Lattices, Garside structures and weakly modular graphs

Thomas Haettel* and Jingyin Huang†

November 8, 2022

Abstract. In this article we study combinatorial non-positive curvature aspects of various simplicial complexes with natural $\widetilde{A}_n$ shaped simplicies, including Euclidean buildings of type $\widetilde{A}_n$ and Cayley graphs of Garside groups and their quotients by the Garside elements. All these examples fit into the more general setting of lattices with order-increasing $\mathbb{Z}$-actions and the associated lattice quotients proposed in a previous work by the first named author. We show that both the lattice quotients and the lattices themselves give rise to weakly modular graphs, which is a form of combinatorial non-positive curvature. We also show that several other complexes fit into this setting of lattices/lattice quotients, hence our result applies, including Artin complexes of Artin-Tits groups of type $\widetilde{A}_n$, a class of arc complexes and weak Garside groups arising from a categorical Garside structure in the sense of Bessis. Along the way, we also clarify the relationship between categorical Garside structure, lattices with $\mathbb{Z}$ action and different classes of complexes studied this article.

The authors would like to acknowledge the very deep influence of Jacques Tits in many topics relating algebra and geometry, notably buildings and Artin-Tits groups, which are both at the core of this article.

The topic of combinatorial non-positive curvature (CNPC) lies in the intersection of metric graph theory and geometric group theory. We refer to [CCHO21] for a detailed discussion of the context and motivation for CNPC. The basic idea is to identify local combinatorial patterns of graphs or complexes that lead to standard consequences of non-positive curvature, e.g. propagation of these combinatorial patterns from local to global in the spirit of the classical Cartan-Hadamard theorem, control of isoperimetric inequalities, existence of nice combings, fixed point properties, asphericity etc. Pioneering examples of CNPC include small cancellation theory, Gromov's flagness condition [Gro87], and systolic complexes [JS06]. For a group, being able to act on graphs or complexes that satisfy some form of CNPC usually has strong implications on the structure of this group. Thus it is of great interest to construct such actions - there are many works in this direction, and we simply mention a few which are closely related to this article [Bes99, HO20, HO19, CCG+20, Mun19, Hod20, CMV20, Hae21a, Soe21, Hae21c, Blu21, HO21].

There is a strong connection between affine Coxeter groups and forms of CNPC. Each such Coxeter group gives a Euclidean polyhedron which is the fundamental domain of

*Thomas Haettel, IMAG, Univ Montpellier, CNRS, France, and IRL 3457, CRM-CNRS, Université de Montréal, Canada.

†Jingyin Huang, Department of Mathematics, The Ohio State University, 100 Math Tower, 231 W 18th Ave, Columbus, OH 43210, U.S.

Keywords : Buildings, Artin-Tits groups, Garside groups, nonpositive curvature, weakly modular graphs. **AMS codes** : 20E42, 20F36, 20F55, 05B35, 06A12, 20F65, 05C25

the action of this group on the associated Euclidean space. Then one can try to build more complicated spaces using such Euclidean polyhedra, and ask whether there is a specific local combinatorial pattern of assembling these Euclidean polyhedrons such that the resulting space is non-positively curved in some sense. For example, systolic complexes and bridged graphs [Che00, JS06] describe a form of CNPC for spaces made of equilateral triangles; CAT(0) cube complexes and median graphs describe another form of CNPC for spaces made of unit cubes, quasi-medians graphs and bucolic complexes can be viewed as forms of CNPC for spaces made with prisms [BCC$^+$13, Gen17], and CNPC of spaces made of orthoschemes is closely related to swm-graphs [CCHO21] and Helly graphs [Hae21c]. This raises two questions. First, whether there is a form of CNPC which is a common generalization of all these notions, hence can be applied to spaces made of mixed types of shapes. Second, whether each affine Coxeter group hints a particular form of CNPC, which is applicable to complexes built with fundamental domains of such Coxeter group. This question is already unknown for Coxeter group of type $\widetilde{A}_n$ with $n \geq 3$, which partially motivates this article.

Attempting to answer the first question leads to the notion of *weakly modular graph*. This notion was initially introduced in [Che89, BC96], whose definition demands metric balls in the graph satisfy a weak form of convexity (see Section 1.2 for a detailed definition). It is a common generalization of bridge graphs, median graphs and Helly graphs mentioned in the previous paragraph, and serves as a mother notion to study its various sub-classes in [CCHO21]. Though it is widely open that whether weak modularity is compatible with other Coxeter shapes. Notable features of weakly modular graphs are that they enjoy a local-to-global characterization, and that they admit Euclidean isoperimetric inequalities.

Back to the $\widetilde{A}_n$ case of the second question with $n \geq 3$, as an initial step, it is shown by Munro [Mun19] that while the 1-skeleton of the Coxeter complex of type $\widetilde{A}_n$ fails most form of CNPC, it does satisfy weak modularity. He also proved 1-skeleton of a 3-dimensional Euclidean building is weakly modular, though the high dimensional case remains open. In this article we prove weak modularity for a much wider class of simplicial complexes with $\widetilde{A}_n$ simplices, including the higher dimensional $\widetilde{A}_n$ buildings.

It turns out that many simplicial complexes may be endowed with natural $\widetilde{A}_n$ simplices, notably $\widetilde{A}_n$ Euclidean buildings, the Artin complex of the Artin-Tits group of type $\widetilde{A}_n$, and quotients of Garside groups. One common feature in all these examples is that the complex in question may be realized as the quotient of a (poset-theoretic) lattice under an action by $\mathbb{Z}$, as in [Hae21c] and [Hae21a]. The geometry of the corresponding lattice can be turned into a Helly graph by thickening, i.e. adding extra edges. However, there are no results about the original simplicial complex, i.e. the quotient of the lattice.

For instance, Hoda [Hod20] proved that affine Coxeter groups of type $\widetilde{A}_n$ are not Helly, and Haettel [Hae21b] proved that if $n \geqslant 4$, then Euclidean buildings of type $\widetilde{A}_n$, even after equivariant thickening, are not Helly graphs.

In this article, we prove the following.

Theorem A. *(Theorem 5.1) The 1-skeleton of any Euclidean building of type $\widetilde{A}_n$ is a weakly modular graph.*

This theorem, as well as several other theorems below is a consequence of more general theorems on weak modularity of graphs coming from lattices with increasing $\mathbb{Z}$-action and the associated quotients, see Theorem 2.1 and Theorem 3.1. We also note that the 1-skeletons considered in the above theorem also satisfy a stronger version of weak modular graph, as discussed in the end of Section 2.

On the other hand, the affine Coxeter complexes of type $\widetilde{C}_2$ and $\widetilde{G}_2$ are not weakly modular. Nevertheless, up to equivariantly adding edges, they become weakly modular. Concerning more general buildings, we formulate the following.

Conjecture B. *Any building has an equivariant thickening which is a weakly modular graph.*

To be precise, we say that a graph Γ is an equivariant thickening of a Euclidean building X if Γ contains X as a subgraph, X is quasi-isometric to Γ, and the automorphism group of X extends as an automorphism group of Γ. This is motivated by the following particular cases.

Theorem C. *(Theorem 5.3) Conjecture B holds for the following buildings:*

- *Any spherical building.*
- *Any Euclidean building of type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$ or $\widetilde{G}_2$.*
- *Any right-angled building.*
- *Any rank 3 building.*
- *Any Gromov-hyperbolic building.*

We are also able to apply the same techniques to various classes of groups and complexes, first with Garside groups and weak Garside groups. Garside structures are essentially structures which locally look like a lattice, see Section 4 for precise definitions.

Theorem D. *(Theorem 5.5) Let (G, Δ, S) denote a (weak) Garside group. Then $\mathrm{Cay}(G, S)$, and its quotient by $\langle \Delta \rangle$, are weakly modular graphs.*

This applies notably to finite type Garside groups, such as braid groups and spherical type Artin-Tits groups. Recall that Artin-Tits groups, defined by Tits in [Tit66], are natural generalizations of Coxeter groups and braid groups (see Section 1.1). In this case of braid groups, one statement of the theorem is that the Cayley graph of braid groups with respect to simple braids is weakly modular. This also applies to infinite type Garside groups, such as braided crystallographic group [MS17], some Euclidean type Artin-Tits groups [Dig06, Dig12, McC15] and some braid groups of imprimitive complex reflection groups [CLL15]. Among weak Garside groups of finite types, one has all braid groups of complex reflection groups [BC06, Bes15, CP11] except possibly the exceptional complex braid group of type G_{31}, all fundamental groups of complements of complexified real simplicial arrangements of hyperplane [Del72], some extensions of Artin-Tits groups of type B_n [CP05].

The techniques also apply to some Artin complexes. Recall that Artin-Tits groups have a natural candidate analogue of the curve complex known as the Artin complex, see [CD95, CMV20]. It is the flag simplicial complex with vertices being cosets of maximal proper standard parabolic subgroups, with an edge for non-trivial intersection. In the case of Euclidean type Artin-Tits groups, the Artin complex is closely related to the Deligne complex.

Theorem E. *(Theorem 5.6) Let A denote the Artin-Tits group of Euclidean type $\widetilde{A}_n$, and let X denote the Artin complex of A. Then X is a weakly modular graph.*

We observe the Artin complex of the Artin-Tits group of Euclidean type $\widetilde{A}_n$ has a topological interpretation as the complex of certain collection of arcs in a surface (cf. Proposition 5.8). Thus we give two treatments of Theorem 5.6, one is more in the Artin-Tits group side, another uses surface topology and factors though the following theorem.

Theorem F. *(Theorem 5.7) Let $n \geqslant 0$, and let Σ be the 2-sphere with $n+2$ punctures $N, S, p_1, \ldots, p_n$. Let $\mathcal{A}(\Sigma)$ denote the subcomplex of the arc complex consisting of arcs between N and S. Then $\mathcal{A}(\Sigma)$ is a weakly modular graph.*

The key objects in this article are lattices. In some applications, the corresponding lattice will be transparent: the lattice of norms in the case of $\widetilde{A}_n$ buildings, and the lattice of a Garside group. In some other examples, the lattice will be revealed through a mere local lattice property, as in the case of Artin complexes and arc complexes. To this end, we also reformulate work of Bessis into a simple local-to-global property for lattices, in the framework of Garside categories (see Section 1.3 for definitions).

