Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors

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Outline

Laplace eigenvalue problem

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, polygon/polyhedron

Energy minimization

Find $u_1 \in V := H_0^1(\Omega)$ such that $(u_1, 1) > 0$ and

$$u_1 := \arg \min_{v \in V, \|v\|=1} \left\{ \frac{1}{2} \|\nabla v\|^2 \right\}.$$

Strong formultion

Find eigenvector & eigenvalue pair (u_1, λ_1) such that

$$\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 && \text{in } \Omega, \\ u_1 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Full problem

Weak formulation of the full problem

Find $(u_k, \lambda_k) \in V \times \mathbb{R}^+$, $k \geq 1$, with $\|u_k\| = 1$, such that

$$(\nabla u_k, \nabla v) = \lambda_k (u_k, v) \qquad \forall v \in V.$$

Comments

- take $v = u_k$ as test function $\Rightarrow \|\nabla u_k\|^2 = \lambda_k$
- $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \to \infty$
- u_k, $k \geq 1$, form an orthonormal basis of $L^2(\Omega)$

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Previous results, Laplace eigenvalue bounds

- Plum (1997), Goerisch and He (1989), Still (1988), Kuttler and Sigillito (1978), Moler and Payne (1968), Fox and Rheinboldt (1966), Bazley and Fox (1961), Weinberger (1956), Forsythe (1955), Kato (1949)
- ...

Previous results, guaranteed lower bounds on λ_1

- Carstensen and Gedicke (2014) & Liu (2015): $\oplus$ guaranteed bound, arbitrarily coarse mesh; $\ominus$ a priori arguments (largest mesh element diameter), only lowest-order nonconforming FEs
- Hu, Huang, Lin (2014): $\oplus$ bounds in nonconforming FEs; $\ominus$ saturation assumption may be necessary
- Armentano and Durán (2004): $\oplus$ bounds in nonconforming FEs; $\ominus$ only asymptotic
- Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013): $\oplus$ general guaranteed bounds; $\ominus$ condition on applicability, suboptimal convergence speed
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- Rannacher, Westenberger, Wollner (2010), Grubišić and Ovall (2009), Durán, Padra, Rodríguez (2003), Heuveline and Rannacher (2002), Larson (2000), Maday and Patera (2000), Verfürth (1994) ...
- ...typically contain uncomputable terms, higher-order on fine enough meshes
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The game

Assumption A (Conforming variational solution)

There holds

- $(u_h, \lambda_h) \in V \times \mathbb{R}^+$
- $\|u_h\| = 1$
- $\|\nabla u_h\|^2 = \lambda_h$ $(\Rightarrow \lambda_h \geq \lambda_1)$

We want to estimate

1. first eigenvalue error

2. first eigenvector energy error

$$\|\nabla(u_1 - u_h)\| \quad (u_h, \lambda_h)$$

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$$\|\nabla(u_1 - u_h)\| \leq \eta(u_h, \lambda_h) \leq C_{\text{eff}} \|\nabla(u_1 - u_h)\|$$

- C_{eff} only depends on mesh shape regularity and on $(1 - \frac{\lambda_1}{\lambda_2})$
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The pathway

1. estimate the $L^2(\Omega)$ error:

$$\|u_1 - u_h\| \leq \alpha_h$$

2. prove equivalence of the eigenvalue & eigenvector errors:

$$C\|\nabla(u_1 - u_h)\|^2 \leq \lambda_h - \lambda_1 \leq \|\nabla(u_1 - u_h)\|^2$$

3. prove equivalence of the eigenvector error & of the dual norm of the residual:

$$\underline{C}\|\mathrm{Res}(u_h, \lambda_h)\|_{-1} \leq \|\nabla(u_1 - u_h)\| \leq \overline{C}\|\mathrm{Res}(u_h, \lambda_h)\|_{-1},$$

where

$$\langle \mathrm{Res}(u_h, \lambda_h), v\rangle_{V',V} := \lambda_h(u_h, v) - (\nabla u_h, \nabla v) \qquad v \in V$$

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4. prove equivalence of the dual residual norm & its estimate:

$$\tilde{C}\eta(u_h, \lambda_h) \leq \overline{C}\|\mathrm{Res}(u_h, \lambda_h)\|_{-1} \leq \eta(u_h, \lambda_h)$$

