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Collapsing irreducible 3-manifolds with nontrivial fundamental group

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Abstract Let M be a closed, orientable, irreducible, non-simply connected 3-manifold. We prove that if M admits a sequence of Riemannian metrics which volume-collapses and whose sectional curvature is locally controlled, then M is a graph manifold. This is the last step in Perelman's proof of Thurston's Geometrisation Conjecture.

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1 Introduction

Thurston's Geometrisation Conjecture states that any closed, orientable, irreducible 3-dimensional manifold M is hyperbolic, Seifert fibred, or contains an incompressible torus. This conjecture has been proved recently by G. Perelman [31–33] (see also [6, 23, 26]) using R. Hamilton's Ricci flow. In this paper, we shall be concerned with the case where $\pi_1 M$ is nontrivial.

The last step of Perelman's proof in this case relies on a "collapsing theorem" which is independent of the Ricci flow part. This result is stated without proof as Theorem 7.4 in [33]. A version of this theorem for closed 3-manifolds is given in the appendix of the paper [36] by Shioya and Yamaguchi using deep results of Alexandrov space theory, including Perelman's stability theorem [30] (see also the paper [21] by V. Kapovitch) and a fibration theorem for Alexandrov spaces, proved by Yamaguchi [39]. A different approach has been proposed by Morgan and Tian [27] and Cao and Ge [5]. Yet another proof has been announced by Kleiner and Lott [22].

Our main result, Theorem 1.1 below, is the special case of Theorem 7.4 of [33] where the manifold M is assumed to be closed, irreducible and not simply-connected. This is sufficient to complete the proof of the Geometrisation Conjecture since the case where M is simply-connected can be proved by the so-called extinction argument, which does not use the collapsing result. The proof of Theorem 1.1 combines arguments from Riemannian geometry, algebraic topology, and 3-manifold theory. It uses Thurston's hyperbolisation theorem for Haken manifolds [25, 28, 29, 38], but avoids the stability and fibration theorems for Alexandrov spaces (which are not used by Perelman in the simply connected case).

In the next two definitions, M is a 3-manifold.

Definition Let g be a Riemannian metric on M and $\varepsilon > 0$ be a real number. A point $x \in M$ is *ε-thin with respect to g* if there exists $0 < \rho \leq 1$ such that on the ball $B(x, \rho)$ the sectional curvature is greater than or equal to $-\rho^{-2}$, and the volume of this ball is less than $\varepsilon \rho^3$.

A sequence of Riemannian metrics g_n on M is said to *collapse* if there exists a sequence $\varepsilon_n \to 0$ such that, for every n, every point of M is ε_n-thin with respect to g_n.

The following is a technical condition which guarantees the regularity of certain limits of Riemannian manifolds. The Riemann tensor is denoted by Rm.

Definition Let $\{g_n\}$ be a sequence of Riemannian metrics on M. We say that $\{g_n\}$ has *locally controlled curvature in the sense of Perelman* if it has the following property: for all $\varepsilon > 0$ there exist $\bar{r}(\varepsilon) > 0$, $K_0(\varepsilon) > 0$, $K_1(\varepsilon) > 0$, such that for n large enough, for each $0 < r \leq \bar{r}(\varepsilon)$, if $x \in (M, g_n)$ satisfies $\mathrm{vol}(B(x,r))/r^3 \geq \varepsilon$, and if the sectional curvature on $B(x,r)$ is greater than or equal to $-r^{-2}$, then $|\mathrm{Rm}(x)| < K_0 r^{-2}$ and $|\nabla \mathrm{Rm}(x)| < K_1 r^{-3}$.

Theorem 1.1 *Let M be a closed, orientable, irreducible, non-simply connected 3-manifold. If M admits a sequence of Riemannian metrics that collapses and has locally controlled curvature in the sense of Perelman, then M is a graph manifold.*

Sequences of metrics satisfying the hypotheses of Theorem 1.1 are provided by Perelman's construction of Ricci flow with surgery and its study of the long time behaviour of the Ricci flow with surgery (see Theorem 7.4 of [33]).

Before ending this introduction, we would like to mention the book [1], which presents a proof of the Geometrisation Conjecture that uses Perelman's work and the arguments of the present paper.

2 Sketch of proof of Theorem 1.1

For classical 3-manifold theory, we use [20], [17] as main references, as well as [3] for post-Thurston results. To avoid any confusion between metric balls and topological balls, we shall call 3-*ball* a 3-manifold homeomorphic to the closed unit ball in $\mathbf{R}^3$. By contrast, our metric balls $B(x,r)$ are open.

Throughout the paper we work in the smooth category. Recall that a *Haken manifold* is a connected, compact, orientable, irreducible 3-manifold which contains an incompressible surface. Any connected, compact, orientable, irreducible 3-manifold whose boundary is not empty is Haken (its boundary may be compressible). It follows from deep work of W. Thurston and earlier work of Jaco-Shalen and Johannson that every Haken manifold has a canonical decomposition along incompressible tori into Seifert fibred and hyperbolic pieces (see e.g. the references given in [3]). We call this decomposition the *geometric decomposition* of the Haken manifold M. Moreover, a Haken manifold is a graph manifold if and only if all pieces in its geometric decomposition are Seifert fibred.

Another key notion used in the proof of Theorem 1.1 is the *simplicial volume,* sometimes called Gromov norm, introduced by M. Gromov in [13]. Our proof relies on an additivity result for the simplicial volume under gluing along incompressible tori (see [13, 24, 37]) which implies that the simplicial volume of a 3-manifold admitting a geometric decomposition is proportional to the sum of the volumes of the hyperbolic pieces. In particular, a Haken manifold has zero simplicial volume if and only if it is a graph manifold.

We also use in an essential way Gromov's vanishing theorem [13, 18, 19]: if a n-dimensional closed manifold M can be covered by open sets U_i such that the covering has dimension less than n and the image of the canonical homomorphism $\pi_1(U_i) \to \pi_1(M)$ is amenable for all i, then the simplicial volume of M vanishes. (Recall that the dimension of a finite covering U_i is the dimension of its nerve, i.e. the minimal d such that every point in M belongs to at most $d+1$ open subsets of the covering.)

Below we outline the proof of Theorem 1.1.

Before discussing the proof proper, we give an example of a *covering argument* which can be used to deduce topological information on M (namely that M has zero simplicial volume) from the collapsing hypothesis.

For n large enough, thanks to the local control on the curvature, each point has a neighbourhood in (M, g_n) which is close to a metric ball in some manifold of nonnegative sectional curvature (Proposition 3.1), and whose volume is small compared to the cube of the radius (by the collapsing hypothesis). These metric balls will be called *local models* throughout the paper and by extension it will also refer to the neighbourhoods which are homeomorphic to these balls. From the classification of manifolds with nonnegative sectional curvature, we deduce that these local models have virtually abelian, hence amenable fundamental groups. A technique introduced by Gromov [13] yields a covering of M, whose dimension is at most 2, by open sets contained in these neighbourhoods. As a consequence, Gromov's vanishing theorem implies that the simplicial volume of M vanishes.

The previous scheme, together with the additivity of the simplicial volume under gluing along incompressible tori, shows that a manifold which admits a geometric decomposition and a sequence of collapsing metrics is a graph manifold. This is however insufficient to prove Theorem 1.1 since we do not assume that M admits a geometric decomposition! Hence we need a trick similar to those of [4] and [2], which we now explain.

In the first step, Proposition 4.1, we find a local model U such that all connected components of $M \setminus U$ are Haken. This requirement is equivalent to irreducibility of each component of $M \setminus U$, because each component of $M \setminus U$ has nonempty boundary. Since M is irreducible, it suffices to show that U is not contained in a 3-ball. This is in particular the case if U is *homotopically nontrivial*, i.e. the homomorphism $\pi_1(U) \to \pi_1(M)$ has nontrivial image.

The proof of the existence of a homotopically nontrivial local model U is done by contradiction: assuming that all local models are homotopically trivial, we construct a covering of M of dimension less than or equal to 2 by homotopically trivial open sets (Assertion 4.4). By a result of J.C. Gómez-Larrañaga and F. González-Acuña [11], corresponding to Corollary 4.3 here, a closed, irreducible 3-manifold admitting such a covering must have trivial fundamental group. This is where we use the hypothesis that M is not simply connected.

The second step (Section 4.2) is again a covering argument, but done relatively to some fixed homotopically nontrivial local model U. It shows that any manifold obtained by Dehn filling on $Y := M \setminus U$ has a covering of dimension less than or equal to 2 by virtually abelian open sets, and therefore has vanishing simplicial volume. We conclude using Proposition 4.15, which states that if Y is a Haken manifold with boundary a collection of tori and such that the simplicial volume of every Dehn filling on Y vanishes, then Y is a graph manifold.[1] This finishes the sketch of proof of Theorem 1.1.

3 Local structure

Throughout the paper, we consider a 3-manifold M and a sequence of Riemannian metrics g_n satisfying the hypotheses of Theorem 1.1. For the sake of simplicity, in the sequel we use the notation $M_n := (M, g_n)$. It is implicit that any quantity depending on a point $x \in M_n$ is computed with respect to the metric g_n and thus depends also on n.

By hypothesis, there exists a sequence $\varepsilon_n \to 0$ such that M_n is ε_n-thin. For all $x \in M_n$, we choose a radius $0 < \rho(x) \leq 1$, such that on the ball $B(x, \rho(x))$ the curvature is not smaller than $-\rho(x)^{-2}$ and the volume of this ball is greater than $\varepsilon_n \rho(x)^3$.