Theorem G. *(Theorem 1.3) Suppose $(P, \leq)$ is a homogeneous weakly ordered set. If there exists an automorphism $\varphi : P \to P$ which generates $\leq$ such that*

1. *$\varphi(x) > x$ for any $x \in P$;*
2. *X_φ is simply connected;*
3. *$[x, \varphi(x)]$ is a lattice for any $x \in P$.*

Then $\leq$ generates a partial order $\leq_t$ on P, and $(P_{\geq_t x}, \leq_t)$ and $(P_{\leq_t x}, \leq_t)$ are lattices for any $x \in P$.

Interestingly, the framework of Garside categories in Bessis [Bes06] and the framework of lattices with $\mathbb{Z}$-action in Haettel [Hae21c] have many connections. Actually the former is a special case of the latter in an appropriate sense, and the latter contains a "continuous" version of the former (see the example and remark after Theorem 5.2). To this end, we record the following theorem, which gives a dictionary between different settings, under appropriated assumptions. See Section 4.3 for precise definitions for terms in the following theorem.

Theorem H. *(Theorem 4.6) The following objects are equivalent:*

- *A categorical Garside structure.*
- *A Garside lattice.*
- *A Garside flag complex.*

Acknowledgements: The authors would like to thank Andrew Putman for precise references concerning the arc complex on the punctured sphere. The authors would also like to thank Owen Garnier and Ivan Marin for many precisions concerning complex braid groups. The authors thank Anthony Genevois for many helpful comments. The authors thank Damian Osajda for stimulating discussions.

Contents

1 Preliminary

1.1 Artin-Tits groups and Coxeter groups

Let Γ be a finite simple graph with each edge labeled by an integer ≥ 2. The *Artin-Tits group with defining graph* Γ, also known as Artin group, denoted A_Γ, is given by the following presentation. Generators of A_Γ are in one to one correspondence with vertices of Γ, and there is a relation of the form

$$\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$$

whenever two vertices a and b are connected by an edge labeled by m. The *Coxeter group with defining graph* Γ, denoted W_Γ, has the same generator sets and the same relators as the Artin-Tits group, with extra relations $v^2 = 1$ for each vertex $v \in \Gamma$.

In this article we are mostly interested in Artin groups and Coxeter groups of type $\widetilde{A}_{n-1}$, in which case the defining graph Γ is a complete graph on n vertices such that there exists an embedded n-cycle in Γ with each edge contained in the cycle are labeled by 3 and all other edges are labeled by 2.

Let A_Γ be an Artin-Tits group of type $\widetilde{A}_{n-1}$ with consecutive generators in its Dynkin diagram labeled by $\{s_1, s_2, \ldots, s_n\}$. Let A_i be the subgroup of A_Γ all the generators except s_i. Let X be the *Artin complex* of A_Γ (cf. [CD95, CMV20]), namely vertices of X are in 1-1 correspondence with left cosets of form $\{gA_i\}_{g\in A_\Gamma, 1\leq i\leq n}$. Two vertices are joined by an edge if the associated left cosets have non-empty intersection. Then X is defined to be the flag completion of its 1-skeleton.

Note that the barycentric subdivision of X coincides with the modified Deligne complex of A_Γ as defined in [CD95], since in type $\widetilde{A}_{n-1}$ the spherical parabolic subgroups are exactly the proper parabolic subgroups.

1.2 Weakly modular graphs and local to global

Let Γ be a simplicial graph. We endow Γ with the path metric such that each edge has length $= 1$. Recall that Γ is *weakly modular* if for every vertex $x \in \Gamma$, and every positive integer $n \geq 2$, the following two conditions hold:

1. (*triangle condition* $\mathrm{TC}(x)$) for any two adjacent vertices $x_1, x_2 \in \Gamma$ such that $d(x, x_1) = d(x, x_2) = n$, there exists vertex y with $d(y, x) = n-1$ such that $d(y, x_1) = d(y, x_2) = 1$;

2. (*quadrangle condition* $\mathrm{QC}(x)$) for for any two vertices $x_1, x_2 \in \Gamma$ with $d(x, x_1) = d(x, x_2) = n$ and $d(x_1, x_2) = 2$ such that x_1 and x_2 are adjacent to a common vertex at distance $n+1$ from x, there exists a vertex y with $d(y, x) = n-1$ such that $d(y, x_1) = d(y, x_2) = 1$.

A graph Γ is *local weakly modular* if for every vertex $x \in \Gamma$, the triangle condition $\mathrm{TC}(x)$ and quadrangle condition $\mathrm{TC}(x)$ hold with $n = 2$.

An *induced* subgraph of Γ is a subgraph which contains all edges of Γ that join two vertices of the subgraph. A *square* of Γ is an induced embedded 4-cycle of Γ.

Given a graph Γ, the *triangle-square complex* of Γ is a two-dimensional cell complex with 1-skeleton Γ such that we fill in a solid triangle for each embedded 3-cycle in the graph and fill in a solid square for each square of Γ.

Theorem 1.1. *([CCHO21, Theorem 3.1]) If Γ is a local weakly modular graph, and its triangle-square complex is simply-connected, then Γ is a weakly modular graph.*

1.3 Posets, Lattices and local to global

Let P be a poset, i.e. a partially ordered set. Let $S \subset P$. An *upper bound* (resp. lower bound) for S is an element $x \in P$ such that $s \leq x$ (resp. $s \geq x$) for any $s \in S$. The *join* of S is an upper bound x of S such that $x \leq y$ for any other upper bound y of S. The *meet* of S is a lower bound of x of S such that $x \geq y$ for any other lower bound y of S. We will write $x \vee y$ for the join of two elements x and y, and $x \wedge y$ for the meet of two elements (if the join or the meet exists). We say P is *lattice* if P is a poset and any two elements in P have a join and have a meet. For $a, b \in P$ with $a \leq b$, the *interval* between a and b, denoted by $[a, b]$, is the collection of all elements x of P such that $a \leq x$ and $x \leq b$. A poset P is homogeneous if there is a function ℓ from each comparable pair in P to the non-negative integers such that if $a \leq b \leq c$, then $\ell(a \leq c) = \ell(a \leq b) + \ell(b \leq c)$; and $\ell(a \leq b) = 0$ if and only if $a = b$. Note that if P is homogeneous lattice, then any upper bounded subset of P has a join and any lower bounded subset of P has a meet. We will also need the following notion of *weakly ordered* set which generalizes the notion of poset by allowing the transitivity to fail.

Definition 1.2. A *weakly ordered* set P is a set with a binary relation $\leq$ over P which is reflexive and antisymmetric. Moreover, while transitivity may fail, we do require the following associativity law for transitivity. Define $(a, b, c) \in P^3$ to be a *transitive triple* if $a \leq b$, $b \leq c$ and $a \leq c$. We require $\leq$ satisfies the following condition:
$(*)$: for any quadruple $a, b, c, d \in P$ with $a \leq b, b \leq c$ and $c \leq d$, we have (b, c, d) and (a, b, d) are transitive triples if and only if (a, b, c) and (a, c, d) are transitive triples.

The notions of upper bound, join, lower bound, meet, interval and homogeneity can be defined for a weakly ordered set in the same way. Let $(P, \leq)$ be a weakly ordered set. For $x \in P$, let $P_{\geq x}$ be the collection of all elements which are $\geq x$. Similarly we define $P_{\leq x}$. Note that $P_{\geq x}$ and $P_{\leq x}$ are actually posets by $(*)$. A *weak chain* in a weakly ordered set P is a sequence of elements $c_1 < c_2 < c_3 < \cdots < c_n$ such that any two adjacent elements in the sequence are comparable. A *chain* is a weak chain such that any two elements in the weak chain are comparable. Note that if P does not contain non-trivial weak chains which start and end at the same element, then the weak order $\leq$ on P actually generated a partial order $\leq_t$, where $a \leq_t b$ if a and b are the first and the last member of a weak chain in P with respect to $\leq$.

An *automorphism* of $(P, \leq)$ is a bijection of P preserving the relation $\leq$. Suppose $(P, \leq)$ is a weakly ordered set with an automorphism $\varphi : P \to P$ such that $\varphi(x) > x$ for any $x \in P$. Let $\leq_\varphi$ be the subrelation of $\leq$ made of all possible $a \leq b$ such that there exists $x \in P$ such that $a, b \in [x, \varphi(x)]$. We say φ *generates* $\leq$ if a and b fit into the first element and the last element of a weak chain with respect to $\leq_\varphi$ whenever $a \leq b$.

Let X_φ be the simplicial complex whose vertex set is P and whose edges correspond to morphisms of form $a \leq b$ such that $a, b \in [x, \varphi(x)]$ for $x \in P$. Then X_φ is defined to be the flag completion of its 1-skeleton. By condition $(*)$ above, simplices of X_φ correspond to weak chains in P that are contained in an interval of form $[x, \varphi(x)]$ for some $x \in P$. Note that though a weak chain contained in $[x, \varphi(x)]$ is automatically a chain.

The following is a consequence of work of Bessis [Bes06].

Theorem 1.3. *Suppose $(P, \leq)$ is a homogeneous weakly ordered set. If there exists an automorphism $\varphi : P \to P$ which generates $\leq$ such that*

1. $\varphi(x) > x$ for any $x \in P$;

2. X_φ is simply connected;

3. $[x, \varphi(x)]$ is a lattice for any $x \in P$.

Then $\leq$ generates a partial order $\leq_t$ on P, and $(P_{\geq_t x}, \leq_t)$ and $(P_{\leq_t x}, \leq_t)$ are lattices for any $x \in P$.

