NORMA

Research report

T4-D3: deliverable Time-advancing algorithmics and parallelism:

A space and time fixed point adaptation method.

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September 28th 2022

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1 Abstract

We introduce a space-time metric-based approach for the best set of spatial meshes $\mathcal{M}_{opt}(t)$, of complexity $n_{opt}(t)$ combined with a timestep $\tau_{opt}(t)$ for the calculation of unsteady flows with implicit time advancing. Both types of error, feature-based and Goal Oriented are considered. The case of a mesh which is adapted at each time step, and the case where the mesh is constant during a subinterval of the time interval are analysed. This approach then takes place inside a Global Transient Fixed Point mesh adaptation algorithm. Application to flows with vortex shedding are then described.
This work was supported by the ANR project NORMA under grant ANR-19-CE40-0020-01 and is delivered as T4-D3 in replacement of a study on parallel multiscale.

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RESEARCH REPORT

N° 7003

April 2023

Project-Teams Gamma3 and Ecuador

ISSN 0249-6399 ISRN INRIA/RR--7003--FR+ENG

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Research Report n° 7003 — April 2023 — 29 pages

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Key-words: compressible flow, mesh adaptation, anisotropic, space-time, implicit

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§ This work was supported by the ANR NORMA project, grant ANR-19-CE40-0020-01 of the French National Research Agency.

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Une méthode de point fixe en adaptation de maillage espace-temps

Résumé : Nous introduisons une approche basée sur les métriques spatio-temporelles pour le meilleur ensemble de maillages spatiaux $\mathcal{M}_{opt}(t)$, de complexité $n_{opt}(t)$ combiné avec un pas de temps $\tau_{opt}(t)$ pour le calcul des écoulements instationnaires avec avancement implicite du temps. Les deux types d'erreur, basées sur les fonctionnalités et orientées vers une fonctionnelle, sont pris en compte. Le cas d'un maillage adapté à chaque pas de temps, et le cas où le maillage est constant pendant un sous-intervalle de l'intervalle de temps sont analysés. Cette approche prend place dans un algorithme de point fixe transitoire global d'adaptation de maillage. L'application aux écoulements avec détachement tourbillonnaire est ensuite décrite.

Mots-clés : écoulement compressible, adaptation de maillage, anisotrope, espace-temps, implicite

Contents

Contents

1 Introduction

Metric based mesh adaptation allows to address anisotropic meshes, and can be done either in a feature-based mode, relying on the Hessian of one or several sensors (=features), or in a goal-oriented mode in which the error committed on a scalar output of a PDE is minimized, with the help of an adjoint state. We are considering the discretization of a PDE on the space-time cylinder $Q = \Omega \times]0, T[$. We advance in time and want to adapt the spatial mesh to the solution.

An important option is the *all-time mesh adaptation*, which consists in building *at each time step* an anisotropic mesh $\mathcal{H}_{\mathcal{M}}$ by defining a metric $\mathcal{M}$ on computational domain Ω, which is optimal for this time step. But this standpoint is expansive in terms of computational cost and may be of low accuracy due to errors committed in transfering solutions between too many successive meshes.

A somewhat better strategy consists in *freezing the adapted mesh during several time steps*. Then it is mandatory that the mesh anticipates the flow behavior during these several time steps. In [3, 2] a transient fixed point (TFP) mesh adaptation algorithm has been introduced in order to master this issue. The TFP approach has been extended to a goal-oriented adaptation in [5] where several analyses where proposed for evaluating the convergence order of the TFP. A more complete analysis, for a feature-based adaptation is proposed in [4].

In those works, either the time step is directly specified by the user, or the time step is assumed to be defined via a CFL stability condition related to an explicit time advancing, an option useful when explicit time advancing is performed, and which more or less adapts the timestep to the solution.

The choice of a time step is defined in other terms when an implicit time advancing is used. Indeed, the size of the time step is no more directly related to a stability condition. The time steps which are used can be notably larger than those permitted with an explicit time advancing. Large time steps induce a higher CPU efficiency. But the approximation accuracy becomes an important issue. Too large time steps degrade the prediction, too small time steps increases the computational cost.

Many attempts to control the timestep size on an accuracy basis can be found in the litterature. An example for a context rather close to our is [6]. But the authors adapt separately time and space and none addresses the global space-time adaptation.

The present study adresses the problem of the space-time adaptation in TFP in which meshes and time steps have to be defined in a coupled manner for minimizing an error functional, in order to give the best accuracy given a measure of the global space-time computational effort.

For the feature-based approach, the problem of optimal simultaneous adaptation of spatial mesh and time step can be formulated under the form of a severely nonlinear -tightly coupled- optimization problem and we propose a slight simplification of it -loosely coupling- in order to design a tractable accurate and efficicent timestep and mesh adaptative TFP algorithm.

In the case of a Goal Oriented criterion, we show that the proposed formulation applies in a natural way.

Numerical examples demonstate the validity of the new mesh/timestep adaptation algorithms.

2 Time-advancing adapted mesh

A *time-advancing adapted mesh* of a cylindric space-time computational domain $\Omega \times]0, T[$ is defined by:
(i)- a time discretization of the computational time interval $]0, T[$

$$t_1 = 0 \ < \ t_2 \ < \ ... \ < \ t_{kmax} = T$$

and
(ii)- for each *time level* k a spatial mesh $\mathcal{H}_k$ of the spatial computational domain Ω.

The number of *time intervals*, that is the number of *time steps* for advancing from 0 to T is $nstep = kmax - 1$. For any k we consider a metric $\mathcal{M}(k)$ from which at a valid unit mesh ([8]) $\mathcal{H}_k$ of metric $\mathcal{M}(k)$ can be derived. Then defining the metrics $(\mathcal{M}(k), k = 1, ..., kmax)$ is sufficient for building (ii). For any k the spatial complexity $n(k) = \mathcal{C}_{spatial}(\mathcal{M}(k))$ ([8]) defines the analog of the number of cells but can be any positive real number. From the knowledge of the $n(k)$'s, assuming an implicit time advancing of algorithmic complexity $n(k)$, we get the *space-time complexity* of the so-defined space-time mesh:

$$\mathcal{C}\big(t_1, ...t_{max}, \mathcal{M}(1), ...\mathcal{M}(k)\big) = \sum_{k=1}^{k=nstep} n(k+1). \tag{1}$$

The above definition is discrete in time. We propose to replace the usual time discretization by a continuous representation of it. The discrete time-step is defined by:

$$\tau_k = t_{k+1} - t_k \quad \text{for} \quad k = 1, nstep.$$

The following continuous time-step is considered:

$$\tau : \ t \in]0, T[\ \mapsto \ \tau(t) = \tau_k \ \ if \ \ t_k \ \leq \ t \ < \ t_{k+1}.$$

Then:

$$\begin{aligned} \int_0^T (\tau(t))^{-1} dt \ &= \ \sum_{k=1}^{k=nstep} \int_{t_k}^{t_{k+1}} (\tau(t))^{-1} dt \\ = \ \sum_{k=1}^{k=nstep} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^{-1} dt \ &= \ nstep. \end{aligned} \tag{2}$$

In other words, τ^{-1} is the time discretization density, i.e. the number of time levels per time unit.

Similarly, when we extend the spatial complexity, [1]

$$n : \ t \in]0, T[\ \mapsto \ n(t) = n_k \ \ if \ \ t_k \ < \ t \ \leq \ t_{k+1}$$

the space-time complexity writes as follows:

$$\begin{aligned} \mathcal{C}(\mathcal{M}, n, \tau) = \mathcal{C}(n, \tau) = \ \sum_{k=1}^{k=nstep} n(k+1) \\ = \ \sum_{k=1}^{k=nstep} \int_k^{k+1} n(t)\tau^{-1} dt \ = \ \int_0^T n(t)\tau^{-1} dt. \end{aligned} \tag{3}$$

This motivates the definition of a *space-time continuous mesh* as follows:

Definition 2.1 *Space-time continuous mesh. We call a valid space-time continuous mesh* $(\mathcal{M}, \tau, nstep)$ *the knowledge of the following ingredients:*
- a number nstep of time intervals.
- a timestep function: $t \in]0, T[\mapsto \tau(t) \in]0, T[$ *valid in the sense that* $\int_0^T (\tau(t))^{-1} dt = nstep$.
- for every $t \in]0, T[$ *a spatial metric* $\mathcal{M}(t)$ *of spatial complexity* $n(t) = \mathcal{C}_{spatial}(\mathcal{M}(t))$.

[1] The motivation in basing the k-th step (from k to $k+1$) on the $k+1$-th mesh is the implicit time advancing, solved at level $k+1$, with typically data interpolated from the mesh of level k.

Definition 2.2 *Complexity. The space-time complexity $\mathcal{C}(\mathcal{M}, \tau, nstep)$ of a space-time continuous mesh $(\mathcal{M}, \tau, nstep)$ is:*

$$\mathcal{C}(\mathcal{M}, \tau, nstep) \;=\; \int_0^T \mathcal{C}_{spatial}(\mathcal{M}(t))(\tau(t))^{-1} dt. \tag{4}$$

Definition 2.3 *Unit space-time mesh. Given a (sufficiently smooth) space-time continuous mesh, a discrete space-time mesh which is (approximatively) unit for it can be derived as follows:*
(i)- Since the time density $(\tau(t))^{-1}$ is positive and satisfies $\int_0^T (\tau(t))^{-1} dt = nstep$ we can successively build time levels t_k by putting:

$$t_k \text{ such that } \int_{t_{k-1}}^{t_k} (\tau(t))^{-1} dt = 1,$$

stopping when it does not holds, for $nstep = integer(\int_0^T \tau^{-1} dt)$.
(ii)- At any time level t_k, $\mathcal{M}(t)$ is used for building a unit spatial mesh.

