# Lectures on hyperbolic groups and convergence groups

Sylvain Maillot

July 20, 2005

## Lecture 1

### Motivation

We start by stating an important theorem in 3-manifold topology.

**Theorem 1 ([28, 24, 21, 25, 11, 6]).** *Let $M$ be a closed, orientable, irreducible 3-manifold. If $\pi_1 M$ has an infinite cyclic normal subgroup, then $M$ is Seifert fibered.*

Since Seifert manifolds are easily seen to be geometric in Thurston's sense, this theorem can be viewed as a partial answer to the Geometrization Conjecture. It was known in some circles as the 'Seifert Conjecture' (although I doubt that Seifert ever conjectured this.)

Here is an outline of proof: let $Z$ be an infinite cyclic normal subgroup of $\pi_1 M$ and set $\Gamma := \pi_1 M/Z$. Let $\hat{M}$ be the covering space of $M$ such that $\pi_1 \hat{M} = Z$. Then $\Gamma$ acts on $\hat{M}$ with quotient space $M$.

**Step 1** The 3-manifold $\hat{M}$ is *tame,* i.e. homeomorphic to $S^1 \times \mathbf{R}^2$.

**Step 2** The group $\Gamma$ is *quasi-isometric* to a complete Riemannian metric on $\mathbf{R}^2$. (The definition of quasi-isometric will be given later.)

**Step 3** Any finitely generated group quasi-isometric to a complete Riemannian metric on $\mathbf{R}^2$ is *virtually* $\mathbf{Z}^2$ (i.e. has a finite index subgroup isomorphic to $\mathbf{Z}^2$,) or is quasi-isometric to the hyperbolic plane $\mathbf{H}^2$.

**Step 4** Any finitely generated group quasi-isometric to $\mathbf{H}^2$ *acts geometrically* on $\mathbf{H}^2$. (This means that the action is isometric, properly discontinuous, and cocompact.)

**Step 5** The fundamental group of $M$ is isomorphic to the fundamental group of some irreducible Seifert fibered space $N$.

**Step 6** $M$ is homeomorphic to $N$, hence Seifert fibered.

The goal of these lectures is to explain Step 4 in some detail, putting it into the more general context of *hyperbolic groups* and *convergence groups.* Before that, we make some brief comments about the other steps, in the chronological order of their proofs.

Step 5 is classical: the class of groups that are virtually $\mathbf{Z}^2$ or act geometrically on $\mathbf{H}^2$ has been well-known for some time: they are called 'planar discontinuous groups' in [31]. Extensions of $\mathbf{Z}$ by these groups can be classified using group cohomology [32]. When such a group is torsion free, it is not hard to construct a Seifert fiber space realizing it.

Step 6 is due to Peter Scott [24]: he proves that if $M, N$ are closed, orientable, irreducible 3-manifolds with isomorphic fundamental groups, and one of them is Seifert, then $M$ and $N$ are homeomorphic. Hence at that time the Seifert Conjecture was reduced to proving that $\Gamma$ is planar discontinuous.

Steps 1–3 are due to Mess [21]. In fact, the conclusion of Step 2 and hypothesis of Step 3 in his original approach is that $\Gamma$ is quasi-isometric to a *quasihomogeneous* metric on $\mathbf{R}^2$. That Step 3 is valid without this additional hypothesis is proved in [19].

Step 1 uses many important results in 3-manifold topology: the Meeks-Simon-Yau theorem about irreducibility of covering spaces, the Scott compact core theorem, the theory of characteristic submanifolds due to Jaco-Shalen and Johannson... Step 2 uses minimal surfaces. Complete proofs of more general results can be found in [20] (see [1] for an introduction). I also refer you to Juan Souto's lectures in this summer school for an introduction to open 3-manifolds.

Step 3 consists of intricate 2-dimensional arguments. A simpler proof was given in [19].

Step 4 can be further decomposed into 2 substeps: if $\Gamma$ is quasi-isometric to $\mathbf{H}^2$, then Mess observed that $\Gamma$ acts on the circle $S^1$ as a *convergence group*. More generally, if $\Gamma$ is a *hyperbolic group* in the sense of Gromov, then it acts in a natural way on a compact topological space $\partial\Gamma$ called its *boundary*, and this action is a convergence action. This will be explained in Lectures 1 and 2.

