Volumetric expressions of the shape gradient of the compliance in structural shape optimization

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October 6, 2016 - IHP quarter on Numerical Methods for PDEs
Advanced Numerical Methods: Recent Developments, Analysis, and Applications

Joint work with **O. Pantz** (LJAD Université Nice-Sophia Antipolis) and **K. Trabelsi** (IPSA)

Outline

1. The linear elasticity equation
   - A problem in structural shape optimization
   - Surface and volumetric expressions of the shape gradient
   - The Boundary Variation Algorithm

2. Minimization of the compliance under a volume constraint
   - Shape gradient using the pure displacement formulation
   - Shape gradient using a dual mixed variational formulation
   - A preliminary experimental comparison

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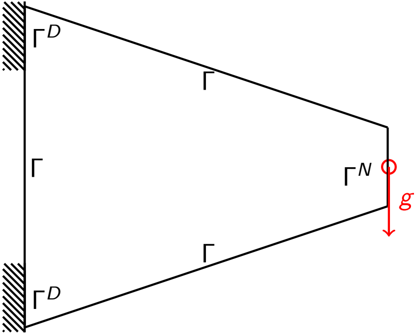
The linear elasticity equation

- $\Omega \subset \mathbb{R}^d$ open connected domain;
- $\partial\Omega = \Gamma^N \cup \Gamma \cup \Gamma^D$, $\mathcal{H}^{d-1}(\Gamma^D) > 0$;
- Γ^N, Γ and Γ^D are disjoint.

$$\begin{cases} -\nabla \cdot \sigma_\Omega = f & \text{in } \Omega \\ \sigma_\Omega = Ae(u_\Omega) & \text{in } \Omega \\ \sigma_\Omega n = g & \text{on } \Gamma^N \\ \sigma_\Omega n = 0 & \text{on } \Gamma \\ u_\Omega = 0 & \text{on } \Gamma^D \end{cases}$$



Hooke's law for a linear elastic material: $Ae(u_\Omega) = 2\mu e(u_\Omega) + \lambda \operatorname{tr}(e(u_\Omega)) \operatorname{Id}$

Compliance:

$$J(\Omega) := j(\Omega, \sigma_\Omega) = \int_\Omega A^{-1}\sigma_\Omega : \sigma_\Omega \, dx$$

A problem in structural shape optimization

Minimization of the compliance under a volume constraint: $\min_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$

$$\mathcal{U}_{ad} = \{\Omega \subset \mathbb{R}^d \ : \sigma_\Omega \text{ is the stress tensor fulfilling the linear elasticity equation on } \Omega \text{ and } V(\Omega) = |\Omega| \text{ is fixed}\}.$$

Gradient-based shape optimization:

A problem in structural shape optimization

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Gradient-based shape optimization:

```
Given the domain Ω_0, set j = 0 and iterate:
1. Compute the solution of the state equation;
2. Compute a descent direction θ_j and an admissible step μ_j;
3. Update the domain Ω_{j+1} = (Id + μ_j θ_j)Ω_j;
4. Until the stopping criterion is not fulfilled, j = j + 1 and repeat.
```

A problem in structural shape optimization

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```

Classical optimization

$$\min_{x \in \mathbb{R}^n} f(x) \quad , \quad f : \mathbb{R}^n \to \mathbb{R}$$

Gradient-based descent direction in x: $\boxed{v \text{ s.t. } (\nabla f(x), v) < 0}$

$$x_{j+1} = x_j + \mu_j \ v_j$$

Shape optimization

$$\min_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$$

Gradient-based descent direction at Ω: $\boxed{\theta \text{ s.t. } \langle dJ(\Omega), \theta \rangle < 0}$

$$\Omega_{j+1} = (\mathrm{Id} + \mu_j \theta_j)\Omega_j$$

Shape gradient

Let $\theta \in X$ be an admissible smooth deformation of Ω. The objective functional J is said to be X-differentiable at $\Omega \in \mathcal{U}_{ad}$ if there exists a continuous linear form $dJ(\Omega)$ on X such that $\forall \theta \in X$ we have:

$$J((\mathrm{Id} + \theta)\Omega) = J(\Omega) + \langle dJ(\Omega), \theta \rangle + o(\theta)$$

Shape gradient

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Surface expression

$$\langle dJ(\Omega), \theta \rangle = \int_{\partial\Omega} h_1(u_\Omega, \sigma_\Omega, \cdot)\theta \cdot n \; ds$$

Volumetric expression

$$\langle dJ(\Omega), \theta \rangle = \int_{\Omega} h_2(u_\Omega, \sigma_\Omega, \cdot, \theta) \; dx$$