Proof. By definition of $\varphi(x)$, we have $x \leq a \leq \varphi(x)$ if and only if $x \leq_\varphi a \leq_\varphi \varphi(x)$. Thus the interval $[x, \varphi(x)]$ with respect to $\leq$ and the same interval with respect to $\leq_\varphi$ are the same. Thus we can assume without loss of generality that $\leq = \leq_\varphi$. We claim for any $x \in P$, if $x \leq a$, then $a \leq \varphi(x)$. Indeed, $x \leq a$ implies there exists $y \in P$ such that $x, a \in [y, \varphi(y)]$. As $y \leq x$, we have $\varphi(y) \leq \varphi(x)$. We now consider the quadruple $x, a, \varphi(y), \varphi(x)$. Then $(x, a, \varphi(y))$ and $(x, \varphi(y), \varphi(x))$ are transitive triples as $x \in [y, \varphi(y)]$. Thus $a \leq \varphi(x)$ and the claim is proved. Similarly, we know for any $x \in P$, if $a \leq x$, then $\varphi^{-1}(x) \leq a$.

Note that $(P, \leq)$ is a special example of a germ in the sense of [Bes06, Definition 1.1]. More precisely, the objects of the germ are elements in P, and there is a morphism from a to b if $a \leq b$. This germ is homogeneous Garside in the sense of [Bes06, Definition 3.2]. Indeed, [Bes06, Definition 3.2] (i) is clear. By the above claim, the collection of morphism starting at x has a maximal element, which is $x \leq \varphi(x)$, thus [Bes06, Definition 3.2] (ii) holds true. Now [Bes06, Definition 3.2] (iv) follows from assumption (3). [Bes06, Definition 3.2] (iii) translates into the following: we consider the map from the set of morphisms starting at x to the set of morphisms ending at $\varphi(x)$ sending $x \leq a$ to $a \leq \varphi(x)$. By the above claim, this map is well-defined and is a bijection. Now [Bes06, Definition 3.2] (iii) is clear.

Consider the category $\mathcal{C}$ generated by $\leq$, whose objects are P and whose morphisms are of form $a_1 \leq a_2 \leq \cdots \leq a_n$ (we require adjacent members in the sequence are comparable, however, non-adjacent members might not be comparable). By [Bes06, Theorem 3.3], $\mathcal{C}$ is a categorical Garside structure in the sense of [Bes06, Definition 2.4]. In particular, the collection of all morphism starting from $x \in P$, endowed with the prefix order (i.e. $f \leq_p g$ if $g = fh$ for a morphism h of $\mathcal{C}$), is a lattice; and the collection of all morphism ending at $x \in P$, endowed with the suffix order (i.e. $f \geq_s g$ if $hg = f$ for a morphism h of $\mathcal{C}$), is a lattice. Thus we are done as long as we can show there are no non-trivial weak chains in P which start and end at the same element.

Let $\mathcal{G}$ be the groupoid obtained by adding formal inverses to all morphisms in $\mathcal{C}$. It follows from the discussion in [Bes06, Section 2] that the map $\mathcal{C} \to \mathcal{G}$ is injective. Take an object $x \in P$, let $\mathcal{G}_x$ be the collection of morphisms in $\mathcal{G}$ starting at x. Let $|\mathcal{G}_x|$ be the simplicial complex defined by Bessis as follows. The vertices are corresponding to elements in $|\mathcal{G}_x|$. Two different vertices are joined by an edge if the corresponding two morphisms f and g satisfies that $f = gh$ or $g = fh$ for some simple morphism h in $\mathcal{C}$ (a *simple* morphism of $\mathcal{C}$ is a morphism of form $a < b$ such that $b < \varphi(a)$). Then $|\mathcal{G}_x|$ is the flag complex its 1-skeleton. By [Bes06, Corollary 7.6], $|\mathcal{G}_x|$ is contractible, hence simply-connected. Note that there is a map p from $\mathcal{G}_x$ to P by sending each morphism to its endpoint. The map p extends to a simplicial map $p : |\mathcal{G}_x| \to X_\varphi$ which is also a covering map. As X_φ is simply connected, we know p is a simplicial isomorphism. Thus there are no non-trivial weak chains in P which start and end at the same element by $(*)$ in Definition 1.2. □

2 The diagonal quotient of a lattice is weakly modular

Assume that L is a lattice, such that each upperly bounded subset of L has a join (as a consequence, each lower bounded subset of L has a meet). Assume that there is an order-preserving increasing action of $\mathbb{Z}$ on L, noted additively (i.e. the image of $x \in L$ under the action $n \in \mathbb{Z}$ is denoted by $x + n$), such that

$$\forall x, y \in L, \exists k \in \mathbb{N}, x - k \leqslant y \leqslant x + k.$$

We will define a graph X from L, with vertex set $L/\mathbb{Z}$. Add an edge between $x, y \in X$ if, for some representatives x_0, y_0 of x, y in L, we have

$$x_0 \leqslant y_0 \leqslant x_0 + 1.$$

We have the following.

Theorem 2.1. *X is a weakly modular graph.*

Note that X is connected by Lemma 2.2 below and our assumption on L. Thus the theorem is a consequence of Lemma 2.3 and Lemma 2.4 below.

Lemma 2.2. *Given any vertices $x, y \in X$ and any representatives $x_0, y_0 \in L$, we have*

$$d(x,y) = \min\{n \geqslant 0 \mid \exists k, h \in \mathbb{Z}, x_0 + k \leqslant y_0 + h \leqslant x_0 + k + n\}.$$

Proof. Let us denote the formula in the statement by d'. If $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n$ is a lift to L of a path in the graph X, then $x_0 \leqslant x_n \leqslant x_0 + n$, so $d'(x_0 + \mathbb{Z}, x_n + \mathbb{Z}) \leqslant n$. Hence $d' \leqslant d$.

Conversely, we will prove by induction on $n \geqslant 0$ that, if $x, y \in L$ are such that $d'(x,y) = n$, then $d(x,y) = n$. When $n \leqslant 1$ it is obvious. Assume that the statement holds for values smaller than n, and fix vertices $x, y \in X$ and representatives $x_0, y_0 \in L$ such that $x_0 \leqslant y_0 \leqslant x_0 + n$, where $n = d'(x,y)$.

Let $z_0 = (x_0 + 1) \wedge y_0$: we have $z_0 + n - 1 = (x_0 + n) \wedge (y_0 + n - 1)$. As $y_0 \leqslant x_0 + n$ and $y_0 \leqslant y_0 + n - 1$, we have $z_0 \leqslant y_0 \leqslant y_0 + n - 1$, so $d'(y,z) \leqslant n - 1$. By induction, we deduce that $d(y,z) = d'(y,z) \leqslant n - 1$. Furthermore, we have $d(x,z) \leqslant 1$, so as $d(x,y) \geqslant d'(x,y) = n$ we conclude that $d(x,y) = n = d'(x,y)$. $\square$

Lemma 2.3. *X satisfies the triangle condition: fix $n \geqslant 2$, and let $x, y, z \in X$ such that $d(x,y) = d(x,z) = n$ and $d(y,z) = 1$. There exists $u \in X$ such that $d(x,u) = d(u,y) = 1$ and $d(u,z) = n - 1$.*

Proof. Consider representatives x_0, y_0, z_0 such that $x_0 \leqslant y_0 \leqslant x_0 + n$ and $y_0 \leqslant z_0 \leqslant y_0 + 1$. We will prove that we can furthermore assume that $z_0 \leqslant x_0 + n$.

Since $x_0 \leqslant z_0 \leqslant y_0 + 1 \leqslant x_0 + n + 1$ and $d(x,z) = n$, By Lemma 2.2 there are two possibilities:

- either $x_0 \leqslant z_0 \leqslant x_0 + n$
- or $x_0 + 1 \leqslant z_0 \leqslant x_0 + n + 1$.

In the former case, we have indeed $z_0 \leqslant x_0 + n$. In the latter case, we have $x_0 \leqslant z_0 - 1 \leqslant y_0 \leqslant x_0 + n$ and $z_0 - 1 \leqslant y_0 \leqslant z_0$, so up to replacing the pair (y_0, z_0) by the pair $(z_0 - 1, y_0)$, we can always assume that $z_0 \leqslant x_0 + n$.

Let $u_0 = (x_0 + n - 1) \wedge y_0$: we have $x_0 \leqslant u_0 \leqslant x_0 + n - 1$, so $d(x,u) \leqslant n - 1$.

Furthermore, since $z_0 \leqslant x_0 + n$ and $z_0 \leqslant y_0 + 1$, we deduce that $u_0 + 1 = (x_0 + n) \wedge (y_0 + 1) \geqslant z_0$, hence $u_0 \leqslant y_0 \leqslant z_0 \leqslant u_0 + 1$. Hence $d(u,y) \leqslant 1$ and $d(u,z) \leqslant 1$.

Hence we have $d(x,u) = n - 1$ and $d(u,y) = d(u,y) = 1$. $\square$

Lemma 2.4. *X satisfies the quadrangle condition: let $n \geqslant 2$ and $x, y, z, t \in X$ such that $d(x,y) = d(x,z) = n$, $d(y,t) = d(z,t) = 1$ and $d(x,t) = n + 1$. There exists $u \in X$ such that $d(x,u) = n - 1$ and $d(u,y) = d(u,z) = 1$.*

Proof. We first prove that there exists $s \in X$ such that $d(y,s) = d(z,s) = 1$ and $d(x,s) \leqslant n$. Consider representatives x_0, y_0, z_0, t_0 such that $x_0 \leqslant y_0, z_0 \leqslant x_0+n$ and $y_0 \leqslant t_0 \leqslant y_0+1$.

We will prove that also $z_0 \leqslant t_0 \leqslant z_0+1$. Since $x_0 \leqslant y_0 \leqslant t_0 \leqslant y_0+1 \leqslant x_0+n+1$ and $d(x,t) = n+1$, we deduce from Lemma 2.2 that t_0 is not comparable to x_0+1 nor x_0+n. Since $z_0 \leqslant x_0+n$, we deduce that t_0 is not inferior to z_0, hence $t_0 \geqslant z_0$ as $d(z,t) = 1$. Similarly, since $z_0+1 \geqslant x_0+1$, we deduce that $t_0 \leqslant z_0+1$. So we have $z_0 \leqslant t_0 \leqslant z_0+1$.