To finish the proof of Theorem 1, one needs:

**Theorem 2 (Convergence Group Theorem [25, 11, 6]).** *Let $\Gamma$ be a group. If $\Gamma$ acts as a convergence group on $S^1$, then $\Gamma$ has a finite normal subgroup $F$ such that $\Gamma/F$ is Fuchsian.*

Recall that a Fuchsian group is a discrete subgroup of isometries of $\mathbf{H}^2$. Hence the conclusion is equivalent to saying that $\Gamma$ acts properly discontinuously and isometrically on $\mathbf{H}^2$. If $\Gamma$ is quasi-isometric to $\mathbf{H}^2$, then one can see that such an action has to be cocompact, hence we get the conclusion of Step 4 as stated above.

In lectures 3 and 4, I will discuss the proof of Theorem 2, following Tukia and Casson-Jungreis.

For other (more group-theoretic) approaches to the Seifert Conjecture and related questions, see e.g. [9, 4, 18].

## Hyperbolicity according to Rips and Gromov

Let $(X, d)$ be a metric space. A *geodesic segment* in $X$ is an isometric embedding of a compact interval $[a, b] \subset \mathbf{R}$ into $X$. We say that $X$ is *geodesic* if for all $x, y \in X$, there is a geodesic segment $c : [a, b] \to X$ such that $c(a) = x$ and $c(b) = y$.

Basic examples of geodesic spaces are complete Riemannian manifolds, in particular Euclidean space $\mathbf{E}^n$ and hyperbolic space $\mathbf{H}^n$. Note however that our notion of 'geodesic segment' is commonly called 'minimizing geodesic' in Riemannian geometry. A nonminimizing geodesic (such as a large portion of the equator in the round 2-sphere) is *not* a geodesic segment in our sense.

Other basic examples are *graphs*, i.e. connected 1-dimensional cellular complexes: let $X$ be a graph. The distance between two vertices $x, y$ is defined to be the minimal number of edges in an edge-path connecting $x$ to $y$. This defines a distance on the set of vertices of $X$. It can be extended to a distance on all of $X$ by fixing a parametrization of each edge, and counting the length of the pieces of edges in a path connecting the two points.

Next we define hyperbolicity in the sense of Rips: the definition involves looking at *geodesic triangles*. Before this, we give some general definitions in metric spaces: if $A$ is a subset of $X$ and $C \geq 0$, the *$C$-neighborhood* of $A$ is the set $N_C(A) := \{x \in X \mid d(x, A) \leq C\}$. If $x \in X$, the $C$-neighborhood of $\{x\}$ is also called the *ball* of radius $C$ around $x$, and denoted by $B_C(x)$.

A *geodesic triangle* in $X$ is a triple $(\alpha_1, \alpha_2, \alpha_3)$ of geodesic segments such that there exist points $x_1, x_2, x_3$ such that $\alpha_i$ connects $x_{i+1}$ to $x_{i+2}$ (with indices in $\mathbf{Z}/3$.) It is *$\delta$-thin* if $\alpha_i$ lies in the $\delta$-neighborhood of $\alpha_{i+1} \cup \alpha_{i+2}$.

**Definition.** Let $\delta \geq 0$. A geodesic metric space is *$\delta$-hyperbolic* if all its geodesic triangles are $\delta$-thin. It is *hyperbolic in the sense of Rips*, or simply *hyperbolic*, if it is $\delta$-hyperbolic for some $\delta$.

*Examples.*

- $\mathbf{H}^n$ is hyperbolic; $\mathbf{E}^n$ is not hyperbolic for $n \geq 2$. Pinched negatively curved manifolds are hyperbolic, as well as their convex subsets.
- Simplicial trees (i.e. 1-connected graphs) are 0-hyperbolic.
- Any bounded geodesic space is hyperbolic. Any product of a hyperbolic space and a bounded space, e.g. $S^2 \times \mathbf{H}^2$ is hyperbolic. These examples show that the definition of Rips hyperbolicity captures the geometry of negative curvature only 'in the large'.

**Exercice 1.** If $X$ is 0-hyperbolic, then for all $x, y \in X$, there is a unique topological arc connecting $x$ to $y$, i.e. topological embedding $c : [a, b] \to X$ such that $c(a) = x$ and $c(b) = y$. (Of course, the uniqueness is up to parameterization.)

A metric space is *proper* if all metric balls are compact. Basic examples of proper spaces are complete Riemannian manifolds and locally finite metric graphs.

A group action is called *geometric* if it is isometric, properly discontinuous, and cocompact.

