Implementation of the $k-\omega$ SST model in Aironum

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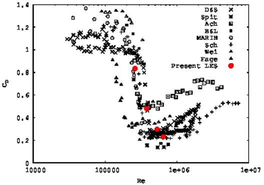


(Drag crisis : Experimental data as of 2014)

Slide-0: k-ω SST model

The k-ω SST model
is a hybrid model obtained by blending the standard k-ϵ and k-ω models to eliminate the disadvantages of each model when applied separately, which are:

- k-ϵ model: poor performance with strong adverse pressure gradients.
- k-ω model: main disadvantage, sensitivity to the far-field BC on ω.

Pressure gradient	k-ω	k-ϵ
favorable	2.6%	7.2%
mild adverse	3.2%	27.2%
moderate adverse	5.9%	39.8%
strong adverse	2.4%	41.8%
all	3.5%	29.0%

Table 1: Wilcox(1998) Computed skin friction errors using the k-ϵ and k-ω models.
What are the standard k-ϵ and k-ω models?
We start by discussing the k-ϵ and the k-ω models, then the k-ω SST model.

Slide-1: k-ϵ model - early development

Is there a standard k-ϵ model? Not really, many contributors
Early contributors to the k-ϵ model - Wilcox(1998)

- Chou(1945)
- Davidov(1961)
- Harlow and Nakayama(1968)
- Jones and Launder(1972) (often referred to as the standard k-ϵ model)
- Launder and Sharma(1974) (strictly speaking, the standard k-ϵ model.)
  1) added viscous damping terms absent in the Jones and Launder model
  2) retuned model coefficients.

Slide-2: Standard k-ϵ model of Jones and Launder(1972)

Wilcox (1998, p 124) written in non-conservative variables k,ϵ

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \epsilon$$
$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right] + C_{\epsilon 1}\frac{\epsilon}{k}P_k - C_{\epsilon 2}\frac{\epsilon^2}{k} \tag{0}$$

where k = the turbulent kinetic energy

ϵ = the turbulence dissipation rate

P_k = the turbulence production term

ν_t = the turbulent eddy viscosity

$$P_k = \tau_{ij}\frac{\partial U_i}{\partial x_j}, \quad \nu_t = C_\mu \frac{k^2}{\epsilon}, \tag{1}$$

The constants C_μ, σ_k, σ_ϵ, $C_{\epsilon 1}$, and $C_{\epsilon 2}$ will be discussed later.

Slide-2 continued: Standard k-ϵ model of Jones and Launder(1972)

Closure Coefficients and Auxiliary Relations

$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

$$\omega = \epsilon/(C_\mu k) \;\text{ and }\; l = C_\mu \frac{k^{3/2}}{\epsilon}$$

$$\begin{aligned} \text{where } k &= \text{the turbulent kinetic energy} \\ \epsilon &= \text{the turbulence dissipation rate} \\ l &= \text{the turbulence length scale} \\ \omega &= \text{definition-1: the RMS fluctuation vorticity} \\ \omega &= \text{definition-2: simply the ratio } \epsilon/k \end{aligned}$$

Slide-3: Standard k-ϵ model - conservative variables

Wilcox (1998) conservative variables ρk,$\rho\epsilon$

$$\frac{D\rho k}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \rho\epsilon$$

$$\frac{D\rho\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right] + C_{\epsilon 1}\frac{\epsilon}{k}P_k - C_{\epsilon 2}\rho\frac{\epsilon^2}{k}$$

where

k = the turbulent kinetic energy

ϵ = the turbulence dissipation rate

P_k = the turbulence production term

ν_t = the turbulent eddy viscosity

$$P_k = \tau_{ij}\frac{\partial U_i}{\partial x_j}, \quad \nu_t = C_\mu \frac{k^2}{\epsilon} \tag{1}$$

Slide-4: Original k-ω model - Wilcox(1998)

Wilcox (1998) conservative variables ρk,$\rho\omega$

$$\frac{D\rho k}{Dt} = P_k - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{k1}\mu_t\right)\frac{\partial k}{\partial x_j}\right]$$
$$\frac{D\rho\omega}{Dt} = \frac{\gamma_1}{\nu_t} P_k - \beta_1 \rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{\omega 1}\mu_t\right)\frac{\partial \omega}{\partial x_j}\right]$$

where $\beta^* = 0.09$, $\beta_1 = 0.075$, $\nu_t = \mu_t/\rho$, and P_k the turbulence production term

$$P_k = \tau_{ij}\frac{\partial U_i}{\partial x_j} \qquad (2)$$

IMPORTANT: Menter's notation will be used in the slides that follow. $\frac{\mu_t}{\sigma_{k1}}$ re-written as: $\sigma_{k1}\mu_t$, $\frac{\mu_t}{\sigma_{\omega 1}}$ re-written as: $\sigma_{\omega 1}\mu_t$.

