

A link condition for simplicial complexes, and CUB spaces

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November 16, 2022

ABSTRACT. We motivate the study of metric spaces with a unique convex geodesic bicombing, which we call CUB spaces. These encompass many classical notions of nonpositive curvature, such as CAT(0) spaces and Busemann-convex spaces. Groups having a geometric action on a CUB space enjoy numerous properties.

We want to know when a simplicial complex, endowed with a natural polyhedral metric, is CUB. We establish a link condition, stating essentially that the complex is locally a lattice. This generalizes Gromov's link condition for cube complexes, for the ℓ^∞ metric.

The link condition applies to numerous examples, including Euclidean buildings, simplices of groups, Artin complexes of Euclidean Artin groups, (weak) Garside groups, some arcs and curve complexes, and minimal spanning surfaces of knots.

Introduction

Nonpositive curvature has proven to be a very rich way to study large families of groups. Classical Riemannian sectional nonpositive curvature has been well-studied, and we are interested in developing analogous tools in the setting of cell complexes. More precisely, assume that X is a cell complex, where each cell is identified with a convex polytope in \mathbb{R}^n , endowed with some norm. If X is piecewise Euclidean, one may ask whether X is locally CAT(0), which serves as a perfect analogue of nonpositive curvature, and has many examples and applications ([BH99], [BGS85], [Bal95], [Bal95], [BB08], [AB98], [CM09b], [CM09a], [CM13], [CL10], [Ham09], [Ham12], [BB08], [BB95], [BB00], [BL12], [BM10], [Duc18], [McC09] among many others). However, even if there is a metric link criterion ensuring that a space is locally CAT(0), it is not a combinatorial criterion. The only general situation where such a criterion becomes combinatorial is the setting of CAT(0) cube complexes, i.e. cell complexes made out of unit Euclidean cubes.

Theorem (Gromov's link condition). *[Gro87],[Lea13, Theorem B.8]*

A cube complex X is locally CAT(0) if and only if the link of every vertex of X is a flag simplicial complex.

The theory of CAT(0) cube complexes, notably due to this extremely simple criterion, knows a huge success, popularized by Agol's and Wise's works (notably leading to the solution to the virtual Haken conjecture for 3-manifolds ([HW12], [HW08], [Wis21],[Ago13])). There are also recent numerous works on CAT(0) cube complexes ([Sag95], [CS11], [CN05],

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Keywords : Nonpositive curvature, link condition, simplicial complexes, injective metric spaces, lattices, buildings, braid groups, Artin groups, Garside groups, curve complexes, complexes of groups. **AMS codes** : 20F65,20F67,20F36,05B35,06A12,20E42,52A35

[NR98], [Rol98], [NR03], [NR98], [Hag07], [CDH10], [CFI16], [Gen21], [Hua17b], [Wri12], [Gen20], [BHS17], [HJP16], [Hae21c], [Hae21a], [Cha07], [CMW19], [CM16], [Hua17a], [Hua16], [Mar15], [LU21], [Osa18] among many others).

However, outside of the world of cube complexes, it becomes excessively hard to decide whether a given cell complex is $CAT(0)$. For instance, braid groups act properly and cocompactly by isometries on a nice simplicial complex, the dual Garside complex, which is conjectured to be $CAT(0)$. However, this question appears to be quite hard to answer ([BM10], [HKS16], [Jeo20]). And one cannot replace this space with a $CAT(0)$ cube complex for braid groups ([Hae21c]).

One problem is that the class of $CAT(0)$ cube complexes, though very rich, remains smaller than the class of spaces and groups that we want to call nonpositively curved. For instance, any group with Kazhdan's property (T) may only act with a fixed point on any such $CAT(0)$ cube complex, whereas many of them, such as cocompact lattices in higher rank simple Lie groups, deserved to be called nonpositively curved (for instance, they admit a geometric action on a $CAT(0)$ space).

We are therefore interested in describing criteria ensuring that a simplicial complex, with a norm on each simplex, has nonpositive curvature in some sense. We suggest the following notion of nonpositive curvature: we call a metric space CUB, or Convexly Uniquely Bicomvable, if it admits a unique convex geodesic bicombing. It encompasses some other notions of nonpositive curvature, such as $CAT(0)$ and Busemann-convex, and it retains many important properties of classical nonpositive curvature, see Section 1 for precisions. The study of geodesic bicomblings is quite interesting, and has recently developed (see [Wen05],[KR17],[Lan13], [DL16], [DL15], [Des16], [KL20], [Mie17], [Bas20] notably).

Note that one would like to allow such a theory to encompass the case of a simplicial tiling of \mathbb{R}^n , for all $n \geq 1$, which is the "zero curvature" case.

One simple combinatorial notion of nonpositive curvature is that of systolic complexes ([JS06], [Prz08], [OP09]); however, one cannot tile \mathbb{R}^n by a systolic complex, for $n \geq 3$. There is not even a proper action of \mathbb{Z}^3 on a systolic complex.

A naive approach would be to consider the Euclidean metric of a regular simplex for each simplex, and wonder whether the resulting metric is $CAT(0)$. However, it only works well in dimensions 1 and 2: indeed, according to [KPP12], any regular $CAT(0)$ simplicial complex is systolic. And if \mathbb{R}^n is tiled by Euclidean simplices with acute angles, then $n \leq 3$.

These negative results suggest that we need to consider simplicial complexes where simplices have a particular symmetry. We will consider two such symmetries:

- The case where simplices have a cyclic order symmetry, which we call type A, which corresponds for instance to the case of the \tilde{A}_n simplex tiling of \mathbb{R}^n , see Figure 1.
- The case where simplices have a total order symmetry, which we call type C, which corresponds for instance to the case of the \tilde{C}_n simplex tiling of \mathbb{R}^n , see Figure 2.

In type A, we define a specific polyhedral metric in Section 3, which turns out to behave exceptionally well for applications. In type C, we define the ℓ^∞ orthoscheme metric, which is recalled in Section 3 and has already been studied in [BM10] (for the ℓ^2 version) and in [Hae22a] and [Hae21b]. This set-up allows us to state our main results in a loose way, just in order to get the general philosophy. We refer the reader to Section 4 for the precise (yet simple) statements of Theorem A, Theorem B and Theorem C.

Theorem (Link condition for simplicial complexes). *Let X denote a simplicial complex with (cyclically) ordered simplices. Then X is locally CUB if and only if X is locally a lattice.*

Amazingly enough, it turns out that a large number of simplicial complexes appear to satisfy this local lattice property. In particular, this applies to classical Euclidean buildings, to Artin complexes of some Euclidean Artin groups, to some arc complexes of punctured spheres, to Garside and weak Garside groups, to general simplices of groups, to the complex of homologous multicurves on a surface, to the Kakimizu complex of minimal Seifert surfaces of a link. See Section 9 for all these applications. Also note that the lattice property is actually simple to verify: one has to check that the poset does not contain any bowtie, see Section 4.

Note that this theory applies to the Garside complex of the braid group for either the standard or the dual Garside structure; whereas with the Euclidean metric, the standard Garside complex is not $CAT(0)$, and the dual Garside complex is only conjectured to be $CAT(0)$.

The idea of considering spaces which are locally lattices appears in [BM10], [CCHO21], [HKS16], and [Hir19] and [Hir20]. It is also one of the key components of [Hae22a], [Hae21b] and [HH22].

One key ingredient for the main theorem comes from the theory of injective metric spaces, applied to various spaces constructed from lattices, see [Hae22a] and [Hae21b] which essentially provide the existence of a convex bicombing. The uniqueness part is new, and constitutes the main technical part of this article. It relies notably on work of Descombes and Lang on the combinatorial dimension ([DL15]). Interestingly, one of the arguments involves the horofunction boundary of a vector space with a polyhedral norm.

In addition to the new ideas involved in the proofs of the main theorems, there are several motivations for this article. One is to advertise the study of convex geodesic bicomblings on spaces, which looks rich and promising. Another one is to illustrate how results from [Hae21b], [DL15] and [Mie17] can be rendered accessible through simple combinatorial criteria for simplicial complexes. A third motivation is to illustrate the simplicity of these criteria through numerous examples. We believe these combinatorial link conditions will be useful in future works.

Here is the organization of the article. In Section 1 and 2, we define CUB spaces and review examples and properties of CUB spaces and groups acting on them. In Section 3, we make precise which polyhedral norms on simplicial complexes we will consider. In Section 4, we state precisely the various link conditions for locally CUB simplicial complexes. In Section 5, we recall definitions of orthoschemes complexes of lattices and their relationship with injective metric spaces. In Section 6, we focus on the uniqueness of the convex bicombing in the diagonal quotient of the orthoscheme complex of a graded lattice. In Section 7, we show that this result extends to a possibly non-graded lattice. Finally in Section 8, we complete the proofs of the link conditions. In Section 9, we list numerous applications of these link conditions.

Acknowledgments: The author would like to thank Pierre-Emmanuel Caprace, Victor Chepoi, Anthony Genevois, Jingyin Huang, Dan Margalit, Jon McCammond, Piotr Przytycki, Andrew Putman and Constantin Vernicos for very interesting discussions.

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1 Bicomblings and CUB spaces

Let X denote a geodesic metric space. Typically X will not be uniquely geodesic, hence the need to select, for each pair of points, a geodesic between them. This is called a geodesic bicombing.

In this article, a *convex bicombing* on X will be a convex, consistent, reversible geodesic bicombing, i.e. a map $\sigma : X \times X \times [0, 1] \rightarrow X$ such that:

- (Bicombing) For each $x, y \in X$, the map $t \in [0, 1] \mapsto \sigma(x, y, t)$ is a constant speed reparametrized geodesic from $\sigma(x, y, 0) = x$ to $\sigma(x, y, 1) = y$.
- (Reversible) For each $x, y \in X$, for each $t \in [0, 1]$, we have $\sigma(x, y, t) = \sigma(y, x, 1 - t)$.
- (Consistent) For each $x, y \in X$, for each $s, t \in [0, 1]$, we have $\sigma(x, y, st) = \sigma(x, \sigma(x, y, t), s)$.
- (Convex) For each $x, y, x', y' \in X$, the map $t \in [0, 1] \mapsto d(\sigma(x, y, t), \sigma(x', y', t))$ is convex.

A weaker but useful notion is that of a *conical* bicombing, i.e. satisfying

$$\forall x, x', y, y' \in X, \forall t \in [0, 1], d(\sigma(x, y, t), \sigma(x', y', t)) \leq (1 - t)d(x, x') + td(y, y').$$

Note that any conical, consistent bicombing is convex.

A *local convex bicombing* on X is, for every $x \in X$, the choice of a neighbourhood U of X such that U , with the induced metric, has a convex bicombing. It is called *unique* if

U can be chosen to have a unique convex bicombing. Note that U can be chosen to be a small ball centered at x .

Definition 1.1 (CUB).

We say that X is *convexly uniquely bicombable* (CUB) if X admits a unique convex bicombing.

We say that X is *locally convexly uniquely bicombable* (locally CUB) if X admits a unique local convex bicombing.

The fact that CUB serves as notion of nonpositive curvature is first justified by the following Cartan-Hadamard type theorem.

Theorem 1.2. [Mie17] *Let X denote a complete, simply connected locally CUB metric space. Then X is CUB.*

Note that Miesch's result is not stated for CUB spaces, but for spaces with a consistent local convex bicombing. The local CUB property is one way to ensure the consistency of a local convex bicombing.

This notion also encompasses numerous spaces of nonpositive curvature.

Examples.

- Any CAT(0) space is CUB.
- A uniquely geodesic metric space is CUB if and only if it is Busemann convex.
- A Riemannian manifold is locally CUB if and only if it has nonpositive sectional curvature.
- A finite-dimensional piecewise Euclidean cell complex, endowed with the length metric, is CUB if and only if it is CAT(0).
- A finite-dimensional piecewise hyperbolic cell complex, endowed with the length metric, is CUB if and only if it is CAT(-1).
- Any proper, finite-dimensional injective metric space is CUB ([DL15]).
- Any Banach space is CUB, where the unique convex bicombing is the affine bicombing ([Bas20, Corollary 1.3]).

Let us comment on two examples. If X is a Riemannian manifold, then it is locally uniquely geodesic. Therefore it is locally CUB if and only if it is Busemann convex. According to the second variation formula, this is equivalent to having nonpositive sectional curvature (see [Bus48], and also [Bal95], [BGS85] or [BH99] for instance).

If X is a piecewise Euclidean or piecewise hyperbolic cell complex, assume the local CUB property, we want to see that X is CAT(0) or CAT(-1). In particular, X has a continuous geodesic bicombing. Then, by induction on dimension, we can use Bowditch's criterion ([Bow95], or [BH99] in the case X has finitely many shapes) to reduce to proving that the link of every vertex has no locally geodesic loop of length smaller than 2π . The existence of such a loop would give rise to a discontinuity of geodesics, so vertex links are CAT(1), and X is CAT(0) or CAT(-1).

Let us state very general properties of CUB metric spaces, which are typical of non-positive curvature.

Theorem 1.3. *Let X denote any CUB metric space. We have the following.*

1. X is contractible.

2. X admits Euclidean isoperimetric inequalities.
3. X admits a Z -boundary and a Tits boundary. Moreover, any action on X by isometries extends continuously to the boundaries.

Proof. Let us give a reference for each statement.

1. Since X admits a convex bicombing, it retracts continuously to any point, and so X is contractible.
2. According to [Wen05], any metric space with a convex geodesic bicombing admits Euclidean isoperimetric inequalities.
3. According to [DL15, Theorem 1.4], any metric space X with a convex geodesic bicombing admits a Z -boundary ∂X , defined as equivalence classes of rays with the cone topology. See [Bes96] for the definition of a Z -boundary. If we endow this boundary with the finer Tits topology, we can use results from [KL20].

□

As we will see in the sequel, CUB spaces behave almost as good as general CAT(0) spaces, notably regarding all arguments using uniqueness and convexity of geodesics. They also happen to be much more frequent than CAT(0) spaces, as the list of examples above suggests.

2 Groups acting on CUB spaces

We now turn to properties of isometric group actions on CUB spaces. First we have a simple fixed point result.

Proposition 2.1. *Let X denote any complete CUB metric space. If G is a finite group of isometries of X , then G has a fixed point.*

Moreover, let G is a group of isometries of X . If the fixed point set of G is not empty, it is contractible.

Proof. According to [Des16, Theorem 2.1], there is a well-defined notion of barycenter on X , which is equivariant under isometries. In particular, if G is a finite group of isometries, the barycenter of any orbit is a fixed point.

If G acts on X with a fixed point, let $x, y \in X^G$. Then by uniqueness of the convex bicombing σ on X , we know that $g \cdot \sigma(x, y) = \sigma(x, y)$. In particular $\sigma(x, y)$ is entirely contained in X^G , so σ restricts to a convex bicombing on X^G . In particular, X^G is contractible. □

In particular, we deduce the following.

Corollary 2.2. *If a group G acts properly by isometries on a complete CUB space X , then X is a classifying space for proper actions of G .*

Moreover, in case we have a proper and cocompact action, we can deduce numerous properties for the group.

Theorem 2.3. *Let G denote a group acting properly and cocompactly by isometries on a CUB space.*

1. G is semihyperbolic in the sense of Alonso-Bridson, and in particular:
 - Any polycyclic subgroup of G is virtually abelian.