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Lemma ($L^2(\Omega)$ bound via a quadratic residual inequality)

Let Assumption A hold and let

$$\lambda_h < \lambda_2$$

and

$$(u_1, u_h) \geq 0.$$

Then

$$\|u_1 - u_h\| \leq \alpha_h := \sqrt{2}\left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \|\mathscr{r}_{(h)}\|.$$

Riesz representation of the residual $\mathscr{r}_{(h)} \in V$

$$(\nabla \mathscr{r}_{(h)}, \nabla v) = \langle \mathrm{Res}(u_h, \lambda_h), v\rangle_{V',V}$$

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weak solution, residual, and Riesz representation definitions:

$$(\mathfrak{r}_{(h)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla \mathfrak{r}_{(h)}) = \frac{1}{\lambda_k}\left(\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)\right)$$
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Parseval equality for $\mathfrak{r}_{(h)}$

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assumption $\lambda_h < \lambda_2$:

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $u_h - u_1$, $(u_h - u_1, u_1) = -\frac{1}{2}\|u_1 - u_h\|^2$:

$$\|r_{(h)}\|^2 \geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h\|u_1 - u_h\|^2 - \frac{C_h}{4}\|u_1 - u_h\|^4$$

dropping the first term above, $e_h := \|u_1 - u_h\|^2$:

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|r_{(h)}\|^2 \geq 0$$

assumption $(u_1, u_h) \geq 0$, employing $\|u_1\| = \|u_h\| = 1$:

$$e_h = \|u_1 - u_h\|^2 = 2 - 2(u_1, u_h) \leq 2,$$

conclusion:

$$\frac{C_h}{2} e_h \leq \|r_{(h)}\|^2 \quad \Leftrightarrow \quad \|u_1 - u_h\| \leq \sqrt{2} C_h^{-\frac{1}{2}} \|r_{(h)}\|$$

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$$\frac{C_h}{4} e_h^2 - C_h e_h + \|\mathscr{r}_{(h)}\|^2 \geq 0$$

assumption $(u_1, u_h) \geq 0$, employing $\|u_1\| = \|u_h\| = 1$:

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conclusion:

$$\frac{C_h}{2} e_h \leq \|\mathscr{r}_{(h)}\|^2 \quad \Leftrightarrow \quad \|u_1 - u_h\| \leq \sqrt{2} C_h^{-\frac{1}{2}} \|\mathscr{r}_{(h)}\|$$

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $u_h - u_1$, $(u_h - u_1, u_1) = -\frac{1}{2}\|u_1 - u_h\|^2$:

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Outline

First eigenvalue error equivalences

Theorem (Eigenvalue error – eigenvector error equivalence)

Under the above assumptions, there holds

$$\frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\left(1-\frac{\alpha_h^2}{4}\right)\|\nabla(u_1-u_h)\|^2 \le \lambda_h-\lambda_1 \le \|\nabla(u_1-u_h)\|^2,$$

as well as

$$\|\nabla(u_1-u_h)\|^2-\lambda_1\alpha_h^2 \le \lambda_h-\lambda_1.$$

Key arguments of the proof

- there holds

$$\lambda_h-\lambda_1 = \|\nabla(u_h-u_1)\|^2-\lambda_1\|u_1-u_h\|^2$$

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Under the above assumptions, there holds

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Outline

How to bound the dual residual norm?

Dual norm of the residual

$$\|\mathrm{Res}(u_h,\lambda_h)\|_{-1} = \sup_{v\in V,\, \|\nabla v\|=1} \langle \mathrm{Res}(u_h,\lambda_h), v\rangle_{V',V}$$
$$= \sup_{v\in V,\, \|\nabla v\|=1} \{\lambda_h(u_h,v) - (\nabla u_h, \nabla v)\}$$

Guaranteed upper bound: $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\sigma}_h = \lambda_h u_h$

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Equilibrated flux construction

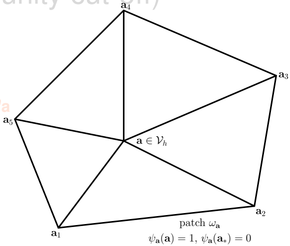
Ideal equilibrated flux reconstruction $(-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega))$

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \lambda_h u_h} \|\nabla u_h + \mathbf{v}_h\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega) \Rightarrow$ **global** minimization, too expensive

