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RESIDUAL ERROR ESTIMATORS FOR COULOMB FRICTION*

PATRICK HILD† AND VANESSA LLERAS†

Abstract. This paper is concerned with residual error estimators for finite element approximations of Coulomb frictional contact problems. A recent uniqueness result by Renard in [*SIAM J. Math. Anal.*, 38 (2006), pp. 452–467] for the continuous problem allows us to perform an a posteriori error analysis. We propose, study, and implement numerically two residual error estimators associated with two finite element discretizations. In both cases the estimators permit us to obtain upper and lower bounds of the discretization error.

Key words. Coulomb friction, a posteriori error estimates, residuals

AMS subject classifications. 65N30, 74M15

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1. Introduction and notation. The numerical approximation of frictional contact problems occurring in structural mechanics is generally achieved using the finite element method (see [38, 41, 53, 58, 83]). In order to evaluate and control the quality of a finite element approximation, a current choice consists in developing adaptive procedures using a posteriori error estimators. The aim of the estimators is to provide the user with global and local quantities which represent in the best way the true error committed by the finite element approximation. Actually there exist various classes of error estimators, any one showing its specificities and advantages. Some currently used estimators are, e.g., those introduced in [7] based on the residual of the equilibrium equations, the estimators linked to the smoothing of finite element stresses (see [85]), and the estimators based on the errors in the constitutive relation, also called "equilibrated fluxes" (see [57]). A review of different a posteriori error estimators can be found in, e.g., [3, 8, 37, 79, 80].

The frictionless unilateral contact problem (or the equivalent scalar valued Signorini problem) shows a nonlinearity on the boundary corresponding to the nonpenetration of the materials on the contact area which leads to a variational inequality of the first kind. For this model the residual based method was first considered and studied in [21, 39, 84] using a penalized approach and in [12] by using the error measure technique developed in [9]. More recently the analysis without penalization term was achieved in [46] and in [47] for the corresponding mixed finite element approximation (see also [10]). Besides the study of error in the constitutive relation was performed in [27, 81, 82] for the contact problem, and a posteriori estimates for the boundary element method are studied in [63, 64]. More generally, we mention that the analysis of residual error estimators for variational inequalities generally leads to important technical difficulties for any model. Note also that an important amount of work has been devoted to the obstacle (or obstacle-type) problem in which the inequality condition holds on the entire domain (see [1, 4, 15, 16, 17, 22, 35, 49, 52, 55, 60, 68, 69, 77, 78]).

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†Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon, France (patrick.hild@univ-fcomte.fr, vanessa.lleras@univ-fcomte.fr). The work of the first author was supported by l'Agence Nationale de la Recherche, project ANR-05-JCJC-0182-01.

Other a posteriori error analyses involving inequalities linked to plasticity were considered in [20, 70, 71, 75], and the Bingham fluid problem is studied in [76].

When considering friction in addition to the contact model, there are supplementary nonlinearities which have to be taken into account. The currently used friction model is the one of Coulomb (although there exist simplified and/or different models: Tresca's friction, normal compliance, smoothed Coulomb friction, etc.; see [53, 74]), whose associated partial differential equation shows numerous mathematical difficulties which remain unsolved. In our work we consider the so-called static friction problem introduced in [30, 31] which roughly speaking corresponds to an incremental problem in the time discretized quasi-static model. For this model, existence of solutions hold when the friction coefficient is small enough; see [32, 33] and the references quoted therein. When the friction coefficient is large, neither an existence nor a nonexistence result is available. Besides the solutions are generally nonunique when the friction coefficient is large enough; see [43, 44]. More recently a first uniqueness result has been obtained in [72] with the assumption that a "regular" solution exists and that the friction coefficient is sufficiently small. From a numerical point of view it is well known that the finite element problem, associated with the continuous static Coulomb friction model, always admits a solution and that the solution is unique if the friction coefficient is small enough (unfortunately the denomination "small" depends on the discretization parameter and the bound ensuring uniqueness vanishes as the mesh is refined; see, e.g., [41]). Concerning the a posteriori error analysis for the Coulomb model, several studies have been achieved: error in the constitutive relation in [25, 62] as well as an heuristic residual based error estimator for BEM-discretizations in [34]. A simpler model, the so-called Tresca friction problem, is considered in [13] (see also the study in [14] for a similar problem where residual estimators are analyzed). Note that the latter model is governed by a variational inequality of the second kind (see [6]). Finally an a posteriori error analysis is performed for the friction model with normal compliance in [59].

Our purpose in this paper is to carry out a residual a posteriori error analysis for the Coulomb friction model and to obtain an error estimator with upper and lower bounds involving the discretization error. As far as we know, such a result is not available in the literature.

The paper is organized as follows. In section 2 we introduce the equations modelling the frictional unilateral contact problem between an elastic body and a rigid foundation. We write the problem using a mixed formulation where the unknowns are the displacement field in the body and the frictional contact pressures on the contact area. In section 3, we choose a classical discretization involving continuous finite elements of degree one and continuous piecewise affine multipliers on the contact zone. Section 4 is concerned with the study of the residual estimator which can be seen as the natural one arising from the discrete problem. Thanks to Renard's uniqueness result we obtain a global upper bound of the error. Then local lower bounds of the error are proved. In section 5 we consider a residual estimator resulting from another discrete model. This second approach has two interesting properties in comparison with the previous one: First, it involves fewer terms coming from the frictional contact, and these terms have quite simple expressions. Second, the error analysis we achieve leads to better error bounds. Section 6 is concerned with the numerical experiments and the comparison of both approaches.

Finally we introduce some useful notation and several functional spaces. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$ indicate vector valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K}, \ldots$) represent functional sets involving vector fields. As usual, we denote by $(L^2(.))^d$ and by $(H^s(.))^d$, $s \geq 0$, $d = 1, 2$, the Lebesgue and Sobolev spaces in one and two space dimensions (see [2]). The usual norm of $(H^s(D))^d$ is denoted by $\| \cdot \|_{s,D}$, and we keep the same notation when $d = 1$ or $d = 2$. For shortness the $(L^2(D))^d$-norm will be denoted by $\| \cdot \|_D$ when $d = 1$ or $d = 2$. In what follows the symbol $|\cdot|$ will denote either the Euclidean norm in $\mathbb{R}^2$, or the length of a line segment, or the area of a plane domain. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b (and of the meshsize of the triangulation) such that $a \leq C\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

2. The frictional contact problem in elasticity. We consider the deformation of an elastic body occupying, in the initial unconstrained configuration, a domain Ω in $\mathbb{R}^2$ where plane strain assumptions are assumed. The Lipschitz boundary $\partial\Omega$ of Ω consists of Γ_D, Γ_N, and Γ_C, where the measure of Γ_D does not vanish. The body Ω is clamped on Γ_D and subjected to surface traction forces $\mathbf{F}$ on Γ_N; the body forces are denoted $\mathbf{f}$. In the initial configuration, the part Γ_C is a straight line segment considered as the candidate contact surface on a rigid foundation for the sake of simplicity, which means that the contact zone cannot enlarge during the deformation process. The contact is assumed to be frictional, and the stick, slip, and separation zones on Γ_C are not known in advance. We denote by $\mu \geq 0$ the given friction coefficient on Γ_C. The unit outward normal and tangent vectors of $\partial\Omega$ are $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (-n_2, n_1)$, respectively.

The contact problem with Coulomb's friction law consists in finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ satisfying (1)–(6):

$$\mathbf{div}\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{3}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \quad \text{on } \Gamma_N. \tag{4}$$

The notation $\boldsymbol{\sigma}(\mathbf{u}) : \Omega \to \mathcal{S}_2$ represents the stress tensor field lying in $\mathcal{S}_2$, the space of second order symmetric tensors on $\mathbb{R}^2$. The linearized strain tensor field is $\boldsymbol{\varepsilon}(\mathbf{u}) = (\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}^T\mathbf{u})/2$, and $\mathcal{C}$ is the fourth order symmetric and elliptic tensor of linear elasticity. Afterwards we adopt the following notation for any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on Γ_C:

$$\mathbf{v} = v_n\mathbf{n} + v_t\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_t(\mathbf{v})\mathbf{t}.$$

On Γ_C, the three conditions representing unilateral contact are given by

$$u_n \leq 0, \qquad \sigma_n(\mathbf{u}) \leq 0, \qquad \sigma_n(\mathbf{u})\, u_n = 0, \tag{5}$$

and the Coulomb friction law is summarized by the following conditions (see, e.g., [33]):

$$\begin{cases} u_t = 0 \implies |\sigma_t(\mathbf{u})| \leq \mu|\sigma_n(\mathbf{u})|, \\ u_t \neq 0 \implies \sigma_t(\mathbf{u}) = -\mu|\sigma_n(\mathbf{u})| \dfrac{u_t}{|u_t|}. \end{cases} \tag{6}$$

The variational formulation of problem (1)–(6) in its mixed form consists in finding $(\mathbf{u}, \boldsymbol{\lambda}) = (\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{V} \times M_n \times M_t(\mu\lambda_n) = \mathbf{V} \times \mathbf{M}(\mu\lambda_n)$ which satisfy (see [48, 72])

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\boldsymbol{\lambda}, \mathbf{v}) = \; L(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \\ \qquad b(\boldsymbol{\nu} - \boldsymbol{\lambda}, \mathbf{u}) \leq \; 0 \qquad \forall \boldsymbol{\nu} = (\nu_n, \nu_t) \in \mathbf{M}(\mu\lambda_n), \end{cases} \tag{7}$$

where

$$\mathbf{V} = \left\{ \mathbf{v} \in (H^1(\Omega))^2;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}$$

and $\mathbf{M}(\mu\lambda_n) = M_n \times M_t(\mu\lambda_n)$ is defined next. We set

$$M_n = \{\nu \in X_n' :\ \nu \geq 0 \text{ on } \Gamma_C\}$$

and, for any $g \in M_n$

$$M_t(g) = \{\nu \in X_t' :\ -g \leq \nu \leq g \text{ on } \Gamma_C\},$$

where X_n' (resp., X_t') is the dual space of X_n (resp., X_t) with $X_n = \{v_{n|_{\Gamma_C}} : \mathbf{v} \in \mathbf{V}\}$ (resp., $X_t = \{v_{t|_{\Gamma_C}} : \mathbf{v} \in \mathbf{V}\}$). Note that $H^{1/2}_{00}(\Gamma_C) \subset X_n \subset H^{1/2}(\Gamma_C)$, $H^{1/2}_{00}(\Gamma_C) \subset X_t \subset H^{1/2}(\Gamma_C)$ and that the inequality conditions incorporated in the definitions of M_n and $M_t(g)$ have to be understood in the dual sense.

Remark 1. Note that the previous mixed method is a nonstandard formulation since there is a bootstrap: find $(\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{V} \times M_n \times M_t(\mu\lambda_n)$ such that (7) holds. This weak formulation could be written in a different way without the bootstrap and by adding a condition: find $(\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{V} \times M_n \times X_t'$ such that $\lambda_t \in M_t(\mu\lambda_n)$ and (7) holds.

In (7), $\mathbf{f} \in (L^2(\Omega))^2$, $\mathbf{F} \in (L^2(\Gamma_N))^2$, and the standard notations are adopted:

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\Omega, \qquad L(\mathbf{v}) = \int_\Omega \mathbf{f}.\mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{F}.\mathbf{v}\, d\Gamma,$$

$$b(\boldsymbol{\nu}, \mathbf{v}) = \langle \nu_n, v_n \rangle_{X_n', X_n} + \langle \nu_t, v_t \rangle_{X_t', X_t}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $(H^1(\Omega))^2$ and $\boldsymbol{\nu} = (\nu_n, \nu_t)$ in $X_n' \times X_t'$. In these definitions the notations $\cdot$ and $:$ represent the canonical inner products in $\mathbb{R}^2$ and $\mathcal{S}_2$, respectively. It is easy to see that if $(\mathbf{u}, \lambda_n, \lambda_t)$ is a solution of (7), then $\lambda_n = -\sigma_n(\mathbf{u})$ and $\lambda_t = -\sigma_t(\mathbf{u})$. The space X_n is equipped with the norm

$$\|w\|_{X_n} = \inf_{\mathbf{v} \in \mathbf{V} : v_n = w \text{ on } \Gamma_c} \|\mathbf{v}\|_{1,\Omega},$$

and a similar expression holds for $\|.\|_{X_t}$. The dual space of $X_n \times X_t$ is endowed with the norm

$$\|\boldsymbol{\nu}\|_{-\frac{1}{2},\Gamma_C} = \sup_{\mathbf{w} \in \mathbf{V} \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{\nu}, \mathbf{w})}{\|\mathbf{w}\|_{1,\Omega}} \qquad \forall \boldsymbol{\nu} = (\nu_n, \nu_t) \in X_n' \times X_t'.$$

To avoid more notation, we will skip over the regularity aspects of the functions defined on Γ_C which are beyond the scope of this paper and afterwards write integral terms instead of duality pairings. Another classical weak formulation of problem (1)–(6) is an inequality problem: find $\mathbf{u}$ such that

$$\text{(8)} \quad \mathbf{u} \in \mathbf{K}, \ \ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \mu \int_{\Gamma_C} \sigma_n(\mathbf{u})(|v_t| - |u_t|)\, d\Gamma \geq L(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in \mathbf{K},$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the nonpenetration conditions:

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} :\ v_n \leq 0 \text{ on } \Gamma_C\}.$$

When friction is omitted (i.e., $\mu = 0$) the condition (6) simply reduces to $\sigma_t(\mathbf{u}) = 0$ and the frictionless contact problem admits a unique solution according to Stampacchia's theorem (see, e.g., [36, 54]). The existence of a solution to (8) was first proved for small friction coefficients in [67] (in two space dimensions), and the bounds ensuring existence have been improved and generalized in [51] and [32] (see also [33]). More precisely existence holds if $\mu \leq \sqrt{3-4P}/(2-2P)$, where $0 \leq P < 1/2$ denotes Poisson's ratio. Recently some multisolutions of the problem (1)–(6) are exhibited for triangular or quadrangular domains. These multiple solutions involve either an infinite set of slipping solutions (see [43]), or two isolated (stick and separation) configurations (see [44]), or two isolated (stick and grazing contact) solutions in [45]. Note that these examples of nonuniqueness involve large friction coefficients (i.e., $\mu > \sqrt{(1-P)/P}$) and tangential displacements with a constant sign on Γ_C. Actually, it seems that no multisolution has been detected for an arbitrary small friction coefficient in the continuous case, although such a result exists for finite element approximations in [42], but for a variable geometry. The forthcoming partial uniqueness result is obtained in [72]: it defines some cases where it is possible to affirm that a solution to the Coulomb friction problem is in fact the unique solution. More precisely, if a "regular" solution to the Coulomb friction problem exists (here the denomination "regular" means, roughly speaking, that the transition is smooth when the slip direction changes) and if the friction coefficient is small enough, then this solution is the only one.

We now introduce the space of multipliers M of the functions ξ defined on Γ_C such that the following norm $\|\xi\|_M$ is finite:

$$\|\xi\|_M = \sup_{v_t \in X_t \setminus \{0\}} \frac{\|\xi v_t\|_{X_t}}{\|v_t\|_{X_t}}.$$

Since Γ_C is assumed to be straight, M contains for any $\varepsilon > 0$ the space $H^{1/2+\varepsilon}(\Gamma_C)$ (see [65] for a complete discussion on the theory of multipliers in a pair of Hilbert spaces). The partial uniqueness result is given assuming that $\lambda_t = \mu\lambda_n\xi$, with $\xi \in M$. It is easy to see that it implies $|\xi| \leq 1$ a.e. on the support of λ_n. More precisely, this implies that $\xi \in \mathrm{Dir}_t(u_t)$ a.e. on the support of λ_n, where $\mathrm{Dir}_t(.)$ is the subdifferential of the convex map $x_t \longmapsto |x_t|$. This means that it is possible to assume that $\xi \in \mathrm{Dir}_t(u_t)$ a.e. on Γ_C.

