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A metric-based mesh adaptation for hybrid RANS/LES flow calculations

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1 Introduction

The Franco-Russian contribution T3-D4 planned in the Norma program included the cooperative development of sufficiently industrialized methods and tools to process the acoustic calculation of a multi-rotor UAV. Due to the cessation of the collaboration with the Russian team, this aspect of the Norma program was abandoned. Research and development on the French side has been redirected towards a more general task, related to mesh adaptation for LES flows. This report is the first version of an article in preparation on this theme.

Large Eddy Simulations and hybrid flow calculations are today still very computer consuming CFD activities. More importantly, the user has difficuties in deciding if the mesh used allows the expected accuracy. Upgrading mesh adaptation methods to a better treatment of LES/hybrid flows is then an important issue in the Norma research.

Large Eddy Simulation (LES) is a numerical tool for predicting turbulent flows. Unlike a statistical Navier-Stokes model which tends to neglect fluctuations by applying a statistical average, which in most cases damps all these fluctuations, LES can be interpreted as damping only a part of unsteady turbulent structures, typically structures with a scale smaller than a prescribed filter size. The smallest structures being the most difficult to solve, LES consumes much less computational resources than a Direct Numerical Simulation which computes all turbulence structures. LES modelling relies on two steps :

- (i)- defining the neglected scales as those which are smaller than a filter width Δ_f and adding a model of the action of neglected scales on the non-neglected ones. This defines a continuous model parameterized by the filter width Δ_f ,
- (ii)- using, in order to approximate the continuous model built in (i), a mesh-based approximation with local mesh size Δ_q , typically :

$$\Delta_g = (\xi \eta \zeta)^{\frac{1}{3}} \tag{1}$$

(expressed directly in terms of the local mesh sizes (ξ, η, ζ) measured in three orthogonal directions).

While, by construction, Δ_f should be larger than Δ_g , in order to approximate accurately the non-neglected scales, the research of the lowest computational cost motivates the practitioner to set

$$\Delta_f = \Delta_g,$$

with the consequence that the smallest unfiltered scales are the smallest scales

computed on the grid and are then very poorly approximated, whatever be the accuracy of the numerical scheme.

When using a second-order accurate approximation, an important disadvantage comes from the fact that many LES models of Smagorinsky type are similar to second-order accurate truncation terms, in such a way that approximation errors are of same order as the filter model. According to an analysis of Ghosal [15] and to the outputs of many numerical computations, see e.g. [18], using a second-order accurate approximation may result in errors larger than the effect of LES modelling. It remains that second-order accurate approximations are much used and very useful for computing LES flows in engineering. A good practise for increasing the confidence in second-order accurate LES computations is to compare (a) the LES-based computations with (b) their no-model counterpart in order two, see e.g. [20].

Let assume that an approximation with a truncation error of order α^{1} is used. An important issue is the fact that the convergence at truncation order is subject to the condition of using a sufficiently refined mesh : the mesh should be in any point sufficiently fine for capturing the smallest local detail of the flow computed, in order to start second-order or higher order convergence.

- Already in steady CFD, the convergence at truncation order is difficult to attain. A very efficient tool for obtaining this convergence is the convergent mesh adaptive double loop as described in [1, 11]. The inner loop is an anisotropic metric-based fixed-point *adaptation* working with a fixed number of unknowns. The outer loop is an anisotropic metric-based *enrichment* increasing progressively the total number of unknowns and controlling the actual convergence to the continuous solution. Thanks to this double loop, steady second-order RANS calculations are reaching a higher level of accuracy and fiability.

- As concerns unsteady RANS, mesh convergence with a double mesh adaptive loop is more difficult to apply, but effective in many cases. See for examples [4], [23], [22]. - As concerns mesh convergence with LES, it is a much more difficult issue. The scenario consisting of a brute-force strategy which increases simply the number of nodes generally may not succeed for the following reason. The filter term can be considered as a second order error. However, refining the mesh diminishes the subgrid-scale (SGS) term and introduces the arising of smaller and smaller new unstable scales in the solution which therefore cannot be accurately approximated until the process simply solves the corresponding DNS flow. Therefore, in contrast to laminar and RANS modeling, mesh adaptation for LES and hybrid models cannot have as goal the faster/fastest convergence to a continuous field, except the exact solution of Navier-Stokes.

^{1.} In the sense of the usual order of convergence which can be observed when no singularity occurs, typically second-order convergence when second-order codes are used with extremely fine meshes.

The designing of a mesh adaptation criterion for LES is an important and difficult issue, addressed by many publications, among which we have selected the following typical ones.

In [5], the approach is a numerical one, related to truncature. The error estimator identifies the regions lacking in accuracy, improving their resolution by either decreasing the size of the element or increasing the polynomial degree which approximates locally the solution. A smoothness indicator guides the hp-decision, leading to p-enrichment for smooth regions and h-refinement for non-smooth regions.

The work in [17] compares three indicators. The first indicator is based on the unsteady residual. The second indicator is based on a local smoothness indicator. The third indicator is based on an estimate for small scale turbulent kinetic energy. Comparisons with DNS tend to show that the first indicator is the best.

Similarly, in [16] several indicators more or less related to discretization and modeling error are compared with a wall jet as main test case.

In [13], a field-inversion machine-learning (FIML) framework is introduced. It only requires unsteady primal solutions. Two error estimates are compared in this work, a time-averaged unsteady residual weighted by a time-averaged adjoint, and an augmented-system residual weighted by the augmented-system adjoint.

In [21] the approach relies on a Discontinuous Galerkin (DG) high-order approximation. It does not really propose a mesh adaptation, but defines the ideal DG-LES solution as the result of the application of two successive filtering operations. A first convolution filter is applied to the DNS data which filters out frequencies beyond the LES grid cut-off. Next, a L^2 -projection of this filtered field is performed on the hp-discretization space.

The physical approach is better addressed in [8]: arguments are based on the ratio of subgrid to viscous dissipation or viscosity. They are meaningful only in the buffer layer of wall-bounded turbulence while LES should be applicable to free shear flows at any Reynolds number.

A mesh adaptive strategy needs a satisfactory measure of the actual modelling error induced by LES. We discuss now an interesting analysis of this error. We have observed that in most models the local filter size introduced in practical LES models is generally taken identical to the local mesh size. Therefore informations concerning the improvement of the filter size can be useful for the improvement of mesh size. This is why we discuss now the work of Germano and co-workers [14] which have proposed a method for improving the filter size. We use the notations of [28] where this method is also explained. If a filter $W \mapsto \overline{W}$ of size $\overline{\Delta}$ is applied to the continuous (incompressible) Navier-Stokes equations for the exact field $(\mathcal{U}, \mathcal{P})$:

$$\mathcal{N}(\mathcal{U}) = \frac{\partial \mathcal{U}_i}{\partial t} + \frac{\partial \mathcal{U}_i \mathcal{U}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial x_j} - \nu \frac{\partial^2 \mathcal{U}_i}{\partial x_j \partial x_j} = 0,$$
(2)

then the coarse-grained representation $(\overline{\mathcal{U}}, \overline{\mathcal{P}})$ of the exact field satisfies :

$$\mathcal{N}_{\bar{\Delta}}^{\text{exact}}(\overline{\mathcal{U}}) = \frac{\partial \overline{\mathcal{U}}_i}{\partial t} + \frac{\partial \overline{\mathcal{U}}_i \overline{\mathcal{U}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \overline{\mathcal{P}}}{\partial x_j} - \nu \frac{\partial^2 \overline{\mathcal{U}}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\bar{\Delta}}^{\text{exact}}}{\partial x_j} = 0,$$
(3)
with $\tau_{ij,\bar{\Delta}}^{\text{exact}} = \overline{\mathcal{U}_i \mathcal{U}_j} - \overline{\mathcal{U}}_i \overline{\mathcal{U}}_j.$

The basic idea of Germano's analysis is to introduce a second filter, not used directly in the model, the *filter test* with a size $\tilde{\Delta} > \bar{\Delta}$ slightly larger than the LES filter. If we apply successively filter $\bar{\cdot}$ and test filter $\tilde{\cdot}$ to \mathcal{U} , we have

$$\mathcal{N}_{\widetilde{\Delta}}^{\text{exact}}(\widetilde{\overline{\mathcal{U}}}) = \frac{\partial \widetilde{\overline{\mathcal{U}}}_i}{\partial t} + \frac{\partial \widetilde{\overline{\mathcal{U}}}_i \widetilde{\overline{\mathcal{U}}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \widetilde{\overline{\mathcal{P}}}}{\partial x_j} - \nu \frac{\partial^2 \widetilde{\overline{\mathcal{U}}}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\widetilde{\Delta}}^{\text{exact}}}{\partial x_j} = 0$$

while applying $\tilde{\cdot}$ to $\mathcal{N}_{\overline{\Delta}}^{\text{exact}}(\overline{\mathcal{U}})$ gives :

$$\widetilde{\mathcal{N}_{\Delta}^{\text{exact}}(\overline{\mathcal{U}})} = \frac{\partial \widetilde{\overline{\mathcal{U}}}_i}{\partial t} + \frac{\partial \widetilde{\overline{\mathcal{U}}}_i \widetilde{\overline{\mathcal{U}}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \widetilde{\overline{\mathcal{P}}}}{\partial x_j} - \nu \frac{\partial^2 \widetilde{\overline{\mathcal{U}}}_i}{\partial x_j \partial x_j} + \frac{\partial \widetilde{\tau_{ij,\overline{\Delta}}^{\text{exact}}}}{\partial x_j} = 0,$$

then

$$\mathcal{N}_{\widetilde{\bar{\Delta}}}^{\mathrm{exact}}(\widetilde{\overline{\mathcal{U}}}) - \mathcal{N}_{\widetilde{\bar{\Delta}}}^{\mathrm{exact}}(\overline{\mathcal{U}}) = \frac{\partial}{\partial x_j} \left(\widetilde{-\tau_{ij,\overline{\Delta}}^{\mathrm{exact}}} + \tau_{ij,\overline{\bar{\Delta}}}^{\mathrm{exact}} - \widetilde{\overline{\mathcal{U}}_i \overline{\mathcal{U}}_j} + \widetilde{\overline{\mathcal{U}}}_i \widetilde{\overline{\mathcal{U}}}_j \right).$$
(4)