Let $s_0 = y_0 \wedge z_0$. Since $x_0 \leqslant s_0 \leqslant x_0+n$, we know that $d(x,s) \leqslant n$. Furthermore, since $t_0 - 1 \leqslant y_0, z_0 \leqslant t_0$, we deduce that also $t_0 - 1 \leqslant s_0 \leqslant t_0$. Thus if we take $s \in X$ to be the vertex associated with s_0, then $d(s,y) \leqslant 1$ and $d(s,z) \leqslant 1$.

Let $u_0 = (x_0-1) \wedge s_0$: we have $d(x,u) \leqslant n-1$. Moreover, $u_0 \leq y_0, z_0$. We will prove that $y_0, z_0 \leqslant u_0+1$. Since $s_0 = y_0 \wedge z_0$, we have $u_0 = (x_0+n-1) \wedge y_0 \wedge z_0$, so $u_0+1 = (x_0+n) \wedge (y_0+1) \wedge (z_0+1)$. Note that $y_0 \leqslant x_0+n$ and $y_0 \leqslant y_0+1$. Furthermore, we have $y_0 \leqslant t_0 \leqslant z_0+1$. As a conclusion, we have $y_0 \leqslant u_0+1$, and similarly $z_0 \leqslant u_0+1$. We conclude that $d(u,y) \leqslant 1$ and $d(u,z) \leqslant 1$.

Hence $d(x,u) = n-1$ and $d(u,y) = d(u,z) = 1$. □

Remark. The above argument implies that X satisfies the following stronger versions of the triangle condition and the quadrangle condition, namely:

1. for any vertex $x \in X$ and any complete subgraph $Y \subset X$ such that each vertex of Y is at distance n from x, there exists a vertex $z \in X$ such that $d(x,z) = n-1$ and z is adjacent to each vertex in Y;

2. for any vertex $x \in X$ and any vertex $t \in X$ with $d(x,t) = n+1$, let $Y = \{y \in X : d(y,t) = 1 \text{ and } d(y,x) = n\}$, then there exists a vertex u at distance $n-1$ from x such that s is adjacent to each vertex in Y, moreover, there exists a vertex $s \in Y$ such that s is adjacent to each vertex in $Y \setminus \{s\}$.

The proof of property (2) is identical to Lemma 2.4. The proof of property (1) is a small adjustment of Lemma 2.3. Namely take $y \in Y$ and choose representatives x_0, y_0 of x, y such that $x_0 \leq y_0 \leq x_0+n$. Then for any $z \in Y$, the proof of Lemma 2.3 implies that there exists a representative z_0 of z such that either $y_0 - 1 \leq z_0 \leq y_0$ and $x_0 \leq z_0 \leq x_0+n$ or $y_0 \leq z_0 \leq y_0+1$ and $x_0 \leq z_0 \leq x_0+n$. Let Y_0 be the collection of all such representatives of elements in Y. We claim each pair of elements z_0, z_0' in Y_0 are comparable, and if $z_0 \geq z_0'$, then $z_0 \leq z_0'+1$. First we consider the case $z_0 \geq y_0 \geq z_0'$. The first part of the claim is clear. As z and z' are adjacent, z_0 and $z_0'+1$ are comparable. If $z_0 > z_0'+1$, then $x_0+n-1 \geq z_0-1 \geq z_0' \geq x_0$, which contradicts that $d(z,x) = n$. Thus $z_0 \leq z_0'+1$. Now we consider the case where both $z_0, z_0' \geq y_0$, then $z_0, z_0' \leq y_0+1$. This, together with the fact that z and z' are adjacent imply the claim. The case $z_0, z_0' \leq y_0$ is similar. Thus the claim is proved. Let v_0 be the meet of Y_0 and $u_0 = (x_0+n-1) \wedge v_0$. By the claim, for any $z_0 \in Y_0$, we have $z_0 - 1 \leq v_0 \leq z_0$. In particular, $z_0 \leq v_0+1$. The argument in Lemma 2.3 implies that $u_0 \leq v_0 \leq z_0 \leq u_0+1$. Hence $d(u,z) \leq 1$ for any $z \in Y$.

3 A lattice with a diagonal action is weakly modular

Assume that L is a lattice, such that each upperly bounded subset of L has a join. Assume that there is an order-preserving increasing action of $\mathbb{Z}$ on L, noted additively, such that

$$\forall x, y \in L, \exists k \in \mathbb{N}, x - k \leqslant y \leqslant x + k.$$

We will define a graph X from L, with vertex set L. Add an edge between $x, y \in X$ if $x \leqslant y \leqslant x+1$ or $y \leqslant x \leqslant y+1$.

Theorem 3.1. *X is a weakly modular graph.*

This theorem is a consequence of Theorem 1.1, as well as Lemma 3.2, Lemma 3.4, Lemma 3.5 and Lemma 3.6 below.

Lemma 3.2. *The graph X is connected.*

Proof. Fix $x, y \in X$. Since there is an edge between x and $x+1$, we may assume that $x \leqslant y$. For each $n \in \mathbb{N}$, let $x_n = (x+n) \wedge y$. For each $n \in \mathbb{N}$, we have $x_n = (x+n) \wedge y \leqslant (x+n+1) \wedge y = x_{n+1}$, and also $x_{n+1} = (x+n+1) \wedge y \leqslant (x+n+1) \wedge (y+1) = ((x+n) \wedge y) + 1 = x_n + 1$. So x_n and x_{n+1} are adjacent in X.

By assumption, there exists $n \in \mathbb{N}$ such that $y \leqslant x+n$. So we deduce that $x_n = y$, and x and y are connected by the path $x_0 = x, x_1, \ldots, x_n = y$ in X. □

Lemma 3.3. *Given any vertices $x, y \in X$, we have*

$$d(x,y) = \min\{n+m \mid n, m \in \mathbb{N}, x \leqslant y+n, y \leqslant x+m\}.$$

Moreover, $x \vee y$ and $x \wedge y$ each belong to a geodesic between x and y.

Proof. Fix $x, y \in X$, and consider minimal $n, m \in \mathbb{N}$ such that $x \leqslant y+n, y \leqslant x+m$.

According to the previous proof, there is a length n path from $x \wedge y$ to $x \wedge (y+n) = x$. There is also a length m path from $x \wedge y$ to $(x+m) \wedge y = y$. Hence $d(x,y) \leqslant n+m$.

Conversely, we will prove by induction on $n+m$ that $d(x,y) = n+m$. Let $y' \in X$ adjacent to y, we will prove that the corresponding integers for x and y' satisfy $n'+m' \leqslant n+m+1$.

If $y \leqslant y' \leqslant y+1$, then $x \leqslant y+n \leqslant y'+n$ and $y' \leqslant y+1 \leqslant x+m+1$, so $n' \leqslant n$ and $m' \leqslant m+1$.

If $y' \leqslant y \leqslant y'+1$, then $x \leqslant y+n \leqslant y'+n+1$ and $y' \leqslant y \leqslant x+m$, so $n' \leqslant n+1$ and $m' \leqslant m$.

We conclude that $d(x,y) = n+m$. □

Lemma 3.4. *X satisfies the triangle condition: fix $n \geqslant 2$, and let $x, y, z \in X$ such that $d(x,y) = d(x,z) = n$ and $d(y,z) = 1$. There exists $u \in X$ such that $d(x,u) = d(u,y) = 1$ and $d(u,z) = n-1$.*

Proof. Assume, without loss of generality, that $y \leqslant z \leqslant y+1$. Let $n, m \in \mathbb{N}$ minimal such that $x \leqslant y+n$ and $y \leqslant x+m$.

Since $d(x,z) = d(x,y) = n+m$, there are two possibilities:

- either $x \leqslant z+n$ and $z \leqslant x+m$. Let $u = y \wedge x+m-1$: we have $d(u, y \wedge x) \leqslant m-1$ and $d(y \wedge x, x) \leqslant n$, so $d(u,x) \leqslant n+m-1$. Also $u \leqslant y \leqslant z$, and $y, z \leqslant y+1, x+m$, so $y, z \leqslant u+1$: we have $d(u,y) \leqslant 1$ and $d(u,z) \leqslant 1$. By the triangular inequality we have $d(x,u) = n+m-1$ and $d(u,y) = d(u,z) = 1$.

- or $x \leqslant z+n-1$ and $z \leqslant x+m+1$.

  Let $u = z \wedge (x+m)$: we have $d(u, z \wedge x) \leqslant m$ and $d(z \wedge x, x) \leqslant n-1$, so $d(u,x) \leqslant n+m-1$. Also $y \leqslant z, x+m$ so $y \leqslant u$. And $u \leqslant z \leqslant y+1$, so $d(u,y) \leqslant 1$. We also have $u \leqslant z$ and $z \leqslant (z+1), (x+m+1)$ so $z \leqslant u+1$: we have $d(u,z) \leqslant 1$. By the triangular inequality we have $d(x,u) = n+m-1$ and $d(u,y) = d(u,z) = 1$.

□

Lemma 3.5. *X satisfies the local quadrangle condition. More precisely, for any $x, y, z, t \in X$ such that $d(x,y) = d(x,z) = 2$, $d(x,t) = 3$ and $d(y,t) = d(z,t) = 1$, there exists $u \in X$ such that $d(x,u) = d(y,u) = d(z,u) = 1$.*

Proof. Note that, if $d(x,y)=2$, there are three possibilities: $x \leqslant y+1$ and $y \leqslant x+1$, $x \leqslant y \leqslant x+2$ and $y \leqslant x \leqslant y+2$. In this proof, we will call the last two possibilities of type $(2,0)$.