Computing a descent direction

Descent direction: θ such that $\langle dJ(\Omega), \theta \rangle < 0$

Surface expression of the shape gradient:

$$\langle dJ(\Omega), \theta \rangle = \int_{\partial\Omega} h_1(u_\Omega, \sigma_\Omega, \cdot)\theta \cdot n \, ds$$

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The Boundary Variation Algorithm

Given the domain Ω_0, set tol > 0, $j = 0$ and iterate:
1. Compute the solution of the state equation;
2. Compute the solution of the adjoint equation;
3. Compute a descent direction $\theta_j \in X$;
4. Identify an admissible step μ_j;
5. Update the domain $\Omega_{j+1} = (\mathrm{Id} + \mu_j \theta_j)\Omega_j$;
6. While $\langle dJ(\Omega_j), \theta_j \rangle >$ tol, $j = j + 1$ and repeat.

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Descent direction: θ such that $\langle dJ(\Omega), \theta \rangle < 0$

Surface expression of the shape gradient:

$$\langle dJ(\Omega), \theta \rangle = \int_{\partial\Omega} h_1(u_\Omega, \sigma_\Omega, \cdot)\theta \cdot n \; ds \quad \Longrightarrow \quad \theta = -h_1(u_\Omega, \sigma_\Omega, \cdot) n \text{ on } \partial\Omega$$

Volumetric expression of the shape gradient:

$$\langle dJ(\Omega), \theta \rangle = \int_{\Omega} h_2(u_\Omega, \sigma_\Omega, \cdot, \theta) \; dx \qquad \Longrightarrow \quad \theta = ?$$

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1. Compute the solution of the state equation;
2. Compute the solution of the adjoint equation;
3. Compute a descent direction $\theta_j \in X$ s.t. $(\theta_j, \delta\theta)_X + \langle dJ(\Omega_j), \delta\theta\rangle = 0 \ \forall \delta\theta \in X$;
4. Identify an admissible step μ_j;
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6. While $\langle dJ(\Omega_j), \theta_j\rangle >$ tol, $j = j + 1$ and repeat.

Outline

Minimization of the compliance under a volume constraint

We introduce a transformation $X_\theta : \mathbb{R}^d \to \mathbb{R}^d$ and we define the open subset $\Omega_\theta \subset \mathbb{R}^d$ as $\Omega_\theta = X_\theta(\Omega)$ where $\Gamma^N_\theta = X_\theta(\Gamma^N)$, $\Gamma_\theta = X_\theta(\Gamma)$ and $\Gamma^D_\theta = X_\theta(\Gamma^D)$.

Perturbation of the identity map: $X_\theta = \mathrm{Id} + \theta + o(\theta) \quad , \quad \theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$.

Jacobian of X_θ: $D_\theta = \mathrm{Id} + \nabla\theta + o(\nabla\theta) \quad , \quad I_\theta := \det D_\theta$

Under the assumption of a small perturbation θ, X_θ is a diffeomorphism and belongs to the space

$$\mathcal{X} := \left\{ X_\theta \ : \ (X_\theta - \mathrm{Id}) \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d) \text{ and } (X_\theta^{-1} - \mathrm{Id}) \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d) \right\}.$$

Minimization of the compliance under a volume constraint: $\min_{\Omega_\theta \in \mathcal{U}_{ad}} J(\Omega_\theta)$

$$\mathcal{U}_{ad} = \{\Omega_\theta \ : \exists X_\theta \in \mathcal{X} \ , \ \Omega_\theta = X_\theta(\Omega) \ , \ \sigma_{\Omega_\theta} \text{ is the stress tensor fulfilling the linear elasticity equation on } \Omega_\theta \text{ and } V(\Omega_\theta) = |\Omega|\}$$

$$\Downarrow$$

$$\boxed{\min_{\Omega_\theta \in \mathcal{U}_{ad}} L(\Omega_\theta) \quad , \quad L(\Omega_\theta) = J(\Omega_\theta) + \gamma V(\Omega_\theta)}$$

Minimization of the compliance under a volume constraint

We introduce a transformation $X_\theta : \mathbb{R}^d \to \mathbb{R}^d$ and we define the open subset $\Omega_\theta \subset \mathbb{R}^d$ as $\Omega_\theta = X_\theta(\Omega)$ where $\Gamma_\theta^N = X_\theta(\Gamma^N)$, $\Gamma_\theta = X_\theta(\Gamma)$ and $\Gamma_\theta^D = X_\theta(\Gamma^D)$.