In [14] the Germano identity is shown :

$$\widetilde{-\tau_{ij,\bar{\Delta}}^{\text{exact}}} + \tau_{ij,\bar{\tilde{\Delta}}}^{\text{exact}} - \widetilde{\overline{\mathcal{U}}_i \overline{\mathcal{U}}_j} + \widetilde{\overline{\mathcal{U}}}_i \overline{\widetilde{\mathcal{U}}}_j = 0$$

and the RHS of (4) is zero. The Germano identity is true for continuous solutions of the exact Navier-Stokes system but will not apply if the exact Navier-Stokes system is replaced by a continuous formulation of a LES model with one filter with \bar{u} as solution :

$$\mathcal{N}_{\bar{\Delta}}^{\text{model}}(\overline{u}) = \frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i \overline{u}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_j} - \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\bar{\Delta}}^{\text{model}}(\overline{u})}{\partial x_j} = 0.$$
(5)

To fix the ideas, for the Smagorinsky model (incompressible case), it writes :

$$\tau_{ij,\bar{\Delta}}^{\text{model}}(\overline{u}) = -(C_s\bar{\Delta})^2 |S| P_{ij},$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right),$$

$$P_{ij} = 2S_{ij} - \frac{2}{3} S_{kk} \delta_{ij},$$
(6)

 C_s being the Smagorinsky constant. A second formulation is the formulation with both filters with \tilde{v} as solution :

$$\mathcal{N}_{\tilde{\Delta}}^{\text{model}}(\tilde{\overline{v}}) = \frac{\partial \tilde{\overline{v}}_i}{\partial t} + \frac{\partial \tilde{\overline{v}}_i \tilde{\overline{v}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \tilde{\overline{p}}}{\partial x_j} - \nu \frac{\partial^2 \tilde{\overline{v}}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\tilde{\Delta}}^{\text{model}}(\tilde{\overline{v}})}{\partial x_j} = 0.$$
(7)

which implies that :

$$\widetilde{\mathcal{N}_{\overline{\Delta}}^{\text{model}}(\overline{u})} - \widetilde{\mathcal{N}_{\overline{\Delta}}^{\text{model}}(\overline{u})} = \frac{\partial}{\partial x_j} \left(\widetilde{\tau_{ij,\overline{\Delta}}^{\text{model}}(\overline{u})} - \tau_{ij,\overline{\Delta}}^{\text{model}}(\overline{\overline{u}}) + \widetilde{u}_i \overline{u}_j - \widetilde{\overline{u}}_i \overline{\overline{u}}_j \right)$$
(8)

which in general is not zero.

The interest of 8 is that it does not give an equation for the *model error* $u - \overline{u}$, but an equation for a quantity $\overline{u} - \tilde{\overline{u}}$ which is quite close to the model error.

Although discarding the divergence $\frac{\partial}{\partial x_j}$, the Germano identity error

$$\mathcal{G}_{ij} = \widetilde{\tau_{ij,\bar{\Delta}}^{\text{model}}(\overline{u})} - \widetilde{\tau_{ij,\bar{\tilde{\Delta}}}^{\text{model}}(\widetilde{u})} + \widetilde{\overline{u}_i \overline{u}_j} - \widetilde{\overline{u}}_i \widetilde{\overline{u}}_j$$

is a (tensorial) measure of the error between the (non-discretized) Navier Stokes flow field and the (non-discretized) LES flow field. Therefore, the Dynamic Germano-Piomelli procedure, by minimizing in some sense the Germano identity error, allows to find either the *optimal coefficient* C_s , or the *optimal product* $C_s\bar{\Delta}$ (both used in (6)). In the Dynamic Germano-Piomelli practice, $C_s = 0.1$, \bar{u} is not known and is replaced by its discrete analog \bar{u}_h computed on a given mesh, the filter size $\bar{\Delta}$ is generally chosen as the local mesh size, the filter test is chosen as $2\bar{\Delta}$, and the method produces a new product $(C_s\bar{\Delta})_{opt}$, which is usually interpreted as giving a new value $(C_s)_{opt} = (C_s\bar{\Delta})_{opt}/\bar{\Delta}$.

Assuming that we do not want to change C_s , the same computation proposes a new value for $\overline{\Delta}$:

$$(\bar{\Delta})_{\rm opt} = (C_s \bar{\Delta})_{\rm opt} / C_s$$

(

which seems to give informations concerning the filter, and then concerning the mesh to use.

The dynamic Germano analysis has inspired Toosi and Larsson [27] in proposing a method for adapting the mesh to a LES formulation. Toosi and Larsson identify the source of LES-modelling error as the residual of the governing LES equation applied to the coarse-grained exact Navier-Stokes solution. For filter level $\overline{\Delta}$:

$$\mathcal{R}_{i,\bar{\Delta}} \equiv \mathcal{N}_{\bar{\Delta}}^{\text{model}}(\overline{\mathcal{U}}) = \frac{\partial \overline{\mathcal{U}}_i}{\partial t} + \frac{\partial \overline{\mathcal{U}}_i \overline{\mathcal{U}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \overline{\mathcal{P}}}{\partial x_j} - \nu \frac{\partial^2 \overline{\mathcal{U}}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\bar{\Delta}}^{\text{model}}(\overline{\mathcal{U}})}{\partial x_j}$$

and, using (3), we get

$$\mathcal{R}_{i,\bar{\Delta}} = \frac{\partial}{\partial x_j} \Big[\tau_{ij,\bar{\Delta}}^{\text{model}}(\overline{\mathcal{U}}) - \tau_{ij,\bar{\Delta}}^{\text{exact}} \Big].$$

Similarly, for filter test $\overline{\Delta}$:

$$\mathcal{R}_{i,\widetilde{\Delta}} \equiv \mathcal{N}_{\widetilde{\Delta}}^{\text{model}}(\widetilde{\overline{\mathcal{U}}}) = \frac{\partial}{\partial x_j} \Big[\tau_{ij,\widetilde{\Delta}}^{\text{model}}(\widetilde{\overline{\mathcal{U}}}) - \tau_{ij,\widetilde{\Delta}}^{\text{exact}} \Big].$$
(9)

The residual $\mathcal{R}_{i,\widetilde{\Delta}}$ is chosen as the *error source of the LES modelling* but is expressed in terms of the unknown exact solution \mathcal{U} . In order to use it in practice, we need to replace \mathcal{U} by an approximate evaluation of it. The *fundamental approximation* done by Toosi and Larsson is replacing in $\mathcal{N}_{\widetilde{\Delta}}^{\text{model}}$ the coarse-grained exact Navier-Stokes solution $\widetilde{\mathcal{U}}$ by the LES one $\widetilde{\overline{u}}$:

$$\mathcal{R}_{i,\widetilde{\Delta}} \equiv \mathcal{N}_{\widetilde{\Delta}}^{\text{model}}(\widetilde{\overline{\mathcal{U}}}) \approx \mathcal{N}_{\widetilde{\overline{\Delta}}}^{\text{model}}(\widetilde{\overline{u}})$$
(10)

thus :

$$\mathcal{R}_{i,\widetilde{\Delta}} \approx \frac{\partial \widetilde{u}_i}{\partial t} + \frac{\partial \widetilde{u}_i \widetilde{u}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \widetilde{p}}{\partial x_j} - \nu \frac{\partial^2 \widetilde{u}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij,\widetilde{\Delta}}^{\text{model}}(\widetilde{\overline{u}})}{\partial x_j}$$

which is transformed applying the test-filtering to (5):

$$\frac{\partial \widetilde{\overline{u}}_i}{\partial t} + \frac{\partial \widetilde{\overline{u}_i \overline{u}_j}}{\partial x_j} + \frac{1}{\rho} \frac{\partial \widetilde{\overline{p}}}{\partial x_j} - \nu \frac{\partial^2 \widetilde{\overline{u}_i}}{\partial x_j \partial x_j} + \frac{\partial \widetilde{\tau}_{ij,\overline{\Delta}}^{\text{model}}(\overline{u})}{\partial x_j} = 0,$$

and then substracting the result from (10). This gives :

$$\mathcal{R}_{i,\widetilde{\Delta}} \approx \mathcal{F}_{i,\widetilde{\Delta}} = \frac{\partial}{\partial x} \Big[\tau_{ij,\widetilde{\Delta}}^{\text{model}}(\widetilde{\overline{u}}) - \widetilde{\tau_{ij,\widetilde{\Delta}}^{\text{model}}}(\overline{\overline{u}}) - \widetilde{\overline{u}_i \overline{u}_j} + \widetilde{\overline{u}_i} \widetilde{\overline{u}}_j \Big].$$
(11)

In order to minimize the RHS of (11), the authors consider a directional test filter $\tilde{\mathbf{n}}_x^{\mathbf{n}_x}$ of size $\bar{\Delta}_{\mathbf{n}_x}$, in direction \mathbf{n}_x :

$$\widetilde{\phi}^{(\mathbf{n}_x)} \approx \left(I + \frac{\overline{\Delta}_{\mathbf{n}_x}^2}{4} \mathbf{n}_x^T \nabla \nabla^T \mathbf{n}_x \right) \overline{\phi}.$$
(12)

