Ricci flow, scalar curvature and the Poincaré Conjecture

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1 Introduction

One of the most important results of the past few years in geometry is Grisha Perelman's proof of the Poincaré Conjecture using Richard Hamilton's Ricci flow. Ricci flow is an evolution equation for a riemannian metric which can sometimes be used in order to deform an arbitrary metric into a 'nice' one, from which one can determine the topology of the underlying manifold. The main goal of this article is to discuss some results concerning Ricci flow on compact 3-manifolds and their application to the Poincaré Conjecture.

Section 2 is a warm-up section about the evolution of closed planar curves by their curvature vector. This topic has the advantage of being more concrete than Ricci flow, as well as requiring less background. Ricci flow is introduced in Section 3. In Section 4 we come to the main topic, which is Ricci flow in dimension 3 and its relation with the Poincaré conjecture.

Section 5 deals with the evolution of the scalar curvature, sweep-outs, and the width of a riemannian manifold. All this is applied to show that any Ricci flow on a 3-dimensional homotopy sphere develops a singularity. We give a fairly detailed treatment of these topics since it seems to us that they convey very well the interplay between topology, geometry, and analysis which is characteristic of the subject. Section 6 tackles canonical neighborhoods and metric surgery, and explains how to prove the Poincaré conjecture using these ideas together with those discussed in Section 5.

We have endeavored to make the exposition as accessible as possible, sometimes at the expense of precision and mathematical rigor. For instance, in Section 5 we treat the functions $R_{\min}(t)$ and $W(t)$ as if they were differentiable, whereas they are only continuous, and the derivatives should be interpreted as limsups of forward difference quotients. For the same reason, we have left out the ε's from Section 6. A more detailed treatment of the material of that section is given in [Mai08], where one can also find a discussion of William Thurston's geometrization conjecture.

2 Prelude: the curve shortening flow

Let α_0 be a smooth simple closed parametrized curve in the plane. We wish to construct a deformation of α_0 by its curvature vector. Mathematically, α_0 is a map from an interval $[a, b]$ to $\mathbf{R}^2$ such that $\alpha_0(a) = \alpha_0(b)$, or equivalently a map from the unit circle S^1 to $\mathbf{R}^2$. We consider a family of curves α_t, where $t \geq 0$ is a real parameter, usually called *time*, satisfying the evolution equation

$$\frac{\partial \alpha}{\partial t} = K \cdot \nu, \tag{1}$$

where K is the curvature of α_t, and ν the inward unit normal vector (Figure 1).

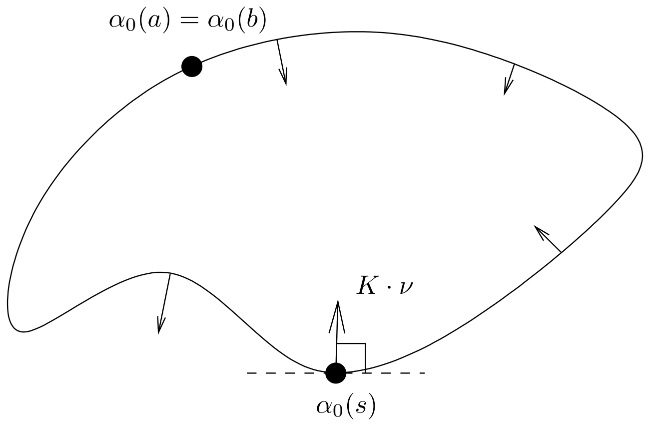

Figure 1: A closed parametrized curvature and its curvature vectors.

The simplest example is when α_0 is a circle: then for every $t > 0$, α_t is also a circle, with the same center, and the length is a strictly decreasing function of time. It is easy to write down an explicit formula for the solution, from which one sees that there exists a finite time $T_{\max}$ such that the solution is defined only for $t \in [0, T_{\max})$. We say that there is a *singularity* at time $T_{\max}$. As t tends to $T_{\max}$, the curve shrinks to a point, and its curvature goes to infinity everywhere (Figure 2).

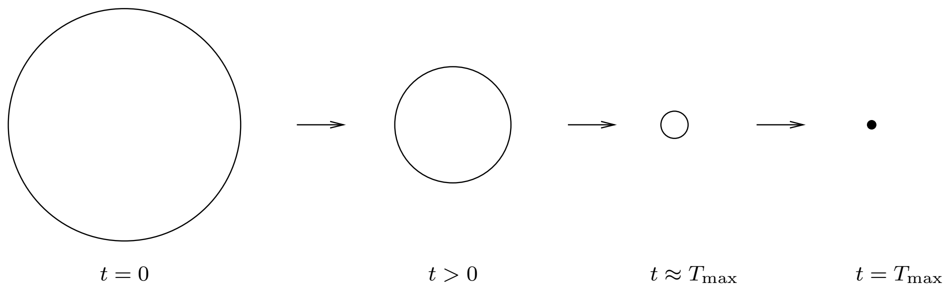

Figure 2: A circle shrinking by the curve shortening flow. (All circles are in fact concentric; they have been drawn separately for ease of visualization.)

More generally, when α_0 bounds a convex domain, all points move toward the interior of this domain, those of higher curvature moving faster. Heuristically, this tends to even up the curvature differences, making the curve look more and more like a circle (Figure 3).