PROPOSITION 2.1 (see [72]). *Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *be a solution to problem* (7) *such that* $\lambda_t = \mu\lambda_n\xi$, *with* $\xi \in M$, $\xi \in \mathrm{Dir}_t(u_t)$ *a.e. on* Γ_C, *and* $\mu\|\xi\|_M$ *is small enough. Then* $(\mathbf{u}, \boldsymbol{\lambda})$ *is the unique solution to problem* (7).

The case $\xi \equiv 1$ corresponds to a homogeneous sliding direction, and the previous result is complementary with the nonuniqueness results obtained in [43, 44, 45]. The multiplier ξ has to vary from -1 to $+1$ each time the sign of the tangential displacement changes from negative to positive. The set M does not contain any multiplier having a discontinuity of the first kind. Consequently, in order to satisfy the assumptions of Proposition 2.1, the tangential displacement of the solution $\mathbf{u}$ cannot pass from a negative value to a positive value and be zero only at a single point of Γ_C. For a more precise discussion concerning the assumption $\lambda_t = \mu\lambda_n\xi$, $\xi \in M$, $\xi \in \mathrm{Dir}_t(u_t)$ and the cases where the assumption cannot be fulfilled independently of the regularity of the solution, we refer the reader to [48, Remark 2].

3. Mixed finite element approximation.

We approximate this problem with a standard finite element method. Namely we fix a regular family of meshes $T_h, h > 0$ (see [18, 19, 23]), made of closed triangles. For $K \in T_h$, let h_K be the diameter of

K and $h = \max_{K\in T_h} h_K$. The regularity of the mesh implies in particular that for any edge E of K one has $h_E = |E| \sim h_K$. Let us define E_h (resp., $\mathcal{N}_h$) as the set of edges (resp., nodes) of the triangulation and set $E_h^{int} = \{E \in E_h : E \subset \Omega\}$ the set of interior edges of T_h (the edges are supposed to be relatively open). We denote by $E_h^N = \{E \in E_h : E \subset \Gamma_N\}$ the set of exterior edges included in the part of the boundary where we impose Neumann conditions, and similarly $E_h^C = \{E \in E_h : E \subset \Gamma_C\}$. Set $\mathcal{N}_h^D = \mathcal{N}_h \cap \overline{\Gamma_D}$ (note that the extreme nodes of $\overline{\Gamma_D}$ belong to $\mathcal{N}_h^D$). For an element K, we will denote by E_K the set of edges of K and, according to the above notation, we set $E_K^{int} = E_K \cap E_h^{int}$, $E_K^N = E_K \cap E_h^N$, $E_K^C = E_K \cap E_h^C$. For each interior edge E we fix one of the two normal vectors and denote it by $\mathbf{n}_E$. The jump of some vector valued function $\mathbf{v}$ across an edge $E \in E_h^{int}$ at a point $\mathbf{y} \in E$ is defined as

$$[\![\mathbf{v}]\!]_E(\mathbf{y}) = \lim_{\alpha\to 0^+} \mathbf{v}(\mathbf{y} + \alpha\mathbf{n}_E) - \mathbf{v}(\mathbf{y} - \alpha\mathbf{n}_E) \qquad \forall E \in E_h^{int}.$$

Note that the sign of $[\![\mathbf{v}]\!]_E$ depends on the orientation of $\mathbf{n}_E$. Finally we introduce the patches: denoting by $\mathbf{x}$ a node, by E an edge, and by K an element, let $\omega_{\mathbf{x}} = \cup_{\{K:\mathbf{x}\in K\}} K$, $\omega_E = \cup_{\{\mathbf{x}:\mathbf{x}\in\overline{E}\}}\omega_{\mathbf{x}}$, and $\omega_K = \cup_{\{\mathbf{x}:\mathbf{x}\in K\}}\omega_{\mathbf{x}}$. The finite element space used in Ω is then defined by

$$\mathbf{V}_h = \left\{\mathbf{v}_h \in (C(\overline{\Omega}))^2 : \forall K \in T_h, \quad \mathbf{v}_h|_K \in (\mathbb{P}_1(K))^2, \quad \mathbf{v}_h|_{\Gamma_D} = \mathbf{0}\right\}.$$

We recall that the contact area is a straight line segment to simplify. The extension to a contact area which is a broken line can be made without additional technical difficulties (see, e.g., [47]). In order to express the contact constraints by using Lagrange multipliers on the contact zone, we have to introduce the range of $\mathbf{V}_h$ by the normal trace operator on Γ_C:

$$W_h = \left\{\nu_h \in C(\overline{\Gamma_C}) : \exists \mathbf{v}_h \in \mathbf{V}_h \text{ s.t. } \mathbf{v}_h \cdot \mathbf{n} = \nu_h \text{ on } \Gamma_C \right\},$$

which coincides with the range of $\mathbf{V}_h$ by the tangent trace operator on Γ_C. The choice of the space W_h allows us to define the following closed convex cones:

$$M_{hn} = \left\{\nu_h \in W_h : \int_{\Gamma_C} \nu_h\psi_h \, d\Gamma \geq 0 \qquad \forall \psi_h \in W_h, \psi_h \geq 0 \right\}$$

and, for $g \in M_{hn}$,

$$M_{ht}(g) = \left\{\nu_h \in W_h : \left|\int_{\Gamma_C} \nu_h\psi_h \, d\Gamma\right| \leq \int_{\Gamma_C} g\psi_h \, d\Gamma \qquad \forall \psi_h \in W_h, \psi_h \geq 0 \right\}.$$

Remark 2. It is easy to check that the functions in M_{hn} are not necessarily nonnegative. In the same way the functions in $M_{ht}(g)$ do not satisfy $|\nu_h| \leq g$ everywhere.

The discretized mixed formulation of the frictional contact problem is to find $\mathbf{u}_h \in \mathbf{V}_h$ and $\boldsymbol{\lambda}_h \in \mathbf{M}_h(\mu\lambda_{hn}) = M_{hn} \times M_{ht}(\mu\lambda_{hn})$ satisfying

$$\text{(9)} \qquad \begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda}_h, \mathbf{v}_h) = & L(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ \qquad b(\boldsymbol{\nu}_h - \boldsymbol{\lambda}_h, \mathbf{u}_h) \leq & 0 \qquad \forall\, \boldsymbol{\nu}_h = (\nu_{hn}, \nu_{ht}) \in \mathbf{M}_h(\mu\lambda_{hn}). \end{cases}$$

Problem (9) could also be written without a bootstrap like the continuous problem (see Remark 1). Using a fixed point argument it can be proved that problem (9) admits at least a solution and that there is a unique solution when $\mu \leq C(h)$ (see [25]). Unfortunately the constant $C(h)$ vanishes when h vanishes ($C(h) \sim h^{1/2}$). The following result proved in [25] gives explicitly the discrete frictional contact conditions.

PROPOSITION 3.1 (see [25]). *Let* $(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ *be a solution of* (9). *Suppose that* $\dim(W_h) = p$ *and let* $\psi_{\mathbf{x}_i}, 1 \leq i \leq p$, *denote the basis functions of* W_h *on* Γ_C. *The* p*-by-*p *mass matrix* $\mathcal{M} = (m_{ij})_{1 \leq i,j \leq p}$ *on* Γ_C *is given by* $m_{ij} = \int_{\Gamma_C} \psi_{\mathbf{x}_i} \psi_{\mathbf{x}_j}$. *Let* U_N *and* U_T *denote the vectors whose components are the nodal values of* u_{hn} *and* u_{ht}, *respectively, and let* L_N *and* L_T *denote the vectors whose components are the nodal values of* λ_{hn} *and* λ_{ht}, *respectively. Then the discrete frictional contact conditions in* (9) *are as follows: for any* $1 \leq i \leq p$

$$\begin{gathered}
(\mathcal{M}L_N)_i \geq 0, \qquad (U_N)_i \leq 0, \qquad (\mathcal{M}L_N)_i (U_N)_i = 0, \\
|(\mathcal{M}L_T)_i| \leq \mu (\mathcal{M}L_N)_i, \\
|(\mathcal{M}L_T)_i| < \mu (\mathcal{M}L_N)_i \implies (U_T)_i = 0, \\
(\mathcal{M}L_T)_i (U_T)_i \geq 0.
\end{gathered}$$

Remark 3. The a priori error analysis of (9) remains an open problem, although an error estimate is obtained in [48] for a slightly different approximation of the frictional contact conditions (see also [40] for an early convergence result). When friction is absent, an important number of a priori error analyses have been achieved (see, e.g., [11, 26, 50] and the references therein). Note that even in this simpler case, the proof of an estimate of order h in the $(H^1(\Omega))^2$-norm with only $(H^2(\Omega))^2$ regularity (without any additional assumption) remains an open problem.

We consider the quasi-interpolation operator π_h: for any $v \in L^1(\Omega)$, we define $\pi_h v$ as the unique element in $V_h = \{v_h \in C(\overline{\Omega}) : \ \forall K \in T_h, \ v_h|_K \in \mathbb{P}_1(K), \ v_h|_{\Gamma_D} = 0\}$ such that

$$\pi_h v = \sum_{\mathbf{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D} \left(\frac{1}{|\omega_{\mathbf{x}}|} \int_{\omega_{\mathbf{x}}} v(\mathbf{y}) \, d\mathbf{y} \right) \psi_{\mathbf{x}}, \tag{10}$$

where for any $\mathbf{x} \in \mathcal{N}_h$, $\psi_{\mathbf{x}}$ is the standard basis function in V_h satisfying $\psi_{\mathbf{x}}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$ $\forall \mathbf{x}' \in \mathcal{N}_h$. Note that we could also consider other quasi-interpolation operators like the ones in [22] or in [24]. The following estimates hold (see, e.g., [80]): for any $v \in H^1(\Omega)$ vanishing on Γ_D, we have $\|v - \pi_h v\|_K \lesssim h_K \|\nabla v\|_{\omega_K}$ $\forall K \in T_h$, and $\|v - \pi_h v\|_E \lesssim h_E^{1/2} \|\nabla v\|_{\omega_E}$ $\forall E \in E_h$. Since we deal with vector valued functions we can define a vector valued operator (which we again denote by π_h for the sake of simplicity) whose components are defined above. So we get the following lemma.

LEMMA 3.2. *For any* $\mathbf{v} \in \mathbf{V}$ *the following estimates hold:*

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_K \lesssim h_K \|\mathbf{v}\|_{1,\omega_K} \qquad \forall K \in T_h, \tag{11}$$

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_E \lesssim h_E^{1/2} \|\mathbf{v}\|_{1,\omega_E} \qquad \forall E \in E_h. \tag{12}$$

4. The residual error estimator $\boldsymbol{\eta}$.

4.1. Definition of the residual error estimator. The element residual of the equilibrium equation (1) is defined by $\mathbf{div}\boldsymbol{\sigma}(\mathbf{u}_h) + \mathbf{f} = \mathbf{f}$ on K. As usual this element residual can be replaced with some simple finite dimensional approximation $\mathbf{f}_K \in (\mathbb{P}_k(K))^2$ and the difference $\mathbf{f} - \mathbf{f}_K$ will be treated as data oscillation. A current choice is to take $\mathbf{f}_K = \int_K \mathbf{f}(\mathbf{x}) / |K|$. In the same way $\mathbf{F}$ can be approximated by a simple quantity denoted $\mathbf{F}_E$ on any $E \in E_h^N$.

DEFINITION 4.1. *The global residual estimator* η *and the local residual error estimators* η_K *are defined by*

$$\eta = \left(\sum_{K\in T_h} \eta_K^2\right)^{1/2},$$

$$\eta_K = \left(\sum_{i=1}^{8} \eta_{iK}^2\right)^{1/2},$$

$$\eta_{1K} = h_K \|\mathbf{f}_K\|_K,$$

$$\eta_{2K} = h_K^{1/2} \left(\sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(\mathbf{u}_h)\|_E^2\right)^{1/2},$$

$$\eta_{3K} = h_K^{1/2} \|\lambda_{hn} + \sigma_n(\mathbf{u}_h)\|_{K\cap\Gamma_C},$$

$$\eta_{4K} = h_K^{1/2} \|\lambda_{ht} + \sigma_t(\mathbf{u}_h)\|_{K\cap\Gamma_C},$$

$$\eta_{5K} = \left(\int_{K\cap\Gamma_C} -\lambda_{hn+}u_{hn}\right)^{1/2},$$

$$\eta_{6K} = \|\lambda_{hn-}\|_{K\cap\Gamma_C},$$

$$\eta_{7K} = \left(\int_{K\cap\Gamma_C} (|\lambda_{ht}| - \mu\lambda_{hn+})_-|u_{ht}| + \int_{K\cap\Gamma_C} (\lambda_{ht}u_{ht})_-\right)^{1/2},$$

$$\eta_{8K} = \|(|\lambda_{ht}| - \mu\lambda_{hn+})_+\|_{K\cap\Gamma_C},$$

where the notations $_+$ *and* $_-$ *denote the positive and negative parts, respectively;* $J_{E,n}(\mathbf{u}_h)$ *means the constraint jump of* $\mathbf{u}_h$ *in the normal direction, i.e.,*

$$J_{E,n}(\mathbf{u}_h) = \begin{cases} [\![\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_E]\!]_E & \forall E \in E_h^{int}, \\ \boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n} - \mathbf{F}_E & \forall E \in E_h^N. \end{cases} \tag{13}$$

The local and global data oscillation terms are defined by

$$\zeta_K = \left(h_K^2 \sum_{K'\subset\omega_K} \|\mathbf{f} - \mathbf{f}_{K'}\|_{K'}^2 + h_E \sum_{E\subset E_K^N} \|\mathbf{F} - \mathbf{F}_E\|_E^2\right)^{1/2}, \quad \zeta = \left(\sum_{K\in T_h} \zeta_K^2\right)^{1/2}. \tag{14}$$

Remark 4. From the previous definition, we see that there are eight contributions for any local estimator η_K. There are only two classical contributions (η_{1K}: equilibrium residual and η_{2K}: interior and Neumann jumps) for all the elements which do not have an edge belonging to Γ_C. The remaining elements on the contact area have six supplementary terms. The terms η_{3K} and η_{4K} represent the deviation of the traction from the equilibrium in the mixed finite element approximation; the terms η_{5K} and η_{6K} (resp., η_{7K} and η_{8K}) represent the nonfulfillment of the unilateral contact conditions (5) (resp., of the friction conditions (6)).

4.2. Upper error bound. We now give an upper bound of the discretization error. In the forthcoming theorem we assume that the solution to the continuous problem satisfies the uniqueness criterion of [72].