Applying the directional test filter to equation (5) gives the following evolution equation for the filtered instantaneous fields at the filter test level :

$$\frac{\partial \widetilde{\overline{u}}_{i}^{(\mathbf{n}_{x})}}{\partial t} + \frac{\partial \widetilde{\overline{u}_{i}}\widetilde{\overline{u}}_{j}^{(\mathbf{n}_{x})}}{\partial x_{j}} + \frac{1}{\rho} \frac{\partial \widetilde{\overline{p}}^{(\mathbf{n}_{x})}}{\partial x_{j}} - \nu \frac{\partial^{2} \widetilde{\overline{u}}_{i}^{(\mathbf{n}_{x})}}{\partial x_{j} \partial x_{j}} + \frac{\partial \tau_{ij}^{\widetilde{\mathrm{model}}}(\overline{u})}{\partial x_{j}} = 0.$$
(13)

Following the above calculation in this directional context, the following source term analog to (11) is obtained :

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{x})}(\mathbf{x}) = \frac{\partial}{\partial x_{j}} \left(\tau_{ij}^{\text{model}}(\widetilde{\overline{u}}^{(\mathbf{n}_{x})}) - \tau_{ij}^{\widetilde{\text{model}}}(\overline{\overline{u}})^{(\mathbf{n}_{x})} - \widetilde{\overline{u}_{i}}\widetilde{\overline{u}_{j}}^{(\mathbf{n}_{x})} + \widetilde{\overline{u}_{i}}^{(\mathbf{n}_{x})}\widetilde{\overline{u}_{j}}^{(\mathbf{n}_{x})} \right)$$
(14)

and the Toosi-Larsson method proposes to minimize with respect to $\bar{\Delta}_{\mathbf{n}_x}, \bar{\Delta}_{\mathbf{n}_y}$, and $\bar{\Delta}_{\mathbf{n}_z}$ the error functional

$$e(\bar{\Delta}_{\mathbf{n}_x}, \bar{\Delta}_{\mathbf{n}_y}, \bar{\Delta}_{\mathbf{n}_z}) = \int_{\Omega} \left(\langle \overline{\mathcal{F}}_i^{(\mathbf{n}_x)}, \overline{\mathcal{F}}_i^{(\mathbf{n}_x)} \rangle + \langle \overline{\mathcal{F}}_i^{(\mathbf{n}_y)}, \overline{\mathcal{F}}_i^{(\mathbf{n}_y)} \rangle + \langle \overline{\mathcal{F}}_i^{(\mathbf{n}_z)}, \overline{\mathcal{F}}_i^{(\mathbf{n}_z)} \rangle \right)^{\frac{1}{2}}$$
(15)

in order to improve the mesh size in each direction.

The approach presented in this paper starts from an existing adaptation method for the numerical approximation errors for steady RANS flows, see [1] and the monograph [11]. In order to extend it to LES/hybrid, we try to combine (a) the existing adaptation of mesh for RANS or non-turbulent steady flow with (b) a special adaptation for LES error model built according to the Toosi-Larsson method. Focusing on hybrid modeling, we could hope (i) a rather good quasi-convergent capturing in RANS regions, to be combined with (ii) a sufficiently predictive resolution in LES region.

In this work, we focus on flows which are of a somewhat intermediate difficulty. These turbulent flows are assumed to be quasi-periodic with a rather well identified Strouhal number, and possibly quasi-steady in a large part of the computational domain. For a thin airfoil at small angle of attack for example, RANS calculation will produce a steady flow. VLES and hybrid RANS/LES calculations with medium meshes will produce a flow which is mainly steady, generally modeled with RANS, for which a deterministic adaptation criterion will be applied, but which presents also an unsteady region with vortices, where LES modeling applies, and for which a LES-based adaptation criterion must be applied.

2 Modeling

2.1 Navier-Stokes model

The compressible Navier-Stokes equations for mass, momentum and energy conservation read :

$$\mathcal{NS}(W) = 0 + \text{Initial and boundary conditions}$$
 (16)

where $\mathcal{NS}(W) = 0$ holds for :

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \frac{\partial (\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \operatorname{div} \mathcal{T} = 0, \\ \frac{\partial (\rho E)}{\partial t} + \operatorname{div}((\rho E + p)\mathbf{u}) - \operatorname{div}(\mathcal{T} \cdot \mathbf{u}) + \operatorname{div}(\lambda \nabla T) = 0, \end{cases}$$
(17)

where ρ denotes the density (kg/m^3) , **u** the velocity (m/s), E the total energy per mass $(m^2.s^{-2})$, p the pressure (N/m^2) , given by : $p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho |\mathbf{u}|^2\right)$ with $\gamma = 1.4$, T the temperature (K) such that $\rho C_v T = E - \frac{1}{2}\rho(u^2 + v^2 + w^2)$, $\lambda = \mu C_p/Pr$ (C_v being the specific heat at constant volume, μ the dynamic viscosity, C_p the specific heat at constant pressure, Pr the Prandtl number). \mathcal{T} is the laminar stress tensor :

$$\mathcal{T} = \mu \left[(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2}{3} \mathrm{div} \, \mathbf{u} \, \mathbb{I}_3 \right] \,,$$

where (in 3D) $\mathbf{u} = (u_1, u_2, u_3)$, μ the laminar dynamic viscosity (kg/(m.s)) and λ the laminar thermal conductivity.

2.2 LES model

We have now to recall the LES analysis for the compressible model. For this we recall some notations. The filtered Navier-Stokes equations are considered. The density Favre filter $\hat{f} = \overline{(\rho f)}/(\bar{\rho})$ (where the over-line denotes the gris filter) is applied and its solution is denoted $\bar{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{e}$.

The filtering of compressible Navier-Stokes equations gives the motion of large structures :

$$\begin{cases} \frac{\partial\bar{\rho}}{\partial t} + \frac{\partial(\bar{\rho}\hat{u}_{j})}{\partial x_{j}} = 0, \\ \frac{\partial(\bar{\rho}\hat{u}_{i})}{\partial t} + \frac{\partial(\bar{\rho}\hat{u}_{i}\hat{u}_{j})}{\partial x_{j}} = -\frac{\partial\bar{p}}{\partial x_{i}} + \frac{\partial(\mu\hat{P}_{ij})}{\partial x_{j}} - \frac{\partial M_{ij}^{(1)}}{\partial x_{j}} + \frac{\partial M_{ij}^{(2)}}{\partial x_{j}}, \\ \frac{\partial(\bar{\rho}\hat{E})}{\partial t} + \frac{\partial[(\bar{\rho}\hat{E} + \bar{p})\hat{u}_{j}]}{\partial x_{j}} = \frac{\partial(\hat{u}_{j}\hat{\sigma}_{ij})}{\partial x_{i}} - \frac{\partial\hat{q}_{j}}{\partial x_{j}} + \frac{\partial}{\partial x_{j}} \left(E_{j}^{(1)} + E_{j}^{(2)} + E_{j}^{(3)}\right). \end{cases}$$

in which :

- \hat{q} is the resolved heat vector flux, $\hat{q} = -k \nabla(\hat{T})$ where k is the heat conductivity and \hat{T} is the Favre filtered temperature,
- $E_j^{(1)}, E_j^{(2)}, E_j^{(3)}$ are defined e.g. in [9], $E_j^{(2)}, E_j^{(3)}$ are negligible compared to $E_j^{(1)}$ and the effect of $E_j^{(1)}$ will be neglected in the sequel,
- $M_{ij}^{(1)}$ is defined from the filtering of the convective term in the moment equation :

$$\frac{\overline{\partial(\rho u_i u_j)}}{\partial x_j} = \frac{\partial(\bar{\rho}\hat{u}_i\hat{u}_j)}{\partial x_j} + \frac{\partial M_{ij}^{(1)}}{\partial x_j} \quad \Rightarrow \quad M_{ij}^{(1)} = \overline{\rho u_i u_j} - \bar{\rho}\hat{u}_i\hat{u}_j.$$
(18)

- $M_{ij}^{(2)}$ is defined by :

$$M_{ij}^{(2)} = \overline{\mu P_{ij}} - \mu \widehat{P}_{ij} \quad \text{with} \quad P_{ij} = 2S_{ij} - \frac{2}{3}S_{kk}\delta_{ij} \tag{19}$$

where $S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$

As a consequence the resolved strain tensor is denoted by :

$$\widehat{S_{ij}} = \frac{1}{2} \left(\frac{\partial \widehat{u}_i}{\partial x_j} + \frac{\partial \widehat{u}_j}{\partial x_i} \right)$$

and the viscous term is written

$$\widehat{\sigma} = \mu \widehat{P_{ij}}$$
 with $\widehat{P_{ij}} = 2\widehat{S_{ij}} - \frac{2}{3}\widehat{S_{kk}}\delta_{ij}.$ (20)

 $M_{ij}^{(2)}$ is negligible compared to $M_{ij}^{(1)}$. The isotropic part of $M_{ij}^{(1)}$, namely $\frac{1}{3}M_{kk}^{(1)}\delta_{ij}$, can be neglected, [12], and its deviatoric part is defined by :

$$T_{ij} = M_{ij}^{(1)} - \frac{1}{3}M_{kk}^{(1)}\delta_{ij} = -\mu_{SGS}\widehat{P_{ij}} = -\mu_{SGS}\left(2\widehat{S_{ij}} - \frac{2}{3}\widehat{S_{kk}}\delta_{ij}\right).$$