Recall that a diffeomorphism $f : X \to Y$ is $(1+\delta)$-*bi-Lipschitz* if f and f^{-1} are $(1+\delta)$-Lipschitz. Two Riemannian manifolds X and Y are said to be δ-*close* if there exists a $(1+\delta)$-bi-Lipschitz diffeomorphism between them.

In the following proposition we use Cheeger-Gromoll's soul theorem [8].

Proposition 3.1 *For all $D > 1$ there exists $n_0(D)$ such that if $n > n_0(D)$, then we have the following alternative*:

(a) *Either M_n is $\frac{1}{D}$-close to some closed nonnegatively curved 3-manifold, or*
(b) *for all $x \in M_n$ there exists a radius $\nu(x) \in (0, \rho(x))$ and a complete noncompact Riemannian 3-manifold X_x, with nonnegative sectional curvature and soul S_x, such that the following properties are satisfied*:
   (1) *$B(x, \nu(x))$ is $\frac{1}{D}$-close to a metric ball in X_x.*
   (2) *There exists a map $f_x : B(x, \nu(x)) \to X_x$ which is a $(1 + \frac{1}{D})$-bi-Lipschitz diffeomorphism onto its image and such that*

$$\max\{d(f_x(x), S_x), \operatorname{diam} S_x\} \leq \frac{\nu(x)}{D}.$$

[1] As already mentioned in [2, 4], Proposition 4.15 is a consequence of the geometrisation of Haken manifolds, additivity of the simplicial volume mentioned above, and Thurston's hyperbolic Dehn filling theorem.

(3) $\mathrm{vol}(B(x, \nu(x))) \leq \frac{1}{D}\nu^3(x)$.

Remark Since $\nu(x) < \rho(x)$, the sectional curvature on $B(x, \nu(x))$ is greater than or equal to $-\frac{1}{\rho^2(x)}$, which is in turn bounded below by $-\frac{1}{\nu^2(x)}$. We shall say, by extension, that the metric balls $B(x, \nu(x))$ are the *local models*.

Remark The only closed, orientable, irreducible 3-manifold containing a projective plane is RP^3, which is a graph manifold. Therefore if the manifold M is not homeomorphic to RP^3, then the soul S_x can be homeomorphic to a point, a circle, a 2-sphere, a 2-torus or a Klein bottle. In this case, the ball $B(x, \nu(x))$ is homeomorphic to $\mathbf{R}^3$, $S^1 \times \mathbf{R}^2$, $S^2 \times \mathbf{R}$, $T^2 \times \mathbf{R}$ or to the twisted $\mathbf{R}$-bundle on the Klein bottle respectively.

Before starting the proof of this proposition, we prove the following lemma and its consequence:

Lemma 3.2 *There exists a small universal constant $C > 0$ such that for all $\varepsilon > 0$, for all $x \in M_n$, and for all $r > 0$, if the ball $B(x, r)$ has volume $\geq \varepsilon r^3$ and curvature $\geq -r^{-2}$, then for all $y \in B(x, \frac{1}{3}r)$ and all $0 < r' < \frac{2}{3}r$, the ball $B(y, r')$ has volume $\geq C \cdot \varepsilon (r')^3$ and curvature $\geq -(r')^{-2}$.*

We use the function $v_{-\kappa^2}(r)$ to denote the volume of the ball of radius r in the 3-dimensional hyperbolic space of curvature $-\kappa^2$. Notice that $v_{-\kappa^2}(r) = \kappa^{-3} v_{-1}(\kappa r)$.

Proof The lower bound on the curvature is a consequence of the monotonicity of the function $-r^{-2}$ with respect to r. In order to estimate from below the volume we apply Bishop-Gromov's inequality twice (cf. [35, Lemma 9.1.6]). First to the ball around y, increasing the radius r' to $\frac{2}{3}r$:

$$\mathrm{vol}(B(y, r')) \geq \mathrm{vol}\left(B\left(y, \frac{2}{3}r\right)\right) \frac{v_{-r^{-2}}(r')}{v_{-r^{-2}}(\frac{2}{3}r)}.$$

Using that $v_{-r^{-2}}(r') = r^3 v_{-1}(\frac{r'}{r}) \geq r^3 (\frac{r'}{r})^3 C_1$ for $C_1 > 0$ uniform, $v_{-r^{-2}}(\frac{2}{3}r) = r^3 v_{-1}(\frac{2}{3})$, and that the ball $B(y, \frac{2}{3}r)$ contains $B(x, \frac{1}{3}r)$, we have

$$\mathrm{vol}(B(y, r')) \geq \mathrm{vol}\left(B\left(x, \frac{1}{3}r\right)\right) \left(\frac{r'}{r}\right)^3 C_2.$$

Applying again the Bishop-Gromov inequality:

$$\mathrm{vol}\left(B\left(x, \frac{1}{3}r\right)\right) \geq \mathrm{vol}(B(x, r)) \frac{v_{-r^{-2}}(\frac{1}{3}r)}{v_{-r^{-2}}(r)} \geq r^3 \varepsilon \frac{v_{-1}(\frac{1}{3})}{v_{-1}(1)} = r^3 \varepsilon\, C_3.$$

Hence $\operatorname{vol}(B(y, r')) \geq (r')^3 \varepsilon C_4$. $\square$

We next deduce an "improvement" of the controlled curvature in the sense of Perelman, in which the conclusion is valid at each point of some metric ball, not only the centre. The only price to pay is that the constants can be slightly different.

Corollary 3.3 *For all $\varepsilon > 0$ there exist $\bar{r}'(\varepsilon) > 0$, $K_0'(\varepsilon)$, $K_1'(\varepsilon)$ such that for n large enough, if $0 < r \leq \bar{r}'(\varepsilon)$, $x \in M_n$ and the ball $B(x, r)$ has volume $\geq \varepsilon r^3$ and sectional curvature $\geq -r^{-2}$ then we have for all $y \in B(x, \frac{1}{3}r)$, $|\operatorname{Rm}(y)| < K_0' r^{-2}$ and $|\nabla \operatorname{Rm}(y)| < K_1' r^{-3}$.*

Proof It suffices to apply Lemma 3.2, setting $\bar{r}'(\varepsilon) = \bar{r}(C\varepsilon)$, $K_0'(\varepsilon) = K_0(C\varepsilon)$ and $K_1'(\varepsilon) = K_1(C\varepsilon)$, so that we can apply the controlled curvature condition on $y \in B(x, \frac{1}{3}x)$. $\square$

Proof of Proposition 3.1 Let us assume that there exist $D_0 > 1$ and, after re-indexing, a sequence $x_n \in M_n$ such that neither of the conclusions of Proposition 3.1 holds with $D = D_0$.

Set $\varepsilon_0 := \frac{v_0(1)}{v_{-1}(1)} \frac{1}{D_0} < 1$. We shall rescale the metrics using the following radii:

Definition For $x \in M_n$, define

$$\operatorname{rad}(x) := \inf\{r > 0 \mid \operatorname{vol}(B(x, r))/r^3 \leq \varepsilon_0\}.$$

Notice that $\operatorname{rad}(x) < \infty$ because M_n has finite volume, and that $\operatorname{rad}(x) > 0$, because $\operatorname{vol}(B(x, \delta))/\delta^3 \to \frac{4}{3}\pi$ when $\delta \to 0$. We gather in the following lemma some properties of $\operatorname{rad}(x)$ which will be useful for the proof:

Lemma 3.4

(i) *For n large enough and $x \in M_n$, one has $0 < \operatorname{rad}(x) < \rho(x)$.*

(ii) *For $x \in M_n$, one has*

$$\frac{\operatorname{vol}(B(x, \operatorname{rad}(x)))}{\operatorname{rad}(x)^3} = \varepsilon_0.$$

(iii) *For $L > 1$, there exists $n_0(L)$ such that for $n > n_0(L)$ and for $x \in M_n$ we have*

$$L \operatorname{rad}(x) \leq \rho(x).$$

In particular $\lim_{n\to\infty} \operatorname{rad}(x_n) = \lim_{n\to\infty} \frac{\operatorname{rad}(x_n)}{\rho(x_n)} = 0$.

Proof Property (i) holds as soon as $\varepsilon_n < \varepsilon_0$, since rad is defined as an infimum.

Assertion (ii) is proved by continuity.

We prove (iii). For $L > 1$, we choose $n_0(L)$ so that $\varepsilon_n < \frac{\varepsilon_0}{L^3}$ for $n > n_0(L)$. Then, for $x \in M_n$ and $n \geq n_0(L)$,

$$\frac{\operatorname{vol}(B(x, \frac{\rho(x)}{L}))}{(\frac{\rho(x)}{L})^3} \leq L^3 \frac{\operatorname{vol}(B(x, \rho(x)))}{\rho(x)^3} \leq L^3 \varepsilon_n \leq \varepsilon_0.$$

Hence $\rho(x)/L \geq \operatorname{rad}(x)$. □

Remark For n large enough, from Lemma 3.4 (iii), we have $\operatorname{rad}(x_n) < \bar{r}'(\varepsilon_0)$, where $\bar{r}'(\varepsilon_0)$ (independent of n) is the parameter from Corollary 3.3.

Corollary 3.5 *There exists a constant $C > 0$ such that any sequence of points $x_n \in M_n$ satisfies*

$$\frac{\operatorname{inj}(x_n)}{\operatorname{rad}(x_n)} \geq C$$

for n large enough.

