisfy the Helly property: any family of pairwise intersecting balls has a non-empty global intersection.

Concerning actions of groups on injective metric spaces, we will distinguish three families:

- A group G is called *coarsely Helly* if it acts properly and coboundedly by isometries on an injective metric space, or equivalently it acts properly and cocompactly by isometries on a coarsely Helly metric space (see [CCG$^+$20, Proposition 3.12]).
- A group G is called *metrically injective* if it acts properly and cocompactly by isometries on an injective metric space.
- A group G is called *Helly* if it acts properly and cocompactly by automorphisms on a Helly graph.

Any Helly group is metrically injective, by considering the injective hull of a Helly graph. And obvisouly, any metrically injective group is coarsely Helly.

We now list examples of such groups.

According to [BvdV91] (see also [HW09, Corollary 3.6]), the thickening of any CAT(0) cube complex is a Helly graph: in particular, any group acting properly and cocompactly on a CAT(0) cube complex is Helly. More generally, any group acting properly and cocompactly on a finite rank metric median space is metrically injective (see [Bow20]). Urs Lang motivated the interest in group actions on injective metric spaces in [Lan13], notably proving that any Gromov-hyperbolic group is Helly (see also [CE07]), and acts properly and cocompactly on the injective hull of any Cayley graph. Chalopin et al. proved (see [CCG$^+$20, Corollary 6.2]) that any type-preserving uniform lattice in a Euclidean building of type $\widetilde{C}_n$ is Helly. Huang and Osajda proved that any Artin group of type FC is Helly (see [HO19]).

The author, Hoda and Petyt proved in [HHP20] that any hierarchically hyperbolic group, including any mapping class group of a surface, is coarsely Helly.

The existence of such actions on injective metric spaces enables us to deduce many properties reminiscent of non-positive curvature, let us list some of them:

Theorem 1.1. *Assume that a finitely generated group G is coarsely Helly. Then:*

- *G is semi-hyperbolic in the sense of Alonso-Bridson, which has many consequences ([BH99]).*

- *G has finitely many conjugacy classes of finite subgroups ([Lan13, Proposition 1.2]).*
- *G satisfies the coarse Baum-Connes conjecture ([CCG$^+$20, Theorem 1.5]).*
- *Asymptotic cones of G are contractible ([CCG$^+$20, Theorem 1.5]).*

Assume furthermore that G is metrically injective. Then:

- *G admits an EZ-boundary ([CCG$^+$20, Theorem 1.5]).*
- *G satisfies the Farrell-Jones conjecture (see [KR17]).*

Assume in addition that G is a Helly group. Then:

- *G is biautomatic ([CCG$^+$20, Theorem 1.5]).*

Note that all consequences are already known for CAT(0) groups, except the biautomaticity (which does not hold for all CAT(0) groups, see [LM21]).

However, not all non-positively curved groups are coarsely Helly: for instance, Hoda proved that the $(3, 3, 3)$ triangle Coxeter group, which is virtually $\mathbb{Z}^2$, is not Helly (see [Hod20]).

2 An injective distance on the space of all norms

Let $\mathbb{K}$ denote a field (or a division algebra) with an absolute value $|\cdot| : K \to e^H$, where H is a non-zero additive subgroup of $\mathbb{R}$. Let V denote a $\mathbb{K}$-vector space. Recall that a norm on V is a map $\eta : V \to e^H$ that satisfies the following.

- $\forall v \in V, \eta(v) = 0 \iff v = 0$.
- $\forall v \in V, \forall \alpha \in \mathbb{K}, \eta(\alpha v) = |\alpha|\eta(v)$.
- $\forall u, v \in V, \eta(u + v) \leqslant \eta(u) + \eta(v)$.

Note that there is a natural partial order on the set of all norms on V: we say that $\eta \leqslant \eta'$ if $\forall v \in V, \eta(v) \leqslant \eta'(v)$. If $\eta \leqslant \eta'$, let us denote the interval $I(\eta, \eta')$ as the set of all norms θ such that $\eta \leqslant \theta \leqslant \eta'$.

Proposition 2.1. *Let X denote a non-empty set of norms on V satisfying the following properties.*

- *for every $\eta \in X$ and every $a \in H$, we have $e^a\eta \in X$.*
- *for every $\eta, \eta' \in X$, there exist $a \in H$ such that $e^{-a}\eta' \leqslant \eta \leqslant e^a\eta'$.*
- *the set X is a join-semilattice: for every non-empty subset $F \subset X$ such that there exists $\eta \in X$ with $F \leqslant \eta$, the set $\{\eta' \in X \,|\, F \leqslant \eta'\}$ has a unique minimum $\wedge F \in X$.*

For any two elements η, η' in X, let us define the Goldman-Iwahori distance

$$d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left| \log \frac{\eta(v)}{\eta'(v)} \right|.$$

Then the family of closed balls in the metric space (X, d) satisfies the Helly property.