- Any finitely generated abelian subgroup of G is quasi-isometrically embedded.
- The word and conjugacy problems are soluble for G .
- The centralizer of any element of G is finitely generated, quasi-isometrically embedded and semihyperbolic.

2. G has finitely many conjugacy classes of finite subgroups.

3. G satisfies the Farrell-Jones conjecture.

4. G has type F_∞ . If G is torsion-free, it has type F .

5. G has at most Euclidean Dehn functions.

6. G satisfies the coarse Baum-Connes conjecture.

7. G has an EZ-boundary.

8. G has contractible asymptotic cones.

Proof. Let X denote the CUB space of which G acts properly and cocompactly by isometries. We will give references for all statements.

1. Since X admits a convex bicombing and G acts properly coboundedly on X , we deduce by [BH99, Theorem III.Γ.4.7] that G is semihyperbolic. For the consequences, see [BH99, Theorem III.Γ.4.9, Theorem III.Γ.4.10, Proposition III.Γ.4.15, Proposition III.Γ.4.17], .
2. According to Proposition 2.1, any finite subgroup has a fixed point. Since the action of G is proper and cocompact, there exist finitely many conjugacy classes of finite subgroups.
3. According to [KR17], any group acting geometrically on a space with a convex bicombing satisfies the Farrell-Jones conjecture.
4. The proofs of [BH99, Proposition II.5.13, Lemma I.7A.15] adapt from CAT(0) spaces to CUB spaces immediately. Hence the quotient X/G has the homotopy type of a finite CW-complex.
5. According to Theorem 1.3, X has Euclidean isoperimetric inequalities. These translate to Dehn functions of G being at most Euclidean.
6. According to [FO20, Theorem 1.3], groups acting geometrically on spaces with convex bicomings satisfy the coarse Baum-Connes conjecture.
7. According to Theorem 1.3, X admits a Z-boundary ∂X , that is G -equivariant. This defines an EZ-boundary as in [FL05].
8. The group G is quasi-isometric to the CUB space X . And any asymptotic cone of X admits a bicombing, and is then contractible.

□

Examples. Here is a list of groups acting properly and cocompactly by isometries on a CUB space.

- CAT(0) groups, which include fundamental groups of compact nonpositively curved Riemannian manifolds, and uniform lattices in semisimple Lie groups over local fields.

- Gromov-hyperbolic groups ([Lan13]), and groups hyperbolic relative to Helly groups ([OV20]).
- Helly groups ([CCHO21]), which include FC type Artin groups and weak Garside groups ([HO21]), and more generally injective groups ([Hae22a],[Hae21b]).
- If we replace the assumption of cocompactness by that of coboundedness, then mapping class groups of surfaces and more generally hierarchically hyperbolic groups are examples ([HHP21]).

It turns out that many arguments about CAT(0) spaces rely only on the fact that there are uniquely geodesic with associated convex bicombing. These arguments often carry on to the CUB setting, as the following nonpositive criterion for complexes of groups.

Theorem 2.4. [BH99, Theorem III.C.4.17]

Let $G(\mathcal{Y})$ denote a complex of groups over a scwol \mathcal{Y} such that the geometric realization $|\mathcal{Y}|$ has a metric such that, for each $\sigma \in \mathcal{Y}$, the local development at σ is locally CUB. Then $G(\mathcal{Y})$ is developable, and the simply connected development of \mathcal{Y} is CUB.

3 Norms on simplicial complexes

We are interested in finding norms on simplices that allow to tile \mathbb{R}^n , for any $n \geq 1$. We noted in the introduction that the regular Euclidean n -simplex does not tile \mathbb{R}^n if $n \geq 3$ ([KPP12]). This suggest considering simplices with a particular symmetry.

Note that there are 4 infinite families of Euclidean tilings of \mathbb{R}^n , for all $n \geq 1$, with fundamental domain a simplex. They correspond to the Coxeter complexes of the affine Coxeter groups of types \tilde{A}_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n . Note that both \tilde{B}_n and \tilde{D}_n tilings may be refined into the \tilde{C}_n tiling. See Figures 1 and 2 for pictures of the \tilde{A}_2 tiling and the \tilde{C}_2 tiling.

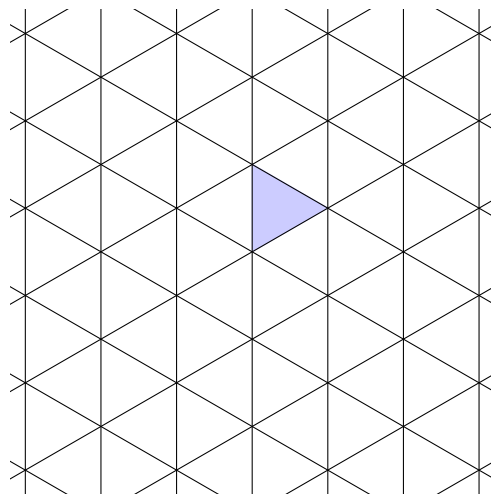


Figure 1: The \tilde{A}_2 tiling of the plane, with the \tilde{A}_2 simplex in blue.

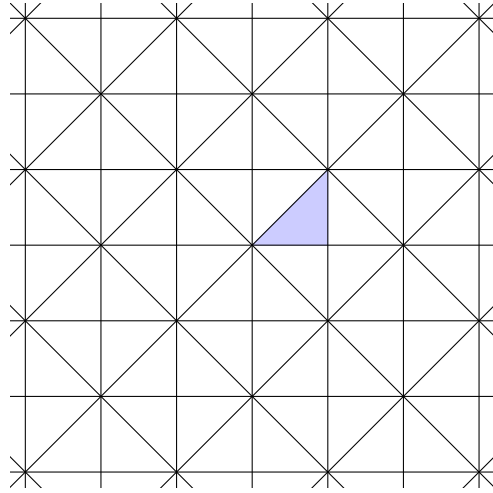


Figure 2: The \tilde{C}_2 tiling of the plane, with the \tilde{C}_2 simplex in blue.

We will therefore focus on one hand on the simplices coming from the \tilde{A}_n tiling, whose vertices have a total cyclic order, which we will call type A. On the other hand, we will focus on the simplices coming from the \tilde{C}_n tiling, whose vertices have a total order, which we will call type C. We give details below, starting with the type C case which is a bit simpler.

We will see in the applications (see Section 9) that the assumption that simplices have an order on their vertices is often quite natural and geometric. Oftentimes, these orders will come from a naturally defined rank or type of vertices.

3.1 Type C

Consider an n -dimensional simplex σ , with a total order on its vertices, which we could then label $v_0 < v_1 < \dots < v_n$. Then one can naturally identify σ with the standard n -simplex of type \tilde{C}_n , which is also called the standard orthosimplex. It may be defined as the convex hull in \mathbb{R}^n of the set of points $v_0 = (0, 0, \dots, 0), v_1 = (1, 0, \dots, 0), \dots, v_n = (1, 1, \dots, 1)$, see Figure 3. It also coincides with a simplex of the barycentric subdivision of the n -cube $[0, 2]^n$, see Figure 4.

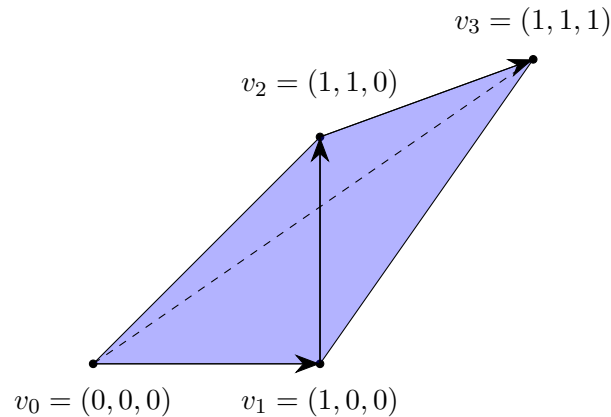


Figure 3: The standard 3-orthosimplex, with the total order on vertices.

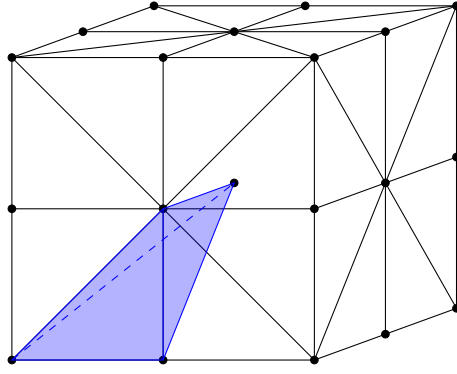


Figure 4: The partition of a cube in \mathbb{R}^3 into standard orthosimplices.

As a subset of \mathbb{R}^n , the standard orthosimplex may naturally be endowed with one of the following two norms:

- The standard ℓ^2 Euclidean norm.
- The standard ℓ^∞ norm, given by $\|x\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$.

While many works have focused on the ℓ^2 Euclidean norm on orthoscheme complexes (see [BM10], [HKS16], [Jeo20], [DMW20]), in this article we will only consider the standard ℓ^∞ norm. Let us be precise about the metric simplicial complexes we will be considering.

Definition 3.1. A simplicial complex X is said to have *ordered simplices* if each simplex of X has a total order on its vertex set, which is consistent with respect to inclusions of simplices.

For such complexes, we are able to define the standard ℓ^∞ metric.

Definition 3.2. Let X denote a simplicial complex with ordered simplices, with finite simplices. Endow each d -simplex of X with the standard ℓ^∞ norm of the standard orthosimplex in \mathbb{R}^d , and endow X with the associated length metric: this is called the *standard ℓ^∞ metric*.

A very important remark is that this metric is well-defined: indeed if τ is a face of a simplex σ with totally ordered vertices, then the standard ℓ^∞ metric on τ (with the induced order) is an isometric subspace of the standard ℓ^∞ metric on σ . Note that this would not be the case with the Euclidean metric: if σ is a triangle with vertices $v_0 < v_1 < v_2$, then the edge between v_0 and v_2 has length $\sqrt{2}$ for the standard Euclidean metric of σ . For the ℓ^∞ metric, all edges have length 1.

Also note that if X is a simplicial complex with ordered simplices, and if G is a group of simplicial automorphisms which either preserve or reverse the orders on vertices of simplices, then G acts by isometries on X with the standard ℓ^∞ metric.

3.2 Type A

Consider an n -dimensional simplex σ , with a total cyclic order on its vertices, which we could then label $(v_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$. Then one can naturally identify σ with the standard n -simplex of type \tilde{A}_n . It may be defined as the convex hull in $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \dots + x_{n+1} = 0\}$ of the set of points

$$v_0 = (0, 0, \dots, 0), v_1 = \left(\frac{n}{n+1}, \frac{-1}{n+1}, \dots, \frac{-1}{n+1}\right), \dots, v_n = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}, \frac{-n}{n+1}\right).$$

More precisely, for $0 \leq i \leq n$, let $j = n + 1 - i$, then the first i coordinates of v_i are equal to $\frac{j}{n+1}$ and the last j coordinates of v_i are equal to $\frac{-i}{n+1}$, see Figure 5.

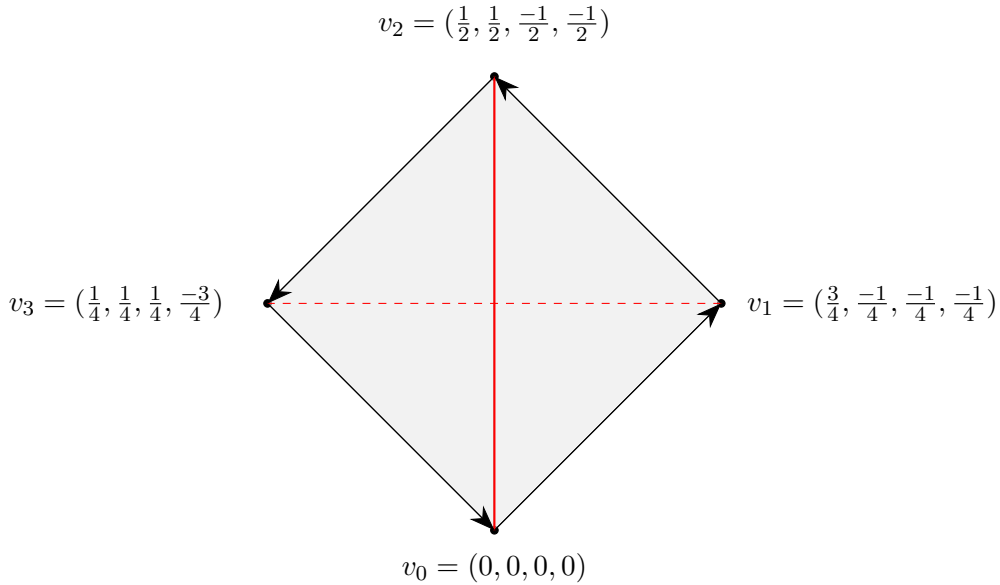


Figure 5: The standard 3-simplex of type \tilde{A}_3 , with the cyclic order on vertices. The two red edges have dihedral angle $\frac{\pi}{2}$, the other four edges have dihedral angle $\frac{\pi}{3}$.

In this description, the cyclic symmetry is not transparent: the standard simplex of type \tilde{A}_n may equivalently be defined as

$$\sigma = \{x \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \dots + x_{n+1} = 0, x_1 \geq x_2 \geq \dots \geq x_n \geq x_{n+1} - 1\}.$$

Therefore, we can also describe the standard simplex of type \tilde{A}_n as the image, in the quotient under the diagonal translation action of \mathbb{R} , of the standard column C in \mathbb{R}^{n+1} :

$$C = \{x \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \dots + x_{n+1} = 0, x_1 \geq x_2 \geq \dots \geq x_n \geq x_{n+1} - 1\}.$$

Note that the column C has a natural description as a simplicial complex, with linearly ordered $(n+1)$ -dimensional simplices with totally ordered vertices. Roughly speaking, these simplices appear when looking at chambers obtained by cutting C using the hyperplanes $\{x_i \in \mathbb{Z}\}$ in \mathbb{R}^{n+1} .

More precisely, we have $C = \bigcup_{q \in \mathbb{Z}, 0 \leq r \leq n} \tau_{q,r}$, where

$$\tau_{q,r} = \{x \in \mathbb{R}^{n+1} \mid q \geq x_r \geq x_{r+1} \geq x_{r+2} \geq \dots \geq x_{n+1} \geq x_1 - 1 \geq x_2 - 1 \geq \dots \geq x_r - 1 \geq q - 1\},$$

see Figure 6. The study and use of columns is fundamental in [BM10], [DMW20] and [Hae21b].