Equilibrated flux reconstruction (partition of unity cut-off)

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\overbrace{\psi_{\mathbf{a}}}^{\text{hat function}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$



- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}$, **local** minimizations
- $\boldsymbol{\sigma}_h$ is a $\mathbf{H}(\mathrm{div}, \Omega)$-conforming lifting of the residual

Destuynder & Métivet (1999), Braess & Schöberl (2008)

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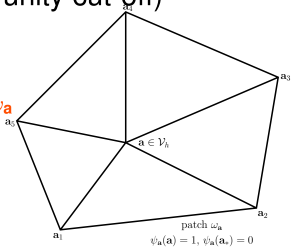

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$H_0^1(\Omega)$- and $\mathbf{H}(\mathrm{div},\Omega)$-conforming local residual liftings

Definition (Mixed local Neumann problems: equilibrated flux)

For all $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ by solving

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Definition (Conforming local Neumann problems: lifted residual)

For each $\mathbf{a} \in \mathcal{V}_h$, define $r_h^{\mathbf{a}} \in X_h^{\mathbf{a}} \subset H^1(\omega_{\mathbf{a}})$ by

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \mathrm{Res}(u_h,\lambda_h), \psi_{\mathbf{a}} v_h\rangle_{V',V} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

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Babuška & Strouboulis (2001), Repin (2008)

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Numerical assumptions

Assumption B (Galerkin orthogonality of the residual to $\psi_{\mathbf{a}}$)

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Assumption C (Shape regularity & piecewise polynomial form)

The meshes $\mathcal{T}_h$ are shape regular. There holds

$u_h \in \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, *and spaces* $\mathbf{V}_h \times Q_h$ *are of degree* $p+1$.

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Theorem (Dual norm of the residual equivalences)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ verifying Assumption B be arbitrary. Then

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Moreover, under Assumption C, there holds

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- C_{st} and $C_{\mathrm{cont,PF}}$ independent of the polynomial degree p
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$$\frac{\langle \mathrm{Res}(u_h, \lambda_h), r_h \rangle_{V',V}}{\|\nabla r_h\|} \leq \|\mathrm{Res}(u_h, \lambda_h)\|_{-1} \leq \|\nabla u_h + \boldsymbol{\sigma}_h\|.$$

Moreover, under Assumption C, there holds

$$\|\nabla u_h + \boldsymbol{\sigma}_h\| \leq (d+1) C_{\mathrm{st}} C_{\mathrm{cont,PF}} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}.$$

- C_{st} and $C_{\mathrm{cont,PF}}$ independent of the polynomial degree p
- we can compute upper bounds on C_{st} and $C_{\mathrm{cont,PF}}$

Ern & Vohralík (2015)

Outline

Bounds on the Riesz representation of the residual

Lemma (Poincaré–Friedrichs bound on $\|r_{(h)}\|$)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ be arbitrary. There holds

$$\|r_{(h)}\| \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla r_{(h)}\|.$$

Lemma (Elliptic regularity bound on $\|r_{(h)}\|$)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ satisfy Assumption B and let the solution $\zeta_{(h)}$ of

$$(\nabla \zeta_{(h)}, \nabla v) = (r_{(h)}, v) \qquad \forall v \in V$$

belong to $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, with

$$\inf_{v_h \in V_h} \|\nabla(\zeta_{(h)} - v_h)\| \leq C_{\mathrm{I}} h^{\delta} |\zeta_{(h)}|_{H^{1+\delta}(\Omega)},$$

$$|\zeta_{(h)}|_{H^{1+\delta}(\Omega)} \leq C_{\mathrm{S}} \|r_{(h)}\|.$$

Then

$$\|r_{(h)}\| \leq C_{\mathrm{I}} C_{\mathrm{S}} h^{\delta} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}.$$

European Research Council

Bounds on the Riesz representation of the residual

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Bounds on the Riesz representation of the residual

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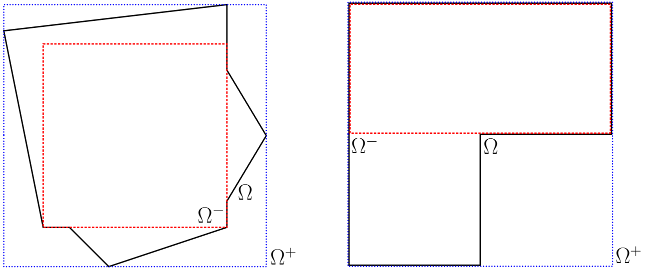
Then

$$\|\mathscr{r}_{(h)}\| \leq C_{\mathrm{I}} C_{\mathrm{S}} h^{\delta} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}.$$

How to guarantee $\lambda_h < \lambda_2$?