- Let us first assume that $x \leqslant y+1$, $y \leqslant x+1$, $x \leqslant z+1$, $z \leqslant x+1$, $y \leqslant z+1$ and $z \leqslant y+1$. Let $u = x \wedge y \wedge z$. We have $u \leqslant x$ and, since $x \leqslant x+1, y+1, z+1$, we have $x \leqslant u+1$. So $d(x,u) \leqslant 1$. Similarly $d(y,u) \leqslant 1$ and $d(z,u) \leqslant 1$. By the triangular inequality we have $d(x,u) = d(y,u) = d(z,u) = 1$.

- Assume now that $x \leqslant y+1$, $y \leqslant x+1$, $x \leqslant z$, $z \leqslant x+1$ and $y \leqslant z \leqslant z+2$. We will show that this contradicts the existence of t. Since $d(x,t)=3$, there are three possibilities:

  1. If $x \leqslant t \leqslant x+3$, then since $d(y,t)=d(z,t)=1$ we deduce that $t \leqslant z+1 \leqslant x+2$, which is a contradiction.
  2. If $x-1 \leqslant t \leqslant x+2$, then since $z \leqslant x+1$, we have $t \not\leqslant z$, so $y \leqslant z \leqslant t$. Hence $t \leqslant y+1$, so $z \leqslant y+1$, which is a contradiction.
  3. If $x-1 \leqslant t \leqslant x+2$, then since $x-1 \leqslant y$, we have $t \not\geqslant y$, so $t \leqslant y \leqslant z$. Hence $t \geqslant z-1$, so $z \leqslant y+1$, which is a contradiction.
  4. If $x-3 \leqslant t \leqslant x$, then since $d(y,t)=d(z,t)=1$ we deduce that $t \geqslant z-1 \leqslant x-2$, which is a contradiction.

- Assume now that $x \leqslant y \leqslant x+2$, $x \leqslant z+1$, $z \leqslant x+1$, $y \leqslant z+1$ and $z \leqslant y+1$. We will show that this contradicts the existence of t. We know that $t \leqslant z+1 \leqslant x+2$ and $t \geqslant y-1 \geqslant x-1$. Hence $x-1 \leqslant t \leqslant x+2$. So $t \not\leqslant x+1$, hence $t \not\leqslant z$: $z \leqslant t$. Similarly $t \not\geqslant x$, hence $t \not\geqslant y$: $t \leqslant y$. We deduce that $z \leqslant t \leqslant y$, which contradicts $d(y,z)=2$.

- Assume now that there are two distances of type $(2,0)$, e.g. $x \leqslant y \leqslant x+2$, $x \leqslant z \leqslant x+2$, $y \leqslant z+1$ and $z \leqslant y+1$. We will show the existence of u independently of the assumption on t. Let $u = (x+1) \wedge y \wedge z$. Then $x \leqslant u \leqslant x+1$, so $d(u,x) \leqslant 1$. Also $u \leqslant y \leqslant u+1$, so $d(u,y) \leqslant 1$. Similarly $d(u,z) \leqslant 1$. By the triangular inequality we have $d(x,u) = d(y,u) = d(z,u) = 1$.

- Assume now that the three distances are of type $(2,0)$. Without loss of generality, we may assume that $y \leqslant z$, so $y \leqslant t \leqslant z$.

  1. If $x \leqslant y \leqslant z$, then $x \leqslant t \leqslant x+2$, which is a contradiction.
  2. If $y \leqslant x \leqslant z$, then $t \leqslant y+1 \leqslant x+1$ and $t \geqslant z-1 \geqslant x-1$, so $x-1 \leqslant t \leqslant x-1$, which is a contradiction.
  3. If $y \leqslant z \leqslant x$, then $x-2 \leqslant y \leqslant t \leqslant x$, which is a contradiction.

□

Lemma 3.6. *The triangle-square complex of X is simply connected.*

Proof. Assume that ℓ is a combinatorial loop in X, and fix $x \in X$ such that $x \leqslant \ell$. Then, for each $n \in \mathbb{N}$, let ℓ_n denote the loop $\ell \wedge (x+n)$: more precisely, if y is a vertex of ℓ, then $y \wedge (x+n)$ is a vertex of ℓ_n. This actually defines a loop since, if $d(y,z)=1$, for instance $y \leqslant z \leqslant y+1$, then $y \wedge x \leqslant z \wedge x \leqslant (y+1) \wedge x \leqslant (y \wedge x)+1$, so $d(y \wedge x, z \wedge x) \leqslant 1$. Since also $d(y \wedge (x+n), y \wedge (x+n+1)) \leqslant 1$, we deduce that, for each $n \in \mathbb{N}$, the loops ℓ_n and ℓ_{n+1} are homotopic in the triangle-square complex of X.

If $N \in \mathbb{N}$ is such that $\ell \leqslant x+N$, the loop ℓ_N is constant. Hence the triangle-square complex of X is simply connected. □

4 Garside categories, Garside lattices and Garside flag complexes

In this section, we explicit a dictionary between categorical Garside structures and certain lattices and simplicial complexes, following Bessis ([Bes15]). In particular, we make connections between categorical Garside structures and the type of lattices studied in Section 2 and Section 3.

4.1 Definition of Garside category and an example

Let $\mathcal{C}$ be a small category. One may think of $\mathcal{C}$ as of an oriented graph, whose vertices are objects in $\mathcal{C}$ and oriented edges are morphisms of $\mathcal{C}$. Arrows in $\mathcal{C}$ compose like paths: $x \xrightarrow{f} y \xrightarrow{g} z$ is composed into $x \xrightarrow{fg} z$. For objects $x, y \in \mathcal{C}$, let $\mathcal{C}_{x\to}$ denote the collection of morphisms whose source object is x. Similarly we define $\mathcal{C}_{\to y}$ and $\mathcal{C}_{x\to y}$.

For two morphisms f and g, we define $f \preccurlyeq g$ if there exists a morphism h such that $g = fh$. Define $g \succcurlyeq f$ if there exists a morphism h such that $g = hf$. Then $(\mathcal{C}_{x\to}, \preccurlyeq)$ and $(\mathcal{C}_{\to y}, \succcurlyeq)$ are posets. A nontrivial morphism f which cannot be factorized into two nontrivial factors is an *atom*.

The category $\mathcal{C}$ is *cancellative* if, whenever a relation $afb = agb$ holds between composed morphisms, it implies $f = g$. $\mathcal{C}$ is *homogeneous* if there exists a length function l from the set of $\mathcal{C}$-morphisms to $\mathbb{Z}_{\geq 0}$ such that $l(fg) = l(f) + l(g)$ and $(l(f) = 0) \Leftrightarrow (f$ is a unit).

We consider the triple $(\mathcal{C}, \mathcal{C} \xrightarrow{\phi} \mathcal{C}, 1_{\mathcal{C}} \overset{\Delta}{\Rightarrow} \phi)$ where ϕ is an automorphism of $\mathcal{C}$ and Δ is a natural transformation from the identity function to ϕ. For an object $x \in \mathcal{C}$, Δ gives morphisms $x \xrightarrow{\Delta(x)} \phi(x)$ and $\phi^{-1}(x) \xrightarrow{\Delta(\phi^{-1}(x))} x$. We denote the first morphism by Δ_x and the second morphism by Δ^x. A morphism $x \xrightarrow{f} y$ is *simple* if there exists a morphism $y \xrightarrow{f^*} \phi(x)$ such that $ff^* = \Delta_x$. When $\mathcal{C}$ is cancellative, such f^* is unique.

Definition 4.1 ([Bes06])**.** A *homogeneous categorical Garside structure* is a triple $(\mathcal{C}, \mathcal{C} \xrightarrow{\phi} \mathcal{C}, 1_{\mathcal{C}} \overset{\Delta}{\Rightarrow} \phi)$ such that:

1. ϕ is an automorphism of $\mathcal{C}$ and Δ is a natural transformation from the identity function to ϕ;
2. $\mathcal{C}$ is homogeneous and cancellative;
3. all atoms of $\mathcal{C}$ are simple;
4. for any object x, $\mathcal{C}_{x\to}$ and $\mathcal{C}_{\to x}$ are lattices.

It has *finite type* if the collection of simple morphisms of $\mathcal{C}$ is finite.

A fundamental property of $\mathcal{C}$ is that the natural map $\mathcal{C} \to \mathcal{G}$ is an embedding, where $\mathcal{G}$ denotes the enveloping groupoid, as follows from the discussion in [Bes06, Section 2].

Definition 4.2. A *Garside category* is a category $\mathcal{C}$ that can be equipped with ϕ and Δ to obtain a homogeneous categorical Garside structure. A *Garside groupoid* is the enveloping groupoid of a Garside category. Informally speaking, it is a groupoid obtained by adding formal inverses to all morphisms in a Garside category.

Let x be an object in a groupoid $\mathcal{G}$. The *isotropy* group $\mathcal{G}_x$ at x is the group of morphisms from x to itself. A *weak Garside group* is a group isomorphic to the isotropy group of an object in a Garside groupoid.

A *Garside monoid* is a Garside category with a single object and a *Garside group* is a Garside groupoid with a single object.

Example. Let $\mathcal{A}$ be a finite central arrangement in $\mathbb{R}^n$, i.e. a finite collection of linear hyperplanes in $\mathbb{R}^n$. Let $ch(\mathcal{A})$ be the set of chambers (connected components of the complement of the hyperplanes in $\mathcal{A}$). We consider an oriented graph Γ, whose vertices are in 1-1 correspondence with the collection of chambers, and we draw a pair of oriented edges going in opposite directions between two vertices if the associated chambers are adjacent along a hyperplane.

A *positive path* on Γ is an edge path from one vertex to another vertex which goes along the positive orientation on each edge. Take a positive path f_1 and a subpath g of f_1 which is a geodesic between its endpoints with respect to the path metric on Γ. An *elementary homotopy* of f_1 is the procedure of replacing the subpath g of f_1 by another positive subpath which is a geodesic between the two endpoints of g. Two positive paths are *equivalent* if they differ by a finite sequence of elementary homotopy.