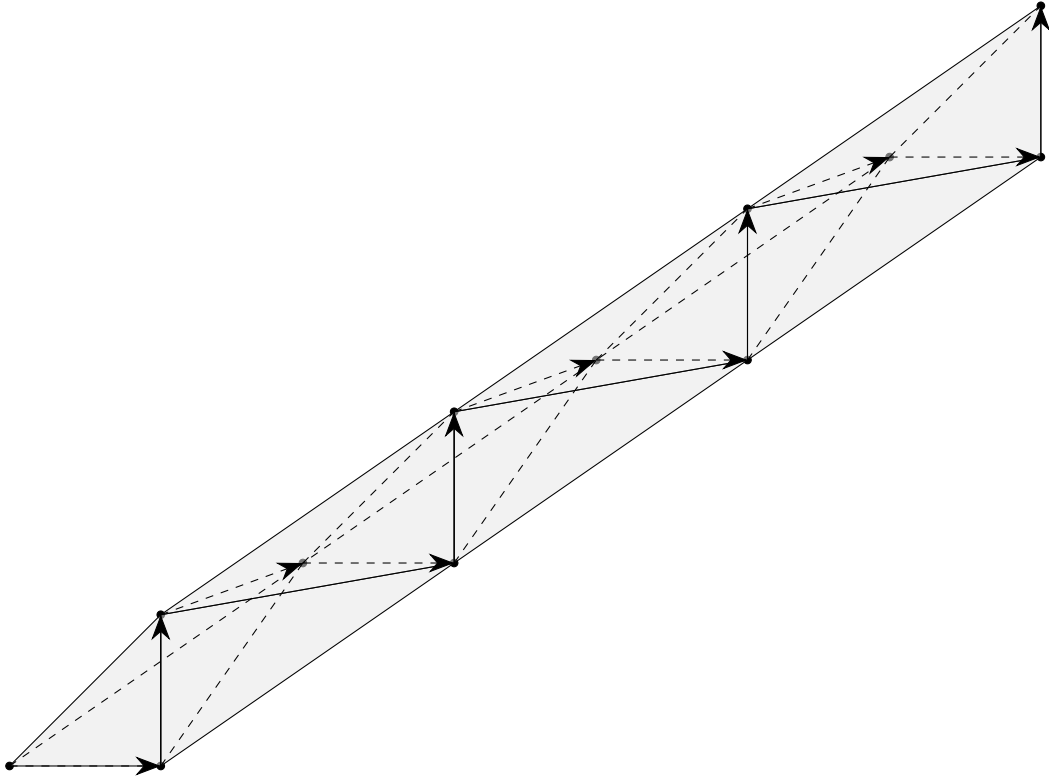


Figure 6: The standard column C in \mathbb{R}^3 , tiled by standard 3-orthosimplices.

It turns out that there are two natural norms on the standard \tilde{A}_n simplex:

- The standard ℓ^2 Euclidean norm.
- The norm $\|x\| = \sup_{1 \leq i \neq j \leq n+1} |x_i - x_j|$, which we will call *standard polyhedral norm*.

Note that the standard polyhedral norm can also be described more naturally as the quotient of the standard column $C \subset \mathbb{R}^{n+1}$, endowed with the standard ℓ^∞ metric, under the diagonal translation action by \mathbb{R} . The standard Euclidean metric is also the quotient of the standard Euclidean metric of $C \subset \mathbb{R}^{n+1}$ by the diagonal translation action by \mathbb{R} .

Also note that both metrics are invariant under a (possibly order-reversing) cyclic permutation of the vertices of the standard \tilde{A}_n simplex. This is easier to see in the column picture, since for instance $(x_1, x_2, \dots, x_{n+1}) \in C \mapsto (x_2, x_3, \dots, x_{n+1}, x_1 - 1) \in C$ is an isometry for both the standard Euclidean and the standard polyhedral norm. However, an order-reversing permutation of the vertices of the standard \tilde{A}_n simplex does not lift to an isometry of the standard column C , but it still induces an isometry of the standard \tilde{A}_n simplex for either norm.

The unit ball of the standard polyhedral norm in \mathbb{R}^n is the projection of the standard $(n+1)$ -cube (with edge lengths 2) in \mathbb{R}^{n+1} by the diagonal translation action of \mathbb{R} . In dimension $n=2$, it is a regular hexagon, and in dimension $n=3$ it is called the rhombic dodecahedron, see Figure 7.

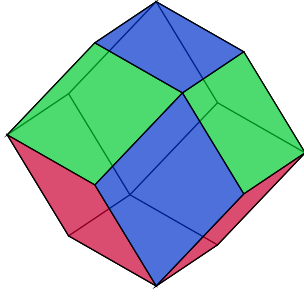


Figure 7: The unit ball of the standard polyhedral norm in \mathbb{R}^3 : the rhombic dodecahedron.

One should also note that the n -simplex, endowed with the Hilbert metric ([Nus88], [LW11]), is isometric to the normed space

$$\left(\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \cdots + x_{n+1} = 0\}, \|x\| = \sup_{1 \leq i \neq j \leq n+1} |x_i - x_j| \right).$$

So the norm is the same as the standard polyhedral norm we defined. However, the embedding of an n -simplex in this model is not immediate: in particular, it depends on the particularly chosen cyclic order on the vertices on the n -simplex.

We now make precise how to define a metric if there is a cyclic order on the vertices of each simplex of a simplicial complex.

Definition 3.3. A simplicial complex X is said to have *cyclically ordered simplices* if each simplex of X has a total order on its vertex set, which is consistent with respect to inclusions of simplices.

For such complexes, we are able to define the standard polyhedral metric.

Definition 3.4. Let X denote a simplicial complex with cyclically ordered simplices, with finite simplices. Endow each d -simplex of X with the standard polyhedral norm of the standard d -simplex, and endow X with the associated length metric: this is called the *standard polyhedral metric*.

Note that, as in the type C case, this metric is well-defined: if τ is a face of a simplex σ with cyclically ordered vertices, then the standard polyhedral metric on τ (with the induced cyclic order) is an isometric subspace of the standard polyhedral metric on σ .

Also note that if X is a simplicial complex with cyclically ordered simplices, and if G is a group of simplicial automorphisms which either preserve or reverse the cyclic orders on vertices of simplices, then G acts by isometries on X with the standard polyhedral metric.

4 Statements of the link conditions

We will present the link conditions for nonpositive curvature. These are local criteria ensuring the CUB property in the case of simplicial complexes with (cyclically) ordered simplices. In order to state the criteria precisely, we will recall basic definitions on lattices and bowties.

4.1 Lattices and bowties

We start by recalling necessary definitions of posets, lattices and bowties.

Definition 4.1.

A *lattice* is a poset L such that any two $x, y \in L$ have a minimal upper bound denoted $x \vee y$, called the *join* of x and y , and a maximal lower bound denoted $x \wedge y$, called the *meet* of x and y .

A (meet)-*semilattice* is a poset where one only requires the existence of meets.

A poset L is *bounded* if it has a minimum (usually denoted 0), and a maximum (usually denoted 1).

A poset L is *graded* if, given any $x < y$ in L , every maximal chain from x to y has the same length (depending on x, y).

A poset L is *homogeneous* if, given any $x < y$ in L , there is a bound on the length of chains from x to y (depending on x, y).

A graded lattice L has rank n if every maximal chain in L has $n + 1$ elements.

A *bowtie* in a poset L consists in 4 elements $a, b, c, d \in L$ such that $a, b < c, d$, and such that there exists no $x \in L$ such that $a, b \leq x \leq c, d$. If L is graded, we say that a bowtie $a, b < c, d$ is *balanced* if $\text{rk}(a) = \text{rk}(b)$ and $\text{rk}(c) = \text{rk}(d)$.

Example. Here are simple examples of such posets.

- The boolean poset $\mathcal{P}(\{1, \dots, n\})$, ordered by inclusion, is a bounded graded lattice of rank n .
- The poset of vector subspaces of an n -dimension vector space, ordered by inclusion, is a bounded graded lattice of rank n .
- The poset of finite-dimensional vector subspaces of an arbitrary vector space, ordered by inclusion, is a graded semilattice.
- The poset of partitions of $\{1, \dots, n\}$, ordered by refinement, is a bounded graded lattice of rank $n - 1$.
- Fix $n \geq 2$, and consider the set vertex set $U_n \subset \mathbb{R}^2$ of a regular n -gon in the plane. Say that a partition P of U_n is noncrossing if, for any distinct $A, B \in P$, the convex hulls of A and B do not intersect. Then the poset of noncrossing partitions of U_n , ordered by refinement, is a bounded graded lattice of rank $n - 1$. Note that his example generalizes to any finite Coxeter group ([BW02], [Bes03]).

It turns out that bowties are quite efficient to decide whether a poset is a lattice.

Proposition 4.2. *Let L denote a graded poset. Then $L \cup \{0, 1\}$ is a lattice if and only if L has no balanced bowtie.*

Proof. Assume that $L \cup \{0, 1\}$ is a lattice, and consider $a, b < c, d$ in L . Then the meet x of c, d is such that $a, b \leq x \leq c, d$. So L has no bowties.

Conversely, assume that L has no bowtie $a, b < c, d$ with $\text{rk}(a) = \text{rk}(b)$ and $\text{rk}(c) = \text{rk}(d)$.

Fix $c, d \in L$ with $\text{rk}(c) = \text{rk}(d)$, we will prove that c, d have a meet in $L \cup \{0\}$. It is sufficient to prove that no bowtie in L contain c, d as upper elements. By contradiction, assume that there exists a bowtie $a, b < c, d$. By assumption, we know that $\text{rk}(a) \neq \text{rk}(b)$, say $\text{rk}(a) < \text{rk}(b)$. Pick $b' \in L$ such that $b' < b$ and $\text{rk}(b') = \text{rk}(a)$. By assumption, there exists $x \in L$ such that $a, b' \leq x \leq c, d$. Since $a, b < c, d$ is a bowtie, we deduce that $a = x$. So $a \leq b$, which contradicts that $a, b < c, d$ is a bowtie.

By symmetry, we also know that any $a, b \in L$ with $\text{rk}(a) = \text{rk}(b)$, have a join in $L \cup \{1\}$.

Now fix any $c, d \in L$, we will prove that c, d have a meet in $L \cup \{0\}$. As before, it is sufficient to prove that no bowtie in L contain c, d as upper elements. By contradiction, assume that there exists a bowtie $a, b < c, d$.

If $\text{rk}(a) = \text{rk}(b)$, then since a, b have a join this contradicts that $a, b < c, d$ is a bowtie.

So we know that $\text{rk}(a) \neq \text{rk}(b)$, say $\text{rk}(a) < \text{rk}(b)$. Pick $b' \in L$ such that $b' < b$ and $\text{rk}(b') = \text{rk}(a)$. Now let x denote the join of a, b' : it is such that $a, b' \leq x \leq c, d$. Since $a, b < c, d$ is a bowtie, we deduce that $a = x$. So $a \leq b$, which contradicts that $a, b < c, d$ is a bowtie.

Hence any $c, d \in L$ have a meet in $L \cup \{0\}$.

By symmetry, we also know that any $a, b \in L$ have a join in $L \cup \{1\}$. Hence $L \cup \{0, 1\}$ is a lattice. \square

4.2 Type A

Given a flag simplicial complex X with cyclically ordered simplices, for each vertex $x \in X$, there is a natural antisymmetric relation \leq_x on $\text{St}(x)$, with minimal element x , obtained by declaring that $y \leq_x z$ iff x, y, z form a triangle, and x, y, z are cyclically ordered.

Definition 4.3. We say that a simplicial complex with cyclically ordered simplices is a *local poset* if, for any vertex $x \in X$, the $(\text{St}(x), \leq_x)$ is a poset. In other words:

$$\forall x, y, z, t \in X, (x \leq y \leq z \leq x) \wedge (x \leq z \leq t \leq x) \Rightarrow (x \leq y \leq t \leq x).$$

We are now able to state precisely the link condition for simplicial complexes with cyclically ordered simplices.

Theorem A. *Let X denote a locally finite-dimensional flag simplicial complex with cyclically ordered simplices, which is a local poset. Then X , endowed with the standard polyhedral norm, is locally CUB if and only if, for every vertex $x \in X$, the poset $(\text{St}(x), \leq_x)$ is a semilattice.*

Note that, in the case $(\text{St}(x), \leq_x)$ is not a semilattice, we prove that X does not even have one local conical bicombing.

Remark. The lattice condition has several equivalent formulations:

1. $(\text{St}(x), \leq_x)$ is a semilattice.
2. $(\text{St}(x) \cup \{1\}, \leq_x)$ is a lattice.
3. $(\text{St}(x), \leq_x)$ has no bowtie.

In the case where all maximal simplices containing x have the same dimension, the poset $(\text{St}(x), \leq_x)$ is graded, and these conditions are also equivalent to:

4. $(\text{St}(x), \leq_x)$ has no balanced bowtie.

4.3 Type C

Given a flag simplicial complex X with ordered simplices, for each vertex $x \in X$, there is a natural induced antisymmetric relation \leq_x on $\text{St}(x)$.

Definition 4.4. We say that a simplicial complex with ordered simplices is a *local poset* if, for any vertex $x \in X$, the set $(\text{St}(x), \leq_x)$ is a poset. In other words:

$$\forall x \in X, \forall y, z, t \in \text{St}(x), (y \leq z) \wedge (z \leq t) \Rightarrow (y \leq t).$$

A simplicial complex X is said to have *ordered simplices* if each simplex of X has a total order on its vertex set, which is consistent with respect to inclusions of simplices.

Given a simplicial complex X with ordered simplices, for each vertex $x \in X$, there is a natural antisymmetric relation \leq_x on $\text{St}(x)$, with distinguished element x , obtained by declaring that $y \leq_x z$ iff x, y, z form a triangle, and the edge between y and z is oriented from y to z . We will also consider the subposet $\text{St}^+(x) = \{y \in \text{St}(x) \mid y \geq x\}$, with minimum x , and the subposet $\text{St}^-(x) = \{y \in \text{St}(x) \mid y \leq x\}$, with maximum x .

We are now able to state precisely the link condition for simplicial complexes with cyclically ordered simplices.

Theorem B. *Let X denote a locally finite-dimensional flag simplicial complex with ordered simplices, which is a local poset. Then X , endowed with the standard ℓ^∞ norm, is locally CUB and is locally injective if and only if, for any vertex $x \in X$, we have:*

- (Lattice condition) *The poset $(\text{St}(x), \leq_x)$ has no bowtie.*
- (Flag condition) *Any $a, b, c \in \text{St}(x)$ which are pairwise upperly bounded (resp. lowerly bounded) have a common upper bound (resp. lower bound).*

Remark that, in the case $(\text{St}(x), \leq_x)$ does not satisfy any of the two conditions, we prove that X is not locally injective and does not even have one local convex bicombing. Note that this flag condition is called "lattice-theoretic flag condition" in [Hir19].

Remark. The lattice condition has several equivalent formulations:

1. $(\text{St}(x), \leq_x)$ has no bowtie.
2. $(\text{St}^+(x), \leq_x)$ is a meet-semilattice and $(\text{St}^-(x), \leq_x)$ is a join-semilattice.
3. $(\text{St}(x) \cup \{0, 1\}, \leq_x)$ is a lattice.

In the case where all maximal simplices containing x have the same dimension, the poset $(\text{St}(x), \leq_x)$ is graded, and these conditions are also equivalent to:

4. $(\text{St}(x), \leq_x)$ has no balanced bowtie.
5. $(\text{St}^+(x), \leq_x)$ and $(\text{St}^-(x), \leq_x)$ have no balanced bowtie.

Similarly, the flag condition is equivalent to asking that the following two conditions hold:

- Any $a, b, c \in \text{St}^+(x)$ which are pairwise upperly bounded have a common upper bound.
- Any $a, b, c \in \text{St}^-(x)$ which are pairwise lowerly bounded have a common lower bound.