Slide-5: Transformed k-ϵ model

Obtained by replacing ϵ in the standard k-ϵ model with

$$\boxed{\epsilon = \beta^* k\omega, \quad \text{Wilcox(1988)}} \tag{2}$$

and solving for $\frac{\partial \rho k}{\partial t}$ and $\frac{\partial \rho \omega}{\partial t}$ to obtain

$$\frac{D\rho k}{Dt} = P_k - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{k2}\mu_t\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma_2}{\nu_t} P_k - \beta_2 \rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{\omega 2}\mu_t\right)\frac{\partial \omega}{\partial x_j}\right] + \boxed{2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}}$$

Slide-6: Comparison of k-ω and transformed k-ϵ models

Standard k-ω model - Wilcox(1998)

$$\frac{D\rho k}{Dt} = P_k - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{k1}\mu_t\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma_1}{\nu_t}P_k - \beta_1 \rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{\omega 1}\mu_t\right)\frac{\partial \omega}{\partial x_j}\right]$$

Transformed k-ϵ model

$$\frac{D\rho k}{Dt} = P_k - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{k2}\mu_t\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma_2}{\nu_t}P_k - \beta_2 \rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_{\omega 2}\mu_t\right)\frac{\partial \omega}{\partial x_j}\right] + \boxed{2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}}$$

Slide-7: The k-ω SST model

k-ω model SST model

$$\frac{D\rho k}{Dt} = P_k - \beta^* \rho \omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_k \mu_t\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{D\rho \omega}{Dt} = \frac{\gamma}{\nu_t} P_k - \beta \rho \omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_\omega \mu_t\right)\frac{\partial \omega}{\partial x_j}\right] + \phi \boxed{2\rho\sigma_\omega \frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}}$$

The blended constants in the SST model are obtained as follows:

$$\sigma_k = \phi\sigma_{k1} + (1-\phi)\sigma_{k2}$$
$$\sigma_\omega = \phi\sigma_{\omega_1} + (1-\phi)\sigma_{\omega_2}$$
$$\beta = \phi\beta_1 + (1-\phi)\beta_2$$

$$\gamma = \beta/\beta^* - \sigma_\omega \kappa^2/\sqrt{\beta^*}$$

where the subscripts 1,2 denote:
1= k-ω values and, 2= k-ϵ values and ϕ the blending function.

Slide-8: Blending function ϕ constants

Set 1 constants for σ_k,σ_ω,β (k-ω model):

$$\sigma_{k1} = 0.85, \quad \sigma_{\omega 1} = 0.5, \quad \beta_1 = 0.0750,$$

Set 2 constants for σ_k,σ_ω,β (k-ϵ model):

$$\sigma_{k2} = 1.0, \quad \sigma_{\omega 2} = 0.856, \quad \beta_2 = 0.0828,$$

The turbulent eddy viscosity viscosity ν_t:

$$\nu_t = \frac{a_1 k}{max(a_1\omega; \Omega F_2)}$$

where $a_1 = 0.31$, $\kappa = 0.41$, $\beta^* = 0.09$.

Slide-9: Blending function ϕ

Blending function ϕ (k-ω SST model):

The function ϕ is given by

$$\phi = \tanh(arg_1^4)$$

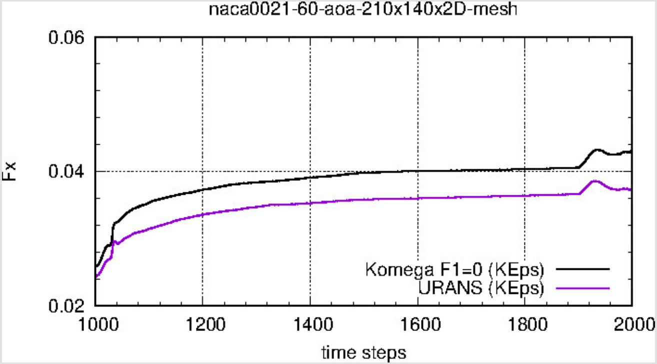
$$arg_1 = min\left[max\left(\frac{\sqrt{k}}{0.09\omega y}, \frac{500\nu}{y^2\omega}\right), \frac{4\rho\sigma_{\omega 2}k}{\mathrm{CD}_{k\omega}y^2}\right]$$

where

$$\mathrm{CD}_{k\omega} = max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}, 10^{-20}\right)$$

Slide-10: Validation of k-ω SST model

Validation step 1: set $\phi = 0$ to recover KEps solution:

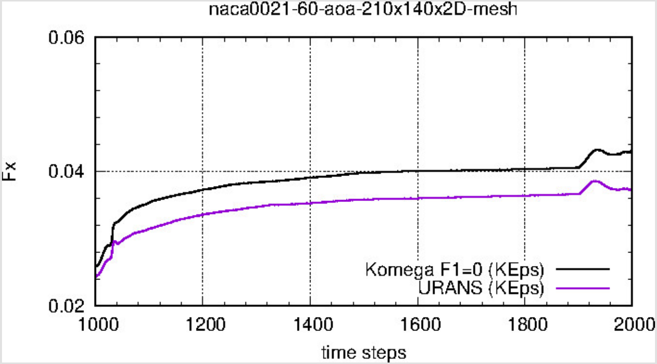


Validation step 2: set $\phi = 1$ to recover k-ω solution:

Validation step 3: ϕ varies across the field

Slide-11: Validation of k-ω SST model

Validation step 2: set $\phi = 1$ to recover k-ω solution:



Slide-12: Validation of k-ω SST model

Validation step 3: ϕ = varies across the field: