Schemes for Flows in Porous Media with Fractures

Konstantin Brenner[1], Jérome Droniou[2], Julian Hennicker[1,3], Roland Masson[1], Pierre Samier[3]

[1] Laboratoire J.A. Dieudonné, Université Nice Sophia Antipolis, and team COFFEE, INRIA Sophia Antipolis, France
[2] School of Mathematical Sciences, Monash University, Melbourne, Australia
[3] Total SA, CSTJF, Pau, France

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Outline

- Discrete Fracture Model (DFM)
- The Gradient Discretization Framework; Convergence Result
- Numerical Results

Discrete Fracture Model (DFM)

generic (non reduced) model

Find phase pressures u^1, u^2 and velocities $\mathbf{q}^1, \mathbf{q}^2$:

$$\left\{ \begin{array}{ll} \text{Darcy law:} & \mathbf{q}^\alpha = -k^\alpha(\mathbf{x}, S^\alpha(\mathbf{x}, p))\ \Lambda(\mathbf{x})\nabla u^\alpha \\ \text{Volume conservation:} & \phi(\mathbf{x})\partial_t S^\alpha(\mathbf{x}, p) + \mathrm{div}(\mathbf{q}^\alpha) = 0 \\ \text{Closure equations:} & p = u^2 - u^1, \\ & S^1(\mathbf{x}, p) + S^2(\mathbf{x}, p) = 1 \end{array} \right.$$

α : phase parameter $(1 = w,\ 2 = nw)$

p : capillary pressure

Λ : permeability tensor

k^α : phase mobility

S^α : phase saturation

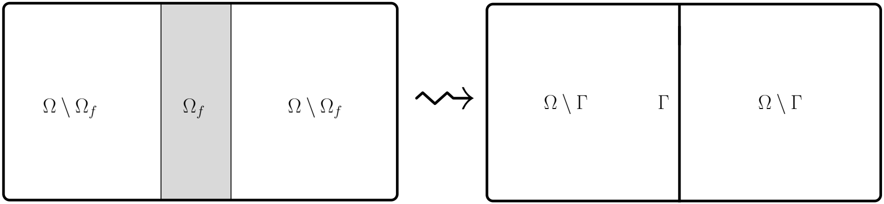
ϕ : porosity

ρ^α : phase mass density

Dimensional Hybridizing: Geometry

- Dimensional hybridizing = Reducing the fracture dimension by 1

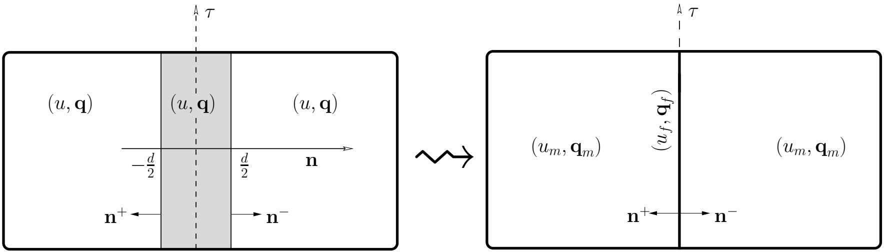
equi-dimensional model: hybrid-dimensional model:



Dimensional Hybridizing: Averaging over the fracture width

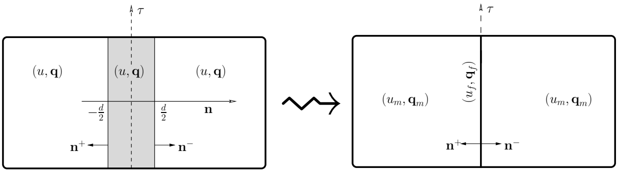
- Dimensional hybridizing = Averaging the model equations over the fracture width

equi-dimensional model: hybrid-dimensional model:



[Alboin et al. 02], [Masson et al. 03], [Jaffré et al. 05]

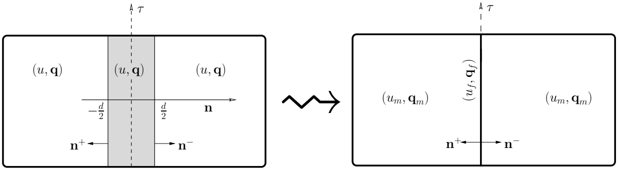
Dimensional Hybridizing: Matrix Equations