In our theoretical development, the turbulent viscosity is defined according to Smagorinsky model [24]:

$$\mu_{SGS} = \bar{\rho} (C_s \bar{\Delta})^2 |\hat{S}| \quad \text{with} \quad |\hat{S}| = \sqrt{2 \hat{S}_{ij} \hat{S}_{ij}} \tag{21}$$

and the model stress tensor is analogous to the one introduced in (5) for the incompressible case :

$$\tau_{ij}^{\text{model}} = T_{ij} = -\bar{\rho} (C_s \bar{\Delta})^2 |\hat{S}| \widehat{P_{ij}}.$$
(22)

In order to define a Large Eddy Simulation (LES) model from the above one, we restrict to the usual definition of the grid size Δ_g (cf. (1)). We replace in the viscous term of moment equation the viscosity μ by the incremented viscosity $\mu + \mu_{SGS}$. We restrict to quasi-isothermal flows and then we do not introduce a model in the energy equation. Our LES model writes :

$$\mathcal{NS}(W) = \begin{pmatrix} 0 \\ \operatorname{div}\left(\mu_{SGS}\left[\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right) - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}_{3}\right]\right) \\ 0 \end{pmatrix}.$$
 (23)

2.3 Spalart-Allmaras model and DDES

Various forms of the Spalart-Allmaras (SA) turbulence model exist. The original Spalart-Allmaras one equation turbulence model writes [26]:

$$\frac{\partial \tilde{\nu}}{\partial t} + \mathbf{u} \cdot \nabla \tilde{\nu} = c_{b1} [1 - f_{t2}] \tilde{S} \tilde{\nu} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2} \right] \left(\frac{\tilde{\nu}}{d} \right)^2 + \frac{1}{\sigma} \left[\nabla \cdot \left((\nu + \tilde{\nu}) \nabla \tilde{\nu} \right) + c_{b2} \| \nabla \tilde{\nu} \|^2 \right] + f_{t1} \Delta \mathbf{u}^2.$$

where $\tilde{\nu}$ is the kinematic eddy turbulent viscosity. In this paper we consider the simplified formulation considering $f_{t1} = 0$ and $f_{t2} = 0$:

$$\frac{\partial \rho \tilde{\nu}}{\partial t} + \frac{\partial u_j \rho \tilde{\nu}}{\partial x_j} = \rho c_{b1} \tilde{S} \tilde{\nu} - \rho c_{w1} f_w \left(\frac{\tilde{\nu}}{d}\right)^2 + \frac{\rho}{\sigma} \left[\frac{\partial}{\partial x_j} \left((\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j}\right) + c_{b2} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}\right].$$

The turbulent eddy viscosity is computed from :

$$\mu_t = \rho \tilde{\nu} f_{v1}$$

where

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$
 and $\chi = \frac{\tilde{\nu}}{\nu}$ with $\nu = \frac{\mu}{\rho}$.

Additional definitions are given by the following equations :

$$\tilde{S} = \Omega + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$
 where $\Omega = \|\nabla \times \mathbf{u}\|.$

The magnitude of the vorticity is computed from the vorticity tensor where each component is given by $\omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$ and $\Omega = \sqrt{2 \sum_{i,j=1..3} \omega_{ij} \omega_{ij}}$. Symbol *d* holds for the distance from the field point to the nearest wall and

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

.

Notice that we have the following relations :

$$\chi f_{v1} = \frac{\tilde{\nu} f_{v1}}{\nu} = \frac{\nu_t}{\nu} \implies 1 - \frac{\chi}{1 + \chi f_{v1}} = 1 - \frac{\tilde{\nu}}{\nu + \nu_t}.$$

The constants are

$$\sigma = \frac{2}{3} \qquad c_{b1} = 0.1355 \quad c_{b2} = 0.622 \quad \kappa = 0.41$$
$$c_{w1} = \frac{c_{b1}}{\kappa} + \frac{1 + c_{b2}}{\sigma} \quad c_{w2} = 0.3 \qquad c_{w3} = 2 \qquad c_{v1} = 7.1.$$

Finally, the function f_w is computed as :

$$f_w = g \left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6} \quad \text{with} \quad g = r + c_{w2} \left(r^6 - r\right) \quad \text{and} \quad r = \min\left(\frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}, 10\right).$$

The standard SA one-equation model reads in pseudo-vector notations :

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1} [1 - f_{t2}] \tilde{S} \tilde{\nu} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2} \right] \left(\frac{\tilde{\nu}}{d} \right)^2 + \frac{1}{\sigma} \left[\frac{\partial}{\partial x_j} \left((\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right) + c_{b2} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i} \right]$$

where $f_{t2} = c_{t3} \exp(-c_{t4}\chi^4)$ with $c_{t3} = 1.2$ and $c_{t4} = 0.5$.

The DDES model [25] replaces the distance d by :

$$d = d - f_d \max[0, (d - C_{DES}\Delta)]$$

in which $C_{DES} = 0.65$ and with ($\kappa = 0.41$)

$$f_d = 1 - tanh\left([8r_d]^3\right)$$
 where $r_d = \frac{\nu + \nu_t}{\sqrt{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\kappa^2 d^2}.$

 $f_d = 0$ yields RANS for which we shall use a truncation error analysis while $r_d \ll 1$ yields the LES region for which we shall use the Germano-type error analysis due to Toosi and Larsson. The above DDES system is denoted in a compact way as :

$$\Psi_{\text{DDES}}(W) = 0 \tag{24}$$

where $W = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho E, \rho \tilde{\nu})$ and

$$\left(\Psi_{\text{DDES}}(W),\varphi\right) = \int_{0}^{T} \int_{\Omega} \varphi \left[W_{t} + \mathcal{S}_{\text{DDES}}(W) + \operatorname{div}\left(\mathcal{F}_{\text{DDES}}(W)\right)\right] \mathrm{d}\mathbf{x} \mathrm{d}t \qquad (25)$$

$$+ \int_{0}^{T} \int_{\partial \Omega} \varphi \overline{\mathcal{F}}_{\text{DDES}}(W) \, \mathrm{d}\sigma \mathrm{d}t \tag{26}$$

where $\mathcal{S}_{\text{DDES}}(W)$, $\mathcal{F}_{\text{DDES}}(W)$, $\overline{\mathcal{F}}_{\text{DDES}}(W)$ hold respectively for the source term, the flux and the boundary flux of the DDES model.

3 Mesh-adaptation

In this first study, the focus is on flows which are essentially steady, with a rather small region of the computational domain in which we have a quasi periodic vortex shedding.

In the goal oriented option, we observe that the numerical error $\delta W = W - W_h$ on state variable W is solution of a linearised system

$$A \,\delta W_{\text{space-time}} = S_{\text{space-time}}$$

where :

- the linear operator A is the derivative of the Navier-Stokes (resp. URANS, or DDES) residual with respect to the state,
- the right-hand side $S_{\text{space-time}}$ of the system is the local error resulting from the space-time discretization (expressed in terms of the metric).

For minimizing the numerical error on the functional j = (g, W) with respect to the mesh metric \mathcal{M} , it is sufficient to minimize the product of the adjoint state W^* with the right-hand side,

$$\delta j = (W^*, S_{\text{space-time}})$$

which will be made possible by the transformation of the local error into a quadratic function of the metric :

$$\min_{\mathcal{M}} \delta j = \min_{\mathcal{M}} (W^*, S_{\text{space-time}}(\mathcal{M}))$$

We obtain the minimum \mathcal{M}_1 .

Similarly, assume that we have also a linear equation for the (test-filtered) error resulting from the LES modeling

$$A \,\delta W_{\rm LES} = S_{\rm LES}$$

where :

- the linear operator A is again the derivative of the Navier-Stokes residual with respect to the state,
- the right-hand side S_{LES} of the system, the local error of LES, is the divergence of the Germano expression.

As a result, the optimal mesh metric minimizing the total numerical *and* LES error on the functional writes :

$$\mathcal{M}_{\text{both}} = Arg \min_{\mathcal{M}} (W^*, S_{\text{space-time}}(\mathcal{M}) + S_{\text{LES}}(\mathcal{M})).$$
(27)

In this formulation, $S_{\text{space-time}}(\mathcal{M})$ and $S_{\text{LES}}(\mathcal{M})$ are not necessarily positive and therefore compensations of one error by the other one are possible and can be taken into account in the minimization.

However, up to now, we have not found a satisfying way for expressing explicitly S_{LES} as a function of \mathcal{M} and therefore we cannot apply the strategy producing (27).