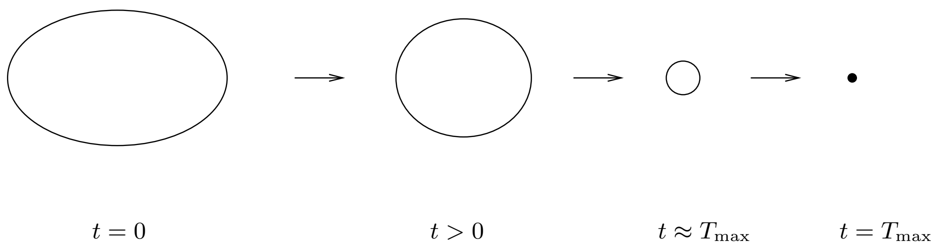

Figure 3: An ellipse shrinking by the curve shortening flow: since the points of higher curvature move faster, the excentricity of the ellipse decreases; near the singularity, the naked eye cannot distinguish it from a circle.

In general, it is difficult to develop intuition about the behavior of α_t. We gather some known results in the next theorem:

Theorem 2.1 (Gage-Hamilton [GH86], Grayson [Gra87, Gra89])**.**
Let $\{\alpha_t\}_{t\in[0,T_{\max})}$ *be a maximal solution of the curve shortening flow equation. Then:*

i. The length of α_t *is decreasing in* t*;*

ii. There is a singularity, i.e. $T_{\max} < +\infty$*;*

iii. As t *goes to* $T_{\max}$*, the diameter of* α_t *goes to* 0*, and the ratio between the maximal curvature and the minimal curvature goes to* 1*.*

Ricci flow, which is the main topic of this article, is not, strictly speaking, a generalization of the curve shortening flow, since it uses an intrinsic notion of curvature. However, Theorem 2.1 gives good hints about the type of results we should (or should not) expect to obtain on Ricci flow. Namely, we should not expect a closed formula for general initial data, or even a precise description of a general solution, except for rough qualitative information (e.g. monotonicity of some geometric quantity as in Conclusion (i) of Theorem 2.1.) By contrast, we can hope to prove the existence or inexistence of a singularity, and obtain information about the behavior of the solution near the maximal time.

This construction can be generalized in many ways: one can allow the curve to have singularities and/or infinite length, or replace the plane by other surfaces. There are also several important and useful generalizations to higher dimensions, the most well-known being perhaps *mean curvature flow*. For more information, see e.g. [HP99] or the introduction of the book [Eck04].

The curve shortening flow is often considered as a kind of nonlinear heat equation. In fact, if the curve α_t is parametrized by arclength s, then Equation (1) takes the form $\frac{\partial \alpha}{\partial t} = \frac{\partial^2 \alpha}{\partial^2 s}$, which is formally the classical heat equation. This analogy is one of the guiding principles in the study of the curve shortening flow. One of the main tools is the *maximum principle:* for the heat equation, it says that heat flows from the parts of highest temperature to the parts of lowest temperature, making the heat distribution more uniform.

Let us give an example for the curve shortening flow: let α_t and β_t be two solutions such that for some time t_0, α_{t_0} is surrounded by β_{t_0} and tangent to it at some point x. Comparing the curvatures of α_{t_0} and β_{t_0} at x, we see that α_t moves faster, so it cannot be caught up by β_t; hence this configuration is in fact impossible! This discussion leads to the important *avoidance principle:* if two curves evolving by the curve shortening flow have no point in common at time 0, then this remains true for all times $t > 0$.

One can deduce the existence of a singularity (i.e. Conclusion (ii) of Theorem 2.1) from the avoidance principle by simply remarking that any closed curve lies inside some large circle. This idea of using the maximum principle to obtain information on a general solution by comparing it to a particular solution whose behavior is known is also useful for Ricci flow, although it takes more sophisticated forms.

3 Ricci Flow

3.1 Generalities

Let M be a manifold. We assume throughout that M is *closed*, i.e. compact and without boundary. A *Ricci flow* on M is a one-parameter family $\{g_{ij}(t)\}$ of riemannian metrics on M solving the equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}(t), \tag{2}$$

where $R_{ij}(t)$ is the Ricci tensor of $g_{ij}(t)$. This equation was introduced by R. Hamilton in the early 80's. One can think of it as a kind of a nonlinear tensorial heat equation. To keep the notation simple, we will often write g for a riemannian metric, and $g(t)$ for a one-parameter family of metrics, although they are in fact tensors. Likewise, we sometimes write Ric for the Ricci tensor.

Hamilton [Ham82] showed that the pure initial value problem for Equation (2) always has a unique solution for small time, i.e. for every metric g_0 on M, there is a number $\varepsilon > 0$ such that there exists a unique family of metrics $\{g(t)\}_{t\in[0,\varepsilon)}$ which solves the Ricci flow equation and satisfies $g(0) = g_0$. Hence we can talk about the maximal solution $\{g(t)\}_{t\in[0,T_{\max})}$ with given initial data, with $T_{\max} \in (0, +\infty]$. When $T_{\max}$ is finite, we say that Ricci flow develops a *singularity*. In this case, a result of W.-X. Shi [Shi89] implies that the norm of the Riemannian tensor is unbounded as $t \to T_{\max}$; hence one also says that the curvature *blows up* at time $T_{\max}$.

Unlike the curve shortening flow in the plane, Ricci flow does not always develop a singularity: for instance, if the initial metric is Ricci-flat, i.e. the Ricci tensor vanishes, then the solution is constant, and of course defined for all time! The simplest example of a singularity is the *round shrinking sphere*, where the initial metric has constant sectional curvature, and contracts to a point in finite time. In this case, the sectional curvature of $g(t)$ is constant for each value of t, and tends to $+\infty$ as t tends to $T_{\max}$.

