# Three Lectures on 3-manifolds

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## Introduction

The purpose of these notes is to give an introduction to low-dimensional topology and geometry, with an emphasis on 3-manifolds and Thurston's geometrization program. It is an expanded version of my handwritten notes for a series of lectures given during a Masterclass at IRMA, Strasbourg in January 2016. The primary audience for these lectures consisted of first-year Master's students, who could not be expected to know differential geometry or algebraic topology, hence the mostly self-contained treatment of manifolds and the ad hoc definitions of simple connectedness and the universal cover. In this text however I sometimes use freely words like 'Riemannian metric' or 'Lie group', but most of it should be understandable with only a basic knowledge of these notions.

There are some definitions and statements of theorems, few proofs, and lots of examples and exercises. Some general references are: [Hat02] for algebraic topology, [GHL04] for Riemannian geometry, and [Sco83, Thu97, BMP03] for geometric manifolds.

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# 1 Manifolds of dimension 1 and 2

## 1.1 Topological manifolds

**Definition 1.1.** Let $n$ be a natural number and $M$ be a topological space. We say that $M$ is an *$n$-dimensional manifold* (or *$n$-manifold*) if it has the following three properties:

i. It is *locally $n$-Euclidean*, which means that every point $x \in M$ has a neighborhood $U$ which is homeomorphic to $\mathbf{R}^n$.

ii. It is *Hausdorff*: for every $(x, y) \in M^2$, if $x \neq y$, then there exists a pair $(U, V)$ of open subsets of $M$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

iii. There is a sequence $\{K_p\}$ of compact subsets of $M$ whose union is $M$.

We will often assume in addition that $M$ is *connected*, and sometimes omit this hypothesis when it is obvious.

**Remarks.**

- What we call '$n$-manifold' is sometimes called 'topological $n$-manifold' in order to distinguish this notion from that of PL or differential manifold. For simplicity we won't develop PL topology nor differential topology, although our discussion of geometric structures would fit more naturally in the smooth category, and the Euler characteristic works better in the PL setting (cf. Exercice 3.)

- There is a more general notion of *manifold with boundary*, where the local model in (i) can be a half-space. For instance, a closed metric ball in $\mathbf{R}^n$ is a manifold with boundary, but not a manifold in our sense.

- The purpose of Condition (iii) is to rule out 'big' spaces such as the long line. Together with the Hausdorff condition, this ensures that all manifolds are metrizable. There also exist non-Hausdorff (hence non-metrizable) 'manifolds' which are of interest e.g. in foliation theory. An example of non-Hausdorff locally 1-Euclidean space is the line with two origins $(\mathbf{R} \setminus \{0\}) \cup \{0_a, 0_b\}$ where $\mathbf{R} \setminus \{0\}$ is given the usual topology and a base of open neighborhoods of $0_a$ (resp. $0_b$) is given by the sets $(U \setminus \{0\}) \cup \{0_a\}$ (resp. $(U \setminus \{0\}) \cup \{0_b\}$) with $U$ an open neighborhood of 0 in $\mathbf{R}$.

**Examples.**

- Euclidean $n$-space: $\mathbf{R}^n$

- The $n$-sphere $\mathbf{S}^n = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid \sum_{k=0}^{n} x_k^2 = 1\}$. (It is connected for $n \geq 1$.)

- The $n$-torus: $\mathbf{T}^n = (\mathbf{S}^1)^n$.
- More general Cartesian products, e.g. $\mathbf{S}^k \times \mathbf{R}^l$ is a $(k+l)$-manifold.

The next proposition is a key tool to construct more examples of manifolds as quotients of manifolds under certain group actions. If $X$ is a topological space we let $\mathrm{Homeo}(X)$ denote the set of all self-homeomorphisms of $X$ and $\mathrm{Id}_X$ (or simply Id) denote the identity of $X$.

**Proposition 1.2.** *Let $X$ be an $n$-manifold and $\Gamma$ be a subgroup of* $\mathrm{Homeo}(X)$. *Suppose that*

*i.* $\Gamma$ *acts* freely *on $X$, i.e. for every $g \in \Gamma$ and every $x \in X$ we have the implication $g(x) = x \implies g = \mathrm{Id}_X$, and*

*ii.* $\Gamma$ *acts* properly discontinuously *on $X$, i.e. for each compact $K \subset X$, the set $\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$ is finite.*

*Then the quotient set $X/\Gamma$, i.e. the space of orbits of $X$ under $\Gamma$, endowed with the quotient topology, is an $n$-manifold.*

**Remarks.**

- If $X$ is compact, then a group $\Gamma$ acts properly discontinuously on $X$ if and only if $\Gamma$ is finite.
- In general any finite group $\Gamma$ of homeomorphisms of $X$ acts properly discontinuously on $X$. However, if $X$ is noncompact and $\Gamma$ is finite, then $X/\Gamma$ is noncompact.

**Examples.**

- Real projective space: this is just $\mathbf{R}P^n := \mathbf{S}^n/\{\pm\mathrm{Id}\}$.
- The $n$-torus revisited. Let $X = \mathbf{R}^n$ and $\Gamma = \mathbf{Z}^n$ acting on $X$ by translations, i.e. for all $X = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $W = (w_1, \ldots, w_n) \in \mathbf{Z}^n$ we set $X \cdot W = X + W = (x_1 + w_1, \ldots, x_n + w_n)$. Then define $T^n := \mathbf{R}^n/\mathbf{Z}^n$.

  For $n = 1$, we get $\mathbf{R}/\mathbf{Z}$, which can be described as the interval $[0, 1]$ with 0 and 1 identified. This space is homeomorphic to the unit circle, or 1-sphere $\mathbf{S}^1$.

  For $n = 2$, think of $\mathbf{R}^2$ being tiled by squares of unit length in the obvious way. Let $Q$ be one of these squares. Then the quotient can be obtained from $Q$ by identifying opposite sides together by translations. This space is homeomorphic to the 2-torus $\mathbf{T}^2$.

**Exercise 1.** Show that for each $n \geq 1$, $\mathbf{T}^n$ is homeomorphic to $T^n$.

The basic question which we are interested in is the following: how to classify compact, connected $n$-manifolds for small values of $n$. We will move slowly from dimension 1 to 2 in the remainder of this lecture, and tackle dimension 3 in the next two.

## 1.2 Dimension 1

**Theorem 1.3.** *Every (connected) 1-manifold is homeomorphic to either* $\mathbf{S}^1$ *or* $\mathbf{R}$*.*

It is easy to see that $\mathbf{S}^1$ and $\mathbf{R}$ are not homeomorphic, since the former is compact while the latter is not. It is more difficult (and a bit tedious) to prove that any 1-manifold is homeomorphic to one of the two.

Instead of proving Theorem 1.3, we will take the 'geometric viewpoint'. This means that instead of trying to deal with all topological 1-manifolds, we will focus on those that can be obtained from the simplest one (the real line) by taking the quotient under the action of a group which acts 'reasonably rigidly.' The 1-dimensional may seem trivial, but we will see that this philosophy also works in higher dimensions, which will prove useful.