3 Errors for space-time analysis

3.1 Local error for feature-based analysis

We concentrate on the advection model

$$u_t + c u_x = 0, \quad x \in \Omega \tag{5}$$

where $\Omega \subset \mathbb{R}$ is an interval of $\mathbb{R}$. We consider the usual continuous P_1 FEM approximation on a partition $\mathcal{M}_h$ of Ω in intervals $T_i = [x_i, x_{i+1}]$:

$$V_h = \varphi_h \in \mathcal{C}^0(\mathbb{R}), \quad \forall T_i, \quad \varphi_h|_{T_i} \text{ is affine} \tag{6}$$

$$u_h \in V_h, \forall \varphi_h \in V_h, \quad (\varphi_h, u_{h,t} + c u_{h,x}) = 0. \tag{7}$$

We are first interested by the local spatial error:

$$\varepsilon_{space} = (\varphi_h, u_{h,t} + c u_{h,x} - (u_t + c u_x)) = 0. \tag{8}$$

A rough truncation error estimate writes this error in terms of mesh size $\Delta x (K_x \in \mathbb{R})$:

$$|(\varphi_h, u_{h,t} + c u_{h,x} - (u_t + c u_x))| \leq K_x \Delta x (H_{u_t} + H_u) \Delta x \tag{9}$$

where H_v holds for the absolute value of the Jacobian of v.

In the general case of a metric based mesh parametrization, we have: [2]

$$|(\varphi_h, u_{h,t} + c u_{h,x} - (u_t + c u_x))| \leq K_x det\Big(\mathcal{M}^{-\frac{1}{2}} (H_{u_t} + H_u) \mathcal{M}^{-\frac{1}{2}}\Big). \tag{10}$$

We also have to define for all $t \in [O, T]$ a *time dependant time step* $\Delta t = \tau(t)$, and a discrete time-derivative:

$$u_{h,\tau,t} \approx u_{h,t} \tag{11}$$

and estimate the time error $(K_t \in \mathbb{R})$:

$$|u_{h,\tau,t} - u_{h,t}| \leq K_t \tau^\alpha |\frac{\partial^\alpha u}{\partial t^\alpha}|. \tag{12}$$

[2] In the case of interpolation error (3D), $K_x = \frac{1}{10}$, see [7], Corollary 1 of Theorem 4.2.2.

We can now define a *strongly coupled error functional* [3]:

$$\mathcal{E}_0(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega \Big[(K_t\tau^\alpha|\frac{\partial^\alpha u}{\partial t^\alpha}| + K_x det\Big(\mathcal{M}^{-\frac{1}{2}}(H_{u_t}+H_u)\mathcal{M}^{-\frac{1}{2}}\Big)\Big]^p \mathrm{d}x\mathrm{d}t. \tag{13}$$

Remark 3.1 *One way to specify the contants K_t and K_x in the particular case of $\alpha = 2$, is to decide that we are interested by the P_1 interpolation error in the space-time representation of the unknown: the tetrahedra of the time levels are extended into 4-dimensional simplexes and the interpolatin error main term expresses in terms of the space-time Hessian of the unknown. Then neglecting the hybrid second derivatives between space and time, and neglecting the Hessian of the time derivative, we obtain the reduced expression*

$$\mathcal{E}_0(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega \Big[(K_t\tau^2|\frac{\partial^2 u}{\partial t^2}| + K_x\Delta x\ H_u\ \Delta x\Big]^p dt\,dx. \tag{14}$$

with

$$K_x = K_t = 1.\square$$

The analysis of $\mathcal{E}_0$ is rather complex and we shall prefer in the sequel to analyse the following *loosely coupled error estimate* [4]:

$$\mathcal{E}(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega \Big[(K_t\tau^\alpha|\frac{\partial^\alpha u}{\partial t^\alpha}|\Big]^p + [K_x\Delta x(H_{u_t}+H_u)\Delta x]^p \mathrm{d}t\mathrm{d}x \tag{15}$$

where we have a sum of a temporal part:

$$\mathcal{E}_{temp}(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega [K_t\tau^\alpha|\frac{\partial^\alpha u}{\partial t^\alpha}|]^p \mathrm{d}t\mathrm{d}x \tag{16}$$

and a spatial part:

$$\mathcal{E}_{space}(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega [K_x\Delta x(H_{u_t}+H_u)\Delta x]^p \mathrm{d}t\mathrm{d}x \tag{17}$$

also written:

$$\begin{array}{l} \mathcal{E}_{space}(\mathcal{M}_h,\tau) = \int_0^T \mathcal{E}_{space,t}(t)\mathrm{d}t \quad \text{with} \\ \mathcal{E}_{space,t}(t) = \int_\Omega [K_x\Delta x(H_{u_t}+H_u)\Delta x]^p \mathrm{d}x \end{array} \tag{18}$$

In a *multidimensional and metric-based extension* to a CFD model with state function $W = (\rho, \rho u, \rho v, ...)$ solution of a state equation:

$$W_t + \Psi(W) = 0 \tag{19}$$

this error analysis writes:

$$\mathcal{E}(\mathcal{M}_h,\tau) = \mathcal{E}_{temp}(\mathcal{M}_h,\tau) + \mathcal{E}_{space}(\mathcal{M}_h,\tau) \tag{20}$$

with

$$\mathcal{E}_{temp}(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega [K_t\tau^\alpha|\frac{\partial^\alpha W}{\partial t^\alpha}|]^p \mathrm{d}x\mathrm{d}t \tag{21}$$

and

$$\begin{array}{l} \mathcal{E}_{space}(\mathcal{M}_h,\tau) = \int_0^T \mathcal{E}_{space,t}(t)\mathrm{d}t \quad \text{with} \\ \mathcal{E}_{space,t}(t) = \int_\Omega [\text{trace}\Big(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\,\mathbf{H}(\mathbf{x},t)\,\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\Big)]^p \mathrm{d}x \end{array} \tag{22}$$

where $\mathbf{H} = |H_{u_t} + H_u|)$ depends of a sensor u computed from W.

[3] For simplifying computations we do not minimize an L^p norm but its power p.

[4] Again for simplifying computations we do not minimize an L^p norm but its power p.

3.2 Goal oriented error

We consider the Goal Oriented formulation as introduced in [5] and keep the notations of that paper. We want to minimize the error of the functional:

$$j = (g, W)$$

where W is the solution of the flow system (19). The novelty with respect to [5] is that taking into account the error on the time discretization leads to have an extra term $\mathbf{K}(\mathcal{M})$ in the error estimate:

$$|(g, W_h - W)| \approx \mathbf{K}(\mathcal{M}) + \mathbf{E}(\mathcal{M})$$

with

$$\mathbf{K}(\mathcal{M}) = \int_0^T \int_\Omega |W^*| \, |K_t \tau^\alpha \frac{\partial^\alpha W}{\partial t^\alpha}| \, \mathrm{d}\Omega \, \mathrm{d}t.$$

Let us define the adjoint state W^*, solution of the adjoint system:

$$-W_t^* + \left(\frac{\partial \Psi}{\partial W}\right)^* W = g. \tag{23}$$

We reproduce now in short the error estimate developed in [9] for the unsteady Euler system. The error $\mathbf{E}(\mathcal{M})$ writes:

$$\begin{aligned} \mathbf{E}(\mathcal{M}) \approx & \int_0^T \int_\Omega |W^*| \, |\left(W - \pi_{\mathcal{M}} W\right)_t| \, \mathrm{d}\Omega \, \mathrm{d}t \\ + & \int_0^T \int_\Omega |\nabla W^*| \, |\mathcal{F}(W) - \pi_{\mathcal{M}} \mathcal{F}(W)| \, \mathrm{d}\Omega \, \mathrm{d}t \\ + & \int_0^T \int_\Gamma |W^*| \, |(\bar{\mathcal{F}}(W) - \pi_{\mathcal{M}} \bar{\mathcal{F}}(W)).\mathbf{n}| \, \mathrm{d}\Gamma \, \mathrm{d}t. \end{aligned} \tag{24}$$

Symbol $\mathcal{F}$ and $\bar{\mathcal{F}}$ hold respectively for the internal and boundary Euler flux functions. Neglecting the boundary term, we get ([9]):

$$\mathbf{E}(\mathbf{M}) \approx \int_0^T \int_\Omega \mathrm{trace}\left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\, \mathbf{H}(\mathbf{x},t)\, \mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\right) \mathrm{d}\Omega \, \mathrm{d}t$$

$$\text{with} \quad \mathbf{H}(\mathbf{x},t) = \sum_{j=1}^{5} \left(\Delta t_j(\mathbf{x},t) + \Delta x_j(\mathbf{x},t) + \Delta y_j(\mathbf{x},t) + \Delta z_j(\mathbf{x},t)\right), \tag{25}$$

with the notations

$$\begin{aligned} \Delta t_j(\mathbf{x},t) &= \left|W_j^*(\mathbf{x},t)\right| \cdot \left|H(W_{j,t})(\mathbf{x},t)\right|, \\ \Delta x_j(\mathbf{x},t) &= \left|\frac{\partial W_j^*}{\partial x}(\mathbf{x},t)\right| \cdot \left|H(\mathcal{F}_1(W_j))(\mathbf{x},t)\right|, \\ \Delta y_j(\mathbf{x},t) &= \left|\frac{\partial W_j^*}{\partial y}(\mathbf{x},t)\right| \cdot \left|H(\mathcal{F}_2(W_j))(\mathbf{x},t)\right|, \\ \Delta z_j(\mathbf{x},t) &= \left|\frac{\partial W_j^*}{\partial z}(\mathbf{x},t)\right| \cdot \left|H(\mathcal{F}_3(W_j))(\mathbf{x},t)\right|. \end{aligned} \tag{26}$$

Here, W_j^* denotes the jth component of the adjoint vector W^*, $H(\mathcal{F}_i(W_j))$ the Hessian of the jth component of the vector $\mathcal{F}_i(W)$, and $H(W_{j,t})$ the Hessian of the jth component of the time derivative of W.