Proof Let us first remark that, since $\operatorname{rad}(x_n) < \rho(x_n)$, the sectional curvature on $B(x_n, \operatorname{rad}(x_n))$ is $\geq -\frac{1}{\rho(x_n)^2} > -\frac{1}{\operatorname{rad}(x_n)^2}$. Moreover, as $\operatorname{rad}(x_n) < \bar{r}'(\varepsilon_0)$, Corollary 3.3 shows that the curvature on the ball $B(x_n, \frac{\operatorname{rad}(x_n)}{3})$ is bounded above by $K_0'(\varepsilon_0)/\operatorname{rad}(x_n)^2$. The rescaled ball

$$\frac{1}{\operatorname{rad}(x_n)} B\left(x_n, \frac{1}{3}\operatorname{rad}(x_n)\right)$$

has volume $\geq \varepsilon_0/27$ (because $\operatorname{vol}(B(x_n, \frac{1}{3}\operatorname{rad}(x_n))) \geq \varepsilon_0(\operatorname{rad}(x_n)/3)^3$ by definition of $\operatorname{rad}(x_n)$) and absolute value of curvature $\leq K_0'(\varepsilon_0)$.

We can now apply Proposition A.1 of the Appendix with $R = 1/3$, $K = K_0'(\varepsilon_0)$ and $\varepsilon = \varepsilon_0/27$ to get a lower bound for the injectivity radius at x_n of the rescaled metric which is independent of n. □

Having proved Lemma 3.4 and its corollary, we continue the proof of Proposition 3.1. Let us consider the rescaled manifold $\overline{M}_n = \frac{1}{\operatorname{rad}(x_n)} M_n$. We look for a limit of the sequence $(\overline{M}_n, \bar{x}_n)$, where $\bar{x}_n$ is the image of x_n. The ball $B(\bar{x}_n, \frac{\rho(x_n)}{\operatorname{rad}(x_n)}) \subset \overline{M}_n$ has sectional curvature bounded below by $-(\frac{\operatorname{rad}(x_n)}{\rho(x_n)})^2$, which goes to 0 when $n \to \infty$, as follows from Assertion (iii) of Lemma 3.4.

Given $L > 1$, the ball $B(\bar{x}_n, 3L)$ is obtained by rescaling the ball $B(x_n, 3L\,\mathrm{rad}(x_n))$. Since $3L\,\mathrm{rad}(x_n) < \rho(x_n)$, the sectional curvature on $B(x_n, 3L\,\mathrm{rad}(x_n))$ is $\geq -\frac{1}{\rho(x_n)^2} \geq -\frac{1}{(3L\,\mathrm{rad}(x_n))^2}$. Moreover, by Lemma 3.4 (ii), we have

$$\frac{\mathrm{vol}(B(x_n, 3L\,\mathrm{rad}(x_n)))}{(3L\,\mathrm{rad}(x_n))^3} \geq \frac{\mathrm{vol}(B(x_n, \mathrm{rad}(x_n)))}{\mathrm{rad}(x_n)^3}\frac{1}{(3L)^3} = \frac{\varepsilon_0}{(3L)^3}.$$

By applying Corollary 3.3 for n sufficiently large so that we have $3L\,\mathrm{rad}(x_n) \leq \bar{r}'(\frac{\varepsilon_0}{(3L)^3})$, there exist $K_0' = K_0'(\varepsilon_0, L) > 0$ and $K_1' = K_1'(\varepsilon_0, L) > 0$ independent of $\mathrm{rad}(x_n)$ such that for each $y \in B(x_n, L\,\mathrm{rad}(x_n))$, $|\,\mathrm{Rm}(y)| < K_0' \cdot (3\,L\,\mathrm{rad}(x_n))^{-2}$ and $|\nabla\,\mathrm{Rm}(y)| < K_1' \cdot (3\,L\,\mathrm{rad}(x_n))^{-3}$. Thus the curvature and its first derivative are uniformly bounded above on any rescaled ball $B(\bar{x}_n, L) \subset \overline{M}_n$, where the bounds depend only on the radius $L > 1$ and on $\varepsilon_0 > 0$.

Since the injectivity radius of the basepoint $\bar{x}_n$ is bounded below along the sequence, this upper bound on the curvature allows to use Gromov's compactness theorem (cf. [12, Chap. 8, Theorem 8.28], [34], [16, Theorem 2.3] or [10, Theorems 4.1 and 5.10]). It follows that the pointed sequence $(\overline{M}_n, \bar{x}_n)$ subconverges in the $\mathcal{C}^2$-topology towards a 3-dimensional smooth manifold $(\overline{X}_\infty, x_\infty)$, with a complete Riemannian metric of class $\mathcal{C}^2$ with nonnegative sectional curvature. This limit manifold cannot be closed, because that would contradict the assumption that the conclusion (a) of Proposition 3.1 does not hold.

Hence $\overline{X}_\infty$ is not compact. By pointed convergence, Assertion (b) (1) of Proposition 3.1 holds true. Let $\overline{S}$ be the soul of $\overline{X}_\infty$. Let us choose

$$\nu(x_n) := L\,\mathrm{rad}(x_n), \quad \text{where } L \geq 2\,\mathrm{diam}(\overline{S} \cup \{x_\infty\})\,D_0.$$

For n large (to be specified later) we set

$$X_{x_n} := \mathrm{rad}(x_n)\overline{X}_\infty, \quad \text{and} \quad S_{x_n} := \mathrm{rad}(x_n)\overline{S}.$$

We then have

$$\mathrm{diam}(S_{x_n}) = \mathrm{rad}(x_n)\,\mathrm{diam}(\overline{S}) < \nu(x_n)/D_0.$$

Let $\bar{f}_n : B(\bar{x}_n, L) \to (\overline{X}_\infty, x_\infty)$ be a map which is a $(1+\delta_n)$-bi-Lipschitz diffeomorphism onto its image and such that $d(\bar{f}_n(\bar{x}_n), x_\infty) < \delta_n$, where δ_n is a sequence going to 0. After rescaling, $f_n : B(x_n, L\,\mathrm{rad}(x_n)) \to X_{x_n}$ is also a $(1+\delta_n)$-bi-Lipschitz diffeomorphism onto its image. We get:

$$\begin{aligned} d(f_n(x_n), S_{x_n}) &= \mathrm{rad}(x_n)\,d(\bar{f}_n(\bar{x}_n), \overline{S}) \\ &\leq \mathrm{rad}(x_n)(d(\bar{f}_n(\bar{x}_n), \bar{x}_\infty) + d(\bar{x}_\infty, \overline{S})) \end{aligned}$$

$$\leq \mathrm{rad}(x_n)\delta_n + \frac{\nu(x_n)}{2D_0} \leq \frac{\nu(x_n)}{D_0}.$$

This proves assertion (b) (2) of Proposition 3.1.

Using the facts that $\nu(x_n) = L\,\mathrm{rad}(x_n) < \rho(x_n)$, the curvature on the ball $B(x_n, \nu(x_n))$ is $\geq -1/\nu(x_n)^2$, $L > 1$ and the Bishop-Gromov inequality, we get:

$$\frac{\mathrm{vol}(B(x_n, \nu(x_n)))}{v_{-\frac{1}{\nu^2(x_n)}}(\nu(x_n))} \leq \frac{\mathrm{vol}(B(x_n, \mathrm{rad}(x_n)))}{v_{-\frac{1}{\nu^2(x_n)}}(\mathrm{rad}(x_n))} = \varepsilon_0 \frac{\mathrm{rad}(x_n)^3}{v_{-\frac{1}{\nu^2(x_n)}}(\mathrm{rad}(x_n))}$$
$$= \varepsilon_0 \left(\frac{\mathrm{rad}(x_n)}{\nu(x_n)}\right)^3 \frac{1}{v_{-1}(\frac{\mathrm{rad}(x_n)}{\nu(x_n)})} = \varepsilon_0 \frac{1}{L^3} \frac{1}{v_{-1}(\frac{1}{L})}.$$

Since $v_{-1}(\frac{1}{L}) \geq v_0(\frac{1}{L}) = v_0(1)\frac{1}{L^3}$, we find that:

$$\mathrm{vol}(B(x_n, \nu(x_n))) \leq \varepsilon_0 \frac{1}{L^3} \frac{v_{-1}(1)}{v_{-1}(\frac{1}{L})} \nu^3(x_n)$$
$$\leq \varepsilon_0 \frac{v_{-1}(1)}{v_0(1)} \nu^3(x_n) = \frac{1}{D_0} \nu^3(x_n),$$

where the last equality comes from the definition of ε_0.

Hence we get the contradiction required to conclude the proof of Proposition 3.1. □

4 Constructions of coverings

We begin by making some reductions for the proof of Theorem 1.1.

If case (a) of Proposition 3.1 occurs for some D, then M is a closed, orientable, irreducible 3-manifold admitting a metric of nonnegative sectional curvature. By [14, 15], M is spherical or Euclidean, hence a graph manifold. Therefore we may assume that all manifolds X_x produced by Proposition 3.1 are noncompact.

For the same reasons, since lens spaces are graph manifolds, we can also assume that M is not homeomorphic to a lens space, and in particular does not contain a projective plane.

4.1 Existence of a homotopically nontrivial open set

We say that a path-connected subset $U \subset M$ is *homotopically trivial* (in M) if the image of the homomorphism $\pi_1(U) \to \pi_1(M)$ is trivial. More generally, we say that a subset $U \subset M$ is homotopically trivial if all its path-connected components have this property.