In this paper, we shall separate the error analysis into two steps, namely :

- (a) research the RANS-optimal metric and,
- (b) research of a somewhat LES-optimal metric by defining the test filter directions $(\mathbf{n}_k, k = 1, 3 \text{ from the RANS-optimal metric and then follow the Toosi-Larsson for computing the optimal filter widths in these directions.$

3.1 Riemannian metric

In the sequel, any mesh is represented by a Riemannian metric. See [1] and [11] for more details on this approach. The Riemannian metric $(\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ is a symmetric positive matrix 3×3 field defined on the computational domain :

$$\mathcal{M}: \mathbf{x} \in \Omega \mapsto \mathcal{M}(\mathbf{x}) = \mathcal{R}(\mathbf{x}) \Lambda(\mathbf{x}) {}^{t}\mathcal{R}(\mathbf{x}), \qquad (28)$$

where diagonal matrix $\Lambda(\mathbf{x})$ is

$$\begin{pmatrix} \lambda_1(\mathbf{x}) & & \\ & \lambda_2(\mathbf{x}) & \\ & & \lambda_3(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} h_1^{-2}(\mathbf{x}) & & \\ & h_2^{-2}(\mathbf{x}) & \\ & & h_3^{-2}(\mathbf{x}) \end{pmatrix}.$$
(29)

 $\mathcal{R}(\mathbf{x})$ is an orthonormal matrix providing the local orientation of mesh stretching through the eigenvectors $(\mathbf{v}_i(\mathbf{x}))_{i=1,3}$, $(\lambda_i(\mathbf{x}))_{i=1,3}$ are the local eigenvalues. $(h_i(\mathbf{x}))_{i=1,3} = (\lambda_i(\mathbf{x})^{-\frac{1}{2}})_{i=1,3}$ are the local mesh sizes along the *principal directions* $\mathbf{p}_{1,\mathcal{M}}, \mathbf{p}_{2,\mathcal{M}}, \mathbf{p}_{3,\mathcal{M}}$ of \mathcal{M} defined by :

$$\mathbf{p}_{k,\mathcal{M}}(\mathbf{x}) = \mathcal{R}(\mathbf{x}) \mathbf{e}_k {}^t \mathcal{R}(\mathbf{x})$$
(30)

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the three Cartesian unitary vectors in x, y, z directions. The density d of \mathcal{M} is defined from its eigenvalues as

$$d(\mathbf{x}) = \det\left(\mathcal{M}(\mathbf{x})\right)^{\frac{1}{2}} = (\lambda_1(\mathbf{x})\,\lambda_2(\mathbf{x})\,\lambda_3(\mathbf{x}))^{\frac{1}{2}} = (h_1(\mathbf{x})\,h_2(\mathbf{x})\,h_3(\mathbf{x}))^{-1}$$

We decompose \mathcal{M} as follows :

$$\mathcal{M}(\mathbf{x}) = d^{\frac{2}{3}}(\mathbf{x}) \,\mathcal{R}(\mathbf{x}) \begin{pmatrix} r_1^{-\frac{2}{3}}(\mathbf{x}) & & \\ & r_2^{-\frac{2}{3}}(\mathbf{x}) & \\ & & r_3^{-\frac{2}{3}}(\mathbf{x}) \end{pmatrix}^t \mathcal{R}(\mathbf{x})$$

where the r_i 's define the stretching strength and where the density d controls the local level of accuracy of \mathcal{M} . The *complexity* \mathcal{C} of \mathcal{M} is defined by :

$$C(\mathcal{M}) = \int_{\Omega} d(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\Omega} \sqrt{\det(\mathcal{M}(\mathbf{x}))} \, \mathrm{d}\mathbf{x}$$

This real-value parameter quantifies the global level of accuracy of $(\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$.

A discrete tetrahedrization \mathcal{H} is *a unit mesh* for the metric \mathcal{M} if any of its edges **ab** has a length in the metric sufficiently close to unity :

$$\frac{1}{\sqrt{2}} \le \int_0^1 \sqrt{t \mathbf{ab}} \, \mathcal{M}(\mathbf{a} + t \, \mathbf{ab}) \, \mathbf{ab}} \, \mathrm{d}t \le \sqrt{2}$$

Then the complexity can also be interpreted as the continuous counterpart of the *number of vertices of a discrete unit mesh* while d can be interpreted as the continuous counterpart of the *number of vertices per volume unit of a discrete unit mesh*.

Lastly, we call *refinement* the process which replaces a unit mesh of a given metric \mathcal{M} with local mesh size $(h_1(\mathbf{x}) h_2(\mathbf{x}) h_3(\mathbf{x}))$ by a unit mesh of \mathcal{M}/β^2 with local mesh size $(\beta h_1(\mathbf{x}) \beta h_2(\mathbf{x}) \beta h_3(\mathbf{x}))$ and complexity $\mathcal{C}(\mathcal{M}/\beta^2) = \beta^3 \mathcal{C}(\mathcal{M})$ where refinement factor β is smaller than one².

3.2 Adaptation sensor for compressible flow

Feature-based adaptation sensor

In the case of a steady RANS calculation of a compressible flow, an efficient approach is to minimize the L^4 interpolation error on the Mach number [1][11]. The local interpolation error $e_{\mathcal{M}}$ (31) is evaluated in terms of the Hessian H_M of Mach number M and of the metric \mathcal{M} used for generating the mesh :

$$e_{\mathcal{M}}(\mathbf{x}) = (M - \pi_{\mathcal{M}}M)(\mathbf{x}) = \frac{1}{10} \operatorname{trace} \left(\mathcal{M}(\mathbf{x})^{-\frac{1}{2}} |H_M(\mathbf{x})| \,\mathcal{M}(\mathbf{x})^{-\frac{1}{2}} \right), \quad (31)$$

^{2.} For $\beta = 2$ this refinement is equivalent to dividing mesh size by a factor 2 and multiplying the number of vertices in 3D by a factor 8.

in which $\mathbf{x} \in \Omega$, $(\mathbf{v}_i)_{i=1,3}$ are the local eigen-directions of \mathcal{M} , and $(h_i)_{i=1,3}$ are the local sizes of \mathcal{M} along these directions. This local error³ is a spatially second-order error. In order to take into account boundary layers in good conditions, it has been observed (see again [1] [11]) that the error norm to be minimized is an L^4 norm. The mesh adaptation problem is then written :

Find
$$\mathcal{M}_1 = \min_{\mathcal{M}} \int_{\Omega} \left(\operatorname{trace} \left(\mathcal{M}(\mathbf{x})^{-\frac{1}{2}} | H_M(\mathbf{x}) | \mathcal{M}(\mathbf{x})^{-\frac{1}{2}} \right) \right)^{\frac{1}{4}} \mathrm{d}\mathbf{x}$$
 (32)

under the constraint that the complexity, or integral of the metric density is equal to a specified number \mathcal{N} :

$$C(\mathcal{M}) = \int_{\Omega} \det(\mathcal{M}) d\Omega = \mathcal{N}$$

Expressing via (31) the functional (32) in terms of H_M and assuming that this Hessian is sufficiently smooth, the solution of this constrained optimisation problem can be explicitly computed [1][11] :

$$\mathcal{M}_1 = D_1 \,\det(|H_M|)^{\frac{-1}{11}} \,|H_M|, \text{ with } D_1 = N^{\frac{2}{3}} \left(\int_{\Omega} \,\det(|H_M|)^{\frac{4}{11}} \mathrm{d}\mathbf{x} \right)^{-\frac{2}{3}}.$$
 (33)

Goal-oriented adaptation sensor

We consider the Goal Oriented unsteady formulation as introduced in [7] and its extension to RANS as in [6, 11]. We keep the notations of these papers. We want to minimize the error $(g, W - W_h)$ committed in the approximation of the functional (or scalar output) :

$$j = (g, W),$$

where W is the exact solution of the state equation (24,26) and W_h the approximate solution. Let us introduce the adjoint state W^* , solution of the adjoint system :

$$\left(\frac{\partial\Psi}{\partial W}\right)^* W^* = g. \tag{34}$$

We reproduce now in short the error estimate developed in [19], [7], [6] [2]. The functional error estimates writes :

$$|(g, W_h - W)| \approx \mathcal{E}_{\text{space}}^p(\mathcal{M}) \tag{35}$$

^{3.} $e_{\mathcal{M}}$ is an *a priori* error when we consider that M is the exact Mach number field. In practice, it will be an *a posteriori* error since $e_{\mathcal{M}}$ will be computed from a discrete solution through a recovery technique.

with

$$\begin{aligned} \mathcal{E}_{\text{space}}^{p}(\mathcal{M}) &\approx \int_{0}^{T} \int_{\Omega} |W^{*}| \left| \left(W - \pi_{\mathcal{M}} W \right)_{t} + \mathcal{S}_{\text{DDES}}(W) - \pi_{\mathcal{M}} \mathcal{S}(W)_{\text{DDES}} \right) \right| d\mathbf{x} dt \\ &+ \int_{0}^{T} \int_{\Omega} \left| \frac{\partial \mathcal{F}_{\text{DDES}}}{\partial W} \cdot \nabla W^{*} \right| |W - \pi_{\mathcal{M}} W | d\mathbf{x} dt \\ &+ \int_{0}^{T} \int_{\Gamma} |W^{*}| \left| \left(\bar{\mathcal{F}}_{\text{DDES}}(W) - \pi_{\mathcal{M}} \bar{\mathcal{F}}_{\text{DDES}}(W) \right) \cdot \mathbf{n} \right| d\Gamma dt. \end{aligned}$$

Neglecting the boundary term, we get :

$$\mathcal{E}_{\text{space}}(\mathcal{M}) \approx \int_{0}^{T} \int_{\Omega} \operatorname{trace} \left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t) \mathbf{H}(\mathbf{x},t) \mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t, \tag{36}$$

with
$$\mathbf{H}(\mathbf{x},t) = \left| W_t^* + W^* \frac{\partial S}{\partial W} + \frac{\partial F}{\partial W} \cdot \nabla W^* \right| \left| H(W) \right|,$$
 (37)

where H(W) is the Hessian of W.

Lemma 3.1 Unified numerical error criterion. The two numerical error criteria, namely (31, 32, 33) for the feature-based option and (35, 36, 37) for the Goal-Oriented option, examined previously, and the related optimization problems can be unified as follows :

$$\mathcal{M}_{num}(\mathbf{x}) = \mathcal{R}_{num}(\mathbf{x}) \begin{bmatrix} h_{1,num}(\mathbf{x}) \\ h_{2,num}(\mathbf{x}) \\ \\ h_{3,num}(\mathbf{x}) \end{bmatrix}^{t} \mathcal{R}_{num}(\mathbf{x})$$

$$\mathcal{M}_{num} = Argmin \, \mathcal{E}(\mathcal{M}) = Argmin \int_{0}^{T} \mathcal{E}_{\text{space},t}(\mathcal{M}, t) \, dt \quad with$$

$$\mathcal{E}_{\text{space},t}(\mathcal{M}, t) = \int_{\Omega} \left[\text{trace} \left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t) \, \mathbf{H}(\mathbf{x}, t) \, \mathcal{M}^{-\frac{1}{2}}(\mathbf{x}, t) \right) \right]^{p} dx,$$
(38)

where :

- in the Feature-Based case, a sensor M is computed from W, setting $\mathbf{H} = H_M$.