Also note that Theorem B applies to all cube complexes. Indeed if X is a locally finite-dimensional cube complex, consider the barycentric subdivision X' of X . It is a simplicial complex with ordered simplices, and the standard ℓ^p metric on orthosimplices coincides with the standard ℓ^p metric on cubes (up to a factor 2). If x is a vertex of X , the lattice condition for $\text{St}_{X'}(x)$ is equivalent to requiring that the link of x in X is simplicial, and the flag condition for $\text{St}_{X'}(x)$ is equivalent to requiring that the link of x in X is a flag simplicial complex. Hence we recover Gromov's link condition for the ℓ^∞ norm. Note that, for a finite-dimensional cube complex, requiring that the piecewise Euclidean metric is CAT(0) is equivalent to requiring that the piecewise ℓ^∞ metric is CUB (see [Mie14]).

The analogous statement for the piecewise Euclidean metric is false, see [Hae21b, Theorem 3.10]. However, according to [Hir19] (see also [HKS16]), it is true under the extra (very restrictive) assumption that $(\text{St}(e), \leq_e)$ is a semimodular lattice.

4.4 Garside flag complexes

A particular class of examples encompassing both types of shapes of simplices are Garside flag complexes (see [HH22]), which we now define.

Definition 4.5. A *Garside flag complex* is a pair (X, φ) , where X is a simply connected flag simplicial complex with finite simplices, with ordered simplices, and φ is an order-preserving automorphism of X , such that the following holds:

- For any simplex σ of X , we have that $\sigma \cup \varphi(\min \sigma)$ is a simplex of X .
- For any vertex $x \in X$, we have $\varphi(x) > x$, and the interval $[x, \varphi(x)]$ is a homogeneous lattice.

If (X, φ) is a Garside flag complex, the quotient X/φ is defined as the flag simplicial complex with vertex set $X^{(0)}/\langle \varphi \rangle$, whose k -simplices correspond to images of chains $x_0 < x_1 < \dots < x_k < \varphi(x_0)$. Note that the quotient X/φ has cyclically ordered simplices.

For this class of simplicial complexes, we are now able to state precisely the link condition.

Theorem C. *Let (X, φ) denote a Garside flag complex. Then X , endowed with the standard ℓ^∞ metric, is CUB and injective. Moreover the quotient X/φ , endowed with the standard polyhedral metric, is CUB.*

Note that the lattice property is also necessary, as in the proof of Theorem B.

5 Lattices, orthoscheme complexes and injective metric spaces

We will review here the relationship between lattices, orthoscheme complexes and injective metric spaces developed in [Hae22a] and [Hae21b].

5.1 Injective metric spaces and combinatorial dimension

A geodesic metric space is called *injective* if any family of pairwise intersecting balls has a non-empty global intersection. See for example [Lan13] for an introduction to injective metric spaces. It turns out that injective metric spaces are ubiquitous:

Theorem 5.1 ([Isb64]). *Any metric space X embeds isometrically in a unique minimal injective metric space EX , its injective hull.*

Injective metric spaces are relevant for CUB spaces because Lang proved that any injective metric space admits a canonical conical geodesic bicombing. Under properness assumption, Descombes and Lang improved the result to an actual convex geodesic bicombing.

Theorem 5.2. [DL15, Theorem 1.1] *Let X denote a proper injective metric space. Then X admits a convex geodesic bicombing.*

Concerning uniqueness, Descombes and Lang provided a criterion relying on the notion of combinatorial dimension.

Definition 5.3. The *combinatorial dimension* of a metric space X is the topological dimension of its injective hull EX .

Example. For instance, if X is a CAT(0) cube complex with the piecewise ℓ^∞ metric, the combinatorial dimension of X coincides with its dimension as a cube complex.

Note that, if X is an isometric subspace of Y , the combinatorial dimension of X is bounded above by the combinatorial dimension of Y .

The combinatorial dimension of a metric space is usually hard to compute. However, we have the following criterion due to Dress.

Theorem 5.4. [DL15, Theorem 4.1] *Let X denote a metric space, and let $n \geq 1$ be an integer. The space X has combinatorial dimension at most n if and only if, for every finite subset $Z \subset X$ with $|Z| = 2(n+1)$ and for every fixed point free involution $i : Z \rightarrow Z$, there exists a fixed point free bijection $j : Z \rightarrow Z$ distinct from i such that*

$$\sum_{z \in Z} d(z, i(z)) \leq \sum_{z \in Z} d(z, j(z)).$$

Descombes and Lang proved that, for metric spaces with finite combinatorial dimension, there could be at most one convex geodesic bicombing.

Theorem 5.5. [DL15, Theorem 1.2] *Let X denote a metric space with finite combinatorial dimension. Then X admits at most one convex geodesic bicombing.*

Combining both results, we immediately get the following.

Corollary 5.6 (Descombes-Lang). *Let X denote a proper, finite-dimensional, injective metric space. Then X is CUB.*

We also record, for later use, the following elementary decomposition result.

Proposition 5.7. *Let L denote a poset, such that x is comparable to every element of L . Let $L^+ = \{y \in L \mid y \geq x\}$ and $L^- = \{y \in L \mid y \leq x\}$. Then the geometric realization $|L|$ of L , with the standard ℓ^∞ metric, is locally isometric at x to the ℓ^∞ product $|L^+| \times |L^-|$ of the geometric realizations of L^+ and L^- with the standard ℓ^∞ metrics.*

Proof. It suffices to notice that this statement is true for orthosimplices, with the ℓ^∞ metric. Consider the standard n -orthosimplex $\sigma \subset \mathbb{R}^n$, with vertices $v_0 = (0, 0, \dots, 0) < v_1 = (1, 0, \dots, 0) < \dots < v_n = (1, \dots, 1)$. More precisely, σ is defined by the following inequalities:

$$\sigma = \{x \in \mathbb{R}^n \mid 1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}.$$

Given any $0 < k < n$, note that v_k does not lie on the hyperplane $\{x_k = x_{k+1}\}$ supporting a face of σ . So a neighbourhood of v_k in σ is isometric to a neighbourhood of v_k in

$$E = \{x \in \mathbb{R}^n \mid 1 \geq x_1 \geq x_2 \geq \dots \geq x_k \geq 0, 1 \geq x_{k+1} \geq x_{k+2} \geq \dots \geq x_n \geq 0\}.$$

The space E , with the ℓ^∞ norm, is isometric to the ℓ^∞ product of two standard orthosimplices of dimensions k and $n - k$. \square

5.2 Orthoscheme complexes of lattices

We now explain that many examples of injective metric spaces come from geometric realizations of posets.

Let L denote a poset. Then the geometric realization X of L is the simplicial complex whose simplices are chains in L . Note that each simplex of X has thus an induced total order on its set of vertices, so it can be endowed with the standard ℓ^∞ norm. We will then endow X with the induced length metric.

One of the main interest of lattices lies in the following result.

Theorem 5.8. [Hae21b, Theorem 3.9] *Let L denote a bounded graded lattice. Then the geometric realization of L , with the standard ℓ^∞ norm, is injective and CUB.*

In the sequel, we will need more details about the result in Theorem 5.8. If L is a bounded, graded lattice of rank n , we start by describing what we call the affine version of L .

We will define a new poset M , which will be called the *affine version* of L . Let $C(L)$ denote the set of maximal chains $c_{0,1} = 0 <_L c_{1,2} <_L \dots <_L c_{n-1,n} <_L c_{n,n+1} = 1$ in L . We will use the convention that the element denoted $c_{i,i+1}$ has rank i .

Let us consider the subspace of \mathbb{R}^n :

$$\Sigma = \{u \in \mathbb{R}^n \mid u_1 \leq u_2 \leq \dots \leq u_n\}.$$

For each maximal chain $c \in C(L)$, let Σ_c denote a copy of Σ .

Let us consider the space

$$M = \bigcup_{c \in C(L)} \Sigma_c / \sim,$$

where for each $c, c' \in C(L)$, if we denote $I = \{1 \leq i \leq n-1 \mid c_{i,i+1} \neq c'_{i,i+1}\}$, we identify Σ_c and $\Sigma_{c'}$ along the subspaces

$$\{u \in \Sigma_c \mid \forall i \in I, u_i = u_{i+1}\} \simeq \{u \in \Sigma_{c'} \mid \forall i \in I, u_i = u_{i+1}\}.$$

We can describe M as a quotient of the space $M_0 = C(L) \times \Sigma$. If $c \in C(L)$ and $u \in \Sigma$, let us denote by $[c, u]$ the equivalence class of $(c, u) \in M_0$ in M .

Example. One illustrating example is the following: consider the boolean lattice L of rank n , i.e. the lattice of subsets of the finite set $\{1, \dots, n\}$, with the inclusion order. Maximal chains in L correspond to permutations of $\{1, \dots, n\}$. The space M may be identified with \mathbb{R}^n , where for each permutation w of $\{1, \dots, n\}$, the subspace Σ_w is

$$\Sigma_w = \{x \in \mathbb{R}^n \mid x_{w(1)} \leq x_{w(2)} \leq \dots \leq x_{w(n)}\}.$$

We will endow M with the length metric induced by the standard ℓ^∞ metric on each subspace $\Sigma_c \subset \mathbb{R}^n$, for $c \in C(L)$. Note that the geometric realization X of L is naturally a subspace of M :

$$X = \{[c, u] \in M \mid c \in C(L), u \in \Sigma, 0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq 1\}.$$

This metric space M satisfies the following. Recall that a subset X of a metric space M with a bicombing σ is called σ -convex if, given $x, y \in X$, we have $\sigma(x, y) \subset X$.

Theorem 5.9. [Hae21b, Theorem 3.8] *The metric space M is injective, and has a unique convex bicombing σ . Moreover, the subspace $X \subset M$ is isometric and σ -convex.*

There is also a diagonal isometric action of \mathbb{R} on M , given by $s \cdot [c, u] = [c, u_1 + s, u_2 + s, \dots, u_n + s]$.

Proposition 5.10. *The convex bicombing σ on M satisfies the following:*

$$\forall x, x' \in M, \forall t \in [0, 1], \forall s, s' \in \mathbb{R}, \sigma(s \cdot x, s' \cdot x', t) = ((1-t)s + ts') \cdot \sigma(x, x', t).$$

Proof. This is a consequence of the proof of [Hae21b, Theorem 3.8], since the midpoint-nonincreasing conical geodesic bicombing defined there satisfies this equality. \square

We also show that stars of simplices are convex in X .

Lemma 5.11. *Let c denote a chain in L containing 0 and 1, and let $A \subset X$ denote the corresponding simplex of X . Then the star of A in X is σ -convex.*

Proof. For each $a \in c \setminus \{0, 1\}$, let $A_a \subset A \subset X$ denote the corresponding 2-simplex of X . Note that the star of A in X equals the intersection of the stars of A_a , for all $a \in c \setminus \{0, 1\}$. Therefore it is sufficient to consider the case $c = (0 < a < 1)$.

Fix $x, x' \in \text{St}(A)$. By considering $\text{St}(A) \subset X \subset M$, we may find $s \geq 0$ such that $s \cdot x \in \text{St}(A)$ and $s \cdot x \geq a$. Similarly, we may find $s' \geq 0$ such that $s' \cdot x' \in \text{St}(A)$ and $s' \cdot x' \geq a$. Note that the interval $\{x \in X \mid x \geq a\}$ is σ -convex in X . So we deduce that, for any $t \in [0, 1]$, we have $\sigma(s \cdot x, s' \cdot x', t) \geq a$, so in particular $\sigma(s \cdot x, s' \cdot x', t) \in \text{St}(A)$. According to Proposition 5.10, we deduce that $\sigma(x, x', t) \in \mathbb{R} \cdot \text{St}(A) \cap X = \text{St}(A)$. Hence $\text{St}(A)$ is σ -convex. \square

5.3 Unique bicombings in orthoscheme complexes of semilattices

We now turn to the case of orthoscheme complexes of semilattices. Here, in order to obtain the injectivity and the CUB property, we need to ask for the flag condition.

Theorem 5.12. *Let L denote a graded semilattice with minimum 0. Assume that any $a, b, c \in L$ pairwise upperly bounded have a common upper bound. Then the orthoscheme complex of L , with the standard ℓ^∞ metric, is injective and CUB.*

Proof. According to [Hae21b, Theorem 6.1], we know that the orthoscheme complex $|L|$ of L , with the standard ℓ^∞ metric, is injective. Since it is finite-dimensional, according to Theorem 5.5 we know that it admits at most one convex bicombing. It remains to prove that it admits a convex bicombing.

Let $L' = L \cup \{1\}$ denote the bounded poset obtained by adding a maximal element 1. Let $\pi : |L'| \rightarrow |L|$ denote the simplicial map obtained by sending each $x \in L$ to itself, and sending 1 to 0. It is a 1-Lipschitz retraction.

Note that L' is a bounded graded lattice, so according to Theorem 5.8 we know that its geometric realization $|L'|$, endowed with the standard ℓ^∞ metric, is injective and admits a unique convex bicombing σ' .

Let us define $\sigma : L \times L \times [0, 1] \rightarrow L$ by $\sigma(x, y, t) = \pi(\sigma'(x, y, t))$: since π is a 1-Lipschitz retraction, we deduce that σ is geodesic, and that σ is conical. Furthermore, according to Proposition 5.10, we know that σ is consistent, hence σ is convex. In conclusion, σ is a convex bicombing on $|L|$. So $|L|$ is CUB. \square

6 Bicombings on quotients of orthoscheme complexes

We will now explain how unique convex bicombings on orthoscheme complexes give rise to unique convex bicombings on their diagonal quotient.

Let L denote a bounded graded lattice of rank n , and let X denote the geometric realization of L with the standard ℓ^∞ metric.

Let Y denote the geometric realization of the semilattice $L \setminus \{1\}$, with the standard polyhedral norm.

Consider the affine version M of L , with the diagonal isometric action of \mathbb{R} . Then in the quotient metric space $\overline{M} = M/\mathbb{R}$, a neighbourhood of 0 is isometric to a neighbourhood of 0 in Y .

We know the following. It is stated in [Hae21b] for Euclidean buildings or Deligne complexes of Artin groups, but it relies in fact only on Theorem 5.8 stating that the orthoscheme complex of a bounded graded lattice is injective.

Theorem 6.1. *[Hae21b, Theorem 4.6] The geometric realization Y of $L \setminus \{1\}$, with the standard polyhedral norm, admits a convex bicombing.*

We will improve that result to say that this convex bicombing is in fact unique.

We start by proving that combing lines are contained in finitely many chambers, before proving a similar uniform statement.

Lemma 6.2. *Let L denote a bounded graded lattice of rank n , let X denote the orthoscheme complex of L , and let σ denote the unique convex bicombing on X . For any $x, y \in L$, $\sigma(x, y)$ is contained in finitely many chambers of X .*

Proof. Fix $x, y \in X$. For each $z \in \sigma(x, y)$, there exists $\varepsilon > 0$ such that the ball $B(z, \varepsilon)$ is a simplicial cone. Therefore we know that $\sigma(x, y) \cap B(z, \varepsilon)$ is contained in a union of 2 chambers of X . The statement follows from compactness of $\sigma(x, y)$. \square

We will now prove that there is a uniform bound on the possible number of chambers for a combing line.