Option 1: estimates of eigenvalues via domain inclusion



$$\begin{aligned} \Omega \subset \Omega^+ \quad &\Rightarrow \quad \lambda_k \geq \lambda_k(\Omega^+), \\ \Omega \supset \Omega^- \quad &\Rightarrow \quad \lambda_k \leq \lambda_k(\Omega^-), \end{aligned} \qquad \forall k \geq 1$$

Option 2: computational estimates

- Carstensen and Gedicke (2014)
- Liu (2015)

Sign condition and practical $L^2(\Omega)$ bound

Lemma (Sign condition and practical $L^2(\Omega)$ bound (no elliptic regularity))

Let $\lambda_h < \underline{\lambda_2} \le \lambda_2$ *and* $(u_h, 1) > 0$. *Let*

$$\overline{\alpha}_h := \sqrt{2}\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-1} \underline{\lambda_2}^{-\frac{1}{2}} \|\nabla u_h + \boldsymbol{\sigma}_h\| \le \min\left\{\sqrt{2}, |\Omega|^{-\frac{1}{2}}(u_h, 1)\right\}.$$

Then $(u_1, u_h) \ge 0$ *and* $\|u_1 - u_h\| \le \overline{\alpha}_h$.

Outline

Outline

Guaranteed bounds for the first eigenvalue

Theorem (Eigenvalue bounds)

Let $0 < \underline{\lambda}_2 \le \lambda_2$, $0 < \underline{\lambda}_1 \le \lambda_1$, $\lambda_h < \underline{\lambda}_2$, *and* $(u_h, 1) > 0$. *Let Assumptions A and B hold, and construct* σ_h *and* r_h. *Then*

$$\tilde{\eta} \le \sqrt{\lambda_h - \lambda_1} \le \eta,$$

where

$$\eta^2 := \overbrace{\left(1 - \frac{\lambda_h}{\underline{\lambda}_2}\right)^{-2}}^{= 1 \text{ under elliptic reg.}} \overbrace{\left(1 - \frac{\overline{\alpha}_h^2}{4}\right)^{-1}}^{\searrow 1} \|\nabla u_h + \sigma_h\|^2,$$

$$\tilde{\eta}^2 := \frac{1}{2}\left(1 - \frac{\lambda_h}{\underline{\lambda}_2}\right)\left(1 - \frac{\overline{\alpha}_h^2}{4}\right)\frac{\lambda_1}{2}\left(\sqrt{1 + \frac{4}{\underline{\lambda}_1}\frac{\langle \mathrm{Res}(u_h, \lambda_h), r_h\rangle_{V',V}^2}{\|\nabla r_h\|^2}} - 1\right).$$

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Outline

1. Introduction
2. Laplace eigenvalue problem equivalences
   - Generic equivalences
   - Dual norm of the residual equivalences
   - Representation of the residual and eigenvalue bounds
3. A posteriori estimates
   - Eigenvalues
   - Eigenvectors
4. Application to conforming finite elements
5. Numerical experiments
6. Extension to nonconforming discretizations
7. Conclusions and future directions

Guaranteed bounds for the first eigenvector

Theorem (Eigenvector bounds)

Under the assumptions of the eigenvalue theorem,

$$\|\nabla(u_1 - u_h)\| \leq \eta.$$

Moreover, under Assumption C,

$$\eta \leq (d+1) C_{\mathrm{cont,PF}} C_{\mathrm{st}} \underbrace{\left(\frac{\|\nabla(u_1-u_h)\|^2}{\lambda_1} + 1 \right)^{\frac{1}{2}}}_{\searrow 1}$$

$$\underbrace{\left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1}}_{\searrow \left(1 - \frac{\lambda_1}{\lambda_2}\right)^{-1}} \underbrace{\left(1 - \frac{\underline{\alpha}_h^2}{4}\right)^{-\frac{1}{2}}}_{\searrow 1} \|\nabla(u_1 - u_h)\|.$$