Proof. We will first describe balls in (X, d). Fix $\eta \in X$ and $a \in R_+$. Then $\eta' \in B(\eta, a)$ if and only if, for every $v \in V$, we have $-a \leqslant \log \frac{\eta'(v)}{\eta(v)} \leqslant a$, hence $e^{-a}\eta(v) \leqslant \eta'(v) \leqslant e^a\eta(v)$. As a consequence, the ball $B(\eta, a)$ coincides with the interval $I(e^{-a}\eta, e^a\eta)$.

We will now prove that the intervals in X satisfy the Helly property. Consider a family $(I_s = I(\eta_s, e^{2a_s}\eta_s))_{s\in S}$ of pairwise intersecting intervals in X, where $a_s \in H$ for each $s \in S$. Let $F = \{\eta_s\}_{s\in S} \subset X$: for any $s, t \in S$, since I_s and I_t are intersecting, we have $\eta_t \leqslant e^{2a_s}\eta_s$. According to the assumption on X, we can consider the join $\eta = \wedge F \in X$. For each $s, t \in S$, since $\eta_t \leqslant e^{2a_s}\eta_s$, we deduce that $\eta \leqslant e^{2a_s}\eta_s$. In particular, for each $s \in S$, we have $\eta_s \leqslant \eta \leqslant e^{2a_s}\eta_s$, so $\eta \in I_s$. We have proved that the global intersection $\bigcap_{s\in S} I_s$ is non-empty. □

3 Bruhat-Tits (extended) buildings are injective

We will now apply Proposition 2.1 to define an injective metric on classical Bruhat-Tits buildings.

3.1 The standard and extended Bruhat-Tits buildings of $\mathrm{GL}(n, \mathbb{K})$

Let $\mathbb{K}$ be a field, with a non-Archimedean absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$. Assume that $\mathbb{K}$ is a local field, or more generally that $\mathbb{K}$ is spherically complete: any decreasing intersection of balls in $\mathbb{F}$ has non-empty intersection. Let V denote a n-dimensional vector space over $\mathbb{K}$.

Let us say that a map $\eta : V \to \mathbb{R}_+$ is an ultrametric norm on V if it satisfies the following.

- $\forall v \in V, \eta(v) = 0 \iff v = 0$.
- $\forall v \in V, \forall \alpha \in \mathbb{K}, \eta(\alpha v) = |\alpha|\eta(v)$.
- $\forall u, v \in V, \eta(u + v) \leqslant \max(\eta(u), \eta(v))$.

An ultrametric norm η on V is called *diagonalizable* if there exists a basis $(v_1, \dots, v_n)$ of V such that

$$\forall v = \sum_{i=1}^{n} x_i v_i \in V, \eta(v) = \max_{1\leqslant i\leqslant n} |x_i|.$$

According to [RTW12, Proposition 1.20], if $\mathbb{K}$ is a local field, any ultrametric norm on V is diagonalizable. This holds more generally if $\mathbb{K}$ is spherically complete, see [RTW12, Remark 1.24].

Say that two ultrametric norms $\eta, \eta' : V \to \mathbb{R}_+$ are homothetic if there exists $a \in \mathbb{R}$ such that $\eta' = e^a\eta$. The set $\overline{X}$ of homothety classes of ultrametric norms on V is called the Bruhat-Tits building of $\mathrm{SL}(n, \mathbb{K})$ (see [Par99] for instance).

Let X denote the space of all (diagonalizable) ultrametric norms on V, it has been studied by Goldman and Iwahori (see [GI63]) and can be identified with the extended Bruhat-Tits building of $\mathrm{GL}(n, \mathbb{K})$. It is homeomorphic to the product $\overline{X} \times \mathbb{R}$.

For any two elements η, η' in X, let us define the Goldman-Iwahori distance

$$d(\eta, \eta') = \sup_{v\in V\setminus\{0\}} \left|\log \frac{\eta(v)}{\eta'(v)}\right|.$$

We have an explicit description of the distance d in terms of apartments of X. This description can also be found in [GI63] without the building point of view, but we will give here a simple description using the building.

Let us recall the description of apartments in the Bruhat-Tits building $\overline{X}$ of $\mathrm{GL}(n, \mathbb{K})$. For each basis $v_1, \dots, v_n$ of V (up to homotheties and permutations), there is an associated apartment in $\overline{X}$. For each $m \in \mathbb{R}^n$, let us consider the following ultrametric norm on V:

$$\forall v = \sum_{i=1}^{n} x_i v_i \in V, \eta_m(v) = \max_{1 \leqslant i \leqslant n} e^{m_i} |x_i|.$$

Then the set of such homothety classes identifies with $\{x \in \mathbb{R}^n \,|\, x_1 + x_2 + \cdots + x_n = 0\} \simeq \mathbb{R}^{n-1}$. It is a model of the standard Euclidean apartment of type $\widetilde{A_{n-1}}$.