**Definition.** A *hyperbolic group* is a group that acts geometrically on some proper hyperbolic space.

*Examples.*

- Convex cocompact Kleinean groups, in particular surface groups.
- Free groups, because they act geometrically on trees.

*Remark.* Any finitely generated group $\Gamma$ acts geometrically on some proper geodesic space. The simplest construction is called the Cayley graph: let $S$ be a finite generating set of $\Gamma$. Then the *Cayley graph* is the graph whose vertices are elements of $\Gamma$, and there is an edge between $\gamma_1$ and $\gamma_2$ if and only if $\gamma_1^{-1}\gamma_2 \in S$.

**Exercice 2.** Let $F_2$ be the free group on two generators. Describe the Cayley graph of $F_2$ with respect to a generating system of your choice.

## Quasi-isometries; quasi-isometry invariance of hyperbolicity

**Definition.** Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces. We say that a map $f : X_1 \to X_2$ is a *quasi-isometric embedding* if there exist $\lambda \geq 1$ and $C \geq 0$ such that the inequality

$$\lambda^{-1} d_1(x, x') - C \leq d_2(f(x), f(x')) \leq \lambda d_1(x, x') + C$$

holds for any $x, x' \in X_1$.

If in addition, the $C$-neighborhood of the image of $f$ is all of $X_2$, we say that $f$ is a *quasi-isometry.*

We say that $(X_1, d_1)$ is *quasi-isometric* to $(X_2, d_2)$ if there exists a quasi-isometry $f : X_1 \to X_2$. Sometimes we just say that $(X_1, d_1)$ and $(X_2, d_2)$ are quasi-isometric.

**Exercice 3.** Prove that if $(X_1, d_1)$ is quasi-isometric to $(X_2, d_2)$, then $(X_2, d_2)$ is quasi-isometric to $(X_1, d_1)$.

Prove that if $(X_1, d_1)$ is quasi-isometric to $(X_2, d_2)$ and $(X_2, d_2)$ is quasi-isometric to $(X_3, d_3)$, then $(X_1, d_1)$ is quasi-isometric to $(X_3, d_3)$.

**Theorem 3.** *Let $X$ be a hyperbolic space. Let $Y$ be a geodesic metric space. If $X$ and $Y$ are quasi-isometric, then $Y$ is hyperbolic.*

The proof of Theorem 3 relies on the following fundamental property of hyperbolic spaces.

A $(\lambda, C)$-*quasigeodesic segment* in a space $X$ is a quasi-isometric embedding of a compact interval $[a, b]$ in $X$.

**Theorem 4 (Morse Lemma on quasigeodesic stability).** *Let $\delta, \lambda \geq 0$. There is a number $D \geq 0$ such that if $X$ is a $\delta$-hyperbolic space, then any $(\lambda, \delta)$-quasigeodesic segment lies in the $D$-neighborhood of a geodesic segment with the same endpoints.*

**Exercice 4.** Deduce Theorem 3 for spaces from Theorem 4.
(Hint: take a geodesic triangle in $B$; its image by a quasi-isometry is a "quasi-geodesic triangle" in $A$, which by Theorem 4 is close to a geodesic triangle; then use the hyperbolicity of $A$ to conclude. You will need to prove that sides of the geodesic triangle lie in the $D'$-neighborhood of the sides of the quasi-geodesic triangle for some constant $D'$.)

**Exercice 5.** Prove Theorem 4 when $X$ is a simplicial tree.

## Consequences for hyperbolic groups

Let $\Gamma$ be a finitely generated group. Let $S$ be a finite generating set. Without loss of generality, we may assume that $S$ is *symmetric*, i.e. $S = S^{-1}$. We define the *word metric* $d_S$ on $\Gamma$ by setting $d_S(\gamma_1, \gamma_2)$ equal to the least integer $n \geq 0$ such that $\gamma_1^{-1}\gamma_2$ can be written as a product of $n$ elements of $S$. Note that this is the metric obtained by viewing $\Gamma$ as the 0-skeleton of its Cayley graph and restricting the graph metric.

**Exercice 6.** Let $S_1, S_2$ be two symmetric generating sets. Show that the identity map $(\Gamma, d_{S_1}) \to (\Gamma, d_{S_2})$ is a quasi-isometry. (Hence it makes sense to say that a group $\Gamma$ is quasi-isometric to some metric space $X$.)