Set $X = \mathbf{R}$. The usual distance on $X$ is given by the formula $d(x, y) = |y - x|$. Let $G = \mathrm{Isom}(\mathbf{R})$ be the group of affine bijections from $X$ to itself which preserve $d$. For every constant $C$, the group $G$ contains the translation $x \mapsto x + C$ and the reflection $x \mapsto C - x$.

**Exercise 2.**

1. Prove that every element of $\mathrm{Isom}(\mathbf{R})$ is a translation or a reflection.
2. Classify subgroups of $\mathrm{Isom}(\mathbf{R})$ that act freely and properly discontinuously on $\mathbf{R}$.

**Remark.** It turns out that every 1-manifold is homeomorphic to the quotient of $\mathbf{R}$ by a group of isometries acting freely and properly discontinuously. In other words, 'every 1-manifold is geometric'. This is no big deal, but in dimension 2 already we will see that the interplay between geometry and topology is more interesting.

## 1.3 Dimension 2

In the sequel, a *surface* is a connected 2-manifold. We will focus on the compact ones, since the classification of noncompact surfaces is much more difficult [Ric63].

We already know three surfaces: the 2-sphere $\mathbf{S}^2$, the 2-torus $\mathbf{T}^2$, and the real projective plane $\mathbf{R}P^2$. It turns out that they are pairwise nonhomeomorphic, though this is probably not obvious to a beginner in topology. In fact, $\mathbf{S}^2$ and $\mathbf{T}^2$ are the first two of a series called the 'orientable surfaces', and $\mathbf{R}P^2$ is the first of another series called the 'nonorientable surfaces'.

In order to define the other surfaces, we first give a definition:

**Definition 1.4.** Let $F, F'$ be two connected (not necessarily compact) surfaces. Then the *connected sum* $F \# F'$ is defined as follows: pick an open disk $D \subset F$ and an open disk $D' \subset F'$; then glue $F \setminus D$ and $F' \setminus D'$ together along the boundary.

**Proposition 1.5.** *The connected sum $F \# F'$ is well-defined, i.e. up to homeomorphism it does not depend on the choice of $D, D'$ or the gluing homeomorphism.*

**Definition 1.6.**

i. For each $g \in \mathbf{N}$ the *orientable surface of genus $g$*, denoted by $F_g$, is defined inductively as follows: $F_0 := \mathbf{S}^2$, $F_{g+1} := F_g \# \mathbf{T}^2$.

ii. For each $g \in \mathbf{N}$ the *nonorientable surface of genus $g$*, denoted by $\tilde{F}_g$, is defined inductively as follows: $\tilde{F}_0 := \mathbf{R}P^2$, $\tilde{F}_{g+1} := \tilde{F}_g \# \mathbf{R}P^2$.

It is easy to see that taking a connected sum with $\mathbf{S}^2$ does not change the surface. In particular, the orientable surface of genus 1 is $\mathbf{T}^2$. The nonorientable surface of genus 1 is called the *Klein bottle*. We will denote it by $\mathbf{K}^2$.

**Theorem 1.7.** *(Classification of surfaces) Every compact surface is homeomorphic to exactly one member of the collection $\{F_g\} \cup \{\tilde{F}_g\}$. Hence compact surfaces are classified by genus and orientability class.*

We will not explain why every compact surface appears on the list. This can be done by PL methods (using triangulations) or differential methods (e.g. Morse functions). In the following two exercises, you will learn a little of the theory needed to prove that the surfaces above are pairwise nonhomeomorphic. The most important invariant is the *Euler characteristic* of a surface.

**Exercise 3.** A *polyhedral surface* is a surface obtained from a finite collection of polygons by gluing them together (or possibly to themselves) along edges. A *polyhedral decomposition* of surface $F$ is a homeomorphism from some polyhedral surface to $F$. One can prove that any surface admits a polyhedral decomposition.

Let $F$ be a compact surface. Fix a polyhedral decomposition of $F$. Let $V$ (resp. $E$, resp. $F$) be the number of vertices (resp. edges, resp. faces.) One can show that the number $V - E + F$ is independent of the choice of polyhedral decomposition of $F$. This number is called the *Euler characteristic* of $F$ and denoted by $\chi(F)$. Furthermore, if $F$ is homeomorphic to $F'$, then $\chi(F) = \chi(F')$.

1. Compute $\chi(F)$ for $F = \mathbf{S}^2, \mathbf{R}P^2, \mathbf{T}^2, \mathbf{K}^2$.
2. If $F, F'$ are compact surfaces, find a formula for $\chi(F\#F')$ in terms of $\chi(F)$ and $\chi(F')$.
3. Use this to compute the Euler characteristics of all surfaces.
4. Check that the formula obtained in question 2 is consistent with the following two facts: for every $F$, $F\#\mathbf{S}^2$ is homeomorphic to $F$; for every nonorientable $F$, $F\#\mathbf{T}^2$ is homeomorphic to $F\#\mathbf{R}P^2\#\mathbf{R}P^2$.

**Exercise 4** (Topological definition of orientability)**.** A *1-sided loop* in a surface $F$ is an injective continuous map $\alpha : S^1 \to F$ with image $C$ such that for every open set $U$ containing $C$ there exists an open set $V$ such that $C \subset V \subset U$ and $V \setminus C$ is connected.

1. Show that for every $g \in \mathbf{N}$, the surface $\tilde{F}_g$ contains a 1-sided loop.
2. It can be proven that none of the orientable surfaces $F_g$ contains a 1-sided loop. Assuming this, show that the surfaces in $\{F_g\} \cup \{\tilde{F}_g\}$ are pairwise nonhomeomorphic.

## 1.4 Geometric manifolds

At this point, you can try to guess the sequel by tackling the following Challenge:

**Challenge.** Find as many compact 3-manifolds as you can. Try to guess which ones are homeomorphic to each other...

In these lectures we won't give a formal definition of 'geometry' or 'geometric structure'. For us, a geometry will be an ordered pair $(X, G)$ where $X$ is a manifold with 'simple topology' (more precisely, simply-connected) and $G$ is a subgroup of $\mathrm{Homeo}(X)$ consisting of 'rigid transformations' (in fact a Lie group.) We will focus on geometries in the sense of Thurston, rather than more general notions of geometries (affine, real projective etc.) For brevity we give the following meta-definition:

**Definition 1.8.** We say that a manifold *admits a* $(X, G)$*-structure* if it is homeomorphic to a quotient $X/\Gamma$ where $\Gamma$ is a subgroup of $G$ that acts freely and properly discontinuously on $X$.

We've already seen the 1-dimensional geometry $(\mathbf{R}, \mathrm{Isom}(\mathbf{R}))$. Let's generalize this to arbitrary dimensions.