- in the Goal-Oriented case, p = 1, and W^* is the adjoint state, **H** is defined as in (37).

Since the flow of interest is quasi-steady in a large part of the computational domain, we keep the metric \mathcal{M}_{num} resulting from the optimization of the numerical error functional $\mathcal{E}(\mathcal{M})$ as numerical criterion of the proposed extension to DDES.

3.3 Toosi-Larsson adaptation step

A simplified way to explain the Toosi-Larsson method [27] is to consider :

- an anisotropic Cartesian mesh available with Cartesian directions $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$,
- three test filters evaluated on the flow in these three directions,
- then new local mesh sizes in these directions $\Delta x(\mathbf{x}, \mathbf{n}_x)$, $\Delta y(\mathbf{x}, \mathbf{n}_x)$, $\Delta z(\mathbf{x}, \mathbf{n}_x)$, $\mathbf{x} \in \Omega$, are defined, defining a new adapted mesh, with variation of mesh size and stretching, but with the same principal directions.

We propose here an equivalent approach for the case of a metric-based anisotropic mesh. Let \mathcal{M}_{num} be one of the two metrics defined in (38) with local mesh sizes $(h_{1,num}(\mathbf{x}), h_{2,num}(\mathbf{x}), h_{3,num}(\mathbf{x}))$ according to (29) and \mathcal{H}_{num} a unit mesh of \mathcal{M}_{num} . The LES or hybrid simulation is performed on \mathcal{H}_{num} with a filter size :

$$\bar{\Delta}_0(\mathbf{x}) = (h_{1,\text{num}}(\mathbf{x}) h_{2,\text{num}}(\mathbf{x}) h_{3,\text{num}}(\mathbf{x}))^{\frac{1}{3}}.$$

Then three test filters are built in principal directions $(\mathbf{p}_k(\mathbf{x}))$, k = 1, 3 of \mathcal{M}_{num} (according to the definition in (30)),

$$\mathbf{n}_{k}(\mathbf{x}) = \widehat{\Delta}_{\mathbf{n}_{k}}(\mathbf{x})\mathbf{p}_{k,\mathcal{M}_{0}}(\mathbf{x})$$
(39)

having the length

$$||\mathbf{n}_k(\mathbf{x})|| = \widetilde{\Delta}_{\mathbf{n}_k}(\mathbf{x}) = 2\overline{\Delta}(\mathbf{x}) \quad k = 1, 3.$$
(40)

The test filter width $= \widetilde{\Delta}_{\mathbf{n}_k}$ will vary in the neighborhood of a reference width :

$$||\mathbf{n}_{k,0}(\mathbf{x})|| = \widetilde{\Delta}_{\mathbf{n}_{k,0}}(\mathbf{x}) = 2\overline{\Delta}_0(\mathbf{x}) \quad k = 1, 3.$$
(41)

We consider the effect of the test filter, which we denote $\tilde{\cdot}^{(\mathbf{n}_k)}$. The error source term (14) becomes in the compressible case :

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k})}(\mathbf{x}) = \frac{\partial}{\partial x_{j}} \bigg(\tau_{ij}^{\text{model}}(\widetilde{\overline{u}}^{(\mathbf{n}_{k})}) - \tau_{ij}^{\widetilde{\text{model}}}(\overline{\overline{u}})^{(\mathbf{n}_{k})} - \mathcal{M}_{ij} \bigg).$$
(42)

where

$$\mathcal{M}_{ij} = \widetilde{\overline{\rho u_i u_j}}^{(\mathbf{n}_k)} - \left(\frac{\widetilde{\overline{\rho u_i}}^{(\mathbf{n}_k)} \widetilde{\overline{\rho u_j}}^{(\mathbf{n}_k)}}{\widetilde{\overline{\rho}}^{(\mathbf{n}_k)}}\right)$$
(43)

It can be shown [14] that the SGS and the sub-test stress tensors are related by the following identity :

$$\mathcal{L}_{ij} = \widetilde{\rho}\widetilde{\widetilde{u}_i}\widetilde{\widetilde{u}_j}^{(\mathbf{n}_k)} - \frac{1}{\widetilde{\rho}^{(\mathbf{n}_k)}} \left(\widetilde{\rho}\widetilde{\widetilde{u}_i}^{(\mathbf{n}_k)} \widetilde{\rho}\widetilde{\widetilde{u}_j}^{(\mathbf{n}_k)} \right) = \mathcal{M}_{ij} - \widetilde{M_{ij}^{(1)}}^{(\mathbf{n}_k)}.$$
(44)

Let us denote

$$L_{ij} = \mathcal{L}_{ij} - \frac{1}{3} \mathcal{L}_{kk} \delta_{ij}.$$
(45)

Then, to each test filter \mathbf{n}_k corresponds an error source term

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k})}(\mathbf{x}) = \frac{\partial}{\partial x_{j}} \bigg(\tau_{ij}^{\text{model}}(\widetilde{\overline{u}}^{(\mathbf{n}_{k})}) - \tau_{ij}^{\widetilde{\text{model}}(\overline{\overline{u}})}^{(\mathbf{n}_{k})} - L_{ij} \bigg).$$
(46)

From (44), (45) and (46), we deduce :

Lemma 3.2

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k})}(\mathbf{x}) = \frac{\partial}{\partial x_{j}} \left((C\bar{\Delta}^{2})\widetilde{\rho}|\widehat{S}|\widehat{P}_{ij}^{(\mathbf{n}_{k})} - (C\bar{\Delta}^{2}) \left(\frac{\widetilde{\Delta}^{(\mathbf{n}_{k})}}{\bar{\Delta}}\right)^{2} \widetilde{\rho}^{(\mathbf{n}_{k})}|\widetilde{S}^{(\mathbf{n}_{k})}| \widetilde{P}_{ij}^{(\mathbf{n}_{k})} - \widetilde{\rho}\widetilde{u}_{i}\widetilde{u}_{j}^{(\mathbf{n}_{k})} + \frac{1}{\widetilde{\rho}^{(\mathbf{n}_{k})}} \left(\widetilde{\rho}\widetilde{u}_{i}^{(\mathbf{n}_{k})}\widetilde{\rho}\widetilde{u}_{j}^{(\mathbf{n}_{k})}\right) + \frac{1}{3} \left(\widetilde{\rho}\widetilde{u}_{l}\widetilde{u}_{l}^{(\mathbf{n}_{k})} - \frac{1}{\widetilde{\rho}^{(\mathbf{n}_{k})}} \left(\widetilde{\rho}\widetilde{u}_{l}^{(\mathbf{n}_{k})}\widetilde{\rho}\widetilde{u}_{l}^{(\mathbf{n}_{k})}\right) \right) \delta_{ij} \right) \quad (47)$$

$$i = 1, \dots, 3, \quad k = 1, \dots, 3 \quad \Box$$

In the Piomelli dynamic formulation, C (or $C\bar{\Delta}^2$) is the unknown scalar parameter to be choosen at each point of the computational domain in order to optimize the efficiency of the LES model. In the present analysis, $C\bar{\Delta}^2$ is known, and the optimization variables are the three filter sizes $\tilde{\Delta}^{(\mathbf{n}_1)}, \tilde{\Delta}^{(\mathbf{n}_2)}, \tilde{\Delta}^{(\mathbf{n}_3)}$. We observe that, when we freeze all the filtered terms, $\tilde{\Delta}^{(\mathbf{n}_k)}$ appears explicitly in (47) under the form of a quadratic term. But for all "hat" filtered quantity, we remark that they approximatively differ from the non-filtered quantity by a rest proportional to the product of square of filter width times a second derivative of unfiltered quantity. However, following [27], and for the sake of simplicity, we work in the neighborhood of a LES filter width $\bar{\Delta}_0$:

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k,0})} = \widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k})} \big|_{\overline{\Delta} = \overline{\Delta}_{0}, \ \widetilde{\Delta}^{(\mathbf{n}_{k})} = \widetilde{\Delta}^{(\mathbf{n}_{k,0})} = 2\overline{\Delta}_{0}}, \quad i = 1, ..., 3, \quad k = 1, ..., 3.$$
(48)

In the neighborhood of $\overline{\Delta} = \overline{\Delta}_0$, $\widetilde{\Delta}^{(\mathbf{n}_k)} = 2\overline{\Delta}_0$, we approximate the variation of $\widetilde{\overline{\mathcal{F}}}_i^{(\mathbf{n}_k)}$ in the neighborhood of $\mathbf{n}_{k,0}$ with the following quadratic formula :

$$\widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k})} \approx \left(\frac{\widetilde{\Delta}^{(\mathbf{n}_{k})}}{2\,\overline{\Delta}_{0}}\right)^{2} \widetilde{\overline{\mathcal{F}}}_{i}^{(\mathbf{n}_{k,0})} \quad i = 1, ..., 3, \quad k = 1, ..., 3, \quad (49)$$