3.2 Ricci flow in two dimensions

In dimension two, the simplest examples are the round shrinking 2-sphere, and the (constant) flat 2-torus. Remarkably, if one starts with an *arbitrary* metric on the 2-sphere or the 2-torus, the behavior of Ricci flow is asymptotically the same as in those very simple examples:

Theorem 3.1 (Hamilton [Ham88], Chow [Cho91])**.**

i. *If* g_0 *is any metric on the* 2*-sphere, then* $T_{\max}$ *is finite, and as* $t \to T_{\max}$*, Ricci flow converges, up to rescaling, to the round metric.*

ii. *If* g_0 *is any metric on the* 2*-torus, then* $T_{\max}$ *is infinite, and as* $t \to +\infty$*, Ricci flow converges to a flat metric.*

The key point for us is that Ricci flow is sensitive to topology: its qualitative behavior depends only on the presence or absence of handles on the surface, rather than the fine properties of the initial metric (Figure 4).

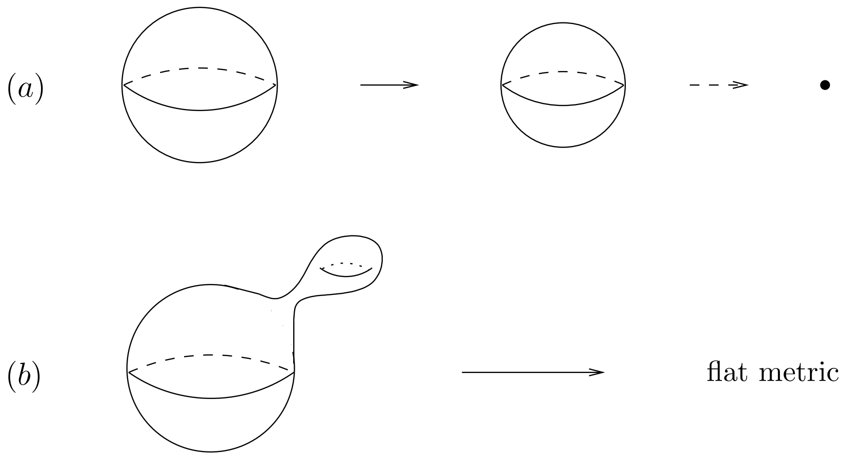

Figure 4: (a) Ricci flow shrinks a round sphere to a round point. (b) If you add a handle, however, it converges to a flat metric in infinite time.

Theorem 3.1 can be extended to surfaces of higher genus as follows: if F is a surface of genus at least 2, then for any initial condition, Ricci flow is defined for all time, and as t goes to infinity, $g(t)$ converges, up to rescaling, to a hyperbolic metric.

4 Ricci flow in three dimensions

Recall that a manifold M is *simply-connected* if any closed curve on M can be continuously shrunk to a point. In dimension 2, it follows from the classification of surfaces, which was already known in the nineteenth century, that the only closed simply-connected surface is the 2-sphere. In 1904, H. Poincaré [Poi04] asked whether the corresponding statement in dimension

3 also holds. This question became known as the *Poincaré Conjecture* and remained open for almost a century[1] until its positive solution by G. Perelman using Ricci flow:

Theorem 4.1 (Perelman [Per02, Per03a, Per03b]). *Let M be a closed* 3*-manifold. If M is simply-connected, then M is (homeomorphic to) the* 3*-sphere.*

The starting point of the Ricci flow approach to the Poincaré conjecture is the following result of Hamilton's:

Theorem 4.2 ([Ham82]). *Let M be a closed* 3*-manifold. Consider a maximal solution $\{g(t)\}_{t\in[0,T_{\max})}$ of the Ricci flow on M. If the initial metric has positive Ricci curvature, then $T_{\max}$ is finite, and $g(t)$ converges, up to rescaling, to a metric of constant sectional curvature, as $t \to T_{\max}$.*

It is a classical fact from Riemannian geometry that the only closed, simply-connected n-dimensional manifold which admits a riemannian metric of constant sectional curvature is the n-sphere.

At this point of the discussion, there seems to be an obvious program for proving the Poincaré conjecture: letting g_0 be an arbitrary metric on M, we consider the maximal Ricci flow solution $\{g(t)\}_{t\in[0,T_{\max})}$ with initial condition g_0. Step 1 would be to prove that there is a singularity; Step 2 would be to prove that an appropriate rescaling of $g(t)$ converges to a round metric as t tends to $T_{\max}$.[2]

As it stands, this simple plan does *not* work, because the result expected in Step 2 is false. However, Step 1 does work:

Theorem 4.3 (Perelman [Per03b],Colding-Minicozzi [CM05, CM07]). *Let M be a simply-connected closed* 3*-manifold, and g_0 be an arbitrary riemannian metric on M. Then Ricci flow with initial condition g_0 develops a singularity.*

The main difficulty for proving Theorem 4.3 is that the hypothesis is purely topological, rather than geometric as e.g. in Theorem 4.2. In fact, no blow-up result under purely topological hypotheses is known in higher dimensions.