3.3 Unified Feature/Goal Oriented criterion

The two error criteria examined previously can be unified as follows:

$$\mathcal{E}(\mathcal{M}_h,\tau) = \mathcal{E}_{temp}(\mathcal{M}_h,\tau) + \mathcal{E}_{space}(\mathcal{M}_h,\tau) \tag{27}$$

with

$$\mathcal{E}_{temp}(\mathcal{M}_h,\tau) = \int_0^T \int_\Omega [K_t \tau^\alpha |W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|]^p \mathrm{d}x\mathrm{d}t \tag{28}$$

and

$$\begin{array}{l} \mathcal{E}_{space}(\mathcal{M}_h,\tau) = \int_0^T \mathcal{E}_{space,t}(t)\mathrm{d}t \quad \text{with} \\ \mathcal{E}_{space,t}(t) = \int_\Omega [\text{trace}\left(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\,\mathbf{H}(\mathbf{x},t)\,\mathcal{M}^{-\frac{1}{2}}(\mathbf{x},t)\right)]^p \mathrm{d}x \end{array} \tag{29}$$

where :

- in the Feature-Based case, a sensor u is computed from W, allowing to replace $W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}$ by a time derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ of the sensor, and setting $\mathbf{H} = H_{u_t} + H_u$.

- in the Goal-Oriented case, $p = 1$, and W^* is the adjoint state, $\mathbf{H}$ is defined as in (25).

4 Loosely coupled Analysis for all-time adaptation

In the sequel we assume that the spatial mesh depends of time:

$$\mathcal{M} : t \mapsto \mathcal{M}(t) \tag{30}$$

in such a way that for any time t we known the number $n(t)$ of nodes

$$n(t) = \mathcal{C}_{spatial}(\mathcal{M}(t)) \tag{31}$$

of the mesh $\mathcal{M}(t)$ [5].

Let N be an integer prescribed by the user, we call *mesh adaptation problem* the following problem:

Find $(\mathcal{M}_h, \tau, nstep)$ which minimizes $\mathcal{E}(\mathcal{M}_h,\tau)$ under the constraint that

$$\mathcal{C}(\mathcal{M}_h, \tau, nstep) = N. \tag{32}$$

In the case where the spatial mesh $\mathcal{M}_h(t)$ is defined via a Riemannian metric $\mathcal{M}(t)$, we have the number of nodes defined by:

$$n_{\mathcal{M}}(t) = \int_\Omega \sqrt{det(\mathcal{M}(x,t)}\mathrm{d}x. \tag{33}$$

Then the global complexity of the space-time mesh is defined as:

$$\mathcal{C}(\mathcal{M}_h, \tau, nstep) = \int_0^T \tau(t)^{-1}\ n_{\mathcal{M}}(t)\ \mathrm{d}t. \tag{34}$$

[5] Assuming that this number is finite for any t and globally bounded over $[0, T]$.

4.1 Analysis for all-time mesh adaptation at a given time

Given $n(t)$ we know the best metric $\mathcal{M}_{opt,n}(t)$ under constraint

$$\mathcal{C}(\mathcal{M}) = n_{\mathcal{M}}(t) = n(t)$$

by minimizing the spatial part of the error:

$$\mathcal{M}_{opt,n}(t) = Arg \min_{\mathcal{M}, n_{\mathcal{M}}=n(t)} \mathcal{E}_{space,t}(t). \tag{35}$$

According to Theorem 4.4.1 in [7], we have:

$$\mathcal{M}_{opt,n}(t) = D_{\mathbf{L}^p} \ \det(\mathbf{H})^{\frac{-1}{2p+3}} \mathbf{H}, \text{ with } D_{\mathbf{L}^p} = n(t)^{\frac{2}{3}} \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \right)^{-\frac{2}{3}}. \tag{36}$$

In the 3D case, the optimal L^p norm of error is given by (4.38) in [7]:

$$\mathcal{E}_{space,t,opt}^{\frac{1}{p}}(t) = 3\, K_x n(t)^{-\frac{2}{3}} \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3p}}. \tag{37}$$

and, recalling that our error functional is the power p of the L^p norm, our corresponding optimal fonctional is given by (in 3D):

$$\mathcal{E}_{space,t,opt}(t) = 3^p\, K_x^p n(t)^{-\frac{2p}{3}} \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \ \mathrm{d}x \right)^{\frac{2p+3}{3}}. \tag{38}$$

We need to define it as a mapping:

$$\begin{array}{l} \mathcal{E}_{space,t,opt} : n \in \mathcal{C}^0[0,T;\mathbb{R}] \mapsto \mathcal{E}_{space,t,opt}(n) \in \mathcal{C}^0[0,T;\mathbb{R}], \\ \mathcal{E}_{space,t,opt}(n) = n^{-\frac{2p}{3}} \ \tilde{h} \\ \tilde{h} = 3^p\, K_x^p \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \ \mathrm{d}x \right)^{\frac{2p+3}{3}} \end{array} \tag{39}$$

where n and $\tilde{h}$ are in $\mathcal{C}^0[0,T;\mathbb{R}]$.

4.2 Analysis for all-time adaptation over time interval

We are now interested in minimizing the space-time error:

$$\mathcal{E}_{space-time} = \int_0^T \left[\mathcal{E}_{space,t,opt}(n) + (K_t \tau^\alpha)^p \int_\Omega \left| W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha} \right|^p \mathrm{d}x \right] \mathrm{d}t.$$

We simplify the second term as follows:

$$\begin{array}{l} (K_t \tau^\alpha)^p \int_\Omega \left| W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha} \right|^p \mathrm{d}x = \tau^{\alpha p} \ \tilde{u}, \\ \text{with } \ \tilde{u} \ = \ K_t^p \int_\Omega \left| W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha} \right|^p \mathrm{d}x \end{array} \tag{40}$$

With these notations, the space-time mesh adaptation problem becomes:

$$\begin{array}{l} \text{Find } \ (n_{opt}, \tau_{opt}) = Arg \min_{n,\tau} \int_0^T n^{-\frac{2p}{3}} \ \tilde{h} + \tau^{\alpha p} \ \tilde{u} dt \\ \text{under the constraint: } \int_0^T n \tau^{-1} \mathrm{d}t = N. \end{array} \tag{41}$$

Let us avoid nonlinear constraints. We decide to search $(n, \zeta = n\tau^{-1})$.

$$\begin{array}{l} \text{Find } \ (n_{opt}, \zeta_{opt}) = Arg \min\limits_{n,\zeta} \mathcal{G}(n,\zeta) \\ \text{with } \ \mathcal{G}(n,\zeta) = \int_0^T f(n) + g(n,\zeta)\mathrm{d}t \\ \text{under the constraint: } \displaystyle\int_0^T \zeta \mathrm{d}t = N \end{array} \tag{42}$$

with:

$$\begin{array}{l} f(n) = n^{-\frac{2p}{3}} \tilde{h} \\ g(n,\zeta) = n^{\alpha p} \zeta^{-\alpha p} \tilde{u}. \end{array} \tag{43}$$

The derivatives of functions f and g are given by:

$$\begin{array}{l} \frac{\partial f}{\partial n} \delta n = -\frac{2p}{3} n^{-\frac{2p+3}{3}} \ \tilde{h} \quad \delta n \\ \frac{\partial g}{\partial n} \delta n = \alpha p \zeta^{-\alpha p} n^{\alpha p - 1} \ \tilde{u} \quad \delta n \\ \frac{\partial g}{\partial \zeta} \delta \zeta = -\alpha p \zeta^{-\alpha p - 1} n^{\alpha p} \ \tilde{u} \ \ \delta \zeta. \end{array} \tag{44}$$

The optimality condition writes:

$$\frac{\partial \mathcal{G}}{\partial (n,\zeta)} \begin{pmatrix} \delta n \\ \delta \zeta \end{pmatrix} = 0 \quad \forall \ \ (\delta n, \delta \zeta) \text{ s.t. } \int_0^T \delta \zeta \mathrm{d}t = 0. \tag{45}$$

In other words:

$$\begin{array}{l} \int_0^T (\frac{\partial f}{\partial n} + \frac{\partial g}{\partial n}) \delta n \mathrm{d}t = 0 \ \ \forall \ \ \delta n \\ \int_0^T (\frac{\partial g}{\partial \zeta}) \delta \zeta \mathrm{d}t = 0 \ \ \forall \ \ \delta \zeta \ \ \text{s.t.} \ \ \int_0^T \delta \zeta \mathrm{d}t = 0 \end{array} \tag{46}$$

or:

$$\begin{array}{l} \frac{\partial f}{\partial n} + \frac{\partial g}{\partial n} = 0 \\ \frac{\partial g}{\partial \zeta} = -C. \end{array} \tag{47}$$

where, according to (44), C is a positive constant not depending in time .