- Hybrid dimensional matrix domain $\Omega \setminus \Gamma$:

$$\begin{cases} \text{Darcy law:} & \mathbf{q}_m^\alpha = -k_m^\alpha(S_m^\alpha(p_m))\, \Lambda_m \nabla u^\alpha \\ \text{Volume conservation:} & \phi_m \partial_t S_m^\alpha(p_m) + \mathrm{div}(\mathbf{q}_m^\alpha) = 0 \end{cases}$$

Dimensional Hybridizing: Averaging over the fracture width



$$\Lambda_f = \begin{pmatrix} \Lambda_{f,\tau} & 0 \\ 0 & \Lambda_{f,n} \end{pmatrix} \text{ in } (\tau, n) \text{ coordinates}$$

$$\mathbf{q} = -k(S(p))\,\Lambda \nabla u$$

$$= \underbrace{-k(S(p))\,\Lambda_{f,\tau} \nabla_\tau u}_{\substack{\text{tangential flux } \mathbf{q}_\tau \\ \rightsquigarrow\ \mathbf{q}_f}} \quad \underbrace{-k(S(p))\,\Lambda_{f,n} \partial_n u \mathbf{n}}_{\substack{\text{normal flux } \mathbf{q}\cdot\mathbf{n}\mathbf{n} \\ \rightsquigarrow \\ \text{interfacial transmission conditions}}}$$

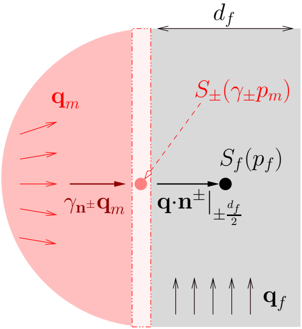
Dimensional Hybridizing: Fracture Equations

$$\mathbf{q} = \underbrace{-k(S(p))\,\Lambda_{f,\tau}\nabla_\tau u}_{\mathbf{q}_\tau \;\rightsquigarrow\; \mathbf{q}_f} \quad \underbrace{-k(S(p))\,\Lambda_{f,n}\partial_n u\mathbf{n}}_{\mathbf{q}\cdot\mathbf{nn} \;\rightsquigarrow\; \text{interf. transmission cond.}}$$

- $u_f = \dfrac{1}{d_f}\displaystyle\int_{-\frac{d_f}{2}}^{\frac{d_f}{2}} u \,\mathrm{d}\mathbf{n}$
- $\mathbf{q}_f = \displaystyle\int_{-\frac{d_f}{2}}^{\frac{d_f}{2}} \mathbf{q}_\tau \,\mathrm{d}\mathbf{n} \approx -d_f k_f(S_f(p_f))\,\Lambda_f \nabla_\tau u_f$ (Darcy Law)
- $d_f\phi_f\partial_t S_f(p_f) + \mathrm{div}_\tau(\mathbf{q}_f) + \mathbf{q}\cdot\mathbf{n}^+\Big|_{+\frac{d_f}{2}} + \mathbf{q}\cdot\mathbf{n}^-\Big|_{-\frac{d_f}{2}} = 0$ (Conservation Equation)

Transmission conditions at the matrix fracture interface

$$\mathbf{q} = \underbrace{-k(S(p))\,\Lambda_{f,\tau}\nabla_\tau u}_{\mathbf{q}_\tau \;\rightsquigarrow\; \mathbf{q}_f} \quad \underbrace{-k(S(p))\,\Lambda_{f,n}\partial_n u\mathbf{n}}_{\mathbf{q}\cdot\mathbf{n}\mathbf{n} \;\rightsquigarrow\; \text{interf. transmission cond.}}$$



- $\mathbf{q}\cdot\mathbf{n}^\pm\Big|_{\pm\frac{d_f}{2}} \approx$

$$k_\pm(S_\pm(\gamma_\pm p_m))\Lambda_{f,\mathbf{n}}\frac{[\![u]\!]_\pm^+}{d_f/2} + k_f(S_f(p_f))\Lambda_{f,\mathbf{n}}\frac{[\![u]\!]_\pm^-}{d_f/2}$$

- $\gamma_{\mathbf{n}^\mp}\mathbf{q}_m + \mathbf{q}\cdot\mathbf{n}^\pm\Big|_{\pm\frac{d_f}{2}} = \eta\partial_t S_\pm(\gamma_\pm p_m)$

$\gamma_\pm$ trace operators at Γ; $\quad [\![u]\!]_\pm = \gamma_\pm u_m - u_f$

$\eta \in \mathbb{R}^+$; $\quad S_\pm = \theta S_m + (1-\theta)S_f$, i.e.