Proposition 6.3. *Let L denote a bounded graded lattice of rank n , let X denote the orthoscheme complex of L , and let σ denote the unique convex geodesic bicombing on X . For any $x, y \in X$, $\sigma(x, y)$ is contained in a union of at most 2^n chambers.*

Proof. We will prove the following stronger statement. Consider any simplex C of X containing $\{0, 1\}$, and let $k \geq 0$ denote the codimension of C in X . Then for any $x, y \in \text{St}(C)$, $\sigma(x, y)$ is contained in a union of at most 2^k chambers.

We will prove this statement for all $n \geq 0$, by induction on k . For $k = 0$, we have $\text{St}(C) = C$, and x, y are contained in $2^0 = 1$ chamber, C .

Let us now fix $k \geq 1$, and assume that the statement is true for all bounded graded lattices, and for all simplices of codimension at most $k - 1$.

Let L_0 denote a bounded graded lattice, and let c_0 denote the chain corresponding to a simplex C of codimension k . For each maximal chain c of L_0 containing c_0 , let B_c denote a copy of the Boolean lattice $\mathcal{P}(\{1, \dots, n\})$, and consider the following poset $L_1 = \theta(L_0)$:

$$L_1 = L_0 \cup \bigcup_{c \text{ maximal chain in } L_0 \text{ containing } c_0} B_c / \sim,$$

where, for each maximal chain c containing c_0 , the chain $c \subset L_0$ is identified with the chain $(\emptyset, \{1\}, \{1, 2\}, \dots, \{1, \dots, n\})$.

Endow L_1 with the order relation generated by L_0 and each B_c , for c a maximal chain containing c_0 . Then L_0 is a sublattice of L_1 , such that any chain in L_0 is contained in a Boolean sublattice of L_1 .

Let us consider the lattice $L = \bigcup_{j \in \mathbb{N}} \theta^j(L_0)$. Since the orthoscheme complex X_0 of L_0 is an isometric subcomplex of L , it is sufficient to prove that, for any $x, y \in \text{St}(C)$, we have that $\sigma(x, y)$ is contained in a union of at most 2^k chambers of X .

Let M denote the affine version of L , and consider X realized isometrically as a σ -convex subset of Y according to Theorem 5.9.

Fix any $x, y \in \text{St}(C) \subset X$, we will extend the σ -line $\sigma(x, y)$ to a biinfinite σ -line in M . Indeed if a σ -line hits the boundary of a sector in $\theta^j(L)$, then one can extend it in a sector in $\theta^{j+1}(L)$. The finiteness statement of Lemma 6.2 ensures that this process can be repeated until we obtain a biinfinite σ -line ℓ in M through x and y .

Let c, c' denote maximal chains in L_0 such that x, y belong to the sectors corresponding to c, c' respectively. Let $A, A' \subset X$ denote the cells corresponding to the chains $c \setminus \{0, 1\}$ and $c' \setminus \{0, 1\}$ respectively. Up to the diagonal action of \mathbb{R} , we may assume that ℓ intersects A and A' . Let C_0 denote the simplex corresponding to the chain $c_0 \setminus \{0, 1\}$: by assumption, we have $C_0 \subset A \cap A'$.

So we can assume that x, y belong to the cells A, A' respectively. Assume that $\sigma(x, y) \cap [0, 1] = \emptyset$, otherwise two chambers are sufficient to cover $\sigma(x, y)$.

There exists a sequence $A_0 = A, A_1, \dots, A_p = A'$ of faces in L such that $\sigma(x, y)$ is contained in the union, according to Lemma 6.2. Moreover by Lemma 5.11, we know that, for each $0 \leq i \leq p$, we have $C_0 \subset A_i$.

We claim that, if A_i is such that $A \cap A_i = C_0$, we have $d(x, A) \geq \frac{1}{2}$. Indeed let us assume that A, B are chambers of X , let $C = A \cap B$, and consider a point x in the face A' of A which is the complement of C , and let $z \in B$. The complex $A \cup B$ has a 1-Lipschitz map to the orthoscheme complex associated to the rank 2 lattice $0 < C < A, B < 1$, where x maps to the vertex A and z maps to a point in 2-cell $[0, B, 1]$. Since the distance between A and the 2-cell $[0, C, 1]$ equals $\frac{1}{2}$, we deduce that $d(x, z) \geq \frac{1}{2}$.

Let $0 \leq i \leq p$ denote the maximal number such that $A \cap A_i \neq \emptyset$, and let $0 \leq j \leq p$ denote the minimal number such that $A' \cap A_j \neq \emptyset$. If $j > i + 1$, then $d(x, y) > d(x, A_i) + d(A_j, y) \geq 1$, which is a contradiction. Hence $j \leq i + 1$.

Note that, as $A \cap A_i \neq \emptyset$, since the star of $A \cap A_i$ is σ -convex according to Lemma 5.11, we deduce that for all $0 \leq i' \leq i$, we have $A \cap A_{i'} \neq \emptyset$. Therefore $A \cap A_i \neq \emptyset$ and $A' \cap A_{i+1} \neq \emptyset$: let $z \in \sigma(x, y) \cap A_i \cap A_{i+1}$.

Since $d(x, A_i) < d(x, z) \leq \frac{1}{2}$, we know that $C_0 \subsetneq A \cap A_i$. As a consequence, $A \cap A_i$ has codimension at most $k - 1$ in X , so by induction we deduce that $\sigma(x, z)$ is contained in a union of at most 2^{k-1} chambers. Similarly, we know that $\sigma(y, z)$ is contained in a union of at most 2^{k-1} chambers. We conclude that $\sigma(x, y)$ is contained in a union of at most $2 \times 2^{k-1} = 2^k$ chambers of X . \square

We now prove a local inductive criterion ensuring uniqueness of a convex bicombing.

Proposition 6.4. *Let X denote a finite-dimensional piecewise normed simplicial complex, which is the star of a face F_0 , with a convex bicombing σ . Assume that:*

- *There exists $C \in \mathbb{N}$ such that, for each $x, y \in X$, the combing line $\sigma(x, y)$ is contained in a union of at most C cells.*
- *For any two faces F, F' of X such that $F \cup F'$ is σ -convex, we ask that the interior of $F \cup F'$, with the induced length metric, has a unique convex bicombing.*
- *For any face F of X such that $F_0 \subsetneq F$, the star of F is σ -convex, and σ is the only convex bicombing in the star of F .*

Then σ is the only convex bicombing on X .

Proof. By contradiction, let us assume the existence of another convex geodesic bicombing σ' on X . Let us consider a pair of points $x, y \in X$ such that $\sigma(x, y) \neq \sigma'(x, y)$. By continuity of the bicombings, up to moving y , we may assume that the minimal sequence of closed cells C_1, \dots, C_n in X such that $\sigma(x, y) \subset C_1 \cup \dots \cup C_n$ is also the minimal sequence for $\sigma'(x, y)$. Moreover, we may assume that, for each x' close to x and each y' close to y , we have $\sigma(x', y')$ and $\sigma'(x', y')$ both contained in $C_1 \cup \dots \cup C_n$.

If $\sigma(x, y) \cap F_0 \neq \emptyset$, then $n \leq 2$. So x and y are both contained in C_1 or in $C_1 \cup C_2$. By assumption, the interior of C_1 or $C_1 \cup C_2$ is σ -convex and has a unique convex bicombing, so $\sigma(x, y) = \sigma'(x, y)$, which is a contradiction.

Assume now that $\sigma(x, y) \cap F_0 = \emptyset$. Up to moving y towards x along $\sigma(x, y)$, we may further assume that, for each $t \in (0, 1)$, there exists $1 \leq i \leq n - 1$ such that $\sigma(x, y, t)$ and $\sigma'(x, y, t)$ are contained in the interior of $C_i \cup C_{i+1}$. Fix $t \in (0, 1)$, and let $1 \leq i \leq n - 1$ such that $\sigma(x, y, t)$ and $\sigma'(x, y, t)$ are contained in the interior of $C_i \cup C_{i+1}$. Let $s < t < s'$

such that, for any $t' \in [s, s']$, we have that $\sigma(x, y, t')$ and $\sigma'(x, y, t')$ are contained in $C_i \cup C_{i+1}$. Note that $C_i \cup C_{i+1}$ is contained in the star of $D_i = C_i \cap C_{i+1}$ in X . By assumption, σ and σ' coincide in the star of D_i , with $F_0 \subsetneq D_i$. In particular, the function $t' \mapsto d(\sigma(x, y, t'), \sigma'(x, y, t'))$ is convex at t .

We deduce that the function $t \mapsto d(\sigma(x, y, t), \sigma'(x, y, t))$ is convex on $(0, 1)$, hence it is constant equal to 0. This contradicts that $\sigma(x, y) \neq \sigma(x', y')$. \square

Before stating the next result, we need to remind the definition of horoboundary and Busemann points. The idea of representing points of a metric space as distance functions is due to Gromov, and has then been quite studied from various perspectives (see [BJ07], [GJT98], [Bri06], [JS17], [HSWW17], [Wal07], [Wal08], [CKS20]).

Recall that if X is a metric space, its *horoboundary* ∂X is defined as the boundary of the image of the embedding

$$\begin{aligned} X &\mapsto \mathbb{R}^X / \mathbb{R} \\ x &\mapsto d(x, \cdot) + \mathbb{R}, \end{aligned}$$

where \mathbb{R} acts on \mathbb{R}^X by postcomposition by the standard addition. Note that, if $\xi \in \partial X$, then for any $x, y \in X$, the quantity $\xi(x) - \xi(y)$ is well-defined.

A horofunction $\xi \in \partial X$ is called a *Busemann point* ([Wal07]) if there exists a geodesic ray $c : [0, \infty) \rightarrow X$ such that

$$\forall x, y \in X, \lim_{t \rightarrow \infty} d(x, c(t)) - d(y, c(t)) = \xi(x) - \xi(y).$$

When we restrict to a finite-dimensional real vector space with a polyhedral norm, we have the following.

Theorem 6.5. [Bri06, Wal07, JS17] *Let X denote a finite-dimensional real vector space, with a polyhedral norm. Then its horofunction compactification $X \cup \partial X$ is naturally homeomorphic to the dual unit ball.*

We immediately deduce the following.

Proposition 6.6. *Let X denote a finite-dimensional real vector space, with a polyhedral norm, and fix $x_0 \in X$. The affine half-lines issued from $x_0 \in X$ correspond to finitely many Busemann points in ∂X .*

Proof. According to Theorem 6.5, there are only finitely many horofunctions in ∂X , up to translation. And each of these translation classes has a unique representative given by an affine half-line issued from a fixed $x_0 \in X$. \square

We now show that a particular space with finitely many Busemann points has finite combinatorial dimension.

Theorem 6.7. *Assume that X is a proper metric space with a geodesically complete conical geodesic bicombing σ . Assume that there exists $x_0 \in X$ such that there are at most $N \in \mathbb{N}$ Busemann points corresponding to σ -rays from x_0 . Then X has combinatorial dimension at most N .*

Proof. Let B denote the finite set of Busemann points corresponding to σ -rays from x_0 . Fix $x, y \in X$ distinct, we will prove that there exists $\beta \in B$ such that $d(x, y) = D = |\beta(x) - \beta(y)|$. We will then explain why this bounds the combinatorial dimension of X by $|B|$.

Consider an infinite σ -geodesic ray $c : [0, +\infty) \rightarrow X$ starting from $c(0) = x$, with $c(D) = y$. Since X is proper, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ going to $+\infty$ such that

$\sigma(x_0, c(t_n))$ converges to an infinite σ -geodesic ray $c_0 : [0, +\infty) \rightarrow X$ starting from $c_0(0) = x_0$ to some $\beta \in B$.

We know that the two rays c, c_0 are at Hausdorff distance at most $C \geq 0$. For each $n \in \mathbb{N}$ large enough, let $z_n \in c_0$ such that $d(x, z_n) = t_n$: we have $d(z_n, x) - d(z_n, y) \xrightarrow{n \rightarrow +\infty} \beta(x) - \beta(y)$. For each $n \in \mathbb{N}$, let $y_n = \sigma(x, z_n, D)$. Since $d(c(t_n), z_n) \leq 2C$, we deduce that $d(y, y_n) \xrightarrow{n \rightarrow +\infty} 0$ by conicality. As a consequence, we know that

$$\lim_{n \rightarrow +\infty} d(z_n, x) - d(z_n, y) = \lim_{n \rightarrow +\infty} d(z_n, x) - d(z_n, y_n) = D = \beta(x) - \beta(y).$$

We now explain how this helps to bound the combinatorial dimension of X . Fix representatives B_0 of B . Consider the map

$$\begin{aligned} \varphi : X &\rightarrow \mathbb{R}^{B_0} \\ x &\mapsto (\beta \in B_0 \mapsto \beta(x)). \end{aligned}$$

If we endow \mathbb{R}^{B_0} with the standard ℓ^∞ distance, we just proved that the map φ is an isometric embedding. Indeed every horofunction is 1-Lipschitz. Since \mathbb{R}^{B_0} is an injective metric space with dimension $|B|$, we deduce that X has combinatorial dimension at most $|B|$. \square

We are now able to gather all ingredients and prove that diagonal quotients of orthoscheme complexes have a unique convex bicombing.

Theorem 6.8. *Let L denote a bounded graded lattice of rank n , let X denote the orthoscheme complex of L , and let Y denote the diagonal quotient of X endowed with the standard polyhedral metric. Then Y is CUB.*

Proof. We will prove it by induction on the cardinality m of the intersection of all maximal chains in L , where the rank n is fixed.

If $m = n + 1$, then L consists in a unique maximal chain, hence Y a simplex. In particular, Y admits a unique convex bicombing according to [Bas20, Corollary 1.3].

Assume now that $m < n + 1$. According to Theorem 6.1, there exists a convex bicombing σ on Y . According to Proposition 6.3, for any $x, y \in Y$, the combing line $\sigma(x, y)$ is contained in a union of at most 2^n chambers.

Let F_0 denote the face of Y of dimension $m - 2$ consisting of the intersections of all chambers of Y .

Consider two faces F, F' of Y containing F_0 , such that $F \cup F'$ is σ -convex. We will apply Theorem 6.7 to the interior of the union $Z = F \cup F'$, with the induced length metric. It is sufficient to consider the tangent space Z_0 to Z at some point interior point x_0 in $F \cap F'$. The bicombing σ on Z_0 is complete, and since F and F' have polyhedral norms, according to Proposition 6.6 there are finitely many Busemann points corresponding to σ -rays issued from x_0 . According to Theorem 6.7, we deduce that the interior of Z has a unique convex bicombing.

Let F denote a face of Y strictly containing F_0 . According to Lemma 5.11, we know that the star of F is σ -convex. Moreover the star of F is isometric to the diagonal quotient of the orthoscheme complex of a sublattice L' of L , such that the intersection of all maximal chains in L has cardinality at least $\dim F + 2 > \dim F_0 + 2 = m$. By induction, we know that the star of F has a unique convex geodesic bicombing.

According to Proposition 6.4, we deduce that Y has a unique convex geodesic bicombing. \square

7 Completion of posets

We now show that we can relax the assumption that the (semi)lattice is graded, and replace it with the assumption that lengths of chains are uniformly bounded.