The second equation writes:

$$\tilde{u} n^{\alpha p} \zeta^{-\alpha p - 1} = C. \tag{48}$$

We get:

$$n = \left(\frac{C}{\tilde{u}}\right)^{\frac{1}{\alpha p}} \zeta^{\frac{\alpha p + 1}{\alpha p}} = b \zeta^{\frac{\alpha p + 1}{\alpha p}} \text{ with } b = \left(\frac{C}{\tilde{u}}\right)^{\frac{1}{\alpha p}}. \tag{49}$$

The first equation becomes:

$$-\frac{2p}{3} n^{-\frac{2p+3}{3}} \tilde{h} + \alpha p \ n^{\alpha p - 1} \zeta^{-\alpha p} \tilde{u} \ = \ 0. \tag{50}$$

Putting the value of $n = b\zeta^{\frac{\alpha p + 1}{\alpha p}}$ from (49):

$$\frac{2p}{3} \ b^{-\frac{2p+3}{3}} \zeta^{-\frac{2p+3}{3} \frac{\alpha p + 1}{\alpha p}} \tilde{h} \ = \ \alpha p \ b^{\alpha p - 1} \zeta^{(\alpha p - 1)\frac{\alpha p + 1}{\alpha p} - \alpha p} \tilde{u} \tag{51}$$

then:

$$\zeta^{(\alpha p-1)\frac{\alpha p+1}{\alpha p}-\alpha p+\frac{2p+3}{3}\frac{\alpha p+1}{\alpha p}} = \frac{2}{3\alpha}\frac{b^{-\frac{2p+3}{3}}\tilde{h}}{b^{\alpha p-1}\tilde{u}}. \tag{52}$$

which simplifies as:

$$\zeta^{\frac{(2p+3)\alpha p+2p}{3\alpha p}} = b^{-\frac{3\alpha p+2p}{3}}\frac{2\tilde{h}}{3\alpha\tilde{u}} \tag{53}$$

giving:

$$\zeta = b^{-\frac{3\alpha^2p^2+2\alpha p^2}{(2p+3)\alpha p+2p}}\left(\frac{2\tilde{h}}{3\alpha\tilde{u}}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}} \tag{54}$$

using $b=\left(\frac{C}{\tilde{u}}\right)^{\frac{1}{\alpha p}}$ we get it in terms of C:

$$\zeta = \left[\frac{C^{\frac{1}{\alpha p}}}{\tilde{u}^{\frac{1}{\alpha p}}}\right]^{-\frac{3\alpha^2p^2+2\alpha p^2}{(2p+3)\alpha p+2p}}\left(\frac{2\tilde{h}}{3\alpha\tilde{u}}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}} \tag{55}$$

thus:

$$\zeta = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}C^{-\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}}\ \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{-\frac{3\alpha p}{(2p+3)\alpha p+2p}+\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}} \tag{56}$$

and:

$$\zeta = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}C^{-\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}}\ \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}}. \tag{57}$$

In order to complete this computation, it remains to identify the value of C. This is done by using the prescribed complexity $\mathcal{C}(\mathcal{M},\tau,nstep)$:

$$\int_0^T \zeta\ \mathrm{d}t = N. \tag{58}$$

The general case writes:

$$N = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}C^{-\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}}\int_0^T \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}}\ \mathrm{d}t. \tag{59}$$

$$C^{\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}} = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}N^{-1}\int_0^T \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}}\ \mathrm{d}t, \tag{60}$$

thus:

$$C = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}N^{-\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}\left(\int_0^T \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}}\ \mathrm{d}t\right)^{\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}. \tag{61}$$

$$C = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{3\alpha p+2p}}N^{-\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}\left(\int_0^T \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}}\ \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}}\ \mathrm{d}t\right)^{\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}. \tag{62}$$

The above computations are synthetized in the following lemma:

Lemma 4.1 (Adaptation at each time step) *Let us define the real constant const and the two time dependant functions $\tilde{h}, \tilde{u}$:*

$$\tilde{h}(t) = 2p3^{p-1}K_x^p \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3p}} \quad ; \quad \tilde{u}(t) = \alpha p K_t^p \int_\Omega |(W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha})|^p \, dx \tag{63}$$

the constant C:

$$C = \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{3\alpha p+2p}} N^{-\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}} \left(\int_0^T \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}} \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}} \, dt \right)^{\frac{(2p+3)\alpha p+2p}{3\alpha p+2p}}. \tag{64}$$

and the third time dependant function:

$$b(t) = \left(\frac{C}{\tilde{u}}\right)^{\frac{1}{\alpha p}}. \tag{65}$$

Then for any t the spatial optimal metric is given by:

$$\mathcal{M}_{opt,n}(t) = D_{\mathbf{L}^p} \det(\mathbf{H})^{\frac{-1}{2p+3}} \mathbf{H}, \text{ with } D_{\mathbf{L}^p} = n(t)^{\frac{2}{3}} \left(\int_\Omega \det(\mathbf{H})^{\frac{p}{2p+3}} \right)^{-\frac{2}{3}} \tag{66}$$

with $\mathbf{H} = H_{u_t} + H_u$.
The time-dependant scalar mesh complexity n_{opt} and timestep τ_{opt} are given by:

$$\begin{aligned} \zeta_{opt}(t) &= \left(\frac{2}{3\alpha}\right)^{\frac{3\alpha p}{(2p+3)\alpha p+2p}} C^{-\frac{3\alpha p+2p}{(2p+3)\alpha p+2p}} \tilde{h}^{\frac{3\alpha p}{(2p+3)\alpha p+2p}} \tilde{u}^{\frac{2p}{(2p+3)\alpha p+2p}} \\ n_{opt}(t) &= b\, \zeta^{\frac{\alpha+1}{\alpha p}} \\ \tau_{opt}(t) &= n_{opt}(t)/\zeta_{opt}(t). \square \end{aligned} \tag{67}$$

5 Loosely coupled analysis with time subintervals (GTFP)

In the GTFP ([4, 7]), the time interval $[0.T]$ is split into n_{adap} subintervals:

$$[0.T[= \bigcup_1^{n_{adap}} [t_{i-1}, t_i[, \quad t_0 = 0, t_{n_{adap}} = T.$$

Inside each subinterval the mesh is not changed, but solely when passing from $[t_{i-1}, t_i[$ to $[t_i, t_{i+1}[$. This section presents how to adapt the above results to the GTFP. We assume that the partition $[0.T[= \cup_i [t_{i-1}, t_i[$ is prescribed by the user, e.g. a uniform partition with a specified n_{adap}. The mesh complexity is therefore a constant n_i over each interval $[t_{i-1}, t_i[$:

$$\forall\ i = 1, ..., n_{adap}, \ \forall\ t \in [t_{i-1}, t_i[, \ n(t) = n_i \in \mathbb{R}. \tag{68}$$

We assume that the time-dependant metric is frozen on $[t_{i-1}, t_i[$, in other word, the previous context is restricted to a time-dependant metric expressed in terme of a set of n_{adap} metrics $(\mathcal{M}^i, i = 1, n_{adap})$:

$$\mathcal{M}(t) = \mathcal{M}^{i(t)}$$

where the index $i(t)$ varies in time t in such a way that:

$$\text{If } t \in [t_{i-1}, t_i[, \text{ then } i(t) = i.$$

We are again interested in minimizing the *loosely coupled error*

$$\begin{array}{l}\mathcal{E}(\mathcal{M}_h,\tau,nstep)=\mathcal{E}_{temp}(\mathcal{M}_h,\tau,nstep)+\mathcal{E}_{space}(\mathcal{M}_h,\tau,nstep)\\ \mathcal{E}_{temp}(\mathcal{M}_h,\tau,nstep)=\int_0^T\int_{-\infty}^{+\infty}[K_t\tau^\alpha|W^*\cdot\frac{\partial^\alpha W}{\partial t^\alpha}|]^p\mathrm{d}t\mathrm{d}x\\ \mathcal{E}_{space}(\mathcal{M}_h,\tau,nstep)=\\ \int_0^T\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i(t)})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i(t)})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}.\mathrm{d}t\end{array}\tag{69}$$

the second term is also written as:

$$\begin{array}{l}\mathcal{E}_{space}(\mathcal{M},\tau,nstep)=\sum_{i=1}^{n_{adap}}\int_{t_{i-1}}^{t_i}\mathcal{E}^i(t)\mathrm{d}t,\\ \mathcal{E}^i(t)=\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i(t)})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i(t)})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}\\ \quad=\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}x.\end{array}\tag{70}$$

5.1 Spatial mesh optimization on a subinterval

Given i, $1\le i\le n_{adap}$, and a mesh complexity n_i for the adapted mesh used during time sub-interval $[t_{i-1},t_i]$, we seek for the optimal continuous mesh $\mathcal{M}^i_{opt}$ solution of the following problem:

$$\begin{array}{l}\min\limits_{\mathcal{M}^i}\mathcal{E}^i(\mathcal{M}^i)=\displaystyle\int_{t_{i-1}}^{t_i}\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{t}\\ \quad\text{such that}\quad\mathcal{C}(\mathcal{M}^i)=n_i\,,\end{array}\tag{71}$$

in which solely $\mathbf{H}$ depends of time. A conservative option for bounding this error functional is to introduce the matrix $\mathbf{H}^i_{inter}$ is defined by

$$H^i_{inter}(\mathbf{x})=(t_i-t_{i-1})\max_{t\in[t_{i-1},t_i]}\mathbf{H}(\mathbf{x},t),$$

the max being obtained by *metric intersection* (see chapter 3 of [7] or see [1]). Then each $\mathcal{E}^i(t)$ depends on i but not of t.

$$\begin{array}{l}\int_{t_{i-1}}^{t_i}\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{t}\\ \le\ \int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\ \max_{t\in[t_{i-1},t_i]}\mathbf{H}(\mathbf{x},t)\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}\int_{t_{i-1}}^{t_i}\mathrm{d}t\\ =\ \int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}^i_{inter}(\mathbf{x})\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}.\end{array}\tag{72}$$