It is instructive to consider *open* manifolds instead of closed ones. The only simply-connected open surface is the plane, but the corresponding result is false in dimension 3: J. H. C. Whitehead [Whi35] constructed an example of open, contractible 3-manifold which is not homeomorphic to $\mathbf{R}^3$. The question of understanding what Ricci flow does on such a manifold is wide open.

[1] For a discussion of the historical significance of the Poincaré conjecture, see e.g. [Mil03] or [Mor07].

[2] that this is overoptimistic is suggested by the existence $S^2 \times S^1$, which, however, is not simply-connected...

5 Finite time blow-up

5.1 Evolution of the scalar curvature

Let $\{g(t)\}$ be a Ricci flow. The Ricci flow equation (2) implies a formula for the evolution of the Riemann tensor R_{ijkl}. This formula, which was computed by Hamilton [Ham82], is fairly complicated, and we will not write it down here. From it one deduces the formula for the evolution of the Ricci tensor R_{ij}, and finally, for its trace, i.e. the scalar curvature R:

$$\frac{\partial R}{\partial t} = \Delta R + 2|\operatorname{Ric}|^2, \tag{3}$$

where Δ is the Laplace-Beltrami operator associated to the metric $g(t)$, and $|\operatorname{Ric}|$ is the norm of its Ricci tensor.

This type of equation is often called a *reaction-diffusion* equation. The diffusion term ΔR tends to make R more uniform, while the reaction term $2|\operatorname{Ric}|^2$ tends to make it more positive. Letting Ric^0 be the trace-free part of the tensor Ric, we obtain:

$$\begin{align} \frac{\partial R}{\partial t} &= \Delta R + \frac{2}{3}R^2 + 2|\operatorname{Ric}^0|^2 \tag{4}\\ &\geq \Delta R + \frac{2}{3}R^2. \tag{5} \end{align}$$

Let $R_{\min}(t)$ denote the minimum of the scalar curvature of $g(t)$. Applying the maximum principle, we get the differential inequality

$$\frac{dR_{\min}}{dt} \geq \frac{2}{3}R_{\min}^2, \tag{6}$$

which in turn yields

$$R_{\min}(t) \geq \frac{R_{\min}(0)}{1 - 2tR_{\min}(0)/3}. \tag{7}$$

If $R_{\min}(0) > 0$, the lower bound given by (7) is positive, and goes to infinity as $t \to 3/(2R_{\min}(0))$. Hence there must be a singularity before that time, and Theorem 4.3 is proved in this case.

If $R_{\min}(0) \leq 0$, the lower bound goes to zero as $t \to \infty$. This is a useful piece of information, but does not suffice to deduce finite time blow-up. Hence we shall need another quantity, called the *width*, which relies on a topological construction called a *sweep-out*.

We shall ultimately need to define these notions in dimension 3, but for simplicity we first introduce them in dimension 2.

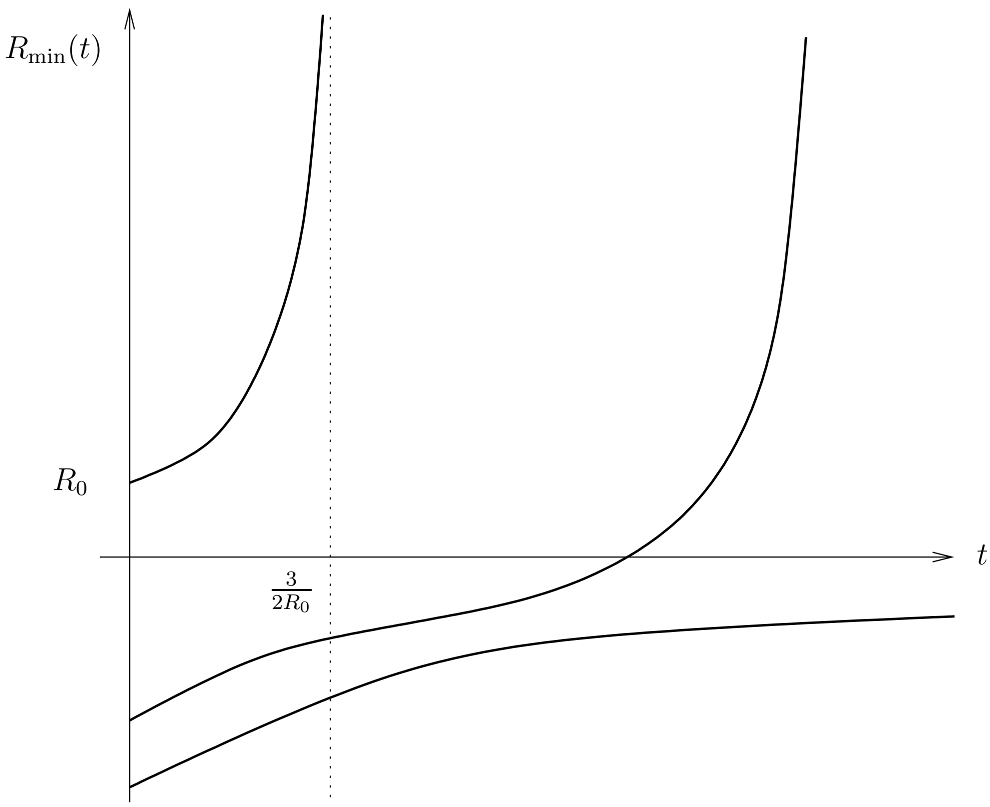

Figure 5: Possible scenarios for $R_{\min}(t)$. If the initial value is strictly positive, then it blows up in finite time.