We observe that this approximation of an error term for LES is coherent with the fact that the LES model is obtained by perturbating the Navier-Stokes model by the SGS term which is of second order with respect to the mesh size $\overline{\Delta}$. We introduce the notations :

$$\Delta_k = \frac{\widetilde{\Delta}^{(\mathbf{n}_k)}}{2} \quad ; \quad \mathcal{G}_k = \overline{\Delta}^{-2} \left(\langle \overline{\overline{\mathcal{F}}}^{(\mathbf{n}_{k,0})}, \overline{\overline{\mathcal{F}}}^{(\mathbf{n}_{k,0})} \rangle \right)^{\frac{1}{2}} \quad k = 1, ..., 3, \tag{50}$$

where the scalar product $\langle a, b \rangle = \sum_{i=1,3} a_i b_i$. We shall now find the following metric (aligned with numerical metric \mathcal{M}_{num} defined in (38)) :

$$\mathcal{M}(\mathbf{x}) = \mathcal{R}_{num}(\mathbf{x}) \begin{pmatrix} \left(\frac{h_{1,num}}{\overline{\Delta}} \Delta_1\right)^{-2}(\mathbf{x}) & \\ & \left(\frac{h_{2,num}}{\overline{\Delta}} \Delta_2\right)^{-2}(\mathbf{x}) & \\ & \left(\frac{h_{3,num}}{\overline{\Delta}} \Delta_3\right)^{-2}(\mathbf{x}) \end{pmatrix} {}^t \mathcal{R}_{num}(\mathbf{x})$$

which minimizes the functional :

$$e(\Delta_1, \Delta_2, \Delta_3) = \int_{\Omega} \left(\mathcal{G}_1^2 \Delta_1^4 + \mathcal{G}_2^2 \Delta_2^4 + \mathcal{G}_3^2 \Delta_3^4 \right)^{\frac{1}{2}} \mathrm{d}\mathbf{x}$$
(51)

under the constraint :

$$\mathcal{C}(\mathcal{M}) = \mathcal{N}_{\mathcal{A}}$$

where N is prescribed by the user. We observe that

$$\mathcal{C}(\mathcal{M}) = \int_{\Omega} \bar{\Delta}^3 (h_1 h_2 h_3 \Delta_1 \Delta_2 \Delta_3)^{-1} \mathrm{d}\mathbf{x},$$

which reduces the problem to a minimization with respect to $(\Delta_1, \Delta_2, \Delta_3)$ under the constraint that the integral of their product is specified. Let us denote by $(\Delta_1^{\text{opt}}, \Delta_2^{\text{opt}}, \Delta_3^{\text{opt}})$ the solution of the minimization problem and :

$$\zeta^{\rm opt} = \left(\Delta_1^{\rm opt} \Delta_2^{\rm opt} \Delta_3^{\rm opt}\right)^{-1}.$$

In any point **x**, knowing solely ζ^{opt} , we can deduce the Δ_k^{opt} 's since the (squared) value of the integrand

$$g(\mathbf{x}) = \mathcal{G}_1^2(\mathbf{x})\Delta_1^4(\mathbf{x}) + \mathcal{G}_2^2(\mathbf{x})\Delta_2^4(\mathbf{x}) + \mathcal{G}_3^2(\mathbf{x})\Delta_3^4(\mathbf{x})$$

must be minimized with respect to the $((\Delta_1^{\text{opt}})^4, (\Delta_2^{\text{opt}})^4, (\Delta_3^{\text{opt}})^4)$, with the constraint that their product $(\zeta^{\text{opt}})^{-4}$ is given. This implies that

$$\mathcal{G}_{1}^{2}(\mathbf{x})(\Delta_{1}^{\text{opt}})^{4}(\mathbf{x}) = \mathcal{G}_{2}^{2}(\mathbf{x})(\Delta_{2}^{\text{opt}})^{4}(\mathbf{x}) = \mathcal{G}_{3}^{2}(\mathbf{x})(\Delta_{3}^{\text{opt}})^{4}(\mathbf{x})$$
$$= \left((\zeta^{\text{opt}})^{-4}\mathcal{G}_{1}^{2}(\mathbf{x})\mathcal{G}_{2}^{2}(\mathbf{x})\mathcal{G}_{3}^{2}(\mathbf{x})\right)^{\frac{1}{3}},$$
(52)

$$\Delta_k^{\text{opt}}(\mathbf{x}) = (\zeta^{\text{opt}})^{-\frac{1}{3}} \Big(\mathcal{G}_1(\mathbf{x}) \mathcal{G}_2(\mathbf{x}) \mathcal{G}_3(\mathbf{x}) \Big)^{\frac{1}{6}} \Big(\mathcal{G}_k(\mathbf{x}) \Big)^{-\frac{1}{2}}$$

It remains to minimize

$$\mathcal{K}(\zeta) = \int_{\Omega} K(\mathbf{x}) \zeta^{-\frac{2}{3}}(\mathbf{x}) \mathrm{d}\mathbf{x},$$

with

$$K(\mathbf{x}) = 3 \left(\mathcal{G}_1(\mathbf{x}) \mathcal{G}_2(\mathbf{x}) \mathcal{G}_3(\mathbf{x}) \right)^{\frac{1}{3}},$$

under the constraint $\int_{\Omega} \zeta(\mathbf{x}) d\mathbf{x} = \mathcal{N}$. The resolution of this minimization problem gives :

$$K(\mathbf{x})\zeta^{-\frac{5}{3}}(\mathbf{x}) = const. \Rightarrow \zeta(\mathbf{x}) = const.^{-\frac{3}{5}}K(\mathbf{x})^{\frac{3}{5}},$$
$$\mathcal{N} = \int_{\Omega} \zeta \, \mathrm{d}\mathbf{x} = const.^{-\frac{3}{5}} \int_{\Omega} K^{\frac{3}{5}} \, \mathrm{d}\mathbf{x} \Rightarrow const.^{-\frac{3}{5}} = \mathcal{N}\left(\int_{\Omega} K^{\frac{3}{5}} \, \mathrm{d}\mathbf{x}\right)^{-1},$$
$$\zeta^{\mathrm{opt}}(\mathbf{x}) = \left(\int_{\Omega} K^{\frac{3}{5}}(\mathbf{x}') \, \mathrm{d}x'\right)^{-1} K(\mathbf{x})^{\frac{3}{5}} \, \mathcal{N},$$
(53)

and

$$\Delta_k^{\text{opt}}(\mathbf{x}) = \left(\mathcal{G}_1(\mathbf{x})\mathcal{G}_2(\mathbf{x})\mathcal{G}_3(\mathbf{x})\right)^{\frac{1}{6}} \left(\mathcal{G}_k(\mathbf{x})\right)^{-\frac{1}{2}} \left(\int_{\Omega} K^{\frac{3}{5}}(\mathbf{x}') \mathrm{d}\mathbf{x}'\right)^{\frac{1}{3}} K(\mathbf{x})^{-\frac{1}{5}} \mathcal{N}^{-\frac{1}{3}}.$$
 (54)

In practice we think it is better that the initial metric \mathcal{M}_0 defining the filter should be the metric defining the previous mesh while the test filter directions will be chosen from the novel mean flow adaptation metric \mathcal{M}_1 . then the metric adapting to both Mach field and LES would write :

$$\mathcal{M}_{\mathrm{H,DDES}}(\mathbf{x}) = \mathcal{R}_{1}(\mathbf{x}) \begin{bmatrix} \left(\frac{h_{1}^{1}\Delta_{1}}{\overline{\Delta}}\right)^{-2}(\mathbf{x}) & \\ & \left(\frac{h_{2}^{1}\Delta_{2}}{\overline{\Delta}}\right)^{-2}(\mathbf{x}) & \\ & & \left(\frac{h_{3}^{1}\Delta_{3}}{\overline{\Delta}}\right)^{-2}(\mathbf{x}) \end{bmatrix} {}^{t}\mathcal{R}_{1}(\mathbf{x}).$$

This step will probably increase the complexity $\mathcal{C}(\mathcal{M}_{H,DDES})$ which will be larger than N. A final step of renormalization is then necessary :

$$\mathcal{M}_{_{\mathrm{H,DDES}}}^{\mathrm{new}} = N^{rac{2}{3}} \Big(\mathcal{C}(\mathcal{M}_{_{\mathrm{H,DDES}}}) \Big)^{-rac{2}{3}} \mathcal{M}_{_{\mathrm{H,DDES}}}.$$

4 Space and time approximation

In the previous sections, space and time discretizations are not considered. Indeed, a continuous optimization problem is introduced and solved under the form of an optimality system. It is this optimality system which is finally discretized by replacing the direct dependancy of the flow field to the metric by a chain of mapping :

$$\mathcal{M} \mapsto \mathcal{H} \mapsto W_{\mathcal{H}} \tag{55}$$

from the metric \mathcal{M} to the discrete flow field $W_{\mathcal{H}}$ via the construction of a unit mesh \mathcal{H} of the metric.

4.1 CFD numerics

The software used in this paper is the NiceFlow software and relies on the *vertex-centered* MUSCL approximation for *tetrahedrizations* described in details in [11]. In particular, NiceFlow involves mesh adaptation functionalities described in [11] : computation of metric-based adaptation criteria, regeneration of adapted anisotropic meshes.