Theorem 7.1. *Let L denote a poset with minimum 0 , which is a meet-semilattice, with a bound on the length of chains. Assume that any $a, b, c \in L$ pairwise upperly bounded have a common upper bound. Then the orthoscheme complex of L , with the standard ℓ^∞ metric, is injective and CUB.*

Proof. Let $n \in \mathbb{N}$ such that each chain of L has length at most n . For each $x \in L$, let us define $r(x) \in \{0, 1, \dots, n\}$ to be the length of a maximal chain from 0 to x . For each pair of elements $x < y$ in L such that x is covered by y and $r(y) - r(x) = k \geq 2$, add a chain $x_0 = x, x_1, \dots, x_k = y$ with $r(x_i) = r(x) + i$ for all $1 \leq i \leq k - 1$. Let L_1 denote the corresponding poset obtained from L .

Let L_2 denote a copy of L' , and let $L' = L_1 \cup_L L_2$ denote the union of L_1 and L_2 , where the copies of the subposet L are identified. Note that there is a natural involution θ on L' exchanging L_1 and L_2 whose fixed point set is L .

The poset L' has a minimum 0 , it is still a meet-semilattice, and is still such that any $a, b, c \in L'$ upperly bounded have a common upper bound in L' . Now L' is also graded of rank at most n . According to Theorem 5.12, the geometric realization of L' , with the orthoscheme ℓ^∞ metric, is injective and CUB.

Let σ denote the unique convex bicombing on $|L'|$. By uniqueness, σ is equivariant with respect to the involution θ . In particular, the subspace $|L|$, which is the fixed-point set of θ , is σ -convex. We deduce that $|L|$ admits a convex bicombing. Since $|L|$ is injective and finite-dimensional, according to Theorem 5.5 we deduce that it is CUB. \square

Theorem 7.2. *Let L denote a bounded lattice, with a bound on the length of chains. Let X denote the orthoscheme complex of L , and let Y denote the diagonal quotient of X endowed with the standard metric. Then Y is CUB.*

Proof. The proof is quite similar to the proof of Theorem 7.1: let us write $L' = L_1 \cup_L L_2$, where L_1, L_2 are isomorphic bounded graded lattices containing L , and there is an involution θ on L' exchanging L_1 and L_2 whose fixed point set is L .

The poset L' is bounded and graded, so according to Theorem 6.8, the diagonal quotient Y' of $|L'|$, endowed with the standard polyhedral metric, is CUB. Note that the involution θ induces an involution θ of $Y' = Y_1 \cup_{Y'} Y_2$ whose fixed-point set is Y .

Let σ' denote the unique convex bicombing on Y' . By uniqueness, σ' is equivariant with respect to the involution θ . In particular, the subspace Y , which is the fixed-point set of θ , is σ' -convex. In particular, Y admits a convex bicombing σ . As in the proof of Theorem 6.8, we can apply inductively Proposition 6.4 and deduce that σ is the only convex bicombing on Y . As a consequence, Y is CUB. \square

8 Proof of the link conditions

We are now able to give a proof of the link conditions.

We start with the proof of Theorem A, for type A simplices.

Proof. Let X denote a locally finite-dimensional flag simplicial complex with cyclically ordered simplices. Assume that, for every vertex $x \in X$, the set $(\text{St}(x), \leq_x)$ is a semilattice.

Fix a vertex $x \in X$, and consider the poset $L = (\text{St}(x), \leq_x) \cup \{1\}$, where 1 is the maximal element and x is the minimal element. By assumption L is a lattice. Furthermore,

the length of chains of L is bounded above by the maximal dimension of simplices of X containing x . So we can apply Theorem 7.2 and deduce that the diagonal quotient Y of $|L|$, with the standard polyhedral metric, is CUB. Since X and Y are locally isometric at x , we deduce that X is locally CUB.

We now turn to the proof of the converse statement: if there exists a vertex $x \in X$ such that $(\text{St}(x), \leq_x)$ is not a semilattice, then X is not locally CUB. Let us call rank of an element $y \in \text{St}(x)$ the length of a maximal chain from x to y . By assumption, there exists a bowtie $a, a' <_x b, b'$ in $\text{St}(x)$. Assume furthermore that the ranks of such b and b' are minimal. Among such bowties, assume furthermore that the meet $m = a \wedge a'$ has maximal rank. And among such bowties, assume finally that the ranks of a and a' are maximal.

By contradiction, assume that X has a conical bicombing σ . Note that the star of b coincides with the 1-ball centered at b , so the star of b is σ -convex. Similarly, the stars of b' and m are σ -convex.

Since $a, a' < b, b'$ is a bowtie, a and a' are not comparable, so a, a' do not lie in a common chamber. Let us consider the two minimal simplices C, C' of $\text{St}(x)$ containing the beginning of $\sigma(a, a')$ starting from a, a' respectively. We know that C and C' are each adjacent to b, b', m . Let $c \in C \cap C'$ denote the maximal element of $C \cap C'$, and let a'' denote the maximal element of C' .

If $c > m$, then $a, a'' < b, b'$ is a bowtie in $\text{St}(x)$ such that the meet of a, a'' has greater rank than m : this is a contradiction. So $c = m$. As a consequence, $C = \{a, m\}$, and $m \in \sigma(a, a')$. We deduce that $\sigma(a, a') = [a, m] \cup [m, a']$. Now let $p \in [a, b]$ denote the midpoint of the edge $[a, b]$. We have $d(p, a) = \frac{1}{2}$, $d(p, m) = 1$ and $d(p, a') \leq 1$, which contradicts the conicality property. \square

We now turn to the proof of Theorem B, for type C simplices.

Proof. Let X denote a locally finite-dimensional flag simplicial complex with ordered simplices, and assume that for every vertex $x \in X$, the poset $(\text{St}(x), \leq_x)$ has no bowtie and, for any $a, b, c \in \text{St}(x)$ which are pairwise upperly bounded (resp. lowerly bounded), they have a common upper bound (resp. lower bound).

Endow X with the standard ℓ^∞ metric, and fix a vertex $x \in X$. According to Proposition 5.7, a neighbourhood of x in X is isometric to a neighbourhood of x in the ℓ^∞ product $|\text{St}^+(x)| \times |\text{St}^-(x)|$, where $\text{St}^+(x) = \{y \in \text{St}(x) \mid x \leq_x y\}$ and $\text{St}^-(x) = \{y \in \text{St}(x) \mid x \geq_x y\}$.

The poset $(\text{St}^+(x), \leq_x)$ is a meet-semilattice with minimum x , whose length of chains is bounded above by the dimension of simplices of X containing x . Moreover, any three pairwise upperly bounded elements have a common upper bound. According to Theorem 7.1, the geometric realization $|\text{St}^+(x)|$ is injective and CUB.

Similarly the geometric realization $|\text{St}^-(x)|$ is injective and CUB.

We deduce that the ℓ^∞ product $|\text{St}^+(x)| \times |\text{St}^-(x)|$ is injective and CUB, so in particular a neighbourhood of x in X is injective and CUB. Finally X is locally injective and locally CUB.

We now turn to the proof of the converse statement: assume first that there exists a vertex $x \in X$ such that $(\text{St}^+(x), \leq_x)$ has a bowtie, we will prove that X has no bicombing. We argue as in the proof of Theorem B, with only slight modifications. Let us call rank of an element $y \in \text{St}^+(x)$ the length of a maximal chain from x to y .

By assumption, there exists a bowtie $a, a' <_x b, b'$ in $\text{St}(x)$. Assume furthermore that the ranks of such b and b' are minimal. Among such bowties, assume furthermore that the meet $m = a \wedge a'$ has maximal rank. And among such bowties, assume finally that the ranks of a and a' are maximal.

By contradiction, assume that X has a convex bicombing σ . Note that the map $y \mapsto y \wedge b$ induces a 1-Lipschitz retraction from $\text{St}(x)$ to the interval $I(x, b)$. Now the map $y \mapsto y \vee m$ induces a 1-Lipschitz retraction from the interval $I(x, b)$ to the interval $I(m, b)$. We can define similarly a projection from $I(m, b)$ to $I(m, b) \cap I(m, b')$. As a consequence, the projection of the geodesic $\sigma(a, a')$ to $I(m, b) \cap I(m, b')$ is a geodesic denoted $\sigma'(a, a')$, with the property that the distance to every point in Y is convex.

Since $a, a' < b, b'$ is a bowtie, a and a' are not comparable, so a, a' do not lie in a common chamber. Let us consider the two minimal simplices C, C' of $\text{St}(x)$ containing the beginning of $\sigma'(a, a')$ starting from a, a' respectively. We know that C and C' are each adjacent to b, b', m . Let $c \in C \cap C'$ denote the maximal element of $C \cap C'$, and let a'' denote the maximal element of C' .

If $c > m$, then $a, a'' < b, b'$ is a bowtie in $\text{St}(x)$ such that the meet of a, a'' has greater rank than m : this is a contradiction. So $c = m$. As a consequence, $C = \{a, m\}$, and $m \in \sigma'(a, a')$. We deduce that $\sigma'(a, a') = [a, m] \cup [m, a']$. Now let $p \in [a, b] \subset Y$ denote the midpoint of the edge $[a, b]$. We have $d(p, a) = \frac{1}{2}$, $d(p, m) = 1$ and $d(p, a') \leq 1$, which contradicts the convexity of the distance to p .

Assume now that there exists a vertex $x \in X$ such that $(\text{St}(x), \leq_x)$ has no bowtie, but there exist $a, b, c \in \text{St}^+(x)$ which are pairwise upperly bounded, but have no common upper bound. We will prove that X has no convex bicombing.

By contradiction, assume that X has a convex bicombing σ . Note that $\sigma(a, b) = [a, m] \cup [m, b]$, where m denotes the midpoint of the edge $[a \wedge b, a \vee b]$. We also have $\sigma(a, c) = [a, m'] \cup [m', c]$, where m' denotes the midpoint of the edge $[a \wedge c, a \vee c]$. Since $d(b, c) = 1$ and σ is convex, we deduce that $d(m, m') \leq \frac{1}{2}$. Since m, m' are midpoints of edges, this implies that m, m' are in a chamber of $\text{St}(x)$. The maximal element d of such a chamber is such that $a \vee b, a \vee c \leq d$, so d is an upper bound to a, b, c : this is a contradiction. \square

Finally, we turn to the proof of Theorem C for Garside flag complexes.

Proof. Let (X, φ) denote a Garside flag complex, and endow X with the standard ℓ^∞ metric. Note that k -simplices of X come in columns, whose vertices come in chains of the form $\dots < \varphi^{-1}(x_k) < x_1 < x_2 < \dots < x_k < \varphi(x_1) < \varphi(x_2) < \dots$.

In this sequence, any $k + 1$ consecutive vertices form the vertices of a k -simplex of this column. In particular, one sees that there is an isometric action $(f_t)_{t \in \mathbb{R}}$ of \mathbb{R} on X , such that $f_1 = \varphi$.

As a consequence, given any point p of X , up to translating p using the action of \mathbb{R} to a generic point, there exists a vertex $x \in X$ such that p is contained in the interior of the geometric realization of the interval $[x, \varphi(x)]$. Since this interval is a homogeneous lattice, according to Theorem 7.1, we deduce that its geometric realization is injective and CUB. So X is locally injective and locally CUB.

Now consider the quotient $Y = X/\varphi$, endowed with the standard polyhedral metric. Fix any vertex $y \in Y$, corresponding to the image of a vertex $x \in X$. Then a neighbourhood of y in Y is isometric to a neighbourhood of y in the diagonal quotient of the lattice $[x, \varphi(x)]$. According to Theorem 7.2, we deduce that a neighbourhood of y in Y is CUB. So Y is locally CUB. \square

9 Applications

We now show that the link conditions can be applied to numerous situations.

9.1 Buildings

We refer the reader to [AB08] and [Ron89] for references on buildings.

Consider a Euclidean building X of type \tilde{A}_n : it is an n -dimensional simplicial complex, such that each vertex has a well-defined type in $\mathbb{Z}/(n+1)\mathbb{Z}$, and such that the vertices of each simplex have different types. Hence X has cyclically ordered simplices. Moreover, for each vertex $x \in X$, the link L of x in X is a spherical building of type A_n .

For instance, in case X is the Euclidean building of $\mathrm{SL}(n+1, \mathbb{Q}_p)$, then L is the spherical building of $\mathrm{SL}(n, \mathbb{F}_p)$. In other words, L is the poset of non-trivial vector subspaces of \mathbb{F}_p^n , which is a lattice (up to adding $\{\{0\}, \mathbb{F}_p^n\}$).

More generally, L is the poset of non-trivial subspaces of a projective geometry, and it is always a lattice (up to adding $\{0, 1\}$). According to Theorem A we deduce the following.

Theorem 9.1. *Any Euclidean building X of type \tilde{A}_n , with the standard polyhedral metric, is CUB.*

We can also consider a Euclidean building X of type extended type \tilde{A}_n : it is an $(n+1)$ -dimensional simplicial complex, such that simplices have a well-defined total order. Similarly, we have the following.

Theorem 9.2. *Any extended Euclidean building X of type \tilde{A}_n , with the standard ℓ^∞ metric, is CUB and injective.*

Proof. Given an extended Euclidean building X of type \tilde{A}_n , there is a well-defined canonical automorphism φ of X , see [Hir20]. For instance, if X is the Bruhat-Tits building of $\mathrm{GL}(n, \mathbb{Q}_p)$, then φ is the homothety of p . For each vertex $x \in X$, the interval $[x, \varphi(x)]$ is the lattice of subspaces of a projective geometry of dimension n .

Then (X, φ) is a Garside flag complex (see Definition 4.5): according to Theorem C, we deduce that X , with the standard ℓ^∞ metric, is CUB and injective. \square

Consider a Euclidean building X of type \tilde{B}_n , \tilde{C}_n or \tilde{D}_n : note that we may always consider X as a (possibly non-thick) Euclidean building of type \tilde{C}_n . It is an n -dimensional simplicial complex, such that each vertex has a well-defined type in $\{0, 1, \dots, n\}$, and such that the vertices of each simplex have different types. So X has ordered simplices. Moreover, for each vertex $x \in X$ of type k , the link L of x in X is the join of two spherical buildings of types B_k and B_{n-k} .

Then L is the poset of non-trivial subspaces of a polar geometry: so $L \cup \{0\}$ is a meet-semilattice, and if three elements have a pairwise upper bound, they have a global upper bound. According to Theorem B, we deduce the following.

Theorem 9.3. *Any Euclidean building X of type \tilde{B}_n , \tilde{C}_n or \tilde{D}_n , with the standard ℓ^∞ metric, is CUB.*

9.2 Simplices of groups

We refer the reader to [BH99] for the theory of complexes of groups. The link condition from Theorem A can be used to prove that certain simplices of groups are developable, and that their development cover is CUB, as noticed in Theorem 2.4.

Consider a simple $(n-1)$ -dimensional simplex of groups S , with cyclically ordered vertices $(s_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$. For each non-empty subset $I \subset \mathbb{Z}/n\mathbb{Z}$, let G_I denote the group associated to the face I . When $I \subset J$, we will consider G_J as a subgroup of G_I .