Also the subinterval optimality problem becomes:

$$\begin{array}{l}\min\limits_{\mathcal{M}^i}\mathcal{E}^i(\mathcal{M}^i)=\displaystyle\int_\Omega\left[\mathrm{trace}\left((\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\,\mathbf{H}^i_{inter}(\mathbf{x})\,(\mathcal{M}^{i})^{-\frac{1}{2}}(\mathbf{x})\right)\right]^p\mathrm{d}\mathbf{x}\\ \quad\text{such that}\quad\mathcal{C}(\mathcal{M}^i)=n_i\,,\end{array}\tag{73}$$

Minimizing as previously, but in interval $[t_{i-1},t_i[$, we get:

Lemma 5.1

$$\begin{array}{l}\mathcal{M}^i_{opt}(\mathbf{x})=(n_i)^{\frac{2}{3}}\ \mathcal{M}^i_1(\mathbf{x})\\ \quad\textit{with}\ \mathcal{M}^i_1(\mathbf{x})=\left(\int_\Omega(\det\mathbf{H}^i_{inter}(\bar{\mathbf{x}}))^{\frac{p}{2p+3}}\mathrm{d}\bar{\mathbf{x}}\right)^{-\frac{2}{3}}\ (\det\mathbf{H}^i_{inter}(\mathbf{x}))^{-\frac{1}{2p+3}}\ \mathbf{H}^i_{inter}(\mathbf{x}).\end{array}\tag{74}$$

The corresponding optimal error $\mathcal{E}^i(\mathcal{M}^i)$ writes:

$$\mathcal{E}^i(\mathcal{M}^i_{opt})=3^p\,K_x^p\,(n_i)^{-\frac{2p}{3}}\left(\int_\Omega(\det\mathbf{H}^i_{inter}(\mathbf{x}))^{\frac{p}{2p+3}}\mathrm{d}\mathbf{x}\right)^{\frac{2p+3}{3}}.\square$$

5.2 Temporal optimization over the time subintervals

Now $\mathbf{n} = n_1, ..., n_{adap}$ is a vector of n_{adap} scalar components, while τ remains a time-dependant function. For our formulation, we use the time-dependant function $n(t)$ defined from vector $\mathbf{n}$ via (68). We also define the spatial error as parameterized by n:

$$\mathcal{E}_{space,opt}(n) = \sum_{i=1}^{n_{adap}} \mathcal{E}^i(\mathcal{M}^i_{opt}) \tag{75}$$

or:

$$\mathcal{E}_{space}(\mathcal{M}, \tau, nstep) = \sum_{i=1}^{n_{adap}} \mathcal{E}^i(\mathcal{M}^i_{opt})$$

The space-time mesh adaptation problem becomes:

$$\begin{array}{l} \text{Find} \quad (\mathbf{n}_{opt}, \tau_{opt}) = Arg \min\limits_{\mathbf{n},\tau} \overline{\mathcal{G}}(\mathbf{n}, \tau) \\ \text{with} \quad \overline{\mathcal{G}}(\mathbf{n}, \tau) = \mathcal{E}_{space,opt}(n) + \int_0^T (K_t \tau^\alpha)^p \int_\Omega \left(|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}| \right)^p \mathrm{d}x\mathrm{d}t \\ \text{under the constraint:} \displaystyle\int_0^T n\tau^{-1}\mathrm{d}t = N, \quad \text{and} \quad n \quad \text{being defined by (68).} \end{array} \tag{76}$$

Let us change our variables:

$$\begin{array}{l} (\mathbf{n}, \zeta) = (\mathbf{n}, n\tau^{-1}) \\ \mathcal{G}(\mathbf{n}, \zeta) = \overline{\mathcal{G}}(\mathbf{n}, n/\zeta) \end{array} \tag{77}$$

then

$$\mathcal{G}(\mathbf{n}, \zeta) = f(\mathbf{n}) + g(\mathbf{n}, \zeta). \tag{78}$$

with:

$$\begin{array}{l} f(n) = \mathcal{E}_{space,opt}(n) \\ g(n, \zeta) = (K_t n^\alpha \zeta^{-\alpha})^p \int_\Omega (|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|)^p \mathrm{d}x. \end{array} \tag{79}$$

Then we have to:

$$\begin{array}{l} \text{Find} \quad (\mathbf{n}_{opt}, \zeta_{opt}) \;=\; Arg \min\limits_{\mathbf{n}_{opt},\zeta} \mathcal{G}(\mathbf{n}_{opt}, \zeta) \\ \text{with} \quad \mathcal{G}(\mathbf{n}, \zeta) \;=\; f(n) \;+\; \int_0^T g(n, \zeta)\mathrm{d}t \\ \text{under the constraint:} \int_0^T \zeta \mathrm{d}t = N. \end{array} \tag{80}$$

The derivatives of these functions are given by:

$$\begin{array}{l} \frac{\partial f}{\partial n}\delta n = -2\, 3^{p-1}\; p\; K_x^p \; \sum_{i=1}^{n_{adap}} \; n_i^{-\frac{2p+3}{3}} \left(\int_\Omega \det \left(|\mathbf{H}^i_{inter}| \right)^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3}} \quad \delta n_i \\ \frac{\partial g}{\partial n}\delta n = \alpha p n^{\alpha p-1} (K_t \zeta^{-\alpha})^p \int_\Omega (|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|)^p \mathrm{d}x \quad \delta n \\ \frac{\partial g}{\partial \zeta}\delta\zeta = -\alpha p \zeta^{-\alpha p-1} (K_t n^\alpha)^p \int_\Omega (|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|)^p \mathrm{d}x \quad \delta\zeta. \end{array} \tag{81}$$

Introducing:

$$\tilde{h}^i_{inter} = 2\; 3^{p-1} p K_x^p \left(\int_\Omega \det \left(|\mathbf{H}^i_{inter}| \right)^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3}} \tag{82}$$

and:

$$\tilde{u}(t) = \alpha p K_t^p \int_\Omega (|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|)^p \mathrm{d}x \tag{83}$$

The derivatives in (81) become:

$$\begin{array}{lll} \frac{\partial f}{\partial n}\delta n = & - \sum_{i=1}^{n_{adap}} n_i^{-\frac{2p+3}{3}} \tilde{h}^i_{inter} \delta n_i \\ \frac{\partial g}{\partial n}\delta n = & \tilde{u}(t)\, n^{\alpha p-1}\zeta^{-\alpha p} \quad \delta n \\ \frac{\partial g}{\partial \zeta}\delta\zeta = & - \tilde{u}(t)\, \zeta^{-\alpha p-1}\, n^{\alpha p} \quad \delta\zeta. \end{array} \tag{84}$$

The optimality condition writes:

$$\begin{array}{l} \dfrac{\partial \mathcal{G}}{\partial(n,\zeta)} \begin{pmatrix} \delta n \\ \delta\zeta \end{pmatrix} = 0 \quad \forall \ (\delta n, \delta\zeta) \text{ s.t.} \\ \int_0^T \delta\zeta \mathrm{d}t = 0, \\ \forall i = 1, ..., n_{adap}, \forall t \in [t_{i-1}, t_i[, \quad \delta n(t) = \delta n(t_{i-1}). \end{array} \tag{85}$$

Or:

$$\begin{array}{l} \frac{\partial f}{\partial n}\delta n + \int_0^T \frac{\partial g}{\partial n}\delta n \mathrm{d}t = 0 \ \ \forall \ \ \delta n, \ \delta n(t) = \delta n(t_{i-1}) \\ \frac{\partial g}{\partial \zeta} = -const. \end{array} \tag{86}$$

Second equation. Using the notation:

$$b(t) = \Big(\frac{const.}{\tilde{u}}\Big)^{\frac{1}{\alpha p}} \tag{87}$$

we solve the second equation

$$\tilde{u}\zeta^{-\alpha p-1} n^{\alpha p} = const.$$

and get:

$$n = b\ \zeta^{\frac{\alpha p+1}{\alpha p}} \quad \text{or} \quad \zeta = (b^{-1}\ n)^{\frac{\alpha p}{\alpha p+1}}. \tag{88}$$

Remark 5.1 *For a smooth function u, b is smooth and therefore, if* $\mathbf{n}$ *is not a vector with components all equal, which is necessary for ensuring that $t \mapsto n(t)$ is continuous function, the time dependant function ζ will present discontinuities at subinterval limits t_i.*

First equation. The first equation writes:

$$\begin{array}{l} \displaystyle\sum_{i=1}^{i=n_{adap}} (\frac{\partial f}{\partial n_i} + \int_{t_{i-1}}^{t_i} \frac{\partial g}{\partial n})\delta n_i \mathrm{d}t = 0 \ \ \forall \ \ \delta n_i \in \mathbb{R}, \\ \Leftrightarrow \ \ (\displaystyle\frac{\partial f}{\partial n_i} + \int_{t_{i-1}}^{t_i} \frac{\partial g}{\partial n})\mathrm{d}t = 0 \ \ \forall i = 1, ..., n_{adap}. \end{array} \tag{89}$$

It comes:

$$-\, n_i^{-\frac{2p+3}{3}} \tilde{h}_{inter} + \int_{t_{i-1}}^{t_i} n_i^{\alpha p-1}\zeta^{-\alpha p}\tilde{u}\ \mathrm{d}t = 0 \ \ \forall i = 1, ..., n_{adap} = \ 0. \tag{90}$$