Weak Formulation of the Hybrid Dim Model

$$
\sum_{\mu\in\{\Omega,\Gamma\}} \Big\{ -\int_0^T \int_\mu d_\mu \phi_\mu S_\mu(p_\mu) \partial_t \varphi_\mu + \int_0^T \int_\mu d_\mu k_\mu(S_\mu(p_\mu))\, \Lambda_\mu \nabla u_\mu \cdot \nabla \varphi_\mu \Big\}
$$

$$
+ \sum_{\pm} \Big\{ -\int_0^T \int_\Gamma \eta S_\pm(\gamma_\pm p_m) \partial_t \gamma_\pm \varphi_m
$$

$$
+ \int_0^T \int_\Gamma \Lambda_{f,\mathbf{n}} \Big(k_\pm(S_\pm(\gamma_\pm p_m)) \frac{[\![u]\!]_\pm^{+}}{d_f/2} + k_f(S_f(p_f)) \frac{[\![u]\!]_\pm^{-}}{d_f/2} \Big) [\![\varphi]\!]_\pm \Big\}
$$

$$
- \sum_{\mu\in\{\Omega,\Gamma,\pm\}} \int_\mu \phi_\mu S_\mu(p_\mu^0) \varphi_\mu^0 \quad = \quad 0
$$

The Gradient Discretization Framework

Gradient discretization framework: spacial discretization

- Vector space of discrete unknowns: $X_{\mathcal{D}}^0$
- Matrix and Fracture gradient reconstruction operators:
  - $\nabla_{\mathcal{D}}^m : X_{\mathcal{D}}^0 \to L^2(\Omega)^d$ and $\nabla_{\mathcal{D}}^f : X_{\mathcal{D}}^0 \to L^2(\Gamma)^{d-1}$
- Matrix and Fracture function reconstruction operators:
  - $\Pi_{\mathcal{D}}^m : X_{\mathcal{D}}^0 \to L^2(\Omega)$ and $\Pi_{\mathcal{D}}^f : X_{\mathcal{D}}^0 \to L^2(\Gamma)$
- Trace and Jump reconstruction operators at Γ:
  - $\mathbb{T}_{\mathcal{D}}^{\pm} : X_{\mathcal{D}}^0 \to L^2(\Gamma)$ and $[\![\cdot]\!]_{\pm,\mathcal{D}} : X_{\mathcal{D}}^0 \to L^2(\Gamma)$

[Eymard et al. 2010], [Droniou et al. 2013], [Eymard et al. 2012]

Space-time gradient discretizations

- Time discretization: $0 = t^0 < t^1 < \cdots < t^N = T$
- Backward Euler method

Weak Formulation of the Hybrid Dim Model

$$\sum_{\mu\in\{\Omega,\Gamma\}} \left\{ -\int_0^T \int_\mu d_\mu \phi_\mu S_\mu(p_\mu)\partial_t \varphi_\mu + \int_0^T \int_\mu d_\mu k_\mu(S_\mu(p_\mu))\, \Lambda_\mu \nabla u_\mu \cdot \nabla \varphi_\mu \right\}$$

$$+ \sum_{\pm} \left\{ -\int_0^T \int_\Gamma \eta S_\pm(\gamma_\pm p_m)\partial_t \gamma_\pm \varphi_m \right.$$

$$\left. + \int_0^T \int_\Gamma \Lambda_{f,\mathbf{n}} \left(k_\pm(S_\pm(\gamma_\pm p_m)) \frac{[\![u]\!]_\pm^{+}}{d_f/2} + k_f(S_f(p_f)) \frac{[\![u]\!]_\pm^{-}}{d_f/2} \right) [\![\varphi]\!]_\pm \right\}$$

$$- \sum_{\mu\in\{\Omega,\Gamma,\pm\}} \int_\mu \phi_\mu S_\mu(p_\mu^0)\varphi_\mu^0 \quad = \quad 0$$

Discrete Model

$$
\begin{aligned}
&\sum_{\mu\in\{\Omega,\Gamma\}} \left\{ \int_0^T \int_\mu d_\mu \phi_\mu \delta_t S_\mu(\Pi_\mathcal{D}^\mu p_\mathcal{D}) \Pi_\mathcal{D}^\mu v_\mathcal{D} \right. \\
&\qquad \left. + \int_0^T \int_\mu d_\mu k_\mu (S_\mu(\Pi_\mathcal{D}^\mu p_\mathcal{D}))\, \Lambda_\mu \nabla_\mathcal{D}^\mu u_\mathcal{D} \cdot \nabla_\mathcal{D}^\mu v_\mathcal{D} \right\} \\
&+ \sum_{\pm} \left\{ \int_0^T \int_\Gamma \eta \left[\delta_t S_\pm(\mathbb{T}_\mathcal{D}^\pm p_\mathcal{D}) \right] \mathbb{T}_\mathcal{D}^\pm v_\mathcal{D} \right. \\
&\qquad + \int_0^T \int_\Gamma \Lambda_{f,\mathbf{n}} \Bigg(k_\pm (S_\pm(\mathbb{T}_\mathcal{D}^\pm p_\mathcal{D})) \frac{[\![u_\mathcal{D}]\!]_{\pm,\mathcal{D}}^{+}}{d_f/2} \\
&\qquad\qquad \left. + k_f(S_f(\Pi_\mathcal{D}^f p_\mathcal{D})) \frac{[\![u_\mathcal{D}]\!]_{\pm,\mathcal{D}}^{-}}{d_f/2} \Bigg) [\![v_\mathcal{D}]\!]_{\pm,\mathcal{D}} \right\} \\
&= \quad 0
\end{aligned}
$$

Coercivity

- **Coercivity**: (discrete Poincaré inequality)
  - $C_{\mathcal{D}} = \max_{0 \neq v_{\mathcal{D}} \in X^0_{\mathcal{D}}} \dfrac{\|\Pi^m_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)} + \|\Pi^f_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Gamma)} + \sum_{\pm} \|\mathbb{T}^{\pm}_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Gamma)}}{\|v_{\mathcal{D}}\|_{\mathcal{D}}}$
  - For $\{\mathcal{D}^l\}_{l\in\mathbb{N}}$: $\quad C_{\mathcal{D}^l} \leq C_P < \infty$

Norm on $X^0_{\mathcal{D}}$: $\|v_{\mathcal{D}}\|_{\mathcal{D}} = \|\nabla^m_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)^d} + \|\nabla^f_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Gamma)^{d-1}} + \sum_{\pm} \|[\![v_{\mathcal{D}}]\!]_{\pm,\mathcal{D}}\|_{L^2(\Gamma)}$

Consistency

- **Consistency error**: for all $u = (u_m, u_f) \in V^0$
  - $\mathcal{S}_{\mathcal{D}}(u) =$

$$\min_{v_{\mathcal{D}} \in X^0_{\mathcal{D}}} \Big\{ \|\nabla^m_{\mathcal{D}} v_{\mathcal{D}} - \nabla u_m\|_{L^2(\Omega)^d} + \|\nabla^f_{\mathcal{D}} v_{\mathcal{D}} - \nabla_\tau u_f\|_{L^2(\Gamma)^{d-1}}$$

$$+\|\Pi^m_{\mathcal{D}} v_{\mathcal{D}} - u_m\|_{L^2(\Omega)} + \|\Pi^f_{\mathcal{D}} v_{\mathcal{D}} - u_f\|_{L^2(\Gamma)}$$

$$+\sum_{\pm} \Big(\|[\![v_{\mathcal{D}}]\!]_{\pm,\mathcal{D}} - [\![u]\!]_{\pm}\|_{L^2(\Gamma)} + \|\mathbb{T}^{\pm}_{\mathcal{D}} v_{\mathcal{D}} - \gamma_{\pm} u_m\|_{L^2(\Gamma)} \Big) \Big\}$$

  - For $\{\mathcal{D}^l\}_{l\in\mathbb{N}}$ $\quad (l \to \infty)$: $\qquad \mathcal{S}_{\mathcal{D}^l}(u) \to 0$

Limit Conformity

- **Conformity error**: for all $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$, $\varphi_\pm \in C_0^\infty(\Gamma)$