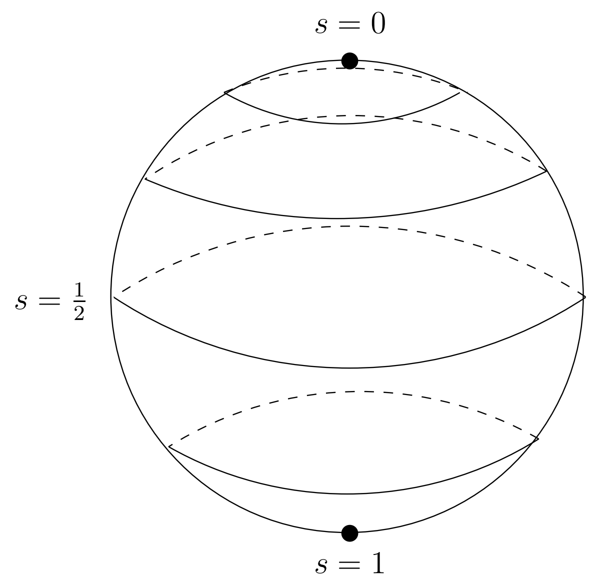

Figure 6: The standard sweep-out of the 2-sphere.

5.2 Sweep-outs in two dimensions

Definition. A *sweep-out* of the 2-sphere S^2 is a continuous map f from the cylinder $S^1 \times [0,1]$ to S^2 such that $f(S^1 \times \{0\})$ and $f(S^1 \times \{1\})$ are constant maps.

One can also think of a sweep-out f as a continuous family of maps $f(\cdot, s)$ from the circle S^1 to S^2, where the parameter s varies between 0 and 1. Figure 6 shows a simple example, called the *standard sweep-out*, which sends $S^1 \times \{0\}$ to the north pole, $S^1 \times \{0\}$ to the south pole, and the circles in-between to parallels in the obvious way.

We say that a sweep-out f is *nontrivial* if it is not homotopic (through sweepouts) to a constant map. Intuitively, this means that the curves $f(S^1 \times \{s\})$ cannot be continuously shrunk to points in a coherent way (although of course, S^2 being simply-connected, every such curve can be individually shrunk to a point.) For instance, it follows from degree theory that the standard sweep-out is nontrivial.

Let us fix a nontrivial sweep-out f_0 (e.g. the standard one.) To each riemannian metric g on S^2, we associate its *width* $W(g)$, defined by the following formula:

$$W(g) := \inf_{f} \max_{s \in [0,1]} \mathrm{Length}_{\mathrm{g}}(f(\cdot, s)), \tag{8}$$

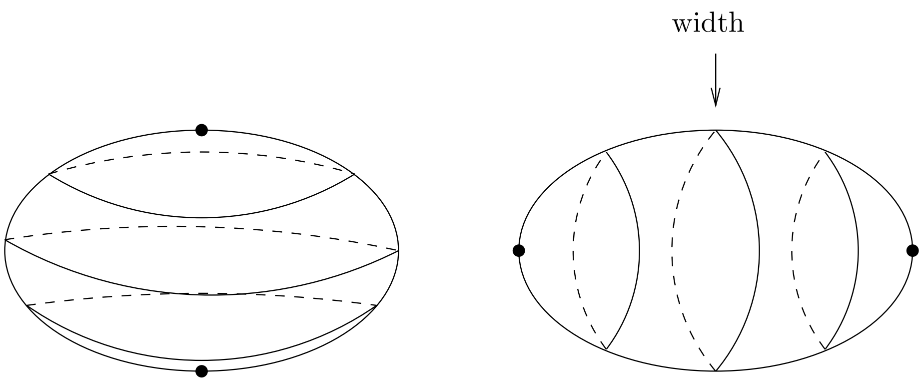

Figure 7: Two ways to sweep out an ellipsoid. The second one is more efficient and realizes the width.

the infimum being taken over all sweep-outs f homotopic to f_0.

In order to understand this definition better, let us first consider the simple example where g is the round metric of unit radius. Then the maximal length of the circles in the standard sweep-out is attained for $s = \frac{1}{2}$, i.e. at the equator, where it equals 2π. One can show that this value cannot be improved by deforming the sweep-out. Hence $W(g) = 2\pi$ in this case.

Suppose now that g is the ellipsoidal metric obtained by rotating an ellipse of half-axes $a < b$ around one of its axes. In this case the width is $2\pi a$ (cf. Figure 7).

5.3 Sweep-outs in three dimensions

It is straightforward to extend the above definitions to dimension 3.

Let M be a closed 3-manifold. A *sweep-out* of M is a continuous map $f : S^2 \times [0,1] \to M$ such that $f(S^2 \times \{0\})$ and $f(S^2 \times \{1\})$ are constant maps. It is *nontrivial* if it is not homotopic to a constant map. In general, a 3-manifold need not have any nontrivial sweep-out. However, it follows from standard results in algebraic topology (Poincaré duality and the Hurewicz isomorphism theorem) that if M is simply-connected, then M does admit nontrivial sweep-outs. Letting f_0 be such a sweep-out, we can define the *width* of a riemannian metric g on M by setting

$$W(g) := \inf_f \max_{s\in[0,1]} \mathrm{Area}_g(f(\cdot, s)), \tag{9}$$

the infimum being again taken over all sweep-outs f homotopic to f_0. Of course, this quantity depends on the choice of f_0, but since f_0 is fixed throughout the argument, we omit to mention it.

