NORMA   RESEARCH REPORT

Higher order methods for compressible CFD. A review.

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1 Introduction

The Norma project [48] is a Russia-France cooperation for improving high-fidelity numerical models in order to better control the noise produced by new and less new aeronefs like drones and helicopters which should move around towns with the smallest sound pollution.

Among Norma's goals is the improvement of numerical approximations based on unstructured meshes, for solving the Navier-Stokes equations (and linearisations). Dissipation and dispersion are the abomination of desolation of second order approximations. The Russian and French teams use extensions of second-order schemes called superconvergent approximation which indeed reduce dissipation and dispersion. Research will tend to further reduce them and examine how the research in high order schemes can help in further improvement of the approximations.

The present paper proposes a short overview of recent progress in these schemes.

Many high-order schemes we are interested with rely on local higher order interpolations, built from data in finite-element-like elements, e.g. tetrahedra.

Having a degree k polynomial approximation in each element will produce a $k+1$-th order accuracy:

$$e \approx h^{k+1}$$

but the cost is related to the number of degrees of freedom:

$$NDOF \approx \frac{1}{h^d}$$

An emblematic approximation, Discontinuous Galerkin, considers these polynomial interpolation independantly in each element:

$$NDOF \approx \frac{m_{k,d}}{h^d} \quad ; \quad m_{k,d} = \frac{1}{d!}\prod_{i=1}^{d}(k+1)$$

where $m_{k,d}$ unknowns are needed to reconstruct a k-polynomial. These features are in common in many recent studies, the way the time derivative (in practise the spatial divergence) is computed from these data makes the main difference between the methods.

2 Analysis of the references

This review is based on a list of papers which we examine now by organizing it in a few sections.

2.1 Reviews and workshops

The paper [80] proposes a review of basic HO methods. In [79] the authors want to "facilitate the adoption" of HO methods, proposing a kind of manual for it. The Hybrid High-Order (HHO) methods reviewed in [7] are formulated in terms of discrete unknowns attached to mesh faces and cells. Hybrid High-Order methods are also reviewed in [24]. The paper [84] reviews stability theories and error estimates. The paper [81] reports on the 1st International Workshop on High-Order CFD Methods, producing also a discussion on why HO are not yet prefered. Test cases involve channel, Ringleb, NACA, flat plate flows, vortex transport, high lift in 2D, and various single wings in 3D. A further publication, [82], contains other comparisons on tests cases.

Let us mention papers presenting computer codes, [50] [69] [83].

2.2 Methodological contributions

Two particular contributions are: [30] proposing TVD-RK schemes for advancing the spatial HO methods.[42] proposes continuous Lagrange Galerkin methods.

2.2.1 Superconvergent approximations

In approximation theory, the term of superconvergence is used for identifying a better convergence when the sequence of mesh satisfy extra regularity property. Designing specially superconvergent approximations can also lead to notably reduce dissipation and dispersion of second-order approximations while not increasing to much their computational cost. A family of superconvergent approximations where designed in [23] [47] [2] where discrete gradients on the vertices of the two upwind and downwind elements at both sides of an edge are used for obtaining the input of the Remann solver. New propositions are presented in [1] [9] [11] [10] where the edge is prolongated both side and its intersection with mesh used for a 1D higher-order flow reconstruction.

2.2.2 A particular third order

A series of contributions aims at proposing improvements to a family of MUSCL-type vertex-centered approximations. In [43], a corrected node-centered scheme is shown to maintain third-order accuracy for the inviscid terms on arbitrary triangular meshes. As main improvement, all gradients are computed using a quadratic least squares method instead of a linear method. Further extensions (including Navier-Stokes) and similar propositions can be found in [25] [44] [61] [55] [64] [67] [66] [65] [63] [45] [70] [62] [29].

2.2.3 Continuous Galerkin

In SUPG/LSG/VMS methods the continuous Galerkin approximation is stabilized by adding to the discrete equations terms which are inspired by the initial residual, but estimated in the element, and are built in order to vanish sufficiently rapidly with mesh refinement for not modifying the asymptotic accuracy. See [35], and [77] for the extension to higher-order.