  - $\mathcal{W}_\mathcal{D}(\mathbf{q}, \varphi_\pm) =$

$$\sup_{\substack{v_\mathcal{D} \in X_\mathcal{D}^0 \\ \|v_\mathcal{D}\|=1}} \Bigg\{ \int_\Omega \Big(\nabla_\mathcal{D}^m v_\mathcal{D} \cdot \mathbf{q}_m + (\Pi_\mathcal{D}^m v_\mathcal{D}) \mathrm{div} \mathbf{q}_m \Big) - \sum_\pm \int_\Gamma \gamma_{\mathbf{n}^\pm} \mathbf{q}_m \mathbb{T}_\mathcal{D}^\pm v_\mathcal{D} \mathrm{d}\tau(\mathbf{x})$$

$$+ \int_\Gamma \Big(\nabla_\mathcal{D}^f v_\mathcal{D} \cdot \mathbf{q}_f + (\Pi_\mathcal{D}^f v_\mathcal{D}) \mathrm{div}_\tau \mathbf{q}_f \Big)$$

$$+ \sum_\pm \int_\Gamma \Big(\mathbb{T}_\mathcal{D}^\pm v_\mathcal{D} - \Pi_\mathcal{D}^f v_\mathcal{D} - [\![v_\mathcal{D}]\!]_{\pm,\mathcal{D}} \Big) \varphi_\pm \mathrm{d}\tau(\mathbf{x}) \Bigg\}$$

  - For $\{\mathcal{D}^l\}_{l \in \mathbb{N}}$ $(l \to \infty)$: $\mathcal{W}_{\mathcal{D}^l}(\mathbf{q}, \varphi_\pm) \to 0$

Compactness

- **Compactness** (in space): $(\boldsymbol{\xi} = (\boldsymbol{\xi}_m, \boldsymbol{\xi}_f) \in \mathbb{R}^d \times \mathbb{R}^{d-1})$
  - $\mathcal{T}_{\mathcal{D}}(\boldsymbol{\xi}) =$

$$\sup_{\substack{v_{\mathcal{D}} \in X^0_{\mathcal{D}} \\ \|v_{\mathcal{D}}\|=1}} \Big\{ \|\Pi^m_{\mathcal{D}} v_{\mathcal{D}}(\cdot + \boldsymbol{\xi}_m) - \Pi^m_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)} + \|\Pi^f_{\mathcal{D}} v_{\mathcal{D}}(\cdot + \boldsymbol{\xi}_f) - \Pi^f_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Gamma)}$$

$$+ \sum_{\pm} \|\mathbb{T}^{\pm}_{\mathcal{D}} v_{\mathcal{D}}(\cdot + \boldsymbol{\xi}_f) - \mathbb{T}^{\pm}_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Gamma)} \Big\}$$

  - For $\{\mathcal{D}^l\}_{l\in\mathbb{N}}$: $\lim_{|\boldsymbol{\xi}|\to 0} \sup_{l\in\mathbb{N}} \mathcal{T}_{\mathcal{D}^l}(\boldsymbol{\xi}) = 0$

Convergence Result

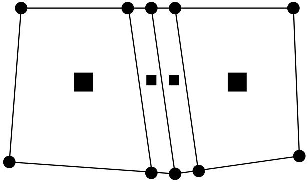
- For coercive, consistant, limit conforming, compact gradient schemes:

$$
\begin{cases}
(\Pi^m_{\mathcal{D}^l}, \Pi^f_{\mathcal{D}^l}) u_{\mathcal{D}^l} \rightharpoonup (u_m, u_f) & \text{in } L^2((0,T)\times\Omega) \times L^2((0,T)\times\Gamma) \\
(\nabla^m_{\mathcal{D}^l}, \nabla^f_{\mathcal{D}^l}) u_{\mathcal{D}^l} \rightharpoonup (\nabla u_m, \nabla_\tau u_f) & \text{in } L^2((0,T)\times\Omega)^d \times L^2((0,T)\times\Gamma)^{d-1} \\
\mathbb{T}^{\pm}_{\mathcal{D}^l} u_{\mathcal{D}^l} \rightharpoonup \gamma_\pm u_m & \text{in } L^2((0,T)\times\Gamma) \\
[\![u_{\mathcal{D}^l}]\!]_{\pm,\mathcal{D}^l} \rightharpoonup [\![u]\!]_\pm & \text{in } L^2((0,T)\times\Gamma) \\
(S_m(\Pi^m_{\mathcal{D}^l} p_{\mathcal{D}^l}), S_f(\Pi^f_{\mathcal{D}^l} p_{\mathcal{D}^l})) & \\
\quad \to (S_m(p_m), S_f(p_f)) & \text{in } L^2((0,T)\times\Omega) \times L^2((0,T)\times\Gamma) \\
\mathbb{T}^{\pm}_{\mathcal{D}^l} S_\pm(p_{\mathcal{D}}) \to S_\pm(\gamma_\pm p_m) & \text{in } L^2((0,T)\times\Gamma)
\end{cases}
$$

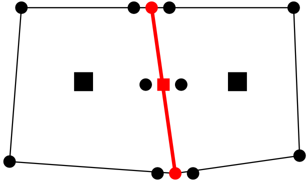
Numerical Tests

Comparison of the Models d.o.f.