Show that the inclusion of $\Gamma$ into its Cayley graph is a quasi-isometry.

The link between groups and spaces is provided by the following fundamental proposition due independently to Efremovič, Švarc and Milnor. (See e.g. [13, 17, 5].)

**Proposition 5.** *Let $X$ be a proper geodesic metric space. Let $\Gamma$ be a group acting geometrically on $X$. Then $\Gamma$ is finitely generated and quasi-isometric to $X$.*

**Corollary 6.** *If a finitely generated $\Gamma$ is quasi-isometric to some hyperbolic space $X$, then $\Gamma$ is hyperbolic.*

In particular:

**Corollary 7.** *If $\Gamma$ is quasi-isometric to $\mathbf{H}^2$, then $\Gamma$ is hyperbolic.*

*Remarks.*

- What we are looking for is a kind of converse to Proposition 5 in the case where $X = \mathbf{H}^2$. In general, given a metric space $X$, one may ask whether all groups quasi-isometric to $X$ act geometrically on $X$, and if not, what are the groups quasi-isometric to $X$. For a survey on this, see Misha Kapovich's notes [16].

- Hyperbolic groups were introduced by Gromov. The standard references are his seminal paper [15] and the books [7, 13, 14]. Hyperbolic groups are also discussed in the more recent book [5].

- There are striking recent applications of $\delta$-hyperbolic spaces to low-dimensional geometry. Among them, the most famous is probably the work of Brock, Canary and Minsky on the Ending Lamination Conjecture (see [22] for an introduction).

# Lecture 2

## The boundary of a hyperbolic space

In all of this lecture, $(X, d)$ is a proper geodesic metric space.

A *geodesic ray* in $X$ is an isometric embedding of $[0, +\infty)$ into $X$. We say that two geodesic rays $c, c'$ are *asymptotic* if the function $t \mapsto d(c(t), c'(t))$ is bounded.

**Definition.** Let $(X, d)$ be a proper hyperbolic space. Fix a basepoint $p \in X$. The *boundary* of $X$, denoted by $\partial_p X$, is the set of equivalence classes of geodesic rays $c$ in $X$ such that $c(0) = p$, where two rays are equivalent if they are asymptotic.

Using properness of $X$ and the Ascoli-Arzela theorem, it is easy to see that if one chooses a different basepoint $q$, then there is a natural bijection $\partial_p X \to \partial_q X$. Hence it is legitimate to drop the basepoint in the notation. We will henceforth denote the boundary of $X$ simply by $\partial X$.

We are going to define a topology on $X \cup \partial X$. For this, it is convenient to make some more definitions: a *generalized ray* is a map $c : [0, +\infty) \to X$ such that either $c$ is a geodesic ray, or there exists $R \geq 0$ such that the restriction of $c$ to $[0, R]$ is a geodesic segment, and the restriction of $c$ to $[R, +\infty)$ is constant. We consider two generalized rays to be equivalent if either they are asymptotic geodesic rays, or they are eventually constant and equal. Hence we can view the set $X \cup \partial X$ as a quotient of the set $\mathcal{R}_p$ of generalized rays starting at $p$.

**Definition.** We endow $\mathcal{R}_p$ with the compact-open topology, i.e. a fundamental system of neighborhoods of a generalized ray $c$ is given by sets of the form

$$V_\epsilon = \{c' \mid d(c(t), c'(t)) \leq \epsilon \quad \forall t \in [0, \epsilon^{-1}]\}.$$

The space $\bar{X} = X \cup \partial X$ is given the quotient topology.

Hence a sequence $c_n \in \mathcal{R}_p$ converges to $c$ if and only if it converges uniformly on compact subsets.

**Proposition 8.** *The topology on $\bar{X}$ is independent of the choice of basepoint. It is a compact Hausdorff space. The inclusion $X \to \bar{X}$ is an embedding with open image. (Hence $\partial X$ is compact.)*

**Exercice 7.** Prove that $\partial \mathbf{H}^n$ is homeomorphic to the $(n-1)$-sphere $S^{n-1}$.

Let $T$ be a Cayley graph of the free group on two generators. Prove that $\partial T$ is a Cantor set.