THEOREM 4.2. *Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *be the solution to problem* (7) *such that* $\lambda_t = \mu\lambda_n\xi$, *with* $\xi \in M$, $\xi \in \mathrm{Dir}_t(u_t)$ *a.e. on* Γ_C, *and* $\mu\|\xi\|_M$ *is small enough. Let* $(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ *be a* *solution to the discrete problem* (9). *Then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_C}\lesssim \eta+\zeta.$$

Proof. To simplify the notation we set $\mathbf{e_u}=\mathbf{u}-\mathbf{u}_h$. Let $\mathbf{v}_h\in\mathbf{V}_h$; from the $\mathbf{V}$-ellipticity of $a(.,.)$ and the equilibrium equations in (7) and (9) we obtain

$$\begin{aligned}\|\mathbf{e_u}\|^2_{1,\Omega}&\lesssim a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h)\\&=a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{v}_h)+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)\\&=L(\mathbf{u}-\mathbf{v}_h)-b(\boldsymbol{\lambda},\mathbf{u}-\mathbf{v}_h)-a(\mathbf{u}_h,\mathbf{u}-\mathbf{v}_h)+b(\boldsymbol{\lambda}_h-\boldsymbol{\lambda},\mathbf{v}_h-\mathbf{u}_h).\end{aligned}$$

Integrating by parts on each triangle K and using the definition of $J_{E,n}(\mathbf{u}_h)$ in (13) and the complementarity conditions $\int_{\Gamma_C}\lambda_n u_n=\int_{\Gamma_C}\lambda_{hn}u_{hn}=0$ gives

$$\begin{aligned}\|\mathbf{e_u}\|^2_{1,\Omega}\lesssim&\int_\Omega \mathbf{f}\cdot(\mathbf{u}-\mathbf{v}_h)+\sum_{E\in E_h^N}\int_E(\mathbf{F}-\mathbf{F}_E)\cdot(\mathbf{u}-\mathbf{v}_h)\\&+b(\boldsymbol{\lambda}_h,\mathbf{v}_h)+b(\boldsymbol{\lambda},\mathbf{u}_h)-\int_{\Gamma_C}\lambda_{ht}u_{ht}-\int_{\Gamma_C}\lambda_t u_t\\&-\sum_{E\in E_h^C}\int_E(\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n})\cdot(\mathbf{u}-\mathbf{v}_h)-\sum_{E\in E_h^{int}\cup E_h^N}\int_E J_{E,n}(\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{v}_h).\end{aligned}$$

Splitting up the integrals on Γ_C into normal and tangential components gives

$$\begin{aligned}\|\mathbf{e_u}\|^2_{1,\Omega}\lesssim&\int_{\Gamma_C}\lambda_n u_{hn}+\int_{\Gamma_C}\lambda_{hn}u_n+\int_{\Gamma_C}(\lambda_t-\lambda_{ht})(u_{ht}-u_t)+\int_\Omega\mathbf{f}\cdot(\mathbf{u}-\mathbf{v}_h)\\&-\sum_{E\in E_h^{int}\cup E_h^N}\int_E J_{E,n}(\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{v}_h)+\sum_{E\in E_h^N}\int_E(\mathbf{F}-\mathbf{F}_E)\cdot(\mathbf{u}-\mathbf{v}_h)\\&+\sum_{E\in E_h^C}\int_E(\lambda_{hn}+\sigma_n(\mathbf{u}_h))(v_{hn}-u_n)+\sum_{E\in E_h^C}\int_E(\lambda_{ht}+\sigma_t(\mathbf{u}_h))(v_{ht}-u_t)\\=&\int_{\Gamma_C}\lambda_n u_{hn}+\int_{\Gamma_C}\lambda_{hn}u_n+\int_{\Gamma_C}(\lambda_t-\lambda_{ht})(u_{ht}-u_t)+I+II+III+IV+V.\end{aligned}\tag{15}$$

We now need to estimate each term of this right-hand side. For that purpose, we take

$$\mathbf{v}_h=\mathbf{u}_h+\pi_h(\mathbf{u}-\mathbf{u}_h),\tag{16}$$

where π_h is the quasi-interpolation operator defined in Lemma 3.2.

We start with the term I. From the definition of $\mathbf{v}_h$ and (11) we get

$$\|\mathbf{u}-\mathbf{v}_h\|_K=\|\mathbf{e_u}-\pi_h\mathbf{e_u}\|_K\lesssim h_K\|\mathbf{e_u}\|_{1,\omega_K}$$

for any triangle K. This estimate together with the Cauchy–Schwarz inequality implies

$$|I|\lesssim(\eta+\zeta)\|\mathbf{e_u}\|_{1,\Omega}.\tag{17}$$

We now consider the interior and Neumann boundary terms in (15): as previously the application of the Cauchy–Schwarz inequality leads to

$$|II|\le\sum_{E\in E_h^{int}\cup E_h^N}\|J_{E,n}(\mathbf{u}_h)\|_E\|\mathbf{u}-\mathbf{v}_h\|_E.$$

Therefore using the expression (16) and estimate (12), we obtain

$$\|\mathbf{u} - \mathbf{v}_h\|_E = \|\mathbf{e_u} - \pi_h \mathbf{e_u}\|_E \lesssim h_E^{1/2} \|\mathbf{e_u}\|_{1,\omega_E}.$$

Inserting this estimate in the previous one, we deduce that

$$|II| \lesssim \eta \|\mathbf{e_u}\|_{1,\Omega}. \tag{18}$$

Moreover

$$|III| \lesssim \zeta \|\mathbf{e_u}\|_{1,\Omega}. \tag{19}$$

The two remaining terms are handled in a way similar to the previous ones so that

$$|IV| + |V| \lesssim \eta \|\mathbf{e_u}\|_{1,\Omega}. \tag{20}$$

Noting that $u_{hn} \leq 0$ on Γ_C, we have

$$\int_{\Gamma_C} \lambda_n u_{hn} \leq 0, \tag{21}$$

and it remains to estimate two terms in (15). Using the discrete complementarity condition $\int_{\Gamma_C} \lambda_{hn} u_{hn} = 0$ implies

$$\begin{aligned}
\int_{\Gamma_C} \lambda_{hn} u_n &= \int_{\Gamma_C} \lambda_{hn}(u_n - u_{hn}) = \int_{\Gamma_C} (\lambda_{hn+} - \lambda_{hn-})(u_n - u_{hn}) \\
&\leq -\int_{\Gamma_C} \lambda_{hn+} u_{hn} - \int_{\Gamma_C} \lambda_{hn-}(u_n - u_{hn}) \\
&\leq \eta^2 - \int_{\Gamma_C} \lambda_{hn-}(u_n - u_{hn}) \\
&= \eta^2 + VI. \qquad (22)
\end{aligned}$$

The last term in the previous expression is estimated using the Cauchy–Schwarz and Young inequalities:

$$|VI| \leq \sum_{E \in E_h^C} \|\lambda_{hn-}\|_E \|u_n - u_{hn}\|_E \leq \sum_{E \in E_h^C} \left(\alpha \|u_n - u_{hn}\|_E^2 + \frac{1}{4\alpha} \|\lambda_{hn-}\|_E^2 \right)$$

for any $\alpha > 0$. A standard trace theorem implies that

$$|VI| \leq \alpha \|u_n - u_{hn}\|_{\Gamma_C}^2 + \frac{1}{4\alpha} \sum_{E \in E_h^C} \|\lambda_{hn-}\|_E^2 \lesssim \alpha \|\mathbf{e_u}\|_{1,\Omega}^2 + \frac{\eta^2}{4\alpha}. \tag{23}$$

Estimates (22) and (23) give

$$\int_{\Gamma_C} \lambda_{hn} u_n \lesssim \alpha \|\mathbf{e_u}\|_{1,\Omega}^2 + \eta^2 \left(1 + \frac{1}{4\alpha}\right) \tag{24}$$

for any $\alpha > 0$.

We now estimate the term corresponding to the friction:

$$\int_{\Gamma_C}(\lambda_{ht}-\lambda_t)(u_t-u_{ht}) = \int_{\Gamma_C}(\lambda_{ht}-\mu\lambda_{hn}\xi)(u_t-u_{ht}) + \int_{\Gamma_C}(\mu\lambda_{hn}\xi-\lambda_t)(u_t-u_{ht})$$
$$= \int_{\Gamma_C}(\lambda_{ht}-\mu\lambda_{hn}\xi)(u_t-u_{ht}) + \int_{\Gamma_C}\mu(\lambda_{hn}-\lambda_n)\xi(u_t-u_{ht}), \quad (25)$$

where $\xi \in M, \xi \in \mathrm{Dir}_t(u_t), \lambda_t = \mu\lambda_n\xi$. The second term in (25) is bounded as follows:

$$\left|\int_{\Gamma_C}\mu(\lambda_{hn}-\lambda_n)\xi(u_t-u_{ht})\right| \le \mu\|\xi\|_M\|u_t-u_{ht}\|_{X_t}\|\lambda_n-\lambda_{hn}\|_{X_t'}$$
$$\lesssim \mu\|\xi\|_M\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_C}$$
$$\lesssim \mu\|\xi\|_M\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}(\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}+\eta+\zeta).$$

In the last inequality, we have used (30). We deduce from Young's inequality that

$$\left|\int_{\Gamma_C}\mu(\lambda_{hn}-\lambda_n)\xi(u_t-u_{ht})\right| \lesssim (1+\alpha)\mu\|\xi\|_M\|\mathbf{e_u}\|^2_{1,\Omega} + \frac{\mu\|\xi\|_M}{2\alpha}\left(\eta^2+\zeta^2\right) \quad (26)$$

for any positive α.

Besides the first term in (25) is handled next:

$$\int_{\Gamma_C}(\lambda_{ht}-\mu\lambda_{hn}\xi)(u_t-u_{ht})$$
$$= \int_{\Gamma_C}\lambda_{ht}u_t - \int_{\Gamma_C}\mu\lambda_{hn+}\xi u_t + \int_{\Gamma_C}\mu\lambda_{hn+}\xi u_{ht} + \int_{\Gamma_C}\mu\lambda_{hn-}\xi(u_t-u_{ht}) - \int_{\Gamma_C}\lambda_{ht}u_{ht}$$
$$= \int_{\Gamma_C}(\lambda_{ht}u_t - \mu\lambda_{hn+}|u_t|) + \int_{\Gamma_C}\mu\lambda_{hn+}\xi u_{ht} + \int_{\Gamma_C}\mu\lambda_{hn-}\xi(u_t-u_{ht}) - \int_{\Gamma_C}\lambda_{ht}u_{ht}$$
$$\le \int_{\Gamma_C}(|\lambda_{ht}|-\mu\lambda_{hn+})_+|u_t| + \int_{\Gamma_C}(\mu\lambda_{hn+}|u_{ht}|-\lambda_{ht}u_{ht}) + \int_{\Gamma_C}\mu\lambda_{hn-}|u_t-u_{ht}|$$
$$\le \int_{\Gamma_C}(|\lambda_{ht}|-\mu\lambda_{hn+})_+|u_t-u_{ht}| + \int_{\Gamma_C}(|\lambda_{ht}|-\mu\lambda_{hn+})_+|u_{ht}|$$
$$+ \int_{\Gamma_C}(\mu\lambda_{hn+}|u_{ht}|-|\lambda_{ht}||u_{ht}|) + \int_{\Gamma_C}\mu\lambda_{hn-}|u_t-u_{ht}| + \int_{\Gamma_C}(|\lambda_{ht}||u_{ht}|-\lambda_{ht}u_{ht})$$
$$\lesssim \|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}(\|(|\lambda_{ht}|-\mu\lambda_{hn+})_+\|_{\Gamma_C}+\mu\|\lambda_{hn-}\|_{\Gamma_C})$$
$$+ \int_{\Gamma_C}[(|\lambda_{ht}|-\mu\lambda_{hn+})_+|u_{ht}|-(|\lambda_{ht}|-\mu\lambda_{hn+})|u_{ht}|] + 2\int_{\Gamma_C}(\lambda_{ht}u_{ht})_-$$
$$\lesssim \|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}(\|(|\lambda_{ht}|-\mu\lambda_{hn+})_+\|_{\Gamma_C}+\mu\|\lambda_{hn-}\|_{\Gamma_C})$$
$$+ \int_{\Gamma_C}(|\lambda_{ht}|-\mu\lambda_{hn+})_-|u_{ht}| + \int_{\Gamma_C}(\lambda_{ht}u_{ht})_-. \quad (27)$$

From (26) and (27), we obtain for any $\alpha > 0$

$$\int_{\Gamma_C}(\lambda_{ht}-\lambda_t)(u_t-u_{ht}) \lesssim (\alpha+(1+\alpha)\mu\|\xi\|_M)\|\mathbf{e_u}\|^2_{1,\Omega} + \frac{\mu\|\xi\|_M+2\alpha+1+\mu^2}{2\alpha}\left(\eta^2+\zeta^2\right). \quad (28)$$

Putting together the estimates (17), (18), (19), (20), (21), (24), and (28) with α small enough in (15) and using Young's inequality, we deduce that if $\mu\|\xi\|_M$ is small enough, then

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega} \lesssim \eta+\zeta. \quad (29)$$

We now search for an upper bound on the discretization error $\boldsymbol{\lambda}-\boldsymbol{\lambda}_h$ corresponding to the multipliers. Let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_h \in \mathbf{V}_h$. From the equilibrium equations in (7) and (9) we get

$$\begin{aligned} b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \mathbf{v}) &= b(\boldsymbol{\lambda}, \mathbf{v} - \mathbf{v}_h) - b(\boldsymbol{\lambda}_h, \mathbf{v} - \mathbf{v}_h) + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \mathbf{v}_h) \\ &= L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v} - \mathbf{v}_h) - b(\boldsymbol{\lambda}_h, \mathbf{v} - \mathbf{v}_h) + a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \\ &= L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) - b(\boldsymbol{\lambda}_h, \mathbf{v} - \mathbf{v}_h). \end{aligned}$$

An integration by parts on each element K gives

$$\begin{aligned} b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{v} - \mathbf{v}_h) \\ + \sum_{E \in E_h^N} \int_E (\mathbf{F} - \mathbf{F}_E) \cdot (\mathbf{v} - \mathbf{v}_h) - \sum_{E \in E_h^C} \int_E (\lambda_{hn} + \sigma_n(\mathbf{u}_h))(v_n - v_{hn}) \\ - \sum_{E \in E_h^C} \int_E (\lambda_{ht} + \sigma_t(\mathbf{u}_h))(v_t - v_{ht}). \end{aligned}$$

Choosing $\mathbf{v}_h = \pi_h \mathbf{v}$, where π_h is the quasi-interpolation operator defined in Lemma 3.2, and achieving a similar calculation as in (17), (18), (19), and (20) we deduce that

$$|b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \mathbf{v})| \lesssim (\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \eta + \zeta) \|\mathbf{v}\|_{1,\Omega}$$

for any $\mathbf{v} \in \mathbf{V}$. As a consequence

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \eta + \zeta. \tag{30}$$

Putting together the two estimates (29) and (30) ends the proof of the theorem. ∎

4.3. Lower error bound.

THEOREM 4.3. *Let $(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ be a solution to the discrete problem (9) and let $\eta = \eta(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ be the corresponding estimator. Let $(\mathbf{u}, \boldsymbol{\lambda})$ be a solution to problem (7) such that $\boldsymbol{\lambda} \in (L^2(\Gamma_C))^2$. For all elements K, the following local lower error bounds hold:*

$$\eta_{1K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{31}$$

$$\eta_{2K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,\omega_K} + \zeta_K. \tag{32}$$

For all elements K having an edge in Γ_C (i.e., $K \cap \Gamma_C = E$), the following local lower error bounds hold:

$$\eta_{iK} \lesssim h_K^{1/2} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \quad i = 3, 4, \tag{33}$$

$$\begin{aligned} \eta_{jK} \le 2(1+\mu)\Big(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E^{1/2} \|\mathbf{u}\|_E^{1/2} \\ + \|\mathbf{u} - \mathbf{u}_h\|_E^{1/2} \|\boldsymbol{\lambda}\|_E^{1/2} \Big), \quad j = 5, 7, \end{aligned} \tag{34}$$

$$\eta_{lK} \le (1+\mu) \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E, \quad l = 6, 8. \tag{35}$$

Proof. We mention that we do not suppose that the solution to the continuous problem is unique. Of course our result holds when $(\mathbf{u}, \boldsymbol{\lambda})$ is the unique solution given by Proposition 2.1. Note also that the solution to the discrete problem is not supposed to be unique.