Let us now describe the apartments of the extended Bruhat-Tits building X of $\mathrm{GL}(n, \mathbb{K})$. For each basis $v_1, \dots, v_n$ of V (up to homotheties and permutations), there is an associated apartment in X: the set of all norms $\{\eta_m \,|\, m \in \mathbb{R}^n\}$ identifies with $\mathbb{R}^n$, which is a model of the extended Euclidean apartment of type $\widetilde{A_{n-1}}$.

Proposition 3.1. *The metric d on X coincides with the length metric associated to the ℓ^∞ metric on each extended apartment.*

Proof. Let d_∞ denote the length metric on X associated to the standard ℓ^∞ metric on each extended apartment. Since the building X admits retractions onto apartments, and as the two metrics d and d_∞ are invariant under the action of $\mathrm{GL}(n, \mathbb{K})$, it is sufficient to prove that the two metrics coincide on a given apartment.

Fix a basis $v_1, \dots, v_n$ of V, and the associated apartment $A = \{\eta_m, m \in \mathbb{R}^n\}$ in X. Fix any $m \in \mathbb{R}^n$. Let $1 \leqslant i \leqslant n$ such that $|m_i| = \|m\|_\infty$, then we have

$$\left|\log \frac{\eta_m(v_i)}{\eta_0(v_i)}\right| = |\log e^{m_i}| = |m_i| = \|m\|_\infty,$$

hence $d_\infty(\eta_0, \eta_m) = \|m\|_\infty \leqslant d(\eta_0, \eta_m)$.

On the other hand, for any $v = \sum_{i=1}^n x_i v_i \in V$, we have

$$\begin{aligned} \left|\log \frac{\eta_m(v)}{\eta_0(v)}\right| &= \left|\log \frac{\max_{1 \leqslant i \leqslant n} e^{m_i} |x_i|}{\max_{1 \leqslant i \leqslant n} |x_i|}\right| \\ &\leqslant \left|\log \frac{\max_{1 \leqslant i \leqslant n} e^{\|m\|_\infty} |x_i|}{max_{1 \leqslant i \leqslant n} |x_i|}\right| = \|m\|_\infty, \end{aligned}$$

so we deduce that $d(\eta_0, \eta_m) \leqslant d_\infty(\eta_0, \eta_m)$.

So we have proved that $d(\eta_0, \eta_m) = d_\infty(\eta_0, \eta_m)$, for any $m \in \mathbb{R}^n$. Hence we deduce that $d = d_\infty$. □

We can now apply Proposition 2.1 to prove that the metric d is injective.

Theorem 3.2. *The extended Bruhat-Tits building X of* $\mathrm{GL}(n, \mathbb{K})$*, endowed with the metric d, is injective.*

Proof. We first have to check that X satisfies the three assumptions of Proposition 2.1.

- For every $\eta \in X$ and every $a \in \mathbb{R}$, we know that $e^a \eta$ is an ultrametric norm on V, hence $e^a \eta \in X$.
- For every $\eta, \eta' \in X$, let $a = d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left|\log \frac{\eta(v)}{\eta'(v)}\right| \in \mathbb{R}_+$. For each $v \in V$, we have $\eta(v) \leqslant e^a \eta'(v)$ and $\eta'(v) \leqslant e^a \eta(v)$, hence $e^{-a} \eta' \leqslant \eta \leqslant e^a \eta'$.

- For every non-empty subset $F \subset X$ such that there exists $\eta \in X$ with $F \leqslant \eta$, let $\theta = \sup F$. It it cleat that θ is a well-defined norm on V, we will check that it is ultrametric: fix $u, v \in V$. For every $\varepsilon > 0$, there exists $\eta' \in F$ such that $\theta(u+v) \leqslant \eta'(u+v) + \varepsilon$. Then

$$\theta(u+v) \leqslant \eta'(u+v) + \varepsilon \leqslant \max(\eta'(u), \eta'(v)) + \varepsilon \leqslant \max(\theta(u), \theta(v)) + \varepsilon.$$

  This holds for any $\varepsilon > 0$, hence $\theta(u+v) \leqslant \max(\theta(u), \theta(v))$. So θ is an ultrametric norm on V: $\theta \in X$, and it is the unique minimum of the set $\{\eta' \in X \mid F \leqslant \eta'\}$. Also recall that, since $\mathbb{K}$ is spherically complete, any ultrametric norm on V is diagonalizable.

According to Proposition 2.1, the balls in (X, d) satisfy the Helly property.

We also know by Proposition 3.1 that the metric space (X, d) is geodesic. So we deduce that the metric space (X, d) is injective. $\square$

3.2 Case of a discrete valuation

We will show that, if we further assume that the valuation is discrete, we can improve Theorem 3.2 by finding a Helly graph.

Assume now that the absolute value is discrete: $|\cdot|(\mathbb{K}) = q^{\mathbb{Z}} \subset \mathbb{R}^+$, where q is the cardinality of the residue field. Then the Bruhat-Tits building $\overline{X}$ of $\mathrm{GL}(n, \mathbb{K})$ has a natural simplicial structure, where the vertex set $\overline{X}^{(0)}$ is given by the homothety classes of ultrametric norms with values in $q^{\mathbb{Z}}$.