Theorem 9.4. *Assume that the simplex of groups satisfies the following.*

- Given any $i \in \mathbb{Z}/n\mathbb{Z}$ and $I, J \subset \mathbb{Z}/n\mathbb{Z}$ containing i , we have $G_I \cap G_J = G_{I \cup J}$.

- Given any $i < j < k < \ell$ in $\mathbb{Z}/n\mathbb{Z}$, we have $G_{ik} \subset G_{ij}G_{i\ell}$.
- Given any $i < j < k$ in $\mathbb{Z}/n\mathbb{Z}$, if $a, a' \in G_{ij}$ and $b, b' \in G_{ik}$ are such that $aba'b' = e$, we have $a, a' \in G_{ik}$ or $b, b' \in G_{ij}$ or there exists $i < j < \ell < k$ such that $a, a', b, b' \in G_{i\ell}(G_{ij} \cap G_{ik})$.

Then the simplex of groups is developable. Moreover its development, endowed with the standard polyhedral norm, is CUB.

Proof. Since S is a simple complex of groups, we know that S is locally developable at each vertex. Fix a vertex G_i , and we will consider the local development L_i at G_i : according to the first condition, it is the flag simplicial complex with vertex set $\{G_i\} \cup \bigcup_{j \neq i} G_i \setminus G_{ij}$, where aG_{ij} is adjacent to bG_{ik} if and only if there exists $c \in G_i$ such that $cG_{ij} = aG_{ij}$ and $cG_{ik} = bG_{ik}$. Note that edges of L_i have a well-defined orientation, where $aG_{ij} < bG_{ik}$ if $i < j < k$ (and G_i is the minimal element of L_i).

We will first check that L_i is the geometric realization of the corresponding poset. Assume that in L_i we have two consecutive ordered edges between aG_{ij} , bG_{ik} and $cG_{i\ell}$, where $a, b, c \in G_i$ and $i < j < k < \ell$. Without loss of generality, we may assume up to translation that $a = e$. Since G_i, G_{ij}, bG_{ik} form a simplex, we have $b \in G_{ij}$, so we may assume up to translation that $b = e$. Since $G_i, G_{ik}, cG_{i\ell}$ form a simplex, we have $c \in G_{ik}$.

By assumption, we have $c = xy$, where $x \in G_{ij}$ and $y \in G_{i\ell}$. Hence $(G_i, G_{ij}, cG_{i\ell}) = x \cdot (G_i, G_{ij}, G_{i\ell})$, so it is a simplex. Hence there is an ordered edge between aG_{ij} and $cG_{i\ell}$ in the star of G_i . So L_i is the geometric realization of the corresponding poset.

We will now check that the vertex set of L_i is a semilattice. According to Proposition 4.2, we only need to check that the vertex set of L_i has no balanced bowtie. Assume that there are $i < j < k$ and vertices such that $gG_{ij}, gabG_{ij} < gaG_{ik}, gaba'G_{ik}$, where $g \in G_i$, $a, a' \in G_{ij}$ and $b \in G_{ik}$. Up to translation, there exists $b' \in G_{ik}$ such that $aba'b' = e$ and the vertices are $G_{ij} = aba'b'G_{ij}, abG_{ij} < aG_{ik}, aba'G_{ik}$.

By assumption, there are three possibilities. If $a, a' \in G_{ik}$, then $aG_{ik} = aba'G_{ik}$, so it is not a bowtie. If $b, b' \in G_{ij}$, then $G_{ij} = abG_{ij}$, so it is not a bowtie. Assume then that there exists $i < j < \ell < k$ such that $a, a', b, b' \in G_{i\ell}(G_{ij} \cap G_{ik})$. So the element $G_{i\ell}$ is such that $G_{ij} = aba'b'G_{ij}, abG_{ij} \leq G_{i\ell} \leq aG_{ik}, aba'G_{ik}$. This shows that there is no balanced bowtie in L_i .

According to Proposition 4.2, we deduce that the vertex set of L_i is a semilattice. So according to Theorem A, we see that the local development at each vertex is CUB. According to Theorem 2.4, we deduce that the simplex of groups S is developable. Moreover its development, endowed with the standard polyhedral norm, is CUB according to Theorem 1.2. \square

Note that in the case of a triangle of groups, the criterion is the same as the one to ensure nonpositive curvature (with the equilateral triangle Euclidean norm), namely that the links of local developments have girth at least 6.

However, already in the case of a 3-simplex of groups, this is to our knowledge the first general combinatorial criterion ensuring developability. Here are very simple examples of such 3-simplices of groups where the assumptions hold.

Example. For each $i \in \mathbb{Z}/4\mathbb{Z}$, let G_i be the symmetric group \mathfrak{S}_4 . For each $j \in \{1, 2, 3\}$, define the image of $G_{i, i+j}$ in G_i to be the stabilizer of $\{1, \dots, j\}$. For each $I \subset \{1, \dots, 4\}$ containing i , define the image of G_I in G_i to be the intersection of all $G_{i, j}$, for $j \in I$. Then the conditions of Theorem 9.4 are easily seen to hold, and in fact the development of S is the standard \tilde{A}_3 tiling of \mathbb{R}^3 .

Example. One can describe the Bruhat-Tits building of $\mathrm{SL}(n, \mathbb{K}((X)))$, for $n \geq 2$ and for any field \mathbb{K} , as follows. Consider the $(n - 1)$ -simplex of groups T with vertices indexed by $\mathbb{Z}/n\mathbb{Z}$. For each $i \in \mathbb{Z}/n\mathbb{Z}$, let $t_i \in \mathrm{GL}(n, \mathbb{K}(X))$ denote the diagonal matrix with the first i entries equal to X , and the last $n - i$ entries equal to 1 (note that since we will make t_i act by conjugation, this will be well-defined for $i \in \mathbb{Z}/n\mathbb{Z}$). For each $i \in \mathbb{Z}/n\mathbb{Z}$, let $G_i = t_i \mathrm{SL}(n, \mathbb{K}[X]) t_i^{-1} \subset \mathrm{SL}(n, \mathbb{K}(X))$. For each non-empty $I \subset \mathbb{Z}/n\mathbb{Z}$, let $G_I = \bigcap_{i \in I} G_i$. Note that, for each $i \in \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$, one can also describe the image of $G_{0,i}$ in G_0 as the stabilizer of the canonical i -plane $\mathbb{K}e_1 + \cdots + \mathbb{K}e_i$ under the quotient action of $G_0 = \mathrm{SL}(n, \mathbb{K}[X]) \rightarrow \mathrm{SL}(n, \mathbb{K})$ on \mathbb{K}^n . Then the conditions of Theorem 9.4 are easily seen to hold, and in fact the development of T is precisely the Bruhat-Tits building of $\mathrm{SL}(n, \mathbb{K}((X)))$.

Example. Let A denote the Artin group of affine type \tilde{A}_{n-1} , with standard generators $(s_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$. For each non-empty $I \subset \mathbb{Z}/n\mathbb{Z}$, let G_I denote the standard parabolic subgroup generated by $\{s_i \mid i \notin I\}$. Then the corresponding $(n - 1)$ -simplex of groups T satisfies the assumptions of Theorem 9.4, and its development is the Artin complex of A studied in Theorem 9.9.

9.3 Weak Garside groups

We refer the reader to [Deh15], [Bes03], [Bes06b] and [HH22] for references concerning Garside and weak Garside groups.

The classical definition of a Garside group starts with the definition of Garside monoid, as follows.

Definition 9.5. A *Garside monoid* (sometimes called quasi-Garside) is a pair (M, Δ) , where M is a monoid and

- M is left and right cancellative,
- there exists $r : M \setminus \{1\} \rightarrow \mathbb{N} \setminus \{0\}$ such that $r(fg) \geq r(f) + r(g)$,
- any two elements of M have left and right least common multiples and greatest common divisors,
- Δ is a *Garside* element of M , i.e. the family S of left and right divisors of Δ coincide and generate M .

If one further require that the set S of simple elements is finite, (M, Δ) is called of finite type.

Now we can define a Garside group.

Definition 9.6. A group G is called a *Garside group* if there exists a Garside monoid (M, Δ) such that G is the group of left fractions of M .

Examples of Garside groups of finite type include finite rank free abelian groups, braid groups, and more generally spherical type Artin groups. Tree products of cyclic groups also form a nice family of examples, see [Pic22]. The standard Garside structure for a spherical type Artin group is associated to the standard Artin monoid. Note that, according to Bessis ([Bes03]), spherical type Artin groups actually admit another Garside structure, called the dual Garside structure. See notably [Pao21],[PS21], [Hae22b] and [DPS22] for the importance of the dual approach for the study of Artin groups.

Examples of Garside groups (with an infinite set of simple elements) include finite rank free groups (see [Bes06a]) and Euclidean Artin groups of type \tilde{A}_n , \tilde{C}_n and \tilde{G}_2 (see [BM15],

[MS17] and [McC15]). Note that the dual structure of Euclidean type Artin groups is not Garside; nevertheless, McCammond and Sulway have defined a way to "complete" these groups to obtain crystallographic braid groups ([MS17]), which are fundamental in the proof of the $K(\pi, 1)$ conjecture for Euclidean type Artin groups by Paolini and Salvetti ([PS21]).

Another closely related notion is that of weak Garside group. Rather than giving the definition, we prefer stating the following geometric characterization of Garside and weak Garside groups.

Theorem 9.7. *[HH22, Theorem 4.7] A group G is a Garside group (resp. weak Garside group) if and only if there exists a Garside flag complex (X, φ) such that G can be realized as a group of automorphisms of X commuting with φ , acting freely and transitively (resp. freely) on vertices of X .*

Moreover, the group G is a (weak) Garside group of finite type if and only if X can be chosen such that the action of G is cocompact.

For instance, finite index subgroups of Garside groups are weak Garside groups. Other examples include the fundamental groups of complements of real simplicial arrangements of hyperplanes ([Del72]), all braid groups of complex reflection groups ([BC06],[Bes15],[CP11]) except possibly the exceptional complex braid group of type G_{31} , and some extensions of Artin-Tits groups of type B_n ([CP05]).

As a consequence of Theorem C, we have the following.

Corollary 9.8. *Let (G, Δ) denote a weak Garside group of finite type, and let $k \geq 1$ such that $\Delta^k \in Z(G)$. Then G and $G/\langle \Delta^k \rangle$ both act properly and cocompactly by isometries on a CUB space.*

Proof. If we denote by (X, φ) the Garside flag complex associated to G , we know that G acts geometrically on X , and $G/\langle \Delta^k \rangle$ acts geometrically on X/φ . \square

Let A be a spherical type Artin group and Δ is a Garside element of A . If we denote by (X, φ) the associated Garside flag complex, then X/φ has also been described by Bestvina ([Bes99]) as the normal form complex of A , which exhibits some form of combinatorial nonpositive curvature. We strengthen this claim by remarking that, when we endow Bestvina's complex with the standard polyhedral metric, it is CUB.

9.4 Artin complexes for some Euclidean type Artin groups

We refer the reader to [Par14], [Par14], [GP12], [CD95a], [CD95b], [Cha], [McC17] for references on Artin groups.

Let A denote any Artin group. The Artin complex is the flag simplicial complex X whose vertex set consists in left cosets of maximal proper standard parabolic subgroups of A , with an edge between gP and $g'P'$ if and only if $gP \cap g'P' \neq \emptyset$, see [CMV20] and [CD95b, Remark (i), p. 606]. Note that, from the presentation of A , the complex X is simply connected.

We will be interested in the case where A is of Euclidean type \tilde{A}_n or \tilde{C}_n .

In type \tilde{A}_n , vertices of X have a well-defined type in $\mathbb{Z}/(n+1)\mathbb{Z}$, so simplices of X have a well-defined cyclic order, and the link L of any vertex is isomorphic to the Artin complex of the Artin group of type A_n . Bessis ([Bes06a]), and independently Crisp and McCammond (unpublished), proved that $L \cup \{0, 1\}$ is isomorphic to the lattice of cut-curves (see [Hae21b] for a proof, following Crisp and McCammond). According to Theorem A, we deduce the following.

Theorem 9.9. *The Artin complex of type \widetilde{A}_n , with the standard polyhedral metric, is CUB.*

In type \widetilde{C}_n , vertices of X have a well-defined type in $\{0, \dots, n\}$, so simplices of X have a well-defined total order. Moreover, the link L_k of any vertex of type k is the join of the Artin complexes of types B_k and B_{n-k} . According to [Hae21b, Proposition 6.3], the Artin complex of type B_k , with 0, is a semilattice such that any a, b, c which are pairwise upperly bounded have an upper bound. According to Theorem B, we deduce the following.

Theorem 9.10. *The Artin complex of type \widetilde{C}_n , with the standard ℓ^∞ metric, is CUB.*

9.5 Some arc complexes

The question of finding nonpositive curvature metrics on some curve complexes or arc complexes on surfaces is quite intriguing and difficult, raised notably by Masur and Minsky ([MM99]). For instance, Webb proved that many such complexes do not admit CAT(0) metrics ([Web20]).

The following is nothing more than a topological description of the Artin complex of the Artin-Tits group of type \widetilde{A}_{n-1} (see [HH22, Proposition 5.8]). Let Σ denote a 2-sphere with $n+2$ punctures $\{N, S, p_1, \dots, p_n\}$ with two distinguished punctures N, S which could thought of as the North pole and the South pole of Σ . The punctures p_1, \dots, p_n may be thought as cyclically ordered on the equator of the 2-sphere.

Let $\mathcal{A}(\Sigma)$ denote the following simplicial complex. Its vertex set consists of isotopy classes of arcs in Σ from N to S . Two vertices are adjacent if they can be realized disjointly. Then $\mathcal{A}(\Sigma)$ is the associated flag simplicial complex, see Figure 8. According to [Wah13, Lemma 2.5], this arc complex $\mathcal{A}(\Sigma)$ is contractible.

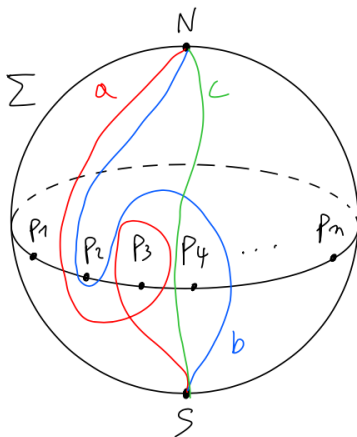


Figure 8: Arcs on the punctured sphere Σ : a is adjacent to b and c

There is a canonical total cyclic order on vertices of simplices of $\mathcal{A}(\Sigma)$, when one fixes an orientation on Σ : let σ denote a simplex of $\mathcal{A}(\Sigma)$. Then any two distinct $a, b \in \sigma$ are such that $\Sigma \setminus \{a, b\}$ has two connected components, associated to a fixed orientation of Σ . So given any three $a, b, c \in \sigma$, we say that $a < b < c$ if a is on the left of b , and c is on the right of b .

Theorem 9.11. *The complex $\mathcal{A}(\Sigma)$, with the standard polyhedral norm, is CUB.*

Proof. Fix an arc $a \in \mathcal{A}(\Sigma)$, and let us consider the surface $D = \Sigma \setminus a$: it is homeomorphic to a disk with n punctures, and with two marked points N, S on the boundary ∂D . The link L of a in $\mathcal{A}(\Sigma)$ is isomorphic to the complex of arcs in D from N to S . Let us call

a_W, a_E the two connected components of $\partial D \setminus \{N, S\}$, where E stands for East and W for West.