**Proposition 9 (Quasi-isometry invariance of the boundary).** *Let $X$ and $Y$ be proper hyperbolic spaces. If $X$ and $Y$ are quasi-isometric, then $\partial X$ and $\partial Y$ are homeomorphic.*

*Remark.* One can show that the boundary of a proper hyperbolic space is always metrizable. This will be assumed implicitly in the sequel. In fact one can construct explicitly a family of metrics on $\partial X$ that induce the topology. This is important, but will not be discussed in these lectures. See [7] or [13].

## From hyperbolic groups to convergence groups

**Definition.** Let $M$ be an infinite compact metrizable topological space and $\Gamma$ be a group acting by homeomorphisms on $M$.

We say that $\Gamma$ *acts as a convergence group* if for each sequence $\{g_n\}$ of distinct elements of $\Gamma$, there exist points $a, b \in M$ and a subsequence $\{g_{n_k}\}$ such that $\lim g_{n_k} \cdot x = a$ uniformly on compact subsets not containing $b$.

**Proposition 10 ([3, 10, 27]).** *Let $X$ be a proper hyperbolic space. Let $\Gamma$ be a group acting properly discontinuously by isometries on $X$. Then $\Gamma$ acts as a convergence group on $\partial X$.*

**Corollary 11.** *Let $\Gamma$ be a group quasi-isometric to $\mathbf{H}^2$. Then $\Gamma$ acts as a convergence group on $S^1$.*

## Basics of convergence groups

Next we discuss an important characterization of convergence groups. Denote by $\Theta(M)$ the set of triples $(x, y, z) \in M^3$ such that $x, y, z$ are pairwise distinct, topologized as a subset of $M^3$.

**Proposition 12 ([3]).** *Let $M$ be an infinite compact metrizable topological space and $\Gamma$ be a group acting by homeomorphisms on $M$. Then $\Gamma$ acts as a convergence group if and only if the induced action of $\Gamma$ on $\Theta(M)$ is properly discontinuous.*

**Exercice 8.** Show that $\Theta(S^1)$ is homeomorphic to two copies of $S^1 \times \mathbf{R}^2$. (Hint: construct a map from $\Theta(S^1)$ to the unit tangent bundle of $\mathbf{H}^2$.)

Let $\Gamma$ act as a convergence group on $M$. An element $\gamma \in \Gamma$ is *elliptic* (resp. *parabolic*, resp. *hyperbolic*) if it has finite order (resp. has a unique fixed point, resp. has exactly two fixed points).

**Proposition 13.** *Any element of $\Gamma$ is either elliptic, parabolic, or hyperbolic.*

*Remarks.*

- We have seen that hyperbolic groups are convergence groups. It is a natural question to characterize hyperbolic groups among convergence groups. Such a characterization has been given by Brian Bowditch [2]. This theorem has been extended by Asli Yaman [29] to a characterization of *relatively hyperbolic groups*. For more on this, see François Dahmani's thesis [8].

- Convergence groups on spheres were introduced by Gehring and Martin [12]. Their original definition is more general: the convergence groups considered in these lectures are what they call *discrete* convergence groups.

# Lecture 3

In this lecture, I will explain the main ideas of the proof of the following 'half' of the Convergence Group Theorem.

**Theorem 14 ([25]).** *Let $\Gamma$ be a group acting as a convergence group on $S^1$. Assume that $\Gamma$ is torsion free, and acts preserving orientation and without parabolic elements. Then $\Gamma$ has a finite normal subgroup $F$ such that $\Gamma/F$ is Fuchsian.*

An *axis* is a pair $(a, b)$ of distinct points of $S^1$. We say that two axes $(a_1, b_1)$, $(a_2, b_2)$ *cross* if $a_2$ and $b_2$ belong to different components of $S^1 \setminus \{a_1, b_1\}$. An axis $A$ is *simple* if for every $g \in \Gamma$, $gA$ does not cross $A$. An axis is *hyperbolic* if it consists of the fixed points of a hyperbolic element of $\Gamma$.

Here is the key lemma:

**Lemma 15.** *Let $\Gamma$ be as in Theorem 14. Then there is a simple hyperbolic axis.*

*Remarks.*

- The idea of finding a simple axis goes back to the work of Nielsen [23], and was further developped by Zieschang [30] in connection with the Nielsen Realization Problem for mapping classes of homeomorphisms of surfaces.

- The paper [25] is rather long and technical. The author has given a nice outline in [26].

# Lecture 4

This lecture will be devoted to the other 'half' of the Convergence Group Theorem.