Similarly, the extended Bruhat-Tits building X of $\mathrm{GL}(n, \mathbb{K})$ has a natural simplicial structure, where the vertex set $X^{(0)}$ is given by the ultrametric norms with values in $q^{\mathbb{Z}}$. To be consistent, we will in this case define the metric d on $X^{(0)}$ as

$$d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left| \log_q \frac{\eta(v)}{\eta'(v)} \right| \in \mathbb{N}.$$

Let us define the thickening X' of X as the graph with vertex set $X^{(0)}$, and with an edge between two vertices η, η' if they satisfy $d(\eta, \eta') = 1$.

Theorem 3.3. *The thickening X' of the extended Bruhat-Tits building of $\mathrm{GL}(n, \mathbb{K})$ is a Helly graph.*

Proof. Following the same proof as Theorem 3.2, with $H = \log(q)\mathbb{Z}$, we prove that the integer-valued metric space $(X^{(0)}, d)$ has the Helly property for balls.

It now suffices to prove that the distance d is a graph distance. According to Proposition 3.1, on each extended apartment, the metric d coincides with the standard ℓ^∞ metric on $\mathbb{R}^n$. Since the restriction of the ℓ^∞ metric on $\mathbb{R}^n$ to the vertex set $\mathbb{Z}^n$ is a graph distance, we deduce that d is a graph distance on $X^{(0)}$. This proves that the thickening X' is a Helly graph. $\square$

3.3 Classical Euclidean buildings

We now show how to apply the previous results concerning the general linear group to the other classical groups.

Fix a local non-Archimedean field $\mathbb{K}$ with residual characteristic different from 2, and consider a classical connected semisimple group G over $\mathbb{K}$, realized as the identity component of the fixed point set of an involution Φ in a general linear group $\mathrm{GL}(n, \mathbb{K})$. According to Bruhat and Tits (see [BT84] and [PY02]), the Bruhat-Tits building X of G identifies with the set of Φ-fixed points in the Bruhat-Tits extended building Y of $\mathrm{GL}(n, \mathbb{K})$.

More generally, we may consider a finite group F of automorphisms of $\mathrm{GL}(n, \mathbb{K})$ such that the residual characteristic of $\mathbb{K}$ does not divide the order of F. Then, according to [PY02], the Bruhat-Tits building X of $G = (\mathrm{GL}(n, \mathbb{K})^F)^o$ identifies with the F-fixed points in the Bruhat-Tits extended building Y of $\mathrm{GL}(n, \mathbb{K})$.

Endow X with the induced piecewise ℓ^∞ metric d from Y.

Theorem 3.4. *The Bruhat-Tits building X of G, with the metric d, is injective.*

Proof. According to [Lan13, Proposition 1.2], the fixed point set $X = Y^F$ of any finite group action on an injective metric space is non-empty and injective. So the metric space (X, d) is injective. □

We can also strengthen this result by looking for an action of G on a Helly graph.

Theorem 3.5. *The group G acts properly and cocompactly by automorphisms on a Helly graph.*

Proof. Let Y' denote the thickening of the 0-skeleton of Y, which is a Helly graph according to Theorem 3.3. Let $F(Y')$ denote the face complex of Y': it is the simplicial complex with vertex set the set of cliques of Y', and with simplices the set of cliques contained in a given clique of Y'. According to [CCG+20, Lemma 5.30], the face complex $F(Y')$ is clique-Helly (i.e. the family of maximal cliques satisfies the Helly property).

The group $\mathrm{GL}(n, \mathbb{K})$ acts properly and cocompactly on Y'. Let X' denote the fixed point set of F inside $F(Y')$: according to [CCG+20, Theorem 7.1, Corollary 7.4], it is a non-empty clique-Helly graph. According to [CCHO21], the underlying graph of X' is Helly, and G acts properly and cocompactly on X'. □

The following is immediate.

Corollary 3.6. *Let G denote a classical reductive Lie group over a non-Archimedean local field of characteristic different from 2, and let $a \geqslant 0$ denote the number of almost simple factors of type A. Then $G \times \mathbb{Z}^a$ acts properly and cocompactly by automorphisms on a Helly graph.*

For any cocompact lattice Γ in G, the group $\Gamma \times \mathbb{Z}^a$ acts properly and cocompactly by automorphisms on a Helly graph, and the group Γ is biautomatic.

Proof. This is a direct consequence of Theorem 3.5. According to [CCG+20, Theorem 1.5], any Helly group is biautomatic. And according to [Mos97, Theorem B], every direct factor of a biautomatic group is biautomatic. □

Swiatkowski proved that any group acting properly and cocompactly on any Euclidean building is biautomatic (see [Ś06, Theorem 6.1]). So we obtain another point of view on this result, for uniform lattices in classical groups.