Suppose now that $\{g(t)\}$ is a Ricci flow, and let $W(t)$ denote the width of $g(t)$. Using the theory of harmonic mappings, Colding and Minicozzi [CM07] proved the following inequality, which gives control on the evolution of $W(t)$:

$$\frac{dW}{dt} \leq -4\pi - \frac{R_{\min} W}{2}. \tag{10}$$

Plugging in the lower bound (7) for $R_{\min}$ and integrating, one obtains an upper bound for $W(t)$ which asymptotically decreases linearly (Figure 8). Since by definition, W is always nonnegative, it follows that Ricci flow cannot be defined for all time. This proves Theorem 4.3.

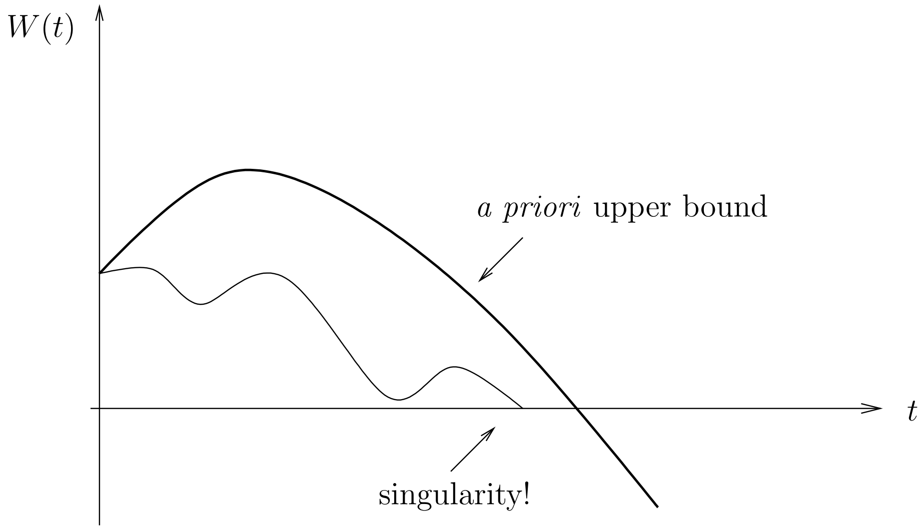

Figure 8: Evolution of width under Ricci flow.

6 Singularity analysis and surgery

Let M be a closed 3-manifold, and $\{g(t)\}_{t\in[0,T_{\max})}$ be a maximal Ricci flow solution. We are mainly interested in the case where M is simply-connected, but this is not essential for the results discussed in this section. For technical reasons, we assume that M is *irreducible*, i.e. any embedding of the 2-sphere into M can be extended to an embedding of the 3-ball. This hypothesis will spare us the need to discuss connected sums; it is not a serious restriction, because a classical theorem of H. Kneser [Kne29] reduces the Poincaré conjecture to the irreducible case. We also suppose that M is orientable, which is not a serious restriction either since every simply-connected manifold is orientable.

By Theorem 4.3, we know that $T_{\max}$ is finite. For this to be useful, we need to study the behavior of the solution for t close to $T_{\max}$. Several results put together (the derivative estimates of Shi, and the Hamilton-Ivey pinching estimate) imply that as long as the scalar curvature remains bounded, Ricci flow is defined. Hence the maximum of the scalar curvature is unbounded as t goes to $T_{\max}$. As we already saw, the scalar curvature is bounded from below, so there must be a sequence of times $t_k \to T_{\max}$ such that the maximum of the scalar curvature of $g(t_k)$ goes to $+\infty$. It follows that the key point of the singularity analysis is understanding points of large scalar curvature.

6.1 Canonical neighborhoods

One of Perelman's main achievements is a precisely a theorem about the local structure of Ricci flow at points where the scalar curvature is large. This theorem says that for any time $t < T_{\max}$, if $x \in M$ is a point such that the scalar curvature of $g(t)$ at x is sufficiently large, then (x, t) has a so-called *canonical neighborhood*. This neighborhood is a subset U of M, which, when endowed with the metric $g(t)$, is of one of the following three types (Figure 9(a)(b)):

- a *neck* (almost homothetic to the product of the round 2-sphere of unit radius with a long interval);
- a *cap* (a metric on the 3-ball such that a collar neighborhood of the boundary is a neck), or
- a closed manifold of positive sectional curvature.

If the third case ever occurs for some $x \in M$, $t < T_{\max}$, then U is equal to the whole of M. In particular, it implies that $g(t)$ has positive Ricci curvature, so that Theorem 4.2 applies. In the sequel, we shall assume that this does not happen.

Let us consider another nice situation: suppose that there exists a time $t_0 < T_{\max}$ such that all points of M have sufficiently large scalar curvature at time t_0. Then at time t_0 all points have canonical neighborhoods, either necks or caps. The topological assumptions made at the beginning of this section are easily shown to imply that M is obtained by gluing together two caps, possibly with a long *tube* (i.e. a subset of M homeomorphic to $S^2 \times [0, 1]$ which is a union of necks) connecting them. It follows immediately that M is homeomorphic to the 3-sphere.