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Minimization of the compliance under a volu

$$\mathcal{U}_{ad} = \{ \Omega_\theta \ : \exists X_\theta \in \mathcal{X} \ , \ \Omega_\theta = X_\theta(\Omega) \ , \ \sigma_{\Omega_\theta}$$

elasticity equation on Ω_θ and

$$\Downarrow$$

$$\min_{\Omega_\theta \in \mathcal{U}_{ad}} L(\Omega_\theta) \quad , \quad L(\Omega_\theta) =$$

The volume of Ω_θ is a purely geometrical quantity and does not depend on the solution of the state problem.

$$V(\Omega_\theta) = \int_{\Omega_\theta} dx_\theta = \int_\Omega I_\theta \ dx =: \upsilon(\theta)$$

$$\det(\text{Id} + C) = 1 + \text{tr}(C) + o(C)$$

$$\langle dV(\Omega), \theta \rangle = \lim_{\theta \searrow 0} \frac{V(\Omega_\theta) - V(\Omega)}{\theta} = \lim_{\theta \searrow 0} \frac{\upsilon(\theta) - \upsilon(0)}{\theta} =: \upsilon'(0)$$

$$\Downarrow$$

$$\langle dV(\Omega), \theta \rangle = \int_\Omega \nabla \cdot \theta \ dx = \int_{\partial\Omega} \theta \cdot n \ ds$$

The pure displacement formulation

Data: $f \in H^1(\mathbb{R}^d;\mathbb{R}^d)$, $g \in H^2(\mathbb{R}^d;\mathbb{R}^d)$
Functional space for the variational formulation:

$$V_\Omega := H^1_{0,\Gamma^D}(\Omega;\mathbb{R}^d) = \{v \in H^1(\Omega;\mathbb{R}^d) \ : \ v = 0 \text{ on } \Gamma^D\}$$

State problem: We seek $u_\Omega \in V_\Omega$ s.t. $a_\Omega(u_\Omega, \delta u) = F_\Omega(\delta u) \quad \forall \delta u \in V_\Omega$

$$a_\Omega(u_\Omega, \delta u) := \int_\Omega Ae(u_\Omega) : e(\delta u)\ dx \quad , \quad F_\Omega(\delta u) := \int_\Omega f \cdot \delta u\ dx + \int_{\Gamma^N} g \cdot \delta u\ ds.$$

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Compliance:

$$J(\Omega) = \int_\Omega A^{-1}\sigma_\Omega : \sigma_\Omega\ dx = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\ dx = \int_\Omega f \cdot u_\Omega\ dx + \int_{\Gamma^N} g \cdot u_\Omega\ ds$$

$$\Downarrow$$

$$J_1(\Omega_\theta) := -\min_{u_{\Omega_\theta} \in V_{\Omega_\theta}} \int_{\Omega_\theta} Ae(u_{\Omega_\theta}) : e(u_{\Omega_\theta})\ dx_\theta - 2\int_{\Omega_\theta} f \cdot u_{\Omega_\theta}\ dx_\theta - 2\int_{\Gamma^N_\theta} g \cdot u_{\Omega_\theta}\ ds_\theta =: j_1(\theta)$$

Shape gradient of the compliance:

$$\langle dJ_1(\Omega), \theta \rangle := \lim_{\theta \searrow 0} \frac{J_1(\Omega_\theta) - J_1(\Omega)}{\theta} = \lim_{\theta \searrow 0} \frac{j_1(\theta) - j_1(0)}{\theta} =: j_1'(0)$$

Shape gradient using the pure displacement formulation

Transformation: $\mathcal{P}_\theta : H^1_{0,\Gamma^D}(\Omega;\mathbb{R}^d) \to H^1_{0,\Gamma^D_\theta}(\Omega_\theta;\mathbb{R}^d) \quad , \quad v_{\Omega_\theta} = \mathcal{P}_\theta(v_\Omega) = v_\Omega \circ X_\theta^{-1}$

Lemma

Let $u_\Omega \in H^1_{0,\Gamma^D}(\Omega;\mathbb{R}^d)$. We consider $u_{\Omega_\theta} = \mathcal{P}_\theta(u_\Omega)$. It follows that

$$\frac{1}{2}\left(\nabla_{x_\theta} u_{\Omega_\theta} + \nabla_{x_\theta} u_{\Omega_\theta}^T\right) =: e_{x_\theta}(u_{\Omega_\theta}) = \frac{1}{2}\left(\nabla_x u_\Omega D_\theta^{-1} + D_\theta^{-T}\nabla_x u_\Omega^T\right)$$

where ∇_{x_θ} (respectively ∇_x) represents the gradient with respect to the coordinate of the deformed (respectively reference) domain.