We recall that the *dimension* of a finite covering $\{U_i\}_i$ of M is the dimension of its nerve, hence the dimension plus one equals the maximal number of U_i's containing a given point.

Proposition 4.1 *There exists $D_0 > 1$ such that for all $D > D_0$, for every $n \geq n_0(D)$ (where $n_0(D)$ is given by Proposition 3.1), there exists $x \in M_n$ such that the image of $\pi_1(B(x, \nu(x))) \to \pi_1(M_n)$ is not homotopically trivial, where $\nu(x)$ is also given by Proposition 3.1.*

In [11] J.C. Gómez-Larrañaga and F. González-Acuña compute the 1-dimensional Lusternik-Schnirelmann category of a closed 3-manifold. One step of their proof gives the following proposition (cf. [11, Proof of Proposition 2.1]):

Proposition 4.2 *Let X be a closed, connected 3-manifold. If X has a covering of dimension 2 by open subsets which are homotopically trivial in X, then there is a connected 2-dimensional complex K and a continuous map $f : X \to K$ such that the induced homomorphism $f_\star : \pi_1(X) \to \pi_1(K)$ is an isomorphism.*

Standard homological arguments show the following, cf. [11, §3]:

Corollary 4.3 *Let X be a closed, connected, orientable, irreducible 3-manifold. If X has a covering of dimension 2 by open subsets which are homotopically trivial in X, then X is simply connected.*

Proof Following Proposition 4.2, let $f : X \to K$ be a continuous map from X to a connected 2-dimensional complex K, such that the induced homomorphism $f_\star : \pi_1(X) \to \pi_1(K)$ is an isomorphism. Let Z be a $K(\pi_1(X), 1)$ space. Let $\phi : X \to Z$ be a map from X to Z realising the identity homomorphism on $\pi_1(X)$ and let $\psi : K \to Z$ be the map from K to Z realising the isomorphism $f_\star^{-1} : \pi_1(K) \to \pi_1(X)$. Then ϕ is homotopic to $\psi \circ f$ and the induced homomorphism $\phi_* : H_3(X; \mathbb{Z}) \to H_3(Z; \mathbb{Z})$ factors through $\psi_* : H_3(K; \mathbb{Z}) \to H_3(Z; \mathbb{Z})$. Since $H_3(K; \mathbb{Z}) = \{0\}$, the homomorphism ϕ_* must be trivial.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & Z \\ K \arrow[ur, "\psi"'] & \end{tikzcd}$$

If $\pi_1(X)$ is infinite, then X is aspherical and $\phi_*: H_3(X;\mathbb{Z}) = \mathbb{Z} \to H_3(Z;\mathbb{Z})$ is an isomorphism, contradicting that it is trivial. Therefore $\pi_1(X)$ is finite.

If $\pi_1(X)$ is finite of order $d > 1$, then let $\widetilde{X}$ be the universal covering of X. The covering map $p: \widetilde{X} \to X$ induces an isomorphism between the homotopy groups $\pi_k(\widetilde{X})$ and $\pi_k(X)$ for $k \geq 2$. Since $\pi_2(X) = \{0\}$ by irreducibility and the sphere theorem, $\pi_2(\widetilde{X}) = \{0\}$, and by the Hurewicz theorem, the canonical homomorphism $\pi_3(\widetilde{X}) \to H_3(\widetilde{X};\mathbb{Z}) = \mathbb{Z}$ is an isomorphism. It follows that the canonical map $\pi_3(X) = \mathbb{Z} \to H_3(X;\mathbb{Z}) = \mathbb{Z}$ is the multiplication by the degree $d > 1$ of the covering $p: \widetilde{X} \to X$. It is well known that one can construct a $K(\pi_1(X), 1)$ space Z by adding a 4-cell to X in order to kill the generator of $\pi_3(X) = \mathbb{Z}$, and adding further cells of dimension ≥ 5 to kill the higher homotopy groups. Then the inclusion $\phi: X \to Z$ induces the identity on $\pi_1(X)$ and a surjection $\phi_*: H_3(X;\mathbb{Z}) = \mathbb{Z} \to H_3(Z;\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$. Since ϕ_* is trivial, X must be simply connected. □

In the proof of Proposition 4.1 we argue by contradiction using Corollary 4.3 and the fact that $\pi_1(M)$ is not trivial. Namely, with the notation of Proposition 3.1, let us assume the following:

Assumption A *For arbitrarily large D there exists $n \geq n_0(D)$ such that the image of $\pi_1(B(x,\nu(x))) \to \pi_1(M_n)$ is trivial for all $x \in M_n$.*

Then for all $x \in M_n$ we set:

$$\operatorname{triv}(x) := \sup \left\{ r \;\middle|\; \begin{array}{l} \pi_1(B(x,r)) \to \pi_1(M_n) \text{ is trivial and} \\ B(x,r) \text{ is contained in } B(x',r') \text{ with} \\ \text{curvature } \geq -\frac{1}{(r')^2} \end{array} \right\}.$$

Notice that $\operatorname{triv}(x) \geq \nu(x)$, by Proposition 3.1.

The proof of Proposition 4.1 follows by contradiction with the following assertion.

Assertion 4.4 *There exists a covering of M_n by open sets $U_1, \ldots, U_p$ such that*:

- *Each U_i is contained in some $B(x_i, \operatorname{triv}(x_i))$. In particular, U_i is homotopically trivial in M.*
- *The dimension of this covering is at most* 2.

Since M is irreducible and non-simply connected, this contradicts Corollary 4.3, hence Assumption A does not hold.

To prove Assertion 4.4, we define

$$r(x) := \min\left\{\frac{1}{11}\operatorname{triv}(x),\, 1\right\}.$$

Lemma 4.5 *For every $x \in M_n$:*

(1) *$B(x, 11\, r(x))$ is contained in some ball $B(x', r'(x))$ with curvature $\geq -\frac{1}{(r')^2}$ and satisfying $r(x) \leq \frac{1}{11}\operatorname{triv}(x) \leq \frac{2}{11} r'(x)$.*
(2) *$\frac{\nu(x)}{11} \leq r(x)$, where $\nu(x) < \rho(x) \leq 1$ is given by Proposition 3.1.*

Proof To prove (1) we shall show that the ball $B(x, \operatorname{triv}(x))$ is contained in another ball $B(x', r')$ with curvature $\geq -\frac{1}{(r')^2}$ and satisfying $r' \geq \frac{1}{2}\operatorname{triv}(x)$. By definition, there exists a sequence of radii $r_k \nearrow \operatorname{triv}(x)$ satisfying that $\pi_1(B(x, r_k)) \to \pi_1(M_n)$ is trivial and that $B(x, r_k)$ is contained in some $B(x'_k, r'_k)$ with curvature $\geq -\frac{1}{(r'_k)^2}$. Since $\pi_1(M_n)$ is nontrivial, $M_n \not\subset B(x, r_k)$, therefore there is a point $y_k \in M_n$ such that $d(x, y_k) = r_k$. By applying the triangle inequality to x, y_k and x'_k, we get $r_k \leq 2r'_k$. Then the claim follows by taking a partial subsequence so that both x'_k and r'_k converge, since we are working in a fixed M_n, that has bounded diameter.

Assertion (2) uses Assumption A and the inequality $\nu(x) < \rho(x)$, because the curvature on $B(x, \rho(x))$ is $\geq -\rho(x)^{-2} \geq -\nu(x)^{-2}$. □

Lemma 4.6 *Let $x, y \in M_n$. If $B(x, r(x)) \cap B(y, r(y)) \neq \emptyset$, then*

(a) $3/4 \leq r(x)/r(y) \leq 4/3$;
(b) $B(x, r(x)) \subset B(y, 4r(y))$.

Proof To prove (a), we may assume that $r(x) \leq r(y)$ and that $r(x) = \frac{1}{11}\operatorname{triv}(x) < 1$. Since $B(x, \operatorname{triv}(y) - r(x) - r(y)) \subset B(y, \operatorname{triv}(y))$, we get:

$$\operatorname{triv}(x) \geq \operatorname{triv}(y) - r(x) - r(y),$$

hence

$$11\, r(x) = \operatorname{triv}(x) \geq 11 r(y) - r(x) - r(y) \geq 9 r(y).$$

Consequently, we have $1 \geq r(x)/r(y) \geq 9/11 \geq 3/4$, which shows (a).

Now (b) follows because $2\, r(x) + r(y) \leq (\frac{8}{3} + 1) r(y) < 4 r(y)$. □

We choose a sequence of points $x_1, x_2, \ldots$ in M_n such that the balls $B(x_1, \frac{1}{4} r(x_1))$, $B(x_2, \frac{1}{4} r(x_2)), \ldots$ are pairwise disjoint. Such a sequence is necessarily finite, since M_n is compact, and Lemma 4.6 implies a positive local lower bound for the function $x \mapsto r(x)$. Let us choose a maximal finite sequence $x_1, \ldots, x_p$ with this property.