4 Symmetric spaces are coarsely Helly

We will apply Proposition 2.1 to define a coarsely Helly metric on classical symmetric spaces of non-compact type.

4.1 The symmetric space of $\mathrm{GL}(n, \mathbb{R})$

Fix $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the division algebra of quaternions), fix $n \geqslant 2$, and let V denote a n-dimensional vector space over $\mathbb{K}$.

Say that two Euclidean norms $\eta, \eta' : V \to \mathbb{R}_+$ are homothetic if there exists $a \in \mathbb{R}$ such that $\eta' = e^a \eta$. The set $\overline{X}$ of homothety classes of hermitian norms on V is called the symmetric space of $\mathrm{SL}(n, \mathbb{K})$, and it identifies naturally with the homogeneous space $\mathrm{SL}(n, \mathbb{K}) / \mathrm{SU}(n, \mathbb{K})$.

Let X denote the space of all hermitian norms on V, it is called the symmetric space of $\mathrm{GL}(n, \mathbb{K})$ and it identifies naturally with the homogeneous space $\mathrm{GL}(n, \mathbb{K}) / \mathrm{U}(n, \mathbb{K})$. It is homeomorphic to the product $\overline{X} \times \mathbb{R}$.

Let $\hat{X}$ denote the space of all norms on V, it contains X as the subset of hermitian norms. The space $\hat{X}$ can also be described as the space of all compact convex subsets of V with non-empty interior, which are invariant under the linear diagonal action of the unit group $\mathbb{U}$ of $\mathbb{K}$. We will call it the augmented symmetric space of $\mathrm{GL}(n, \mathbb{K})$. The group $\mathrm{GL}(n, \mathbb{K})$ acts naturally on $\hat{X}$, by precomposing the norms, or by the linear action on convex subsets of V.

For any two elements η, η' in X, let us define the distance

$$d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left| \log \frac{\eta(v)}{\eta'(v)} \right|.$$

It is a lift of the Banach-Mazur distance, which is defined on the set of isometry classes of such norms.

We have an explicit description of the distance d in terms of apartments of X.

Let us recall the description of maximal flats in the symmetric space $\overline{X}$ of $\mathrm{SL}(n, \mathbb{K})$. For each basis $v_1, \dots, v_n$ of V (up to homotheties and permutations), there is an associated maximal flat in $\overline{X}$. For each $m \in \mathbb{R}^n$, let us consider the following hermitian norm on V:

$$\forall x = \sum_{i=1}^{n} x_i v_i \in V, \eta_m(x) = \sqrt{\sum_{i=1}^{n} e^{2m_i} |x_i|^2}.$$

Then the set of such homothety classes identifies with $\{m \in \mathbb{R}^n \,|\, m_1 + m_2 + \dots + m_n = 0\} \simeq \mathbb{R}^{n-1}$. It is a model of the standard Euclidean flat of type $\widetilde{A_{n-1}}$.

Let us now describe the maximal flats of the symmetric space X of $\mathrm{GL}(n, \mathbb{K})$. For each basis $v_1, \dots, v_n$ of V (up to homotheties and permutations), there is an associated maximal flat in X, the set $\{\eta_m \,|\, m \in \mathbb{R}^n\}$ is a model of the extended Euclidean flat of type $\widetilde{\widetilde{A_{n-1}}}$.

Proposition 4.1. *The metric d on X coincides with the Finsler length metric associated to the ℓ^∞ metric on each extended maximal flat.*

Proof. Let d_∞ denote the length metric on X associated to the standard ℓ^∞ metric on each extended apartment. Since the symmetric space X admits retractions onto maximal flats, and as the two metrics d and d_∞ are invariant under the action of $\mathrm{GL}(n, \mathbb{K})$, it is sufficient to prove that the two metrics coincide on a given maximal flat.

Fix a basis $v_1, \dots, v_n$ of V, and the associated maximal flat $A = \{\eta_m, m \in \mathbb{R}^n\}$ in X. Fix any $m \in \mathbb{R}^n$. Let $1 \leqslant i \leqslant n$ such that $|m_i| = \|m\|_\infty$, then we have

$$\left| \log \frac{\eta_m(v_i)}{\eta_0(v_i)} \right| = |\log e^{m_i}| = |m_i| = \|m\|_\infty,$$

hence $d_\infty(\eta_0,\eta_m)=\|m\|_\infty \leqslant d(\eta_0,\eta_m)$.

On the other hand, for any $v=\sum_{i=1}^n x_i v_i \in V$, we have

$$\begin{aligned}\left|\log \frac{\eta_m(v)}{\eta_0(v)}\right| &= \left|\log \frac{\sqrt{\sum_{i=1}^n e^{2m_i}|x_i|^2}}{\sqrt{\sum_{i=1}^n |x_i|^2}}\right| \\ &\leqslant \left|\log \frac{\sqrt{\sum_{i=1}^n e^{2\|m\|_\infty}|x_i|^2}}{\sqrt{\sum_{i=1}^n |x_i|^2}}\right| = \|m\|_\infty,\end{aligned}$$

so we deduce that $d(\eta_0,\eta_m)\leqslant d_\infty(\eta_0,\eta_m)$.