The estimates of η_{1K} and η_{2K} in (31) and (32) are standard (see, e.g., [79]). We now estimate η_{3K}. Writing $\mathbf{w}_E = w_{En}\mathbf{n} + w_{Et}\mathbf{t}$ on $E \in E_K^C$ and denoting by b_E the edge bubble function associated with E (i.e., $b_E = 4\psi_{\mathbf{a}_1}\psi_{\mathbf{a}_2}$, where $\mathbf{a}_1, \mathbf{a}_2$ are the two extremities of E; we recall that $\psi_{\mathbf{x}}$ is the standard basis function at node $\mathbf{x}$ in V_h satisfying $\psi_{\mathbf{x}}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$ for any node $\mathbf{x}'$; see (10)), we choose $w_{En} = (\lambda_{hn}+\sigma_n(\mathbf{u}_h))b_E$ and $w_{Et} = 0$ in the element K containing E (here we make a slight abuse of notation to simplify) and $\mathbf{w}_E = \mathbf{0}$ in $\overline{\Omega} \setminus K$. Therefore

$$\begin{aligned}
\|\lambda_{hn} + \sigma_n(\mathbf{u}_h)\|_E^2 &\sim \int_E (\lambda_{hn} + \sigma_n(\mathbf{u}_h))w_{En}\\
&= b(\boldsymbol{\lambda}_h, \mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E)\\
&= b(\boldsymbol{\lambda}_h, \mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{w}_E)\\
&= b(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \mathbf{w}_E) + L(\mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E)\\
&\lesssim \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{w}_E\|_E + \|\mathbf{f}\|_K \|\mathbf{w}_E\|_K + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} \|\mathbf{w}_E\|_{1,K}.
\end{aligned}$$

An inverse inequality and estimate (31) imply

$$\begin{aligned}
h_K^{1/2} \|\lambda_{hn} + \sigma_n(\mathbf{u}_h)\|_E &\lesssim h_K^{1/2} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + h_K \|\mathbf{f}\|_K\\
&\lesssim h_K^{1/2} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K.
\end{aligned}$$

This estimate gives the bound of η_{3K} in (33). The estimate of η_{4K} in (33) is obtained as previously by choosing $w_{En} = 0$ and $w_{Et} = (\lambda_{ht} + \sigma_t(\mathbf{u}_h))b_E$.

We now consider η_{5K}. If $E \in E_K^C$, let $F \subset E$ be the part of the edge where $\lambda_{hn} = \lambda_{hn+}$. So

$$\begin{aligned}
\int_E -\lambda_{hn+}u_{hn} &= \int_F -\lambda_{hn}u_{hn}\\
&= \int_F (\lambda_{hn} - \lambda_n)(u_n - u_{hn}) - \int_F \lambda_{hn}u_n - \int_F \lambda_n u_{hn}\\
&= \int_F (\lambda_{hn} - \lambda_n)(u_n - u_{hn}) - \int_F (\lambda_{hn} - \lambda_n)u_n - \int_F \lambda_n(u_{hn} - u_n)\\
&\leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u} - \mathbf{u}_h\|_E + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u}\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E \|\boldsymbol{\lambda}\|_E.
\end{aligned}$$

The last estimate implies the bound of η_{5K} in (34) by taking the square root.

The estimate of η_{6K} in (35) is obvious. Since $\lambda_n \geq 0$ we have $0 \leq \lambda_{hn-} \leq |\lambda_n - \lambda_{hn}|$ on Γ_C. So

$$\|\lambda_{hn-}\|_E \leq \|\lambda_n - \lambda_{hn}\|_E \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E.$$

Next we estimate η_{7K}. If $E \in E_K^C$, let $F \subset E$ be the part of the edge where $-(|\lambda_{ht}| - \mu\lambda_{hn+}) = (|\lambda_{ht}| - \mu\lambda_{hn+})_-$. So

$$\begin{aligned}
\int_E (|\lambda_{ht}| - \mu\lambda_{hn+})_- |u_{ht}| + \int_E (\lambda_{ht}u_{ht})_- &= \int_F (-|\lambda_{ht}| + \mu\lambda_{hn+})|u_{ht}| + \int_E (\lambda_{ht}u_{ht})_-\\
&= \int_F (-|\lambda_{ht}| + \mu\lambda_{hn})|u_{ht}| + \int_E (\lambda_{ht}u_{ht})_- + \int_F \mu\lambda_{hn-}|u_{ht}|. \qquad (36)
\end{aligned}$$

The first term in (36) is estimated as follows using (6):

$$\begin{aligned}-\int_F (|\lambda_{ht}| - \mu\lambda_{hn})|u_{ht}| &= -\int_F (|\lambda_{ht}| - |\lambda_t| - \mu(\lambda_{hn} - \lambda_n))(|u_{ht}| - |u_t|)\\ &\quad - \int_F (|\lambda_{ht}| - |\lambda_t| - \mu(\lambda_{hn} - \lambda_n))|u_t|\\ &\quad - \int_F (|\lambda_t| - \mu\lambda_n)(|u_{ht}| - |u_t|)\\ &\le (1+\mu)(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u} - \mathbf{u}_h\|_E\\ &\quad + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u}\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E \|\boldsymbol{\lambda}\|_E).\end{aligned}$$

The second term in (36) is estimated by noting that $\lambda_t u_t \ge 0$ on Γ_C. Hence

$$\begin{aligned}0 \le (\lambda_{ht}u_{ht})_- &\le |\lambda_t u_t - \lambda_{ht}u_{ht}|\\ &= |\lambda_t(u_t - u_{ht}) + (\lambda_t - \lambda_{ht})(u_{ht} - u_t) + (\lambda_t - \lambda_{ht})u_t|.\end{aligned}$$

So

$$\int_E (\lambda_{ht}u_{ht})_- \le \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u} - \mathbf{u}_h\|_E + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E \|\mathbf{u}\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E \|\boldsymbol{\lambda}\|_E.$$

The third term in (36) yields, using the estimate of η_{6K},

$$\int_F \mu\lambda_{hn-}|u_{ht}| \le \int_E \mu\lambda_{hn-}|u_{ht} - u_t| + \int_E \mu\lambda_{hn-}|u_t| \le \mu\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E(\|\mathbf{u} - \mathbf{u}_h\|_E + \|\mathbf{u}\|_E).$$

This proves the bound of η_{7K}. Finally we consider the upper bound of η_{8K}. We have

$$0 \le (|\lambda_{ht}| - \mu\lambda_{hn+})_+ = (|\lambda_{ht}| - \mu\lambda_{hn} - \mu\lambda_{hn-})_+ \le (|\lambda_{ht}| - \mu\lambda_{hn})_+.$$

Since $|\lambda_t| - \mu\lambda_n \le 0$, we have

$$(|\lambda_{ht}| - \mu\lambda_{hn})_+ \le ||\lambda_{ht}| - |\lambda_t| - \mu\lambda_{hn} + \mu\lambda_n| \le |\lambda_{ht} - \lambda_t| + \mu|\lambda_{hn} - \lambda_n|.$$

Hence

$$\|(|\lambda_{ht}| - \mu\lambda_{hn+})_+\|_E \le \|\lambda_t - \lambda_{ht}\|_E + \mu\|\lambda_n - \lambda_{hn}\|_E \le (1+\mu)\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_E. \qquad \square$$

Remark 5. Assume that $\mathbf{u} \in (H^2(\Omega))^2$ (so $\boldsymbol{\lambda} \in (H^{\frac{1}{2}}(\Gamma_C))^2$), and that optimal a priori error estimates hold (note that this question is entirely open and that the only aim of the present remark is to try to illustrate our result), and define

$$\eta_i = \Big(\sum_{K\in T_h} \eta_{iK}^2\Big)^{1/2}, \quad 1 \le i \le 8.$$

Then one would have $\eta_i \lesssim h, 1 \le i \le 4$; $\eta_j \lesssim h^{1/4},\ j = 5, 7$; $\eta_l \lesssim h^{1/2},\ l = 6, 8$. So $\eta \lesssim h^{1/4}$.

5. A second finite element discretization and the corresponding estimator $\tilde{\boldsymbol{\eta}}$. The aim of this section is to consider a finite element discretization of the frictional contact conditions which allows us to obtain a simpler residual error estimator. More precisely a different quadrature formula is used for the frictional contact conditions (see [53] for the early idea).

5.1. Preliminaries. For any $\boldsymbol{\nu} = (\nu_{hn}, \nu_{ht}) \in W_h \times W_h$ and $\mathbf{v}_h \in \mathbf{V}_h$, we define the bilinear form $c(.,.)$ such that

$$c(\boldsymbol{\nu}_h, \mathbf{v}_h) = \int_{\Gamma_C} (I_h(\nu_{hn} v_{hn}) + I_h(\nu_{ht} v_{ht}))\ d\Gamma,$$

where I_h is the classical piecewise affine Lagrange interpolation operator at the nodes of $\overline{\Gamma_C}$. Let $K_{hn} = \{\nu_h \in W_h : \ \nu_h \geq 0\ \}$ be the closed convex cone of nonnegative functions in W_h. For $g \in K_{hn}$, we set $K_{ht}(g) = \{\nu_h \in W_h : \ |\nu_h| \leq g\ \}$.

Next, we consider the problem of finding $\tilde{\mathbf{u}}_h \in \mathbf{V}_h$ and $(\tilde{\lambda}_{hn}, \tilde{\lambda}_{ht}) = \tilde{\boldsymbol{\lambda}}_h \in \mathbf{K}_h(\mu\tilde{\lambda}_{hn}) = K_{hn} \times K_{ht}(\mu\tilde{\lambda}_{hn})$ satisfying

$$(37) \qquad \begin{cases} a(\tilde{\mathbf{u}}_h, \mathbf{v}_h) + c(\tilde{\boldsymbol{\lambda}}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ c(\boldsymbol{\nu}_h - \tilde{\boldsymbol{\lambda}}_h, \tilde{\mathbf{u}}_h) \leq 0 \qquad \forall\, \boldsymbol{\nu}_h = (\nu_{hn}, \nu_{ht}) \in \mathbf{K}_h(\mu\tilde{\lambda}_{hn}). \end{cases}$$

Problem (37) could also be written without a bootstrap like the continuous problem (see Remark 1). Using the same techniques as in [25] for problem (9), one can prove that the problem (37) admits at least a solution and that there is a unique solution when $\mu \leq C(h)$. The proof of this result can be found in the appendix. Besides one can prove that the pointwise discrete frictional contact conditions incorporated in the inequality of (37) are as follows.

PROPOSITION 5.1. *Let* $(\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *be a solution of* (37). *Suppose that* $\dim(W_h) = p$ *and let* $\psi_{\mathbf{x}_i}, 1 \leq i \leq p$, *be the basis functions of* W_h *on* Γ_C. *Let* $\tilde{U}_N$ *and* $\tilde{U}_T$ *denote the vectors whose components are the nodal values of* $\tilde{u}_{hn}$ *and* $\tilde{u}_{ht}$, *respectively, and let* $\tilde{L}_N$ *and* $\tilde{L}_T$ *denote the vectors whose components are the nodal values of* $\tilde{\lambda}_{hn}$ *and* $\tilde{\lambda}_{ht}$, *respectively. Then the discrete frictional contact conditions in* (37) *are as follows: for any* $1 \leq i \leq p$

$$(38) \qquad (\tilde{L}_N)_i \geq 0, \qquad (\tilde{U}_N)_i \leq 0, \qquad (\tilde{L}_N)_i(\tilde{U}_N)_i = 0,$$

$$(39) \qquad |(\tilde{L}_T)_i| \leq \mu(\tilde{L}_N)_i,$$

$$(40) \qquad |(\tilde{L}_T)_i| < \mu(\tilde{L}_N)_i \implies (\tilde{U}_T)_i = 0,$$

$$(41) \qquad (\tilde{L}_T)_i(\tilde{U}_T)_i \geq 0.$$

Proof. From $\tilde{\lambda}_{hn} \in K_{hn}$, we immediately get (38). Condition

$$\int_{\Gamma_C} I_h((\nu_{hn} - \tilde{\lambda}_{hn})\tilde{u}_{hn})\ d\Gamma \leq 0 \qquad \forall \nu_{hn} \in K_{hn}$$

is equivalent to

$$(42) \int_{\Gamma_C} I_h(\nu_{hn}\tilde{u}_{hn})\ d\Gamma \leq 0 \qquad \forall \nu_{hn} \in K_{hn} \qquad \text{and} \qquad \int_{\Gamma_C} I_h(\tilde{\lambda}_{hn}\tilde{u}_{hn})\ d\Gamma = 0.$$

Choosing in (42) $\nu_{hn} = \psi_{\mathbf{x}_i}$ and writing $\int_{\Gamma_C} I_h(\psi_{\mathbf{x}_i}\tilde{u}_{hn}) = \tilde{u}_{hn}(\mathbf{x}_i)\ \int_{\Gamma_C} \psi_{\mathbf{x}_i}$ gives the second inequality in (38). Equality $\int_{\Gamma_C} I_h(\tilde{\lambda}_{hn}\tilde{u}_{hn}) = \sum_{i=1}^{p} \tilde{\lambda}_{hn}(\mathbf{x}_i)\tilde{u}_{hn}(\mathbf{x}_i) \int_{\Gamma_C} \psi_{\mathbf{x}_i} = 0$ implies $(\tilde{L}_N)_i(\tilde{U}_N)_i = 0, 1 \leq i \leq p$.

Inequality (39) follows directly from $\tilde{\lambda}_{ht} \in K_{ht}(\mu\tilde{\lambda}_{hn})$. Since

$$(43) \qquad \int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht})\ d\Gamma \leq 0 \qquad \forall \nu_{ht} \in K_{ht}(\mu\tilde{\lambda}_{hn})$$

we choose ν_{ht} in (43) as follows: $\nu_{ht} = \mu\tilde{\lambda}_{hn}$ at node $\mathbf{x}_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p-1$ other nodes. We obtain

$$(44)\quad \int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht})\, d\Gamma = (\mu\tilde{\lambda}_{hn}(\mathbf{x}_i) - \tilde{\lambda}_{ht}(\mathbf{x}_i))\tilde{u}_{ht}(\mathbf{x}_i) \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma \le 0.$$

Similarly, take $\nu_{ht} = -\mu\tilde{\lambda}_{hn}$ at node $\mathbf{x}_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p-1$ other nodes. We get

$$(45)\quad \int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht})\, d\Gamma = (-\mu\tilde{\lambda}_{hn}(\mathbf{x}_i) - \tilde{\lambda}_{ht}(\mathbf{x}_i))\tilde{u}_{ht}(\mathbf{x}_i) \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma \le 0.$$

Putting together estimates (44) and (45) implies (40).

It remains to prove (41). Define ν_{ht} in (43) as follows: $\nu_{ht} = \frac{1}{2}\tilde{\lambda}_{ht}$ at node $\mathbf{x}_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p-1$ other nodes. Therefore

$$\int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht})\, d\Gamma = -\frac{1}{2}\tilde{\lambda}_{ht}(\mathbf{x}_i)\tilde{u}_{ht}(\mathbf{x}_i) \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma \le 0.$$

Hence we get inequality (41). □

5.2. Definition of the residual error estimator. As for the first discretization the element residual is defined by $\mathbf{div}\boldsymbol{\sigma}(\tilde{\mathbf{u}}_h) + \mathbf{f} = \mathbf{f}$ on K. The data $\mathbf{f}$ can be replaced by $\mathbf{f}_K \in (\mathbb{P}_k(K))^2$, and the difference $\mathbf{f} - \mathbf{f}_K$ will be treated as data oscillation. Similarly $\mathbf{F}$ can be approximated by a simpler quantity denoted $\mathbf{F}_E$ on any $E \in E_h^N$.