Lemma 4.7 *The balls $B(x_1, \frac{2}{3}r(x_1)), \ldots, B(x_p, \frac{2}{3}r(x_p))$ cover M_n.*

Proof Let $x \in M_n$ be an arbitrary point. By maximality, there exists a point x_j such that $B(x, \frac{1}{4}r(x)) \cap B(x_j, \frac{1}{4}r(x_j)) \neq \emptyset$. From Lemma 4.6, we have $r(x) \leq \frac{4}{3}r(x_j)$ and $d(x, x_j) \leq \frac{1}{4}(r(x) + r(x_j)) \leq \frac{7}{12}r(x_j)$, hence $x \in B(x_j, \frac{2}{3}r(x_j))$. □

Let us define $r_i := r(x_i)$. For $i = 1, \ldots, p$, we set

$$V_i := B(x_i, r_i).$$

Since $r_i \leq \frac{1}{11}\operatorname{triv}(x_i)$, the V_i are homotopically trivial. The construction of the open sets V_i and Lemma 4.7 imply the following:

Lemma 4.8 *The open sets $V_1, \ldots, V_p$ cover M_n.*

Let K be the simplicial complex that has one vertex for each open set of the covering $\{V_i\}$, and such that $n + 1$ vertices define an n-simplex of K iff the intersection of the corresponding $n + 1$ open sets of the covering is nonempty. The complex K is called the nerve of the covering $\{V_i\}$. To shrink this covering, we shall start with a map $f : M_n \to K$ constructed by means of a partition of unity, shrink it to the 2-skeleton $K^{(2)}$ and take the pullbacks of open stars of vertices in $K^{(2)}$. This technique is borrowed from Gromov [13, §3.4].

The following lemma shows that the dimension of K is bounded above by a uniform constant.

Lemma 4.9 *There exists a universal upper bound N on the number of open sets V_i which intersect a given V_k.*

Proof If $V_i \cap V_k \neq \emptyset$, then $B(x_i, r_i) \cap B(x_k, r_k) \neq \emptyset$ and $B(x_i, r_i) \subset B(x_k, 4r_k)$, by Lemma 4.6 (b). On the other hand, for all $i_1 \neq i_2$ such that V_{i_1} and V_{i_2} intersect V_k one has $d(x_{i_1}, x_{i_2}) \geq \frac{1}{4}(r_{i_1} + r_{i_2}) \geq \frac{3}{8}r_k$. Thus $B(x_{i_1}, \frac{3}{16}r_k) \cap B(x_{i_2}, \frac{3}{16}r_k) = \emptyset$ and $B(x_i, \frac{3}{16}r_k) \subset B(x_k, 4r_k)$. This motivates the following inequalities:

$$\frac{\operatorname{vol}(B(x_k, 4r_k))}{\operatorname{vol}(B(x_i, \frac{3}{16}r_k))} \leq \frac{\operatorname{vol}(B(x_i, 8r_k))}{\operatorname{vol}(B(x_i, \frac{3}{16}r_k))} \leq \frac{\operatorname{vol}(B(x_i, 11r_i))}{\operatorname{vol}(B(x_i, \frac{r_i}{8}))}.$$

As $B(x_i, 11r_i)$ is included in a ball $B(x', r')$ with curvature $\geq -\frac{1}{(r')^2}$ by Lemma 4.5, by the Bishop-Gromov inequality this quotient is bounded above by:

$$\frac{v_{-\frac{1}{(r')^2}}(11r_i)}{v_{-\frac{1}{(r')^2}}(\frac{r_i}{8})} = \frac{v_{-1}(\frac{11r_i}{r'})}{v_{-1}(\frac{r_i}{8r'})} \leq N.$$

The existence of a uniform N uses $\frac{r_i}{r'} \leq \frac{2}{11}$. This N bounds the number of V_i that intersect a given V_k. □

Let $\Delta^{p-1} \subset \mathbf{R}^p$ denote the standard unit simplex of dimension $p-1$. By using test functions ϕ_i supported on the V_i, that are Lipschitz with gradient $\leq \frac{4}{r_i}$, we construct a map:

$$f = \frac{1}{\sum_i \phi_i}(\phi_1, \ldots, \phi_p) \colon M_n \to \Delta^{p-1} \subset \mathbf{R}^p.$$

In particular, the coordinate functions of f are a partition of unity subordinated to (V_i).

We view K as a subcomplex of Δ^{p-1}, so that the range of f is contained in K, whose dimension is at most N. We first estimate the Lipschitz constant of the map $f \colon M_n \to K$, by choosing the ϕ_i's.

Lemma 4.10 *There exists a uniform $L > 0$ such that the partition of unity can be chosen so that the restriction $f|_{V_k}$ is $\frac{L}{r_k}$-Lipschitz.*

Proof Let $\tau : [0, 1] \to [0, 1]$ be an auxiliary function with Lipschitz constant bounded by 4, which vanishes in a neighbourhood of 0 and satisfies $\tau|_{[\frac{1}{3},1]} \equiv 1$. Let us define $\phi_k := \tau(\frac{1}{r_k} d(\partial V_k, \cdot))$ on V_k and let us extend it trivially on M_n. Then ϕ_k is $\frac{4}{r_k}$-Lipschitz.

Let $x \in V_k$. The functions ϕ_i have Lipschitz constant $\leq \frac{4}{3} \cdot \frac{4}{r_k}$ on V_k, and all ϕ_i vanish at x except at most $N+1$ of them. Since the functions

$$(y_0, \ldots, y_N) \mapsto \frac{y_k}{\sum_{i=0}^N y_i}$$

are Lipschitz on

$$\left\{ y \in \mathbf{R}^{N+1} \mid y_0 \geq 0, \ldots, y_N \geq 0 \text{ and } \sum_{i=0}^N y_i \geq 1 \right\},$$

and each $x \in M_n$ belongs to some V_k with $d(x, \partial V_k) \geq \frac{r_k}{3}$ by Lemma 4.7, the conclusion follows. □

We shall now inductively deform f by homotopy into the 3-skeleton $K^{(3)}$, while keeping the local Lipschitz constant under control on each V_k.

We recall that the *open star* of a vertex of K is the union of the interiors of all simplices whose closures contain the given vertex.

Lemma 4.11 *For all $d \geq 4$ and $L > 0$ there exists $L' = L'(d, L) > 0$ such that the following assertion holds true*:

Let $g : M_n \to K^{(d)}$ be a map that is $\frac{L}{r_k}$-Lipschitz on V_k and such that the pull-back of the open star of the vertex $v_{V_k} \in K^{(0)}$ is contained in V_k. Then g is homotopic rel $K^{(d-1)}$ to a map $\tilde{g} : M_n \to K^{(d-1)}$ which is $\frac{L'}{r_k}$-Lipschitz on V_k and so that the pull-back of the open star of v_{V_k} is still contained in V_k.

Proof It suffices to find a constant $\theta = \theta(d, L) > 0$ such that each d-simplex $\sigma \subset K$ contains a point z whose distance to $\partial\sigma$ and to the image of g is $\geq \theta$. In order to push g into the $(d-1)$-skeleton, we compose it on σ with the radial projection from z. This increases the Lipschitz constant by a multiplicative factor bounded above by a function of $\theta(d, L)$, and decreases the inverse image of the open stars of the vertices.

If θ does not satisfy the required property for some d-simplex σ, then one can tile an open subset of $\sigma \setminus (\theta - \text{neighbourhood of } \partial\sigma)$ with at least $C'(d) \cdot \frac{1}{\theta^d}$ cubes of length 2θ, and each cube contains a point of image(g) in its interior. By choosing one point of image(g) inside each cube whose tiling coordinates are even, we find a subset of cardinality at least $C(d) \cdot \frac{1}{\theta^d}$ of points in image$(g) \cap \operatorname{int}(\sigma)$ whose pairwise distances are $\geq \theta$. Let $A \subset M_n$ be a set containing exactly one point of the inverse image of each of these points. By hypothesis, $A \subset V_k = B(x_k, r_k)$ for any k corresponding to a vertex of σ. As g is $\frac{L}{r_k}$-Lipschitz on V_k, the distance between any two distinct points in A is bounded below by $\frac{r_k}{L} \cdot \theta$. Hence to bound the cardinality of A, we use the following inequality for $y \in A$:

$$\frac{\operatorname{vol}(B(x_k, r_k))}{\operatorname{vol}(B(y, \frac{r_k\theta}{2L}))} \leq \frac{\operatorname{vol}(B(y, 2r_k))}{\operatorname{vol}(B(y, \frac{r_k\theta}{2L}))} \leq \frac{v_{-\frac{1}{(r')^2}}(2r_k)}{v_{-\frac{1}{(r')^2}}(\frac{r_k\theta}{2L})}$$

$$= \frac{v_{-1}(2\frac{r_k}{r'})}{v_{-1}(\frac{\theta}{2L}\frac{r_k}{r'})} \leq C''\left(\frac{L}{\theta}\right)^3.$$

Since $r_k \leq \frac{2}{11}r'$, such a uniform C'' exists. Thus the cardinality of A is at most $C''(L/\theta)^3$. In order to apply Bishop-Gromov, we used the fact that $B(x_k, 11r_k)$ is contained in a ball of radius r' with curvature $\geq -1/(r')^2$. The inequality $C(d) \cdot \frac{1}{\theta^d} \leq C'' \cdot (\frac{L}{\theta})^3$ gives a positive lower bound $\theta_0(d, L)$ for θ. Consequently, any $\theta < \theta_0$ has the desired property. $\square$

Lemma 4.12 *There exists a universal constant C_0 such that the following holds. Let $D > 1$ and $n > n_0(D)$ be as in Proposition 3.1. Then*

$$\mathrm{vol}(B(x_i, r_i)) \leq C_0 \frac{1}{D} r_i^3 \quad \textit{for all } i.$$