So we have proved that $d(\eta_0,\eta_m)=d_\infty(\eta_0,\eta_m)$, for any $m\in\mathbb{R}^n$. Hence we deduce that $d=d_\infty$. $\square$

Proposition 4.2. *The symmetric space X of* $\mathrm{GL}(n,\mathbb{K})$ *is cobounded in $\hat{X}$.*

Proof. Let $K\in\hat{X}$. Let $B\subset K$ denote the unique John-Löwner ellipsoid of maximal volume. Since K is invariant under the linear diagonal action of the unit group $\mathbb{U}$, by uniqueness of B, we deduce that B is also invariant under the linear diagonal action of the unit group $\mathbb{U}$. So the convex B is the unit ball of a hermitian norm on $\mathbb{K}^n$: $B\in X$. According to [Joh48], we know that $d(B,K)\leqslant \log(\sqrt{an})$, where $a=\dim_{\mathbb{R}}(\mathbb{K})$. Therefore any point of $\hat{X}$ is at distance at most $\log(\sqrt{an})$ from X. $\square$

We do not know whether the metric space $(\hat{X},d)$ of all norms on $\mathbb{K}^n$ is geodesic: according to Proposition 4.1, we only know that the subspace (X,d) of all hermitian norms is geodesic. Nevertheless, the Helly property for balls is enough to study the injective hull of (X,d).

Proposition 4.3. *Assume that $(\hat{X},d)$ is a locally compact metric space satisfying the Helly property for balls, and assume that X is a geodesic subspace of $\hat{X}$ that is cobounded in $\hat{X}$. Then X is cobounded in the injective hull EX of (X,d), and EX is locally compact.*

Proof. We first prove that X is cobounded in its injective hull EX, so according to [CCG+20, Proposition 3.12] we will prove that (X,d) is coarsely Helly: consider a family $(x_i)_{i\in I}$ of points in X, and a family $(r_i)_{i\in I}$ of nonnegative real numbers, such that $\forall i,j\in I, r_i+r_j\geqslant d(x_i,x_j)$. Since X is geodesic, the balls $(B_X(x_i,r_i))_{i\in I}$ pairwise intersect in X. Since balls in $(\hat{X},d)$ satisfy the Helly property, we deduce that the balls $(B_{\hat{X}}(x_i,r_i))_{i\in I}$ have a global intersection in $\hat{X}$. Since X is cobounded in $\hat{X}$, there exists a real number $C\geqslant 0$ such that every point of $\hat{X}$ is at distance at most C from a point in X. Therefore the balls $(B_X(x_i,r_i+C))_{i\in I}$ have a global intersection in X. Hence the metric space (X,d) is coarsely Helly.

We now prove that the injective hull EX is locally compact metric space, by proving that any bounded sequence in EX has a convergent subsequence. According to [Lan13], if we denote

$$\Delta X=\{f:X\to\mathbb{R}_+ \,|\, \forall x,y\in X, f(x)+f(y)\geqslant d(x,y)\},$$

then EX can be realized as the minimal set of ΔX.

For each $f\in EX\subset\Delta X$, note that the intersection $K_f=\bigcap_{x\in X} B_{\hat{X}}(x,f(x))$ is non-empty in $\hat{X}$ by injectivity. Furthermore, by minimality of f, we have

$$\forall x\in X, f(x)=\min\{r\geqslant 0 \,|\, K_f\subset B_{\hat{X}}(x,r)\}.$$

Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $EX\subset\Delta X$. We have a corresponding bounded sequence $(K_{f_k})_{k\in\mathbb{N}}$ of non-empty compact subspaces of $\hat{X}$. Since $\hat{X}$ is locally compact, we deduce that, up to passing to a subsequence, the sequence $(K_{f_k})_{k\in\mathbb{N}}$ converges, with respect to the Chabauty topology on the space of closed subsets of $\hat{X}$, to a non-empty compact subset K of $\hat{X}$. For any $x \in X$, let us denote $f(x) = \min\{r \geqslant 0 \,|\, K_f \subset B_{\hat{X}}(x,r)\}$.

For each $k \in \mathbb{N}$, let $d_k \geqslant 0$ denote the Hausdorff distance between K_{f_k} and K. Since the bounded sequence $(K_{f_k})_{k\in\mathbb{N}}$ converges to K, we know that $(d_k)_{k\in\mathbb{N}}$ converges to 0. Also, for any $x \in X$, we have $d(f_k, f) = \sup_{x\in X} |f_k(x) - f(x)| \leqslant d_k$. As a consequence, we deduce that $(f_k)_{k\in\mathbb{N}}$ converges to f in ΔX. Since EX is closed in ΔX, we conclude that $f \in EX$. This concludes that EX is locally compact. $\square$

Theorem 4.4. *The symmetric space X of $\mathrm{GL}(n,\mathbb{K})$, endowed with the distance d, is coarsely Helly. Moreover, the injective hull of (X,d) is locally compact.*

Proof. We will first check that $\hat{X}$ satisfies the three assumptions of Proposition 2.1.