DEFINITION 5.2. *The global residual estimator $\tilde{\eta}$ and the local residual error estimators $\tilde{\eta}_K$ are defined by*

$$\begin{aligned}
\tilde{\eta} &= \left(\sum_{K\in T_h} \tilde{\eta}_K^2\right)^{1/2},\\
\tilde{\eta}_K &= \left(\sum_{i=1}^{6} \tilde{\eta}_{iK}^2\right)^{1/2},\\
\tilde{\eta}_{1K} &= h_K\|\mathbf{f}_K\|_K,\\
\tilde{\eta}_{2K} &= h_K^{1/2}\left(\sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(\tilde{\mathbf{u}}_h)\|_E^2\right)^{1/2},\\
\tilde{\eta}_{3K} &= h_K^{1/2}\|\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h)\|_{K\cap\Gamma_C},\\
\tilde{\eta}_{4K} &= h_K^{1/2}\|\tilde{\lambda}_{ht} + \sigma_t(\tilde{\mathbf{u}}_h)\|_{K\cap\Gamma_C},\\
\tilde{\eta}_{5K} &= \left(\int_{K\cap\Gamma_C} -\tilde{\lambda}_{hn}\tilde{u}_{hn}\right)^{1/2},\\
\tilde{\eta}_{6K} &= \left(\int_{K\cap\Gamma_C} (\mu\tilde{\lambda}_{hn}\,|\tilde{u}_{ht}| - \tilde{\lambda}_{ht}\tilde{u}_{ht})\right)^{1/2},
\end{aligned}$$

where we recall that $J_{E,n}(\tilde{\mathbf{u}}_h)$ is the constraint jump of $\tilde{\mathbf{u}}_h$ in the normal direction defined by (13). *As in the previous section, the local and global data oscillation terms ζ_K and ζ are defined by* (14).

Remark 6. From the previous definitions we have $\tilde{\eta}_{1K} = \eta_{1K}$. We mention that there are no terms η_6 and η_8 in $\tilde{\eta}$ since $\tilde{\lambda}_{hn} \ge 0$ and $|\tilde{\lambda}_{ht}| \le \mu\tilde{\lambda}_{hn}$.

5.3. Upper error bound. As in the statement of Theorem 4.2 we need to assume that the solution to the continuous problem satisfies the uniqueness criterion of [72] in order to obtain the upper bound of the discretization error.

THEOREM 5.3. *Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *be the solution to problem* (7) *such that* $\lambda_t = \mu\lambda_n\xi$, *with* $\xi \in M$, $\xi \in Dir_t(u_t)$ *a.e. on* Γ_C, *and* $\mu\|\xi\|_M$ *is small enough. Let* $(\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *be a solution to the discrete problem* (37)*. Then*

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,\Omega} + \|\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \tilde{\eta} + \zeta.$$

Proof. We adopt the following notations for the error term in the displacement: $\tilde{\mathbf{e}}_{\mathbf{u}} = \mathbf{u} - \tilde{\mathbf{u}}_h$. As in Theorem 4.2, we obtain for any $\mathbf{v}_h \in \mathbf{V}_h$

$$\begin{aligned}
\|\tilde{\mathbf{e}}_{\mathbf{u}}\|^2_{1,\Omega} \lesssim & \int_\Omega \mathbf{f}\cdot(\mathbf{u}-\mathbf{v}_h) - \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\tilde{\mathbf{u}}_h)\cdot(\mathbf{u}-\mathbf{v}_h) \\
& + \sum_{E\in E_h^C} \int_E (\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h))(v_{hn} - u_n) + \sum_{E\in E_h^C} \int_E (\tilde{\lambda}_{ht} + \sigma_t(\tilde{\mathbf{u}}_h))(v_{ht} - u_t) \\
& + \int_{\Gamma_C} (I_h(\tilde{\lambda}_{hn}(v_{hn} - \tilde{u}_{hn})) - \tilde{\lambda}_{hn}(v_{hn} - \tilde{u}_{hn})) \\
& + \int_{\Gamma_C} (I_h(\tilde{\lambda}_{ht}(v_{ht} - \tilde{u}_{ht})) - \tilde{\lambda}_{ht}(v_{ht} - \tilde{u}_{ht})) + \sum_{E\in E_h^N} \int_E (\mathbf{F} - \mathbf{F}_E)\cdot(\mathbf{u}-\mathbf{v}_h) \\
& + \int_{\Gamma_C} (\tilde{\lambda}_{hn} - \lambda_n)(u_n - \tilde{u}_{hn}) + \int_{\Gamma_C} (\tilde{\lambda}_{ht} - \lambda_t)(u_t - \tilde{u}_{ht})
\end{aligned}$$

$$= \tilde{I} + \tilde{II} + \tilde{III} + \tilde{IV} + \tilde{V} + \tilde{VI} + \tilde{VII} + \tilde{VIII} + \int_{\Gamma_C} (\tilde{\lambda}_{ht} - \lambda_t)(u_t - \tilde{u}_{ht}). \tag{46}$$

As in Theorem 4.2, we take $\mathbf{v}_h$ of the form (16). So

$$\left|\tilde{I}\right| + \left|\tilde{II}\right| + \left|\tilde{III}\right| + \left|\tilde{IV}\right| + \left|\tilde{V}\right| \lesssim (\tilde{\eta} + \zeta)\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\Omega}. \tag{47}$$

Now we estimate the two terms in (46) with the interpolation operator using a basic error estimate of numerical integration (trapezoidal formula):

$$\begin{aligned}
\left|\tilde{VI}\right| &= \left|\int_{\Gamma_C} \left(I_h(\tilde{\lambda}_{hn}(\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n) - \tilde{\lambda}_{hn}(\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n\right)\right| \\
&= \left|\sum_{E\in E_h^C} \int_E \left(I_h(\tilde{\lambda}_{hn}(\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n) - \tilde{\lambda}_{hn}(\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n\right)\right| \\
&\lesssim \sum_{E\in E_h^C} h_E^3 |(\tilde{\lambda}_{hn}(\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n)''| \\
&\lesssim \sum_{E\in E_h^C} h_E^3 |\tilde{\lambda}'_{hn}((\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n)'| \\
&\le \sum_{E\in E_h^C} h_E^2 \|\tilde{\lambda}'_{hn}\|_E \|((\pi_h\tilde{\mathbf{e}}_{\mathbf{u}})_n)'\|_E \\
&\lesssim \sum_{E\in E_h^C} h_E^{3/2} \|\tilde{\lambda}'_{hn}\|_E \|\pi_h\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,K}
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\lesssim \sum_{E\in E_h^C} h_E^{3/2}\|\tilde{\lambda}'_{hn}\|_E\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\omega_K}\\
&= \sum_{E\in E_h^C} h_E^{3/2}\|(\tilde{\lambda}_{hn}+\sigma_n(\tilde{\mathbf{u}}_h))'\|_E\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\omega_K}\\
&\lesssim \sum_{E\in E_h^C} h_E^{1/2}\|\tilde{\lambda}_{hn}+\sigma_n(\tilde{\mathbf{u}}_h)\|_E\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\omega_K}\\
&\lesssim \tilde{\eta}\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\Omega},
\end{aligned}$$

where K is the element containing E. Above we have used the Cauchy–Schwarz inequality, the H^1 stability of π_h, proved in Lemma 3.1 of [22] (see also [79]), and the trace inequality on an element (see [79]). In a similar way, we obtain

$$\left|\tilde{VII}\right| \lesssim \sum_{E\in E_h^C} h_E^{1/2}\|\tilde{\lambda}_{ht}+\sigma_t(\tilde{\mathbf{u}}_h)\|_E\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\omega_K} \le \tilde{\eta}\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\Omega}. \tag{49}$$

Noting that $\tilde{u}_{hn}\le 0$ and $\tilde{\lambda}_{hn}\ge 0$ on Γ_C, we have $\int_{\Gamma_C}\tilde{\lambda}_{hn}u_n\le 0$, $\int_{\Gamma_C}\lambda_n u_n = 0$, and $\int_{\Gamma_C}\lambda_n\tilde{u}_{hn}\le 0$. Consequently, we obtain

$$\tilde{VIII} \le \int_{\Gamma_C} -\tilde{\lambda}_{hn}\tilde{u}_{hn} \le \tilde{\eta}^2. \tag{50}$$

It remains to estimate one term in (46): the one coming from the friction approximation. As in (25) and (26), we obtain

$$\int_{\Gamma_C}(\tilde{\lambda}_{ht}-\lambda_t)(u_t-\tilde{u}_{ht}) = \int_{\Gamma_C}(\tilde{\lambda}_{ht}-\mu\tilde{\lambda}_{hn}\xi)(u_t-\tilde{u}_{ht}) + \int_{\Gamma_C}\mu(\tilde{\lambda}_{hn}-\lambda_n)\xi(u_t-\tilde{u}_{ht}), \tag{51}$$

where $\xi\in M, \xi\in \mathrm{Dir}_t(u_t), \lambda_t = \mu\lambda_n\xi$, and

$$\left|\int_{\Gamma_C}\mu(\tilde{\lambda}_{hn}-\lambda_n)\xi(u_t-\tilde{u}_{ht})\right| \lesssim (1+\alpha)\mu\|\xi\|_M\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\Omega}^2 + \frac{\mu\|\xi\|_M}{2\alpha}\left(\tilde{\eta}^2+\zeta^2\right) \tag{52}$$

for any positive α. The first term in (51) is handled as follows:

$$\begin{aligned}
\int_{\Gamma_C}(\tilde{\lambda}_{ht}-\mu\tilde{\lambda}_{hn}\xi)(u_t-\tilde{u}_{ht}) &= \int_{\Gamma_C}\tilde{\lambda}_{ht}u_t - \int_{\Gamma_C}\mu\tilde{\lambda}_{hn}\xi u_t + \int_{\Gamma_C}\mu\tilde{\lambda}_{hn}\xi\tilde{u}_{ht} - \int_{\Gamma_C}\tilde{\lambda}_{ht}\tilde{u}_{ht}\\
&= \int_{\Gamma_C}(\tilde{\lambda}_{ht}u_t - \mu\tilde{\lambda}_{hn}\,|u_t|) + \int_{\Gamma_C}(\mu\tilde{\lambda}_{hn}\xi\tilde{u}_{ht}-\tilde{\lambda}_{ht}\tilde{u}_{ht})\\
&\le \underbrace{\int_{\Gamma_C}(|\tilde{\lambda}_{ht}|-\mu\tilde{\lambda}_{hn})\,|u_t|}_{\le 0} + \int_{\Gamma_C}(-\tilde{\lambda}_{ht}\tilde{u}_{ht}+\mu\tilde{\lambda}_{hn}\,|\tilde{u}_{ht}|)\\
&\le \tilde{\eta}^2.
\end{aligned} \tag{53}$$

By (52) and (53), we obtain for any positive α

$$\int_{\Gamma_C}(\tilde{\lambda}_{ht}-\lambda_t)(u_t-\tilde{u}_{ht}) \lesssim (1+\alpha)\mu\|\xi\|_M\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{1,\Omega}^2 + \frac{\mu\|\xi\|_M+2\alpha}{2\alpha}\left(\tilde{\eta}^2+\zeta^2\right). \tag{54}$$

Putting together the estimates (47), (49), (49), (50), and (54) in (46) and using Young's inequality, we come to the conclusion that if $\mu\|\xi\|_M$ is small enough (see also (29)),

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,\Omega} \lesssim \tilde{\eta} + \zeta. \tag{55}$$

As in the proof of Theorem 4.2, we obtain

$$\|\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim +\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,\Omega} + \tilde{\eta} + \zeta. \tag{56}$$

Putting together the two estimates (55) and (56) ends the proof of the theorem. ∎

5.4. Lower error bound.

THEOREM 5.4. *Let* $(\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *be a solution to the discrete problem* (37) *and let* $\tilde{\eta} = \tilde{\eta}(\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *be the corresponding estimator. Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *be a solution to problem* (7) *such that* $\boldsymbol{\lambda} \in (L^2(\Gamma_C))^2$*. For all elements* K*, the following local lower error bounds hold:*

$$\tilde{\eta}_{1K} \lesssim \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,K} + \zeta_K,$$
$$\tilde{\eta}_{2K} \lesssim \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,\omega_K} + \zeta_K.$$

For all elements K *having an edge in* Γ_C *(i.e.,* $K \cap \Gamma_C = E$*), the following local lower error bounds hold:*

$$\tilde{\eta}_{iK} \lesssim h_K^{1/2}\|\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h\|_E + \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{1,K} + \zeta_K, \quad i = 3, 4,$$
$$\tilde{\eta}_{5K} \lesssim \tilde{\eta}_{3K}^{1/2}\|\tilde{\mathbf{u}}_h\|_{1,K}^{1/2}, \tag{57}$$
$$\tilde{\eta}_{6K} \lesssim (\mu\tilde{\eta}_{3K} + \tilde{\eta}_{4K})^{1/2}\|\tilde{\mathbf{u}}_h\|_{1,K}^{1/2}. \tag{58}$$

Proof. As in Theorem 4.3 we need only estimate $\tilde{\eta}_{3K}$, $\tilde{\eta}_{4K}$, $\tilde{\eta}_{5K}$, and $\tilde{\eta}_{6K}$. In addition, the bounds of $\tilde{\eta}_{3K}$, $\tilde{\eta}_{4K}$ are obtained as in Theorem 4.3. So we consider $\tilde{\eta}_{5K}$. If $E \in E_K^C$, one has by the trapezoidal integration formula an inverse inequality and the scaled trace inequality:

$$\begin{aligned}
\int_E -\tilde{\lambda}_{hn}\tilde{u}_{hn} &= \int_E (I_h(\tilde{\lambda}_{hn}\tilde{u}_{hn}) - \tilde{\lambda}_{hn}\tilde{u}_{hn}) \\
&\lesssim h_E^3|(\tilde{\lambda}_{hn}\tilde{u}_{hn})''| \\
&\lesssim h_E^3|\tilde{\lambda}'_{hn}\tilde{u}'_{hn}| \\
&\le h_E^2\|\tilde{\lambda}'_{hn}\|_E\|\tilde{u}'_{hn}\|_E \\
&= h_E^2\|(\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h))'\|_E\|\tilde{u}'_{hn}\|_E \\
&\lesssim h_E\|\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h)\|_E\|\tilde{u}'_{hn}\|_E \\
&\lesssim h_E^{1/2}\tilde{\eta}_{3K}\|\tilde{u}'_{hn}\|_E \\
&\lesssim \tilde{\eta}_{3K}\|\tilde{\mathbf{u}}_h\|_{1,K}.
\end{aligned}$$

The last estimate implies (57) by taking the square root.

Finally we consider $\tilde{\eta}_{6K}$. According to Proposition 5.1 we have for any node $\mathbf{x}_i$ in $\overline{\Gamma_C}$ $(\mu\tilde{\lambda}_{hn}|\tilde{u}_{ht}| - \tilde{\lambda}_{ht}\tilde{u}_{ht})(\mathbf{x}_i) = ((\mu\tilde{\lambda}_{hn} - |\tilde{\lambda}_{ht}|)|\tilde{u}_{ht}|)(\mathbf{x}_i) = 0$. Let $E \in E_K^C$; it is easy to see that either $\tilde{u}_{ht}$ is of constant sign on E (i.e., nonnegative or nonpositive) or $\tilde{u}_{ht}(\mathbf{x}_1)\tilde{u}_{ht}(\mathbf{x}_2) < 0$ (where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the extremities of E) and $\tilde{u}_{ht}$ admits a unique zero denoted $\mathbf{m}$ in E.