**Definition 1.9** (Euclidean geometry)**.** Set $X := \mathbf{R}^n$ endowed with the usual Euclidean metric

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

Let $G := \mathrm{Isom}(\mathbf{R}^n)$ be the group of affine bijections from $\mathbf{R}^n$ to itself which preserve $d$. Manifolds which admit an $(\mathbf{R}^n, \mathrm{Isom}(\mathbf{R}^n))$-structure are called *Euclidean.*

**Example.** Taking $\Gamma = \mathbf{Z}^n$, we see that the $n$-torus is Euclidean.

**Remarks.**

- Let $M = \mathbf{R}^n/\Gamma$ be a Euclidean manifold and $p : \mathbf{R}^n \to M$ be the quotient map. There is a natural metric $d$ on $M$ obtained by letting $d(x, y)$ be the infimum of the distance from $\tilde{x}$ to $\tilde{y}$ over all pairs $(\tilde{x}, \tilde{y}) \in p^{-1}(x) \times p^{-1}(y)$. Note that classifying Euclidean manifolds up to homeomorphism is not the same as classiying them up to isometry, or up to homothety. For instance, $\mathbf{R}^2/\mathbf{Z}^2$ is homeomorphic to $\mathbf{R}^2/(\mathbf{Z}\times 2\mathbf{Z})$ (they are both 2-tori) but these manifolds are not homothetic to each other.

- Groups that act properly discontinuously by isometries on $\mathbf{R}^n$ with compact quotient are classically called *crystallographic groups.* Most of them do not act freely, so they do not give rise to Euclidean manifolds (but to more general objects called orbifolds (see. e.g., [BMP03].) For instance, in dimension 2 there are (up to isomorphism) 17 crystallographic groups, colloquially called 'wallpaper groups'. Only two of them act freely, with quotients the 2-torus and the Klein bottle, respectively. To show that the Klein bottle is Euclidean, we need an alternative description, which we set as an exercise.

**Exercise 5.** Let $\Gamma$ be the subgroup of $\mathrm{Isom}(\mathbf{R}^2)$ generated by the translation $(x, y) \mapsto (x + 1, y)$ and the glide-reflection $(x, y) \mapsto (-x, y + 1)$.

1. Check that $\Gamma$ acts freely and properly discontinuously on $\mathbf{R}^2$. Thus $\mathbf{R}^2/\Gamma$ is a Euclidean surface $F$.

2. Prove that $F$ is homeomorphic to $\mathbf{K}^2$. (This is fairly easy if you assume the classification of surfaces, but you can also try to prove it *without* using this classification.)

**Definition 1.10** (Spherical geometry)**.** Set $X := \mathbf{S}^n$ and $G := \mathbf{O}(n+1)$. Manifolds with this geometry are called *spherical.*

**Examples.**

- The $n$-sphere itself, with $\Gamma$ the trivial group.
- Projective $n$-space, with $\Gamma$ the 2-element group.

**Remark.** Since $\mathbf{S}^n$ is compact, groups that act properly discontinuously on it are finite. Again, the study of finite subgroups of orthogonal groups is a classical topic. For $n = 2$, a well-known theorem states that any finite subgroup of $\mathbf{SO}(3)$ is cyclic, dihedral, or isomorphic to the isometry group of some regular polyhedron (the latter case gives exactly three groups up to isomorphism: $\mathcal{A}_4$, $\mathcal{S}_4$, and $\mathcal{A}_5$.) From this it is easy to deduce the classification of finite subgroups of $\mathbf{O}(3)$ since $-\mathrm{Id}$ is a central element.

**Exercise 6.**

i. Recall (or prove!) that every element of $\mathbf{O}(3)$ is a rotation, a reflection or a rotoreflection (i.e. the product of a rotation $\rho$ with the reflection who plane is orthogonal to the axis of $\rho$.)

ii. For which value of the angle $\alpha$ does a rotoreflection have finite order? Compute this order in terms of $\alpha$.

iii. Let $\Gamma$ be a nontrivial finite subgroup of $\mathbf{O}(3)$ which acts freely on $\mathbf{S}^2$. Show that every element of $\Gamma \setminus \{\mathrm{Id}\}$ is a rotoreflection and has order 2.

iv. Deduce that $\mathbf{S}^2$ and $\mathbf{R}P^2$ are the only spherical surfaces.

The next geometry is *hyperbolic geometry.* It can be defined in all dimensions, but in these notes we content ourselves with dimensions 2 and 3.

First we recall the classical notion of an *inversion in a circle:* let $C$ be a circle in $\mathbf{R}^2$. Choose an affine system of coordinates such that $C$ is the unit circle around the origin. Then the inversion in $C$ is the map $i_C$ from $\mathbf{R}^2 \setminus \{O\}$ to itself which takes a point $P = rv$ (where $v$ is a unit vector and $r$ is a positive real number) to the point $P' = r^{-1}v$. Alternatively, $i_C$ can be defined as the only map from $\mathbf{R}^2 \setminus \{O\}$ to itself which fixes each point of $C$, exchanges the interior and the exerior of $C$, and takes circles orthogonal to $C$ to themselves.

**Definition 1.11** (2-dimensional hyperbolic geometry)**.** Let $X$ be the open unit disk in $\mathbf{R}^2$, which we denote by $\mathbf{H}^2$. Let $G$ be the group generated by (restrictions to $\mathbf{H}^2$ of) inversions in circles orthogonal to $\delta\mathbf{H}^2$ and reflections through diameters of $\mathbf{H}^2$. The group $G$ is called the *2-dimensional Möbius group*, and surfaces which admit an $(X, G)$-structure are called *hyperbolic.*

**Theorem 1.12** (Geometrization of surfaces). *Let $F$ be a compact surface. Then:*

*i. $F$ is spherical if and only if $F$ is homeomorphic to $\mathbf{S}^2$ or $\mathbf{R}P^2$;*

*ii. $F$ is Euclidean if and only if $F$ is homeomorphic to $\mathbf{T}^2$ or $\mathbf{K}^2$;*

*iii. $F$ is hyperbolic if and only if $F$ has genus at least 2.*

**Corollary 1.13.** *Every compact surface is geometric, and has exactly one type of geometric structure.*

A proof of Theorem 1.12 is outside the scope of these notes. We will give a rough outline and mention the key ingredients.