- For every $\eta \in \hat{X}$ and every $a \in \mathbb{R}$, we know that $e^a\eta$ is a norm on V, hence $e^a\eta \in \hat{X}$.
- For every $\eta, \eta' \in \hat{X}$, let $a = d(\eta,\eta') = \sup_{v\in V\setminus\{0\}} \left|\log \frac{\eta(v)}{\eta'(v)}\right| \in \mathbb{R}_+$. For each $v \in V$, we have $\eta(v) \leqslant e^a\eta'(v)$ and $\eta'(v) \leqslant e^a\eta(v)$, hence $e^{-a}\eta' \leqslant \eta \leqslant e^a\eta'$.
- For every non-empty subset $F \subset \hat{X}$ such that there exists $\eta \in \hat{X}$ with $F \leqslant \eta$, let $\theta = \sup F$. It it clear that θ is a well-defined norm on V, so $\theta \in \hat{X}$, and it is the unique minimum of the set $\{\eta' \in \hat{X} \,|\, F \leqslant \eta'\}$.

According to Proposition 2.1, the balls in $(\hat{X}, d)$ satisfy the Helly property.

We also know by Proposition 4.1 that the metric space (X,d) is geodesic, and by Proposition 4.2 we know that X is cobounded in $\hat{X}$, which is locally compact. So we can apply Proposition 4.3 to deduce that (X,d) is coarsely Helly, and that the injective hull of (X,d) is locally compact. $\square$

This raises the following natural questions: is the space $\hat{X}$ of all norms geodesic ? If not, can we describe the injective hull EX inside $\hat{X}$?

4.2 Classical symmetric spaces of non-compact type

We now show how to apply the previous results concerning the general linear group to the other classical groups.

Fix a classical almost simple non-compact real Lie group G over $\mathbb{R}$ which is not of type A, i.e. G is commensurable to one of $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{C})$, $\mathrm{Sp}(n,\mathbb{H})$, $\mathbb{O}(n,\mathbb{C})$, $\mathbb{O}(n,\mathbb{H})$, $\mathbb{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$.

There exists $n \geqslant 1$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and a finite group F of automorphisms of $\mathrm{GL}(n,\mathbb{K})$ such that G embeds in $\mathrm{GL}(n,\mathbb{K})$ and identifies with the fixed point subgroup $\mathrm{GL}(n,\mathbb{K})^F$. Furthermore, if we denote by K a maximal compact subgroup of G, we can assume that $K = \mathrm{U}(n)^F$, and that the corresponding embedding of the symmetric space $X = G/K$ of G into the symmetric space $Y = \mathrm{GL}(n,\mathbb{K})/\,\mathrm{U}(n)$ has image the fixed point set $X = Y^F$ of F. We endow X with the induced piecewise ℓ^∞ metric d from Y. Let us denote $\hat{X} = \hat{Y}^F$.

Proposition 4.5. *Any classical irreducible symmetric space of non-compact type X, which is not of type A, is cobounded in $\hat{X}$.*

Proof. Let $K \in \hat{Y}^F$. Let $B \subset K$ denote the unique John-Löwner ellipsoid of maximal volume. By uniqueness, we deduce that B is invariant under F, i.e. $B \in \hat{X}^F$. According to [Joh48], we know that $d(B,K) \leqslant \log(\sqrt{an})$, where $a = \dim_{\mathbb{R}}(\mathbb{K})$. Therefore any point of $\hat{X}^F$ is at distance at most $\log(\sqrt{an})$ from X. $\square$

Theorem 4.6. *Let X denote a classical irreducible symmetric space of non-compact type, which is not of type A. Then the Finsler metric space (X, d) is coarsely Helly, and its injective hull is locally compact.*

Proof. According to Theorem 4.4, the symmetric space $Y = \mathrm{GL}(n, \mathbb{K}) / \mathrm{U}(n)$, endowed with the piecewise ℓ^∞ distance, is coarsely Helly, and its injective hull EY is locally compact. The isometric action of the finite group F on Y extends to an isometric action on EY.

According to [Lan13, Proposition 1.2], the fixed point set $(EY)^F$ of F on EY is an injective metric space. Therefore, the injective hull EX of X may be realized as an isometric closed subspace of $(EY)^F$, so EX is locally compact.

On the other hand, since $X = Y^F$ is geodesic and cobounded in $\hat{X} = \hat{Y}^F$, which satisfy the Helly property for balls, we deduce by Proposition 4.3 that the metric space X is coarsely Helly. □

The following consequence is immediate.