- Equi dim



- Hybrid dim



Vertex Approximate Gradient (VAG) Discret.: [Eymard et al. 10], [Brenner et al. 16]

Comparison of equi- and hybrid-dimensional models

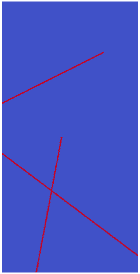
- $\Omega = (0, 400) \times (0, 800)$ m
- Equi-dimensional mesh: 22500 triangles
- Hybrid dimensional mesh: 16900 triangles
- Matrix:

$$\phi_m = 0.2, \quad \Lambda_m \text{ isotropic}$$

- Fractures:

$$d_f = 4m, \quad \phi_f = 0.4, \quad \Lambda_f \text{ isotropic}$$

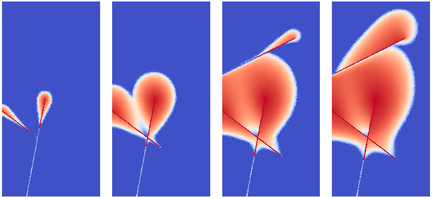
- Injection of oil in the bottom fracture
- Initially saturated with water



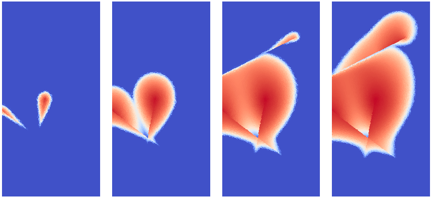
Drains: $\Lambda_f / \Lambda_m = 1000$

Capillary Pressure: $p_m = -10^5 \ln(S_m^w); \; p_f = 0$

- Equi dim



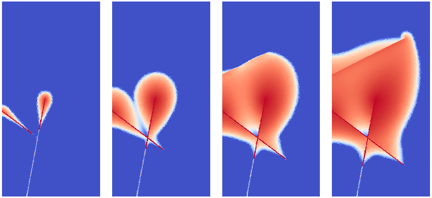
- Hybrid dim



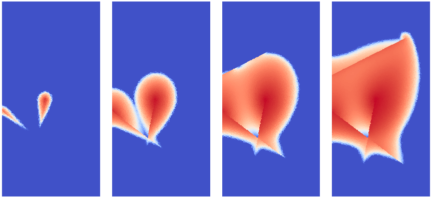
Drain-Barrier: $\Lambda_f^{drain}/\Lambda_m = 1000$; $\Lambda_f^{barrier}/\Lambda_m = 0.01$

Capillary Pressure: $p_m = -10^5 \ln(S_m^w)$; $p_f = 0$

- Equi dim

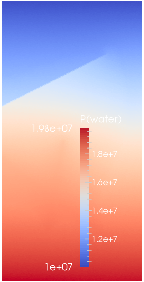


- Hybrid dim

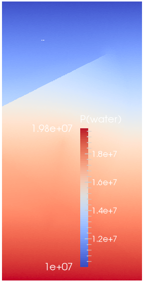


Drain-Barrier: $\Lambda_f^{drain}/\Lambda_m = 1000$; $\Lambda_f^{barrier}/\Lambda_m = 0.01$

Equi dim.



Hybrid dim



Computational Performance

Model	**Nb Cells**	**Nb dof**	**Nb dof el.**
equi dim.	22477	45315	22838
hybrid dim.	16889	35355	18466

Model	$\mathbf{N}_{\Delta t}$	$\mathbf{N}_{Newton}$	$\mathbf{N}_{GMRes}$	$\mathbf{N}_{Chop}$	**CPU**
Test Drains					
equi dim.	3054	18993	425182	406	30697
hybrid dim.	1530	7839	75220	20	4123
Test Drain-Barrier					
equi dim.	2777	15518	227961	376	24199
hybrid dim.	1305	6444	63022	9	3546

Conclusion and Perspectives

- Discrete fracture models + VAG scheme
  - pressure discontinuity at matrix-fracture interfaces
  - discontinuous capillary pressure
  - polyhedral meshes
  - saturation stratification inside the DFN
- Gradient Discretization Framework
  - convergence results

Citations

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Thankings

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