Compliance:

$$j_1(\theta) = -\min_{u_\Omega \in V_\Omega} \int_\Omega A\left(\frac{1}{2}\left(\nabla u_\Omega D_\theta^{-1} + D_\theta^{-T}\nabla u_\Omega^T\right)\right) : \left(\frac{1}{2}\left(\nabla u_\Omega D_\theta^{-1} + D_\theta^{-T}\nabla u_\Omega^T\right)\right) I_\theta \, dx$$
$$- 2\int_\Omega f \circ X_\theta \cdot u_\Omega \; I_\theta \, dx - 2\int_{\Gamma^N} g \circ X_\theta \cdot u_\Omega \,\mathrm{Cof}\, D_\theta \, ds.$$

Shape gradient using the pure displacement formulation

Transformation: $\mathcal{P}_\theta : H^1_{0,\Gamma^D}(\Omega;\mathbb{R}^d) \to H^1_{0,\Gamma^D_\theta}(\Omega_\theta;\mathbb{R}^d) \quad , \quad v_{\Omega_\theta} = \mathcal{P}_\theta(v_\Omega) = v_\Omega \circ X_\theta^{-1}$

Lemma

Let $u_\Omega \in H^1_{0,\Gamma^D}(\Omega;\mathbb{R}^d)$. We consider $u_{\Omega_\theta} = \mathcal{P}_\theta$

$$\frac{1}{2}\left(\nabla_{x_\theta} u_{\Omega_\theta} + \nabla_{x_\theta} u_{\Omega_\theta}^T\right) =: e_{x_\theta}(u_{\Omega_\theta})$$

where ∇_{x_θ} (respectively ∇_x) represents the gra
the deformed (respectively reference) domain.

We recall that $X_\theta = \mathrm{Id} + \theta + o(\theta)$. Hence:

$$D_\theta = \mathrm{Id} + \nabla\theta + o(\nabla\theta)$$
$$D_\theta^T = \mathrm{Id} + \nabla\theta^T + o(\nabla\theta)$$
$$D_\theta^{-1} = \mathrm{Id} - \nabla\theta + o(\nabla\theta)$$
$$\det(\mathrm{Id} + C) = 1 + \mathrm{tr}(C) + o(C)$$
$$\mathrm{Cof}(\mathrm{Id} + C) = \mathrm{Id} + \mathrm{tr}(C)\,\mathrm{Id} - C + o(C)$$

Compliance:

$$j_1(\theta) = -\min_{u_\Omega \in V_\Omega} \int_\Omega A\left(\frac{1}{2}\left(\nabla u_\Omega D_\theta^{-1} + D_\theta^{-T}\nabla u_\Omega^T\right)\right) : \left(\frac{1}{2}\left(\nabla u_\Omega D_\theta^{-1} + D_\theta^{-T}\nabla u_\Omega^T\right)\right) I_\theta \; dx$$
$$- 2\int_\Omega f \circ X_\theta \cdot u_\Omega \; I_\theta \; dx - 2\int_{\Gamma^N} g \circ X_\theta \cdot u_\Omega \,\mathrm{Cof}\, D_\theta \; ds.$$

Shape gradient of the compliance (By differentiating $j_1(\theta)$ w.r.t θ in $\theta = 0$):

$$\langle dJ_1(\Omega), \theta\rangle = \int_\Omega Ae(u_\Omega) : \left(\nabla u_\Omega \nabla\theta + \nabla\theta^T \nabla u_\Omega^T\right) \; dx - \int_\Omega Ae(u_\Omega) : e(u_\Omega)(\nabla\cdot\theta) \; dx$$
$$+ 2\int_\Omega (\nabla f\theta \cdot u_\Omega + f \cdot u_\Omega(\nabla\cdot\theta))\,dx + 2\int_{\Gamma^N} (\nabla g\theta \cdot u_\Omega + g \cdot u_\Omega\,(\nabla\cdot\theta - \nabla\theta n \cdot n))\,ds$$