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Comments

Eigenvalue bounds

- **guaranteed**
- **optimally convergent**
- **improvement of the upper bound**

Eigenvector bounds

- **efficient** and **polynomial-degree robust**
- $\|\nabla u_h + \sigma_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \sigma_h\|_K^2 \Rightarrow$ **adaptivity-ready**
- **maximal overestimation guaranteed**

Three settings

- no applicability condition (fine mesh, approximate solution)
- improvements for explicit, a posteriori verifiable conditions
- multiplicative factor goes to one under elliptic regularity

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Outline

Application to conforming finite elements

Finite element method

Find $(u_h, \lambda_h) \in V_h \times \mathbb{R}^+$ with $\|u_h\| = 1$ and $(u_h, 1) > 0$, where $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V$, $p \geq 1$, such that,

$$(\nabla u_h, \nabla v_h) = \lambda_h(u_h, v_h) \qquad \forall v_h \in V_h.$$

Assumptions verification

- $V_h \subset V$
- $\|u_h\| = 1$ and $(u_h, 1) > 0$ by definition
- $\|\nabla u_h\|^2 = \lambda_h$ follows upon taking $v_h = u_h$ ($\Rightarrow$ Assumption A)
- Assumption B follows upon taking $v_h = \psi_{\mathbf{a}} \in V_h$
- Assumption C satisfied

Application to conforming finite elements

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Outline

Unit square

Setting

- $\Omega = (0,1)^2$
- $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$ known explicitly
- $u_1(x,y) = \sin(\pi x)\sin(\pi y)$ known explicitly

Parameters

- convex domain: $C_{\mathrm{S}} = 1$, $\delta = 1$, $C_{\mathrm{I}} \approx 1/\sqrt{8}$
- $\underline{\lambda_1} = 1.5\pi^2$, $\underline{\lambda_2} = 4.5\pi^2$

Effectivity indices

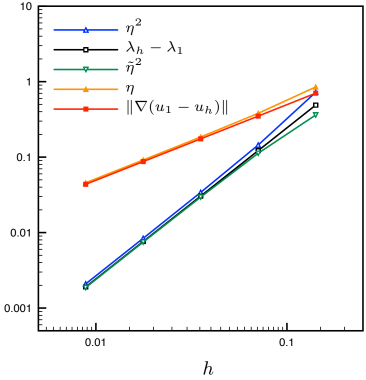
- recall $\tilde{\eta}^2 \leq \lambda_h - \lambda_1 \leq \eta^2$

$$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}} := \frac{\lambda_h - \lambda_1}{\tilde{\eta}^2}, \qquad I^{\mathrm{ub}}_{\lambda,\mathrm{eff}} := \frac{\eta^2}{\lambda_h - \lambda_1}$$

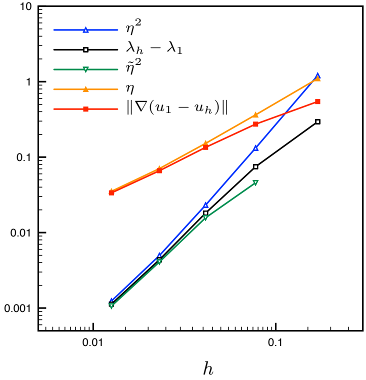
- recall $\|\nabla(u_1 - u_h)\| \leq \eta$

$$I^{\mathrm{ub}}_{u,\mathrm{eff}} := \frac{\eta}{\|\nabla(u_1 - u_h)\|}$$

Eigenvalue and eigenvector errors and estimators



Structured meshes



Unstructured meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Structured meshes

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Unstructured meshes

L-shaped domain

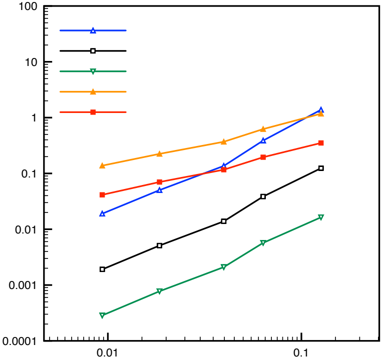
Setting

- $\Omega := (-1,1)^2 \setminus [0,1] \times [-1,0]$
- $\lambda_1 \approx 9.6397238440$