We inject (88):

$$-\, n_i^{-\frac{2p+3}{3}} \tilde{h}^i_{inter} + \int_{t_{i-1}}^{t_i} n_i^{\alpha p-1} b^{+\frac{\alpha^2 p^2}{\alpha p+1}} n_i^{-\frac{\alpha^2 p^2}{\alpha p+1}} \tilde{u}\ \mathrm{d}t = 0 \ \ \forall i = 1, ..., n_{adap} = \ 0 \tag{91}$$

then:

$$n_i^{-\frac{2p+3}{3}} \tilde{h}^i_{inter} \ = \ n_i^{\alpha p-1-\frac{\alpha^2 p^2}{\alpha p+1}} \int_{t_{i-1}}^{t_i} \tilde{u} b^{+\frac{\alpha^2 p^2}{\alpha p+1}}\ \mathrm{d}t$$

$$n_i^{-\frac{2p+3}{3}-\alpha p+1+\frac{\alpha^2p^2}{\alpha p+1}} = \Big[\int_{t_{i-1}}^{t_i} \tilde{u} b^{+\frac{\alpha^2p^2}{\alpha p+1}}\ \mathrm{d}t\Big]\Big[\int_{t_{i-1}}^{t_i} \tilde{h}\ \mathrm{d}t\Big]^{-1}$$

or, using that:

$$\begin{aligned}&(-2p-3)(\alpha p+1)+(-\alpha p+1)(3\alpha p+3)+3\alpha^2p^2 = \\ &-2\alpha p^2-2p-3\alpha p-3-3\alpha^2p^2-3\alpha p+3\alpha p+3+3\alpha^2p^2 = \\ &-2\alpha p^2-3\alpha p-2p = -(2\alpha p^2+3\alpha p+2p)\end{aligned} \tag{92}$$

we get

$$n_i = \Big[\int_{t_{i-1}}^{t_i} \tilde{u} b^{\frac{\alpha^2p^2}{\alpha p+1}}\ \mathrm{d}t\Big]^{-\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}}\Big[\tilde{h}^i_{inter}\Big]^{\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}} \tag{93}$$

and getting rid of the notation "b":

$$n_i = \Big[\int_{t_{i-1}}^{t_i} \Big(\frac{const.}{\tilde{u}}\Big)^{\frac{\alpha p}{\alpha p+1}} \tilde{u}\ \mathrm{d}t\Big]^{-\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}}\Big[\tilde{h}^i_{inter}\Big]^{\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}} \tag{94}$$

and extracting "const." (not depending on time): [6]

$$\begin{aligned}&n_i = D_i\ const.^{-\frac{3\alpha}{2\alpha p+3\alpha+2}} \\ &\text{with:} \\ &D_i = \Big[\textstyle\int_{t_{i-1}}^{t_i} \tilde{u}^{\frac{1}{\alpha p+1}}\ \mathrm{d}t\Big]^{-\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}}\Big[\tilde{h}^i_{inter}\Big]^{\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}}.\end{aligned} \tag{95}$$

In the continuous metric theory, the instantaneous complexity $n(t)$ (deduced from n_i's via (68)) is not necessarily an integer: solely the discretization of the optimal metric system will have integer complexities. In particular, the above values of $n(t)$ can be directly injected in the definition of the timestep $\tau = n/\zeta$ using (88):

$$\tau = n(b^{-1}\ n)^{-\frac{\alpha p}{\alpha p+1}} = b^{+\frac{\alpha p}{\alpha p+1}} n^{\frac{1}{\alpha p+1}} = \Big(\frac{const.}{\tilde{u}(t)}\Big)^{\frac{1}{\alpha p+1}} n^{\frac{1}{\alpha p+1}}. \tag{96}$$

Complexity constraint. In order to complete this computation, it remains to identify the value of *const.*. This is done by using the prescribed complexity $\mathcal{C}(\mathcal{M},\tau,nstep)$ ((4)(32)):

$$\int_0^T n\tau^{-1}\ \mathrm{d}t = N \tag{97}$$

which turns to be:

$$N = \int_0^T n\Big(\frac{const.}{\tilde{u}(t)}\Big)^{-\frac{1}{\alpha p+1}} n^{-\frac{1}{\alpha p+1}}\ \mathrm{d}t = \int_0^T \Big(\frac{const.}{\tilde{u}(t)}\Big)^{-\frac{1}{\alpha p+1}} n^{\frac{\alpha p}{\alpha p+1}}\ \mathrm{d}t \tag{98}$$

where we can introduce the n_i's:

$$N = \sum_{i=1}^{i=n_{adap}} \int_{t_{i-1}}^{t_i} \Big(\frac{const.}{\tilde{u}(t)}\Big)^{-\frac{1}{\alpha p+1}} n_i^{\frac{\alpha p}{\alpha p+1}}\ \mathrm{d}t \tag{99}$$

[6] After simplification:
$-\frac{(\alpha p)(3\alpha p+3)}{(\alpha p+1)(2\alpha p^2+3\alpha p+2p)} = -\frac{3\alpha}{2\alpha p+3\alpha+2}$.

replacing n_i from (95): [7]

$$N = \sum_{i=1}^{i=n_{adap}} \int_{t_{i-1}}^{t_i} \left(\frac{const.}{\tilde{u}(t)}\right)^{-\frac{1}{\alpha p+1}} D_i^{\frac{\alpha p}{\alpha p+1}} const.^{-\frac{3\alpha^2 p}{(\alpha p+1)(2\alpha p+3\alpha+2)}} \, \mathrm{d}t \tag{100}$$

thus: [8]

$$\begin{aligned} & N = E\ const.^{\beta} \\ & \text{with:} \\ & \beta = -\frac{3\alpha+2}{2\alpha p+3\alpha+2} \\ & E = \sum_{i=1}^{i=n_{adap}} \int_{t_{i-1}}^{t_i} \tilde{u}(t)^{\frac{1}{\alpha p+1}} D_i^{\frac{\alpha p}{\alpha p+1}} \, \mathrm{d}t \end{aligned} \tag{101}$$

Then:

$$const. = E^{-\frac{1}{\beta}} \ N^{\frac{1}{\beta}} \tag{102}$$

[7] With the simplification:
$-\frac{(\alpha p)^2(3\alpha p+3)}{(\alpha p+1)^2(2\alpha p^2+3\alpha p+2p)} = -\frac{3\alpha^2 p}{(\alpha p+1)(2\alpha p+3\alpha+2)}$.

[8] Using the simplification:
$-\beta = \frac{1}{\alpha p+1} + \frac{3\alpha^2 p}{(\alpha p+1)(2\alpha p+3\alpha+2)} = \frac{2\alpha p+3\alpha+2+3\alpha^2 p}{(\alpha p+1)(2\alpha p+3\alpha+2)} = \frac{3\alpha+2}{2\alpha p+3\alpha+2}$.

Lemma 5.2 (GTFP with time adaptation) *Let us define the two real constants β, const., and E, the two time-dependant functions $\tilde{h}, \tilde{u}$ and the vector D_i:*

$$\beta = -\frac{1}{\alpha p + 1} - \frac{(\alpha p)^2(3\alpha p + 3)}{(\alpha p + 1)^2(2\alpha^2 p^2 + 3\alpha p + 2p)} \tag{103}$$

$$\begin{aligned} \tilde{h}^i_{inter} &= 2\ 3^{p-1} p K_x^p \left(\int_\Omega \det\left(|\mathbf{H}_{inter}|\right)^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3}} \\ \tilde{u}(t) &= \alpha p K_t^p \int_\Omega (|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}|)^p \, dx \end{aligned} \tag{104}$$

$$D_i = \left[\int_{t_{i-1}}^{t_i} \tilde{u}^{\frac{1}{\alpha p+1}} \ dt \right]^{-\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}} \left[\tilde{h}^i_{inter} \right]^{\frac{3\alpha p+3}{2\alpha p^2+3\alpha p+2p}} \quad i = 1, n_{adap} \tag{105}$$

$$E = \sum_{i=1}^{i=n_{adap}} \int_{t_{i-1}}^{t_i} \tilde{u}(t)^{\frac{1}{\alpha p+1}} D_i^{\frac{\alpha p}{\alpha p+1}} \ dt \tag{106}$$

$$const. = E^{-\frac{1}{\beta}} \ N^{\frac{1}{\beta}}. \tag{107}$$

Then the optimal adaptation is defined by:

- (i) the spatial complexity $n \mapsto n_{opt}(t)$ and the timestep $t \mapsto \tau(t)_{opt}$:

$$\begin{aligned} n_{i,opt} &= D_i \ const.^{-\frac{3\alpha}{2\alpha p+3\alpha+2}} \\ \tau_{opt}(t) &= \left(\frac{const.}{\tilde{u}(t)} \right)^{\frac{1}{\alpha p+1}} (n_{opt}(t))^{\frac{1}{\alpha p+1}}, \end{aligned} \tag{108}$$

- (ii) the spatial metric defined in each i-th time interval $[t_{i-1}, t_i[$:

$$\begin{aligned} &\mathcal{M}^i_{opt}(\mathbf{x}) = (n_i)^{\frac{2}{3}} \ \mathcal{M}^i_1(\mathbf{x}) \\ &\textit{with } \mathcal{M}^i_1(\mathbf{x}) = \left(\int_\Omega (\det \mathbf{H}^i_{inter}(\bar{\mathbf{x}}))^{\frac{p}{2p+3}} \mathrm{d}\bar{\mathbf{x}} \right)^{-\frac{2}{3}} (\det \mathbf{H}^i_{inter}(\mathbf{x}))^{-\frac{1}{2p+3}} \ \mathbf{H}^i_{inter}(\mathbf{x}) \\ &\textit{and } H^i_{inter}(\mathbf{x}) = (t_i - t_{i-1}) \max_{t \in [t_{i-1}, t_i]} \mathbf{H}(\mathbf{x}, t). \square \end{aligned} \tag{109}$$