A dual mixed formulation with weakly-enforced symmetry of the stress tensor

Functional spaces for the variational formulation:

$$H(\mathrm{div},\Omega;\mathbb{M}_d) := \{\tau \in L^2(\Omega;\mathbb{M}_d) \ : \ \nabla\cdot\tau \in L^2(\Omega;\mathbb{R}^d)\}$$

$$\Sigma_\Omega := \{\tau \in H(\mathrm{div},\Omega;\mathbb{M}_d) \ : \ \tau n = g \text{ on } \Gamma^N \text{ and } \tau n = 0 \text{ on } \Gamma\}$$

$$\Sigma_{\Omega,0} := \{\tau \in H(\mathrm{div},\Omega;\mathbb{M}_d) \ : \ \tau n = 0 \text{ on } \Gamma^N \cup \Gamma\}$$

$$V_\Omega := L^2(\Omega;\mathbb{R}^d) \quad , \quad Q_\Omega := L^2(\Omega;\mathbb{K}_d) \quad , \quad W_\Omega := V_\Omega \times Q_\Omega$$

State problem: We seek $(\sigma_\Omega,(u_\Omega,\eta_\Omega)) \in \Sigma_\Omega \times W_\Omega$ such that

$$\begin{aligned} a_\Omega(\sigma_\Omega,\delta\sigma) + b_\Omega(\delta\sigma,(u_\Omega,\eta_\Omega)) &= 0 \qquad && \forall \delta\sigma \in \Sigma_{\Omega,0} \\ b_\Omega(\sigma_\Omega,(\delta u,\delta\eta)) &= F_\Omega(\delta u) && \forall (\delta u,\delta\eta) \in W_\Omega \end{aligned}$$

$$a_\Omega(\sigma_\Omega,\delta\sigma) := \int_\Omega A^{-1}\sigma_\Omega : \delta\sigma \ dx$$

$$b_\Omega(\sigma_\Omega,(\delta u,\delta\eta)) := \int_\Omega (\nabla\cdot\sigma_\Omega)\cdot\delta u \ dx + \frac{1}{2\mu}\int_\Omega \sigma_\Omega : \delta\eta \ dx,$$

$$F_\Omega(\delta u) := -\int_\Omega f\cdot\delta u \ dx.$$

Shape gradient using a dual mixed formulation I

Compliance:

$$J_3(\Omega_\theta) := \inf_{\sigma_{\Omega_\theta} \in \Sigma_{\Omega_\theta}} \sup_{(u_{\Omega_\theta}, \eta_{\Omega_\theta}) \in W_{\Omega_\theta}} \int_{\Omega_\theta} A^{-1} \sigma_{\Omega_\theta} : \sigma_{\Omega_\theta} \, dx_\theta + \int_{\Omega_\theta} (\nabla \cdot \sigma_{\Omega_\theta} + f) \cdot u_{\Omega_\theta} \, dx_\theta + \frac{1}{2\mu} \int_{\Omega_\theta} \sigma_{\Omega_\theta} : \eta_{\Omega_\theta} \, dx_\theta =: j_3(\theta)$$

Mapping $H(\text{div}, \Omega_\theta; \mathbb{M}_d)$ to $H(\text{div}, \Omega; \mathbb{M}_d)$

A key aspect of this transformation is the preservation of the normal traces of the tensors under analysis. $\Longrightarrow$ Special isomorphism known as **contravariant Piola transform**.

Transformations:

$$\mathcal{Q}_\theta : H(\text{div}, \Omega; \mathbb{M}_d) \to H(\text{div}, \Omega_\theta; \mathbb{M}_d) \quad , \quad \tau_{\Omega_\theta} = \mathcal{Q}_\theta(\tau_\Omega) = \frac{1}{I_\theta} D_\theta \tau_\Omega \circ X_\theta^{-1} D_\theta^T$$

$$\mathcal{R}_\theta : L^2(\Omega; \mathbb{R}^d) \to L^2(\Omega_\theta; \mathbb{R}^d) \quad , \quad v_{\Omega_\theta} = \mathcal{R}_\theta(v_\Omega) = D_\theta^{-T} v_\Omega \circ X_\theta^{-1}$$

Lemma

Let $\sigma_\Omega \in H(\text{div}, \Omega; \mathbb{M}_d)$. We consider $\sigma_{\Omega_\theta} = \mathcal{Q}_\theta(\sigma_\Omega)$. It follows that

$$\nabla_{x_\theta} \cdot \sigma_{\Omega_\theta} = \frac{1}{I_\theta} D_\theta \nabla_x \cdot \sigma_\Omega$$

where $\nabla_{x_\theta} \cdot$ (respectively $\nabla_x \cdot$) represents the divergence with respect to the coordinate of the deformed (respectively reference) domain.