**Theorem 16.** *Let $\Gamma$ be a group acting as a convergence group on $S^1$. If the action is orientation-preserving and if $\Gamma$ has a torsion element of order at least 3, then $\Gamma$ has a finite normal subgroup $F$ such that $\Gamma/F$ is Fuchsian.*

Of course, Theorems 14 and 16 do not imply the Convergence Group Theorem. However, the methods discussed in the previous lecture extend to cover all cases not covered by Theorem 16 (see [25]) and even some more. Roughly, the proof of Theorem 14 extends without difficulties if there are orientation-reversing elements or elements of order 2. If there are parabolics, one proves the existence of a simple *regular axis* (which may not consist in the fixed points of a hyperbolic element).

I will follow the Casson-Jungreis approach [6]. For a completely different proof, see Gabai's paper [11].

If time permits, I will discuss a proof of the Morse Lemma using ultralimits, following [17].

# References


[1] M. Boileau, S. Maillot, and J. Porti. *Three-dimensional orbifolds and their geometric structures*, volume 15 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2003.

[2] B. H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[3] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[4] B. H. Bowditch. Planar groups and the Seifert conjecture. *J. Reine Angew. Math.*, 576:11–62, 2004.

[5] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[6] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118:441–456, 1994.

[7] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.

[8] F. Dahmani. *Les groupes relativement hyperboliques et leurs bords*. Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 2003/13. Université Louis Pasteur Département de Mathématique Institut de Recherche Mathématique Avancée, Strasbourg, 2003. Thèse, l'Université Louis Pasteur (Strasbourg I), Strasbourg, 2003.

[9] M. J. Dunwoody and E. L. Swenson. The algebraic torus theorem. *Invent. Math.*, 140(3):605–637, 2000.

[10] E. M. Freden. Negatively curved groups have the convergence property. I. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(2):333–348, 1995.

[11] D. Gabai. Convergence groups are Fuchsian groups. *Annals of Math.*, 136:447–510, 1992.

[12] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)*, 55(2):331–358, 1987.

[13] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Birkhäuser Boston Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[14] É. Ghys, A. Haefliger, and A. Verjovsky, editors. *Group theory from a geometrical viewpoint*, River Edge, NJ, 1991. World Scientific Publishing Co. Inc.

[15] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, New York, 1987.

[16] M. Kapovich. Lecture notes on geometric group theory. available at `http://www.math.ucdavis.edu/%7Ekapovich/eprints.html`.

[17] M. Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.

[18] M. Kapovich and B. Kleiner. Coarse Alexander duality and duality groups. Preprint 1999.

[19] S. Maillot. Quasi-isometries of groups, graphs and surfaces. *Comment. Math. Helv.*, 76(1):29–60, 2001.

[20] S. Maillot. Open 3-manifolds whose fundamental groups have infinite center, and a torus theorem for 3-orbifolds. *Trans. Amer. Math. Soc.*, 355(11):4595–4638 (electronic), 2003.

[21] G. Mess. The Seifert conjecture and groups which are coarse quasiisometric to planes. Preprint.

[22] Y. N. Minsky. Combinatorial and geometrical aspects of hyperbolic 3-manifolds. In *Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001)*, volume 299 of *London Math. Soc. Lecture Note Ser.*, pages 3–40. Cambridge Univ. Press, Cambridge, 2003.

[23] J. Nielsen. *Jakob Nielsen: collected mathematical papers.* Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, MA, 1986. Edited and with a preface by Vagn Lundsgaard Hansen.

[24] P. Scott. There are no fake Seifert fibered spaces with infinite $\pi_1$. *Annals of Math.*, 117:35–70, 1983.

[25] P. Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.

[26] P. Tukia. Homeomorphic conjugates of Fuchsian groups: an outline. In *Complex analysis, Joensuu 1987*, volume 1351 of *Lecture Notes in Math.*, pages 344–353. Springer, Berlin, 1988.

[27] P. Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.

[28] F. Waldhausen. Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten. *Topology*, 6:505–517, 1967.

[29] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[30] H. Zieschang. *Finite groups of mapping classes of surfaces*, volume 875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[31] H. Zieschang, E. Vogt, and H.-D. Coldewey. *Surfaces and planar discontinuous groups*, volume 835 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Translated from the German by John Stillwell.

[32] H. Zieschang and B. Zimmermann. Über Erweiterungen von $\mathbf{Z}$ und $Z_2 * Z_2$ durch nichteuklidische kristallographische Gruppen. *Math. Ann.*, 259(1):29–51, 1982.