4.2 Implementation of LES criterion

LES filter

The SGS term is assembled by element and the filter size $\overline{\Delta} = (h_1 h_2 h_3)^{\frac{1}{3}}$ is evaluated from the volume of the local element jt:

$$\bar{\Delta}_{|jt} = \left(vol(jt) \right)^{\frac{1}{3}}.$$

Test filter

In [20] an isotropic test filter is built at each vertex is from a mean on the elements jt which have is as vertex, which we write $jt \ni is$:

$$\widetilde{w(is)} = \left(\sum_{jt\ni is} \sum_{js\in jt} vol(jt)\right)^{-1} \sum_{jt\ni is} \sum_{js\in jt} vol(jt) \ w(js)$$
(56)

where the $\sum_{js\in jt}$ means the sum over the vertices js of element jt and vol(jt) holds for the volume of element jt. It results that the integration area of the test filter is the P_1 basis function support, or union of elements around a vertex. Equivalently, the test filter is a mean on the set of neighboring elements, of volume $Vol_{is} = \sum_{jt\ni is} vol(jt)$. Defining as N_{is} the number of elements having vertex is as a vertex, and assuming that the volume of elements around vertex is is close to vol(jt) for a element containing is, then $Vol_{is} \approx N_{is}vol(jt)$. Taking vol(jt) as the integration area of the model filter, we can set :

$$\frac{\overline{\Delta}}{\Delta} = N_{is}^{\frac{1}{3}}.$$

For building the anisotropic test filter directions, we use the principal directions $\mathbf{p}_{k,\mathcal{M}}$ (see (30)) of the background metric :

$$\mathbf{n}_k = 2\Delta \mathbf{p}_{k,\mathcal{M}}.$$

The anisotropic filter according to vector \mathbf{n}_k , that is, aligned with the unit vector $\mathbf{p}_{k,\mathcal{M}}$, and of filter width $2\bar{\Delta}$ writes :

$$\widetilde{w(is)}^{k} = \sum_{jt \ni is} \sum_{js \in jt} vol(jt) |\langle \frac{\mathbf{isjs}}{|isjs|}, \mathbf{n}_{k} \rangle | w(js) \Big(\sum_{jt \ni is} \sum_{js \in jt} vol(jt) |\langle \frac{\mathbf{isjs}}{|isjs|}, \mathbf{n}_{k} \rangle | \Big)^{-1}$$
(57)

where **isjs** is the vector from vertex *is* to vertex *js*.

5 Adaptation algorithm

The adaptation algorithm solves the discretization of the optimality conditions for the optimal metric.

The flows under study are unsteady. In order to apply mesh adaptation, we shall use a version of the Transient Fixed Point introduced in [3]. A sketch of this algorithm is given by Algorithm 0.1.

Algorithm 0.1 Transient Fixed Point Given a complexity $N_{\text{prescribed}}$, an initial metric, \mathcal{M}_0 of complexity c_0 , build a unit mesh \mathcal{H}_0 from \mathcal{M}_0 For iadapt = 0, nadapt

- Compute over [0, T] the flow W_{iadapt} from with mesh \mathcal{H}_{iadapt}
- Compute the k_{\max} new metrics $\mathcal{M}_{iadapt+1}^k$ of complexity $N_{prescribed}$ each taking into account the flow over $[t_k, t_{k+1}]$.
- Compute the k_{max} new meshes $\mathcal{H}^k_{iadapt+1}$ from $\mathcal{M}^k_{iadapt+1}$
- iadapt = iadapt + 1

End for *iadapt*

Let us assume that Algorithm 0.1 iteratively converges (when *iadapt* increases to infinity) to a fixed point $(W_{\infty}, \mathcal{M}_{\infty})$. Then this fixed point is a numerical flow

computed on a succession of meshes \mathcal{M}_{∞}^{k} , $k = 1, k_{\max}$, each mesh \mathcal{H}_{∞}^{k} being adapted to the best approximation (in some sense) of the flow on time interval $[t_k, t_{k+1}]$. Further, the sum of the complexities of the different meshes for $k = 1, k_{\max}$ is the global complexity $k_{\max}\mathcal{N}_{\text{prescribed}}$.

Since we are interested by vortex shedding flows past blunt bodies, it is possible to use a single mesh for the whole time interval, i.e. we shall work with the *single-mesh* option of the transient fixed point :

$$k_{\max} = 1 . (58)$$

6 Numerical applications

In this first application of the proposed mesh adaptation method, we concentrate on the flow around a circular cylinder, and in particular on two typical flows, depending of their Reynolds number :

- a typical subcritical flow is the flow at Reynolds number 3900 in which the boundary layer is said a laminar one, while the detached wake is turbulent, we also consider the case of the flow at Reynolds number 20K,

- a typical supercritical flow if Reynolds 1 Million, in which boundary layers and wake pewsent turbulent characteristic.

Here subcritical and supercritical refer to the drag crisis Reynolds numbers around 300000 at which the drag decreases rapidly.

6.1 Subcritical flow calculation with adaptation

The subcritical cases are flows around a circular cylinder at Reynolds number 3900 and 20K. The boundary layer being of laminar regime (although participating to the mechanism of the turbulent global flow), the state-of-art modelisation for this flow is to apply a LES model of medium sophistication. We choose the combination of the VMS formulation with the WALE SGS model, as described in Norma report T1-D5.

A first mesh-adaptive calculation uses the Hessian of Mach number as anisotropic adaptation criterion, with the L^4 norm, according to the results of available experiments, see for example [11].

Re = 3900

First, let us take a look at Figure 1 to see what our algorithm produces for $\Delta_1^{\text{opt}}/\bar{\Delta}$. Note that this value varies enormously, from 10^{-4} to 86. Keeping in mind that this value multiplies the local mesh size h_1 we decide to reduce the range of possible values for $\Delta_1^{\text{opt}}/\bar{\Delta}$ (Figure 2).

In the mesh adaptation method of this work, the detection of vortices is translated in a field of multiplicative factors of the local mesh size. This field is depicted in Figure 2.



FIGURE 1 – Instantaneous value of $\Delta_1^{opt}/\bar{\Delta}$ for flow around a cylinder at Re = 3900. These values are those given by the algorithm without clamping.



FIGURE 2 – Instantaneous value of $\Delta_1^{opt}/\bar{\Delta}$ for flow around a cylinder at Re = 3900. These values are those given by the algorithm with a choosen clamping.

The resulting mesh is presented in Figure 3 where the region one diameter after the cylinder received a supplement of vertices.



FIGURE 3 – Cylindre Re = 3900 VMS-WALE results : Top, mesh obtained with the new LES criterion (around 400K vertices), and bottom, mesh obtained with Mach criterion only (around 300K vertices).



FIGURE 4 – Cylindre Re = 3900 VMS-WALE results : Top, vorticity obtained with the new LES criterion, and bottom, vorticity obtained with Mach criterion only.

$Re = 20\ 000$

For this test case, we compare the results obtained in Norma report T1-D5 with those obtained for our new method. Looking at the meshes obtained in Figure 6, we do not see any drastic change, even if the back body for the mesh obtained with our new theory seems more remeshed. Nevertheless, Figure 1 shows that our vorticity field is captured with much greater precision.



FIGURE 5 – Cylindre Re = 20K VMS-WALE results : Top, vorticity obtained with the new LES criterion, and bottom, vorticity obtained with Mach criterion only.



FIGURE 6 – Cylindre Re = 20K VMS-WALE results : Top, mesh obtained with the new LES criterion (around 1M vertices), and bottom, mesh obtained with Mach criterion only (around 1.1M vertices).



FIGURE 7 – Instantaneous value of $\Delta_1^{opt}/\bar{\Delta}$ for flow around a cylinder at Re = 20K. These values are those given by the algorithm with a choosen clamping.

6.2 Mesh adaptive computation of a supercritical flow

At a Reynolds of 1M, the flow around the circular cylinder is supercritical, with a completely turbulent boundary layer. Computing with LES this boundary layer is very expansive, and therefore such a strategy is not useful in industrial studies. However, this type of flow is frequently met in industry.

As options in NiceFlow, we start from the anisotropic mesh adaptation approach which was proven as very efficient for steady RANS flows, see for example [1],[11], and unsteady RANS, see [4].

For addressing both RANS regions and LES-type regions, we apply the DDES available in NiceFlow, defined initially in [25]. This model is hybrid in the sense that in some "LES" regions, the model viscosity is very small, comparable to a SGS viscosity, while not being a Smagorinsky-like viscosity.

The rest of the algorithm which we apply is identical to the one applied to the two previous subcritical flows, with two stages of adaptation criteria : a Mach-Hessian criterion and its correction with the Toosi-Larsson analysis.

At this Reynolds number and with a limited number of vertices (around 1M), the DDES calculation produces a poorly fluctuating flow, although the model DDES viscosity is much lower in the wake than the RANS viscosity. We estimate that a more fluctuating solution would be obtained by running DDES with something like 3 times mode vertices. Our development was done on a lap top with 4 processors. A more ambitious calculation is starting but will take weeks.

However, the results are reasonable without the LES criterion and improved with the LES criterion.



FIGURE 8 – Cylindre Re = 1M DDES results : Mesh (left) and Mach field (right) in cross-section for the new LES criterion. The mesh is around 1M vertices.



FIGURE 9 – Instantaneous value of $\Delta_1^{opt}/\bar{\Delta}$ for flow around a cylinder at Re = 1M. These values are those given by the algorithm with a choosen clamping.

7 Concluding remarks

Anisotropic mesh adaptive CFD is an arising important progress for applied numerical CFD. Our contribution aims at making more available hybrid RANS/LES calculations for industry, thanks to anisotropic mesh adaptation.

To address the hybrid modelization, we propose a hybrid mesh adaptation. The RANS mesh adaptation contributes to a good capture of the overall flow. For LES regions, we adapt and extend the analysis of Toosi and Larsson which is based on the source term of LES-error equation. The two mesh-adaptation criteria are applied inside a transient fixed point algorithm with the option of using a single adapted mesh for the whole time interval. Numerical examples concerning subcritical and supercritical vortex shedding flows past a circular cylinder demonstrate the interest of the proposed approach. An extremely interesting sequel of this development would be to extend the analysis to the so-called goal-oriented analysis. In [10], the goal oriented analysis permitted to look for the best mesh for propagating an acoustic wave produced by an artificial acoustic source. With the new criterion, noise production by turbulence could be addressed.

The introduction of the space-time adaptation would be also of interest in order to compute with optimally chosen time steps.

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