Shape gradient using a dual mixed formulation II

Compliance:

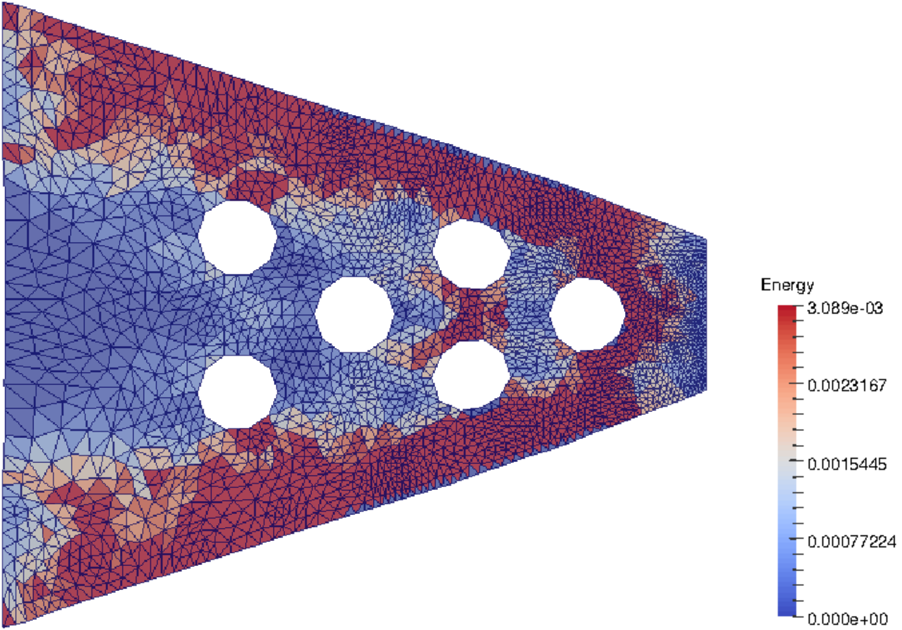
$$
\begin{aligned}
j_3(\theta) = \inf_{\sigma_\Omega \in \Sigma_\Omega} \sup_{(u_\Omega, \eta_\Omega) \in W_\Omega} & \frac{1}{2\mu} \int_\Omega \frac{1}{I_\theta} D_\theta^T D_\theta \sigma_\Omega D_\theta^T D_\theta : \sigma_\Omega \; dx \\
& - \frac{\lambda}{2\mu(d\lambda + 2\mu)} \int_\Omega \frac{1}{I_\theta} \operatorname{tr}\left(D_\theta^T D_\theta \sigma_\Omega\right) \operatorname{tr}\left(D_\theta^T D_\theta \sigma_\Omega\right) dx \\
& + \frac{1}{2\mu} \int_\Omega \frac{1}{I_\theta} D_\theta^T D_\theta \sigma_\Omega D_\theta^T D_\theta : \eta_\Omega \; dx \\
& + \int_\Omega (\nabla \cdot \sigma_\Omega) \cdot u_\Omega \; dx + \int_\Omega f \circ X_\theta \cdot \left(D_\theta^{-T} u_\Omega\right) I_\theta \; dx.
\end{aligned}
$$

Shape gradient of the compliance:

$$
\begin{aligned}
\langle dJ_3(\Omega), \theta \rangle = & \frac{1}{2\mu} \int_\Omega \left(N(\theta)\sigma_\Omega : \sigma_\Omega + \sigma_\Omega N(\theta) : \sigma_\Omega\right) dx \\
& - \frac{\lambda}{2\mu(d\lambda + 2\mu)} \int_\Omega 2 \operatorname{tr}\left(N(\theta)\sigma_\Omega\right) \operatorname{tr}\left(\sigma_\Omega\right) dx \\
& + \frac{1}{2\mu} \int_\Omega \left(N(\theta)\sigma_\Omega : \eta_\Omega + \sigma_\Omega N(\theta) : \eta_\Omega\right) dx \\
& + \int_\Omega \left(\nabla f \theta \cdot u_\Omega + f \cdot u_\Omega (\nabla \cdot \theta) - f \cdot (\nabla \theta^T u_\Omega)\right) dx,
\end{aligned}
$$

where $N(\theta) := \nabla\theta + \nabla\theta^T - \frac{1}{2}(\nabla \cdot \theta)\,\mathrm{Id}\,.$

A cantilever with six holes ($V_0 = 40.59,\ \gamma_0 = 0.13$)



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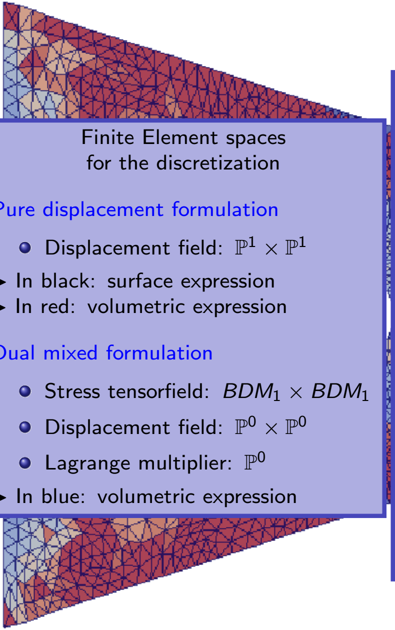