Now we consider that category $\mathcal{C}$ as follows. Objects of $\mathcal{C}$ are vertices of Γ, and morphisms of $\mathcal{C}$ are equivalent classes of positive paths from one vertex to another vertex. There is an orientation-preserving automorphism $\alpha : \Gamma \to \Gamma$ arising from the central symmetry of $\mathbb{R}^n$ with respect to the origin. Note that α induces an automorphism of category $\mathcal{C} \overset{\phi}{\to} \mathcal{C}$, which is a functor sending object x to $\alpha(x)$, and the morphism represented by a path f to the morphism represented by $\alpha(f)$. For each object $x \in \mathcal{C}$, let Δ_x be the morphism from x to $\phi(x)$ represented by a positive geodesic in Γ from x to $\phi(x)$. One readily verifies that the family of morphisms $\{\Delta_x\}_{x\in \mathrm{Obj}(\mathcal{C})}$ gives a natural transformation between the identity function and the functor ϕ, i.e. for each morphism $[f]$ in $\mathcal{C}$ represented by a positive path f from x to y, we have $\Delta_x\phi([f]) = [f]\Delta_y$.

It is shown in [Del72] (see also [Bes06, Example 3.4]) if the arrangement $\mathcal{A}$ is *simplicial*, namely, hyperplanes in $\mathcal{A}$ cuts the unit sphere of $\mathbb{R}^n$ into a simplicial complex, then $\mathcal{C}$ is a homogeneous categorical Garside structure in the above sense (the non-trivial part is property (4) of Definition 4.1). Moreover, the associated weak Garside group is isomorphism to the fundamental group of the complement of the complexification of hyperplanes of $\mathcal{A}$ in $\mathbb{C}^n$.

4.2 From Garside category to Garside lattice

The type of lattices studied in Section 2 and Section 3 are what we call a Garside lattice, and we define now.

Definition 4.3. A *Garside lattice* is a pair (L, φ), where L is a homogeneous lattice and φ is an increasing automorphism of L, such that, for any $x, y \in L$, there exists $k \in \mathbb{N}$ such that $x \leqslant \varphi^k(y)$.

We now see that a categorical Garside structure naturally gives rise to a Garside lattice.

Proposition 4.4. *Let $(\mathcal{C}, \phi, \Delta)$ be a categorical Garside structure, and let $\mathcal{G}$ be the associated Garside groupoid. Let x be an object in $\mathcal{G}$, and let L_x denote the set of morphisms from x in the groupoid of $\mathcal{G}$. Let us consider the map $\psi : f \in L_x \mapsto \Delta_x\phi(f)$ of L_x. We endow L_x with the partial order $\leq$ such that $g \leq h$ if $h = gf$ with $f \in \mathcal{C}$. Then $(L_x, \leq)$ is a lattice such that*

1. *any upper bounded set in L_x has a join;*
2. *ψ is an increasing automorphism of $(L_x, \leq)$;*
3. *for any $g, h \in L_x$, there exists $k \in \mathbb{N}$ such that $\psi^{-k}(g) \leqslant h \leqslant \psi^k(g)$.*

Proof. Recall that the natural map $\mathcal{C} \to \mathcal{G}$ is an embedding. Given $g \in L_x$, by Definition 4.1 (4), the collection $(L_x)_{\geq g}$ of all elements of L_x that is $\geq g$ form a lattice under the order $\leq$. Similarly, $(L_x)_{\leq g}$ is a lattice. To see $(L_x, \leq)$ is a lattice, it suffices to justify the claim that any two elements in L_x has a lower bound. Indeed, given $f, g \in L_x$, by Definition 4.1 (2) and (3), we can write $f = g s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$, and each $s_i \in \mathcal{C}$ is a simple element. We claim $\psi(f)$ is of form $gh(s')_2^{\varepsilon_2} \cdots (s')_n^{\varepsilon_n}$ where $h \in \mathcal{C}$ and each $s'_i \in \mathcal{C}$ is simple. Indeed, as Δ gives a natural transformation from the identity function to ϕ, we know $\psi(f) = \Delta_x \phi(g)\phi(s_1^{\varepsilon_1}) \cdots \phi(s_n^{\varepsilon_n}) = g\Delta_{t(g)}\phi(s_1^{\varepsilon_1}) \cdots \phi(s_n^{\varepsilon_n})$ where $t(g)$ denotes the terminal object of g. Then the claim is clear if $\varepsilon_1 = 1$. If $\varepsilon_1 = -1$, we find s_1^* such that $s_1 s_1^* = \Delta_{t(g)}$. As Δ is a natural transformation, we know $\Delta_{t(g)}\phi(s_1^{\varepsilon_1}) = s_1^{\varepsilon_1}\Delta_{t(s_1^{\varepsilon_1})}$. Let y be the starting point of s_1. Then $\Delta_{t(s_1^{\varepsilon_1})} = \Delta_y = s_1 s_1^*$ for some $s_1^* \in \mathcal{C}$. Thus $\Delta_{t(g)}\phi(s_1^{\varepsilon_1}) = s_1^*$ and the claim follows. By applying the claim several times, we know $\psi^n(f) = gh$ with $h \in \mathcal{C}$. Thus $g \leq \psi^n(f)$. Similarly $f \leq \psi^n(g)$. Thus property (3) of the proposition holds. In particular, any two elements in L_x have a lower bound. Thus L_x is a lattice. As L_x is homogeneous, property (1) holds. Property (2) is clear. □

4.3 Garside categories, Garside lattices and Garside flag complexes

We will relate three Garside notions: categorical Garside structures, Garside lattices, and Garside flag complexes, which we define now. These are flag simplicial complexes with a local lattice structure and a special automorphism.

Definition 4.5. A *Garside flag complex* is a pair (X, φ), where X is a simply connected flag simplicial complex with finite simplices, with a consistent total order on each simplex, and φ is an order-preserving automorphism of X, such that the following hold:

- For any simplex σ of X, we have that $\sigma \cup \varphi(\min \sigma)$ is a simplex of X.
- For any vertex $x \in X$, we have $\varphi(x) > x$, and the interval $[x, \varphi(x)]$ is a homogeneous lattice.

We say that a group G acts by automorphisms on a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$ if G acts by automorphisms on the category $\mathcal{C}$ and its action is compatible with ϕ and Δ in the following sense:

1. the action of G commutes with the action of ϕ;
2. Δ is a natural transformation between the identity functor and each functor in G.

If a group G acts by automorphisms on a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$ such that the restriction on the action to the set of objects is free, then one may consider the quotient categorical Garside structure, whose set of objects is the quotient of the set of objects of $\mathcal{C}$ by G. These two categorical Garside structures, however, have the same set of morphisms starting a given object. We will call two categorical Garside structures *equivalent* if they are quotients of the same categorical Garside structure by actions of automorphisms which are free on the set of objects.

Theses notions enable us to write the following dictionary between categorical Garside structures, Garside lattices and Garside flag complexes.

Theorem 4.6. *The following objects are equivalent:*

- *A connected categorical Garside structure, up to equivalence.*
- *A Garside lattice, up to isomorphism.*
- *A Garside flag complex, up to isomorphism.*

Proof. Given a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$, let $\mathcal{G}$ denote the associated Garside groupoid, fix an objet $x \in \mathcal{G}$, and let G denote the weak Garside group that is the isotropy group of $\mathcal{G}$ at x. Then the gar flag complex $\mathrm{gar}(\mathcal{G}, S, x)$ from [Bes15, Definition B.11], whose vertex set is the set of morphisms from x, is a Garside flag complex. Note that, since $\mathcal{G}$ is connected, $\mathrm{gar}(\mathcal{G}, S, x)$ does not depend on x. We also see that the lattice from Proposition 4.4 consisting of morphisms from x forms a Garside lattice.

Given a Garside flag complex (X, φ), it naturally induces a homogeneous weak order on its vertex set L. The properties from Theorem 1.3 follow from the definition of a Garside flag complex, and ensure that, for any $x \in L$, the posets $L_{\leqslant x}$ and $L_{\geqslant x}$ are lattices. Since X is connected, we deduce that, for any $x, y \in L$, there exists $k \in \mathbb{N}$ such that $x \leqslant \varphi^k(y)$. In particular, one sees that any two elements of L have a lower bound, so L is a lattice. Hence L is a Garside lattice.

Given a Garside lattice (L, φ), one may consider the category $\mathcal{C}'$ whose objects are elements of L, and there is a unique morphism from x to y if and only if $x \leqslant y$. The automorphism φ of L induces an automorphism ϕ' of $\mathcal{C}'$. Given any morphism $x < y$, one may also define $\phi'(x < y)$ as the unique morphism $\varphi(x) < \varphi(y)$. Given any $x \in L$, one may define $\Delta'(x)$ as the unique morphism $x < \varphi(x)$. So $(\mathcal{C}', \phi', \Delta')$ is a categorical Garside structure, which gives a Garside groupoid $\mathcal{G}'$. Let x' denote the object of $\mathcal{G}'$ consisting of the identity morphism $x \leqslant x$. Let G' denote the weak Garside group that is the isotropy group of $\mathcal{G}'$ at x'.

We will check that the composition of these three constructions, starting from a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$, gives an equivalent categorical Garside structure $(\mathcal{C}', \phi', \Delta')$. Note that the weak Garside group G acts on the objects of $\mathcal{C}'$ by precomposition, so G acts by automorphisms on $\mathcal{C}'$. Since ϕ' and Δ' are defined by postcomposition, we deduce that the action of G is compatible with ϕ' and Δ'. Hence G acts by automorphisms on $(\mathcal{C}', \phi', \Delta')$.

The set of objects of the quotient $\mathcal{C}'/G$ then identifies canonically with the set of objects of $\mathcal{C}$, and the set of morphisms of $\mathcal{C}'/G$ identifies canonically with the set of morphisms of $\mathcal{C}$. The quotient of ϕ' and Δ' by G also identify canonically with ϕ and Δ. Hence the categorical Garside structures $(\mathcal{C}, \phi, \Delta)$ and $(\mathcal{C}', \phi', \Delta')$ are equivalent. □

We also have a similar characterization of (weak) Garside groups.