Let us first consider the second case: we denote $E_1 = (\mathbf{x}_1, \mathbf{m})$ and $E_2 = (\mathbf{m}, \mathbf{x}_2)$ and suppose without loss of generality that $\tilde{u}_{ht} > 0$ in E_1 and $\tilde{u}_{ht} < 0$ in E_2. We denote by J_h the piecewise affine Lagrange interpolation operator defined in E at the points $\mathbf{x}_1, \mathbf{m}, \mathbf{x}_2$. Since $(\mu\tilde{\lambda}_{hn}|\tilde{u}_{ht}| - \tilde{\lambda}_{ht}\tilde{u}_{ht})(\mathbf{m}) = 0$ and using the same arguments as for $\tilde{\eta}_{5K}$, we get

$$\begin{aligned}
&\int_E (\mu\tilde{\lambda}_{hn}\,|\tilde{u}_{ht}| - \tilde{\lambda}_{ht}\tilde{u}_{ht}) \\
&= \int_{E_1} (\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) + \int_{E_2} (-\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) \\
&= \int_{E_1} \Big((\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) - J_h(\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) \Big) \\
&\quad + \int_{E_2} \Big((-\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) - J_h(-\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht}) \Big) \\
&\lesssim h_{E_1}^3 |(\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht})''{}_{|_{E_1}}| + h_{E_2}^3 |(-\mu\tilde{\lambda}_{hn}\tilde{u}_{ht} - \tilde{\lambda}_{ht}\tilde{u}_{ht})''{}_{|_{E_2}}| \\
&\le h_E^3 |(\mu\tilde{\lambda}_{hn}\tilde{u}_{ht})''{}_{|_E}| + h_E^3 |(\tilde{\lambda}_{ht}\tilde{u}_{ht})''{}_{|_E}| \\
&\lesssim h_E^3 \mu |\tilde{\lambda}'_{hn}\tilde{u}'_{ht}| + h_E^3 |\tilde{\lambda}'_{ht}\tilde{u}'_{ht}| \\
&\le h_E^2 \mu \|\tilde{\lambda}'_{hn}\|_E \|\tilde{u}'_{ht}\|_E + h_E^2 \|\tilde{\lambda}'_{ht}\|_E \|\tilde{u}'_{ht}\|_E \\
&= h_E^2 \mu \|(\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h))'\|_E \|\tilde{u}'_{ht}\|_E + h_E^2 \|(\tilde{\lambda}_{ht} + \sigma_t(\tilde{\mathbf{u}}_h))'\|_E \|\tilde{u}'_{ht}\|_E \\
&\lesssim h_E \mu \|\tilde{\lambda}_{hn} + \sigma_n(\tilde{\mathbf{u}}_h)\|_E \|\tilde{u}'_{ht}\|_E + h_E \|\tilde{\lambda}_{ht} + \sigma_t(\tilde{\mathbf{u}}_h)\|_E \|\tilde{u}'_{ht}\|_E \\
&\lesssim h_E^{1/2} (\mu\tilde{\eta}_{3K} + \tilde{\eta}_{4K}) \|\tilde{u}'_{ht}\|_E \\
&\lesssim (\mu\tilde{\eta}_{3K} + \tilde{\eta}_{4K}) \|\tilde{\mathbf{u}}_h\|_{1,K}.
\end{aligned}$$

Hence we get (58) by taking the square root. The first case ($\tilde{u}_{ht}$ is either nonnegative or nonpositive in E) is straightforward and handled as previously. ∎

Remark 7. Assume that $\mathbf{u} \in (H^2(\Omega))^2$ (so $\boldsymbol{\lambda} \in (H^{\frac{1}{2}}(\Gamma_C))^2$) and that optimal a priori error estimates hold (as for the first finite element approximation, this question is entirely open and the only aim of the present remark is to try to illustrate our result). We define

$$\tilde{\eta}_i = \Big(\sum_{K \in T_h} \tilde{\eta}_{iK}^2 \Big)^{1/2}, \quad 1 \le i \le 6.$$

Then it is straightforward to check that $\tilde{\eta}_i \lesssim h, 1 \le i \le 4; \eta_j \lesssim h^{1/2},\ j = 5, 6$. So $\tilde{\eta} \lesssim h^{1/2}$. A deeper insight in the estimates of $\tilde{\eta}_{5K}$ and $\tilde{\eta}_{6K}$ (which we prefer to avoid) would show that the estimates in [47, Remark 5.7] could also be applied in our case, and this would lead to the estimate $\tilde{\eta} \lesssim (-\ln(h))^{1/4} h^{3/4}$.

6. Numerical experiments. In this section we achieve the numerical implementation of the residual estimator for both finite element discretizations. The information given by the error estimators is then coupled with a mesh adaptivity procedure. In what follows, we suppose that the bodies are homogeneous isotropic materials so that Hooke's law (2) becomes

$$\boldsymbol{\sigma}(\mathbf{u}) = \frac{EP}{(1-2P)(1+P)} \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + \frac{E}{1+P}\boldsymbol{\varepsilon}(\mathbf{u}),$$

where $\mathbf{I}$ represents the identity matrix, tr is the trace operator, and E and P denote Young's modulus and Poisson's ratio, respectively, with $E > 0$ and $0 \le P < 1/2$.

Our main aim is to discuss the theoretical results by computing the different contributions of the estimators η and $\tilde{\eta}$ and their orders of convergence as h vanishes. In particular we are interested in the following terms (where we adopt the notations of Remarks 5 and 7):

$$\eta_i = \Big(\sum_{K\in T_h} \eta_{iK}^2\Big)^{1/2}, \quad 1 \le i \le 8, \qquad \tilde{\eta}_i = \Big(\sum_{K\in T_h} \tilde{\eta}_{iK}^2\Big)^{1/2}, \quad 1 \le i \le 6.$$

We will also make use of the frictional contact contributions

$$\eta_C = \Big(\sum_{i=3}^{8} \eta_i^2\Big)^{1/2}, \qquad \tilde{\eta}_C = \Big(\sum_{i=3}^{6} \tilde{\eta}_i^2\Big)^{1/2}.$$

In the following we denote by N_C the number of elements of the mesh on Γ_C. In the case of uniform meshes this parameter measures the size of the mesh. Moreover we suppose that the friction coefficient μ and the meshsize h are such that both discrete problems (9) and (37) admit unique solutions $(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ and $(\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\lambda}}_h)$. In such a case it is easy to check that $\mathbf{u}_h = \tilde{\mathbf{u}}_h$ and that $c(\tilde{\boldsymbol{\lambda}}_h, \mathbf{v}_h) = b(\boldsymbol{\lambda}_h, \mathbf{v}_h)\ \forall \mathbf{v}_h \in \mathbf{V}_h$, which implies that $\eta_2 = \tilde{\eta}_2$.

6.1. A first example with slip and separation. We consider the domain $\Omega =]0,1[\times]0,1[$ with material characteristics $E = 10^6$ and $P = 0.3$. The body is clamped on $\Gamma_D = \{0\}\times]0,1[$, it is initially in contact with $\Gamma_C = \{1\}\times]0,1[$, and no force is applied on $\Gamma_N =]0,1[\times(\{0\} \cup \{1\})$. The body Ω is acted on by a uniform vertical force $\mathbf{f} = (0, f_2)$ with $f_2 = -76518$, and the friction coefficient μ equals 0.2. We use criss-cross meshes (this means that the body is divided into identical squares, each of them being divided into four identical triangles). Figure 1 depicts the initial and deformed configurations with $N_C = 32$.

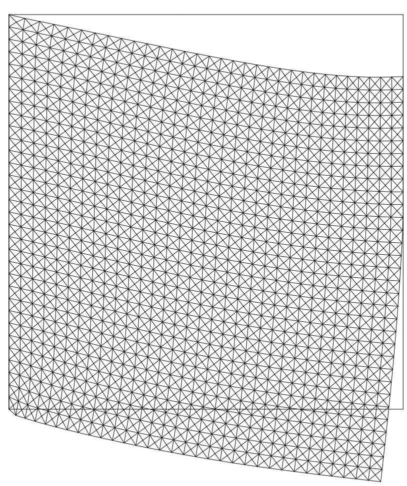

FIG. 1. *First example. Initial and deformed configurations with $\mu = 0.2$ and $N_C = 32$.*

We first observe that all the nodes on Γ_C have a negative tangential displacement and that Γ_C is divided into two parts: an upper part where the body remains in

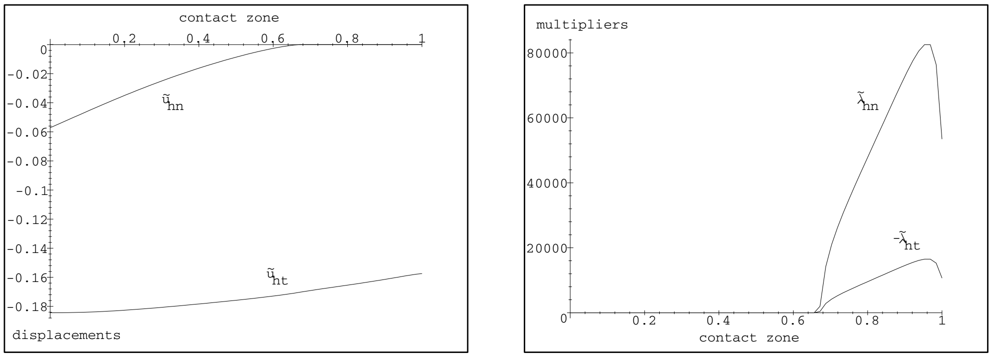

FIG. 2. *First example. Left: normal and tangential displacements* $(\tilde{u}_{hn}, \tilde{u}_{ht})$ *on* Γ_C*. Right: normal and tangential multipliers* $(\tilde{\lambda}_{hn}, -\tilde{\lambda}_{ht})$ *on* Γ_C*.*

TABLE 1
Contributions in η *and* $\tilde{\eta}$ *for the first example.*

Errors	$N_C = 1$	$N_C = 2$	$N_C = 4$	$N_C = 8$	$N_C = 16$	$N_C = 32$	$N_C = 64$	Convergence rates
$\eta_2 = \tilde{\eta}_2$	71943	89950	72476	48412	29533	17687	12504	0.57
η_3	32980	21134	6826.6	2366.7	960.54	565.63	322.29	1.21
$\tilde{\eta}_3$	11092	8681.4	4165.7	1868.4	778.11	391.41	223.87	1.06
η_4	30028	15319	6299.3	2594.8	1012.5	457.45	244.01	1.19
$\tilde{\eta}_4$	29379	14325	6079.3	2542.3	997.58	448.78	239.20	1.18
η_5	13.674	8.1415	3.0994	1.5381	0.50073	0.21377	0.036429	1.56
$\tilde{\eta}_5$	14.503	3.9747	3.1219	0.77988	0.42660	0.13956	0.039897	1.33
η_6	12680	11242	1599.8	1945.9	416.70	385.61	79.730	1.43
$\tilde{\eta}_6$	0	0	0	0	0	0	0	-
η_7	12.121	13.619	4.8373	4.6372	1.8200	1.4717	0.54418	0.93
η_8	2535.9	2248.4	319.95	389.18	83.339	77.122	15.946	1.43

contact with the axis $x = 1$ (slipping nodes) and the lower part of Γ_C where it separates from this axis with a separation point near $(1, 0.65)$ (see Figure 2). In Table 1 we report the convergence rates by averaging the rates between $N_C = 2$ and $N_C = 64$. Note that the convergence rate of the terms $\eta_1 = \tilde{\eta}_1 = h(\sum_{K \in T_h} \|\mathbf{f}_K\|_K^2)^{1/2} \sim h$ is 1.

From the computations we see that all the terms η_i and $\tilde{\eta}_i$ converge towards zero as h vanishes and that $\eta_2 = \tilde{\eta}_2$ is obviously the term converging the slowest towards zero. The main part of the error in η and $\tilde{\eta}$ is located near the singular points $(0, 0)$ and $(0, 1)$. The error terms for which no optimal error analysis is available (i.e., $\eta_5, \eta_6, \eta_7, \eta_8, \tilde{\eta}_5, \tilde{\eta}_6$) vanish faster than all the other ones except η_7, which has a slower convergence rate. Note that $\tilde{\eta}_6 = 0$ since $u_{ht} < 0$ and $\mu\tilde{\lambda}_{hn} = -\tilde{\lambda}_{ht}$ on Γ_C. We note also that the error $\tilde{\eta}_5$ is located on one element near the separation point whereas $\eta_5, \eta_6, \eta_7, \eta_8$ are located on Γ_C, especially in the separation area.

Next we couple the error estimator with a mesh adaptivity procedure. The aim of adaptive procedures is to offer the user a level of accuracy denoted η_0 with a minimal computational cost. We use the h-version in which the size and the topology of the elements are modified but the same kind of basis functions for the different meshes are retained. A mesh T^* is said to be optimal with respect to a measure of the error

η^* if (see [56])

$$\begin{cases} \eta^* = \eta_0, \\ N \text{ minimal } (N\text{: number of unknowns (or degrees of freedom) when using } T^*). \end{cases}$$
(59)

To solve problem (59), the following procedure is applied:

1. an initial analysis is performed on a relatively uniform and coarse mesh T;
2. the corresponding global, error η (resp., $\tilde{\eta}$) and the local contributions η_K (resp., $\tilde{\eta}_K$) are computed;
3. the characteristics of the optimal mesh T^* are determined in order to minimize the computational costs in respect of the global error;
4. a second finite element analysis is performed on the mesh T^*.

The optimal mesh T^* is determined by the computation of a size modification coefficient r_K on each element K of the mesh T: $r_K = h_K^*/h_K$, where h_K^* represents the size that must be imposed to the elements of T^* in the region of K in order to ensure optimality. The computation of the coefficients r_K uses the rate of convergence of the error which depends not only on the used element but also on the regularity of the solution [28]. So, to compute the coefficients r_K, we use a technique detailed in [29] that automatically takes into account the steep gradient regions. The mesh T^* is generated by an automatic mesher able to accurately respect a map of sizes. If the user wishes more accuracy, then the procedure is repeated as far as a precision close to η_0 is reached (see [28]).

Applying this procedure to the example, we obtain a family of adapted meshes which are refined near the singularities $(0,0)$ and (0,1) (see Figure 3). We also observe that the difference between the values of η and $\tilde{\eta}$ is not significant when refining, and we note that the contact contributions η_C (resp., $\tilde{\eta}_C$) are dominated by η_3, η_4 (resp., $\tilde{\eta}_3, \tilde{\eta}_4$), the other terms being small (this observation also holds for the second and third examples considered hereafter). Denoting by N the number of unknowns, we observe that the estimators η and $\tilde{\eta}$, computed on adaptively generated meshes, behave like $N^{-0.5}$ and that the contact contributions behave approximately like $N^{-0.8}$. Figure 3 depicts $\tilde{\eta}$ and $\tilde{\eta}_C$ as functions of N.

6.2. A second example with stick, slip, and separation. Next we study an example where none of the terms η_i and $\tilde{\eta}_i$ vanish ($i \geq 2$), where the three different zones characterizing friction (stick, slip, separation) exist and have softer corner singularities than in the previous example. We consider the geometry $\hat{\Omega} =]0,2[\times]0,1[$ and adopt symmetry conditions (i.e., $u_n = 0, \sigma_t(\mathbf{u}) = 0$) on $\Gamma_S = \{1\}\times]0,1[$. We achieve the computations on the square $\Omega =]0,1[\times]0,1[$. We set $\Gamma_C =]0,1[\times\{0\}$ and $\Gamma_N = (]0,1[\times\{1\}) \cup (\{0\}\times]0,1[)$. A Poisson ratio of $P = 0.2$, a Young modulus of $E = 10^4$, and a friction coefficient $\mu = 0.5$ are chosen. A density of surface forces $\mathbf{F}$ of magnitude 1 oriented inwards Ω is applied on $\{0\}\times]0.5,1[$ and $]0.5,1[\times\{1\}$. Such a configuration corresponds to a $\mathbf{K}$-elliptic case (see [41, Theorem 6.3]). Figure 4 depicts the initial and deformed configurations of the body. Here again Γ_C shows a separation and a contact part with a transition point near $(0.26, 0)$. In addition the contact part is divided into a slip part (on its left) and a stick part (on its right) with a transition point from slip to stick near $(0.47, 0)$ (see Figures 4 and 5).