Corollary 5.1 *The optimal error is given by:*

$$\begin{aligned} \mathcal{E}(\mathcal{M}_{opt}, n_{opt}, \tau_{opt}) &= \sum_{i=1}^{i=n_{adap}} \mathcal{E}^i(\mathcal{M}^i_{opt}) + \mathcal{E}_{time}(\mathcal{M}_{opt}, n_{opt}, \tau_{opt}) \\ \mathcal{E}^i(\mathcal{M}^i_{opt}) &= 3^p \ K_x^p \ (n_{i,opt})^{-\frac{2p}{3}} \left(\int_\Omega (\det \mathbf{H}^i_{inter}(\mathbf{x}))^{\frac{p}{2p+3}} \mathrm{d}\mathbf{x} \right)^{\frac{2p+3}{3}} \\ \mathcal{E}_{time}(\mathcal{M}_{opt}, n_{opt}, \tau_{opt}) &= \int_0^T (K_t \tau^\alpha_{opt})^p \int_\Omega \left(|W^* \cdot \frac{\partial^\alpha W}{\partial t^\alpha}| \right)^p dx \, dt. \square \end{aligned} \tag{110}$$

Restricting to $\alpha = 2$ the above lemma writes:

Lemma 5.3 (GTFP with time adaptation for $\alpha = 2$) *Let us define E, the two time-dependant functions $\tilde{h}, \tilde{u}$ and the vector D_i:*

$$\begin{aligned} \tilde{h}_i &= 2p \ 3^{p-1} K_x^p \left(\int_\Omega \det\left(|\mathbf{H}^i_{inter}|\right)^{\frac{p}{2p+3}} \right)^{\frac{2p+3}{3}} \\ \tilde{u}(t) &= 2p K_t^p \int_\Omega \left(\left| \frac{\partial^2 u}{\partial t^2} \right| \right)^p dx \end{aligned} \tag{111}$$

$$D_i = \tilde{h}_i^{\frac{3(2p+1)}{4p(p+2)}} \left(\int_{t_{i-1}}^{t_i} \tilde{u}^{\frac{1}{2p+1}} \right)^{-\frac{3(2p+1)}{4p(p+2)}} \ , \ E = \sum_{i=1}^{n_{adapt}} \tilde{h}_i^{\frac{3}{2(p+2)}} \left(\int_{t_{i-1}}^{t_i} \tilde{u}^{\frac{1}{2p+1}} \right)^{\frac{2p+1}{2(p+2)}}$$

Then the optimal adaptation is defined by:

- *(i) the spatial complexity $n \to n_{opt}(t)$ and the timestep $t \to \tau(t)_{opt}$:*

$$\begin{aligned} n_{i,opt} &= D_i \, E^{-\frac{3}{4}} N^{\frac{3}{4}} \\ \tau_{opt}(t) &= \left(\frac{D_i}{\tilde{u}(t)} \right)^{\frac{1}{2p+1}} E^{\frac{1}{4}} N^{-\frac{1}{4}}, \end{aligned} \tag{112}$$

- *(ii) the spatial metric defined in each i-th time interval $[t_{i-1}, t_i]$:*

$$\begin{aligned} &\mathcal{M}^i_{opt}(\mathbf{x}) = (n_i)^{\frac{2}{3}} \ \mathcal{M}^i_1(\mathbf{x}) \\ &\textit{with } \mathcal{M}^i_1(\mathbf{x}) = \left(\int_\Omega (\det \mathbf{H}^i_{inter}(\bar{\mathbf{x}}))^{\frac{p}{2p+3}} \mathrm{d}\bar{\mathbf{x}} \right)^{-\frac{2}{3}} (\det \mathbf{H}^i_{inter}(\mathbf{x}))^{-\frac{1}{2p+3}} \, \mathbf{H}^i_{inter}(\mathbf{x}) \\ &\textit{and } H^i_{inter}(\mathbf{x}) = (t_i - t_{i-1}) \max_{t \in [t_{i-1}, t_i]} \mathbf{H}(\mathbf{x}, t). \end{aligned} \tag{113}$$

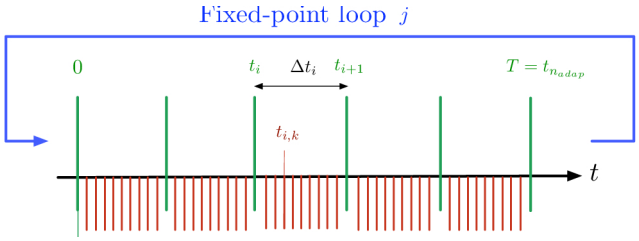

Figure 1: Time splitting of the Transient Fixed Point mesh adaptation algorithm. Sub-intervals (in green) used for the transient process and timesteps (in red).

6 Algorithm

The Transient Fixed Point algorithm was proposed for specifying automatically a *succession of* n_{adap} *meshes* over a decomposition in sub-intervals used for the transient process (Figure 1). The flowchart is presented Figure 2.

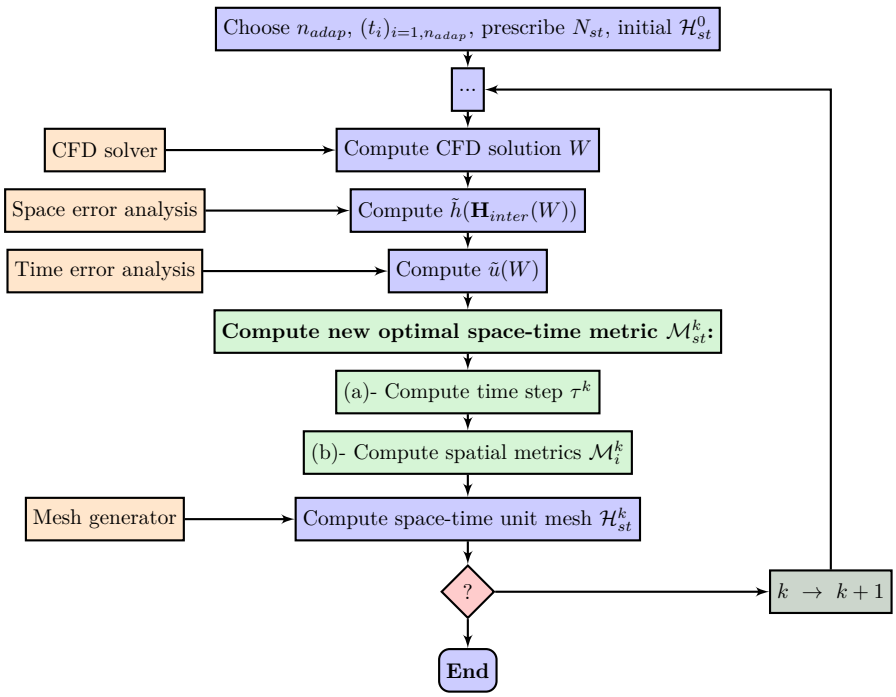

Figure 2: Space-time transient fixed Point algorithm

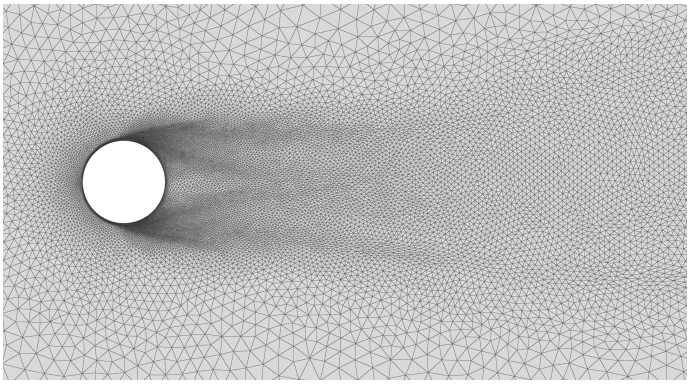

Figure 3: Cylinder at Reynolds number 3900. Mesh.

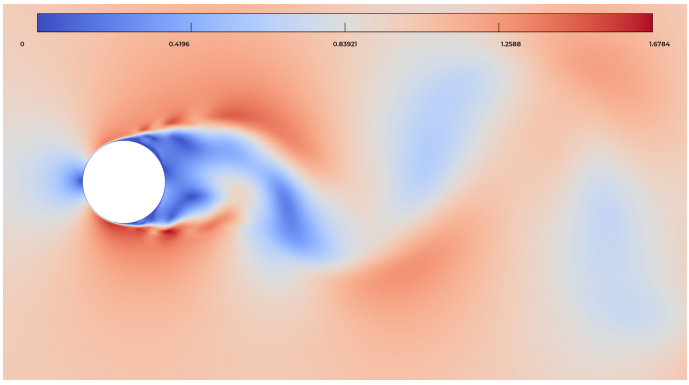

Figure 4: Cylinder at Reynolds number 3900. Velocity norm.

7 Numerical experiments

7.1 First numerical experiment

We consider the 2D computation of a flow around a cylinder at Reynolds number 3900, Mach number 0.1, with Spalart-Allmaras turbulence model.
Mesh adaptation options are :
- only one adapted spatial mesh, i.e. $n_{adap} = 1$
- Space-Time complexity N_{st} is prescribed to 10M, 20M, 100M, 200M.
The evolution of meshes and errors is presented in Table 1

7.2 Second numerical experiment

We consider now the 2D computation of a flow around a cylinder at Reynolds number 1M, Mach number 0.1, with Spalart-Allmaras turbulence model.
Mesh adaptation options are :
- only one adapted spatial mesh,
- Space-Time complexity is prescribed to 10M, 20M, 100M, 200M.