Finite Element spaces
for the discretization

Pure displacement formulation

- Displacement field: $\mathbb{P}^1 \times \mathbb{P}^1$

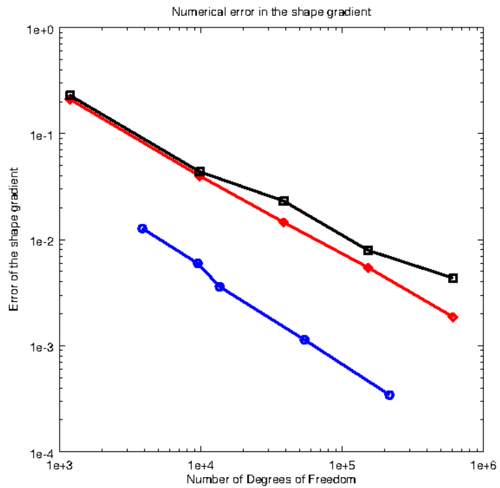
▶ In black: surface expression

▶ In red: volumetric expression

Dual mixed formulation

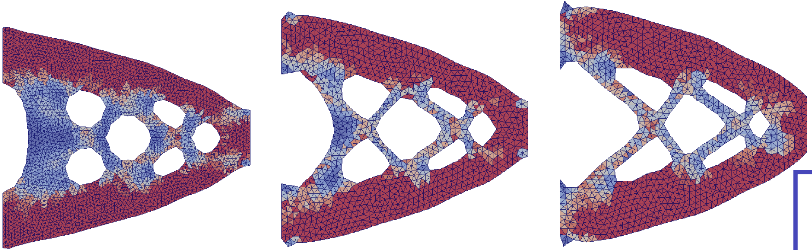
- Stress tensorfield: $BDM_1 \times BDM_1$
- Displacement field: $\mathbb{P}^0 \times \mathbb{P}^0$
- Lagrange multiplier: $\mathbb{P}^0$

▶ In blue: volumetric expression

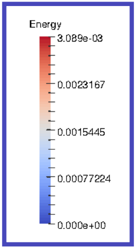


A cantilever with six holes ($V_0 = 40.59,\ \gamma_0 = 0.13$)

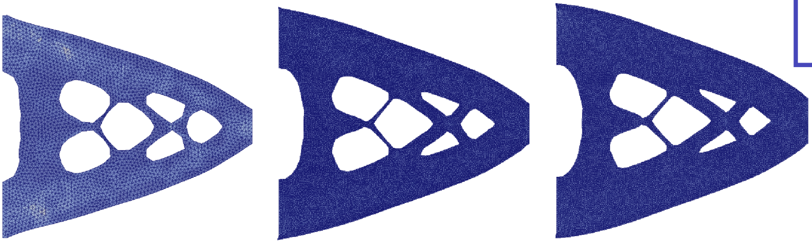
Pure displacement formulation:



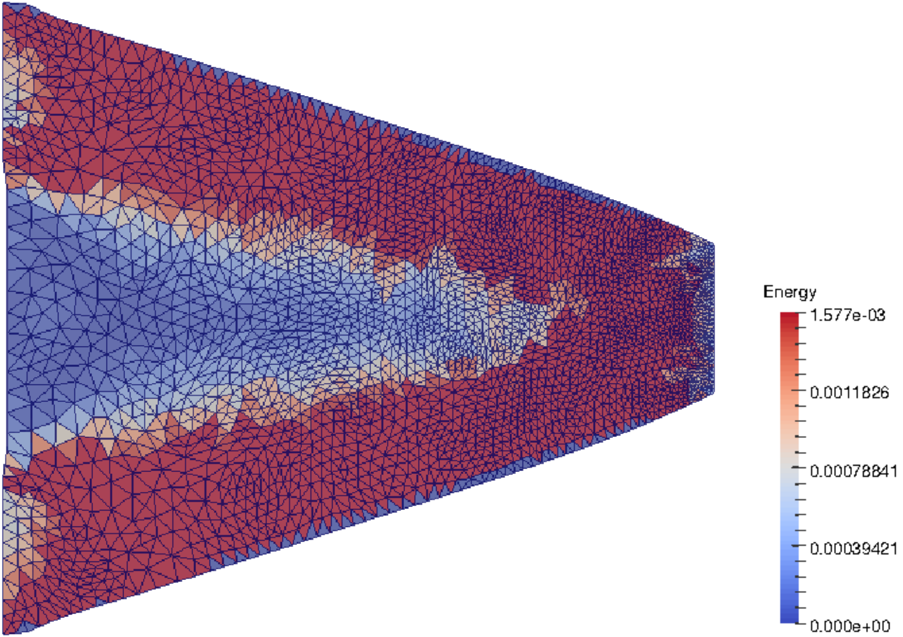
10 iterations 20 iterations 30 iterations