We have already seen that $\mathbf{S}^2$ and $\mathbf{R}P^2$ are spherical, and that $\mathbf{T}^2$ and $\mathbf{K}^2$ are Euclidean. The fact that every surface of genus $\geq 2$ is hyperbolic can be proven by constructing every such surface $F$ from an abstract polygon by identifying edges in pairs, in much the same way as $\mathbf{T}^2$ and $\mathbf{K}^2$ can be constructed from a rectangle. Then the polygon can be realized geometrically as a right-angled 'polygon' $P$ in $\mathbf{H}^2$ whose edges are arcs of circles orthogonal to the boundary. Then $\mathbf{H}^2$ can be tesselated by the images of $P$ under a discrete subgroup $\Gamma$ of the Möbius group is such a way that $F$ is homeomorphic to $\mathbf{H}^2/\Gamma$. An illustration of this construction for the surface of genus 2 is given by the video `https://www.youtube.com/watch?v=G1yyfPShgqw` by Jos Leys. More details are given in Thurston's book [Thu97, p. 16]

Thanks to Theorem 1.7 (the classification of surfaces), there only remains to show that the three types of geometric structures are mutually exclusive. This is a straightforward consequence of the *Gauß-Bonnet formula* from Riemannian geometry. It can also be proved by group-theoretic arguments, cf. Theorem 1.14.

## 1.5 Surface groups

To each compact surface $F$ we can associate a geometry $(X, G)$ and a subgroup $\Gamma$ of $G$ such that $F$ is homeomorphic to $X/\Gamma$. This group is called the *fundamental group* of $F$ and denoted by $\pi_1(F)$. It turns out that this is a topological invariant, i.e. homeomorphic surfaces have isomorphic fundamental groups. This follows from elementary homotopy theory where the fundamental group of a pointed topological space is defined using homotopy classes of loops, and the connection between the two definitions is made using covering theory.

We can readily compute some fundamental groups of surfaces. By definition, $\pi_1(\mathbf{S}^2)$ is trivial; $\pi_1(\mathbf{R}P^2)$ is cyclic of order 2; $\pi_1(\mathbf{T}^2)$ is isomorphic to $\mathbf{Z}^2$. The Klein bottle group is a little more difficult to describe, but it is easy to see that it has an index two subgroup isomorphic to $\mathbf{Z}^2$. (In general, it follows from the first Bieberbach theorem that the fundamental group of any Euclidean manifold has an abelian group of finite index, consisting of translations.)

Let us say that a group is *virtually abelian* if it has an abelian subgroup of finite index. More generally, we will say that a group $\Gamma$ virtually has some property if some finite index subgroup of $\Gamma$ has this property. Then we have the following classification of surface groups:

**Theorem 1.14.** *Let $F$ be a compact surface. Then:*

*i. $F$ is spherical if and only if $\pi_1(F)$ is finite.*

*ii. $F$ is Euclidean if and only if $\pi_1(F)$ is infinite and virtually abelian.*

*iii. $F$ is hyperbolic if and only if $\pi_1(F)$ is not virtually abelian.*

In order to prove Theorem 1.14, the only remaining task is to show that if $F$ is a hyperbolic surface, then $\pi_1(F)$ is not virtually abelian. One way to do this is to compute a presentation of $\pi_1(F)$ using van Kampen's theorem and then use this presentation to find a nonabelian free subgroup of $\pi_1(F)$. Alternatively, use covering theory: if $\pi_1(F)$ were virtually abelian, then $F$ would have a finite cover $\hat{F}$ such that $\pi_1(\hat{F})$ is abelian. Then show that no compact hyperbolic surface can have an abelian fundamental group using hyperbolic geometry (cf. [Sco83, Section 1].)

# 2 Thurston's eight 3-D geometries

## 2.1 Preliminaries

In the sequel we will need the following notions from topology: simple connectedness, fundamental group, universal covering, orientable manifold. If you know the definitions, you can skip ahead to Subsection 2.2. Otherwise, the following discussion will give you enough information so that you can successfully pretend that you know covering theory and differential topology.

There is a general notion of a *simply-connected* topological space. The only thing we will need to know is that Euclidean spaces $\mathbf{R}^n$ for $n \geq 1$, spheres $\mathbf{S}^m$ for $m \geq 2$ and their Cartesian products are simply-connected. In particular, this is the case for the manifolds $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{S}^2$, which appear as the manifold $X$ in the 1-dimensional and 2-dimensional geometries $(X, G)$ considered in the first lecture. (Note that $\mathbf{H}^2$ is homeomorphic to $\mathbf{R}^2$.) This will be the case for $\mathbf{R}^3$, $\mathbf{S}^3$, and $\mathbf{S}^2 \times \mathbf{R}$, which play a similar role in dimension 3.

Let $M = X/\Gamma$ be a manifold which admits an $(X, G)$-structure. Then the group $\Gamma$ is called the fundamental group of $M$ and denoted by $\pi_1(M)$. The space $X$ is called the *universal cover* of $M$. Finally, we say that $M$ is *orientable* if every element of $\Gamma$ is an *orientation-preserving* homeomorphism of $X$. We won't give a general definition of this notion since we will use it only for $X \in \{\mathbf{R}^3, \mathbf{S}^3, \mathbf{S}^2 \times \mathbf{R}\}$, where the meaning should be clear.

## 2.2 The eight three-dimensonal geometries

In the sequel we focus on compact, orientable 3-manifolds. We will define eight types of such manifolds which we will call 'geometric' and, whenever appropriate, give a few examples of geometric manifolds and discuss some of their features.

We start with the three geometries which are analogous to the 2-dimensional ones.

**Euclidean geometry** This is just the special case of Definition 1.9 in dimension three, i.e. $(X, G) = (\mathbf{R}^3, \mathrm{Isom}(\mathbf{R}^3))$.

The simplest example of a compact, orientable Euclidean 3-manifold is the 3-torus $\mathbf{T}^3$.

In dimension 3, there are 230 crystallographic groups. Of these, only 10 act freely, producing 10 compact Euclidean 3-manifolds, among which only 6 are orientable. We shall have more to say about them in the next lecture (cf. Exercice 11.)

**Spherical geometry** This is the special case $(X, G) = (\mathbf{S}^3, \mathbf{O}(4))$ of Definition 1.10.

We have seen two examples of spherical 3-manifolds: $\mathbf{S}^3$ and $\mathbf{R}P^3$. Unlike in dimension 2, there are infinitely many spherical 3-manifolds. In fact, every finite cyclic group is the fundamental group of some spherical 3-manifold, as the following exercice shows:

**Exercise 7** (Lens spaces)**.** Let $p, q$ be relatively prime numbers. We identify $\mathbf{C}^2$ with $\mathbf{R}^4$ so that $\mathbf{O}(4)$ acts on this space. Let $\Gamma \subset \mathbf{O}(4)$ be the cyclic subgroup generated by the map $(z_1, z_2) \mapsto e^{2i\pi/p}, e^{2i\pi q/p}$. Check that this group acts freely on $\mathbf{S}^3$. What is its order? Prove that $\mathbf{R}P^3$ can be defined in this way.