N_{st}	k TFP	$\mathcal{C}_{space}(nodes)$	# timesteps	$\mathcal{E}_{space}$	$\mathcal{E}_{time}$
$10M$	1	27K(63K)	361	$5.2\ 10^{-3}$	$3.1\ 10^{-5}$
$10M$	5	60K(64K)	166	$2.9\ 10^{-3}$	$3.1\ 10^{-3}$
$20M$	1	44K	456	$3.3\ 10^{-3}$	$3.1\ 10^{-5}$
$20M$	5	84K(86K)	237	$2.\ 10^{-3}$	$8.2\ 10^{-4}$
$100M$	1	128K	780	$1.\ 10^{-3}$	$3.1\ 10^{-5}$
$100M$	5	173K(186K)	575	$9.4\ 10^{-4}$	$8.8\ 10^{-5}$
$200M$	1	203K	982	$7.1\ 10^{-4}$	$3.1\ 10^{-5}$
$200M$	5	225K	887	$7.1\ 10^{-4}$	$4.\ 10^{-5}$

Table 1: **Space-time statistics for circular cylinder case at Reynolds number** 3900. The space-time convergence order is about 1.5.

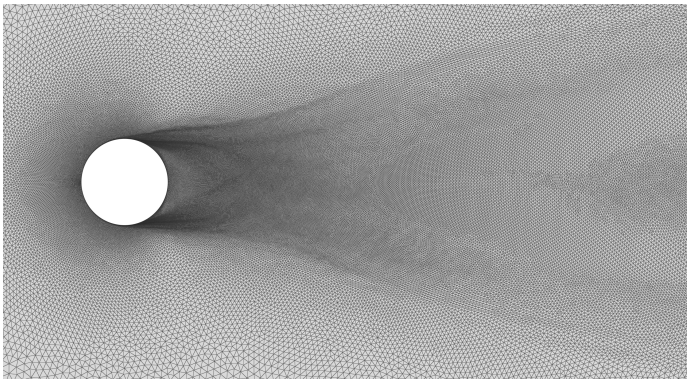

Figure 5: .

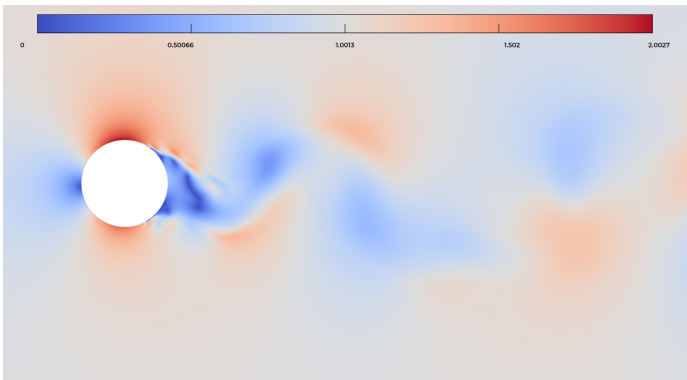

Figure 6: .

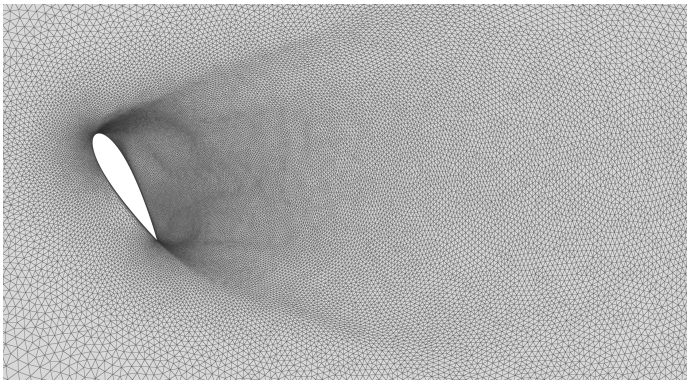

Figure 7: .

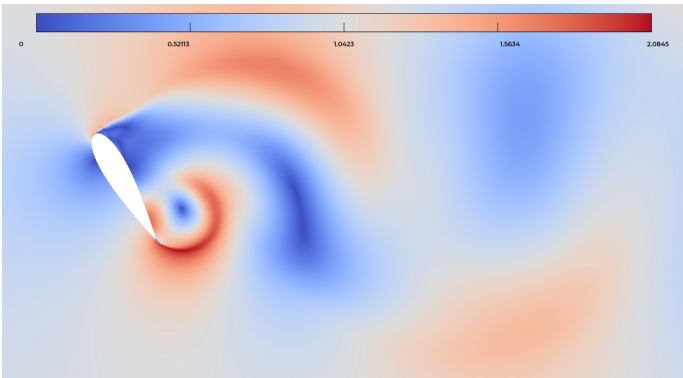

Figure 8: .

7.3 Third numerical experiment

It concerns the 2D computation of a flow around a NACA0021 at Reynolds number 270K, Mach number 0.1, with Spalart-Allmaras turbulence model.
Mesh adaptation options are :
- only one adapted spatial mesh,
- Space-Time complexity is prescribed to 10M, 20M, 100M, 200M.

We observe in Figure 9 the tendance of the algorithm to equilibrate the time and space error. Figure 10 shows the timestep size for the initial computation with a coarse mesh of 1766 vertices and a CFL of 50. It corresponds to 1132 timesteps. The optimal timestep size computed by the algorithm for a much finer mesh of 16K vertices is between three and four times larger (123 timesteps).

N_{st}	k TFP	$\mathcal{C}_{space}(nodes)$	# timesteps	$\mathcal{E}_{space}$	$\mathcal{E}_{time}$
$2M$	1	1766	1132	$1.7\ 10^{-1}$	$2.4\ 10^{-6}$
$2M$	10	16K	123	$2.6\ 10^{-2}$	$1.3\ 10^{-2}$
$4M$	1	2804	1426	$1.\ 10^{-1}$	$2.4\ 10^{-6}$
$4M$	10	23K	172	$1.9\ 10^{-2}$	$4.6\ 10^{-2}$
$8M$	1	4451	1797	$6.8\ 10^{-2}$	$2.4\ 10^{-6}$
$8M$	10	31K	261	$1.4\ 10^{-2}$	$6.\ 10^{-3}$
$16M$	1	7066	2264	$4.3\ 10^{-2}$	$2.4\ 10^{-6}$
$16M$	10	42K	384	$1.1\ 10^{-2}$	$2.3\ 10^{-3}$
$32M$	1	11K	2852	$2.7\ 10^{-2}$	$2.4\ 10^{-6}$
$32M$	20	57K	558	$7.5\ 10^{-3}$	$1.8\ 10^{-3}$

Table 2: **Space-time statistics for NACA0021 at Reynolds number** $270K$.

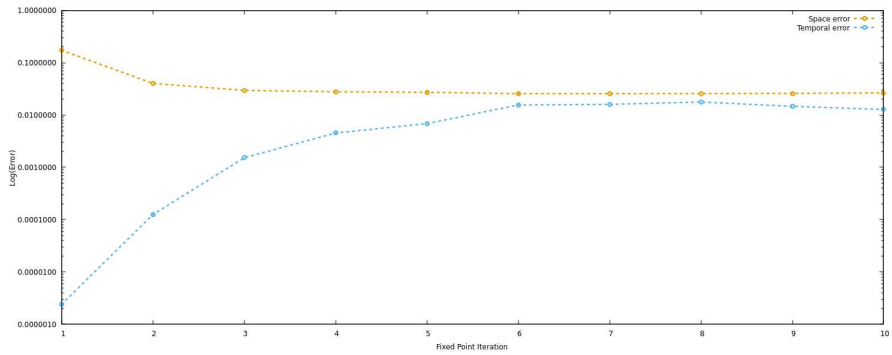

Figure 9: Evolution of theorical space error $\mathcal{E}_{space}$ and time error $\mathcal{E}_{time}$ with TFP iterations.

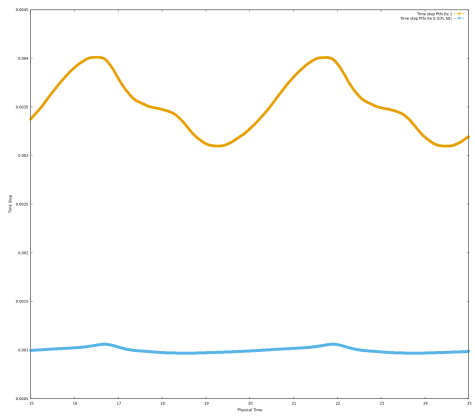

Figure 10: Blue: Timestep lengths of initial flow at CFL=50, 1132 timesteps on 1766 vertices, and orange: first timestep lengths proposed by the adaptation algorithm, 123 timesteps on 16K vertices.

8 Concluding remarks

We have presented a proposition for the space-time metric based mesh optimization for a CFD model. This proposition extends both the methods of Goal-Oriented Transient Fixed Point [5] and Feature-based Global TFP of [4]. In particular, both Feature-Based and Goal-Oriented criteria are taken into account.

Several applications are demonstrated. Best time steps are obtained in a few TFP iterations, with CPU improvement. When the number of spatial meshes is $n_{adap} = 1$, we observe a space-time convergence order of about 1.5. In order to get higher order one needs to increase progressively n_{adap}.

We emphasize that several mesh constraints still limit the accuracy order:
- the choice of a low number n_adap for the different spatial meshes used in the time interval,
- the particular family of space-time meshes imposed by the time-advancing option.

The performance of the proposed algorithme enables to consider in a future work its extension to Large Eddy Simulation.

9 Acknowledgements

This work was supported by the ANR NORMA project, grant ANR-19-CE40-0020-01 of the French National Research Agency.

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ISSN 0249-6399