Dual mixed formulation:

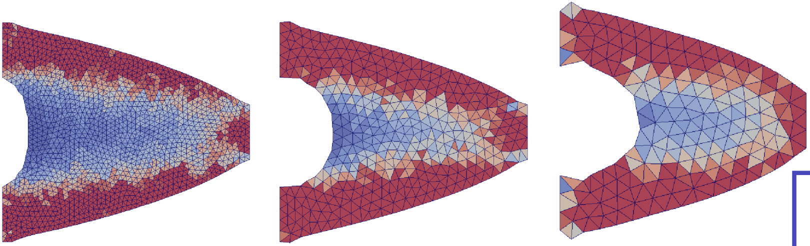


A bulky cantilever ($V_0 = 45$, $\gamma_0 = 0.1$)

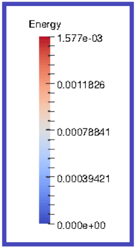


A bulky cantilever ($V_0 = 45$, $\gamma_0 = 0.1$)

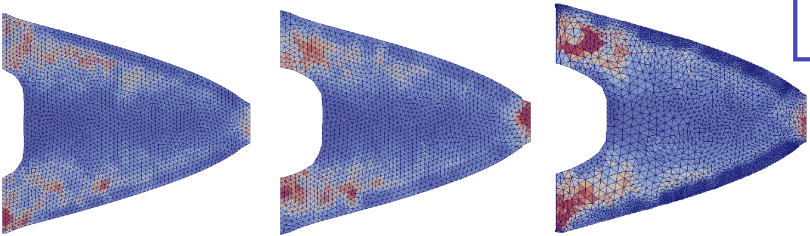
Pure displacement formulation:



10 iterations | 20 iterations | 30 iterations



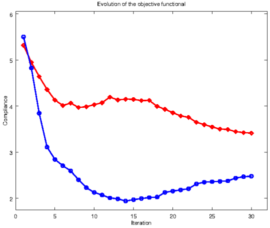
Dual mixed formulation:

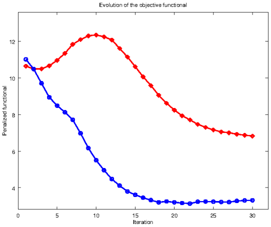


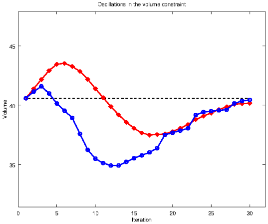
A preliminary experimental comparison

Pure displacement (in red) VS dual mixed (in blue) formulations

Cantilever with six holes:

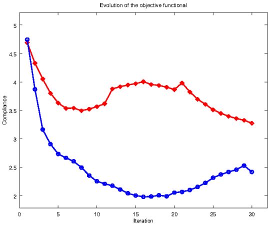


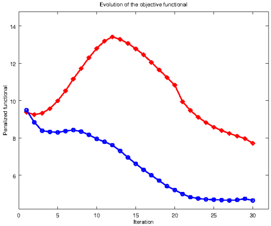


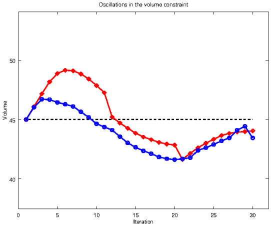


$J(\Omega)$ $\qquad$ $L(\Omega) = J(\Omega) + \gamma V(\Omega)$ $\qquad$ $V(\Omega)$

Bulky cantilever:







Conclusions

From the experimental results:

- Better convergence rate using the volumetric shape gradient.
- More robust approach using the dual mixed formulation of the problem.
- Configurations with lower compliance (and elastic energy) are obtained starting from the dual mixed formulation.
- The dual mixed formulation seems to provide better convergence rate than the pure displacement one.

Ongoing and future investigations:

- Proof of the equivalence of the volumetric expressions in the continuous framework
- A priori estimate of the error due to the numerical approximation of the shape gradient