Theorem 4.7. *A group G is a Garside group (resp. weak Garside group) if and only if there exists a Garside flag complex (X, φ) (or equivalently on a Garside lattice) such that G can be realized as a group of order-preserving automorphisms of X commuting with φ, acting freely and transitively (resp. freely) on vertices of X.*

Moreover, the group G is a (weak) Garside group of finite type if and only if X can be chosen such that the action of G is cocompact.

Proof. If G is a (weak) Garside group, it is clear from Theorem 4.6 that G acts on the associated Garside lattice satisfying these properties.

Conversely, assume that G acts on a Garside lattice X satisfying these properties. Then, if we fix a vertex $x \in X$, we may consider the categorical Garside structure whose objects are elements of X bigger than x as in Theorem 4.6. Its quotient by G is a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$, and the isotropy group at the image of x in the associated groupoid coincides with G. □

Example. As a very simple example, consider the n-strand braid group B_n, with the standard Garside element Δ_n and positive monoid B_n^+. Note that B_n^+ plays the role of the categorical Garside structure (with only one object), and B_n is the associated Garside groupoid. Let L denote the lattice consisting of elements in B_n, where $g \leqslant h$ if and only if $h \in gB_n^+$. The automorphism $\varphi : g \mapsto g\Delta$ of L turns L into a Garside lattice. Now the group B_n acts freely transitively on L by left multiplication, commuting with φ, so we recover that B_n is a Garside group. Moreover, any subgroup of B_n is a weak Garside group, and any finite index subgroup of B_n is a weak Garside group of finite type. This is the case of the pure braid group, in which case the categorical Garside structure coincides with the structure described in the example about simplicial hyperplane arrangements.

5 Applications

5.1 Euclidean buildings

Theorem 5.1. *Let X denote a Euclidean building of type $\widetilde{A}_n$. Then the 1-skeleton of X is a weakly modular graph.*

Proof. Let $m = n+1$. Let us denote the type function $\tau : X^{(0)} \to \mathbb{Z}/m\mathbb{Z}$. Let us consider

$$L = \{(x,k) \in X^{(0)} \times \mathbb{Z} \mid \tau(x) \equiv k[m]\}.$$

Let us consider the order relation on L generated by $(x,k) \leqslant (x',k')$ if x and x' are adjacent in X and $k \leqslant k'$. According to [Hir20], L is a lattice.

Note that this lattice is quite explicit in the Bruhat-Tits case: let X denote the Bruhat-Tits building of $\mathrm{PGL}(n,\mathbb{K})$, where $\mathbb{K}$ is a non-Archimedean local field. The vertex set of X may be described as the set of homothety classes of ultrametric norms on $\mathbb{K}^n$. The extended Bruhat-Tits building $\hat{X}$ of $\mathrm{GL}(n,\mathbb{K})$ may be described as the set of ultrametric norms on $\mathbb{K}^n$. There is a canonical simplicial map $\hat{X} \to X$, such that the preimage of a vertex is isomorphic to $\mathbb{Z}$. The vertex set L of $\hat{X}$, with the natural order on norms on $\mathbb{K}^n$, is in fact a lattice (see also [Hae21a])

Let us consider the action of $\mathbb{Z}$ on L by $p \cdot (x,k) = (x, k+pm)$. This action satisfies the assumptions of Theorem 2.1, so the quotient graph of $L/\mathbb{Z}$, which coincides with $X^{(1)}$, is weakly modular. □

In fact, we can see Euclidean buildings of type $\widetilde{A}_n$ as categorical Garside structures, as follows.

Theorem 5.2. *Let X denote a Euclidean building of type $\widetilde{A}_n$, and let G denote an automorphism group of X. Then $X^{(0)}/G$ is the set of objects of a categorical Garside structure. Moreover, for any subgroup H of G acting freely on X, the group $H \times \mathbb{Z}$ is weakly Garside.*

Proof. According to the proof of Theorem 5.1, the lattice $L \subset X^{(0)} \times \mathbb{Z}/n\mathbb{Z}$ is a Garside lattice. So according to Theorem 4.6, we deduce that $X^{(0)}$ is the set of objects of a categorical Garside structure on which G acts by automorphisms. Hence $X^{(0)}/G$ is the set of objects of a categorical Garside structure.

Moreover, given any subgroup H of G acting freely on X, the group $H \times \mathbb{Z}$ acts freely by automorphisms on L, so according to Theorem 4.7 it is a weak Garside group. □

Example. For instance, the Bruhat-Tits building of $\mathrm{PGL}(n,\mathbb{K})$, for a non-Archimedean valued field $\mathbb{K}$, may be described as the categorical Garside structure associated to the following Garside germ $(\mathcal{C}, S)$. The category $\mathcal{C}$ has one object, and the set of simple morphisms S coincide with the poset of vector subspaces of $\Bbbk^n$, where $\Bbbk$ is the residue field. The partial composition of morphisms coincides with the natural inclusion order on S. If one considers the Bruhat-Tits building of $\mathrm{SL}(n,\mathbb{K})$ instead, one may naturally consider a similar Garside structure, whose objects are $\mathbb{Z}/n\mathbb{Z}$.

Remark. In the view of [Hae21a], one may also consider the space X of all convex symmetric bodies of $\mathbb{R}^n$, which can be understood as the injective hull of the symmetric space $\mathrm{GL}(n,\mathbb{R})/O(n)$, as a "continuous" categorical structure, where the homogeneous assumption should be replaced with a graded with values in $\mathbb{R}$. Hence torsion-free subgroups of $\mathrm{GL}(n,\mathbb{R})$ can therefore be described as "continuous" weak Garside groups.

5.2 Weakly modular thickenings of buildings

Here we prove Theorem C, showing that Conjecture B is true for many buildings.

Let X denote the 1-skeleton of a building. We say that a graph Γ is an equivariant thickening of X if Γ contains X as a subgraph, X is quasi-isometric to Γ, and the automorphism group of X extends as an automorphism group of Γ. Combining our result for Euclidean buildings of type $\widetilde{A}_n$ with other results, we deduce the following.

Theorem 5.3. *The following buildings have equivariant weakly modular thickenings.*

- *Any spherical building.*
- *Any Euclidean building of type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$ or $\widetilde{G}_2$.*
- *Any right-angled building.*
- *Any rank 3 building.*
- *Any Gromov-hyperbolic building.*

Proof.

- The spherical case is trivial, since the full graph on the X is a weakly modular equivariant thickening.
- If X is a Euclidean building of type $\widetilde{A}_n$, Theorem 5.1 states that X is weakly modular, without the need of a thickening. According to [Hae21a], any Euclidean building of type $\widetilde{B}_n$, $\widetilde{C}_n$ or $\widetilde{D}_n$ has an equivariant thickening which is Helly, thus weakly modular.
- Any right-angled building has a 1-skeleton which is a median graph, and is in particular weakly modular.
- According to [PS16], any rank 3 building which is not of type $(2,4,4)$, $(2,4,5)$ or $(2,5,5)$ has an equivariant thickening which is systolic, thus weakly modular. The type $(2,4,4)$ is the affine type $\widetilde{C_2}$, which is already covered. The types $(2,4,5)$ or $(2,5,5)$ are of hyperbolic type, and are covered by the last case.
- Any Gromov-hyperbolic building has a cobounded Helly hull according to [Lan13], and in particular has an equivariant thickening which is weakly modular.

□

5.3 Weak Garside groups

Let $(\mathcal{C},\phi,\Delta)$ be a categorical Garside structure, let x be an object in $\mathcal{C}$, and let L_x denote the set of morphisms from x in the groupoid $\mathcal{G}$ associated with $\mathcal{C}$.

Definition 5.4. Let us consider a weak Garside group G_x associated to an object x in a categorical Garside structure $(\mathcal{C},\phi,\Delta)$ and set of simple morphisms S. The *weak Cayley graph* $\mathrm{Cay}(G_x,S)$ of G_x is the graph with vertex set L_x, the set of morphisms from x, with an edge between f and g if there exists a simple morphism $s\in S$ such that $f=gs$ or $g=fs$. The *Garside automorphism* of $\mathrm{Cay}(G_x,S)$ is the precomposition by Δ_x.

Theorem 5.5. *Let G_x denote a weak Garside group, with set of simple morphisms S. Then the weak Cayley graph* $\mathrm{Cay}(G_x, S)$ *and its quotient* $\mathrm{Cay}(G_x, S)/\langle\Delta_x\rangle$ *are weakly modular graphs.*

Proof. By Proposition 4.4, L_x is a Garside lattice, and one remarks that the edge relation on $\mathrm{Cay}(G_x, S)$ coincides with the one given in Section 3, and the edge relation on the quotient $\mathrm{Cay}(G_x, S)/\langle\Delta_x\rangle$ coincides with the one given in Section 2. □

5.4 Artin complexes

Theorem 5.6. *Let X be the Artin complex of the Artin-Tits group of type $\widetilde{A}_{n-1}$ (cf. Section 1.1). Then the 1-skeleton of X is a weakly modular graph.*

Proof. Let A_i be as defined in Section 1.1. We represent vertices of X by left cosets of form gA_i. Let $P = X^{(0)} \times \mathbb{Z}$. We put a weak order on P as follows. Define $(g_1A_{i_1}, n) \leq (g_2A_{i_2}, m)$ if one of the following holds:

1. $n = m$, $g_1A_{i_1} \cap g_2A_{i_2} \neq \emptyset$ and $i_1 \leq i_2$;
2. $m = n + 1$, $g_1A_{i_1} \cap g_2A_{i_2} \neq \emptyset$ and $i_2 \leq i_1$.