Corollary 4.7. *Let G denote any reductive Lie group over $\mathbb{R}$, with classical non-compact almost simple factors. Let $a \geqslant 0$ denote the number of almost simple factors of type A. Then $G \times \mathbb{R}^a$ acts properly and cocompactly on an injective metric space. In particular, for any cocompact lattice Γ in G, the group $\Gamma \times \mathbb{Z}^a$ acts properly and cocompactly on an injective metric space.*

As we will see below, the factors $\mathbb{R}^a$ and $\mathbb{Z}^a$ are necessary.

5 The special linear group is not coarsely Helly

We now turn to the case of the special linear group. We will prove that it is not coarsely Helly, inspired by the result of Hoda that the $(3, 3, 3)$ triangle Coxeter group W, which is virtually $\mathbb{Z}^2$, is not Helly (see [Hod20]). However, the group W is a subgroup of $\mathbb{Z}^3 \rtimes \mathfrak{S}_3$, which is Helly. This situation is analogous to the inclusion of $\mathrm{SL}(n, \mathbb{K})$ in $\mathrm{GL}(n, \mathbb{K})$:

Theorem 5.1. *Let $\mathbb{K}$ be a local field (with characteric different from 2 if $\mathbb{K}$ in non-Archimedean), and let $n \geqslant 3$. Then $\mathrm{SL}(n, \mathbb{K})$ is not coarsely Helly: $\mathrm{SL}(n, \mathbb{K})$ does not act properly and coboundedly on an injective metric space.*

Proof. By contradiction, assume that $G = \mathrm{SL}(n, \mathbb{K})$ acts properly and coboundedly on an injective metric space X.

Let $A \subset \mathrm{SL}(n, \mathbb{K})$ denote the diagonal subgroup, and let $M \subset \mathrm{PSL}(n, \mathbb{K})$ denote the monomial subgroup of $\mathrm{PSL}(n, \mathbb{K})$: $M \simeq A \rtimes \mathfrak{A}_n$ is the subgroup of matrices with exactly one non-zero entry on each row and each column (and $\mathfrak{A}_n$ denotes the alternating group). Let $F \subset A$ denote the finite diagonal subgroup with entries in $\{-1, 1\}$. Since $\mathbb{K}$ has characteristic different from 2, we know that the subgroup of G fixed by the conjugation by F is $G^F = A$. According to [Lan13, Proposition 1.2], the fixed point set X^F of F in X is non-empty and injective. We will prove that M acts properly and coboundedly on the injective metric space X^F. Firstly, since F is normalized by M, we deduce that M stabilizes X^F, and acts properly on X^F. We will prove that A acts coboundedly on X^F, which will imply that M also acts coboundedly on X^F.

Fix $x_0 \in X^F$, and let $C_X \geqslant 0$ such that any $x \in X$ is at distance at most C_X from a point in $G \cdot x_0$.

Fix $x \in X^F$, there exists $g \in G$ such that $d(x, g \cdot x_0) \leqslant C_X$. So we deduce that, for any $f \in F$, we have $d(g \cdot x_0, fg \cdot x_0) \leqslant 2C_X$. Let d_G denote a proper left-invariant metric on G. Since the action of G on X is proper, we deduce that there exists $C_G \geqslant 0$ such that, for any $f \in F$, we have $d_G(g, fgf^{-1}) \leqslant C_G$. Let Y denote the symmetric space or Bruhat-Tits building of G, endowed with the CAT(0) metric, choose a basepoint $y_0 \in Y$ fixed by F, and let $y = g \cdot y_0$. Then there exists $C_Y \geqslant 0$ such that, for any $f \in F$, we have $d(y, f \cdot y) \leqslant C_Y$. Let $\overline{y} \in Y$ denote the CAT(0) barycenter of the finite orbit $F \cdot y$: it is fixed by F, and also $d(y, \overline{y}) \leqslant C_Y$. Since G acts coboundedly on Y, there exists a constant $C'_G \geqslant 0$ and $\overline{g} \in G^F = A$ such that $d_G(g, \overline{g}) \leqslant C'_G$. Let us denote $\overline{x} = \overline{g} \cdot x_0 \in X^F$: there exists a constant C'_X such that $d(\overline{x}, x) \leqslant C'_X$. This proves that the action of A on X^F is cobounded.

So we have proved that the group $M \simeq A \rtimes \mathfrak{A}_n$ acts properly and coboundedly on an injective metric space. By passing to an asymptotic cone, we deduce that the group $\mathbb{R}^{n-1} \rtimes \mathfrak{A}_n$ has a left-invariant injective metric. In particular, this defines an injective norm on the vector space $\mathbb{R}^{n-1}$, with linear isometry group containing the alternating group $\mathfrak{A}_n$.

According to [Nac50], the only $(n-1)$-dimensional injective normed vector spaces are linearly isometry to ℓ_∞^{n-1}: this is a contradiction.

This concludes the proof that $\mathrm{SL}(n, \mathbb{K})$ is not coarsely Helly. $\square$

However, this leaves the following question open: are uniform lattices in $\mathrm{SL}(n, \mathbb{K})$ coarsely Helly ?

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