Another famous example of a spherical 3-manifold is the *Poincaré sphere*, whose fundamental group is the so-called *binary icosahedral group*, a finite group of order 120 whose abelianization is trivial.[1]

**Hyperbolic geometry** 3-dimensional hyperbolic geometry is defined analogously to the 2-dimensional case. If $S$ is a sphere of center $P$ in $\mathbf{R}^3$, the inversion in $S$ is the only map from $\mathbf{R}^3 \setminus \{P\}$ to itself which fixes each point of $S$, exchanges the interior and the exerior of $S$, and takes spheres orthogonal to $C$ to themselves. We let $X$ be the open unit ball in $\mathbf{R}^3$ and $G$ be the *3-dimensional Möbius group*, which is generated by restrictions to $X$ of inversions in spheres orthogonal to $\delta X$ and of reflections with respect to planes passing through the origin. We denote $X$ by $\mathbf{H}^3$ and call it *hyperbolic 3-space*. A geometric 3-manifold with $\mathbf{H}^3$ geometry is called *hyperbolic*.

Although we shall see in the next section that there are many hyperbolic 3-manifolds, it is not so easy to construct examples by elementary means. One classical example is the *Seifert-Weber dodecahedral space*, which is a hyperbolic cousin of the Poincaré sphere.

Two other geometries are easy to define:

**$\mathbf{S}^2 \times \mathbf{R}$ geometry** We let $X$ be the Cartesian product $\mathbf{S}^2 \times \mathbf{R}$ and set $G := \mathbf{O}(3) \times \mathrm{Isom}(\mathbf{R})$.

An obvious example of compact, orientable 3-manifold with this geometry is $\mathbf{S}^2 \times \mathbf{S}^1$, whose fundamental group is the infinite cyclic group generated by the map $(x, t) \mapsto (x, t + 1)$.

[1]This group can be defined as $\mathbf{SL}_2(\mathbf{F}_5)$, where $\mathbf{F}_5$ is the field with 5 elements. Its only nontrivial quotient is the group $\mathbf{PSL}_2(\mathbf{F}_5)$, which is simple (isomorphic to $\mathcal{A}_5$).

A somewhat less obvious example is obtained by taking $\Gamma$ generated by the maps $(x,t) \mapsto (-x,-t)$ and $(x,t) \mapsto (1-x,-t)$. This manifold can be described as a connected sum of two copies of $\mathbf{R}P^3$ (see Definition 3.4.) The group $\Gamma$ is a free product of two groups of order 2, and has an infinite cyclic subgroup of index 2.

**Exercise 8.** Show that every compact, orientable 3-manifold with $\mathbf{S}^2 \times \mathbf{R}$ geometry is homeomorphic to one of these two examples.

**$\mathbf{H}^2 \times \mathbf{R}$ geometry** Here $X = \mathbf{H}^2 \times \mathbf{R}$, where $\mathbf{H}^2$ is the Poincaré disk, and $G$ is the direct product of the 2-dimensional Möbius group with $\mathrm{Isom}(\mathbf{R})$.

From Lecture 1, we already know infinitely many compact orientable manifolds with $\mathbf{H}^2 \times \mathbf{R}$ geometry: just take $F_g \times \mathbf{S}^1$ with $g \geq 2$.

The remaining three geometries are somewhat more difficult to describe. In each case, $X$ is a 3-dimensional Lie group, in fact homeomorphic to $\mathbf{R}^3$.

**Nilgeometry** Let **Nil** be the *Heisenberg group*, i.e. the group of $3 \times 3$-matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $(x,y,z) \in \mathbf{R}^3$. Pick a left-invariant Riemannian metric on **Nil** and let $G$ be the corresponding group of isometries. In particular, $G$ contains a copy of **Nil** acting on itself by left multiplication.

The *integral Heisenberg group* is the subgroup $\Gamma \subset \mathbf{Nil}$ consisting of the matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for $(a,b,c) \in \mathbf{Z}^3$. This group acts freely and properly discontinuously on **Nil**, and the quotient manifold is compact and orientable. In fact, there are infinitely many examples, though the others are more difficult to construct.

**$\widetilde{\mathbf{SL}_2(\mathbf{R})}$ geometry** We let $X$ be the Lie group $\widetilde{\mathbf{SL}_2(\mathbf{R})}$. This is the universal cover of $\mathbf{SL}_2(\mathbf{R})$, thus it is an extension

$$1 \to \mathbf{Z} \to \widetilde{\mathbf{SL}_2(\mathbf{R})} \to \mathbf{SL}_2(\mathbf{R}) \to 1.$$

Again we let $G$ be the isometry group of $X$ with respect to some left-invariant Riemannian metric.

For every $g \geq 2$, the unit tangent bundle of the surface $F_g$ is a compact, orientable 3-manifold with this geometry.

**Solvegeometry** Let **Sol** be the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}$ where the action of $\mathbf{R}$ on $\mathbf{R}^2$ is given by $t \cdot (x, y) = (e^t x, e^{-t} y)$. Let $G$ be the isometry group of **Sol** with respect to some left-invariant Riemannian metric.

There are infinitely many compact, orientable examples. We will discuss some of them in the next section.

## 2.3 Fundamental groups of geometric 3-manifolds

The purpose of this subsection is to discuss algebraic properties of the fundamental groups of compact geometric 3-manifolds which can be used to distinguish between those groups. In particular, we will see that a compact 3-manifold can have at most one type of geometry. We will not give complete proofs, however, since that would be outside the scope of these notes.

In this subsection, a *geometric 3-manifold* is a compact, orientable 3-manifold which admits a geometric structure modeled on one of the eight geometries described above.[2]

First we make a few remarks. In six of the eight 3-dimensional Thurston geometries, the model space $X$ is homeomorphic to $\mathbf{R}^3$. The other two are spherical geometry (with $X = \mathbf{S}^3$) and $\mathbf{S}^2 \times \mathbf{R}$ geometry (with $X = \mathbf{S}^2 \times \mathbf{R}$ !)

Thus, among geometric 3-manifolds, the spherical ones are exactly those with compact universal cover, hence exactly those with finite fundamental group.

The fundamental groups of the $\mathbf{S}^2 \times \mathbf{R}$ manifolds are $\pi_1(\mathbf{S}^2 \times \mathbf{S}^1) = \mathbf{Z}$ and $\pi_1(\mathbf{R}P^3 \# \mathbf{R}P^3) = (\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z})$, so they are virtually infinite cyclic.[3] Conversely, one can show that a geometric 3-manifold with virtually infinite cyclic fundamental group has $\mathbf{S}^2 \times \mathbf{R}$ geometry and none other. This can be proven by hand, but there is a conceptual way to see this based on the *number of ends* (cf. Subsection 3.2.)

By the Bieberbach theorem (or by the classification) the fundamental group of every Euclidean 3-manifold is virtually $\mathbf{Z}^3$. Again one can show that none of the other geometric manifolds has this property.

[2]Note, however, that in Section 3, we will allow noncompact manifolds, for a reason which will be clear.

[3]Recall that this means that each of these groups has a finite index subgroup isomorphic to $\mathbf{Z}$.

In order to distinguish the other groups, we need to recall some classical definitions from group theory.