It is easy to check that the symmetry conditions on Γ_C lead to supplementary error terms similar to the ones in η_4 and $\tilde{\eta}_4$, and we add these terms to $\eta_2 = \tilde{\eta}_2$. Moreover we have $\eta_1 = \tilde{\eta}_1 = 0$. The results concerning η and $\tilde{\eta}$ are reported in Table 2, where the convergence rates are averaged between $N_C = 2$ and $N_C = 128$.

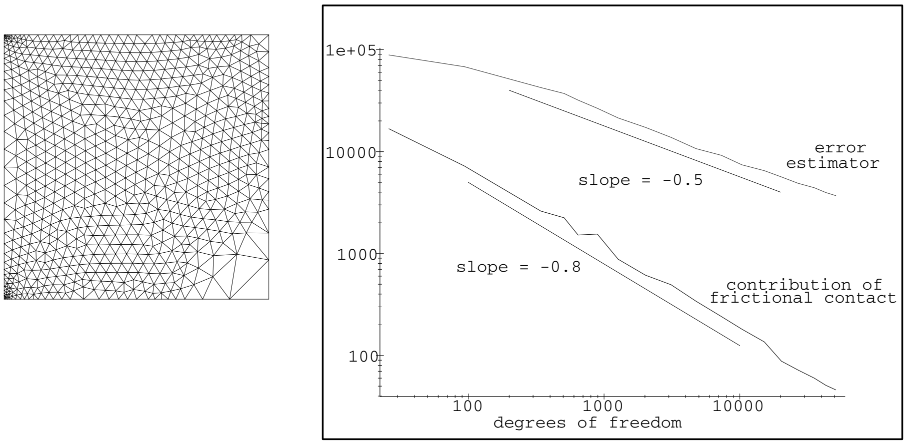

FIG. 3. *First example. Left: adapted mesh. Right: convergence of the error estimator $\tilde{\eta}$ and its frictional contact contribution $\tilde{\eta}_C$ with adaptive refinement.*

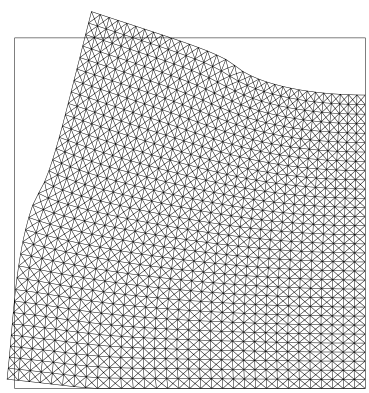

FIG. 4. *Second example. Initial and deformed configurations with $\mu = 0.5$ and $N_C = 32$ (deformation is amplified by a factor 2000).*

We observe that the errors η and $\tilde{\eta}$ are mainly located near the singularities $(0, 0.5)$ and $(0.5, 1)$ and also near the transition point between contact and separation. The error near the transition point between stick and slip is much smaller. As in the previous example, $\eta_2 = \tilde{\eta}_2$ is the main term in the estimator with the lowest (but greater than in the previous example) convergence rate and the error terms for which no optimal convergence result is available (i.e., $\eta_5, \eta_6, \eta_7, \eta_8, \tilde{\eta}_5, \tilde{\eta}_6$) vanish with a higher rate than theoretically expected. The particularity in this example is that many terms (in particular η_6) converge towards 0 with a nonuniform convergence rate.

We then apply the adaptive procedure described before and depict the initial mesh and two refined meshes in Figure 6. As previously the error decay using refinement behaves like $N^{-0.5}$ and is a bit faster than the error decay using refined meshes (near $N^{-0.45}$; see Figure 7). Figure 7 also shows the convergence of the contact contribution $\tilde{\eta}_C$, and we observe that $\tilde{\eta}_C/\tilde{\eta} \sim N^{-0.2}$, which therefore vanishes when $N \to \infty$. The results are similar when considering η instead of $\tilde{\eta}$.

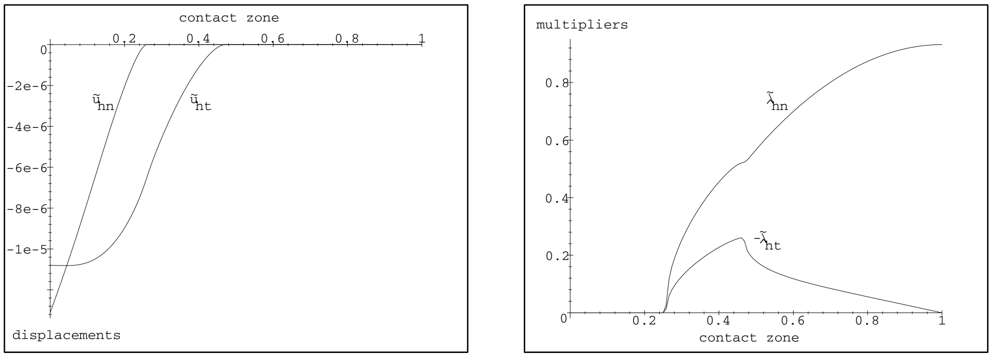

FIG. 5. *Second example. Left: normal and tangential displacements* $(\tilde{u}_{hn}, \tilde{u}_{ht})$ *on* Γ_C*. Right: normal and tangential multipliers* $(\tilde{\lambda}_{hn}, -\tilde{\lambda}_{ht})$ *on* Γ_C*.*

TABLE 2
Contributions in η *and* $\tilde{\eta}$ *for the second example.*

Errors $\times 10^5$	$N_C = 2$	$N_C = 4$	$N_C = 8$	$N_C = 16$	$N_C = 32$	$N_C = 64$	$N_C = 128$	Convergence rates
$\eta_2 = \tilde{\eta}_2$	87774	53444	32022	18577	10449	5740.85	3109.85	0.80
η_3	16925	5164.72	2111.32	857.613	359.365	113.333	43.3032	1.44
$\tilde{\eta}_3$	10176	4448.27	1665.89	642.493	256.814	93.1164	35.8664	1.36
η_4	17166	7553.27	3860.54	1818.65	848.092	388.834	184.881	1.09
$\tilde{\eta}_4$	9237.09	5292.69	2825.89	1376.62	631.021	278.175	127.115	1.03
η_5	39.1890	6.22418	3.65335	3.48613	2.18880	0.873704	0.113440	1.41
$\tilde{\eta}_5$	52.2094	24.4419	9.21759	2.94782	0.220389	0.544605	0.197534	1.34
η_6	8624.25	1500.79	228.240	505.892	647.810	226.079	8.07287	1.68
$\tilde{\eta}_6$	34.2719	16.9743	6.48435	1.95881	0.607762	0.212090	0.0769817	1.47
η_7	33.4342	19.1663	9.98284	7.51870	6.30249	3.09423	0.494072	1.01
η_8	4157.91	780.210	509.431	501.907	323.932	113.182	9.04330	1.47

6.3. Third example: A case with small friction, comparison with an example in the literature. Finally we consider an example from the literature (see [81], "square on a plane") which is somewhat more regular than the previous ones. Namely we consider the geometry $\hat{\Omega} =]0,1[\times]0,1[$ with symmetry conditions on $\Gamma_S = \{0.5\}\times]0,1[$ and compute the solutions on $\Omega =]0,0.5[\times]0,1[$. We set $\Gamma_C =]0,0.5[\times\{0\}$, $\Gamma_N = (]0,0.5[\times\{1\}) \cup (\{0\}\times]0,1[)$, $P = 0.3$, and $E = 10^4$. A density of inward oriented surface forces $\mathbf{F}(x,y) = -x^2(1-x)^2$ (resp., $\mathbf{F}(x,y) = 2y^2(1-y)^2$) is applied on $]0,0.5[\times\{1\}$ (resp., $\{0\}\times]0,1[$). We choose a small friction coefficient $\mu = 0.1$ keeping in mind that the numerical example in [81] is frictionless. Figure 8 depicts the initial and deformed configurations of the body (with $N_C = 64$). The boundary part Γ_C shows a transition point between contact and separation near $(0.08, 0)$. Due to the (small) friction we observe that (only) the last contact element near $(0.5, 0)$ is stuck on the foundation. Figure 9 shows the surface displacements and tractions on Γ_C.

The adaptive procedure is summarized in Figures 10 and 11. The initial mesh and two refined meshes are shown in Figure 10; the refined meshes are more uniform than in the previous examples and contain more small elements near the boundary (except where symmetry holds). Note that the error decay is optimal (like $N^{-0.5}$) when uniform meshes are used and that the frictional contact contribution in the error

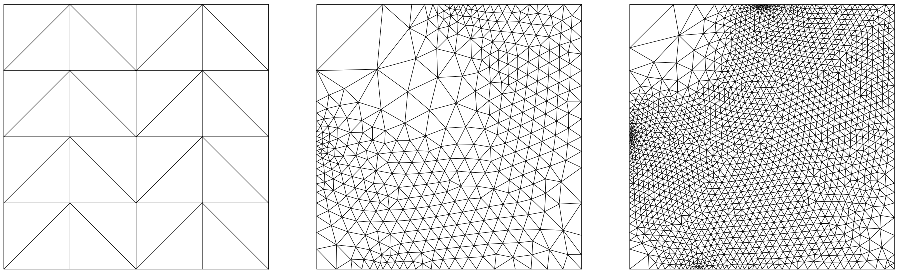

FIG. 6. *Second example. Initial (left) and refined meshes in the adaptive procedure.*

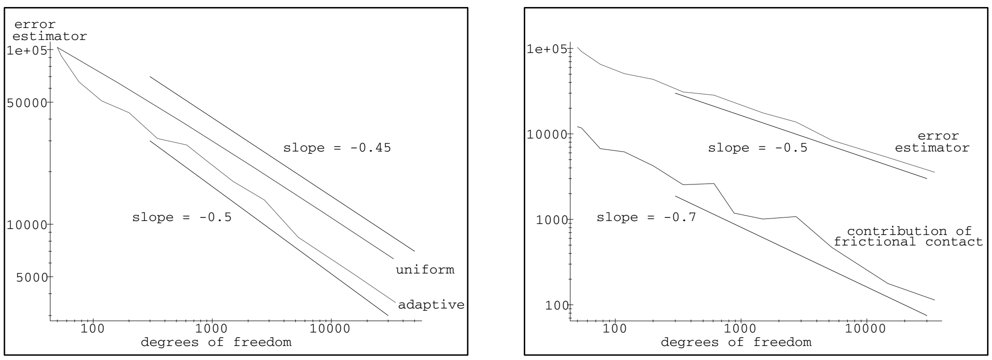

FIG. 7. *Second example. Left: convergence of the error estimator $\tilde{\eta}$ with uniform and adaptive refinement. Right: convergence of the error estimator $\tilde{\eta}$ and its frictional contact contribution $\tilde{\eta}_C$ with adaptive refinement.*

estimator behaves approximately like $N^{-0.85}$; see Figure 11. These results obtained for a small friction coefficient show many similarities with the ones obtained in [81] without friction.

7. Conclusion and perspectives. In this paper we propose, analyze, and implement two residual error estimators η and $\tilde{\eta}$ corresponding to two finite element discretizations of the static Coulomb friction problem by using the partial uniqueness result obtained in [72]. To our knowledge our study yields the first results (for the Coulomb friction problem) involving residual estimators with both upper and lower bounds of the discretization error. From the definitions and the theoretical estimates we observe that $\tilde{\eta}$ is simpler to define and yields better bounds. From the numerical experiments, we observe that all the terms in η and $\tilde{\eta}$ for which no optimal theoretical results can be provided behave better than theoretically expected and that both approaches are worth being considered.

Another line of research could consist in obtaining a uniqueness result for the quasi-static problem by adapting the techniques in [72] and then to perform an a posteriori analysis (note that the existence results obtained in [5, 73] for the quasi-static problem are of the same type as the ones for the static problem).

Another (difficult) study consists in extending the estimators obtained in this paper to the so-called XFEM method for crack problems (see [66]), where frictional contact occurs on the crack lips and where the mesh of the body does not coincide

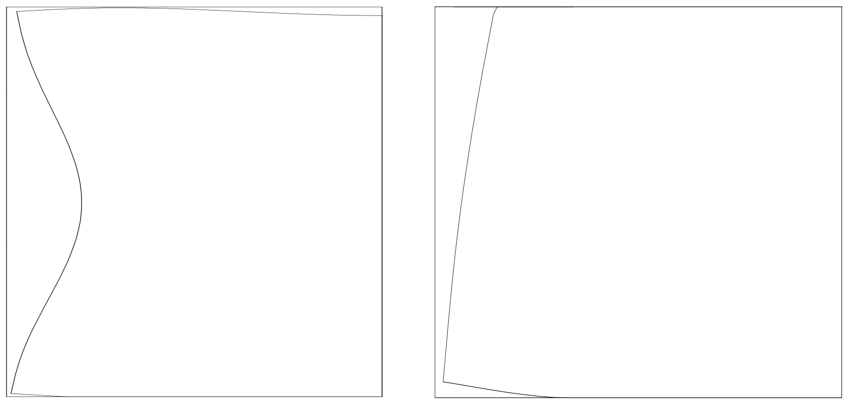

FIG. 8. *Third example. Left: initial and deformed configurations with $\mu = 0.1$. Right: zoom near the separation zone.*

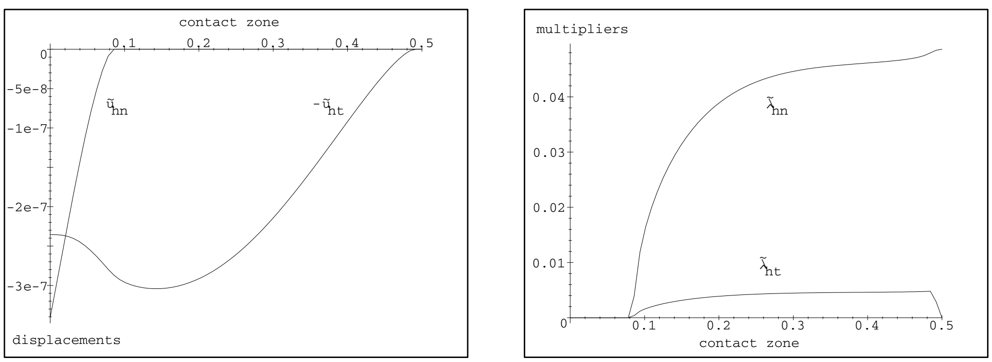

FIG. 9. *Third example. Left: normal and tangential displacements $(\tilde{u}_{hn}, -\tilde{u}_{ht})$ on Γ_C. Right: normal and tangential multipliers $(\tilde{\lambda}_{hn}, \tilde{\lambda}_{ht})$ on Γ_C.*

with the crack. This study is actually under investigation in [61].

Appendix.