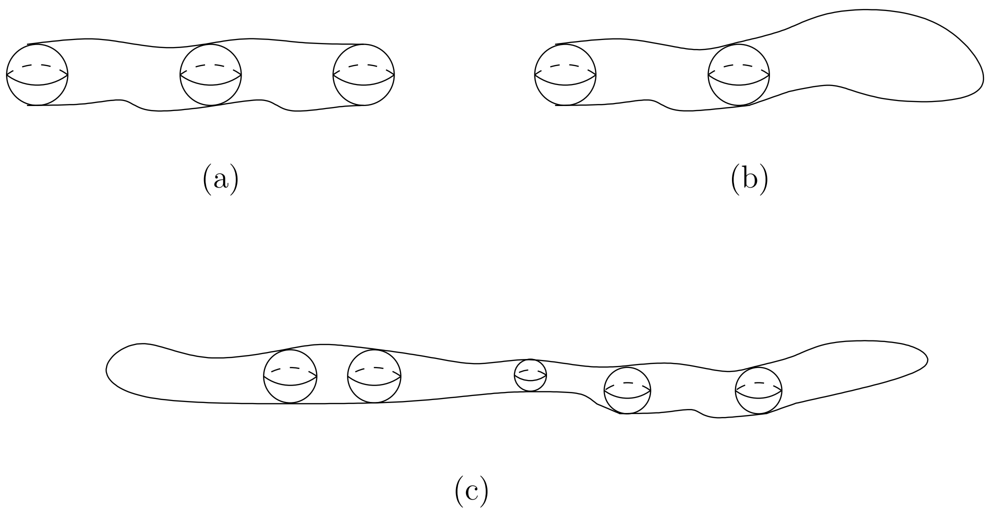

Figure 9: (a) a neck; (b) a cap; (c) the 3-sphere as a union of necks and caps.

6.2 The neck pinch

However, a less nice situation is possible: as one approaches the singular time, the scalar curvature may become large *somewhere*, but not *everywhere*. A famous example is the *neck pinch*, where for t_0 close to $T_{\max}$, the riemannian manifold $(M, g(t_0))$ consists of two regions M_1, M_2 of low curvature connected by a thin tube (Figure 10).

Intuitively, if something like this happens, then it should get worse with time, since if N is a neck near the middle, then Ricci curvature in the S^2 direction is very high and tends to pinch the neck even more (see [AK04] for a rigorous discussion.) At this point, we still have not learned anything about the topology of M.

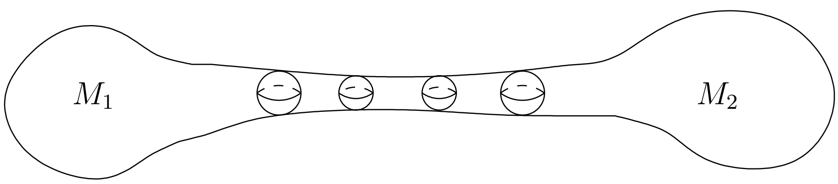

Figure 10: A neck pinch.

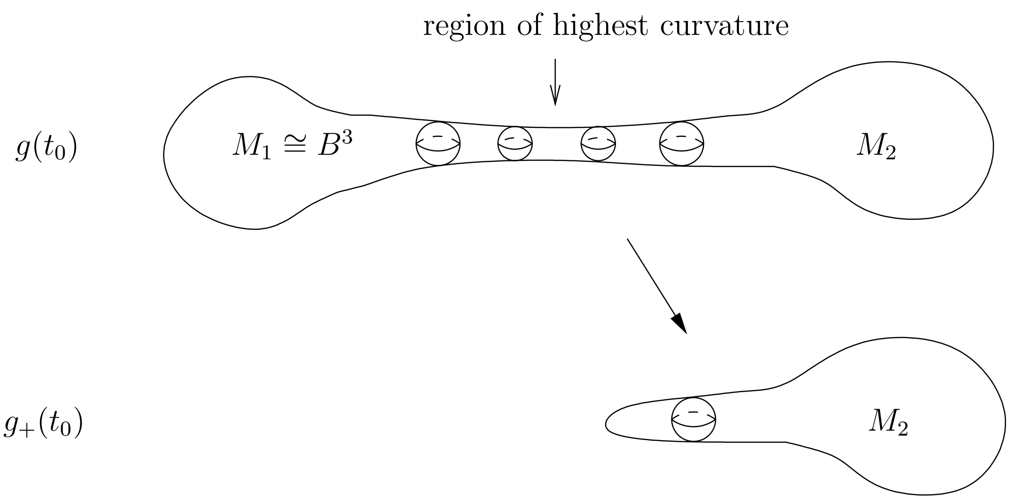

Figure 11: The manifold before and after metric surgery. An important feature is that the maximum of the scalar curvature drops by a definite factor; this ensures that Ricci flow with initial condition $g_+(t_0)$ exists for a sufficiently long time so that surgery times do not accumulate.

6.3 Metric surgery

Suppose that a neck pinch occurs, and let t_0 be a time such that $(M, g(t_0))$ is as described above. Since we are assuming that M is irreducible, we know that one of the M_i's, say M_1, is topologically a 3-ball. Since we have excellent control on the geometry at time t_0 around the boundary of M_1, we can produce a new riemannian metric $g_+(t_0)$ on M which coincides with $g(t_0)$ outside some neighborhood U_1 of M_1, and such that $(U_1, g_+(t_0))$ is close to a round hemisphere. The operation which permits to construct $g_+(t_0)$ from $g(t_0)$ is called *metric surgery*.