**Definition 2.1** (Nilpotent group)**.** Let $G$ be a group. The *lower central series* of $G$ is the sequence of subgroups $\{G_n\}$ defined inductively by setting $G_1 := G$ and $G_{n+1} := [G, G_n]$ (where $[G, H]$ is the subgroup of $G$ generated by commutators $[x, y] = xyx^{-1}y^{-1}$ with $x \in G$ and $y \in H$.) We say that $G$ is *nilpotent* if there exists $n \geq 1$ such that $G_n$ is trivial.

**Definition 2.2** (Solvable group)**.** Let $G$ be a group. The *derived series* of $G$ is the sequence of subgroups $\{G^{(n)}\}$ defined inductively by setting $G^{(0)} := G$ and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$. The group $G$ is called *solvable* (or in some countries *soluble*) if there exists $n \in \mathbf{N}$ such that $G^{(n)}$ is trivial.

**Exercise 9.**

1. Show that every abelian group is nilpotent and that every nilpotent group is solvable.
2. Compute the lower central series and the derived series of the dihedral groups $\mathrm{D}_3$ and $\mathrm{D}_4$. Deduce from these computations that the converse implications do not hold.

We are now ready to state a result which relates algebraic properties of geometric 3-manifold groups to the type of geometry involved.

**Theorem 2.3.** *Let $\Gamma$ be the fundamental group of a geometric 3-manifold $M$.*

*i. Suppose that $\Gamma$ is virtually solvable. Then:*

- *(a) If $\Gamma$ is not virtually nilpotent, then $M$ has Solvegeometry.*
- *(b) If $\Gamma$ is virtually nilpotent, but not virtually abelian, then $M$ has Nilgeometry.*
- *(c) If $\Gamma$ is virtually abelian, then it is finite, virtually $\mathbf{Z}$, or virtually $\mathbf{Z}^3$, and $M$ has spherical, $\mathbf{S}^2 \times \mathbf{R}$, or Euclidian geometry respectively.*

*ii. Suppose that $\Gamma$ is not virtually solvable. Then:*

- *(a) If $\Gamma$ does not admit a normal infinite cyclic subgroup, then $M$ is hyperbolic.*
- *(b) If $\Gamma$ admits a normal infinite cyclic subgroup, then $M$ has $\mathbf{H}^2 \times \mathbf{R}$ geometry if and only if $\Gamma$ is virtually a direct product $\pi_1(F_g) \times \mathbf{Z}$ with $g \geq 2$; otherwise $M$ has $\widetilde{\mathbf{SL}_2(\mathbf{R})}$ geometry.*

To finish this lecture, let us try to understand this result from the perspective of *Geometric Group Theory*, by which we mean the branch of mathematics that studies (typically infinite) finitely generated groups from a geometric viewpoint.

The general idea is that some groups are 'big', other 'small', and among the small (resp. big) ones, some are smaller (resp. bigger) than others. There are several ways to make this precise, which interestingly do not yield quite the same results.

One way to think about this is to consider the *growth function* of a group. The definition is as follows: let $\Gamma$ be a finitely generated group, and let $S$ be a finite generating subset of $\Gamma$. For every nonnegative integer $n$, let $B_n$ be the set of elements $\gamma \in \Gamma$ such that $\gamma$ can be written as a product of at most $n$ elements of $S \cup S^{-1}$. Then the growth function is the function $f_{\Gamma,S} : n \mapsto \#B_n$. We write $f_\Gamma = f_{\Gamma,S}$ when making a statement that does not depend on the choice of the generated set $S$.

Here are simple examples. If $\Gamma$ is finite, then $f_\Gamma(n)$ is eventually constant, equal to $\#\Gamma$. In particular, it is bounded. If $\Gamma$ is infinite, then $f_\Gamma$ is unbounded. For $\Gamma = \mathbf{Z}$ and $S$ a singleton, we have $f_{\Gamma,S}(n) = 2n + 1$. If one changes the generating set, or passes to a finite index subgroup, the growth function changes, but its asymptotic behavior does not (this can be made precise, but we will not do this here.)

The following result gives a rough classification of geometric 3-manifold groups according to growth, ranging from 'small' to 'big'.

**Theorem 2.4.** *Let $\Gamma$ be the fundamental group of a geometric 3-manifold $M$.*

*i. If $M$ has spherical geometry, then $f_\Gamma$ is bounded.*

*ii. If $M$ has $\mathbf{S}^2 \times \mathbf{R}$ geometry, then $f_\Gamma$ is linear.*

*iii. If $M$ has Euclidean geometry, then $f_\Gamma$ is cubic.*

*iv. If $M$ has Nilgeometry, then $f_\Gamma$ is quartic.*

*v. In all other cases, $f_\Gamma$ is exponential.*

Another divide in geometric group theory is between *amenable* and *nonamenable* groups, the former been the 'smaller' ones. We will not give the definition here. If $\Gamma$ a geometric 3-manifold group, then $\Gamma$ is amenable if and only if it is virtually solvable (which as we have seen corresponds to 5 out of 8 geometries.) It is nonamenable if and only if it contains a nonabelian free group.

In general, containing a nonabelian free group implies exponential growth, but the converse is false, since fundamental groups of 3-manifolds with Solve-geometry are amenable, but have exponential growth. Thus these groups can be thought of as 'small' or 'big' depending on the viewpoint...

In general it is true that an amenable group does not contain any nonabelian free group. However, there exist finitely groups which are nonamenable, yet do not contain an nonabelian free group. There also exist groups which are amenable, but not virtually solvable, and groups whose growth function is neither polynomial nor exponential. Thus we see that the class of geometric 3-manifold groups is a relatively well-behaved one, where those exotic phenomena do not occur.

Finally, we cannot leave this topic without a mention of *Gromov hyperbolic groups*. This is a vast class of groups which share some properties with free groups, and includes fundamental groups of compact manifolds with negative curvature. If $\Gamma$ is the fundamental group of a geometric 3-manifold $M$, then $\Gamma$ is a nonelementary[4] Gromov hyperbolic group if and only if $M$ has hyperbolic geometry.

# 3 Geometrizing 3-manifolds

## 3.1 Many hyperbolic 3-manifolds

The goal of this subsection is to state two results that are often quoted as justifying the informal assertion that 'there are many hyperbolic 3-manifolds', although it is not so easy to construct even a single one. The first one is the Thurston-Jørgensen theorem on the order structure of the set of hyperbolic volumes; the second one is the hyperbolization theorem for mapping tori of hyperbolic surfaces.

Let $M$ be an orientable, not necessarily compact hyperbolic 3-manifold. By definition, $M$ can be written as $\mathbf{H}^3/\Gamma$, where $\Gamma$ is a subgroup of the 3-dimensional Möbius group $G_3$ acting freely and properly discontinuously on $\mathbf{H}^3$. It turns out that $G_3$ is the isometry group of $\mathbf{H}^3$ for the *Poincaré metric*

$$ds^2 = 4\frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2}.$$

The latter is a complete Riemannian metric on $\mathbf{H}^3$ with constant sectional curvature equal to $-1$. Thus $M$ inherits a Riemannian metric with the same properties, which may or may not have finite volume.