Proof By Proposition 3.1, we know that $\mathrm{vol}(B(x_i, \nu(x_i))) \leq \frac{1}{D}\nu(x_i)^3$. Furthermore, by Lemma 4.5 we have $r_i \geq \frac{\nu(x_i)}{11}$ and $B(x_i, r_i)$ is included in a ball $B(x', r')$ with curvature $\geq -\frac{1}{r'^2}$. As $r' \geq \frac{11}{2} r_i > r_i$, the curvature on $B(x_i, r_i)$ is $\geq -\frac{1}{r_i^2}$. The Bishop-Gromov inequality gives:

$$\frac{\mathrm{vol}(B(x_i, \frac{\nu(x_i)}{11}))}{v_{-\frac{1}{r_i^2}}(\frac{\nu(x_i)}{11})} \geq \frac{\mathrm{vol}(B(x_i, r_i))}{v_{-\frac{1}{r_i^2}}(r_i)}.$$

Equivalently,

$$\begin{aligned} \mathrm{vol}\left(B\left(x_i, \frac{\nu(x_i)}{11}\right)\right) &\geq \frac{\mathrm{vol}(B(x_i, r_i))}{v_{-1}(1)} v_{-1}\left(\frac{\nu(x_i)}{11 r_i}\right) \\ &\geq \mathrm{vol}(B(x_i, r_i)) \frac{1}{C_0}\left(\frac{\nu(x_i)}{r_i}\right)^3, \end{aligned}$$

for some uniform $C_0 > 0$. Hence, using Proposition 3.1 (b) (3),

$$\mathrm{vol}(B(x_i, r_i)) \leq C_0 \left(\frac{r_i}{\nu(x_i)}\right)^3 \mathrm{vol}\left(B\left(x_i, \frac{\nu(x_i)}{11}\right)\right) \leq C_0 r_i^3 \frac{1}{D}.$$ □

Finally we push f into the 2-skeleton.

Lemma 4.13 *For a suitable choice of $D > 1$, there exists a map $f^{(2)} : M_n \to K^{(2)}$ such that*:

(i) *$f^{(2)}$ is homotopic to f rel $K^{(2)}$.*
(ii) *The inverse image of the open star of each vertex $v_{V_k} \in K^{(0)}$ is contained in V_k.*

Proof The inverse image by f of the open star of the vertex $v_{V_k} \in K^{(0)}$ is contained in V_k. Using Lemma 4.11 several times, we find a map $f^{(3)} : M_n \to K^{(3)}$ homotopic to f and a universal constant $\hat{L}$ such that $(f^{(3)})^{-1}(\mathrm{star}(v_{V_k})) \subset V_k$ and $f^{(3)}_{|V_k}$ is $\frac{\hat{L}}{r_k}$-Lipschitz.

It now suffices to show that no 3-simplex $\sigma \subset K$ can lie entirely in the image of $f^{(3)}$. Indeed, once we know this, we can push $f^{(3)}$ into the 2-skeleton of K using a central projection in each simplex, with centre in the complement of this image. Note that here no metric estimate is required in the conclusion.

Let us thus assume that there exists a 3-simplex σ contained in the image of $f^{(3)}$. The inverse image of $\mathrm{int}(\sigma)$ by $f^{(3)}$ is a subset of the intersection of those V_j's such that v_{V_j} is a vertex of σ. Let V_k be one of them. As $\mathrm{vol}(f^{(3)}(V_k)) \leq \mathrm{vol}(f^{(3)}(B(x_k, r_k)))$, Lemma 4.12 yields:

$$\begin{aligned} \mathrm{vol}(\mathrm{image}(f^{(3)}) \cap \sigma) \leq \mathrm{vol}(f^{(3)}(V_k)) &\leq \left(\frac{\hat{L}}{r_k}\right)^3 \mathrm{vol}(B(x_k, r_k)) \\ &\leq C_0 \hat{L}^3 \frac{1}{D} \end{aligned}$$

with uniform constants C_0 and $\hat{L}$. Hence, if D is sufficiently large, then one has $\mathrm{vol}(\mathrm{image}(f^{(3)}) \cap \sigma) < \mathrm{vol}(\sigma)$. □

The inverse images of the open stars of the vertices v_k satisfy $(f^{(2)})^{-1}(\mathrm{star}(v_{V_k})) \subset V_k$, thus $(f^{(2)})^{-1}(\mathrm{star}(v_{V_k}))$ is homotopically trivial in M. This proves Assertion 4.4 and ends the proof by contradiction of Proposition 4.1.

4.2 End of the proof of Theorem 1.1

The following is a consequence of Proposition 4.1, where the constants D_0 and $n_0(D)$ are provided by Propositions 4.1 and 3.1 respectively. For the sake of simplicity let $X_0 := X_{x_0}$ and $S_0 := S_{x_0}$ be the manifolds given by Proposition 3.1 for some $x_0 \in M$.

Corollary 4.14 *There exists $D_0 > 0$ such that if $D > D_0$ and $n \geq n_0(D)$, then there exists a compact submanifold $\mathcal{W}_0 \subset M_n$ with the following properties*:

(i) *$\mathcal{W}_0$ is $\frac{1}{D}$-close to a tubular neighbourhood of the soul of the manifold X_0 for some point $x_0 \in M_n$.*
(ii) *$\mathcal{W}_0$ is a solid torus, a thickened torus or the twisted I-bundle on the Klein bottle. In particular $\partial\mathcal{W}_0$ is a union of (one or two) tori.*
(iii) *$\mathcal{W}_0$ is homotopically non-trivial in M_n.*

Proof By Proposition 4.1, there exists a point $x_0 \in M_n$ such that $B(x_0, \nu(x_0))$ is homotopically non-trivial; one of the remarks following Proposition 3.1 shows that $B(x_0, \nu(x_0))$ is necessarily a solid torus, a thickened torus or a twisted I-bundle over the Klein bottle. Indeed, the soul S_0 of the manifold X_0 can neither be a point nor a 2-sphere, otherwise $B(x_0, \nu(x_0))$ would be homeomorphic to B^3 or $S^2 \times I$, which have trivial fundamental group. Let $f_{x_0} \colon B(x_0, \nu_0) \to X_0$ denote the $(1 + \frac{1}{D})$-bi-Lipschitz diffeomorphism onto its image, provided by Proposition 3.1. We take $\mathcal{W}_0 = f_{x_0}^{-1}(\overline{\mathcal{N}_\delta(S_0)})$, the inverse image of the closed tubular neighborhood of radius δ of S_0, for some $0 < \delta < \nu_0/D$. □

As $\mathcal{W}_0$ is not contained in any 3-ball, each component Y of its complement is irreducible, hence a *Haken manifold* whose boundary is a union of (possibly compressible) tori. In particular, Y admits a geometric decomposition. Here is an important consequence of Thurston's hyperbolic Dehn filling theorem and Geometrisation for Haken manifolds (cf. [2, Proposition 10.17], [3, Proposition 9.36]):

Proposition 4.15 *Let Y be a Haken 3-manifold whose boundary is a union of tori. Assume that any manifold obtained from Y by Dehn filling has vanishing simplicial volume. Then Y is a graph manifold.*

In order to prove that M_n is a graph manifold, it is sufficient to show that each component of $M_n \setminus \mathrm{int}(\mathcal{W}_0)$ is a graph manifold. To conclude the proof of Theorem 1.1, it suffices to show the following proposition:

Proposition 4.16 *For n large enough, one can find a submanifold $\mathcal{W}_0$ as above such that every Dehn filling on each component Y of $M_n \setminus \mathrm{int}(\mathcal{W}_0)$ has vanishing simplicial volume.*

We choose the set $\mathcal{W}_0$ as follows. There exists a point $x \in M_n$ such that $B(x, \nu(x))$ is homotopically non-trivial in M_n, where $\nu(x) > 0$ is given by Proposition 3.1 (b). We choose $x_0 \in M_n$ such that

$$\nu_0 = \nu(x_0) \geq \frac{1}{2} \sup_{x \in M_n} \{\nu(x) \mid \pi_1(B(x, \nu(x))) \to \pi_1(M_n) \text{ is non-trivial}\}.$$

Let S_0 be the soul of the manifold X_0 of $B(x_0, \nu_0)$, provided by Proposition 3.1. If $f_{x_0} : B(x_0, \nu_0) \to X_0$ is the $(1 + \frac{1}{D})$-bi-Lipschitz diffeomorphism onto its image, we choose

$$\mathcal{W}_0 := f_{x_0}^{-1}(\overline{\mathcal{N}_\delta(S_0)})$$

where $\overline{\mathcal{N}_\delta(S_0)}$ denotes the closed metric δ-neighbourhood of S_0, with $0 < \delta < \frac{\nu_0}{D}$, where $D > D_0$ is given by Corollary 4.14.

Notice that for each $x \in M_n$, $\pi_1(B(x, \nu_x))$ is virtually abelian. We say that a subset $U \subset M_n$ is *virtually abelian relatively to $\mathcal{W}_0$* if the image in $\pi_1(M_n \setminus \mathrm{int}(\mathcal{W}_0))$ of the fundamental group of each connected component of $U \cap (M_n \setminus \mathrm{int}(\mathcal{W}_0))$ is virtually abelian.