Having performed metric surgery, we restart Ricci flow, using $g_+(t_0)$ as new initial condition. If another neck pinch arises, we repeat this construction. If this is done carefully enough, the process either stops when all points have a canonical neighborhood (which as we saw implies that M is the 3-sphere), or goes on for as much time as we want.

The construction outlined above leads to the following result, which is a strengthened version of (a special case of) a theorem of Perelman [Per03a].

Theorem 6.1 ([B^{3}MP])**.** *Suppose that M is a closed, orientable, irreducible 3-manifold which does not admit any metric of constant positive sectional curvature. Then for every metric g_0 on M and every number $T > 0$, there exists a 1-parameter family $\{g(t)\}$ of riemannian metrics on M, defined for* *all $t \in [0,T]$, and such that:*

i. $g(0) = g_0$*;*

ii. $g(t)$ *evolves by Ricci flow except for finitely many values of* t*, called* surgery times*, where the evolution is discontinuous;*

iii. If t_0 *is a surgery time, then letting* $g(t_0)$ *(resp.* $g_+(t_0)$*) denote the pre-surgery metric (resp. the post-surgery metric) we have:*

   (a) $R_{\min}(g_+(t_0)) \geq R_{\min}(g(t_0))$*;*

   (b) $g_+(t_0) \leq g(t_0)$ *(i.e. the identity map from* $(M, g(t_0))$ *to* $(M, g_+(t_0))$ *is distance-nonincreasing.)*

We now explain how to deduce the Poincaré conjecture from Theorem 6.1. Let M be a closed, irreducible, simply-connected 3-manifold. If M has a metric of constant positive sectional curvature, then M is homeomorphic to S^3, so we argue by contradiction and assume it is not the case.

Let g_0 be an arbitrary metric on M, and let R_0 be the minimum of its scalar curvature. First consider the case where $R_0 > 0$, and set $T := 3/(2R_0)$. Applying Theorem 6.1, we get a family of metrics $\{g(t)\}$ defined for all $t \in [0,T]$ satisfying Conclusion (i)–(iii). By (i) we have $R_{\min}(0) = R_0 > 0$. By (ii) and (iiia), the *a priori* lower bound (7) for $R_{\min}$ is satisfied. This contradicts the fact that $\{g(t)\}$ is defined up to time T. Hence in fact M does admit a metric of constant positive sectional curvature, and is therefore homeomorphic to S^3.

Suppose now that $R_0 \leq 0$. Note that (iiib) implies that $W(g_+(t_0)) \leq W(g(t_0))$ at a surgery time. Hence the argument using W to prove finite time blow-up gives a similar contradiction.

On the proof of Theorem 6.1 A thorough discussion of this proof lies outside the scope of this survey. However, we would like to mention briefly a few important issues.

In order to understand the local geometry at points of large scalar curvature, one argues by contradiction, taking a sequence of pointed Ricci flows $(\{g_k(t)\}, x_k, t_k)$ such that $R(x_k, t_k) \to +\infty$ and yet (x_k, t_k) does not have a canonical neighborhood. Using a *compactness theorem,* one finds a subsequence that converges in an appropriate sense. One then shows that the limit, which is also a pointed Ricci flow, belongs to a special class of Ricci flows, called κ*-solutions*, which are known to have canonical neighborhoods. Hence for sufficiently large k, the point (x_k, t_k) has a neighborhood close to an open subset of a κ-solution. This proves that (x_k, t_k) has a canonical neighborhood.

Two crucial ingredients used to establish the existence of a convergent subsequence are the already mentioned *Hamilton-Ivey pinching estimate,* which allows to control the whole Riemann tensor using only the scalar curvature, and Perelman's *no local collapsing estimate.* The latter is known to hold in all dimensions, while the former is special to dimension 3. In fact, the lack of a similar estimate is one of the biggest obstacles to extending this theory to higher dimensions.

Another important consequence of Hamilton-Ivey pinching is that the limit flows coming up in contradiction arguments have nonnegative sectional curvature. Hence several geometric and analytical techniques can be brought to bear, such as the *Cheeger-Gromoll soul theorem*, which ensures that the underlying manifold has a simple topology, or Hamilton's *Harnack inequality* for Ricci flow.

There are some subtleties involving in doing metric surgery. For a more detailed discussion, we refer to [Mai08].

7 Further reading

Among the many expositions of Perelman's work for a general audience of mathematicians, we suggest the talks of John Lott [Lot07] and John Morgan [Mor07] at the Madrid 2006 ICM, Gérard Besson's 2005 Séminaire Bourbaki talk [Bes06], and Laurent Bessières's 2005 article [Bes05]. Earlier expository texts which are still worth reading include [And04, Mor05].

Much useful information can be gathered from Terence Tao's blog for his 2008 course on the Poincaré conjecture [Tao], as well as his survey article [Tao06]. His viewpoint is quite different from the one adopted in the present article; in particular there are nice discussions of Harnack inequalities and Ricci solitons.

There are by now several excellent introductory books on Ricci flow. We recommend the monographs by Bennett Chow and Dan Knopf [CK04] and by Peter Topping [Top06], which complement well each other. Richard Hamilton's 1993 survey [Ham95] remains a highly valuable introduction to Ricci flow in general and the Ricci flow approach to geometrization of 3-manifolds in particular.

For full proofs of the Poincaré conjecture and the geometrization conjecture, the reader may consult [Per02, Per03a, Per03b, CM05, B^{3}MP07, KL06, MT07, CZ06, CM07, B^{3}MP].

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