[4] i.e. infinite and not virtually cyclic

The Mostow rigidity theorem implies that if $M_1$ and $M_2$ are complete hyperbolic manifolds of finite volume, then they are homeomorphic if and only if they are isometric. It follows that the volume of a hyperbolic metric is a topological invariant. We denote by $\mathcal{V}_3$ the set of volumes of hyperbolic 3-manifolds. This is a subset of the real line, hence it has a natural linear order.

**Theorem 3.1.** *The ordered set $\mathcal{V}_3$ is a well-ordered set of type $\omega^\omega$.*

For people unfamiliar with ordinal theory, here is an explanation. An ordered set $V$ is *well-ordered* if it does not contained any strictly decreasing sequence. Saying that $V$ has type $\omega^\omega$ means the following: first $V$ has a smallest element $v_0$. Then $X \setminus \{v_0\}$ has a smallest element $v_1$, the set $V \setminus \{v_0, v_1\}$ has a smallest element $v_2$, and so on. Thus we have a sequence $\{v_n\}$ of points of $V$. The set $X \setminus \{v_n \mid n \in \mathbf{N}\}$ has in turn a smallest element, denoted by $v_\omega$. We then have another sequence $v_{\omega+1}, v_{\omega+2}, \ldots$, an element $v_{2\omega}$, and so on. We leave to the reader to guess what $v_{2\omega+1}$, $v_{3\omega}$, $v_{\omega^2}$, $v_{\omega^3}$ are. Finally, saying that $V$ has type $\omega^\omega$ means that for every $k \in \mathbf{N}$ there is an element $v_{\omega^k}$, but there is no element $v_{\omega^\omega}$.

From the topological viewpoint, $v_k$ is an isolated point of $\mathcal{V}_3$ for each $k \in \mathbf{N}$; the sequence $\{v_k\}_{k\in\mathbf{N}}$ converges to $v_\omega$; the sequence of isolated points $\{v_{\omega+k}\}_{k\in\mathbf{N}}$ converges to $v_{2\omega}$, and so on. The sequence $\{v_{k\omega}\}$ converges to $v_{\omega^2}$. Finally the sequence $\{v_{\omega^k}\}$ goes to infinity.

A key ingredient in the proof is Thurston's hyperbolic Dehn surgery theorem which allows to construct many hyperbolic manifolds from a single one. For instance, the existence of hyperbolic manifolds of volumes $v_k$ for $k \in \mathbf{N}$ can be deduced from that of a hyperbolic manifold of volume $v_\omega$. For more details, see [Gro81, Mai10].

We now turn to a discussion of geometric structures on 3-manifolds belonging to a very interesting class: mapping tori of homeomorphisms of surfaces.

**Definition 3.2.** Let $F$ be a compact, orientable surface. Let $h : F \to F$ be a homeomorphism which preserves orientation. Then the *mapping torus* of $h$ is the 3-manifold $\Sigma_h = F \times [0,1]/\sim$, where $\sim$ is the finest equivalence relation such that $(x, 0) \sim (h(x), 1)$ for every $x \in F$.

**Exercise 10.** Prove that the mapping torus of a surface homeomorphism is indeed a 3-manifold.

For $F = \mathbf{S}^2$ we only get $\mathbf{S}^2 \times \mathbf{S}^1$, which has $\mathbf{S}^2 \times \mathbf{R}$ geometry.

For $F = \mathbf{T}^2$ every mapping torus of a homeomorphism of $F$ is geometric; depending on $h$ the geometry can be Euclidean, Nil or Sol.

**Exercise 11.** Find elements of finite order in $\mathbf{SL}_2(\mathbf{Z})$. Show that their mapping tori (as self-homeomorphisms of $\mathbf{T}^2$) are Euclidean manifolds. Try to get a feeling of what it is like to live inside such a manifold.

Can you get all 6 compact orientable Euclidean manifolds in this way?

When $F$ has genus et least 2, i.e. when $F$ is hyperbolic, the mapping torus of a homeomorphism of $F$ is 'generically' hyperbolic. To state a precise result we need two definitions. We say that $h$ is *periodic* if some power of $h$ is the identity of $F$. We say that $h$ is *reducible* if there is a finite union of disjoint simple closed curves on $F$ that is preserved by $h$.

**Theorem 3.3** (Thurston)**.** *Let $F$ be a compact, orientable surface of genus at least 2. Let $h$ be a homeomorphism of $F$. Assume that $F$ is neither isotopic to a periodic homeomorphism of $F$ nor to a reducible one. Then the mapping torus of $h$ is hyperbolic.*

## 3.2 A nongeometric manifold

At this point you may wonder whether every compact, orientable 3-manifold is geometric. In this subsection we give a counterexample.

**Definition 3.4.** Let $M_1, M_2$ be oriented 3-manifolds. The *connected sum* of $M_1$ and $M_2$, denoted by $M_1\#M_2$, is defined as follows: let $B_1$ (resp. $B_2$) be a subset of $M_1$ (resp. $M_2$) homeomorphic to a 3-ball. Then $M_1\#M_2$ is obtained by gluing $M_1\backslash B_1$ to $M_2\backslash B_2$ along their boundaries by an orientation-reversing homeomorphism.

One can check that the manifold $M_1\#M_2$ is well-defined up to homeomorphism, i.e. does not depend on the choice of $B_1$, $B_2$ and the gluing homeomorphism. It is also true that for every 3-manifold $M$ the manifold $M\#\mathbf{S}^3$ is homeomorphic to $M$, i.e. $\mathbf{S}^3$ is an identity element for the connected sum operation. Likewise, this operation is 'associative', so if $M_1, \ldots M_n$ is a finite collection of 3-manifolds the iterated connected sum $M_1\#\cdots\#M_n$ is well-defined.

Our next task is to show that the manifold $\mathbf{S}^2\times\mathbf{S}^1\#\mathbf{S}^2\times\mathbf{S}^1$ is not geometric. The proof is based on the notion of number of ends of a topological space.

**Definition 3.5.** Let $X$ be a topological space. The *number of ends* of $X$, denoted by $e(X)$ is the supremum (in $\mathbf{N}\cup\infty$) over all compact subsets $K\subset X$ of the number of noncompact connected components of $\overline{X\setminus K}$.

For instance, the number of ends of $\mathbf{R}^2$ is 1, because the closure of the complement of the closed unit ball has a single connected component that is noncompact, but for every compact $K \subset \mathbf{R}^2$ the complement of $K$ has only one unbounded component. Likewise, the number of ends of $\mathbf{R}$ is 2, the supremum being realized by e.g. $[0,1]$.