We set:

$$\mathrm{ab}(x) := \sup\left\{ r \;\middle|\; \begin{array}{l} B(x,r) \text{ is virtually abelian relatively to } \mathcal{W}_0 \\ \text{and } B(x,r) \text{ is contained in a ball } B(x',r') \\ \text{with curvature } \geq -\frac{1}{(r')^2} \end{array} \right\}$$

and

$$r(x) := \min\left\{ \frac{1}{11}\,\mathrm{ab}(x), 1 \right\}.$$

Remark Corollary 4.14 implies $\mathrm{ab}(x_0) \geq \nu(x_0)$, because $B(x_0, \nu(x_0))$ is virtually abelian with respect to $\mathcal{W}_0$ and, since $\nu(x_0) \leq \rho(x_0)$, the ball $B(x_0, \nu(x_0))$ has curvature $\geq -\rho(x_0)^{-2} \geq -\nu(x_0)^{-2}$. Hence

$$r(x_0) \geq \frac{1}{11}\nu(x_0).$$

We are now led to prove the following assertion:

Assertion 4.17 *With this choice of $\mathcal{W}_0$, for n large enough, M_n can be covered by a finite collection of open sets U_i such that*:

- *Each U_i is contained in a $B(x_i, r(x_i))$ for some $x_i \in M_n$. In particular, U_i is virtually abelian relatively to $\mathcal{W}_0$.*
- *The dimension of this covering is not greater than 2, and it is zero on $\mathcal{W}_0$.*

Let us first show why this assertion implies Proposition 4.16.

Proof of Proposition 4.16 The covering described in the assertion induces naturally a covering on every closed, orientable manifold $\hat{Y}$ obtained by gluing solid tori to ∂Y. By the second point of Assertion 4.17, it is a 2-dimensional covering of $\hat{Y}$ by open sets which are virtually abelian and thus amenable in $\hat{Y}$. Gromov's vanishing theorem [13, §3.1], see also [18, 19], then implies that the image of the bounded cohomology of $\hat{Y}$ in the usual cohomology vanishes in dimension 3. Hence the dual to the fundamental class of $\hat{Y}$ (i.e. the generator of $H^3(\hat{Y}; \mathbb{R})$) is not a bounded cohomology class. By duality this fact is equivalent to the vanishing of the simplicial volume of $\hat{Y}$ (see [13, §1.1]) and thus proves Proposition 4.16. □

We now prove Assertion 4.17. The argument for the construction of a 2-dimensional covering by abelian open sets is similar to the one used in the proof of Assertion 4.4, replacing everywhere the triviality radius triv by the abelianity radius ab. The construction of the covering takes care of $\mathcal{W}_0$, in particular we will require Lemma 4.18. Then the direct analogues of Lemmas 4.5 to 4.13 hold with three small fixes. Firstly, the proof of Lemma 4.5

(1) needs to be adapted to this setting (Lemma 4.19). Secondly, Lemma 4.5 (2) must be replaced here by a similar statement with different constants (Lemma 4.20). Finally the analogue of Lemma 4.7 also needs further discussion (Lemma 4.21).

We now make this more precise.

We choose $x_0 \in \mathcal{W}_0 \subset M_n$ and a maximal finite sequence

$$x_0, x_1, x_2, \ldots, x_p$$

such that the balls $B(x_i, \frac{1}{4}r(x_i))$ are disjoint.

We set $r_i = r(x_i)$, for $i = 0, \ldots, p$ and

- $V_0 := B(x_0, r_0)$.
- $V_i := B(x_i, r_i) \setminus \mathcal{W}_0$ for $i = 1, \ldots, p$.

We first need the following result about $\mathcal{W}_0$.

Lemma 4.18 *If n is large enough, then we have*

$$\mathcal{W}_0 \subset B\left(x_0, \frac{4\nu(x_0)}{D}\right) \subset B\left(x_0, \frac{r_0}{9}\right) \subset V_0.$$

Proof The first inclusion uses the properties of Proposition 3.1: $\operatorname{diam}(S_0) < \frac{\nu_0}{D}$, $d(f_{x_0}(x_0), S_0) < \frac{\nu_0}{D}$, the construction $\mathcal{W}_0 = f_{x_0}^{-1}(\overline{\mathcal{N}_\delta(S_0)})$ with $\delta < \frac{\nu_0}{D}$ and the fact that f_{x_0} is a $(1 + \frac{1}{D})$-bi-Lipschitz diffeomorphism onto its image.

The second inclusion follows from $r_0 \geq \frac{1}{11}\nu(x_0)$ (see the remark before Assertion 4.17) by taking $D \geq 4 \cdot 9 \cdot 11$. □

Then we have the following analogue of Lemma 4.5 (1):

Lemma 4.19 *We can assume that for every $x \in M_n$, $B(x, 11\,r(x))$ is contained in some ball $B(x', r'(x))$ with curvature $\geq -\frac{1}{(r')^2}$ and satisfying $r(x) \leq \frac{1}{11}\operatorname{ab}(x) \leq \frac{2}{11}r'(x)$.*

Proof If M_n is virtually abelian relatively to $\mathcal{W}_0$, then the application of Assertion 4.17 holds true by taking a single open set, the whole M_n. Hence we may assume that M_n is not virtually abelian relatively to $\mathcal{W}_0$. Thus for every $x \in M_n$ we have $M_n \not\subset B(x, \operatorname{ab}(x))$. Therefore, by the same argument as in the proof of Lemma 4.5 (1), there exists a ball $B(x', r')$ that contains $B(x, \operatorname{ab}(x))$ and has curvature $\geq -\frac{1}{(r')^2}$. Moreover $r' \geq \frac{1}{2}\operatorname{ab}(x)$. □

The next lemma is the analogue of Lemma 4.5 (2), with a new constant c that just entails a change of constant in the analogue of Lemma 4.12.

Lemma 4.20 *There exists a universal $c > 0$ such that, if n is sufficiently large, then $r_i \geq c\,\nu(x_i)$ for all i.*

Proof One has $r_0 \geq \frac{1}{11}\nu(x_0)$ by construction. For all $i > 0$, if $B(x_i, \nu(x_i)) \cap \mathcal{W}_0 = \emptyset$, then $\mathrm{ab}(x_i) \geq \nu(x_i)$ so $r_i \geq \frac{1}{11}\nu(x_i)$. Hence we assume $B(x_i, \nu(x_i)) \cap \mathcal{W}_0 \neq \emptyset$, and we claim that $d(x_i, \mathcal{W}_0) > c'\,\nu(x_i)$ for a uniform $c' > 0$.

We also assume from now on that $r_i = \frac{1}{11}\mathrm{ab}(x_i) < 1$ (otherwise $r_i = 1 \geq \rho(x_i) > \nu(x_i)$ and we are done). Since $\mathcal{W}_0 \subset B(x_0, \frac{1}{9}r_0)$ and $B(x_i, \nu(x_i))$ has virtually abelian fundamental group and curvature $\geq \frac{-1}{\nu(x_i)^2}$,

$$r_i \geq \frac{1}{11}d(x_i, \mathcal{W}_0) \geq \frac{1}{11}\left(d(x_i, x_0) - \frac{1}{9}r_0\right) \geq \frac{r_0/4 - r_0/9}{11} > \frac{r_0}{88} \geq \frac{\nu(x_0)}{1000}. \tag{1}$$

We distinguish two cases, according to whether $\mathcal{W}_0$ is contained in $B(x_i, \nu(x_i))$ or not.

If $\mathcal{W}_0 \subset B(x_i, \nu(x_i))$, then the image of $\pi_1(B(x_i, \nu(x_i))) \to \pi_1(M_n)$ cannot be trivial, since the image of $\pi_1(\mathcal{W}_0) \to \pi_1(M_n)$ is not. In addition,

$$\nu(x_0) \geq \frac{1}{2}\nu(x_i), \tag{2}$$

by the choice of x_0 and $\nu(x_0)$. Equations (1) and (2) give $r_i \geq \nu(x_i)/2000$.

If $\mathcal{W}_0 \not\subset B(x_i, \nu(x_i))$, then since $\mathcal{W}_0 \cap B(x_i, \nu(x_i)) \neq \emptyset$ and $\mathcal{W}_0 \subset B(x_0, \frac{r_0}{9})$, we have

$$11\,r_i = \mathrm{ab}(x_i) \geq d(x_i, x_0) - r_0/9;$$
$$\nu(x_i) \leq d(x_i, x_0) + r_0/9.$$

Since $d(x_i, x_0) \geq \frac{1}{4}r_0$, this yields $\frac{r_i}{\nu(x_i)} \geq \frac{1}{11} \cdot \frac{1/4 - 1/9}{1/4 + 1/9} \geq \frac{1}{30}$. $\square$

Finally the analogues of Lemmas 4.6 to 4.13 apply with no changes except for Lemma 4.7. The last result is a version of Lemma 4.7 in the context of the new boundary created by $\mathcal{W}_0$ and it is used in the control of the Lipschitz constant of the characteristic map, Lemma 4.10.

Lemma 4.21 *Each $x \in M_n$ belongs to some V_k such that $d(x, \partial V_k) \geq \frac{1}{3}r_k$.*

Proof of Lemma 4.21 If $x \in B(x_0, \frac{2}{3}r_0)$ we may choose $k = 0$. Let us then assume that $x \notin B(x_0, \frac{2}{3}r_0)$. There exists k such that $x \in B(x_k, \frac{2}{3}r_k)$, by the analogue of Lemma 4.7. If V_k and V_0 are disjoint, then $V_k \cap \mathcal{W}_0 = \emptyset$ and we are done. Hence we assume that $V_k \cap V_0 \neq \emptyset$. By Lemma 4.18 and the analogue of Lemma 4.6, one has:

$$d(x, \mathcal{W}_0) \geq d(x, x_0) - \frac{1}{9} r_0 \geq \frac{2}{3} r_0 - \frac{1}{9} r_0 \geq \frac{3}{4} \cdot \frac{5}{9} r_k > \frac{1}{3} r_k.$$

This implies that $d(x, \partial V_k) \geq \frac{1}{3} r_k$. □

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Appendix

In this appendix we give some details for the estimate of the pointwise injectivity radius needed in Corollary 3.5. This is a variation on Cheeger's propeller lemma. It is however different since the propeller lemma is written for closed manifolds and just gives a lower bound for the length of the smooth closed geodesics on M (see [7], Theorem 5.6).