The induced order on L is the following: if b, c are disjoint arcs in D from N to S , then $b \leq_a c$ if b is on the west of c . The poset L can be completed with a minimum element a_W , and a maximum element a_E . Now $L \cup \{a_W, a_E\}$ is isomorphic to the poset of cut-curves. According to [Bes06a] (and [Hae21b] for an account of the unpublished proof due to Crisp and McCammond), this poset is a lattice. See also [HH22], or Figure 9 to see how the lattice property works.

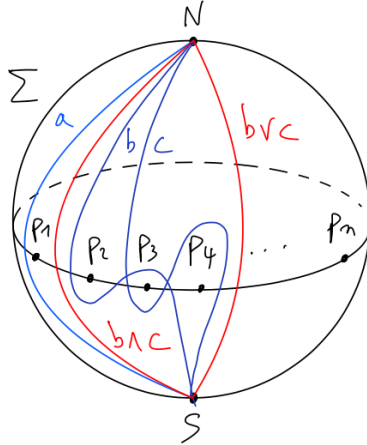


Figure 9: The lattice property: the meet $b \wedge c$ and the join $b \vee c$ of the two arcs b, c in the interval $[a_W, a_E]$.

We can therefore apply Theorem A and deduce that the complex $\mathcal{A}(\Sigma)$, with the standard polyhedral norm, is CUB. \square

9.6 A complex of homologous multicurves

Apart from the complex of arcs on a punctured sphere, there is another natural complex for which for lattice property is straightforward. Note that the complex, as well the tools used for its study, are very similar to the Kakimizu complex studied in [PS12], see also Section 9.7.

Fix a closed surface S of genus $g \geq 1$ with $p \geq 0$ punctures. Fix the homology class $[a] \in H_1(S, \mathbb{Z})$ of a simple closed oriented non-separating curve. Let \tilde{S} denote the associated infinite cyclic cover. If b is an oriented simple closed multicurves on S homologous to a , the lifts of the complement $S \setminus b$ to \tilde{S} are separated by lifts of b denoted $(\tilde{b}_n)_{n \in \mathbb{Z}}$. Also note that each \tilde{b}_n bounds an unbounded subsurface $B(\tilde{b}_n)$ of \tilde{S} which is "below" \tilde{b}_n , i.e. which contains all lifts \tilde{b}_m with $m < n$. If b, b' are two such oriented simple closed multicurves on S homologous to a , we say that a lift \tilde{b}_n of b is *below* of a lift \tilde{b}'_m of b' if $B(\tilde{b}_n)$ is contained in $B(\tilde{b}'_m)$, up to homotopy. We denote $\tilde{b}_n \prec \tilde{b}'_m$.

Let $\mathcal{C}_a(S)$ denote the flag simplicial complex defined as follows:

- Vertices of $\mathcal{C}_a(S)$ are homotopy classes of oriented simple closed multicurves on S homologous to a .
- There is an edge between $b, b' \in \mathcal{C}_a(S)$ if there are lifts $\tilde{b}_n, \tilde{b}'_m$ of b, b' respectively, such that $\tilde{b}_n \prec \tilde{b}'_m \prec \tilde{b}_{n+1}$.

Here is an example of a closed surface S of genus 3, with a simple closed non-separating curve a . There are also three represented simple disjoint multicurves b, c, d which are homologous to a , see Figure 10.

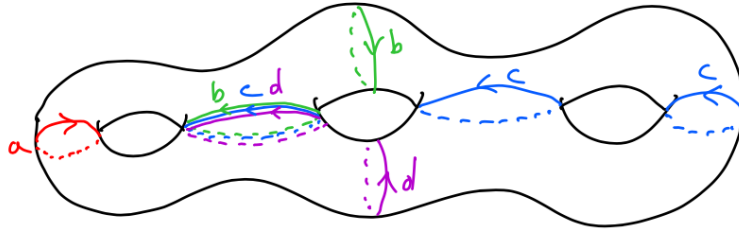


Figure 10: Four pairwise disjoint multicurves a, b, c, d .

When we consider the cyclic cover \tilde{S} of S associated to $[a]$, we obtain a family of lifts of each of the multicurves a, b, c, d . In this example, the lifts satisfy $\tilde{a}_0 \prec \tilde{d}_0 \prec \tilde{c}_0 \prec \tilde{b}_0 \prec \tilde{a}_1$, so the four multicurves a, b, c, d form a simplex of $\mathcal{C}_a(S)$, see Figure 11.

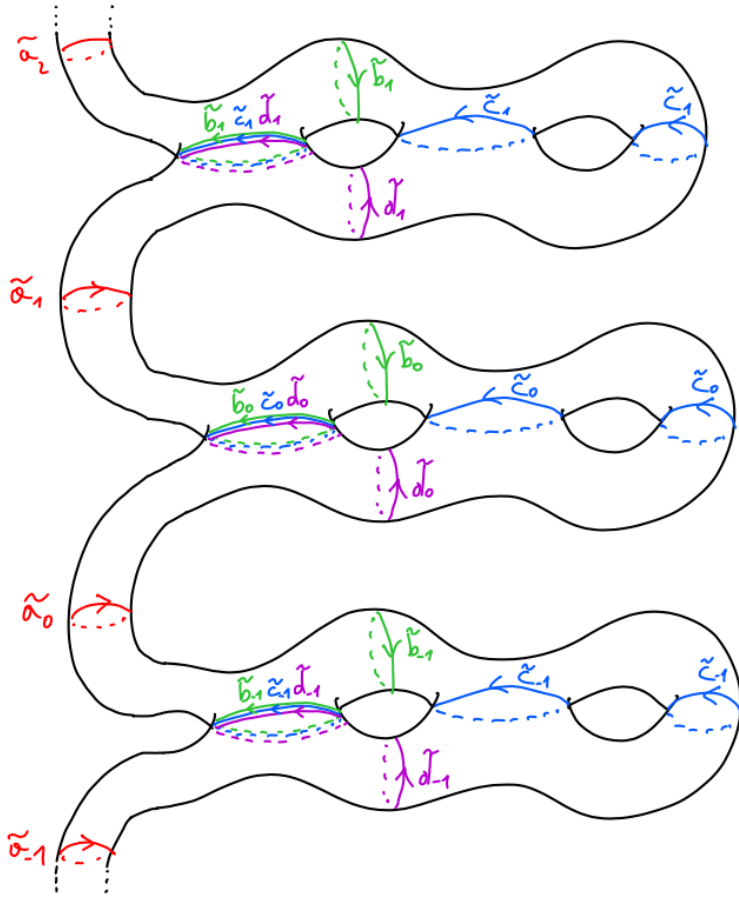


Figure 11: The lifts of the multicurves a, b, c, d in the cyclic cover.

Hatcher and Margalit defined a very similar complex, and proved that it is contractible ([HM12, Proposition 7]). We will prove that the complex $\mathcal{C}_a(S)$ is contractible, following ideas from [PS12].

Proposition 9.12. *The complex $\mathcal{C}_a(S)$ is contractible.*

Proof. Let us introduce the following distance on vertices of $\mathcal{C}_a(S)$. If $b, b' \in \mathcal{C}_a(S)$, the distance $D(b, b')$ is the minimal number $R \in \mathbb{N}$ such that there exist lifts $\tilde{b}_n, \tilde{b}'_m$ of b, b' respectively, such that $\tilde{b}_n \prec \tilde{b}'_m \prec \tilde{b}_{n+R}$.

Note that D is bounded above by the graph distance. In particular, if $D(a, b) = 1$, then b is adjacent to a . We will prove that there is a retraction from $B_D(a, R+1)$ to $B_D(a, R)$ for $R \in \mathbb{N}$.

Fix $b \in B_D(a, R+1)$. Consider the lifts $(\tilde{a}_n)_{n \in \mathbb{Z}}$, and consider the unique lift \tilde{b} of b such that $\tilde{a}_0 \prec \tilde{b} \prec a_{R+1}$. Let us consider the subsurface $T = B(\tilde{b}) \cap B(\tilde{a}_R)$ of \tilde{S} up to homotopy: to be more precise, we may fix a hyperbolic structure on S , its lift to \tilde{S} , and only consider geodesic representatives of multicurves. The boundary $\tilde{c}_0 = \partial T$ is a multicurve in \tilde{S} , whose projection c to S is a simple closed multicurve homologous to a . Moreover, we know that $\tilde{c}_0 \prec \tilde{b} \prec a_{R+1} \prec \tilde{c}_1$, so c is adjacent to b . Moreover, since $\tilde{a}_0 \prec \tilde{c}_0 \prec a_R$, we know that $D(a, c) \leq R$.

So we have defined a simplicial retraction from $B_D(a, R+1)$ to $B_D(a, R)$ for $R \in \mathbb{N}$. As a consequence, the complex $\mathcal{C}_a(S)$ is contractible. \square

Note that if σ is a simplex of $\mathcal{C}_a(S)$, we know that we can find lifts $\tilde{b}_{n_1}^1, \dots, \tilde{b}_{n_p}^p$ of σ to \tilde{S} satisfying $\tilde{b}_{n_1}^1 \prec \dots \prec \tilde{b}_{n_p}^p \prec \tilde{b}_{n_1+1}^1$. Furthermore, the corresponding cyclic order $b^1 < b^2 < \dots < b^p < b^1$ on vertices of σ is well-defined. In the example of Figure 10, the corresponding cyclic order is $a < b < c < d < a$. For this complex, we are able to prove the following.

Theorem 9.13. *The complex $\mathcal{C}_a(S)$, endowed with the standard polyhedral metric, is CUB. In particular, it is contractible.*

Proof. Fix a multicurve $b \in \mathcal{C}_a(S)$, with a lift \tilde{b}_0 in \tilde{S} . Let us consider the set E of lifts \tilde{c} of elements $c \in \mathcal{C}_a(S)$ such that $\tilde{b}_0 \prec \tilde{c} \prec \tilde{b}_1$. The set E may be endowed with the partial order \prec . We want to prove that the star of b , with the induced order, is a semilattice. This poset is isomorphic to $(E \setminus \{\tilde{b}_1\}, \prec)$. So it is equivalent to prove that (E, \prec) is a semilattice.

Consider $\tilde{c}, \tilde{c}' \in E$. Let us consider the subsurface $T = B(\tilde{c}) \cap B(\tilde{c}')$ of \tilde{S} up to homotopy. Its boundary $\tilde{d} = \partial T$ is the lift of a multicurve $d \in \mathcal{C}_a(S)$, such that $\tilde{b}_0 \prec \tilde{d} \prec \tilde{c}, \tilde{c}' \prec \tilde{b}_1$. In particular, $\tilde{d} \in E$. Moreover, by construction, \tilde{d} is the maximal such element of E : this means that \tilde{d} is the meet of \tilde{c} and \tilde{c}' . So E is a semilattice.

The meet of two multicurves in the star of a is depicted in Figure 12. For simplicity, we represented the multicurves in the surface S and not in its cyclic cover.

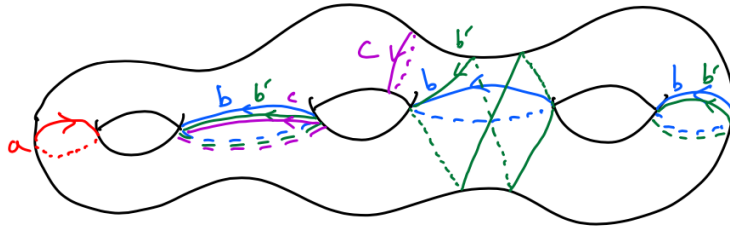


Figure 12: The meet c of the multicurves b and b' .

According to Theorem A, we deduce that the complex $\mathcal{C}_a(S)$, endowed with the standard polyhedral metric, is locally CUB. Since the complex $\mathcal{C}_a(S)$ is simply connected, we deduce by Theorem 1.2 that it is CUB. \square

Note that the stabilizer of the homology class of a in the mapping class group of S acts on $\mathcal{C}_a(S)$ by isometries. In particular, the Torelli group of S acts on $\mathcal{C}_a(S)$ by isometries, for each simple closed nonseparating curve a on S .

9.7 The Kakimizu complex

Let L denote a knot in \mathbb{S}^3 , and let $E = E(L)$ denote the exterior of a tubular neighbourhood of L . A *spanning surface* is a surface properly embedded in E , which is contained in some

Seifert surface for L . Let $MS(L)$ denote the flag simplicial complex whose vertices are isotopy classes of minimal genus spanning surfaces, with an edge between two surfaces if they have disjoint representatives.

This complex can also be defined for the complement of a link in S^3 , or more generally for a compact, connected, orientable irreducible 3-manifold with boundary, see [PS12].

This complex is called the *Kakimizu complex*, and it has been defined in [Kak92]. Scharlemann and Thompson proved that this complex is connected ([ST88]), and Kakimizu gave another proof when L is a link ([Kak92]). Schultens proved that $MS(L)$ is simply connected ([Sch10]), and Przytycki and Schultens proved that it is in fact contractible ([PS12]).

Note that if we fix an orientation of L , there is a well-defined total cyclic ordering on vertices of simplices of $MS(L)$, given informally by the "angle" at which surfaces intersect the tubular neighbourhood E of L . More precisely, consider the infinite cyclic covering \tilde{E} of E corresponding to the meridian of L . If S, R, R' are pairwise disjoint spanning surfaces, then R, R' have disjoint lifts \tilde{R}, \tilde{R}' in \tilde{E} contained between two consecutive lifts \tilde{S}_0, \tilde{S}_1 of S . One then say that $S < R < R'$ if \tilde{R} separates \tilde{S}_0 and \tilde{R}' .

Using this order, one may endow the Kakimizu with the standard polyhedral metric.

Theorem 9.14. *The Kakimizu complex $MS(L)$, endowed with the standard polyhedral metric, is CUB.*

As a consequence, one obtain another proof that the Kakimizu is contractible (assuming that it is simply connected). Applying Proposition 2.1, one also deduces another proof of [PS12, Corollary 1.3] that for any finite subgroup G of the mapping class group of E , there exists a union of pairwise disjoint minimal genus spanning surfaces which is G -invariant up to isotopy.

Proof. We will apply Theorem A. Since $MS(L)$ is simply connected according to [Sch10], it is sufficient to check that the star of a vertex is a semilattice.

Fix three spanning surfaces $S, R, R' \in MS(L)$, with R, R' disjoint from S , and consider the infinite cyclic covering \tilde{E} of E corresponding to the meridian of L . Let τ denote the covering automorphism of \tilde{E} . Fix lifts $\tilde{S}, \tilde{R}, \tilde{R}'$ of S, R, R' to \tilde{E} such that \tilde{R}, \tilde{R}' are disjoint from \tilde{S} , but not from $\tau^{-1}(\tilde{S})$. Let \tilde{P} denote the surface "below $\tilde{R} \cup \tilde{R}'$ ", i.e. obtained from \tilde{R} and \tilde{R}' by a cut-and-paste operation as in [PS12, Section 4]. Let P denote the image of \tilde{P} in E . Then it is clear that we have $S \leq P \leq R, R'$. Furthermore, by construction P is maximal, hence P is the meet of R and R' . So the star of S is a semilattice. \square

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