Note that $(P, \leq)$ is a weakly ordered set. If $(x, n) \leq (y, m)$, then x and y are adjacent in X. Conversely, if x and y are adjacent, then either $(x, n) \geq (y, n)$ or $(x, n) \geq (y, n-1)$. As $X^{(1)}$ is connected, we deduce that for any $(x, n) \in P$ and any $y \in X$, there exists $m \in \mathbb{Z}$ such that we can find a weak chain from (y, m) to (x, n). The weakly ordered set P is homogeneous, indeed, for $a \leq b$ in P, we can define the length function $\ell(a \leq b)$ to be the maximal length of chains in P from a to b.

Let $\varphi : P \to P$ be the map sending (x, n) to $(x, n + 1)$. Then φ is an automorphism of weakly ordered set. Then φ generates the weak order $\leq$. Theorem 1.3 (2) holds true as X_φ is homeomorphic to $X \times \mathbb{R}$ and X is simply-connected. Theorem 1.3 (3) holds true by [Hae21c, Theorem 4.3]. Thus Theorem 1.3 implies $\leq$ generates a partial order $\leq_t$ on P. The previous paragraph implies any two elements in $(P, \leq_t)$ have lower bound, thus $(P, \leq_t)$ is a lattice by Theorem 1.3. As $(P, \leq_t)$ is homogeneous, we know any lower bounded subset in $(P, \leq_t)$ has a meet and any upper bounded subset in $(P, \leq_t)$ has a join. Now the automorphism φ gives an action of $\mathbb{Z}$ on $(P, \leq_t)$. The previous paragraph also implies for any $a, b \in (P, \leq_t)$, there exists $k \in \mathbb{Z}$ such that $-k \cdot a \leq b \leq k \cdot a$. Note that $X^{(1)}$ coincides with the quotient graph of $(P, \leq_t)$ by $\mathbb{Z}$ as defined in Section 2. Thus we are done by Theorem 2.1. □

5.5 Arc complexes

The following is nothing more than a topological description of the Artin complex of the Artin-Tits group of type $\widetilde{A}_{n-1}$. Let Σ denote a 2-sphere with $n + 2$ punctures $\{N, S, p_1, \ldots, p_n\}$ with two distinguished punctures N, S which could thought of as the North pole and the South pole of Σ. The punctures $p_1, \ldots, p_n$ may be thought as cyclically ordered on the equator of the 2-sphere.

Let $\mathcal{A}(\Sigma)$ denote the following simplicial complex. Its vertex set consists of isotopy classes of arcs in Σ from N to S. Two vertices are adjacent if they can be realized disjointly. Then $\mathcal{A}(\Sigma)$ is the associated flag simplicial complex, see Figure 1. According to [Wah08, Lemma 2.5], this arc complex $\mathcal{A}(\Sigma)$ is contractible.

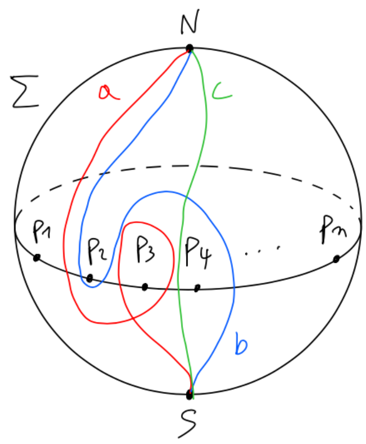

Figure 1: Arcs on the punctured sphere Σ: a is adjacent to b and c

Theorem 5.7. *The arc complex $\mathcal{A}(\Sigma)$ is a weakly modular graph.*

Proof. Let us consider the cyclic cover $\widetilde{\Sigma}$ of Σ over the set of poles $\{N, S\}$. More precisely, consider the group morphism $\psi : \pi_1(\Sigma) \to \mathbb{Z}$ sending a loop around N to $+1$, a loop around S to -1, and a loop around any p_i to 0. The cyclic covering $\widetilde{\Sigma}$ of Σ associated to $\operatorname{Ker} \psi$ is homeomorphic to a 2-sphere, with 2 distinguished punctures denoted N, S, and infinitely many punctures forming n ϕ-orbits $\{\phi^k(\hat{p}_i), 1 \leqslant i \leqslant n, k \in \mathbb{Z}\}$, where ϕ is a generator the of deck transformation group of $\widetilde{\Sigma}$ and $\hat{p}_i$ is a lift of p_i in $\widetilde{\Sigma}$, see Figure 2.

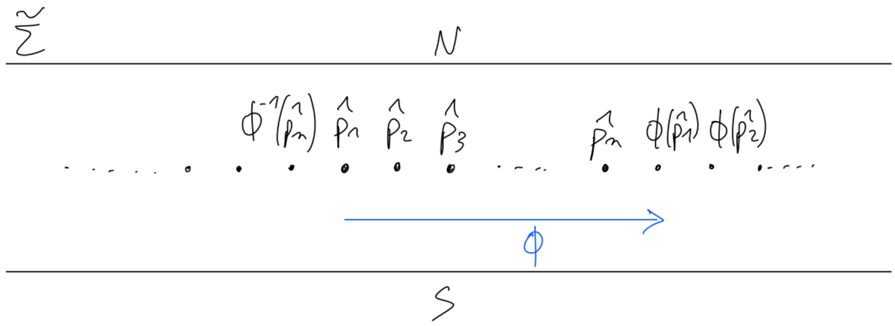

Figure 2: The cyclic cover $\widetilde{\Sigma}$.

Let us consider the set P of isotopy classes of arcs a from N to S in $\widetilde{\Sigma}$, which are lifts of arcs in $\mathcal{A}(\Sigma)$. There is an induced induced action of ϕ on P.

We will put a weak order on P as follows. Say that $a \leqslant a'$ if $a' = a$, $a' = \phi(a)$, or a' is disjoint from $\phi(a)$, and a' separates a and $\phi(a)$. Since the arc complex $\mathcal{A}(\Sigma)$ is connected, we deduce that $\phi : P \to P$ generates $\leqslant$.

We will show that we can apply Theorem 1.3.

1. By definition of the weak order, for any $a \in P$, we have $\phi(a) > a$.
2. Since the arc complex $\mathcal{A}(\Sigma)$ is simply connected, we deduce that X_ϕ is simply connected.
3. For any $a \in P$, the interval $[a, \phi(a)]$ is isomorphic to the lattice of cut-curves, see [Bes03] and [Hae21c] for the proof of the lattice property due to Crisp and McCammond (unpublished). The lattice property is quite geometric in this case: roughly speaking, the meet of two arcs b, c in the interval $[a, \phi(a)]$ may be defined as the "westmost" part of $b \cup c$, see Figure 3.

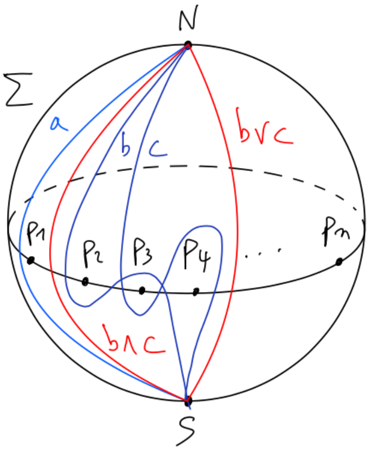

Figure 3: The lattice property: the meet $b \wedge c$ and the join $b \vee c$ of the two arcs b, c in the interval $[a, \phi(a)]$.

According to Theorem 1.3, we deduce that $(P, \leqslant_t)$ is a lattice. Moreover, given any $a, a' \in P$, there exists $k \in \mathbb{N}$ such that $a \leqslant_t \phi^k(a')$. So we can apply Theorem 2.1, and deduce that the quotient graph, which coincides with $\mathcal{A}(\Sigma)$, is a weakly modular graph. □

It turns out that this arc complex coincides with the above Artin complex.

Proposition 5.8. *The arc complex $\mathcal{A}(\Sigma)$ is isomorphic to the Artin complex of the affine Artin-Tits group of type $A(\widetilde{A}_{n-1})$.*

Proof. Note that the mapping class group $G = \mathrm{Mod}(\Sigma, \{N, S\})$ fixing the set of North and South poles act by simplicial automorphisms on $\mathcal{A}(\Sigma)$. According to [CC05], G is isomorphic to the semidirect product $A(\widetilde{A}_{n-1}) \rtimes \mathbb{Z}/n\mathbb{Z}$, where $A(\widetilde{A}_{n-1})$ is the affine Artin group of type $\widetilde{A}_{n-1}$ and $\mathbb{Z}/n\mathbb{Z}$ acts by rotations on the defining graph of $A(\widetilde{A}_{n-1})$.

Note that the subgroup $A = A(\widetilde{A}_{n-1})$ of G acts on $\mathcal{A}(\Sigma)$, with strict fundamental domain the n-simplex consisting of the n meridians $a_1, \dots, a_n$ "separating" the points $p_1, \dots, p_n$ in this cyclic order. More precisely, a_i crosses the equatorial arc between p_i and p_{i+1}, see Figure 4.

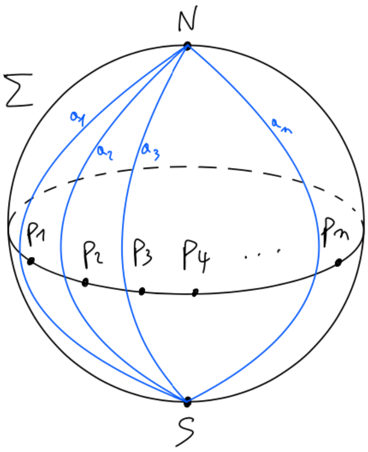

Figure 4: The fundamental simplex $a_1, a_2, \dots, a_n$ of the arc complex.

Note that the stabilizer of each a_i is naturally isomorphic to the n-strand braid group, generated by $n-1$ consecutive standard generators of $A(\widetilde{A}_{n-1})$. Similarly, the stabilizer of $(a_{i_1}, \ldots, a_{i_k})$ is isomorphic to the corresponding product of braid groups. We deduce that both $\mathcal{A}(\Sigma)$ and the Artin complex of affine type $A(\widetilde{A}_{n-1})$ are universal covers of the simplex of groups defining the Artin complex of affine type $A(\widetilde{A}_{n-1})$. $\square$

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