Proposition A.1 *Let $K > 0$, $R > 0$ and $\varepsilon > 0$, then there exists $C = C(K, R, \varepsilon) > 0$ satisfying the following property. Let M be a complete Riemannian manifold and let $x \in M$. We assume that the sectional curvature is bounded above in absolute value by K on the ball $B(x, R)$ and that* $\mathrm{vol}(B(x, R)) > \varepsilon$. *Then, the injectivity radius of M at x satisfies*:

$$\mathrm{inj}(x) \geq C(K, R, \varepsilon).$$

Proof Here M is a complete manifold (*i.e.* without boundary). The injectivity radius of M at x is the largest $\rho > 0$ such that $\exp_x$ is an embedding on the open ball of radius ρ in $T_x M$. Equivalently it is the distance from x to its cut-locus.

Pick a point q in the cut locus of x achieving the minimum distance to x. Lemma 5.6 of [7] shows that q is either conjugate to x along a minimising geodesic or there is a geodesic loop based at x and smooth at q (but not necessarily smooth at x).[2]

[2]Notice that there is a misprint in the statement of the lemma, indeed in the last equality p should be replaced by q. This is clear when looking at the proof that follows.

We now consider the ball $B(x, R)$; the sectional curvature on this ball is bounded above in absolute value by K. If $q \in B(x, R)$ is conjugate to x along a minimising geodesic, then by the Rauch Comparison Theorem, $d(x, q) \geq \pi/\sqrt{K}$ (see Theorem 1.29 of [7] and notice that this just requires a control of the sectional curvature along the geodesic which is entirely contained in the ball). If $q \notin B(x, R)$ is conjugate to x then $d(x, q) \geq R$.

We define

$$r = \min(\pi/\sqrt{K}, R).$$

Note that on $B(x, r)$ the upper bound for the absolute value of the sectional curvature is still valid.

On the other hand, for any $0 < s \leq R$, a lower bound for $\mathrm{vol}(B(x, s))$ in terms of ε, s, R and K is obtained by applying Bishop-Gromov's theorem on $B(x, R)$ (the proof is again a comparison theorem along geodesics starting from x which all remain in the larger ball). The inequality is

$$V_s(x) \geq \varepsilon \frac{v_{-K}(s)}{v_{-K}(R)}$$

where $v_{-K}(\rho)$ denotes the volume of a ball of radius ρ in the simply connected space of sectional curvature $-K$, and $V_s(x)$ is the volume of $B(x, s)$.

We then apply Inequality 4.22 in [9] to $B(x, r)$, with $r_0 = s = r/4$, see Theorem 4.3 of [9]. It shows that if a geodesic loop based at x has length 2ℓ, then

$$\ell \geq \frac{r_0}{2} \frac{1}{1 + v_{-K}(r_0 + s)/V_s(x)} = \frac{r}{8(1 + v_{-K}(r/2)/V_{r/4}(x))}. \tag{3}$$

Hence the length is bounded below by a positive constant computed in terms of ε, R and K. Let us insist on the fact that Theorem 4.3 is local and that we are dealing with geodesic loops based at x that are not necessarily smooth at x (see the discussion on p. 45 of [9]).

Finally, if $q \in B(x, r)$, the first case in the alternative (q being conjugate to x) is ruled out by the fact that $d(x, q) < \pi/\sqrt{K}$. Inequality (3) then gives a bound from below for the length of the geodesic loop based at x, hence for $d(x, q)$, depending on ε, K and R only. □

References

1. Bessières, L., Besson, G., Boileau, M., Maillot, S., Porti, J.: Geometrisation of 3-manifolds (2009, submitted)
2. Boileau, M., Leeb, B., Porti, J.: Geometrization of 3-dimensional orbifolds. Ann. Math. (2) **162**(1), 195–290 (2005)

3. Boileau, M., Maillot, S., Porti, J.: Three-Dimensional Orbifolds and Their Geometric Structures. Panoramas et Synthèses (Panoramas and Syntheses), vol. 15. Société Mathématique de France, Paris (2003)
4. Boileau, M., Porti, J.: Geometrization of 3-orbifolds of cyclic type. Astérisque **272**, 208 (2001). Appendix A by Michael Heusener and Porti
5. Cao, J., Ge, J.: A proof of perelman's collapsing theorem for 3-manifolds, 28 October 2009. arXiv:math.DG/0908.3229
6. Cao, H.-D., Zhu, X.-P.: A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. Asian J. Math. **10**(2), 165–492 (2006)
7. Cheeger, J., Ebin, D.G.: Comparison Theorems in Riemannian Geometry. North-Holland Mathematical Library, vol. 9. North-Holland, Amsterdam (1975)
8. Cheeger, J., Gromoll, D.: On the structure of complete manifolds of nonnegative curvature. Ann. Math. (2) **96**, 413–443 (1972)
9. Cheeger, J., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. J. Differ. Geom. **17**(1), 15–53 (1982)
10. Fukaya, K.: Hausdorff convergence of Riemannian manifolds and its applications. In: Recent Topics in Differential and Analytic Geometry. Adv. Stud. Pure Math., vol. 18, pp. 143–238. Academic Press, Boston (1990)
11. Gómez-Larrañaga, J.C., González-Acuña, F.: Lusternik-Schnirelmann category of 3-manifolds. Topology **31**(4), 791–800 (1992)
12. Gromov, M.: Structures Métriques Pour les Variétés Riemanniennes. Textes Mathématiques (Mathematical Texts), vol. 1. CEDIC, Paris (1981). Edited by J. Lafontaine and P. Pansu
13. Gromov, M.: Volume and bounded cohomology. Inst. Hautes Études Sci. Publ. Math. **56**, 5–99 (1983)
14. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**(2), 255–306 (1982)
15. Hamilton, R.S.: Four-manifolds with positive curvature operator. J. Differ. Geom. **24**(2), 153–179 (1986)
16. Hamilton, R.S.: A compactness property for solutions of the Ricci flow. Am. J. Math. **117**(3), 545–572 (1995)
17. Hempel, J.: 3-Manifolds. Ann. of Math. Studies, vol. 86. Princeton University Press, Princeton (1976)
18. Ivanov, N.V.: Foundations of the theory of bounded cohomology. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **143**, 69–109 (1985). Studies in topology, V
19. Ivanov, N.V.: Foundations of the theory of bounded cohomology. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **143**, 177–178 (1985). Studies in topology, V
20. Jaco, W.: Lectures on Three-Manifold Topology. CBMS Regional Conference Series in Mathematics, vol. 43. American Mathematical Society, Providence (1980)
21. Kapovitch, V.: Perelman's stability theorem. In: Surveys in Differential Geometry. Vol. XI. Surv. Differ. Geom., vol. 11, pp. 103–136. Int. Press, Somerville (2007)
22. Kleiner, B., Lott, J.: Locally collapsed 3-manifolds (in preparation)
23. Kleiner, B., Lott, J.: Notes on Perelman's papers. Geom. Topol. **12**(5), 2587–2855 (2008)
24. Kuessner, T.: Multicomplexes, bounded cohomology and additivity of simplicial volume (2001). math.AT/0109057
25. McMullen, C.: Iteration on Teichmüller space. Invent. Math. **99**(2), 425–454 (1990)
26. Morgan, J., Tian, G.: Ricci Flow and the Poincaré Conjecture. Clay Mathematics Monographs, vol. 3. American Mathematical Society, Providence (2007)
27. Morgan, J., Tian, G.: Completion of the proof of the geometrization conjecture (2008). arXiv:0809.4040

28. Otal, J.-P.: Thurston's hyperbolization of Haken manifolds. In: Surveys in Differential Geometry, vol. III, Cambridge, MA, 1996, pp. 77–194. Interface Press, Boston (1998)
29. Otal, J.-P.: The Hyperbolization Theorem for Fibered 3-manifolds. SMF/AMS Texts and Monographs, vol. 7. American Mathematical Society, Providence (2001). Translated from the 1996 French original by Leslie D. Kay
30. Perelman, G.: Alexandrov spaces with curvatures bounded from below II. Preprint (1991). http://www.math.psu.edu/petrunin/papers/papers.html
31. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). math.DG/0211159
32. Perelman, G.: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds (2003). math.DG/0307245
33. Perelman, G.: Ricci flow with surgery on three-manifolds (2003). math.DG/0303109
34. Peters, S.: Convergence of Riemannian manifolds. Compos. Math. **62**(1), 3–16 (1987)
35. Petersen, P.: Riemannian Geometry. Graduate Texts in Mathematics, vol. 171. Springer, New York (1998)
36. Shioya, T., Yamaguchi, T.: Volume collapsed three-manifolds with a lower curvature bound. Math. Ann. **333**(1), 131–155 (2005)
37. Soma, T.: The Gromov invariant of links. Invent. Math. **64**(3), 445–454 (1981)
38. Thurston, W.P.: Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. Ann. Math. (2) **124**(2), 203–246 (1986)
39. Yamaguchi, T.: A convergence theorem in the geometry of Alexandrov spaces. In: Actes de la Table Ronde de Géométrie Différentielle, Luminy, 1992. Sémin. Congr., vol. 1, pp. 601–642. Soc. Math. France, Paris (1996)