**Exercise 12.**

1. Compute $e(\mathbf{R}^3)$, $e(\mathbf{S}^2 \times \mathbf{R})$ and $e(\mathbf{S}^3)$.
2. Prove that the universal cover of $\mathbf{S}^2 \times \mathbf{S}^1 \# \mathbf{S}^2 \times \mathbf{S}^1$ has infinitely many ends. Conclude that $\mathbf{S}^2 \times \mathbf{S}^1 \# \mathbf{S}^2 \times \mathbf{S}^1$ cannot have a geometric structure.

**Remark.** In fact, the only compact, orientable 3-manifold that is a nontrivial connected sum and admits a geometric structure is $\mathbf{R}P^3 \# \mathbf{R}P^3$, which we have seen to have $\mathbf{S}^2 \times \mathbf{R}$ geometry. A rough outline of the proof is as follows: if $M$ is a geometric manifold that is a nontrivial connected sum, then $M$ contains an embedded 2-sphere $S$ that does not bound a 3-ball. Since $\mathbf{S}^2$ is simply-connected, we can lift it to an embedded 2-sphere $\tilde{S} \in \tilde{M}$. By a combinatorial argument one shows that $\tilde{S}$ does not bound a 3-ball in $\tilde{M}$. By Alexander's theorem, $\tilde{M}$ cannot be diffeomorphic to $\mathbf{R}^3$ or $\mathbf{S}^3$, so it must be $\mathbf{S}^2 \times \mathbf{R}$. Then we conclude using the classification of $\mathbf{S}^2 \times \mathbf{R}$ manifolds.

## 3.3 The geometrization theorem

In the sequel all 3-manifolds are orientable.

**Definition 3.6.** Let $M$ be a 3-manifold. We say that $M$ is *prime* if $M$ is not homeomorphic to $\mathbf{S}^3$ and for all 3-manifolds $M_1, M_2$, if $M$ is homeomorphic to $M_1 \# M_2$, then $M_1$ or $M_2$ is homeomorphic to $\mathbf{S}^3$.

**Remark.** The convention that $\mathbf{S}^3$ is not prime is analogous to the fact that 1 is not a prime number. In particular, it simplifies the statement of uniqueness in the prime decomposition, which we do not give in these notes.

**Theorem 3.7** (Kneser)**.** *Let $M$ be a compact 3-manifold. Then there exists a finite collection of compact, prime 3-manifolds $M_1, \dots M_n$ such that $M$ is homeomorphic to $M_1 \# \cdots \# M_n$.*

**Remark.** If $M = \mathbf{S}^3$ then $n = 0$. Again this is a natural convention.

We are now led to the question of whether every prime compact 3-manifold is geometric. The answer is no, but understanding why is more difficult.

**Example** (Flip manifold)**.** Let $F$ be the complement of an open disk in $\mathbf{T}^2$. Set $X = \mathbf{S}^1 \times F$. Then $X$ is a 3-manifold with boundary a torus $\mathbf{S}^1 \times \delta F$. Let $M$ be obtained from two copies $X_1, X_2$ of $X$ by gluing them along a homeomorphism which sends $\mathbf{S}^1_1 \times \{*\}$ to $\{*\} \times (\delta F)_1$ and $\mathbf{S}^1_2 \times \{*\}$ to $\{*\} \times (\delta F)_2$. Then $M$ is a prime compact manifold (without boundary) that is not geometric. Again, a proof of this statement is too difficult for these notes. It could go as follows: first show that the universal cover of $X$ is contractible, and use this to prove that $X$ is prime. Then by a combinatorial argument show that $M$ is prime. To see that $M$ is not geometric, use van Kampen's theorem to compute $\pi_1(M) = A *_C B$ where $A, B$ are isomorphic to the direct product of $\mathbf{Z}$ by the free group of rank 2. From this deduce that $\pi_1 M$ contains a nonabelian free group, which rules out 5 out of 8 geometries by the discussion in Subsection 2.3. An elementary argument in hyperbolic geometry implies that the fundamental group of a compact hyperbolic 3-manifold cannot contain a subgroup isomorphic to $\mathbf{Z}^2$, so hyperbolic geometry is also ruled out. Finally, if $M$ was geometric with geometry $\mathbf{H}^2 \times \mathbf{R}$ or $\widetilde{\mathbf{SL}_2(\mathbf{R})}$, then $\pi_1 M$ would contain a normal infinite cyclic subgroup, so some index 2 subgroup of $\pi_1 M$ would have infinite center, which is not the case.

**Definition 3.8.** Let $F$ be an embedded orientable surface in a 3-manifold $M$. We say that $F$ is *incompressible* if the group homomorphism from $\pi_1 F$ to $\pi_1 M$ induced by the inclusion map is injective.

The following result was conjectured by Thurston in the mid 1970s, and proven by Perelman in the early aughts.

**Theorem 3.9** (Perelman)**.** *Let $M$ be a compact, orientable 3-manifold. If $M$ is prime, then there is a finite family of pairwise disjoint, incompressible embedded tori $\{T_i\}$ in $M$ such that each connected component of $M \setminus \bigcup T_i$ is geometric.*

In the flip manifold example, the torus along which the two copies of $X$ are glued is incompressible. Its complement has two connected components, both homeomorphic to $\mathbf{S}^1 \times (\mathbf{T}^2 \setminus \{*\})$. The latter manifold is a (noncompact) $\mathbf{H}^2 \times \mathbf{R}$ manifold.

**Remark.** In the case of mapping tori of hyperbolic surfaces, we have seen that $\Sigma_h$ is hyperbolic unless $h$ is isotopic to a periodic homeomorphism or a reducible one. One can show that if $h$ is periodic then $\Sigma_h$ has $\mathbf{H}^2 \times \mathbf{R}$ geometry, whereas if $h$ is reducible, then it contains one or more incompressible tori.

# 4 Additional exercises

**Exercise 13.** Prove that $\mathbf{T}^2$ and $\mathbf{K}^2$ are the only Euclidean surfaces.

**Exercise 14.** Let $H$ be a regular hexagon in the Euclidean plane. Label each edge of $H$ be an arrow together with the letter $a, b$ or $c$ in such a way that opposite edges have the same label, with arrow going in the same direction. Check that the space obtained by identification from such data is a surface. Which one is it?

**Exercise 15.** Compare various ways of representing $\mathbf{R}P^3$. Show that $\mathbf{R}P^3$ is homeomorphic to $\mathbf{SO}(3)$. (Hint: use the model of the closed ball of radius $\pi$ with some points on the boundary identified.)

**Exercise 16.** Classify compact, orientable $\mathbf{S}^2 \times \mathbf{R}$ manifolds.

**Exercise 17.** Let $\mathbf{Nil}_\mathbf{Z}$ be the integral Heisenberg group. Set

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

i. Prove that the subset $\{A, B, C\}$ generates $\mathbf{Nil}_\mathbf{Z}$.

ii. Show that $\mathbf{Nil}_\mathbf{Z}$ is nilpotent. Is it abelian?

**Exercise 18.** Show that the mapping torus of the homeomorphism of $\mathbf{T}^2$ defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is a **Sol** manifold.

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