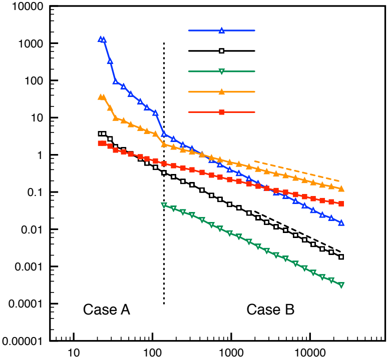
Parameters

- $\underline{\lambda_1} = \pi^2/2$ and $\underline{\lambda_2} = 5\pi^2/4$ by inclusion into the square $(-1,1)^2$

Eigenvalue and eigenvector errors and estimators



Unstructured meshes



Adaptively refined meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
50	0.0634	2286	9.6397	9.6783	9.2904	9.6726	6.77	10.06	4.03e-02	3.19
100	0.0397	8691	9.6397	9.6536	9.5173	9.6515	6.61	9.84	1.40e-02	3.17
200	0.0185	34206	9.6397	9.6448	9.5946	9.6440	6.59	9.85	5.14e-03	3.20
400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

Unstructured meshes

Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	140	9.6397	9.9700	6.3175	9.9260	7.50	11.06	4.44e-01	3.31
15	561	9.6397	9.7207	9.0035	9.7075	6.17	8.86	7.53e-02	2.98
20	2188	9.6397	9.6601	9.4887	9.6566	5.88	8.43	1.75e-02	2.88
25	8513	9.6397	9.6449	9.6019	9.6440	5.77	8.31	4.37e-03	2.75
30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

Adaptively refined meshes

Outline

Nonconforming discretizations

Nonconforming setting

- $u_h \notin V$, $\|u_h\| \neq 1$
- $\|\nabla u_h\|^2 \neq \lambda_h$

Main tool

- conforming eigenvector reconstruction

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in W_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}, \qquad s_h := \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}$$

Unified framework

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin elements
- mixed finite elements

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Unified framework

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin elements
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SIPG: square

N	h	ndof	λ_1	λ_h	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2} - \eta^2$	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1414	600	19.7392	20.0333	19.1803	20.0101	4.23e-02	1.93
20	0.0707	2400	19.7392	19.8169	19.6907	19.8099	6.03e-03	1.50
40	0.0354	9600	19.7392	19.7591	19.7324	19.7572	1.26e-03	1.37
80	0.0177	38400	19.7392	19.7442	19.7378	19.7438	2.99e-04	1.34
160	0.0088	153600	19.7392	19.7405	19.7389	19.7403	7.09e-05	1.33

Structured meshes

N	h	ndof	λ_1	λ_h	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2} - \eta^2$	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1698	732	19.7392	19.9432	17.8788	19.9501	1.10e-01	3.26
20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

Unstructured meshes

SIPG: L-shape

N	h	ndof	λ_1	λ_h	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2} - \eta^2$	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
5	0.7165	90	9.6397	10.7897	-128.5909	11.0700	-2.38e+00	9.32
10	0.3041	492	9.6397	9.9085	-3.4330	9.9928	4.09e+00	6.36
20	0.1670	2058	9.6397	9.7044	8.3596	9.7448	1.53e-01	3.97
40	0.0839	8136	9.6397	9.6576	9.2512	9.6729	4.46e-02	3.90
80	0.0459	33078	9.6397	9.6447	9.5110	9.6506	1.46e-02	3.92
160	0.0234	129342	9.6397	9.6413	9.5929	9.6436	5.27e-03	3.92

Unstructured meshes

Level	ndof	λ_1	λ_h	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2} - \eta^2$	$\frac{\|\nabla s_h\|^2}{\|s_h\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
5	186	9.6397	10.2136	-30.6026	10.3629	-4.05e+00	7.19
10	777	9.6397	9.8154	7.2388	9.8388	3.04e-01	3.75
15	3453	9.6397	9.6865	9.1572	9.6902	5.66e-02	3.38
20	14706	9.6397	9.6509	9.5335	9.6517	1.23e-02	3.23
25	61137	9.6397	9.6425	9.6144	9.6426	2.93e-03	3.00

Adaptively refined meshes

Outline

Conclusions and future directions

Conclusions

- guaranteed upper and lower bounds for the first eigenvalue
- guaranteed and polynomial-degree robust bounds for the associated eigenvector
- general framework

Ongoing work

- extension to nonlinear eigenvalue problems

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Thank you for your attention!