2.2.4 Discontinuous Galerkin

The discontinuous Galerkin method is a family of variational methods applying on finite-element meshes in which the unknown and test functions are polynomial functions inside each element. They are a priori discontinuous through inter-element faces. The integration of first-order hyperbolics is completed by upwind (Riemann solvers) integration at inter-element faces. The integration of second-order elliptic terms is addressed by a set of different approaches, as explained in [8]. Further propositions for elliptic case are given in [19],[21]. The extension to compressible Navier-Stokes is addressed in [13]. A pedagogic presentation can be found in [33].

The initial DG is rather computationally expensive due to the many degrees of freedom. Improvements where introduced through hybrydizable[1]

[1]The mixed finite element method or hybrid finite element method, is a finite element method in which extra independent variables are introduced as variables. The relation

formulations (HDG), [59] [20]. In particular some HDG method have a number of degrees of freedom located on the inter-element faces and not much more important than for continous Galerkin, together with being advanceble with an explicit time stepping, see [60]. See also [46].

A posteriori estimates (for diffusion-convection) are presented in [18], while in [74] is presented an explicit HDG method for numerically solving the acoustic wave equation. Explicit and implicit options for this model are compared in [49].

DG methods have been use in several approaches for HO mesh adaptation, either through purely discrete mesh optimization as in [22] [85], or relying on continuous metrics as in [26] [71] [72].

An extended variant of DG, the $P_N P_M$ scheme is proposed in [27].The first index N indi- cates the polynomial degree of the test functions and the second M is the degree of the polynomials used for flux and source computation. The general $P_N P_M$ schemes contain classical high order accurate finite volume schemes ($N = 0$) as well as standard discontinuous Galerkin methods ($M = N$).

2.2.5 Residual distribution schemes

The RD relies on cells -generally dual of the solution nodes- on which the residual is computed. In a second step, each cell residual is distributed to the solution nodal values. RD is extended to high order in [4], [5].

2.2.6 ENO/WENO/CENO

Initally designed [32] for structured meshes, the Essentially-Non-Oscillatory scheme was soon extended to unstructured meshes, [3],[34]. The most popular extension is Weighted ENO (WENO) introduced in [68]. An extension to unstructured triangular meshes is discussed in [28].

A family of formulation the central ENO (CENO) is close to the Barth-Frederickson reconstruction scheme [12], and uses a unique molecule, [39] [57]

of these extra independent variables with the primary ones are generally constrained by using Lagrange multipliers.

[56] [15] [38] [73] [76] [41] [17] [31] [40] [6] [16] [58] [51].

A low dissipation version and a mesh-adaptive application is studied in [14].

2.2.7 Spectral Volumes

In spectral volumes, as in DG, the unknown is a polynomial inside each element (tetrahadron). Fluxes between elements are also integrated with a Riemann solver. Instead of Lagrange interpolation from nodal values, each element is split in subcell/finite volumes, and the polynomial is constrained to a given mean, the degree of freedom, on each subcell. Fluxes between subcell of an element are integrated with the continuous polynomial interpolation, [54] [52] [75].

2.2.8 Spectral Differences

As in DG and SV, the unknown is a polynomial inside each element (tetrahadron). Fluxes between elements are also integrated with a Riemann solver. In each element(tetrahedron), two sets of grid points are used, solution points (the DOFs) with Lagrange reconstruction (degree k), and flux points. Flux are the reconstructed (degree $k+1$). Solution values are updated in differential form, *i.e.* with the divergence of flux at solution points. [53] [86]

2.2.9 Flux reconstruction

The flux reconstruction idea dates back probably to [36]. The approach amounts to evaluating the derivative of a discontinuous piecewise polynomial function by employing its straightforward derivative estimate together with a correction, which accounts for the jumps at the interfaces. The resulting degree k + 1 approximate total transformed flux is continuous at interfaces and is used to update with a differential form the unknown at solution points. A particular extension to 2D and triangles is the CPR method (Correction Procedure via Reconstruction), [78] [37].

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