PROPOSITION 7.1. *For any positive μ, problem* (37) *admits at least a solution.*

Proof. Let $\mu > 0$ be given. We introduce the problem of friction $P(g_{hn})$ with a given threshold μg_{hn} and $g_{hn} \in K_{hn}$. It consists in finding $\mathbf{u}_h \in \mathbf{V}_h$ and $(\lambda_{hn}, \lambda_{ht}) = \boldsymbol{\lambda}_h \in \mathbf{K}_h(\mu g_{hn}) = K_{hn} \times K_{ht}(\mu g_{hn})$ satisfying

$$(60) \qquad P(g_{hn}) \quad \begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + c(\boldsymbol{\lambda}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ c(\boldsymbol{\nu}_h - \boldsymbol{\lambda}_h, \mathbf{u}_h) \le 0 \qquad \forall\, \boldsymbol{\nu}_h = (\nu_{hn}, \nu_{ht}) \in \mathbf{K}_h(\mu g_{hn}). \end{cases}$$

Problem (60) is equivalent of finding a saddle-point $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht}) = (\mathbf{u}_h, \boldsymbol{\lambda}_h) \in \mathbf{V}_h \times \mathbf{K}_h(\mu g_{hn})$ verifying

$$\mathcal{L}(\mathbf{u}_h, \boldsymbol{\nu}_h) \le \mathcal{L}(\mathbf{u}_h, \boldsymbol{\lambda}_h) \le \mathcal{L}(\mathbf{v}_h, \boldsymbol{\lambda}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,\ \forall \boldsymbol{\nu}_h \in \mathbf{K}_h(\mu g_{hn}),$$

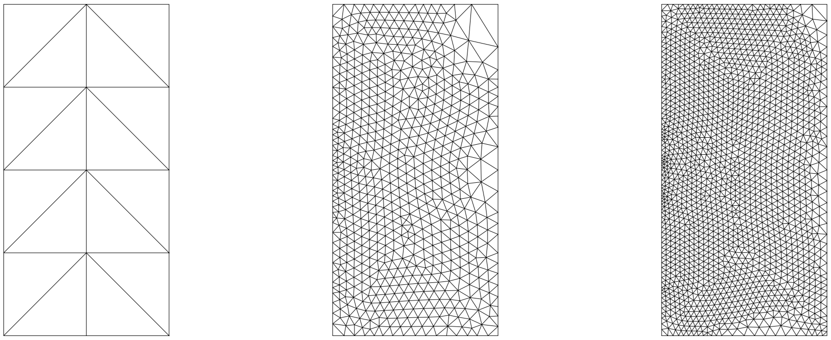

FIG. 10. *Third example. Initial (left) and refined meshes in the adaptive procedure.*

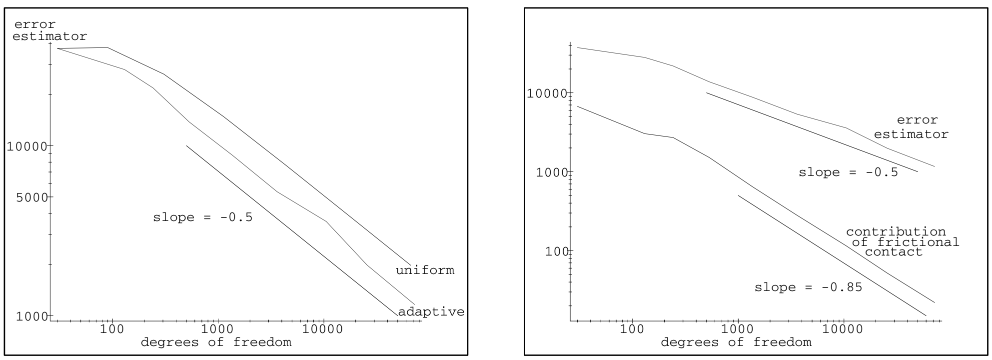

FIG. 11. *Third example. Left: convergence of the error estimator $\tilde{\eta}$ with uniform and adaptive refinement. Right: convergence of the error estimator $\tilde{\eta}$ and its frictional contact contribution $\tilde{\eta}_C$ with adaptive refinement.*

where

$$\mathcal{L}(\mathbf{v}_h, \boldsymbol{\nu}_h) = \frac{1}{2}a(\mathbf{v}_h, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\nu_{hn} v_{hn})\, d\Gamma + \int_{\Gamma_C} I_h(\nu_{ht} v_{ht})\, d\Gamma - L(\mathbf{v}_h).$$

By using standard arguments on saddle-point problems as in [41, Theorem 3.9, p. 339], we deduce that there exists such a saddle-point. The strict convexity of $a(.,.)$ implies that the first argument $\mathbf{u}_h$ is unique. Suppose that the second argument is not unique: then the equality in (60) implies

$$c(\boldsymbol{\lambda}_h^1 - \boldsymbol{\lambda}_h^2, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

The definition of W_h allows us to choose $\mathbf{v}_h = \boldsymbol{\lambda}_h^1 - \boldsymbol{\lambda}_h^2$ on Γ_C. From the definition of $c(.,.)$ we come to the conclusion that $\boldsymbol{\lambda}_h^1 - \boldsymbol{\lambda}_h^2 = \mathbf{0}$. Consequently, the second argument $\boldsymbol{\lambda}_h$ is unique and (60) admits a unique solution. The next lemma is a straightforward consequence of the definition of problems (37) and (60).

LEMMA 7.2. *The solutions of Coulomb's discrete frictional contact problem* (37) *are the solutions of $P(\tilde{\lambda}_{hn})$, where $\tilde{\lambda}_{hn}$ is a fixed point of Φ_h. The functional Φ_h is defined as follows:*

$$\begin{aligned} \Phi_h :\quad K_{hn} &\longrightarrow K_{hn}, \\ g_{hn} &\longmapsto \lambda_{hn}, \end{aligned}$$

where $(\mathbf{u}_h, \boldsymbol{\lambda}_h)$ *is the solution of* $P(g_{hn})$.

To establish existence of a fixed point of Φ_h, we use Brouwer's fixed point theorem. First we prove the mapping Φ_h is continuous. Set $\tilde{\mathbf{V}}_h = \{\mathbf{v}_h \in \mathbf{V}_h : \ v_{ht} = 0 \text{ on } \Gamma_C\}$. From the definition of W_h, it is easy to check that the definition of $\|.\|_{-\frac{1}{2},h}$ given by

$$\|\nu\|_{-\frac{1}{2},h} = \sup_{\mathbf{v}_h \in \tilde{\mathbf{V}}_h} \frac{\int_{\Gamma_C} I_h(\nu v_{hn})\, d\Gamma}{\|\mathbf{v}_h\|_{1,\Omega}}$$

is a norm on W_h. Let $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$ and $(\overline{\mathbf{u}_h}, \overline{\lambda_{hn}}, \overline{\lambda_{ht}})$ be the solutions of $P(g_{hn})$ and $P(\overline{g_{hn}})$, respectively. On the one hand, we get

$$a(\mathbf{u}_h, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\lambda_{hn} v_{hn})\, d\Gamma = L(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h,$$
$$a(\overline{\mathbf{u}_h}, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\overline{\lambda_{hn}} v_{hn})\, d\Gamma = L(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h.$$

Subtracting the previous equalities and using the continuity of the bilinear form $a(.,.)$ gives

$$\int_{\Gamma_C} I_h((\lambda_{hn} - \overline{\lambda_{hn}}) v_{hn})\, d\Gamma = a(\overline{\mathbf{u}_h} - \mathbf{u}_h, \mathbf{v}_h) \lesssim \|\mathbf{u}_h - \overline{\mathbf{u}_h}\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \qquad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h.$$

Hence, we get a first estimate

$$\|\lambda_{hn} - \overline{\lambda_{hn}}\|_{-\frac{1}{2},h} \lesssim \|\mathbf{u}_h - \overline{\mathbf{u}_h}\|_{1,\Omega}. \tag{61}$$

On the other hand, we have from (37)

$$a(\mathbf{u}_h, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\lambda_{hn} v_{hn})\, d\Gamma + \int_{\Gamma_C} I_h(\lambda_{ht} v_{ht})\, d\Gamma = L(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{62}$$
$$a(\overline{\mathbf{u}_h}, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\overline{\lambda_{hn}} v_{hn})\, d\Gamma + \int_{\Gamma_C} I_h(\overline{\lambda_{ht}} v_{ht})\, d\Gamma = L(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{63}$$

Choosing $\mathbf{v}_h = \mathbf{u}_h - \overline{\mathbf{u}_h}$ in (62) and $\mathbf{v}_h = \overline{\mathbf{u}_h} - \mathbf{u}_h$ in (63) implies by addition

$$\begin{aligned} a(\mathbf{u}_h - \overline{\mathbf{u}_h}, \mathbf{u}_h - \overline{\mathbf{u}_h}) = & \int_{\Gamma_C} I_h((\overline{\lambda_{hn}} - \lambda_{hn})(u_{hn} - \overline{u_{hn}}))\, d\Gamma \\ & + \int_{\Gamma_C} I_h((\overline{\lambda_{ht}} - \lambda_{ht})(u_{ht} - \overline{u_{ht}}))\, d\Gamma. \end{aligned} \tag{64}$$

Let us notice that the inequality in (60) is obviously equivalent to the two following conditions:

$$\int_{\Gamma_C} I_h((\nu_{hn} - \lambda_{hn}) u_{hn})\, d\Gamma \leq 0 \qquad \forall \nu_{hn} \in K_{hn}, \tag{65}$$
$$\int_{\Gamma_C} I_h((\nu_{ht} - \lambda_{ht}) u_{ht})\, d\Gamma \leq 0 \qquad \forall \nu_{ht} \in K_{ht}(\mu g_{hn}). \tag{66}$$

According to the definition of K_{hn}, we can choose $\nu_{hn} = 0$ and $\nu_{hn} = 2\lambda_{hn}$ in (65), which gives

$$\int_{\Gamma_C} I_h(\lambda_{hn} u_{hn})\, d\Gamma = 0 \quad \text{ and } \quad \int_{\Gamma_C} I_h(\nu_{hn} u_{hn})\, d\Gamma \leq 0 \qquad \forall \nu_{hn} \in K_{hn},$$

from which we deduce that

$$\int_{\Gamma_C} I_h((\overline{\lambda_{hn}} - \lambda_{hn})(u_{hn} - \overline{u_{hn}})\, d\Gamma \leq 0.$$

Hence (64) becomes

$$\|\mathbf{u}_h - \overline{\mathbf{u}_h}\|_{1,\Omega}^2 \lesssim \int_{\Gamma_C} I_h((\overline{\lambda_{ht}} - \lambda_{ht})(u_{ht} - \overline{u_{ht}}))\, d\Gamma. \tag{67}$$

From the definition of $K_{ht}(\mu g_{hn})$, we get

$$\int_{\Gamma_C} I_h(\lambda_{ht}\overline{u_{ht}})\, d\Gamma \leq \int_{\Gamma_C} I_h(|\lambda_{ht}|\,|\overline{u_{ht}}|)\, d\Gamma \leq \int_{\Gamma_C} I_h(\mu g_{hn}\,|\overline{u_{ht}}|)\, d\Gamma.$$

A similar expression can be obtained when integrating the term $I_h(\overline{\lambda_{ht}} u_{ht})$. Besides from (66)

$$\begin{aligned} -\int_{\Gamma_C} I_h(\lambda_{ht} u_{ht})\, d\Gamma &\leq -\int_{\Gamma_C} I_h(\nu_{ht} u_{ht})\, d\Gamma \\ &= -\sum_{i=1}^{p} \nu_{ht}(\mathbf{x}_i) u_{ht}(\mathbf{x}_i) \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma \qquad \forall \nu_{ht} \text{ s.t. } |\nu_{ht}| \leq \mu g_{hn}. \end{aligned}$$

If $u_{ht}(\mathbf{x}_i) \geq 0$, we choose $\nu_{ht}(\mathbf{x}_i) = \mu g_{hn}(\mathbf{x}_i)$, and if $u_{ht}(\mathbf{x}_i) \leq 0$, we choose $\nu_{ht}(\mathbf{x}_i) = -\mu g_{hn}(\mathbf{x}_i)$. This yields the following bound:

$$-\int_{\Gamma_C} I_h(\lambda_{ht} u_{ht})\, d\Gamma \leq -\mu \sum_{i=1}^{p} g_{hn}(\mathbf{x}_i)\,|u_{ht}(\mathbf{x}_i)| \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma = -\int_{\Gamma_C} I_h(\mu g_{hn}\,|u_{ht}|)\, d\Gamma.$$

A similar expression can be obtained when integrating the term $I_h(\overline{\lambda_{ht}}\,\overline{u_{ht}})$. Finally, (67) becomes

$$\begin{aligned} \|\mathbf{u}_h - \overline{\mathbf{u}_h}\|_{1,\Omega}^2 &\lesssim \mu \int_{\Gamma_C} I_h((g_{hn} - \overline{g_{hn}})(|\overline{u_{ht}}| - |u_{ht}|))\, d\Gamma \\ &\leq \mu \int_{\Gamma_C} I_h(|g_{hn} - \overline{g_{hn}}|\,|\overline{u_{ht}} - u_{ht}|)\, d\Gamma \\ &= \mu \sum_{i=1}^{p} |(g_{hn} - \overline{g_{hn}})(\mathbf{x}_i)|\,|(\overline{u_{ht}} - u_{ht})(\mathbf{x}_i)| \int_{\Gamma_C} \psi_{\mathbf{x}_i}\, d\Gamma \\ &\lesssim \mu \left(\sum_{i=1}^{p} |(g_{hn} - \overline{g_{hn}})(\mathbf{x}_i)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{p} |(\overline{u_{ht}} - u_{ht})(\mathbf{x}_i)|^2\right)^{\frac{1}{2}} \\ &\lesssim \mu C(h) \|g_{hn} - \overline{g_{hn}}\|_{-\frac{1}{2},h}\, \|\overline{\mathbf{u}_h} - \mathbf{u}_h\|_{1,\Omega}, \end{aligned} \tag{68}$$

where the equivalence of norms in finite dimensional spaces has been used as well as the trace theorem. Combining (68) and (61) implies that there exists a constant $C(h)$ such that

$$\|\lambda_{hn} - \overline{\lambda_{hn}}\|_{-\frac{1}{2},h} \lesssim \mu C(h) \|g_{hn} - \overline{g_{hn}}\|_{-\frac{1}{2},h}. \tag{69}$$

Hence Φ_h is continuous.

Let $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$ be the solution of $P(g_{hn})$. Taking $\mathbf{v}_h = \mathbf{u}_h$ in (60) gives

$$a(\mathbf{u}_h, \mathbf{u}_h) + \int_{\Gamma_C} I_h(\lambda_{hn} u_{hn})\, d\Gamma + \int_{\Gamma_C} I_h(\lambda_{ht} u_{ht})\, d\Gamma = L(\mathbf{u}_h). \tag{70}$$

According to

$$\int_{\Gamma_C} I_h(\lambda_{hn} u_{hn})\, d\Gamma = 0 \quad \text{and} \quad \int_{\Gamma_C} I_h(\lambda_{ht} u_{ht})\, d\Gamma \geq 0,$$

we deduce from (70) the $\mathbf{V}$-ellipticity of $a(.,.)$ and the continuity of $L(.)$:

$$\|\mathbf{u}_h\|^2_{1,\Omega} \lesssim a(\mathbf{u}_h, \mathbf{u}_h) \leq L(\mathbf{u}_h) \lesssim \|\mathbf{u}_h\|_{1,\Omega}.$$

So, we deduce that $\|\mathbf{u}_h\|_{1,\Omega}$ is bounded. In other respects

$$a(\mathbf{u}_h, \mathbf{v}_h) + \int_{\Gamma_C} I_h(\lambda_{hn} v_{hn})\, d\Gamma = L(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h$$

leads to

$$\int_{\Gamma_C} I_h(\lambda_{hn} v_{hn})\, d\Gamma \lesssim \|\mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} + \|\mathbf{v}_h\|_{1,\Omega} \qquad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h.$$

Therefore $\|\Phi_h(g_{hn})\|_{-\frac{1}{2},h} = \|\lambda_{hn}\|_{-\frac{1}{2},h} \lesssim \|\mathbf{u}_h\|_{1,\Omega} + 1 \lesssim 1\ \forall g_{hn} \in M_{hn}$. This boundedness of Φ_h together with the continuity of Φ_h proves that there exists at least a solution of Coulomb's discrete frictional contact problem according to Brouwer's fixed point theorem. □

Remark 8. From (69), we obtain a meshsize dependent uniqueness result when $\mu\ C(h) < 1$. This means that uniqueness holds when μ is small enough, where the denomination "small" depends on the discretization parameter. A more detailed study would show that this uniqueness criterion disappears when h vanishes (i.e., $\lim_{h\to 0} C(h) = +\infty$).

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